Continuous Ramsey Theory in Higher Dimensions

Dissertation vorgelegt von

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Chapter 1

Introduction

The main objective of this thesis is continuous colorings on uncountable Polish spaces. The classical Ramsey Theorem states that for every natural number $m \in \omega$ and every coloring $c : [\omega]^m \to 2$ of the m-element subsets of ω there exists an infinite set on which only one color appears – a c-homogeneous set. Admitting higher ordinals instead of ω, nothing of comparable generality may be proven. Let α and β be two ordinals and let m be a natural number. Suppose for every coloring of the m-element subsets of α with $r \in \omega$ colors a homogeneous set $X \subseteq \alpha$ of order type β can be found. In this case we say that α *arrows* β *for* m*-colorings* and write

$$\alpha \to (\beta)^m_r.$$

Hence Ramsey's theorem translates to $\omega \to (\omega)^m_r$ for arbitrary natural numbers m, r. The theorem is proved by induction on m starting with a simple pidgeon-hole argument. In the case of uncountable ordinals the inductive step does not work and one faces multiple problems. First of all, even if we restrict our attention to colorings of pairs, the uncountable cardinalities are much more difficult. Assuming the Axiom of Choice (AC) it is possible to construct a coloring $c : [2^\omega]^2 \to 2$ such that there are no uncountable c-homogeneous sets, the so called Sierpińsky coloring. One might omit those difficulties by talking about continuous or open colorings with respect to some reasonable topology. In the case of colorings with finitely many colors the notions of continuous and open colorings coincide. The first paper in this line of thought is the almost classical paper of Blass which deals with higher dimensional continuous colorings on perfect sets. Later on in [ARS85] an "Open Coloring Axiom" ($\mathrm{OCA}_{[\mathrm{ARS}]}$) is introduced. It was proved that $\mathrm{OCA}_{[\mathrm{ARS}]}$ is relatively consistent with ZFC and that the $\mathrm{OCA}_{[\mathrm{ARS}]}$ is implied by the Proper Forcing Axiom. In [Ge05] a "Dual Open Coloring Axiom" is introduced. This axiom is as well relatively consistent with ZFC. A proof can be found in [GKKS02]. Both axioms deal with continuous pair colorings. That it is impossible to generalize the consistency of $\mathrm{OCA}_{[\mathrm{ARS}]}$ to triple colorings is proven in [Vel92] by providing a counterexample. Fortunately we were able to give a proof for the consistency of the Dual Open Coloring Axiom for triple colorings.

In other parts of uncountable Ramsey theory, it appears that the generalizations to colorings of triples mostly fail dramatically. The theorem of Dushnik-Miller for example states that for every coloring $c : [\omega_1]^2 \to 2$ there is either a c-homogeneous set of color 0 of uncountable cardinality or there is a set of order type $\omega + 1$ which is homogeneous for the other color. We write $\omega_1 \to (\omega_1, \omega + 1)^2$. It turns out that this is an optimal result because under CH it is possible to construct an example where no homogeneous set of order type $\omega + 2$ can be found. The best result for triples which is known to the author only claims that given an $n \in \omega$ either there is a homogeneous set of order type $\omega + \omega + 1$ in one color or a homogeneous n-element set in the other color. This is impressive in the sense that even though these proofs are very elaborate and complicated they only yield finite homogeneous sets [Jo]. Furthermore already in ZFC it is possible to construct a lot of colorings $c : [\omega_1]^3 \to 2$ such that the c-homogeneous sets are very small. One can for example show that $\omega_1 \not\to (\omega + 1, \omega + 2)^3$, $\omega_1 \not\to (\omega_1, 4)$. Anyone really interested in those results should study recent work of Albin Jones. We merely want to quote these results to suggest that the generalization to higher dimensions is much more difficult in the uncountable case than in the countable. Furthermore it should be noted that there is no obvious pattern in the proofs of the positive arrow results about triples. Basically they consist of many case distinctions. Hence it is quite surprising that by moving from arbitrary to continuous colorings some results can be generalized to triple colorings.

The same leap of complexity appears in the study of arrow relations for graphs. Here the quest for homogeneous "huge"(in order type or cardinality) sets is replaced by homogeneous induced copies of a certain graph. In the case of graphs the difficulties already start in the study of countable graphs. In [ErHaPo73] a coloring on the countable random graph was introduced, which does not admit homogeneous countable complete bipartite graphs. The complexity rises again fastly with increasing dimension. Results in this direction can be found in [PoSau96] and [Sau03]. For further explanations on those results compare the introduction to the third chapter.

1.1 Overview

The thesis has basically two parts. The first one contains the generalization of the dual colorings axiom to triple colorings and some other results. The questions and the techniques follow the work of Geschke et al. in [GKKS02][GGK04] and [Ge05]. It should be noticed that the results presented there are consistency results and proficiency in the technique of Forcing is required. The second part is dedicated to the study of higher dimensional colorings of graphs and no Forcing will appear in this part. Here higher dimensional colorings of graph structures are studied. The central technique in both parts is the so called fusion sequence. We tried to avoid to repeat definitions. Hence the following is the basic notation used in both parts. Additionally each part has an extra section with notation only used in the corresponding part.

1.2 Basic Notation

1.2.1 The Baire Space and its Product Space

The *Baire space* ω^ω is the space of all infinite sequences of natural numbers. The *Cantor space* 2^ω is the subspace of infinite binary sequences. This latter space can also be viewed as the power set of ω. The Baire space is a complete metric space for which the basic open sets are of the form $U_s = \{y \in \omega^\omega : s \subseteq y\}$ for a finite sequence of natural numbers s. The Cantor space 2^ω is a compact topological space if we look at it as a subspace of the Baire space. For a set X and $m \in \omega$ the symbol $[X]^m$ refers to the set $\{y \in \mathcal{P}(X) : |y| = m\}$ in which $\mathcal{P}(X)$ is the power set of X. If X has a linear order then we assume such sets $\{x_0, \ldots, x_{m-1}\}$ to be in increasing order without further mentioning it. For a pair $\{x, y\} \in [\omega^\omega]^2$ let $\Delta(x, y) = \min\{n \in \omega : x(n) \neq y(n)\}$ and the longest common initial segment is $x \wedge y = x \restriction \Delta(x, y) = y \restriction \Delta(x, y)$. As usual, $y \restriction n$ means the restriction of the function y to the predecessor set of n. Hence $x \wedge y$ is the largest common initial segment of x and y. The *lexicographic ordering* on ω^ω is given by

$$x <_{\text{lex}} y \iff x(\Delta(x, y)) < y(\Delta(x, y)).$$

There is also the standard (complete) metric on ω^ω which is given by $d(x, y) = 2^{-\Delta(x,y)}$, for distinct $x, y \in \omega^\omega$ and which leads to the topological space as described before. The fact that the standard metric is an *ultrametric* is contained in the first condition of the following statement.

Fact 1.2.1. For all $m > 0$ and $x_0 <_{\text{lex}} x_1 <_{\text{lex}}, \ldots, x_m$ in the Baire space it holds that

(1) $x_0 \wedge x_m \subseteq x_i$ for all $i \leq m$. That is, $\Delta(x_0, x_m) = \min\{\Delta(x_i, x_j) : i < j \leq m\}$.

(2) If $m > 0$, there is some $i < m$ such that $\Delta(x_0, x_m) = \Delta(x_i, x_{i+1})$

Proof. For (1), if for some $0 < i < m$, $x_0 \wedge x_m \not\subseteq x_i$, then $\Delta(x_0, x_i) = \Delta(x_i, x_m) < \Delta(x_0, x_m)$. If $x_i(\Delta(x_0, x_i)) < x_0(\Delta(x_0, x_i))$, then $x_0 \not<_{\text{lex}} x_i$ and otherwise $x_i \not<_{\text{lex}} x_m$.

In order to see (2) assume that $m > 1$. Let i be minimal such that $i + 1 \leq m$ and such that $\Delta(x_0, x_{i+1}) = \Delta(x_0, x_m)$. If $i = 0$, then we are done and otherwise by the minimality of i and item (1) it holds that $\Delta(x_0, x_i) > \Delta(x_0, x_m)$, hence $\Delta(x_0, x_m) = \Delta(x_0, x_{i+1}) = \Delta(x_i, x_{i+1})$. □

For a finite subset $\{x_0 \ldots x_{m-1}\} \in [\omega^\omega]^m$ we assume from now on that the elements are enumerated with increasing lexicographic order. For such a set, we define the highest splitting level as $\Delta^m(\{x_0, \ldots x_{m-1}\}) = \max\{\Delta(x_i, x_{i+1}) : i \in m - 1\}$. Let

$i \in m$ be such that $\Delta(x_i, x_{i+1}) = \Delta^m(\{x_0, \dots x_{m-1}\})$. We call $x_i \wedge x_{i+1}$ the *highest splitting node* of the m-tupel. The set $[\omega^\omega]^m$ can again be viewed as a topological space where a set $X \subseteq [\omega^\omega]^m$ is open if and only if for all $\{x_0, \dots, x_{m-1}\} \in X$ there are disjoint open neighborhoods $U_i \ni x_i$, $i \in m$, such that

$$\text{if } \forall i \in m(x'_i \in U_i) \text{ then } \{x'_0 \dots x'_{m-1}\} \in X. \tag{1.2.1}$$

1.2.2 Trees

In our case a subtree of $\omega^{<\omega}$ is a subset $T \subseteq \omega^{<\omega}$ of finite sequences of natural numbers ordered by $\subseteq$ and such that the set is downward closed with respect to this order. That is, for every $s \in T$ the set $\{t \in \omega^\omega : t \subseteq s\}$ is contained in T. Hence our trees have exactly one root and that is the empty set. For an arbitrary subset A of $\omega^{<\omega}$ we define the downward closure or *the spanned tree* $T_{\leq A}$ as the minimal tree containing A, that is $T_{\leq A} = \{t \in \omega^{<\omega} : \exists s \in A(t \subseteq s)\}$. Given a closed subset of the Baire space $X \subseteq \omega^\omega$ we call the tree $T_{\leq X} \subseteq \omega^{<\omega}$ which satisfies $[T_{\leq X}] = X$ the tree which is *spanned by the branches in* X. Two nodes $s_i, s_j \in 2^{<\omega}$ are called incompatible ($s_j \perp s_i$) iff $s_i \nsubseteq s_j \wedge s_j \nsubseteq s_i$. We call A an *antichain in* T if the elements of A are pairwise incompatible. We call A a *chain* if they are pairwise compatible. The symbol $[T]$ denotes the set of maximal chains (branches) of the tree T and $[[T]]^m$ denotes the collection of all sets which contain exactly $m \in \omega$ branches of T. If all the branches of T are infinite this can be thought of as a subspace of the Baire space and hence again as a topological space. On the other hand, given a closed set $X \subseteq \omega^\omega$ there is a tree T such that $[T] = X$. We call T the *tree generated by the set* X.

Let $T \subseteq \omega^{<\omega}$ a subtree of the Baire space. We call the elements of T *nodes*. For two nodes s, s' such that $s \subseteq s'$ we say that s is a *predecessor of* s' whereas s' is called *a successor of* s. As usual we use the word *proper* to signify that $s \neq s'$. If $|s| + 1 = |s'|$ we say that s' is an immediate successor of s and $\mathrm{IS}(s, T)$ is the set of immediate successors of s in the tree T. Let $\{s'_0, \dots, s'_{m-1}\}$ be an enumeration of the immediate successors of a node s in increasing lexicographical order. For $i \in m$ we will also write $s^\frown i$ instead of the node s'_i. If s has exactly two immediate successors then $s^\frown 0$ refers to the "left" and $s^\frown 1$ to the "right" immediate successor of s. A node $s \in T$ is called a *splitting node in* T if it has at least two immediate successors in T. We call t an *n-th splitting node* in T if t is a splitting node and has exactly $n-1$ proper initial segments in T which are splitting nodes in T. If t is an n-th splitting node in a tree T we say that any immediate successor of t in T has splitting order n. The splitting order of intermediate nodes is undefined and the node $\emptyset$ is by definition a 0th splitting node and of splitting order 0. The order type of the set of predecessors of t or just $|t|$ is called the *level of* t. For $n \in \omega$ the set $\mathrm{LEV}_n(T)$ consists of all nodes on the n-th level. The restriction of T up to level n is defined as $T \upharpoonright n = T_{\leq T}(\mathrm{LEV}_n(T)) = \bigcup_{m<n} \mathrm{LEV}_m(T)$. Analogously $\mathrm{SPL}_n(T)$ consists of all nodes of splitting order n and $\mathbf{sp}(T)$ is the set of all splitting nodes contained in the tree T. The tree T is called a *finitely branching tree* if every $s \in T$ has a finite amount of immediate successors. That means that every level is finite. A finitely branching tree T is called n-branching for an $n \in \omega$ if for all $s \in T$ it holds that $|\mathrm{IS}(s)| \leq n$. We call T an *infinite tree* if all maximal chains are infinite

For a node $t \in T$ we use the following abbreviation $T[t] = \{s \in T : s \subseteq t \vee s \supseteq t\}$. For a subset $A \subseteq T$ we set $T[A] = \bigcup_{s \in A} T[s]$.

A subtree $T \subseteq \omega^{<\omega}$ is *a perfect tree* if every node $s \in T$ has a successor which is a splitting node. In this case the set of branches $[T] \subseteq \omega^\omega$ is a perfect subset (closed without isolated points) of the Baire space. A perfect tree T in which every splitting node has exactly two immediate successors will be called a *perfect binary tree*. In a perfect binary tree T any node $t \in T$ of splitting order n in T can be coded by some $\sigma \in 2^n$ in the following way: if $s \subseteq t$ is a $(k+1)$th splitting node we set $\sigma(k) = 0$ iff t is a successor of $s^\frown 0$. On the other hand relative to a given tree T every element $\sigma \in 2^n$ defines a unique node t_σ in T. We will simply write t_σ and $t_\sigma(T)$ if the tree is not clear from the context. We set $T * \sigma = T[t_\sigma]$.

A decreasing sequence $(T_n)_{n\in\omega}$ of perfect trees is called a *fusion sequence* iff $T_{n+1} \leq_n T_n$. The ordering $\leq_n$ is defined by

$$T' \leq_n T'' \iff \mathrm{SPL}_n(T') = \mathrm{SPL}_n(T'') \wedge T' \subseteq T''.$$

The resulting tree $T_\infty = \bigcap_{n<\omega} T_n$ is again a perfect tree and we call it *the fusion* of the sequence $(T_n)_{n\in\omega}$.
Let $\{s_0, \dots s_{d-1}\}$ be an antichain in T where T is a perfect subtree of ω^ω. Let $T_i \subseteq T[s_i]$ be perfect trees. We call the tree $\bigcup_{i<2^n} T_i$ *the amalgamation of the* $T[s_i]$. For choosing any perfect $T_i \subseteq T[s_i]$ we say that we *thin out above the node* s_i. Accordingly let $\{\sigma_0 \dots \sigma_{2^n}\}$ be an enumeration of all $0,1$-sequences of length n, let T be a perfect binary tree and T' the amalgamation of the $T * \sigma_i$. Then it holds that $T' \leq_n T$. In stage n of a construction of a fusion sequence $(T_n)_{n\in\omega}$ we will choose perfect trees $T_i \subseteq T_n * \sigma_i$ for every $\sigma_i \in 2^n$ and create T_{n+1} as the amalgamation of those T_i. In this situation let t be the lowest splitting node in T_i for an $i \in 2^n$. Then t remains a splitting node in every $T_{n'}$ for $n < n'$ as well as in the resulting fusion of the sequence. We also say, we *fix the splitting node t above σ_i or above the node s_{σ_i}*.

Let $T \subseteq$ be a finitely branching tree and let $A \subseteq T$ be a finite subset, we call a set $A' \subseteq T$ *A-dense* if for each $s \in A$ there is an $s' \in A'$ such that $s' \supseteq s$. If we can furthermore find for every $a \in A$ an $a' \in A'$ such that $a \leq a'$ then we write $A \prec A'$. If there is a $k \in \omega$ such that the set A' is a subset of $\mathrm{LEV}_k(T)$, we say that A' is *A-dense on level k*. Accordingly given a collection $T_0, \dots, T_{d-1}$ such that for $i, i' \in d$ the root $\emptyset$ is the only element in the intersection $T_i \cap T_{i'}$ and given a set $A \subseteq \bigcup_{i\in d} T_i$. We call the set $A' \subseteq \bigcup_{i\in d} T_i$ an A-dense set if for every $i \in d$ the set $A' \cap T_i$ is $A \cap T_i$ -dense.

1.2.3 Colorings and Partitions

Given $m \in \omega + 1$, $k \in \omega$, $X \subseteq \omega^\omega$ and $D \subseteq [\omega^\omega]^m$. A function $c : [X]^m \cap D \to k$ is called *a coloring of the D-sets of X with k colors*. If the function c is continuous (Borel, definable, measurable,...) with respect to the subspace topology on $[X]^m$ we call it a continuous (Borel, definable, measurable) coloring. If $D = [\omega^\omega]^m$, we call c *a coloring of the m-sets of X*. A set $Y \subseteq X$ is called *c-homogeneous* if c is constant on $[Y]^m$. We call a subset Y c-homogeneous on D if c is constant on $[Y]^m \cap D$.

Definition 1.2.2. Let X be a perfect subset of the Baire space and let $m, t \in \omega$. A partition $[X]^m = \dot{\bigcup}_{i\in t} C_i$ is called a *basic perfect partition for continuous colorings* if the following holds:

(1) For any perfect subset $Y \subseteq X$ and all $i \in t$ the intersection $C_i \cap [Y]^m$ is nonempty

(2) For every $i \in t$ and a continuous coloring $c : [X]^m \to 2 \in \omega$ there is a perfect subset $X_i \subseteq X$ which is c-homogeneous on C_i.

If $t \in \omega$ we shall call it *finite basic perfect partition*. Accordingly we define basic partition for another class of colorings (like Borel).

Let X, m be as before, let $[\omega^\omega]^m = \bigcup_{i \in t} C_i$ be a finite basic perfect partition, let c be a coloring with $r \in \omega$ colors. For $i \in t$ we can apply (2) to the following coloring c' which only uses 2 colors: define $c'(x) = 1$ iff $c(x) \in r \setminus \{0\}$ and $c'(x) = 0$ otherwise. Hence there is a perfect $X' \subseteq X$ which is c'-homogeneous on C_i. That means that c restricted to C_i uses only $r - 1$ colors. Thus by induction (2) is equivalent to the statement:

(2)* For every $i \in t, r \in \omega$ and a continuous coloring $c : [X]^m \cap C_i \to r \in \omega$ there is a perfect subset $X_i \subseteq X$ which is c-homogeneous on C_i.

