Chapter 0

Introduction

Freese and Nation ([13]) used a property of partial orders which is now called Freese-Nation property (FN) in order to characterize projective lattices. Projective Boolean algebras have this property. Heindorf ([23]) characterized the Boolean algebras with the FN as the rc-filtered Boolean algebras. These algebras are sometimes called openly generated. In the book by Heindorf and Shapiro ([23]) a generalization of the FN is considered, the weak Freese-Nation property (WFN). Heindorf ([23]) characterized the Boolean algebras with the WFN as being σ-filtered. Fuchino, Koppelberg, and Shelah ([16]) introduced a further generalization of the FN, the κ-Freese-Nation property (κ-FN), for any regular cardinal κ. Their approach is more set-theoretic than Heindorf's, but implicitly they proved that for all partial orders the κ-FN is equivalent to what would be called κ-filteredness. rc-filteredness is $\aleph_0$-filteredness and σ-filteredness is $\aleph_1$-filteredness. Roughly speaking, a partial order is κ-filtered iff it has many nicely embedded suborders. How nice these embeddings are, depends on κ. The smaller κ, the nicer the embeddings. A partial order $(P, \leq)$ has the κ-Freese-Nation property iff there is a function $f : P \to [P]^{<\kappa}$ such that for all $a, b \in P$ with $a \leq b$ there is $c \in f(a) \cap f(b)$ with $a \leq c \leq b$. Every partial order of size $\leq \kappa$ has the κ-FN. FN is $\aleph_0$-FN and WFN is $\aleph_1$-FN. For a partial order P let WFN(P) denote the statement 'P has the WFN'. The study of the κ-FN, especially for $\kappa = \aleph_1$, was continued by Fuchino, Koppelberg, Shelah, and Soukup in [17] and [19].

Koppelberg ([28]) introduced and studied the notion of tight σ-filteredness of a Boolean algebra, which generalizes projectivity. Using this notion, she gave uniform proofs of several mostly known results about the existence of certain homorphisms into countably complete Boolean algebras. Tight σ-filteredness is a strengthening of the WFN, in the same way as projectivity strengthens the FN. Every Boolean algebra of size $\leq \aleph_1$ which has the FN is projective. Similarly, every Boolean algebra of size $\leq \aleph_2$ which has the WFN is tighly σ-filtered.

My research concerning tight σ-filtrations was initiated by a list of questions addressed by Fuchino. The first task was to give a usable characterization of tight σ-filteredness. The relation between tight σ-filteredness and σ-filteredness is very similar to the relation between projectivity and rc-filteredness. However, while projective Boolean algebras are precisely the retracts of free Boolean algebras, a similar characterization of tightly σ-filtered Boolean algebras does not seem to be available. But as it turns out, tightly σ-filtered Boolean algebras can be characterized in a similar way as projective Boolean algebras have been characterized by Ščepin, Haydon, and Koppelberg. (See [23] or [29].) This characterization of tight σ-filteredness can be used to get some results on the Stone spaces of tightly σ-filtered Boolean algebras. The parallel results for projective Boolean algebras were used by Koppelberg ([29]) to show that for every uncountable regular cardinal λ there are only $2^{<\lambda}$ isomorphism types of projective Boolean algebras of size λ. This does not hold for tightly σ-filtered Boolean algebras. For every infinite cardinal λ there are 2^λ pairwise non-isomorphic tightly σ-filtered Boolean algebras of size λ.

One of the main reasons why the WFN and tight σ-filteredness are interesting is that in some models of set theory infinite complete Boolean algebras can have these properties. This is not the case with projectivity or FN. It was shown by Fuchino, Koppelberg, and Shelah ([16]) that adding a small number of Cohen reals to a model of CH results in a model of WFN($\mathfrak{P}(\omega)$). Fuchino and Soukup ([19]) later extended this result showing that adding any number of Cohen reals to a model of CH+$\neg 0^\sharp$ yields a model of WFN($\mathfrak{P}(\omega)$). $\mathfrak{P}(\omega)$ plays an important role considering questions about the WFN of complete Boolean algebras since it is a retract of every infinite complete Boolean algebra and the WFN is hereditary with respect to retracts. In short, if any infinite Boolean algebra has the WFN, then so does $\mathfrak{P}(\omega)$. Using the characterization mentioned above, it turns out that the same is true for tight σ-filteredness. Fuchino, Koppelberg, and Shelah ([16]) observed that WFN($\mathfrak{P}(\omega)$) implies that the unboundness number $\mathfrak{b}$ is $\aleph_1$. It follows that the question whether there are any infinite complete Boolean algebras with the WFN cannot be answered in ZFC alone.