Let $T \subseteq \omega^{<\omega}$ a perfect subtree of the Baire tree. Let $m \in \omega$ and $c : [[T]]^m \to \{0, 1\}$ a continuous coloring. We define a coloring $\overline{c} : [T]^m \to \{\mathbf{nil}, 0, 1\}$. If the s_i are pairwise incompatible we set $\overline{c}(\{s_0, \dots, s_{m-1}\}) = j$ if for every set $\{x_0, \dots, x_{m-1}\} \in [T]^m$ such that for all $i \in m$ which are such that $s_i \subseteq x_i$ the color $c(\{x_0, \dots, x_{m-1}\})$ equals j. In all other cases we set $\overline{c}(\{s_0, \dots, s_{m-1}\}) = \mathbf{nil}$. The function $\overline{c}$ can be viewed as a real number. If c is continuous and we only know the function $\overline{c}$ we can reconstruct c in the following way. Take $\{x_0, \dots, x_{m-1}\}$. Because of continuity there are basic open neighborhoods on which the color assigned by c is constant and hence for all $i \in m$ there are $s_i \subseteq x_i$ such that $\overline{c}(\{s_0, \dots, s_{m-1}\}) = j$ for some $j \in 2$. Hence $c(\{x_0, \dots, x_{m-1}\}) = j$. In this way the original continuous coloring c can be reconstructed from $\overline{c}$, which is to say, it can be coded as a real number.

Observation 1.2.3. Every continuous coloring $c : [\omega^\omega]^m \to 2$ can be coded as a real number.

A continuous coloring $c : [[T]]^2 \to 2$ is called *immediately decided* if for every $\vec{x} \in [[T]]^m$ the following holds: $\overline{c}(\vec{x} \upharpoonright \Delta^m(\vec{x})+1) \in 2$. If $m = 2$ and T is binary perfect this is equivalent to the existence of a coloring on the splitting nodes $d : \mathbf{sp}(T) \to 2$ such that $c(\{x, y\}) = d(x \wedge y)$ for $\{x, y\} \in [[T]]^2$. In particular every coloring of the splitting nodes induces a coloring on pairs of branches.

Chapter 2

Weakly homogeneous subsets of the reals

2.1 The Theorem of Blass

In [Bla81] a continuous coloring $c_{\mathrm{bl}} : [\omega^\omega]^m \to 2$ was introduced for every $m \in \omega$. The picture shows the coloring for the case $m = 3$: Each triple which looks like

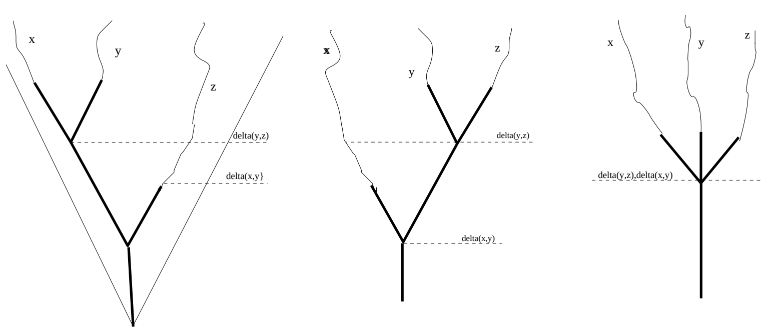

Figure 2.1: Types for $m = 3$

the one in the first picture is colored by 0 and those which look like the one in the second picture are colored by 1. The situation in the third picture can be avoided on a perfect set. Every uncountable subset of the Baire space contains triples in both colors. This is an example for a continuous coloring on an uncountable set which does not admit big homogeneous sets. This coloring can be generalized to every finite dimension and Blass showed in the same paper that this coloring is the worst it can get. That is, he showed that the resulting partition $[\omega^\omega]^m = \bigcup_{i\in(m-1)!} C_i^{\mathrm{bl}}$ is a finite basic perfect partition for arbitrary $m \in \omega$. Since later on we will restrict our study to the case $m = 3$ we will now only talk about triples (The general theorem can be found in [FaTo95]). For $j = 0, 1$ we say that the triples in the set C_j^{bl} are of type j. Hence the theorem of Blass states, that there is a perfect set X such that all triple of the same type are colored by the same color. In this case we say that X is weakly homogeneous and it turns out, that weak homogeneity is the adequate notion for the three dimensional case and should be used instead of homogeneity. The original proof of Blass' theorem for all dimensions uses the Halpern-Läuchli Theorem, which will be stated later on in Theorem 3.2.10. This theorem is not at all trivial and we will present a proof for $m \leq 3$ without using it.

Before we start with the proofs we want to give a formal definition of the partition: let $\{x, y, z\} \in [[T]]^3$ and recall that the elements of the set are ordered lexicographically. Put $\{x, y, z\} \in C_0^{\mathrm{bl}}$ if $\Delta(x, y) < \Delta(y, z)$ and in the other case put $\{x, y, z\}$ into C_1^{bl}. We will prove item (2) of Definition 1.2.2 for this partition. Since in this first section we are only concerned with perfect subsets we might as well assume all occurring trees to be binary and hence without further mentioning it, we take them to be perfect subsets of the Cantor space 2^ω.

Theorem 2.1.1. *Let $Y \subseteq \omega^\omega$ be perfect and let $c : [Y]^3 \to 2$ be a continuous coloring. Then there is a perfect subset $Y' \subseteq Y$ which is c-homogeneous on C_0^{bl} and on C_1^{bl}.*

The 2-dimensional case is due to Galvin. The reader familiar with the technique of fusion sequences and amalgamation should immediately proceed to the 3-dimensional case. Afterwards the same technique of proof will be tacitly used.

Theorem 2.1.2. *Let $Y \subseteq \omega^\omega$ a perfect subset of the Baire space. Then the set $[Y]^2$ is a basic partition.*

Proof. This theorem is proved by taking a perfect tree $T \subseteq \omega^{<\omega}$ such that $[T] = Y$. Consider a continuous coloring $c : [[T]]^2 \to 2$. If Y contains a basic open neighborhood which is c-homogeneous the proof is finished. So suppose this is not the case and construct a fusion sequence starting with $T_0 = T$. In step $n+1$, we can find for each $i \in 2^n$ a pair $\{x, y\} \in [T_n * \sigma_i]^2$ which is colored by 1. Because of continuity there are $s_0^i, s_1^i \in T_n$ such that for every $x' \in [T_n[s_0^i]], y' \in [T_n[s_1^i]]$ it holds that $c(\{x', y'\}) = 1$. Amalgamate all trees $T_n[s_j^i], T_n[s_j^i]$, $i \in 2^n$, $j \in 2$. This leads to the tree T_{n+1}. The fusion of this sequence satisfies the statement. □

The fusion sequence for the proof of Theorem 2.1.1 will be chosen inductively on n and twice repeated use (once for the triple in C_0^{bl} and then for the triple in C_1^{bl}) will prove the theorem. The next lemma enables us to construct the next splitting node with the right properties in order to get the fusion sequence mentioned before. Later on in the consistency result there will be an analogue lemma. The lemma ensures that for a fixed $i \in \omega$ we can move on to a perfect tree such the color of a triple $\{x, y, z\} \in C_i^{\mathrm{bl}}$ only depends on the lowest splitting point of the triple. In this way we will reduce the coloring to the two dimensional case in order to use Theorem 2.1.2
.

Lemma 2.1.3. *Let $T \subseteq 2^{<\omega}$ be a perfect subtree, let $l \in \omega$ be a natural number and let $\{s_0, s_1\} \in [\mathrm{LEV}_l(T)]^2$. Then there is a perfect tree $T' \subseteq T$ such that* $\mathrm{LEV}_l(T) = \mathrm{LEV}_l(T')$ *and the following holds:*

$$\begin{aligned} (2.1.1)\quad \exists j_0, j_1 \in \{0,1\} \forall \{x, y, z\} \in [[T']]^3 ((x \supseteq s_0 \wedge y, z \supseteq s_1 \Rightarrow c(\{x, y, z\}) = j_0) \\ \wedge (x, y \supseteq s_0 \wedge z \supseteq s_1 \Rightarrow c(\{x, y, z\}) = j_1)) \end{aligned}$$

Proof. The tree T' is constructed as an amalgamation of the $T[t]$ where $t \in \mathrm{LEV}_l(T)$. We only have to show how to thin out above s_0, s_1. To establish the condition (2.1.1) for triple in C_0^{bl} we simultaneously construct two fusion sequences $(T_n, S_n)_{n \in \omega}$ such that $T_n \subseteq T[s_0]$ and $S_n \subseteq T[s_1]$ for all $n \in \omega$. Simultaneous construction means that in stage n we will first fix all $(n+1)$th splitting nodes in T_n and then all in S_n. The difficulty lies in the choice of the splitting nodes above s_1. We need two auxiliary sets. For a set $\{t, s\} \in [T]^2$ which consists of two incomparable nodes such that the higher node s is a splitting node in T and a given color $j \in 2$ we formulate the following condition depending on the pair $\{s,t\}$ and $j \in 2$:

$$\forall \{x,y,z\} \in [T]^3 (x \supseteq t \wedge y \supseteq s^\frown 0 \wedge z \supseteq s^\frown 1 \to c(x,y,z)) = j) \tag{2.1.2}$$

Let the calligraphic letter $\mathcal{F}$ range over the finite antichains in $\omega^{<\omega}$. Let $\mathbb{S}$ be the set of all perfect subtrees of 2^ω. For $j \in 2$ we define the sets:

$$\begin{aligned} E_j := \{(T,S) \in \mathbb{S} \times \mathbb{S} : \forall \mathcal{F} \subseteq T \forall s \in S \exists \mathcal{F}' \subseteq T \exists s_* \in S (\mathcal{F} \prec \mathcal{F}' \wedge s_* \supseteq s) \wedge \\ \forall t \in \mathcal{F}' (t, s_* \text{ satisfy (2.1.2) for color } j)\} \end{aligned} \tag{2.1.3}$$

Claim 1. Suppose $(T,S) \notin E_0$. We find perfect trees $T' \subseteq T, S' \subseteq S$ such that $(T', S') \in E_1$.

Proof of Claim 1. If $(T,S) \notin E_0$ we can find an $s \in S$ and an antichain $\mathcal{F} \subseteq T$ such that for every splitting node $s_* \supseteq s$ there is a $t \in \mathcal{F}$ that makes it impossible to find $t_* \supseteq t$, such that t_*, s_* satisfy (2.1.2) for 0: Fix an enumeration $\mathcal{F} = \{t_l : l < m\}$. We define a coloring on $[T[s]]^2$ with m colors:

$$c'(x,y) = i \iff \text{there is no } t_* \supseteq t_i \text{ such that } s_*, t_* \text{ satisfy (2.1.2) for color 0.}$$

By Theorem 2.1.2, there is a perfect subtree S' of $S[s]$ which is c'-homogeneous with color $i \in m$. Then t_i and any $s' \in \mathbf{sp}(S')$ satisfy (2.1.2) for color 1. Hence $(T[t_i], S')$ is an element of E_1. Actually, it even holds that each $t \in T[t_i]$ and $s \in \mathbf{sp}(S')$ satisfies (2.1.2) for 1.

$\square$(Claim 1)

Claim 1 enables us to construct the fusion sequences. We start with $T_0 \subseteq T[s_0]$ and $S_0 \subseteq T[s_1]$ such that $(T_0, S_0) \in E_j$ for a $j \in 2$. Suppose T_n, S_n is defined and let $\Sigma = \{\sigma_l : l \in 2^n\}$ be an enumeration of the possible codes for nodes with splitting order n. We first fix the nth splitting nodes in S_n. For $\tau \in \Sigma$ we use (2.1.3) with $s = s_\tau(S_n)$ and $\mathcal{F} = \{t_{\sigma_l}(T_n) : l \in 2^n\}$. This gives $\mathcal{F}' = \{t_l : l \in 2^n\}$ and a splitting node $s_* \supseteq s$. We set $T_{n+1} * \sigma_l = T_n[t_l]$, $l \in 2^n$ and we set $S_{n+1} * \tau {}^\frown 0 = S_n[s_*^\frown 0]$ and $S_{n+1} * \tau {}^\frown 1 = S_n[s_*^\frown 1]$. If necessary, we thin out the $\tau * S_n$ for every $\tau \in \Sigma$ such that the lowest splitting node in $\tau * S_n$ appears on levels above $|s_*|$. This defines the new S_{n+1}. We thin out the $\tau * T_n$ in order to get the lowest splitting nodes above $|s_*|$. Set T_{n+1} as the amalgamation. Let T_∞, S_∞ be the fusion of those sequences $(T_n)_{n\in\omega}$ and $(S_n)_{n\in\omega}$. Amalgamation of T_∞, S_∞ and the trees $T[t]$ for $t \in \mathrm{LEV}_l(T) \setminus \{s_0, s_1\}$ finishes the proof of the lemma. $\square$

Proof of Theorem 2.1.1. As mentioned before by changing to a binary subtree of $\omega^{<\omega}$ we can assume T to be a subset of $2^{<\omega}$ such that there is a tree embedding $f : T \to \omega^{<\omega}$ and $[\mathrm{rg}(f(T))] \subseteq Y$. We assume that T contains at most one splitting node on each level. We construct a fusion sequence $(T_n)_{n\in\omega}$ such that in stage n we fix an nth splitting node $s \in T_n$ and establish condition (2.1.1) for $\{s^\frown 0, s^\frown 1\}$. Let the corresponding colors be j_0^s, j_1^s. Let T_∞ be the fusion of the sequence. We can define a continuous coloring $c' : [[T_\infty]]^2 \to 2 \times 2$:

$$c_0'(\{x_0, x_1\}) = (j_0^s, j_1^s) \text{ if } s_0 \subseteq x_0 \text{ and } s_1 \subseteq x_1.$$

With Theorem 2.1.2 we get a c'-homogeneous set and hence a perfect subtree $T' \subseteq T_\infty$ which is c-homogeneous on C_0^{bl} and on C_1^{bl}.

$\square$(Theorem 2.1.1)

We conclude this section with a corollary, which can be easily concluded from the last theorem once we have seen that for $j \in 2$ there are ω-sets $S \subseteq 2^\omega$ such that $[S]^3 \subseteq C_j^{\mathrm{bl}}$:

Corollary 2.1.4. Let $c : [2^\omega]^3 \to 2$ be a continuous coloring without infinite homogeneous sets of color 0. Then there is a perfect set $X \subseteq 2^\omega$ which is homogeneous of color 1.

In the same way it is obvious that if the appearing c-homogeneous sets of color 0 are all finite, then we can actually find c-homogeneous sets of color 1 which are not only infinite but of order type $\omega + k$ for any $k \in \omega$ with respect to the lexicographical order.

2.1.1 Introduction

Definition 2.1.5. Let $c : [\omega^\omega]^3 \to r$ be a coloring. We call $X \subseteq \omega^\omega$ *weakly c-homogeneous* if X is c-homogeneous on C_0^{bl} and C_1^{bl}.

Let X be a nonempty set. A set $\mathcal{I} \subseteq \mathcal{P}(X)$ is called *ideal*, if it contains all finite subsets of X, is closed under finite unions and for all $X \in \mathcal{I}$ and subsets $Y \subseteq X$ it follows that $Y \in \mathcal{I}$. We call $\mathcal{I}$ a *σ-ideal* if it closed under countable unions. For a σ-ideal $\mathcal{I}$ on Y we define $\mathrm{cov}(\mathcal{I}) = \min\{|A| : A \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} = Y\}$ as well as $\mathrm{non}(\mathcal{I}) = \min\{|X| : X \subseteq Y \wedge X \notin \mathcal{I}\}$. For a continuous coloring $c : [\omega^\omega]^3 \to 2$ we define $\mathcal{I}^c_{hom}$ to be the smallest σ-ideal on ω^ω which contains all weakly c-homogeneous sets. The cardinal $\mathrm{hom}(c) = \mathrm{cov}(\mathcal{I}^c_{\mathrm{hom}})$ is called the *homogeneity number of c*. Define $\mathrm{hom}^2 = \max\{\mathrm{cov}(\mathcal{I}^c_{\mathrm{hom}}) : c : [\omega^\omega]^2 \to 2 \text{ is continuous}\}$ and respectively hom^3. Let $c : [\omega^\omega]^3 \to 2$ be a continuous coloring and $s \in 2 \times 2$. Consider sets $X \subseteq \omega^\omega$ such that for $j \in 2$ the coloring c has the value $s(j)$ on $[X]^3 \cap C_j^{\mathrm{bl}}$, that is those sets X where the coloring has a value which is described by s. Let $\mathcal{I}^c_s$ be the σ-ideal generated by those sets. The cardinal $\mathrm{chrom}(c) = \min\{\mathrm{cov}(\mathcal{I}^c_s) : s \in 2 \times 2\}$ is also called *chromatic number for c*.

When it comes to continuous colorings of 2-element sets it is known that hom^2 is always uncountable and consistently smaller than the continuum [GKKS02]. We will generalize this result to triple colorings by showing that $\mathrm{hom}^3 < 2^{\aleph_0}$ is true in the Sacks model. Obviously $\mathrm{hom}^2 \leq \mathrm{hom}^3$ is true everywhere because every continuous pair coloring $c : [\omega^\omega]^2 \to 2$ gives rise to a continuous triple coloring $d : [\omega^\omega]^3 \to 2$ with

$$d(\{x_0, x_1, x_2\}) = 0 \iff \{x_0, x_1, x_2\} \text{ is } c\text{-homogeneous.}$$

Hence $\mathrm{hom}(c) \leq \mathrm{hom}(d)$ because the d-homogeneous sets of color 1 are finite (of cardinality 6 at most). A generalization to higher dimensions is as far as we could see not possible with the proof presented here. This proof combines the techniques from the Forcing argument in [GKKS02] with the idea of the sets E_j,$j \in 2$ which were introduced in the proof of Theorem 2.1.1. This idea does not work for higher dimensions. The question if it is consistent to have $\mathrm{hom}^4 < 2^\omega$ or if this statement holds in the Sacks model is still open. It is natural to suspect that in the theory of covering numbers the leap of complexity occurs by proceeding from 3 to 4 and not from 2 to 3 as in large parts of Ramsey theory. This suspicion is further supported by the fact that the theorem of Blass was proved by Galvin [Gal68],[Gal69] already for the cases $m = 2, 3$. That is, the idea of Blass to use the Halpern-Läuchli Theorem was only needed for $m > 3$.

Even if we restrict our attention to continuous colorings the difference between triples and pairs is apparent. There are two axioms in the literature which are called open coloring axioms. The first one was introduced in [ARS85] and is the statement "for every separable metric space X of size $\aleph_1$ and every continuous coloring $c : [X]^2 \to m$ for some $m \in \omega$ the set X can be covered by countably many c-homogeneous sets". From this it can be concluded that $\mathrm{non}(\mathcal{I}^c_{\mathrm{hom}}) > \aleph_1$ for all continuous pair colorings c. Therefore Geschke suggested in [Ge05] to call the statement that the homogeneity number hom^2 is strictly less than the continuum a dual coloring axiom. In [ARS85] $\mathrm{OCA}_{[\mathrm{ARS}]}$ was shown to be independent and its consistency with several statements was established. One might ask if the open coloring axiom $\mathrm{OCA}_{[\mathrm{ARS}]}$ can be consistent for higher dimensions. In [Vel92] they constructed a continuous coloring d_v of the triples of the Baire space such that there is an uncountable set $R \subseteq X$ which contains no uncountable d_v-homogeneous sets. But that amounts to saying that $\mathrm{non}(\mathcal{I}^{d_v}_{\mathrm{hom}}) = \aleph_1$. It should be noted that this coloring is constructed in ZFC and uses the fact that the Baire space is not compact. We will now give a short sketch of the construction of the coloring. We will cite lemmas of the paper, the proofs can be found in [Vel92].

2.1.2 "$\mathrm{non}(\mathcal{I}^3_{\mathrm{hom}}) = \aleph_1$"

For a function $k : \omega^{<\omega} \otimes \omega^{<\omega} \to 2$ and a function $f : D \to 2$ on a finite set $D \subseteq \omega^{<\omega}$ we say that a node $s \in \omega^{<\omega}$ *realizes* f if there is a $j \in \omega$ such that

$$\forall d \in D(k(s^\frown j, d) = f(d)).$$

Let $k : \omega^{<\omega} \otimes \omega^{<\omega} \to 2$ be a function such that every node $s \in \omega^{<\omega}$ realizes every possible f on any D as described in the definition. That it is possible to construct such a k can be easily seen and is spelled out in the proof of Theorem 6.2 of [Vel92]. Now we can define a continuous coloring $d : [\omega^\omega]^3 \to 2$ for a triple $\{x, y, z\}$ in increasing lexicographical order.

$$d(\{x, y, z\}) = k(x \restriction \Delta(y, z), y \restriction \Delta(y, z)).$$

Now it is possible to pick an uncountable set of reals X such that the continuous coloring d acts on X as the discontinuous coloring found by Todorcevic in [Todo87]:

Theorem 2.1.6. *There is a coloring $c_{\mathrm{TO}} : [\omega_1]^2 \to \omega_1$ such that every uncountable subset $X \subseteq \omega_1$ contains pairs of every color.*

Let $\omega_1 = D_0 \dot{\cup} D_1$ be a partition and fix the coloring $c'_{\mathrm{TO}}(\{x, y\}) = 0 \iff c_{\mathrm{TO}}(\{x, y\}) \in D_0$. The missing link is the following lemma which is Lemma 6.1 in [Vel92]:

Lemma 2.1.7. *There is a set of reals $X = \{r_\alpha : \alpha \in \omega_1\}$ such that for every $\alpha, \beta \in \omega_1$ there is an $n \in \omega$ such that for every $m > n$ $k(r_\alpha \restriction m, r_\beta \restriction m) = c'_{\mathrm{TO}}(\alpha, \beta)$.*

Proof. The idea is to pick reals by recursion on $\alpha < \omega_1$. In step α, we take an enumerating function $j : \omega \to \alpha$ and consider the reals picked so far: $\{r_{j(i)} : i \in \omega\}$. Define r_α inductively on $m \in \omega$ such that $k(r_\alpha \restriction m, r_{j(i)} \restriction m) = c(\alpha, j(i))$ for all $i \in m$. Hence at every stage of this construction only finitely many requirements have to be met and this is possible because $r_\alpha \restriction m$ realizes every function on m. $\square$

Observation 2.1.8. Let $j \in 2$ be a fixed color. Any uncountable subset of X contains a triple $\{x, y, z\}$ such that $d(\{x, y, z\}) = j$.

Proof. Take an uncountable subset $I \subseteq \omega_1$ of the index set. By removing all neighborhoods which contain only countable many elements, we may assume that every neighborhood which intersects $Y = \{r_\alpha : \alpha \in I\}$ contains uncountably many elements of Y. Because of Theorem 2.1.6 there is a pair $\alpha, \beta \in I$ such that $c'_{\mathrm{TO}}(\{\alpha, \beta\}) = j$. By Lemma 2.1.7 there is an $n \in \omega$ such that $k(r_\alpha \restriction m, r_\beta \restriction m) = 0$ for all $n \leq m$. Pick any $z \in Y$ such that $r_\beta \restriction n \subseteq z$. Then $d(\{r_\alpha, r_\beta, z\}) = 0$. $\square$

It is important to observe that this coloring d is continuously defined on all triples of the space ω^ω and it does not matter for the argument what values we assign depending on whether a triple is in C_0^{bl} or C_1^{bl}. The construction of the function k cannot be executed on a compact subspace and it remains open if there is an example which can be defined on a compact space. Furthermore it should be noted that the result is actually a lot stronger. Taking the original coloring with uncountably many colors, the resulting continuous coloring is such that none of those uncountable colors can be omitted on any uncountable subset of X.

The drop of $\mathrm{non}(\mathcal{I})$ due to the 3rd dimension is rather drastic and it is in accordance with the above mentioned results in the sense that there seems to be a leap on complexity at the step from 2 to 3 dimensions. Interestingly this is only the case with uncountable sets. Ramsey's theorem holds for all colorings of ω in every dimension and the argument for this is straightforward by induction. Restricting the attention to continuous colorings and especially to the study of covering numbers we will establish the independence of the equality of the homogeneity numbers. We first present a model where $\mathrm{hom}^2 = \mathrm{hom}^3 = \aleph_1 < 2^{\aleph_0}$ and then one with $\mathrm{hom}^2 \neq \mathrm{hom}^3$. At first glance this second result might suggest an increase in complexity which essentially depends on the rise in dimensions. This might not be the case because in [GGK04] a model was presented where the homogeneity numbers for different continuous pair colorings had different values. Hence it is not clear if the homogeneity number mirrors the specific complexity due to a rise in dimension.

2.2 The Consistency of "$\mathrm{hom}^3 < 2^{\aleph_0}$"

2.2.1 Forcing Notation

A complete survey of the theory of Forcing would certainly exceed the framework of this thesis and is not worth it, since the second part of the thesis deals only with results provable in ZFC. An accessible textbook on Forcing and iterated Forcing is the classical book of Kunen. Where Sacks Forcing is concerned we follow, where possible, the notation of [GKKS02] as well as [GeQu]. A *notion of Forcing* is a partial order $(\mathbb{P}, \leq)$ together with a highest element $1_{\mathbb{P}}$. The variables for the elements or conditions of such a Forcing will be p, q, r. For two conditions p, q we say p is stronger than q if $p \leq q$ in the partial order. A subset $D \subseteq \mathbb{P}$ is called *dense in* $\mathbb{P}$ if for every $p \in \mathbb{P}$ there is a stronger condition $q \leq p$ and $q \in D$. A set $G \subseteq \mathcal{P}(\mathbb{P})$ is called a*filter* if for every element $p \in G$ the set $\{q : p \leq q\}$ is a subset of G and for every $p, q \in G$ there is an $r \in G$ such that $r \leq p \wedge r \leq q$. A filter F is called $\mathbb{P}$-generic over V if G intersects every dense subset of $\mathbb{P}$ which is contained in V. For a statement ϕ in the language of Forcing we say that a condition p forces ϕ and we write $p \Vdash \phi$ if ϕ holds in every extension $V[G]$ such that $p \in G$ for a $\mathbb{P}$-generic filter G. For a Forcing notion $\mathbb{P}$ the element $1_{\mathbb{P}}$ is the maximal condition. That is, the condition which is in every $\mathbb{P}$-generic filter. Let $\mathbb{S}$ denote Sacks Forcing, that is, the Forcing notion whose conditions are perfect subtrees of $2^{<\omega}$ ordered by set-theoretic inclusion. For a condition $p \in \mathbb{S}$ we set $\operatorname{stem}(p) \in p$ as the first splitting node. Sacks Forcing is not countably closed, but for infinite decreasing sequences which are fusion sequences the intersection is a condition. Due to this feature any countable function which appears in the extension can be covered (or predicted) with some binary decision tree. This feature is usually referred to as the Sacks property. The countable support iteration of Sacks Forcing of length $\alpha \in \omega_2$, which we refer to by $\mathbb{S}_\alpha$, also satisfies the Sacks property. We will give an adequate definition of fusion sequence for the iteration in a second. For a $\beta \in \alpha$ the condition $p \restriction \beta$ is the iteration up to β and hence $p \restriction \beta \in \mathbb{S}_\beta$ and $p(\beta)$ is an $\mathbb{S}_\beta$-name for a condition in $\mathbb{S}$. The order is given by:

$$p \leq q \iff \forall \beta \in \alpha (p \restriction \beta \Vdash p(\beta) \leq q(\beta)).$$

The *support* of a condition $p \in \mathbb{S}_\alpha$ is defined as

$$\operatorname{supp}(p) = \{\beta \in \alpha : \forall q \leq p (q \restriction \beta \Vdash p(\beta) \neq 1_{\mathbb{S}})\}.$$

Definition 2.2.1. Let $\alpha \in \omega_2$ be the length of an iteration, $(F_n)_{n\in\omega} \subseteq [\alpha]^{<\omega}$ a sequence of finite sets, increasing with respect to set theoretic inclusion and let $(\eta_n : F_n \to \omega)_{n<\omega}$ be a sequence of functions. We call $(p_n)_{n<\omega} \subseteq \mathbb{S}_\alpha$ a *fusion sequence* *with witness* F_n, η_n if the following holds for every $n \in \omega$:

(1) $\forall \beta \in F_n : p_{n+1} \restriction \beta \Vdash p_{n+1}(\beta) \leq_{\eta_n(\beta)} p_n(\beta)$.

(2) $\forall \beta \in F_n(\eta_n(\beta) \leq \eta_{n+1}(\beta))$

(3) $\forall \gamma \in \mathrm{supp}(p_n) \exists m \in \omega(\gamma \in F_m \wedge \eta_m(\gamma) > n)$.

In this situation we call the intersection $\bigcap_{n\in\omega} p_n$ the *fusion of the sequence* $(p_n)_{n\in\omega}$. Observe that the fusion is again an element of $\mathbb{S}_\alpha$. For (1) we will also write $p_{n+1} \leq_{F_n,\eta_n} p_n$. Let $F \in [\alpha]^{<\omega}$, $\eta : F \to \omega$ a function and $\sigma \in \prod_{\gamma\in F} 2^{\eta(\gamma)}$. We define $p * \sigma$ as the condition such that for every $\gamma \in F$ the following holds:

$$p * \sigma \restriction \gamma \Vdash \text{“}p * \sigma(\gamma) = p(\gamma) * \sigma(\gamma)\text{”}.$$

Let $\{\sigma_i : i \in |2^{\eta(\gamma)}|\}$ be an enumeration of the sequences of $0, 1$ of length $\eta(\gamma)$ and let q_i be such that $q_i \leq p * \sigma_i$ for each $i \in |2^{\eta(\gamma)}|$ we define the *amalgamation* $\bigcup\{q_i : \sigma_i \in \prod_{\gamma\in F_n} 2^{\eta(\gamma)}\}$ as the unique condition q such that $q * \sigma_i = q_i$ for all i. Hence q satisfies $q \leq_{F,\eta} p$.

Unless mentioned otherwise, the variable $\dot{z}$ will be a Forcing name for an element in ω^ω. The respective Forcing notion should be clear from the context but most of the time it will be some $\mathbb{S}_\alpha$ for $\alpha \in \omega_2$. For a condition q we define the tree of possibilities for $\dot{z}$.

$$T_{\dot{z},q} = \{s \in \omega^{<\omega} : \exists q' \leq q(q' \Vdash s \subseteq \dot{z})\}.$$

The stem of the tree T_q is the maximal initial segment of the real $\dot{z}$ which is decided by the condition q. We refer to this using the symbol $\dot{z}[q]$. Since $\dot{z}[q]$ is an element of the ground model, we can define the following notion: a set of conditions $\{q_0, \ldots q_m\}$ is called $\dot{z}$-*proper* if $\dot{z}[q_0], \ldots, \dot{z}[q_m]$ forms an antichain, furthermore given an indexed set of conditions we always assume that $\dot{z}[q_i] <_{lex} \dot{z}[q_{i'}]$ holds for indices which satisfy $i < i'$. For a condition $q \in \mathbb{S}_\alpha$, we call a set $\{\sigma_0, \ldots, \sigma_m\} \subseteq \prod_{\gamma\in F} 2^{\eta(\gamma)}$ $\dot{z}$-proper if the set $\{q * \sigma_0, \ldots, q * \sigma_m\}$ is $\dot{z}$-proper.

A condition p "*decides a real up to* n" means that $|\dot{z}[p]| \geq n$. In the iterative case we use the following notation: let $\alpha < \beta < \aleph_2$ be ordinals and let p be a condition of the Forcing notion $\mathbb{S}_\alpha$. Suppose there are names $\dot{p}_0, \dot{p}_1$ such that $p \restriction \beta \Vdash \dot{p}_0, \dot{p}_1 \leq p \restriction [\beta, \alpha)$. We say that $p, \dot{p}_0, \dot{p}_1$ decides $\dot{z}$ up to level n if $|\dot{z}[p^\frown \dot{p}_j]| \geq n$, where $p^\frown \dot{p}_j \restriction \beta = p$ and $p \restriction \beta \Vdash p^\frown \dot{p}_j[\beta, \alpha) = \dot{p}_j$ for $j \in 2$.

The term *Sacks model* refers to any model which is created by iterative Sacks Forcing with countable support of length ω_2 starting from a model which satisfies the Continuum Hypothesis. All these models share a good deal of properties such that it makes sense to refer to them as "one" model. This iteration of Sacks Forcing has no antichains of size $\aleph_2$. Analyzing the names of new reals it turns out that we can restrict ourself to those with *nice names*. That is, they are defined with the help of antichains. Hence the continuum cannot be bigger than $\aleph_2$. On the other hand in every stage of the iteration new reals are added and the generic reals are pairwise different, as we can see with the help of an ordinary density argument. This and the fact that no cardinals are collapsed leads to the conclusion that the continuum equals $\aleph_2$ in the Sacks model. This is a general feature of countable support iterations. Hence a completely different technique is needed if one wants to construct a model where the continuum is bigger than $\aleph_2$. All those facts can be found in [GeQu] and in [Go91]. It should be noticed that the Sacks model is the model where most invariants which can consistently be less than the continuum are equal to $\aleph_1$ and that only at stages of countable cofinality new reals are added [BaJu95][Lemma 1.5.7]. We summarize those features of the Sacks model which are crucial to our proof. The filter G_α is $\mathbb{S}_\alpha$-generic over V, where $\mathbb{S}_\alpha$ refers to the countable support iteration of Sacks Forcing of length $\alpha \leq \omega_2$.

Fact 2.2.2. In the Sacks model the following statements are true:

1. The continuum is $\aleph_2$
2. If $\alpha \in \omega_2$ and $\mathrm{cf}(\alpha) = \aleph_1$ then $V[G_\alpha] \models \mathrm{CH}$.
3. Let $x \in \omega^\omega \cap V[G_{\omega_2}]$ be a real in the Sacks model. Then there is an $\alpha \in \omega_2$ such that $x \in V[G_\alpha]$ and x is not in any $V[G_\beta]$ for $\beta < \alpha$. That means that no new reals are added in the final limit step ω_2.
4. No cardinals are collapsed in the iteration.

2.2.2 The Sacks Model

We will now begin to prove the main theorem of the first part of the thesis:

Theorem 2.2.3. *In the Sacks model the following holds: For a continuous coloring $c : [\omega^\omega]^3 \to 2$ the homogeneity number $\hom^3(c)$ is $\leq \aleph_1$.*

That is, the homogeneity number for three dimensions is strictly less than the size of the continuum. We will first give a rough description of the strategy of the proof: In each step of the iteration we want to add a weakly homogeneous set for some continuous coloring c. Since the iteration has length ω_2 every really number and hence every code for a coloring is added in some initial stage, we thus manage to take care of every coloring. We first argue that we can assume that the coloring c is already in the ground model. Then we capture every element of $V[G_{\omega_2}] \cap \omega^\omega$ in a a weakly homogeneous set in the ground model. Since the ground model satisfies the continuum hypothesis and no cardinals are collapsed this shows that $\hom^3(c)$ equals $\aleph_1$ in the Sacks model. We will construct the weakly homogeneous sets as follows. For $\alpha \in \omega_2$, an $\mathbb{S}_\alpha$-name $\dot{z}$ and $q \in \mathbb{S}_\alpha$ such that $q \Vdash \dot{z} \in \omega^\omega$, we know that $q \Vdash \dot{z} \in [T_q]$. We construct $q_1 \leq q_0 \leq q$, such that $T_{\dot{z},q_0}$ is c-homogeneous on C_0^{bl} and $T_{\dot{z},q_1}$ is weakly c-homogeneous. The most difficult part is to show how to find q_0. The condition q_1 can be obtained in exactly the same way just by switching indices.

Lemma 2.2.4. *Let $c \in V$ be a continuous coloring $c : [\omega^\omega]^3 \to \{0,1\}$, $\alpha < \omega_2$ and let $\dot{z}$ be an $\mathbb{S}_\alpha$-name for an element in ω^ω not added at any initial stage of the iteration $\mathbb{S}_\alpha$. Then the set $\{q \in \mathbb{S}_\alpha : T_q$ is weakly c-homogeneous $\} \in V$ is dense in $\mathbb{S}_\alpha$.*

First we give a technical lemma which will be used on several occasions during the inductive proofs:

Observation 2.2.5 (the argument is exactly like in [GKKS02], Claim 31)**.** Let $\mathbb{Q}$ be any notion of Forcing, $\dot{z}$ a $\mathbb{Q}$-name for a new element of ω^ω, and $c : [\omega^\omega]^3 \to \{0,1\}$ a continuous coloring. Let $\{q_0, q_1\}$ be a $\dot{z}$-proper set of conditions. Then there are $p_0 \leq q_0$ and $p_1, p_2 \leq q_1$ such that $\overline{c}(\{\dot{z}[p_0], \dot{z}[p_1], \dot{z}[p_2]\}) \in \{0,1\}$

Lemma 2.2.6. *For some $\alpha < \omega_2$ let q be a condition in $\mathbb{S}_\alpha$, let $c : [\omega^\omega]^3 \to \{0,1\}$ be a continuous coloring in the ground model and suppose that $\dot{z}$ is not added at an initial stage of the iteration $\mathbb{S}_\alpha$. There is a $q' \leq q$ such that $T_{\dot{z},q'}$ has at most one splitting node on each level and for all $\vec{x} \in [[T_{\dot{z},q'}]]^3$ the value $\overline{c}(\vec{x} \restriction \Delta^3(\vec{x})+1)$ is in 2.*

Proof. We construct a fusion sequence $(p_n)_n$ with witness $(F_n, \eta_n)_{n\in\omega}$ and a function $f : \omega \to \omega$ such that for pairwise different $\sigma_i, \sigma_i', \sigma_j, \sigma_j'$ which are elements in $\prod_{\gamma\in F_n} 2^{\eta_n(\gamma)}$, $n \in \omega$ the following four properties are satisfied:

(1) $\dot{z}[p_n * \sigma_i] \perp \dot{z}[p_n * \sigma_j]$.