One of Fuchino's questions about tight σ-filteredness was whether it is consistent that $\mathfrak{P}(\omega)$ is tightly σ-filtered while the continuum is $\geq \aleph_3$. The only reason for $\mathfrak{P}(\omega)$ being tightly σ-filtered known so far is WFN($\mathfrak{P}(\omega)$) together with $2^{\aleph_0} \leq \aleph_2$. Investigating whether $\mathfrak{P}(\omega)$ is tightly σ-filtered in certain models of set theory, I noticed that it is even difficult to get models of $\neg$ CH + WFN($\mathfrak{P}(\omega)$), apart from starting with a model of CH and extending the continuum by adding Cohen reals. This led to a systematic study of WFN($\mathfrak{P}(\omega)$) in various models of set theory. Together with Fuchino and Soukup, I found that if WFN($\mathfrak{P}(\omega)$) holds, then, as far as the reals are concerned, the universe behaves very similar to a model of set theory that was obtained by adding Cohen reals to a model of CH.

While it is quite easy to see that WFN($\mathfrak{P}(\omega)$) implies WFN($\mathfrak{P}(\omega)/fin$) and WFN($\mathbb{C}(\omega)$), where $\mathbb{C}(\omega)$ is the Cohen algebra, i.e. the completion of the countably generated free Boolean algebra, it is not so clear whether WFN($\mathfrak{P}(\omega)$) also implies WFN($\mathbb{R}(\omega)$), where $\mathbb{R}(\omega)$ is the measure algebra of the Cantor space. It does, however. If the universe is not too bad, that is, if $0^\sharp$ does not exist, then WFN($\mathfrak{P}(\omega)$) even implies that all measure algebras have the WFN and the class of complete Boolean algebras with the WFN has nice closure properties. The argument used here is similar to an argument used by Fuchino and Soukup ([19]) in order to get their result about WFN($\mathfrak{P}(\omega)$) in Cohen extensions and to obtain a nice characterization of partial orders with the WFN. It was shown in [16] that all complete Boolean algebras A with WFN(A) satisfy the c.c.c. In [19] it was proved that if $0^\sharp$ does not exist and CH holds, then WFN(A) holds for all complete c.c.c. Boolean algebras A. Moreover, under CH, for all complete c.c.c. Boolean algebras A of size $< \aleph_\omega$, WFN(A) holds. This together with the fact that under $\neg 0^\sharp$ the class of complete Boolean algebras with the WFN has nice closure properties contrasts with some recent results of Soukup. He proved that if the existence of a supercompact cardinal is consistent with ZFC, then it is also consistent that GCH holds, but there is a complete c.c.c. Boolean algebra without the WFN. Using a similar argument, he also proved that it is consistent with ZFC that WFN($\mathfrak{P}(\omega)$) holds, but there is a complete c.c.c. Boolean algebra of size $\aleph_2$ not having the WFN.

0.1 Overview

In the first chapter I introduce the basic notions for this thesis such as tight κ-filteredness and κ-FN and recall the known results. At some places I give straightforward generalizations of known results. Tight κ-filteredness is a generalization of Koppelberg's tight σ-filteredness. Tight σ-filteredness is tight $\aleph_1$-filteredness.

The second chapter deals with tightly κ-filtered Boolean algebras. κ-FN and tight κ-filteredness are equivalent for Boolean algebras of size $\leq \kappa^+$. Any tightly κ-filtered Boolean algebra has the κ-FN.