(2) $\Delta(\dot{z}[p_n * \sigma_i], \dot{z}[p_n * \sigma_{i'}]) \neq \Delta(\dot{z}[p_n * \sigma_j], \dot{z}[p_n * \sigma_{j'}])$.

(3) $f(n+1) \geq |\dot{z}[p_n * \sigma_i]| \geq f(n)$.

(4) $\overline{c}(\dot{z}[p_n * \sigma_i], \dot{z}[p_n * \sigma_{i'}], \dot{z}[p_n * \sigma_j]) \in 2$.

We start with $F_0 = \emptyset$. The requirements for the witness can be ensured with some trivial bookkeeping. Assuming that p_n is constructed and F_{n+1}, η_{n+1} is given we show how to choose p_{n+1}. There are basically two cases, every other case is obtained by finite combination of the two:

a) $F_{n+1} = F_n \cup \{\beta\}$; $\eta_{n+1} = \eta_n \cup (\beta, 0)$ for $\beta \in \alpha$.

b) $F_{n+1} = F_n, \beta \in F_n$, $\eta_{n+1} \restriction (F_n \setminus \{\beta\}) = \eta_n \restriction (F_n \setminus \{\beta\})$ and $\eta_{n+1}(\beta) = \eta_n(\beta)+1$.

Case a) is trivial. For case b) we go through an enumeration $S = \{\sigma_0, \ldots, \sigma_{l-1}\}$ of the elements in $\prod_{\gamma\in F_n} 2^{\eta_n(\gamma)}$ fixing a splitting node in $T_{\dot{z},p_n*\sigma_i}$ for each $i \in l$. This splitting node will be the $\eta_{n+1}(\beta)$th splitting node in $p_n(\beta)$. We will need the following abbreviation for a $j \in l$:

$$S^+(j) = \{\tau \in \prod_{\gamma\in F_n} 2^{\eta_{n+1}(\gamma)} : \exists i \in j(\tau(\beta) \supseteq \sigma_i(\beta))\}.$$

Let $\delta = \max(F_n)$.

Claim 2. There is a condition $p_n' \leq_{F_n,\eta_n} p_n$ such that for each $\sigma_i \in S$ there exist names $\dot{p}_{i,0}, \dot{p}_{i,1}$ which satisfy

(i) $p_n' * \sigma_i \restriction \delta \Vdash \dot{p}_{i,0}, \dot{p}_{i,1} \leq p_n' * \sigma_i \restriction [\delta, \alpha) \wedge \mathrm{stem}(p_{i,0}(\delta)) \perp \mathrm{stem}(p_{i,1}(\delta))$.

(ii) For every $\tau \in S^+(i)$ and for every triple $\{x_0, x_1, x_2\} \in [\{\dot{z}[p_n * \sigma_i {}^\frown \dot{p}_{i,0}], \dot{z}[p_n * \sigma_i {}^\frown \dot{p}_{i,1}]\} \cup \{\dot{z}[p_n * \tau], \dot{z}[p * \sigma_{i+1}], \ldots, \dot{z}[p * \sigma_{l-1}]\}]$ it holds that $\overline{c}(\{x_0, x_1, x_2\}) \in 2$.

(iii) $g(0) = f(n)$, $\Delta(\dot{z}[p_n' * \sigma_i {}^\frown \dot{p}_{i,0}], \dot{z}[p_n' * \sigma_i {}^\frown \dot{p}_{i,1}]) > g(i)$ and for $i > 0$ the function $g(i)$ equals $\Delta(\dot{z}[p_n' * \sigma_{i-1} {}^\frown \dot{p}_{i-1,0}], \dot{z}[p_n' * \sigma_{i-1} {}^\frown \dot{p}_{i-1,1}])$.

Proof of Claim 2. We construct a $\leq_{F_n,\eta_n}$-decreasing sequence $(q_i)_{i \in |S|}$ which starts with p_n. In step i we will take care of $\sigma_i \in S$ by decreasing $q_i * \sigma_i$ such that an initial part of the real $\dot{z}$ longer than $g(i)$ is decided. Because $\dot{z}$ is not added by any initial part of $\mathbb{S}_\alpha$, we can choose the names $\dot{p}_{i,0}, \dot{p}_{i,1}$ such that $q_i * \sigma_i \restriction \sigma \Vdash \dot{z}[\dot{p}_{i,0}] \perp \dot{z}[\dot{p}_{i,1}]$. Decreasing q_i as well as the names, this can be done such that $V \models \dot{z}[q_i * \sigma_i {}^\frown \dot{p}_{i,0}] \perp \dot{z}[q_i * \sigma_i {}^\frown \dot{p}_{i,1}]$. For every $\tau \in S^+(i) \cup \{\sigma_{i+1}, \ldots \sigma_{l-1}\}$ decrease $q_i * \sigma_i {}^\frown \dot{p}_i^0$, $q_i * \sigma_i {}^\frown \dot{p}_i^1$ and the $q_i * \tau$ until they decide $\dot{z}$ up to the level where the color of any tripel $\{x_0, x_1, x_2\}$ of the form as in (ii) is decided. For an argument why this is possible compare [GKKS02] The only triples which cause problems are those which contain a branch extending $\dot{z}[q_i * \sigma_i {}^\frown \dot{p}_{i,j}]$ for $j = 0, 1$. Repeating this for every choice of $\tau \in S^+(i)$ yields the new q_i. After $|S|$ many steps we can set $p' = \bigcap_{i \in |S|} q_i$ and this proves the claim.

$\square$(Claim 2)

Let $\varrho \in 2^{\eta_n(\delta)}$. Without loss of generality p_n is such that $\dot{p}_{i,0}, \dot{p}_{i,1}$ exist as in the claim. We pick names $\dot{r}^{\varrho^\frown 0}, \dot{r}^{\varrho^\frown 1}$ such that for $\sigma_i \in S$ with $\sigma_i(\delta) = \varrho$ the following holds:

$$p_n * \sigma_i \restriction \delta \Vdash \dot{r}^{\varrho^\frown 0} = \dot{p}_{i,0} \wedge \dot{r}^{\varrho^\frown 1} = \dot{p}_{i,1}.$$

Let $\tau \in \prod_{\gamma \in F_{n+1}} 2^{\eta_{n+1}(\gamma)}$. We have to consider two cases:
Case 1: $\beta = \delta$:
Set $p_{n+1} \restriction \delta = p_n \restriction \delta$ and let $p_{n+1} \restriction [\delta, \alpha)$ be a sequence of names such that

$$p_{n+1} * \tau \restriction \delta \Vdash p_{n+1} * \tau \restriction [\delta, \alpha) = \dot{r}^{\tau(\delta)}.$$

Case 2: $\beta \lneq \delta$:
Set $p_{n+1} \restriction \delta = p_n \restriction \delta$ and let $p_{n+1} \restriction [\delta, \alpha)$ be a sequence of names but this time defined a little different by:

$$p_{n+1} * \tau \restriction \delta \Vdash p_{n+1} * \tau \restriction [\delta, \alpha) = \dot{r}^{\tau(\delta)^\frown j},$$

where $j = \tau(\beta)(\eta_{n+1}(\beta))$.
Set $f(n)$ as the level of the highest splitting point which we have fixed in step $n+1$. This proves the lemma. $\square$

In the sequence we have constructed, the elements in $\prod_{\gamma \in F_n} 2^{\eta_n}$ code splitting nodes in the tree of possibilities. In this case the following observation can be made.

Observation 2.2.7. Let $(p_n)_n$ be a fusion sequence as in the lemma with fusion q. T_n is the tree generated by $\{\dot{z}[p_n * \sigma] : \sigma \in \prod_{\gamma \in F_n} 2^{\eta_n(\gamma)}\}$. Obviously $T_q = \bigcup_{n \in \omega} T_n$.

Definition 2.2.8. Let F, η be as above and $c : [\omega^\omega]^3 \to \{0,1\}$ a continuous coloring. We will call a condition $p \in \mathbb{S}_\alpha$ (F, η, C_0)-*faithful* if for every $\tau_0, \tau_1 \in \prod_{\gamma \in F_n} 2^{\eta(\gamma)}$ the following holds:

(1) $\dot{z}[p * \tau_0] \neq \dot{z}[p * \tau_1]$.

(2) there is a color $j \in \{0,1\}$ such that

$$\forall \{x_0, x_1, x_2\} \in [T_{\dot{z},p}]^3 (x_0 \supseteq \dot{z}[p * \tau_0] \wedge x_1, x_2 \supseteq \dot{z}[p * \tau_1] \to c(x_0, x_1, x_2) = j).$$

Later we want to construct an "outer" fusion sequence $(r_{n'})_{n' \in \omega}$ in $\mathbb{S}_\alpha$ with $\eta_{n'}, F_{n'}$, such that $r_{n'}$ is $(\eta_{n'}, F_{n'}, C_0)$-faithful. The fusion r of this sequence gives a tree $T_{\dot{z},r}$ on which the following coloring $c' : [[T_{\dot{z},r}]]^3 \to 2$ is well-defined:

$$c'(\{x_0, x_1\}) = c(\{x_0, x_1, x_2\}) \text{ where } (x_1 \leq_{lex} x_2 \wedge \Delta(x_1, x_2) > \Delta(x_0, x_1)\}).$$

Usually in fusion arguments we have to choose the splitting nodes carefully. In this case it is not necessary, since by thinning out, the requirements can be met above any node. It is the same procedure as in the proof of Theorem 2.1.1 and will also be implemented by simultaneous construction of two "inner" fusion sequences $(q_n^{0,n'})_{n<\omega}, (q_n^{1,n'})_{n<\omega}$ starting with $q^{0,n'} = \tau_0 * r_{n'}$ and $q^{1,n'} = \tau_1 * r_{n'}$ in the "outer"step n'. Simultaneous construction means in this case that they share the same witnessing sequence and in each step we fix nodes in both trees. We will use F, η for the inner and the outer fusion, but the "splitting nodes" in the outer fusion are denoted by τ whereas the inner are denoted by σ.

Definition 2.2.9. For a Forcing notion $\mathbb{Q}$, an $m \in \omega$, and a condition $q \in \mathbb{Q}$, define $\restriction q = \{q' : q' \leq q\}$. Let $[\restriction p]^m$ denote the collection of all m-element antichains which are subsets of $[\restriction p]$. Accordingly, the collection of sets of finite antichains $[\restriction p]^{<\omega}$ is defined. For a fixed $\mathbb{Q}$-name $\dot{z}$ let $[\restriction q]^m_{\text{prop}}$ denote those elements of $[\restriction q]^m$ which are $\dot{z}$-proper. Let $\mathcal{F}, \mathcal{F}' \in [\restriction q]^{<\omega}$. We say $\mathcal{F} \prec \mathcal{F}'$ if for each $p \in \mathcal{F}$ there is a stronger condition $p' \in \mathcal{F}'$ with $p' \leq p$.

Definition 2.2.10. Let $\dot{z}$ be a fixed name for a new element of ω^ω. For a witness η, F, $q \in \mathbb{S}_\alpha$, we say a set $\{q^0, q^1\} \in [\restriction q]^2_{\text{prop}}$ is *inner-* (F, η, C_0)*-faithful* if there is a color $j \in 2$ such that for every sequence $(\sigma_0, \{\sigma_1, \sigma_2\}) \in \prod_{\gamma \in F} 2^{\eta(\gamma)} \times [\prod_{\gamma \in F} 2^{\eta(\gamma)}]^2$ the following holds: $\overline{c}(\{\dot{z}[q^0 * \sigma_0], \dot{z}[q^1 * \sigma_1], \dot{z}[q^1 * \sigma_2]\}) = j$.

The central notion for the induction step of the inner induction below are the sets E_j where $j \in 2$. Their definition looks very complicated, but the meaning can be split into a combinatorial part which says that we can always find Forcing conditions implementing splitting nodes above any $\dot{z}[q^{\xi(0)}]$ and a Forcing part which says that those splitting nodes can be realized behind any coordinate β. For the combinatorial part compare the proof of Theorem 2.1.1.

Definition 2.2.11.

$$E_j = \{\{p, q\} \in [\restriction 1_{\mathbb{S}_\alpha}]^2_{\text{prop}} : \forall \beta < \alpha \forall q' \leq q \forall \mathcal{F} \in [\restriction p]^{<\omega} \exists q'' \leq q' \exists \dot{q}_0 \exists \dot{q}_1 \leq q'' \exists \mathcal{F}' \succ \mathcal{F} :$$
$$q'' \restriction \beta \Vdash \dot{q}_0, \dot{q}_1 \leq q'' \restriction [\beta, \alpha) \wedge \overline{c}(\{x_0, x_1, x_2\}) = j \text{ for all } \{x_0, x_1, x_2\} \text{ which satisfy:}$$
$$\exists f \in \mathcal{F}' \; (x_0 = \dot{z}[f] \wedge x_1 = \dot{z}[q'' {}^\frown \dot{p}_0], x_2 = \dot{z}[q'' {}^\frown \dot{p}_1]) \tag{2.2.1}$$

Observe that if $\{p,q\} \in E_j$ for a $j \in 2$, then $\{p',q'\} \in E_j$ if $p' \leq p, q' \leq q$.

Lemma 2.2.12. *Suppose $\{p,q\} \in E_j$ and it is inner-$F, \eta, C_0^{\mathrm{bl}}$-faithful.*

a) Let $\beta \in \alpha \setminus F$, $F' = F \cup \{\beta\}$ and $\eta' = \eta \cup \{(\beta, 0)\}$. Then $\{p,q\}$ is inner-$F', \eta', C_0^{\mathrm{bl}}$-faithful.

b) Let $\beta \in F$. Let $\eta' \restriction (F \setminus \{\beta\}) = \eta \restriction (F \setminus \{\beta\})$ and $\eta'(\beta) = \eta(\beta) + 1$. Then there are $r_p \leq_{F,\eta} p, r_q \leq_{F,\eta} q$, such that $\{r_p, r_q\}$ is inner-$\eta', F, C_0^{\mathrm{bl}}$-faithful.

Proof. a) is obvious. For b): We first fix the $(\eta(\beta)+1)$th splitting nodes in q, then all of p and so on. The difficult part is the splitting in q. Let $\delta = \max(F)$. Let $S = (\sigma_l : l \leq l')$ a fixed enumeration of all elements of $\prod_{\gamma \in F} 2^{\eta(\gamma)}$.

Claim 3. There are $q' \leq_{F,\eta} q, p' \leq_{F,\eta} p$ such that for every $\sigma_l \in S$ and an $\mathcal{F} = \{p' * \sigma_k :\leq l'\}$ it is possible to find names $\dot{q}_{l,0}$ and $\dot{q}_{l,1}$ such that

(1) $q' * \sigma_l, \dot{q}_{l,0}, \dot{q}_{l,1}$ decide $\dot{z}$ and

(2) $q' * \sigma_l \restriction \delta \Vdash \dot{q}_{l,0}, \dot{q}_{l,1} \leq q' * \sigma_l \restriction [\delta, \alpha) \wedge c(\{x_0, x_1, x_2\}) = j$ for $\{x_0, x_1, x_2\}$ chosen as in (2.2.1) with $\mathcal{F}$.

(3) $q' * \sigma_l \restriction \delta \Vdash \mathrm{stem}(\dot{q}_{l,0}(\gamma)) \perp \mathrm{stem}(\dot{q}_{l,1}(\gamma))$

Proof of Claim 3. We construct a $\prec$ increasing sequence $(\mathcal{F}_l)_{l<l'}$ starting with $\mathcal{F}_{-1} = \{p * \sigma_k : k < l'\}$. At the same time we construct the condition q' via amalgamation. For each $\sigma_l \in S$ we use the definition of E_j to get a condition $q_l \leq q * \sigma_l$ and an $\mathcal{F}_l$ such that $\mathcal{F}_{l-1} \prec \mathcal{F}_l$ and such that there are names $\dot{q}_{l,0}, \dot{q}_{l,1}$ satisfying $q_l \restriction \delta \Vdash \dot{q}_{l,0}, \dot{q}_{l,1} \leq q_l \wedge \overline{c}(\{x_0, x_1, x_2\}) = j$ for all $\{x_0, x_1, x_2\}$ chosen as in (2.2.1) with $\mathcal{F}_l$. Amalgamation of the q_l yields q'. For p' we amalgamate the conditions contained in $\mathcal{F}_{l'-1}$. Observe that because of the choice of $\mathcal{F}_{-1}$ it is true that $p' \leq_{F,\eta} p$

□(Claim 3)

Dropping the apostrophe we just assume without loss of generality that the p, q are chosen as in Claim 3. We have to construct the new splitting nodes for p, q. First we use Claim 3 and construct r_q exactly as in the last step in the proof of

Lemma 2.2.6 inserting the $\dot{q}_0, \dot{q}_1$. If needed, we further reduce the elements of $\mathcal{F}$ such that $\dot{z}$ is always decided up to the maximal level of splitting nodes we have constructed in r_q. Using the technique of Lemma 2.2.6 we can then add arbitrary splitting nodes to p. Observe that their choice does not change anything in the color of the relevant sets. This finishes the proof of Lemma 2.2.12. $\square$

Lemma 2.2.13. *Let $r \in \mathbb{S}_\alpha$ and $\dot{z}$ such that the color of triples in $T_{\dot{z},r}$ is immediately decided in the sense of Lemma 2.2.6. For any $\dot{z}$-proper set of conditions $\{p, q\} \in [\restriction r]^2_{\mathrm{prop}}$, we can assume without loss of generality (decreasing all conditions) that it is an element of E_j for some $j \in \{0, 1\}$.*

Proof. Suppose $\{p, q\} \notin E_1$. We choose $\gamma < \alpha, q' \leq q, \mathcal{F} = \{f_k : k < l\} \in [\restriction p]^{<\omega}$ such that:

$$\forall q'' \leq q' \forall \dot{q}_0, \dot{q}_1 \leq q'' \forall \mathcal{F}' \succ \mathcal{F} : q'' \restriction \gamma \Vdash \dot{q}_0, \dot{q}_1 \leq q'' \restriction [\gamma, \alpha)$$
$$q'' \restriction \gamma \not\Vdash \overline{c}(\{x_0, x_1, x_2\}) = 1 \text{ if } \{x_0, x_1, x_2\} \text{ is picked according to (2.2.1)}$$

We argue in the ground model and define a function on the splitting nodes$d' : \mathbf{sp}(T_{\dot{z},q'}) \to l + 1$ by setting $d'(s) = l$ if there are no $q'' \leq q'$ and names $\dot{q}_0, \dot{q}_1$ such that:

$$q'' \restriction \gamma \Vdash \dot{q}_0, \dot{q}_1 \leq q'' \restriction [\gamma, \alpha) \wedge s^\frown 0 \subseteq \dot{z}[\dot{q}_0] \wedge s^\frown 1 \subseteq \dot{z}[\dot{q}_1].$$

If $q'', \dot{q}_0, \dot{q}_1$ exist in the described way, then there is a $k \in l$ such that for all $f' \leq f_k$ we get

$$q'' \restriction \gamma \not\Vdash \overline{c}(\{\dot{z}[f'], \dot{z}[\dot{p}_0], \dot{z}[\dot{p}_1]\}) = 1 \tag{2.2.2}$$

We set $d'(s)$ to be the minimal k of this sort.

Claim 4. d' is well-defined.

Proof of Claim 4. Let $s \in T_{\dot{z},q'}$ and assume $d'(s) \in l$. That is, there exist $q'', \dot{q}_0, \dot{q}_1$ realizing the splitting node s behind γ. Let $k \in l$ be minimal with the property in (2.2.2). Because $T_{\dot{z},q}$ is a tree according to Lemma 2.2.6 where the color is decided in the highest splitting point, this is equivalent to the following statement in the ground model:

$$\forall f' \leq f_k(|\dot{z}[f']|) = |s| \rightarrow \overline{c}(\{\dot{z}[f'], s^\frown 0, s^\frown 1\})) = 0.$$

Obviously these statements do not depend on the choice of $q'', \dot{q}_0, \dot{q}_1$.

□(Claim 4)

We define a continuous coloring $d : [[T_{\dot{z},q'}]]^2 \rightarrow l + 1$ by:

$$d(\{x, y\}) = d'(x \restriction \Delta(x, y)).$$

Claim 5. There is a $k \in l$ and a condition $q^{\#} \leq q''$ such that $T_{\dot{z},q^{\#}}$ is d'-homogeneous for the color k.

Proof of Claim 5. In [GKKS02] our Theorem 2.2.3 was proven for the case $m = 2$. It has been shown that for a name $\dot{z}$ of an element of ω^ω added in stage β of the iteration and a continuous coloring $c : [\omega^\omega]^2 \rightarrow \{0, 1\}$ defined in the ground model the set $\{p \in \mathbb{S}_\beta : T_{\dot{z},p}$ is c-homogeneous $\}$ is dense in $\mathbb{S}_\beta$. This can be proved with the same technique as in the end of Lemma 2.2.6. This means, it is first shown that there is a color $j \in 2$ such that a splitting node, which decides the color to be j, can be realized after any coordinate $\gamma < \alpha$. Then a fusion sequence is constructed for the witness $(F_n, \eta_n)_{n\in\omega}$ such that in every step the color is realized in the iteration behind the coordinate $\max(F_n)$. Hence, if we construct the condition $q^\sharp$ in such a way and we start with F_0 which contains γ, our claim is proved.

□(Claim 5)

We set $p^\sharp = f_k$ and show that $\{p^\sharp, q^\sharp\} \in E_0$:

Claim 6. Let $\beta < \alpha$, $p, q \in \mathbb{S}_\alpha$ and $\mathcal{F} \subseteq [\restriction p]^{<\omega}$ be such that for some $j \in \{0, 1\}$ there are $\dot{q}_0, \dot{q}_1$ and a $\mathcal{F}'$ such that $\mathcal{F} \prec \mathcal{F}'$ and for every triple $\{x_0, x_1, x_2\}$ which is chosen as in (2.2.1) the following holds:

$$(q \restriction \beta \Vdash \dot{q}_0, \dot{q}_1 \leq q \restriction [\beta, \alpha) \wedge \overline{c}(\{x_0, x_1, x_2\}) = j).$$

Let $\gamma < \beta$. Then there are $q^* \leq q$ and $\dot{q}_0^*, \dot{q}_1^*$ such that for every triple $\{x_0, x_1, x_2\}$ which is chosen as in (2.2.1) the following holds:

$$(q^* \restriction \gamma \Vdash \dot{q}_0^*, \dot{q}_1^* \leq q'[\gamma, \alpha) \wedge \overline{c}(\{x_0, x_1, x_2\}) = j).$$

Proof of Claim 6. Choose $q^* \leq q$ such that $q^* \restriction \beta$ decides $\dot{z}[\dot{q}_0]$ and $\dot{z}[\dot{q}_1]$ and set $q^* \restriction (\beta, \alpha) = q \restriction (\beta, \alpha)$. Set $\dot{q}_j^* \restriction [\gamma, \beta) = q^* \restriction [\gamma, \beta)$ and $\dot{q}_j^*[\beta, \alpha) = \dot{q}_j$ for $j \in \{0, 1\}$. They will do the job.

□(Claim 6)

We continue the proof of Lemma 2.2.13 observing with Claim 6 that it is enough to consider cofinally many coordinates $\beta < \alpha$ in order to prove $\{p^\sharp, q^\sharp\} \in E_0$. Let $\beta \geq \gamma, q \leq q^\sharp, \mathcal{F} \in [\restriction p^\sharp]^{<\omega}$. The coordinate γ was chosen as a counterexample at the very beginning of the proof of this Lemma. Fixing longer parts of $\dot{z}$ we can choose $\mathcal{F} \prec \mathcal{F}', q' \leq q, \dot{q}_0, \dot{q}_1$ such that a color j is decided. for all $\{x_0, x_1, x_2\}$ chosen according to (2.2.1) the following holds:

$$q' \restriction \beta \Vdash \dot{q}_0, \dot{q}_1 \leq q' \wedge c(\{x_0, x_1, x_2\}) = j.$$

By Claim 6 we can decrease q' and find $\dot{q}_0', \dot{q}_1'$ such that for all $\{x_0, x_1, x_2\}$ chosen according to (2.2.1) it holds that

$$q' \restriction \gamma \Vdash \dot{q}_0', \dot{q}_1' \leq q' \wedge c(\{x_0, x_1, x_2\}) = j.$$

Because of the choice of γ we can conclude $j = 0$. This proves the lemma. □

Now we can prove the induction step for the "outer" fusion. Recall the earlier notion of (F, η, C_0)-faithfulness:

Lemma 2.2.14. *Let r be a condition in $\mathbb{S}_\alpha$, let $\dot{z}$ be a $\mathbb{S}_\alpha$-name for an element of ω^ω not added at any initial stage, let η and $F \in [\beta]^{<\omega}$ be as above, and let $c : [\omega^\omega]^3 \to 2$ be a continuous coloring. Assume r is (F, η, C_0)-faithful.*

a) If $\beta \in \alpha \setminus F$, $F' = F \cup \{\beta\}$, and $\eta' = \eta \cup \{(\beta, 0)\}$, then r is (F', η', C_0)-faithful.

b) If $\beta \in F$, $\eta' = \eta \subseteq F \setminus \{\beta\} \cup \{(\beta, \eta(\beta)+1)\}$, and $F' = F$, then there is a condition $r' <_{\eta,F} r$ which is (F', η', C_0)-faithful.

Proof. Case a) is again obvious by the definition of faithfulness. Using the technique we have been using so far, it is easy to see that by decreasing r we can achieve that all sets of conditions $\{r * \sigma_i : \sigma_i \in \Pi_{\gamma\in F} 2^{\eta'(\gamma)}\}$ are $\dot{z}$-proper sets. With this preparation we will take the next occurring layer of splitting nodes, thin out above them and amalgamate in order to prove b):

Claim 7. Let r be a condition in $\mathbb{S}_\alpha$ and let $\{\sigma_0, \sigma_1\} \in [\Pi_{\gamma\in F} 2^{\eta'(\gamma)}]^2$ be such that the set $\{r * \sigma_0, r * \sigma_1\}$ is $\dot{z}$-proper. Then there is a condition $r' \leq_{F,\eta} r$ and a color j which satisfies the following. If $x_0 \in [T_{\dot{z}, r' * \sigma_0}]$ and $\{x_1, x_2\} \in [[T_{\dot{z}, r' * \sigma_1}]]^2$, then the value of c in $\{x_0, x_1, x_2\}$ is j.

Proof of Claim 7. According to Lemma 2.2.13 we can assume $\{r * \sigma_0, r * \sigma_1\} \in E_j$ for some color $j \in \{0, 1\}$. Now we simultaneously construct fusion sequences $(p_n)_{n<\omega} \leq r * \sigma_0$ and $(q_n)_{n\in\omega} \leq r * \sigma_1$ with witness $(F_n, \eta_n)_{n\in\omega}$ such that the pair $\{p_n, q_n\}$ is inner-(F_n, η_n, C_0-faithful). Let $p = \bigcap_{n\in\omega} p_n$ and $q = \bigcap_{n\in\omega} q_n$ denote the fusions of the sequences and we set $r' * \sigma_0 = p$ and $r' * \sigma_1 = q$.

□(Claim 7)

Using Claim 7 for every 2-element subset of $\Pi_{\gamma\in F} 2^{\eta'(\gamma)}$ and doing amalgamation gives the desired condition. This proves the lemma. □

Now we can prove the induction step which enables us to reduce a 3-dimensional coloring to the 2-dimensional case.

Lemma 2.2.15. *Let $\alpha < \omega_2$ and let $\dot{z}$ be an $\mathbb{S}_\alpha$-name for an element of ω^ω which is not added at any initial stage of the iteration $\mathbb{S}_\alpha$. For a condition $q \in \mathbb{S}_\alpha$, it is possible to find a condition $q' \leq q$ such that every pair $\{s_0, s_1\} \in [T_{\dot{z}, q'}]^2$ satisfies the following condition:*

$$\begin{aligned} (2.2.3) \quad & \exists j_0, j_1 \in \{0,1\} \forall \{x_0, x_1, x_2\} \in [[T_{\dot{z},q'}]]^3 ((x \supseteq s_0 \wedge x_2, x_3 \supseteq s_1 \\ & \Rightarrow c(\{x_1, x_2, x_3\}) = j_0) \wedge (x_0, x_1 \supseteq s_0 \wedge x_2 \supseteq s_1 \Rightarrow c(\{x_0, x_1, x_2\}) = j_1)) \end{aligned}$$

Proof. Lemma 2.2.14 allows us to construct a fusion sequence $(q_n)_{n\in\omega}$ with witness $(F_n, \eta_n)_{n\in\omega}$, such that each q_n is (F_n, η_n, C_0)-faithful. The fusion of this sequence is the desired condition. $\square$

Proof of Lemma 2.2.4. The corresponding statement for pair colorings $d : [2^\omega]^2 \to 2$ has been proved in [GKKS02]. Let $q_0 \in \mathbb{S}_\beta$. Then there is a $q_1 \leq q_0$ according to Lemma 2.2.6 such that $T_{\dot{z},q_1}$ splits at most once on a level and the color of a triple is immediately decided. We begin with triples in the set C_0. Lemma 2.2.15 guarantees that there is a $q_2 \leq q_1$ such that it is possible to define a coloring $c_0 : [[T_{\dot{z},q_2}]]^2 \to \{0,1\}$ in the ground model with $c_0(\{x_0, x_1\}) = c(\{x_0, x_1, x_2\}$ where $\{x_0, x_1, x_2\} \in [[T_{\dot{z},q_2}]]^3 \cap C_0$ and as usually $x_0 \leq_{lex} x_1 \leq_{lex} x_2$. With the above-mentioned result of [GKKS02] we conclude that there is a $q_3 \leq q_2$ such that c_0 is constant on $[T_{\dot{z},q_3}]^2$ and hence c is constant on $[T_{\dot{z},q_3}]^3 \cap C_0^{\mathrm{bl}}$. Now observe, that we get exactly the same Lemmas for C_1^{bl} just by exchanging p and q in the definitions of faithfulness. This leads to a q such that $[T_{\dot{z},q}]^3$ is weakly c-homogeneous. $\square$(Lemma 2.2.4)

Proof of Theorem 2.2.3. Let $c : [\omega^\omega]^3 \to \{0,1\}$ be a continuous coloring in the Sacks model $V[G_{\omega_2}]$. The coloring c can be coded as a real $\overline{c}$. In the limit step ω_2 there are no new reals added. Hence by switching to an intermediate model we can assume without loss of generality that c is an element of the ground model V and $V \models \mathrm{CH}$. Consider an element z of the Baire space ω^ω such that $z \in V[G_{\omega_2}]$. Such z must have been added at an initial stage $\alpha < \omega_2$ of the iteration and is hence an element of $V[G_\alpha]$ with the filter calculated up to α. We choose an $\mathbb{S}_\alpha$-name $\dot{z}$ for z. By Lemma 2.2.4 there is a $q \in G_\alpha$ such that $T_{\dot{z},q} \in V$ is weakly c-homogeneous and $V[G_\alpha] \models \dot{z} \subseteq \check{T}_q$. That means that every z which appears during the iteration is contained in a weakly c-homogeneous perfect set in the ground model. Because the set $T_{\dot{z},q}$ is closed it can be coded as a real number. Since the ground model satisfies CH there are only $\aleph_1$ many such weakly c-homogeneous sets and since neither $\aleph_1$ nor $\aleph_2$ is collapsed we can conclude that the Baire space can be covered by $\leq \aleph_1$ many weakly homogeneous sets.

$\square$(Theorem 2.2.3)

2.3 Separating Homogeneity Numbers

So far we have constructed a model for the statement

$$\text{“}\mathrm{hom}^2 = \mathrm{hom}^3 = \aleph_1 < 2^{\aleph_0}.\text{”}$$

A natural question is, whether these homogeneity numbers can consistently be separated.

Definition 2.3.1. Let $m = 2, 3$. A continuous coloring $c : [\omega^\omega]^m \to 2$ is called *reduced* if no open neighborhood is (weakly) c-homogeneous. We call the coloring *strongly reduced* if every basic open neighborhood contains a perfect (weakly) c-homogeneous subset.

In [GGK04] it was shown that when it comes to covering numbers we only have to look at reduced colorings. Given an arbitrary continuous coloring $c : [\omega^\omega]^m \to 2$, we can remove all (weakly) c-homogeneous basic open neighborhoods and thus obtain a set $X \subseteq \omega^\omega$ on which c is reduced. The remaining set X is closed without isolated points. If X is countable, then the homogeneity number is as well countable. Otherwise X is perfect and the homogeneity number is uncountable. The exact value of $\mathrm{hom}(c)$ only depends on the homogeneity number of the coloring c restricted to $[X]^2$. This can be easily seen using the Baire Category theorem and the fact that the c-homogeneous sets are nowhere dense for reduced c. In the same paper they constructed a continuous coloring $c_{\min} : [2^\omega]^2 \to 2$ which is a lower bound for the homogeneity number hom^2:

$$c_{\min}(\{x, y\}) = \Delta(x, y) \mod 2.$$

It is easy to argue that for every continuous reduced pair coloring c on the Baire space the following holds: $\mathrm{hom}(c) \geq \mathrm{hom}(c_{\min})$.

In the Sacks model all homogeneity numbers for reduced pair or triple colorings are $\aleph_1$. A natural question would be if the generalization of homogeneity number to 4 dimensions yields different values in the Sacks model. In this case for a given coloring

$d : [\omega^\omega]^4 \to 2$ we would call a set X *weakly d -homogeneous* if X is c-homogeneous on $C^{\mathrm{bl}}_{4,i}$ for every $i \in 6$ where the $C^{\mathrm{bl}}_{4,i}$ are derived from the Blass coloring [Bla81]. We could not prove that in the Sacks model the four dimensional homogeneity number $\hom^4$ is $\aleph_1$. This statement is likely to be true, because experience shows that everything which can consistently be less than 2^ω equals $\aleph_1$ in the Sacks model. We have already mentioned that with countable support iteration the continuum necessarily equals $\aleph_2$. Hence in order to allow many different values for homogeneity numbers other methods will be needed. But even if the continuum was very big, there would be only two possible values for homogeneity numbers:

Lemma 2.3.2. [GGK04, Lemma 8] $\hom(c_{\min})^+ \geq 2^{\aleph_0}$ *and hence* $(\hom^3)^+ \geq 2^{\aleph_0}$.

The second part of the lemma follows from the first if we consider the 3-dimensional coloring given by $d(\{x, y, z\}) = 0 \iff \{x, y, z\}$ is $c_{\min}$-homogeneous.

In [GGK04] a second continuous pair coloring $c_{\max} : [X]^2 \to 2$ for a compact space $X \subseteq \omega^\omega$ was constructed and the relative consistency of $\hom(c_{\min}) < \hom(c_{\max})$ was shown. In this model $\hom(c_{\max})$ is equal to the continuum and to all higher dimensional homogeneity numbers $\hom^2, \hom^3, \hom^4, \ldots$. This model was constructed as a ω_2-iteration with countable support of a new tree Forcing $\mathbb{P}_{\max}$. The Forcing $\mathbb{P}_{\max}$ satisfies Axiom A (compare [GGK04],[GeQu]) while adding new reals. Axiom A basically states that the technique of fusion is on hand and hence that the new reals can be captured in certain sets of the ground model. In the constructed model no cardinals are collapsed and the continuum is $\aleph_2$. That is, the techniques are not at all far away from the things we have seen in the chapter before.

In [Ge05] a similar model was constructed in which for reduced continuous colorings $c : [\omega^\omega]^2 \to 2$ the value of $\hom(c)$ is $\aleph_1$, but at the same time the covering number for the ideal generated by the closed sets which can be viewed as n-branching trees $\{[T] \subseteq \omega^\omega : \exists n \in \omega \forall s \in T(|\operatorname{IS}(s, T)| \leq n)\}$ equals $\aleph_2 = 2^\omega$. In this section we will shortly review this model and its properties and conclude with the following theorem.

Theorem 2.3.3. *The following statement is consistent with* ZFC*:*

$$\mathrm{hom}^2 \neq \mathrm{hom}^3.$$

The compact subspace $\prod_{n\in\omega}(n+1) \subseteq \omega^{<\omega}$ is called $\mathbb{R}_C$. Let the compact Miller-Lite-Forcing $\mathbb{ML}$ consist of subtrees $T \subset \mathbb{R}_C$ such that for each $t \in T$ and $n \in \omega$ there is a $t' \in T$ such that $t \subseteq t'$ and t' has at least n immediate successors. The elements of $\mathbb{ML}$ are ordered by set theoretic inclusion and as usual in Forcing we will use variables p, q for them. We assume that if $t \in p \in \mathbb{ML}$ is an n-th splitting node, it has exactly n immediate successors in p, without further mentioning it. This is a dense suborder of the Miller-Lite-Forcing as described in [Ge05]. Let G be a $\mathbb{ML}$-generic filter over a model V. We can define the generic element which is an infinite branch in $\mathbb{R}_C$ by setting $x = \bigcap_{p\in G} p$. We call this real the *the Miller-lite real.* On the other hand we can capture new reals in certain sets of the ground model. The definition of fusion sequence is just the one given in the previous chapter. The interested reader can find the proofs of the following lemmas in [Ge05]. The methods are very much like the ones we applied in the chapter about Sacks Forcing using fusion sequences and the finite Ramsey theorem for the first lemma.

Lemma 2.3.4. *Let $c : [[\mathbb{R}_C]]^2 \to 2$ be a continuous coloring in the ground model which is immediately decided. Let $p \in \mathbb{ML}$ and $\dot{z}$ a $\mathbb{ML}$-name such that $p \Vdash \dot{z} \in \mathbb{R}_C$. Then there is a $p' \leq p$ such that $T_{\dot{z},p'}(\dot{z}) = \{s \in \mathbb{R}_C : \exists q \leq p'(q \Vdash s \subseteq \dot{z})\}$ is c-homogeneous.*

Let $\mathbb{ML}_\alpha$ be the iteration with countable support of length $\alpha \in \omega_2$ and let $V[G_{ML}]$ be the model which is generated by countable support iteration of Miller-Lite-Forcing of lenght ω_2.