I give a characterization of tightly κ-filtered Boolean algebras which is similar to the characterization of projective Boolean algebras developed by to Haydon, Koppelberg, and Ščepin. (See [23] or [29].) I show that for every infinite cardinal κ the number of tightly σ-filtered Boolean algebras of size κ is precisely 2^κ, contrasting the result of Koppelberg ([29]) that there are only $2^{<\kappa}$ projective Boolean algebras of size κ for every regular $\kappa > \aleph_0$.

For every infinite regular cardinal κ, I construct (in ZFC) a Boolean algebra which has the FN but is not tightly κ-filtered. This construction is a generalization of one of Ščepin's constructions of a Boolean algebra which is rc-filtered but not projective. (See [23].)

I show that adding ω_3 Cohen reals to a model of CH yields a model of ZFC where $\mathfrak{P}(\omega)$ is not tightly σ-filtered, even though WFN($\mathfrak{P}(\omega)$) holds. A very similar proof shows (in ZFC) that the Cohen algebra over $(2^{\aleph_0})^{++}$ generators, i.e. the completion of the free Boolean algebra over $(2^{\aleph_0})^{++}$ generators, is not tightly σ-filtered. It follows that no complete Boolean algebra of size $\geq (2^{\aleph_0})^{++}$ is tightly σ-filtered.

The third chapter deals with the WFN, mostly for complete Boolean algebras. I characterize those proper notions of forcing P for which $\mathfrak{P}(\omega)$ of the ground model M is σ-embedded in $\mathfrak{P}(\omega)$ in $M[G]$ for every P-generic G. I observe that many forcing notions fail to have this property. (In fact, all forcing notions I have considered that are generated by a name for a real and do not collapse cardinals, except for Cohen forcing.) It follows that in many iterated forcing extensions WFN($\mathfrak{P}(\omega)$) fails. For example, adding ω_2 random reals to a model of CH yields a model of $\neg$ WFN($\mathfrak{P}(\omega)$). I show that adding a Hechler real over ω_2 Cohen reals to a model of CH also gives a model of $\neg$ WFN($\mathfrak{P}(\omega)$). This shows that even adding one real by some σ-centered forcing can destroy WFN($\mathfrak{P}(\omega)$).

It turns out that WFN($\mathfrak{P}(\omega)$) implies that the covering number of the ideal of meager subsets of ${}^{\omega}2$ is large, by a joint result with Soukup. I prove that the groupwise density number $\mathfrak{g}$ is $\aleph_1$ under WFN($\mathfrak{P}(\omega)$). I show that under the assumption $\neg 0^\sharp$, WFN($\mathfrak{P}(\omega)$) implies the WFN of many complete c.c.c. Boolean algebras, among them all measure algebras. Without $\neg 0^\sharp$, my argument only works for algebras which are completely generated by less than $\aleph_\omega$ elements.

0.2 Sources

The first chapter mainly surveys the known results about κ-embeddings, κ-FN, and tight σ-filteredness from [23], [28], [29], [19], [16] and [17]. The second chapter is quite algebraic, although set-theoretic methods are used in several places. The methods and notions used in this chapter are mainly taken from the books by Heindorf and Shapiro ([23]) and Eklof and Mekler ([11]) and from Koppelberg's articles ([28], [29]). The set theory that is used here can be found in the books by Kunen ([32]) and Jech ([24]) and the reference for Boolean algebras is the first volume of the Handbook of Boolean Algebras ([30]). Everything that is needed about general topology is contained in Engelking's book ([12]). The third chapter heavily uses forcing. I basically rely on the books by Kunen ([32]) and Jech ([24]), but I also use several facts from more modern texts ([1], [21]). For cardinal invariants of the continuum, everything necessary is provided by Blass' article ([4]).

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Lutz Heindorf gave me a copy of [23] and Sabine Koppelberg gave me a copy of [30]. Both books have been very useful. Sakaé Fuchino gave me a copy of his japanese translation ([27]) of Kanamori's book ([26]), which has not been extremely useful yet, for the obvious reason. But I hope sometime I will be able to read it. The LaTeX-document class used for typesetting this thesis is due to Carsten Schultz.