Lemma 2.3.5. *Let $\dot{z}$ an $\mathbb{ML}_\alpha$-name for an element of $\mathbb{R}_C$ and let $c : [\mathbb{R}_C]^2 \to 2$ be a continuous coloring in the ground model. For a $p \in \mathbb{ML}_\alpha$ there is a $p' \leq p$ such that $T_{\dot{z},p'}[\dot{z}]$ is c-homogeneous.*

We just recite the argument from the Sacks model. Every coloring in $V[G_{ML}]$ can be coded as a real and is thus already in an intermediate model V'. Since CH holds in cofinally many models we can assume that it holds in V' and argue now under the assumption that V' actually is the ground model. Hence every element of $\mathbb{R}_C$ in $V[G_{ML}]$ is contained in a ground model c-homogeneous set. This shows the first half of the equation in the following statement.

Theorem 2.3.6. *In $V[G_{ML}]$ for the homogeneity number of continuous pair colorings the following is true:*

$$\hom^2 = \aleph_1 < 2^\omega = \aleph_2 = \hom^3.$$

Proof. In order to prove the second half of the equation consider the following coloring $d : [\mathbb{R}_C]^3 \to 2$ with:

$$d(\{x, y, z\}) = 1 \iff \Delta(x, y) = \Delta(y, z).$$

Hence all d-homogeneous subsets of color 1 are finite, actually of size 3. If $T \subseteq \mathbb{R}_C$ is such that $[T]$ is d-homogeneous of color 0 then every node $s \in T$ has at most 2 immediate successors. Hence $\hom^3(d)$ depends on how many binary sets are needed to cover $\mathbb{R}_C$.

Claim 8. Let $T \subseteq \mathbb{R}_C$ and $T \in V$ be such that $[T]$ is d-homogeneous of color 0. Then the Miller-Lite-Forcing adds a real $\dot{z}$ such that $V[G_{ML}] \models \dot{z} \in \mathbb{R}_C \setminus [T]$.

Proof of Claim 8. Let $p \in \mathbb{ML}$ be a condition, we have to find a $q \leq p$ and a $\dot{z}$ such that $q \Vdash \dot{z} \notin T$. Suppose $|p \cap T| < \omega$. Then the Miller-lite real $\dot{z_m}$ and p prove the claim. Otherwise pick $t \in p \cap T$ such that $\mathrm{IS}(t, p) > 2$. Then $p \setminus T$ contains a $q \in \mathbb{ML}$ and q together with $\dot{z_m}$ shows the claim.

□(Claim 8)

Let $\bigcup_{i \in \omega_2} X_\alpha = \mathbb{R}_C$ be a d-homogeneous cover in $V[G_{ML}]$. Take the closures $\overline{X_i}$, $i \in \omega_1$. Then $\overline{X_i}$ can be coded as reals and hence there is an intermediate stage $V[G_\alpha]$ where all the sets $\overline{X_i}, i \in \omega_1$ already exist. At stage $\alpha + 1$ a Miller-lite real $\dot{z}_m$ is added such that $\mathbb{ML}_\alpha \Vdash \dot{z}_m \notin \bigcup_{i \in \omega_1} X_i$ for all X_i which are homogeneous of color 0. The other X_i are finite and thus anywise avoided by the Miller-lite real. □

As mentioned before in the definition of the partition $C_0^{\mathrm{bl}} \dot\cup C_1^{\mathrm{bl}}$ the triples in which all elements split in a single node could also be introduced as the third type. We did not add them to the basic partition because they can be avoided on a perfect set. Hence it should be observed that since $\mathbb{R}_C$ is a compact subspace of the Baire space there is a continuous tree order preserving function $g : [2^{<\omega}] \to [2^{<\omega}]$ such that the coloring d can be translated into the coloring $d'(x, y) = d(f(x), f(y))$ and $\mathrm{hom}(d') = \mathrm{hom}(d)$. So the homogeneity number raises even for colorings on the Cantor space on which the question about a third type does not at all occur.

2.4 Problems and Perspectives

We want to close this first part by mentioning certain features of triple colorings and the differences to pair colorings. Thus giving a motivation for further study. After its discovery, the coloring introduced by Blass, suggested that no reasonable Ramsey results could be proved for continuous colorings of triples. This turned out to be wrong. The theorem of Blass and the subsequent work, including our own, show that the notion of weak homogeneity is the correct generalization to the three dimensional case and that with this new notion results about the homogeneity number can be obtained. Unfortunately we did not succeed to generalize the other results contained in [GKKS02] to triple colorings. That is, it was not possible to define a reasonable $c_{\min}$ for triples. Hence we still lack a good characterization of what is means to be a genuine triple coloring. Recall that in the two dimensional case we could restrict our attention to reduced colorings.

Observation 2.4.1. For a continuous coloring $d : [2^\omega]^3 \to 2$ and a $j \in 2$ we try to cover the space 2^ω with sets $X \subseteq 2^\omega$ which are d-homogeneous on C_j. The smallest number of d-homogeneous sets which is needed to cover the Cantor space is called homogeneity number on C_j. Hence we get a homogeneity number on C_0 and one on C_1. The higher one equals $\mathrm{hm}(d)$. Hence the amount of sets which is homogeneous in one type is a lower bound for the homogeneity number.

This fact can be easily proved. Given two witnessing coverings $(X_i^j)_{j\in 2, i\in I_j}$ the collection $X_i^0 \cap X_{i'}^1$ for $i \in I_0, i' \in I_1$ are a weakly c-homogeneous cover of size $|I_0 \times I_1| = \max\{|I_0|, |I_1|\}$. Hence within our interest and in this last chapter, it is enough to restrict our study to continuous triple colorings of the set C_0^{bl} and ask for homogeneous sets on C_0^{bl}.

For continuous colorings of pairs $c : [Y]^2 \to 2$ the chromatic number $\mathrm{chrom}(c)$ is somehow degenerate, because it can be easily seen from the following statement that $\mathrm{chrom}(c)$ is either countable or has the size of the continuum.

Lemma 2.4.2. *[GKKS02] Let $c : [Y]^2 \to 2$ be a continuous coloring. If Y cannot be covered by countably many sets from $\mathcal{I}_0^c$ then there is a perfect set in $\mathcal{I}_1^c$.*

This can be proved by removing all basic open sets which are countably coverable. The leftover is closed and such that every basic open neighborhood contains pairs which are colored by 1. Then a perfect set in color 1 can be easily constructed.

The homogeneity number is not degenerate in this sense. Obviously for a continuous pair coloring c it is always the case that $\mathrm{hom}(c) \leq \mathrm{chrom}(c)$. Hence if the chromatic number is countable they are the same. On the other hand it is consistent that for every continuous pair coloring c which satisfies $\mathrm{chrom}(c) > \aleph_0$ the homogeneity number $\mathrm{hom}(c)$ is strictly smaller than $\mathrm{chrom}(c)$. Hence for a continuous pair coloring c the following statements are equivalent:

a) c is reduced

b) $\mathrm{chrom}(c) = 2^\omega$

c) there is a Forcing extension which satisfies $\mathrm{chrom}(c) \neq \mathrm{hom}(c)$

d) every open neighborhood contains perfect sets in both colors (strongly reduced)

These characterizations are equivalent for continuous pair colorings and they characterize what a sufficiently interesting coloring might be. The underlying intuition is that the two colors should somehow equally occur. Item c) for example states that one actually needs homogeneous sets in both colors to cover the space. Trying to generalize those notions to triple colorings failed very much. Lets first consider the following class of colorings which even though it satisfies item a), the two colors are not at all equally spread. Furthermore the homogeneity numbers for this class of colorings are bounded by hom^2.

Example 1. Take a reduced pair coloring $d : [2^\omega]^2 \to \{0,1\}$ and set:

$$c(\{x_0, x_1, x_2\}) = \begin{cases} 0, & \text{if } \{x_0, x_1, x_2\} \text{ is } d\text{-homogeneous} \\ 1, & \text{otherwise} \end{cases}$$

Let $U \subseteq 2^\omega$ be a basic open set. We show that it contains triples in both colors. Since d is reduced there is a pair $\{x_0, x_1\} \in U$ such that $d(\{x_0, x_1\}) = 1$ and there are neighborhoods U_{x_0}, U_{x_1} such that d restricted to pairs in $U_{x_0} \times U_{x_1}$ is constant with value 1. Then there are $x_2, x_3, y_2, y_3 \in U_{x_1}$ such that $d(\{x_2, x_3\}) = 1$ and $d(\{y_2, y_3\}) = 0$. Now c colors the triple $\{x_0, y_2, y_2\} \in [U]^3$ by 1 whereas the triple $\{x_0, x_2, x_3\} \in [U]^3$ is colored by 0. This shows that the homogeneous sets are nowhere dense and that the chromatic number must be uncountable. However, it follows from Ramsey's theorem that all c-homogeneous sets which are homogeneous for 1 contain at most six elements. Let $(X_i)_{i \in I}$ be a c-homogeneous cover, that is $X_i \in \mathcal{I}^c_{\mathrm{hom}}$. Let $I' \subseteq I$ be the index set such that X_i is homogeneous of color 1 for each $i \in I' \subseteq I$. Because singletons are by definition 0-homogeneous the cover $(X_i)_{i \in I \setminus I'} \cup (\cup X_i)_{i \in I'}$ is a c-chromatic cover and hence $\mathrm{chrom}(c) = \mathrm{hom}(c)$. It can be easily seen that for this coloring the inequality $\mathrm{chrom}(c) \leq \mathrm{hom}(d)$ is valid, because every d-homogeneous set is c-homogeneous for color 0.

Hence for pair colorings the notion reduced and strongly reduced coincide. This is not the case when it comes to triple colorings. A strongly reduced coloring can be easily constructed by taking a triple coloring d such that there are perfect sets P, P' such that d colors elements of $[P]^3 \cap C_i^{\mathrm{bl}}$ by 0 and elements of $[P']^3 \cap C_i^{\mathrm{bl}}$ by 1 for $i = 0, 1$. Obviously this implies item (2). Hence this is an example of a strongly reduced coloring satisfying (2) and (3), the latter witnessed by the Sacks model. Suggesting strongly reduced as characterization of continuous triple colorings we can ask the following question, which remains open:

Question 1. Is there a minimal coloring $c_3 : [2^\omega]^3 \to 2$ such that for every strongly reduced $c : [2^\omega]^m \to 2$ the following holds: $\mathrm{hm}(c_3) \leq \mathrm{hm}(c)$.

Obviously the problem is, that a very large class of colorings is strongly reduced. We give a few examples for $m = 3$ in which it can be observed that they do not share a common description as the one for reduced colorings:

a) $c_0(\{x_0, x_1, x_2\}) = \Delta(x_0, x_1) \mod 2$

b) $c_1(\{x_0, x_1, x_2\}) = \min\{\Delta(x_0, x_1), \Delta(x_1, x_2)\} \mod 2$

c) $c_2 \restriction C_0$ such that$c_2(\{x_0, x_1, x_2\}) = \Delta(x_0, x_1) + x_0(\Delta(x_1, x_2))$.

We want to conclude this section with a lemma, which shows even more that even the notion of strongly reduced colorings does not imply that the colors are equally spread in the neighborhood of a point.

Lemma 2.4.3. *Let $c : [2^\omega]^2 \to 2$ be a reduced continuous coloring. Then every $x \in \omega^\omega$ is contained in a homogeneous perfect set $Y \subseteq 2^\omega$.*

Proof. An easy fusion argument shows that there is a $T_0 \subseteq 2^{<\omega}$ such that $x \in [T_0]$, the coloring is immediately decided and for every $s_0 \in T_0$ such that $s_0 \not\subseteq x$, there is an extension $s_1 \in T_0$ such that $2^{<\omega}[s_1]$ is contained in $T_0[s_1]$. Since every basic open neighborhood of the Cantor space contains perfect sets in both colors there is a perfect subtree $T_1 \subseteq T_0$ such that $x \in [T_1]$ and there is a color $j \in 2$ such that $[T_1]$ is c-homogeneous with color j and for all $t \subseteq x$ which are incompatible to s_1 the color of $\{t, s_1\}$ equals j. We construct an infinite sequence of perfect trees $(T_i)_{i\in\omega}$ which starts with $T_0 = T$. Each tree contains x as an infinite branch. Secondly for the sequence $(s_i)_{i\in\omega}$ such that $s_i = x \restriction i^\frown(1 - x(i))$ we construct in such a way that every $T_i[s_i]$ is c-homogeneous. Let $i > 0$ and suppose that $s_i \in T_{i-1}$. If the latter is not the case then we set $T_{i+1} = T_i$. Otherwise there is a c-homogeneous perfect tree $T' \subseteq T_{i-1}[s_i]$ which is c-homogeneous for a color $i_j \in 2$. Set T_i as the tree T_{i-1} where $T_{i-1}[s_i]$ is substituted by the homogeneous T'. After ω many steps set $T_\infty = \bigcup_{i\in\omega} T_i$. There is a color j' and an infinite set I such that for all $i \in I$ the color j_i is defined and $j_i = j'$. The tree $T^* = \cup_{i\in\omega} x \restriction i \bigcup_{i\in I} T_\infty[s_i]$ is perfect and c-homogeneous with color j'. □

Lemma 2.4.4. *In the Sacks model the following statement is true for $m = 2$ but wrong for $m = 3$:*

For a continuous coloring $c : [\omega^\omega]^m \to 2$ the continuum can be covered by $\aleph_1$ many perfect (weakly) c-homogeneous sets.

Proof. We first argue that this statement is true for the case $m = 2$. Recalling the proof of Theorem 2.2.3 we can assume that we argue in a model V which satisfies CH and the continuous coloring $c : [\omega^\omega]^2 \to 2$ is an element of V. As we saw above there is $\omega^\omega = P \cup X$ such that X can be covered with homogeneous basic open sets and c is reduced on P. Let $x \in \omega^\omega$ a real number in the Sacks model. If $x \in V$ we apply Lemma 2.4.3 and the homogeneous perfect set P is still perfect and homogeneous in the Sacks model. If $x \notin V$ the argument from 2.2.4 shows that there is a perfect set in V and hence in the Sacks model which contains x. Since V satisfies CH the statement follows.

In the case $m = 3$ we can construct a reduced continuous coloring: Let $x \in 2^\omega$ and define $c(\{x_0, x_1, x_2\}) = 0$, if $x \restriction \Delta(x_1, x_2) = x_0 \restriction \Delta(x_1, x_2)$ and $x(\{x_0, x_1, x_2\} = 1$ otherwise. Obviously there exist homogeneous perfect sets in both colors in every basic open neighborhood, but there is no perfect homogeneous set which contains x. □

The possibility to separate the value $\hom^2$ from $\hom^3$ does not in itself show that there is a rise in complexity which is due to the dimensions, since in [GKKS02] a model is constructed where the homogeneity numbers for different pair colorings were separated. Furthermore, a coloring $c : C_0^{\mathrm{bl}} \to 2$ which is immediately decided might be interpreted as a coloring on pairs of nodes s, t such that $|s| = |t|$ and $d(\{s, t\}) = c(\{x, y, z\})$ where $s \subseteq x$ and $y \wedge z = t$. This together with the fact that our argument cannot be generalized to dimension 4 might hint towards the conclusion that the sudden increase in complexity for continous colorings appears in the comparison between dimension 3 and 4 and is hence shifted with 1. That amounts to ask if there is a deeper reason why our proof of Theorem 2.1.1 could not be generalized to dimension 4. Furthermore as mentioned before it is open whether or not the statement $\hom^2 = \hom^3 \neq \hom^4$ is true in the Sacks model as well as the following more general statements with which we will conclude this first part of the thesis:

Question 2. Is it consistent to have $\hom^4 < 2^\omega$?

Question 3. Is it consistent to have $\hom^3 \neq \hom^4$?

Chapter 3

Arrow results for $G_{\max}$

3.1 Introduction

In the second part of this thesis we want to study the so called basic partitions of the universal profinite graph $G_{\max}$. This part can be read independently from the first one. Those who have read the first part will realize that we are here using similar techniques since again we are concerned with higher dimensional continuous colorings on compact subspaces of the Baire space.

Let F and G be graphs and let c be a coloring of the edges of F with two colors. An induced subgraph of F simply consists of a subset of the vertex set $F' \subseteq F$ together with the edge set of F restricted to F'. We say F' is a copy of G if F' and G are graph isomorphic with respect to the induced edge relation. If F contains a c-homogeneous induced copy of G for every such coloring c, we say that F *arrows* G *for finite colorings of edges* and write:

$$F \rightarrowtail (G)^2_2.$$

For a finite graph G one can prove that there is always a graph F which arrows G [NesRoed77][NesRoed76]. This is a generalization of the finite Ramsey Theorem which is also true under certain restrictions for higher dimensional colorings of graphs. Higher dimension here means that not just edges, but bigger induced graphs are colored.

Surprisingly, the generalization to edge colorings of higher cardinalities is not true. Hajnal and Komjath [ErHaPo73] showed, that there is a model in which an uncountable graph can be constructed which cannot be arrowed by a bigger graph. This model is obtained by adding a single Cohen real. Of course the graph is not constructed on a very well behaving set of reals. In this context the question of whether there are infinite graphs, which arrow themselves pops up naturally as well as the question whether all countable graphs are arrowed by some bigger (still countable) graph.

The random graph G_ω is the countable universal graph. That means G_ω is the minimal graph which contains every countable graph as an induced subgraph. Therefore it looks like a good candidate to arrow countable graphs which is only true in dimension one. That means that it can be easily proved that for vertex colorings the graph G_ω arrows itself but that Erdös and Posa presented a coloring of edges such that every induced copy of the countable complete bipartite graph contains edges in both colors. Fortunately, this coloring is the worst it can get. That means that this coloring can be viewed as a partition $C_0 \dot\cup C_1$ of the edge set and in [PoSau96] they managed to show that for every coloring there is a copy which is homogeneous on each of the sets C_0 and C_1 independently. That is, the partition forms a basic partition for the random graph. In subsequent papers [Sau03][Sau06] basic partitions of all sets of finite subgraphs were found.

There is a generalization of G_ω to the cardinality of the continuum. The profinite graph $G_{\max}$ can be viewed as a continuous graph on a perfect compact subspace of the Baire space which thus admits a representation as a finitely branching subtree of $\omega^{<\omega}$. This graph does not contain all countable graphs as induced subgraphs. We will generalize the results of Sauer to the graph $G_{\max}$, thus defining a basic partition. Just as in the last section the Sierpińsky Coloring is a limiting example when it comes to arrow results. Again we will restrict our attention to continuous colorings. Then we will generalize the Blass Partition from part one of this thesis to a partition of the induced subgraphs of $G_{\max}$ and finally prove that those partitions are actually basic. That is for continuous colorings we can find induced copies of $G_{\max}$ which are homogeneous on each of the pieces of the partition. Furthermore, it turns out that these partitions can be generalized to closed induced subgraphs of higher cardinalities within certain limits.

The graph $G_{\max}$ appeared in [GGK04] for the first time in the study of finite continuous colorings of pairs of reals. Colorings with 2 colors can be interpreted as a graph structure on the set of reals by taking one color as edges and the other as non-edges. The continuous coloring $c_{\max}$ which produces the graph $G_{\max}$ is the canonical representative for those colorings which have a huge homogeneity number. That is, many $c_{\max}$-homogeneous sets are needed to cover the real line. The canonical representative for continuous colorings which are non trivial but of small homogeneity number produces a graph $G_{\min}$ which is such that a pair of reals is connected if and only if they split on an even level. In his doctoral thesis [Sheu05] Sheu formulates a theorem which comes down to the statement that for every Borel coloring of the pairs of nodes in $G_{\min}$ there is an induced copy of $G_{\min}$ such that all pairs which are edges are colored with the same color. If the coloring is continuous on finite induced subgraphs, the Blass types combined with the graph structure leads to a basic partition. The main difficulty in the present case of $G_{\max}$ stems from the fact that the tree which underlies the node set of $G_{\max}$ has to be non-binary.

3.2 Notation

3.2.1 Ordered Graphs

Unless mentioned otherwise all graphs are ordered. Hence a *graph* is a triple $G = (X, E, <^G)$ such that $E \subseteq [X]^2$ and $<$ is a linear ordering on V. We call X the vertex set of G and E is the set of edges of G. We will use the same letter for the graph and the set of vertices, mainly the capital letters F, G. For $n \in \omega$, $K_n = (n, [n]^2)$ is the complete graph. For a subset $Y \subseteq X$ we call $(Y, E \cap [Y]^2, <\restriction Y)$ the induced ordered subgraph spanned by Y in X. For two ordered graphs $G_1 = (X_1, E_1, <_1)$ and $G_2 = (X_2, E_2, <_2)$. An injection $e : X_1 \to X_2$ which preserves the order and the edge relation is called an *embedding of* G_1 *into* G_2. The symbol $\binom{G}{H}$ refers to the set of all embeddings of the ordered graph H into the ordered graph G. Unless stated otherwise, the word "graph" will always mean "ordered graph" from now on.

If H is finite and the set of vertices of G is a Hausdorff topological space with a topology τ, we can define a topology on the set of embeddings $\binom{G}{H}$ for a finite graph H. A set $X \subseteq \binom{G}{H}$ is *open* if and only if for each $f \in X$ there are disjoint neighborhoods $U_1, \dots U_{m-1} \in \tau$ with $f(i) \in U_i$ for $i \in m-1$ such that

$$\forall f' \in \binom{G}{H} (\forall i \in m-1(f'(i) \in U_i)) \to f' \in X).$$

One can also look at the space of images $\{\mathrm{rg}(f) : f \in \binom{G}{H}\}$ as a subspace of $[\mathbb{R}_C]^{|H|}$ and the topology on $\binom{G}{H}$ is just the subspace topology translated to embeddings instead of points. In this sense we will speak about continuous or Borel functions on the set of embeddings into a given graph.

Let $G = (V, E, <)$ be an ordered graph and for all $v \in V$ let $H_v = (V^v, E^v, <^v)$ be an ordered graph. We define the *ordered graph sum*$\sum_{v \in V} H_v$ as the graph whose set of vertices is the disjoint union $\bigcup_{v \in V} V^v$ ordered by $u < w$ if and only if there either exists a $v \in V$ such that $u, w \in V^v$ and $u <^v w$ or if they come from different graphs $u \in V^{v_1}, w \in V^{v_2}$ and v_1 is smaller than v_2 in the ordering on G. The edge set of $\sum_{v \in V} H_v$ is given by $\bigcup_{v \in V} E^v \cup \{\{u, w\} : (\exists \{v_1, v_2\} \in E)(u \in V^{v_1} \wedge w \in V^{v_2})\}$. The special case in which H_v is a single vertex for all $v \in V$ except for a single distinguished vertex $u \in V$ is called "a blow up of the vertex u by $H = H_u$ ". In a blow up of a vertex $u \in V$ by H it holds that for every $v \in V \setminus \{u\}$ either $\{v, w\} \in E$ for all $w \in H^v$ or $\{v, w\} \notin E$ for all $w \in H^v$.

Definition 3.2.1 (The ordered random Graph on ω)**.** A graph $G = (V, E)$ is *random* if for every two disjoint finite sets $A, B \subseteq V$ there exists $v \in V$ which is connected by edges to every member of A and not connected to any member of B. Any two countable random graphs are isomorphic to each other, therefore there is a unique, up to isomorphism, random graph with vertex set ω. We fix $E \subseteq [\omega]^2$ such that (ω, E) is a random graph and and refer to $G_\omega = (\omega, E_\omega, <_\omega)$ from now on as *the* ordered random graph.

A convenient construction of the random graph is the following. Let $(s_m)_{m\in\omega}$ be an enumeration of the eventually 0 infinite binary sequences. Let $n \in m \in \omega$ put $\{n, m\} \in E$ iff $s_m(n) = 1$. The graph $G = (\omega, E)$ is random and therefore isomorphic to G_ω which can be easily seen as follows: Let $A, B \subseteq \omega$ be finite disjoint sets. Set $s \in \omega^{<\omega}$ a finite sequence of length $\max\{A \cup B\}$ such that $s(j) = 0$ iff $j \in A$. There is a t such that $s \subseteq t$. The *norm* of a finite ordered graph G is the maximal n for which there exists an order preserving graph embedding of $G_\omega \restriction n+1$ into G. We write $\|G\|_{\text{rand}} = n$.

3.2.2 $G_{\max}$ and Graphs on Ordered Trees

If $H \subseteq G$ is an infinite graph and G is a Hausdorff topological and closed space the symbol $\binom{G}{H}$ refers to all embeddings $f : H \to G$, such that $\operatorname{rg}(f) \subseteq H$ is closed. If H is finite this is trivially true. A function $c : \binom{G}{H} \to r$ for $r \in \omega$ is called a coloring of $\binom{G}{H}$ by r colors. An induced subgraph $G' \subseteq G$ is *m-chromatic for c* if $\left|\operatorname{rg}\left(c \restriction \binom{G'}{H}\right)\right| \leq m$. Let G, H, I be arbitrary ordered graphs. If for a coloring $c : \binom{G}{H} \to r$ with r colors there is an m-chromatic induced copy of I we write $G \rightarrowtail (I, \leq)^H_{r,m}$. In case $m = 1$ we skip the index m at all.

The basic induced Ramsey Theorem for ordered graph is the Nesetril-Rödl Theorem.

Lemma 3.2.2. *[NesRoed77],[NesRoed76] For all finite ordered graphs H, F and $r \in \omega$ there exists a finite ordered graph G such that*

$$G \rightarrowtail (H, \leq)^F_r.$$

Since every finite ordered graph can be embedded into $G_\omega \restriction n$ for some $n \in \omega$, an equivalent formulation of this theorem is:

Lemma 3.2.3. *Let G_ω be the random graph on ω, H a finite ordered graph and $n, r \in \omega$. Then there exists $\mathbf{R}(H, n, r) \in \omega$ such that*

$$G_\omega \restriction (\mathbf{R}(H, n, r) + 1) \rightarrowtail (G_\omega \restriction (n + 1), \leq)^H_r.$$

This is a generalization of the finite Ramsey Theorem and $\mathbf{R}(H,n,r)$ corresponds to the Ramsey number of H,n,r. We will write $\mathbf{R}(H,n)$ instead of $\mathbf{R}(H,n,2)$. Observe that whenever there is graph G such that $\|G\|_{\mathrm{rand}} > \mathbf{R}(H,n)$ and there is a coloring $c : \binom{G}{H} \to 2$ then G contains a c-homogeneous copy of $G_\omega \restriction n{+}1$. We turn the lexicographically ordered Baire space (compare chapter 1.2) into an ordered graph as follows: for distinct $x, y \in \omega^\omega$ we let $\{x,y\} \in E \iff \{x(\Delta(x,y)), y(\Delta(x,y))\} \in E_\omega$. In words: two infinite sequences of natural numbers are connected by an edge iff in the first coordinate on which they disagree they assume a pair of natural values which form an edge in the ordered random graph we have fixed on ω. Let G_{random} denote $(\omega^\omega, E, <_{\mathrm{lex}})$, defined just now. Let $T_{\max} = \bigcup_{0<n\in\omega} \prod_{m<n}(m+1)$ be the tree such that $\mathbb{R}_C$ is the set of branches through $[T_{\max}]$. The ordered graph $G_{\max}$ is the compact induced subgraph of G_{random} whose vertices are the elements of $\mathbb{R}_C$. The graph $G_{\max}$ is the main object in this second part and we usually refer to it in this representation, as the set of branches through $\mathbb{R}_C$. The graph $G_{\max}$ can also be presented as the inverse limit of the sequence $F_n, n \in \omega$ such that F_n is defined inductively: F_0 is a single vertex and F_{n+1} is $\sum_{v\in V^{F_n}} G_\omega \restriction (n{+}1)$, compare [FrKo08]. When we say finite ordered graph F without further specification, we assume that the set of vertices is a natural number and the order is the natural order on ω. Let $F \in \omega$ be a finite ordered graph and $f \in \binom{G_{\max}}{F}$ an order preserving graph embedding. We call $\Delta^{|F|}(f) = \Delta^{|F|}(\mathrm{rg}(f))$ the highest splitting level of the embedding f. The highest splitting node of f is defined accordingly. Furthermore for a finite or infinite graph F and an embedding $f \in \binom{G_{\max}}{F}$ we define $T_{\leq f} = T_{\leq \mathrm{rg}(f)}$.

Let $A \subseteq T_{\max}$ be a set of pairwise incompatible nodes (an *antichain*). Identifying each $t \in A$ with some $x_t \in G_{\max}$ such that $t \subseteq x_t$, A is embedded into $T_{\max}$ and inherits an induced subgraph structure from $G_{\max}$. Alternatively, for all distinct nodes $t_1, t_2 \in A$ the set $\{t_1, t_2\}$ is an edge if $\{t_1(\Delta(t_1,t_2), t_2(\Delta(t_1,t_2)\}$ is in $\in E_\omega$. From now on we shall talk about the induced graph structure of antichains of $T_{\max}$ via this definition. For two antichains $A_0, A_1 \subseteq T_{\max}$ we write $A_0 \prec A_1$ if A_0 is a maximal antichain contained in $T_{\leq A_1}$.

Let $T \subseteq T_{\max}$ be a subtree and let t be a splitting node in T. We define the *norm of* $t \in T$ as the norm of the graph $(\mathrm{IS}(t,T), E \restriction \mathrm{IS}(t,T)$ and denote it by $\|t\|_T$. The set $\mathrm{IS}(t, T)$ is indeed an antichain in $T_{\max}$ and therefore has an induced subgraph structure from $G_{\max}$, as described above.

Let $T \subseteq T_{\max}$ be a subtree. A function $f : T \to T_{\max}$ is called *tree graph embedding* if it preserves $\leq_{\text{lex}}$ and respects the edge relation on every induced subgraph $A \subseteq T$.

A perfect tree $T \subseteq T_{\max}$ which satisfies that for every two comparable splitting nodes $t_1 \subset t_2$ it holds that $\|t_1\|_T < \|t_2\|_T$, is called a *tree of infinite norm.* For example $T_{\max}$ itself is a tree of infinite norm. A tree $T \subseteq T_{\max}$ is called *trimmed* iff every two consecutive splitting nodes s_0, s_1 satisfy $\|s_0\|_T + 1 = \|s_1\|_T$. We call T *trimmed above a node* $s \in T$ if the tree $T[s]$ is trimmed. Observe that every tree of infinite norm contains a trimmed subtree of infinite norm. The significance of trees of infinite norm is that they provide graph embeddings of $G_{\max}$ into itself. First

Observation 3.2.4. If $T \subseteq T_{\max}$ has infinite norm, then there exists a tree graph embedding from $T_{\max}$ into T.

Suppose T is a tree of infinite norm and $\mathfrak{g} : T_{\max} \to T$ is a tree graph embedding. Then $\mathfrak{g}$ induces a continuous, order preserving function from $[T_{\max}]$ into $[T]$ which preserves edges and non-edges, namely it induces a continuous order preserving graph embedding from $G_{\max}$ into itself. When searching for homogeneous copies of $G_{\max}$ in itself we shall actually be looking at trees of infinite norm. Tree graph embeddings and their corresponding graph embeddings are usually referred to by the same letter. If necessary we may add an index G or T.

The main tool for constructing trees of infinite norm is again *fusion* which we define a bit different than in the case of Sacks Forcing. The main difference is, that in the relation $\leq_n$ the $n \in \omega$ refers to the level instead of the splitting level. It is easily checked that this is only a notational difference. The simple reason for this is convenience. For two trees $T, S \subseteq T_{\max}$ and $n \in \omega$ we define a relation:

$$T \leq_n S \iff T \subseteq S \wedge (\mathrm{LEV}_n(T) = \mathrm{LEV}_n(S)).$$

Let T_n be subtrees of $T_{\max}$ each with infinite norm. We call a decreasing sequence $(T_n)_{n\in\omega}$ together with an increasing sequence of natural numbers $(l_n)_{n\in\omega}$ *fusion se-*

quence if for every $n \in \omega$ the following conditions hold:

$$(T_{n+1} \leq_{l_n} T_n) \tag{3.2.1}$$

$$\forall t \in T_n \upharpoonright l_n \, \exists n' > n \, \exists s \in T_{n'} \upharpoonright l_{n'} (t \subseteq s \wedge s \in \mathbf{sp}(T_{n'})) \tag{3.2.2}$$

In the constructions of fusion sequences we will again use amalgamation. While thinning out above a given node we sometimes have to assume that the tree under consideration is trimmed above this node. This is done without further mentioning.

Lemma 3.2.5. *If $(T_n)_{n\in\omega}$ with $\{l_n : l_n \in \omega\}$ is a fusion sequence and each T_n is of infinite norm, then the intersection $T_\infty = \bigcap_{n\in\omega} T_n$ is of infinite norm.*

Proof. Let t be a node in T_∞. Then $t \in T_n \upharpoonright l_n$ for an $n \in \omega$ and there is an $n' > n$ and a splitting node $s \in T_{n'} \upharpoonright l_{n'}$ which extends t. Therefore s is in T_∞. Because each T_n is of infinite norm this proves the lemma. □

That means that every $s \in T_n \upharpoonright l_n$ will appear with the same norm, if any in the final tree T_∞, which we also call *the fusion* of the sequence. Hence if $s \in T_n \upharpoonright l_n$ is a splitting node, we also say, that T_n *fixes the splitting node* s.

3.2.3 Colorings and Partitions

Let $T_0, T_1 \subseteq T_{\max}$ be two infinite trees. The two trees have *the same type* iff there is a bijective mapping which respects the lexicographic ordering $f : \mathbf{sp}(T_0) \to \mathbf{sp}(T_1)$ such that the following holds for every $s, t \in \mathbf{sp}(T_0)$:

(1) $|s| < |t| \to |f(s)| < |f(t)|$

(2) the mapping f induces an order preserving graph isomorphism between the induced graphs $\mathrm{IS}(s, T_0)$ and $\mathrm{IS}(f(s), T_1)$. Hence f preserves the norm.

Let G be some graph and let $g_0, g_1 \in \binom{G_{\max}}{G}$ be two embeddings. We say that g_0, g_1 have *the same type* iff the trees $T_{\leq g_0}, T_{\leq g_1}$ have the same type. This is an equivalence relation. Let the variable ξ refer to such a type, both in the sense of equivalence class of embeddings of G into $G_{\max}$ and equivalence classes of trees, the set $\binom{G_{\max}}{G}^{\xi}$ refers to the set of embeddings contained in the class ξ. The finite type which is such that the trees in this class contain only one splitting node has the extra symbol Δ. Obviously for every finite graph F there is an embedding $f \in \binom{G_{\max}}{F}$ of type Δ. This is not the case for most of the types. Hence given a closed subgraph $F \subseteq G_{\max}$ and an embedding $f \in \binom{G_{\max}}{F}$ we call the type of f a *possible type for* F and refer to this set of types by $\mathbf{TP}^F$. The equivalence relation naturally defines a partition which we call the *type partition*. Let $F = (k, E_F)$ be a *path*, that means possible type for F and refer to this set of types by $\mathbf{TP}^F$. The equivalence relation naturally defines a partition which we call the *type partition*. Let $F = (k, E_F)$ be a *path*, that means that only "neighboring" nodes like i and $i + 1$ are connected by edges. It can be easily observed that for the finite paths the only possible type is Δ and that infinite paths cannot be embedded into $G_{\max}$ at all.

Definition 3.2.6. A graph is a closed graph if the vertex set is a closed subset of $G_{\max}$. Let $F \subseteq G_{\max}$ be a closed graph and ξ a type which is possible for F. Let $c : \binom{G_{\max}}{F} \to 2$ be a coloring. The induced subgraph $G \subseteq G_{\max}$ is called *c-homogeneous on* ξ if $c \restriction \binom{G_{\max}}{F}^{\xi}$ is constant. Accordingly, a tree $T \subseteq T_{\max}$is called *c-homogeneous on* ξ if $[T]$ is c-homogeneous on ξ. A tree $T \subseteq T_{\max}$ is *skew* if each level of T contains at most one splitting node.

Observation 3.2.7. For every tree $T \subseteq T_{\max}$ of infinite norm we can find a skew tree $T' \subseteq T$ with infinite norm.

With this in mind we assume from now on, if not mentioned otherwise that all appearing trees of infinite norm $T \subseteq T_{\max}$ as well as $T_{\max}$ are skew and trimmed. Hence certain types cannot occur. From now on, when we say type we mean those which can occur in skew trees.

Definition 3.2.8. Let $F \subseteq G_{\max}$ be a closed subgraph. A partition of $\binom{G_{\max}}{F}$ into disjoint C_i for $i \in I$ for some index set I is called a *basic $G_{\max}$-partition for Borel colorings* if the following holds:

(1) For an induced copy G' of $G_{\max}$ in $G_{\max}$ and all $i \in I$ the intersection $\binom{G'}{F} \bigcap C_i$ is nonempty.

(2) For every Borel coloring $c : \binom{G_{\max}}{F} \to r \in \omega$ and for every $i \in I$ there is an induced copy G of $G_{\max}$ in $G_{\max}$ such that c is constant on $C_i \bigcap \binom{G_{\max}}{F}$.

We will prove that the possible types on skew trees yield a basic $G_{\max}$-partition $\dot{\bigcup}_{\xi \in \mathbf{TP}^F} \binom{G_{\max}}{F}^{\xi}$. Suppose there are two possible types ξ_0, ξ_1 for a given graph F, then every set $X \subseteq G_{\max}$ such that every $f \in \binom{X}{F}$ is of type ξ_0 is at most countable. Hence (1) holds obviously. It should be observed that it is enough to consider colorings with 2 colors in order to establish (2). Furthermore we will state most of the lemmas for continuous colorings. The generalization to Borel colorings follows with the help of Lemma 3.3.7 . Consider an embedding $\mathfrak{g} \in \binom{G_{\max}}{G_{\max}}$. The letter η is reserved for the type of embeddings of $G_{\max}$ into itself. Let η refer to the type of $\mathfrak{g}$ and let $(l_i)_{i \in \omega}$ be an increasing sequence of levels starting with $l_0 = 0$ and such that $T = T_{\leq \mathfrak{g}}$ has exactly one splitting point s_i on the levels in the interval $[l_i, l_{i+1}) \subseteq \omega$. For every $i \in \omega$ put F_i^{η} as the lexicographically ordered graph $(|\operatorname{LEV}_{l_i}(T)|, E_i)$, such that there exists a graph embedding $f_i : F_i^{\eta} \to \operatorname{LEV}_{l_i}(T)$. This defines a sequence $(F_i^{\eta})_{i \in \omega}$. Let $j_i \in F_i^{\eta}$ be such that $f_i(j_i) \subseteq s_i$. The graph F_{i+1}^{η} is obtained from F_i^{η} by blowing up the j_ith vertex by the graph $H_i = G_\omega \restriction \|s_i\|_T + 1$. The following *type function* gives a unique characterization of the type η:

$$\eta(i) = (j_i, H_i).$$

Observe that given a sequence $(f_i)_{i \in \omega}$ of embeddings it is possible to reconstruct the tree $T_\eta = \bigcup_{i \in \omega} T_{\leq f_i}$. We say that f_i is an *initial η-embedding of $G_{\max}$* . It is not enough to know the sequence of graphs F_i^{η} in order to determine the type, because the metric relations are also needed. Let $F \subseteq G_{\max}$ be an arbitrary closed subgraph and let $f \in \binom{G_{\max}}{F}$ be an embedding of type ξ. We may analogously define the type function ξ on the amount of splitting nodes contained in $T_{\leq f}$. This time we have to allow an arbitrary finite graph as the second coordinate. Let f be an initial ξ-embedding for some graph F. The amount of splitting nodes contained in the spanned tree $T_{\leq f}$ is is called the *height of* f and we we write $\mathrm{hght}(f)$.

Let $\mathbb{H}^\infty$ denote the hyperspace which consists of all perfect subsets X of $\mathbb{R}_C$ such that the spanned trees $T_{\leq X}$ have infinite norm. The Hausdorff metric is defined as follows where A and B are elements of the hyperspace and d refers to the usual metric on the Baire space:

$$h_d(A, B) = \sup_{x \in \mathbb{R}_C} |d(A, x) - d(B, x)|.$$

The topology induced by this metric is the *Vietoris topology.* Talking about continuous colorings of closed induced subgraphs of $G_{\max}$ we refer to this topology. Observe that for a closed $F \subseteq G_{\max}$ and a possible type ξ the set $\binom{G_{\max}}{F}^\xi$ is open in the subspace $\binom{G_{\max}}{F}$. There is one ingredient–which was already mentioned– which is the key in all proofs of this second part of the present thesis. That is the theorem of Halpern-Läuchli in [HaLae66]. This theorem has been already mentioned in the context of Blass' theorem and turned out to be a very important tool in any study of finitely branching trees. A proof may be found in [AFK02].

Definition 3.2.9. Let $T_0, \ldots, T_{d-1}$ be a collection of finitely branching trees, and $A \subseteq \omega$. We define the tree product:

$$\bigotimes_{i<d}^{A} T_i = \bigcup_{n \in A} \mathrm{LEV}_n(T_0) \times \cdots \times \mathrm{LEV}_n(T_{d-1}).$$

Theorem 3.2.10 (Halpern-Läuchli)**.** *Let d be a natural number, let $A \subseteq \omega$ be an infinite subset and let the T_i be finitely branching trees for $i \in d$. For a given function (coloring)*

$$e : \bigotimes_{i<d}^{A} T_i \to 2$$

there exist $(t_0, \ldots, t_{d-1}) \in \bigotimes_{i<d} T_i$, an infinite set $A' \subseteq A$ and a color $j \in 2$ such that for every $m \in \omega$ there is a $k \in A'$ and sets D_i, $i < d$ such that each $D_i \subseteq T_i[t_i]$ is $\mathrm{LEV}_m(T_i[t_i])$-dense on level k and $e(\vec{t}) = j$ for every $\vec{t} \in \bigotimes_{i<d}^{A'} T_i$.

3.3 Coloring Finite Graphs

3.3.1 The Finite Basic $G_{\max}$-Partition

We want to begin our study of the arrow relations for G_{max} with Borel colorings of the form

$$c : \binom{G_{\max}}{F} \to 2$$

such that F is a finite graph. Let $\xi_0, \ldots \xi_{m-1}$ be an enumeration of the possible types for F. We will prove that the partition $\binom{G_{\max}}{F} = \dot{\bigcup}_{i \in m} C_i$ such that $C_i = \binom{G_{\max}}{F}^{\xi_i}$ is a basic partition in the sense of Definition 3.2.8. We will argue in two steps. We will first show the theorem for continuous colorings, that is in our case those colorings c which satisfy that $c^{-1}(j)$ is an open set for all $j \in r$. This is the main part of this section and requires most of the work. Afterwards we will argue that from the statement about open colorings one can obtain the same result for Borel colorings.

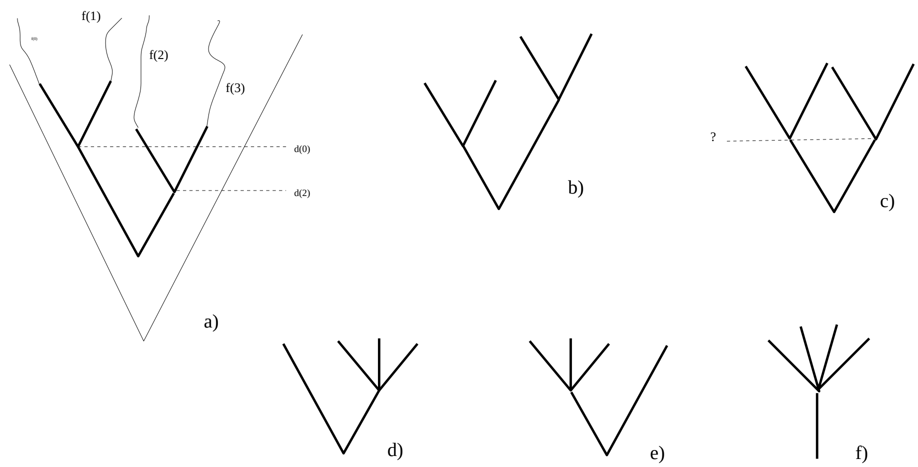

Figure 3.1: Types for $k = 4$

Definition 3.3.1. In the case of the Blass' Theorem the number of possible types for the set $[2^\omega]^n$ is $(n-1)!$. The amount of the different types of subtrees of $T_{\max}$ such that the subtree has n branches is certainly higher than $(n-1)!$, because branching in $T_{\max}$

is unbounded. If we take into account the graph structure and ask for possible types of embeddings, the amount of types decreases again. As mentioned before, not every graph can be embedded with every type. Unfortunately, we have no effective way to calculate the amount of possible types for a given graph. In Figure 3.1 the possible types for trees with 4 branches $\{x_0, x_1, x_2, x_3\}$ can be seen. The notation in the picture is taken from the definition below. As mentioned before we exclude the case c) by assuming that all appearing trees are skew. Let $T \subseteq T_{\max}$ be of infinite norm and consider the graph $[T]$ defined on the branches. This graph is ordered by the lexicographic order $\leq_{\text{lex}}$. Let m be a natural number and let $f : m \to [T]$ be an order preserving function. For $i \in m-1$ we set $d(i) = \Delta(f(i), f(i+1))$. In the special case of finite embeddings and trees with finitely many infinite branches we want to define an alternative type function θ_ξ for a type ξ which is more handy for the here presented proof. We define the following auxiliary sets $A_i \subseteq m-1, l_i \in \{d(j) : j \in m-1\}$:

$$A_0 = m - 1;\ l_i = \max\{d(j) : j \in A_i\};\ \ A_{i+1} = A_i \setminus k(i).$$

Then we set $\theta_\xi(i) = \{j \in A_i : d(j) = l_i\}$ for every $i \in \omega$. This type function $\theta_\xi : m-1 \to \mathcal{P}(m-1)$ describes the type ξ completely.

Definition 3.3.2. Let F_1 be a finite Graph with domain $F_1 \in \omega$ and let $T \subseteq T_{\max}$ be a tree of infinite norm. We define $\binom{\text{LEV}(T)}{F_1} = \{f : \exists l \in \omega (f \in \binom{\text{LEV}_l(T)}{F})\}$. Let $F_2 \subseteq F_1$ be an induced subgraph and $f_j \in \binom{\text{LEV}_l(T)}{F_j}$ for $j = 1, 2$. We write $f_2 \prec f_1 \iff \text{rg}(f_2) \prec \text{rg}(f_1)$. Let $T \subseteq T_{\max}$ be of infinite norm, $r \in \omega$ and $c : \binom{[T]}{F} \to r$ a continuous coloring. We can define a coloring $\overline{c} : \binom{\text{LEV}(T)}{F} \to r \cup \{\textbf{nil}\}$:

$$\overline{c}(f) = \begin{cases} j, & \text{if } \forall f' \in \binom{\text{LEV}(T)}{F} (f \prec f' \to c(f') = j) \\ \textbf{nil} & \text{otherwise.} \end{cases}$$

We call the coloring c *immediately decided in* Tif $\overline{c}$ is nowhere **nil**. In this case we will abuse notation and write c instead of $\overline{c}$.

Lemma 3.3.3. *Let $T \subseteq T_{\max}$ be a tree of infinite norm, F a finite graph, and*

$c : \binom{[T]}{F} \to 2$ a continuous coloring. There is a subtree $T' \subseteq T$ of infinite norm, such that c is immediately decided in T'.

Proof. We construct a fusion sequence: Start with $T_0 = T_{\max}$ and $l_0 = 0$. Suppose T_n and l_n are already defined. For an element of $\mathrm{LEV}_{l_n}(T_n) = \{t_0, \dots, t_N\}$, for example t_N, there is a $t \in T_n$ such that $\|t\|_{T_n} > \max\{\|s\|_{T_n} : s \subseteq t_N\}$ and $t \in T_n[t_N]$. Pick for each t_i with $i \in N$ an extension on level $|t| + 1$. For simplicity we will still call them t_i. Set $T'_n = \bigcup_{i \in N} T_n[t_i] \cup T_n[\mathrm{IS}(t, T_n)]$. Enumerate $\binom{\mathrm{LEV}_{|t|+1}(T'_n)}{F} = \{f^0, \dots f^d\}$. Because of continuity we find $g^0 \succ f^0$ with $g^0 \in \binom{\mathrm{LEV}(T'_n)}{F}$ such that $\overline{c}(g^0) \neq \mathbf{nil}$. Let S^0 be a $\mathrm{LEV}_{|t|+1}(T'_n)$-dense set on the level of $\mathrm{rg}(g^0)$ which contains $\mathrm{rg}(g^0)$. Repeat the procedure with f^1 in the tree $T'_n[S^0]$. After d many steps this yields the tree $T_{n+1} = T'_n[S^d]$ and we set l_{n+1} as the level of the range of g^d. □

Lemma 3.3.4. *Let $T \subseteq T_{\max}$ be a tree of infinite norm, $F \in \omega$ a finite graph, and $c : \binom{[T]}{F} \to 2$ a continuous coloring. Then there is a $T' \subseteq T$ with infinite norm which is c-homogeneous on Δ.*

Proof. Let F, T and c as in the lemma. By thinning out we might assume that the color is immediately decided. Hence for every splitting node $t \in \mathbf{sp}(T)$ the coloring c induces a coloring c_t on $\binom{\mathrm{IS}(t,T)}{F}$. For a color $j \in 2$, a subtree with infinite norm $S \subseteq T_{\max}$ and an $n \in \omega$ we say that a node t satisfies condition $\phi_{j,n}$ in S, or simply $\phi_{j,n}(t, S)$, if $\mathrm{IS}(t, S)$ contains an induced copy of $G_\omega \restriction n+1$ which is c_t-homogeneous of color j. With this notation Lemma 3.2.3 translates to the following:

$$\forall n \in \omega\ \exists n' \in \omega\ \forall t \in T(\|t\|_T > n' \to (\phi_{0,n}(t,T) \vee \phi_{1,n}(t,T))) \tag{3.3.1}$$

We say a $T \subseteq T_{\max}$ satisfies Ψ_j for $j \in 2$ if the following holds:

$$\forall s \in T\ \forall n \in \omega\ \exists t \supseteq s\ (t \in T \wedge \phi_{j,n}(t,T)) \tag{Ψ_j}$$

Claim 9. If $\Psi_0(T)$ is not true then there is a tree $T^* \subseteq T$ of infinite norm such that $\Psi_1(T^*)$ holds.

Proof of Claim 9. Let $s \in T, n \in \omega$ be witness to the failure of $\Psi_0(T)$. Because of (3.3.1) it is possible to substitute s by a node on a higher level such that we can assume $\phi_{1,n}(t, T)$ for all $t \supset s$. The tree $T[s]$ satisfies Ψ_1. This can be seen as follows: Pick $t \in T[s]$ and $m \in \omega$. If m is less or equal to n then t obviously satisfies $\phi_{1,m}$. If $m > n$, there is because of (3.3.1) a $t' \in T[s]$ which satisfies $\phi_{1,m}$ or $\phi_{0,m}$. Because of the choice of s the option $\phi_{0,m}(t', T[s])$ is impossible.

□(Claim 9)

Without loss of generality we assume that T satisfies Ψ_0. Then we can define a fusion sequence: Suppose we have constructed up to T_n and level l_n. For every $t \in \mathrm{LEV}_{l_n}(T_n)$ and $m \in \omega$ we find $t' \supseteq t$ which satisfies $\phi_{0,m}$. Let $H \subseteq \mathrm{IS}(t', T_n)$ be the $c_{t'}$-homogeneous copy of $G_\omega \upharpoonright m + 1$ and let S be the union of the nodes in H together with a $\mathrm{LEV}_{l_n}(T_n) \setminus \{t\}$-dense set on the level $|t'| + 1$. Set $T_{n+1} = T_n[S]$ and $l_{n+1} = |t'| + 1$. The fusion of this sequence witnesses the statement of Lemma 3.3.4. □

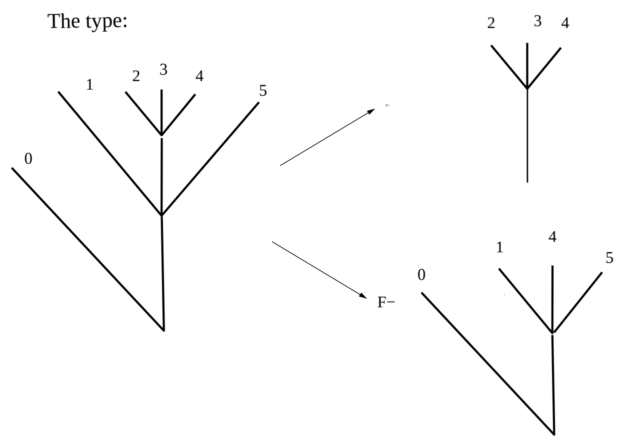

Figure 3.2:

The following first result can be directly deduced:

Theorem 3.3.5. $G_{\max} \rightarrowtail_{\mathrm{cont}} (G_{\max}, \leq_{\mathrm{lex}})^{K_2}_r$

If $f \in \binom{G_{\max}}{F}$ is an ordered graph embedding which has the type ξ, then the interval $\theta_\xi(0)$ describes the highest splitting point of the tree spanned by $\mathrm{rg}(f)$. The embedding f can be decomposed according to the levels of splitting $f = f^+ \cup f^-$ such that f^+ equals $f \restriction \theta_\xi(0) \cup \{\max(\theta_\xi(0))+1\}$ and f^- is $f \setminus \theta_\xi(0)$. Now the ordered graph $\mathrm{rg}(f)$ is a blow up of $\mathrm{rg}(f^-)$ by $\mathrm{rg}(f^+)$ and the ordered graph F is a blow up of $F^- = \mathrm{dom}(f^-)$ by $F^+ = \mathrm{dom}(f^+)$.

Lemma 3.3.6. *Given a finite ordered graph $F \in \omega$, a tree $T \subseteq T_{\max}$ with infinite norm, some type ξ which is possible for F and a continuous coloring $c : \binom{G_{\max}}{F} \to 2$. Fix $h \in \binom{\mathrm{LEV}_l(T)}{F^-}$ for $l \in \omega$. Then there is a $T' \subseteq T$ with infinite norm such that* $\mathrm{LEV}_l(T) = \mathrm{LEV}_l(T')$ *and the following holds:*

$$\exists j \in 2\ \forall g \succ h \left(g \in \binom{G_{\max}}{F}^{\xi} \to c(g) = j \right) \tag{3.3.3}$$

Proof. Let $\Sigma = \{t_0, \dots t_{N_l}\}$ be an enumeration of $\mathrm{LEV}_l(T)$. Set $\Sigma_h = \{i \leq N_l : \mathrm{rg}(h) \cap T[t_i] \neq \emptyset\}$. We define $i_\xi = \max\{\theta_\xi(0)\} + 1 \in \Sigma_h$ as the index of the highest splitting node. That means for an embedding $g \in \binom{G_{\max}}{F}^{\xi}$ with $h \prec g$ the highest splitting point and hence all its successors are in $T[t_{i_\xi}]$.

Claim 10. Without loss of generality we can assume that for the collection $\{T[t_i] : i \in \Sigma_h\}$ and every pair g_0, g_1 with $g_j \in \binom{\bigcup_{i \in \Sigma_h} T[t_i]}{F}$ and $g_j \succ h$ for $j \in 2$ the following holds:

$$\forall i \in F\ (g_0(i) \restriction \Delta^m(g_0) = g_1(i) \restriction \Delta^m(g_1)) \Rightarrow c(g_0) = c(g_1) \tag{3.3.4}$$

That means that if two embeddings g_0, g_1 coincide up to their highest splitting point, then they have the same color.

Proof of Claim 10. We simultaneously construct finitely many fusion sequences $(T_{i,n})_{n\in\omega}$ and $(l_n)_{n\in\omega}$ starting with $T_{i,0} = T[t_i]$ and $l_0 = |t_i|$ for $i \in \Sigma_h$ such that for every $n \in \omega$ and every $i \in \Sigma_h$ the following holds:

(1) each $T_{i,n+1}$ contains exactly one splitting node s_i between level l_n and level l_{n+1} such that $\|s_i\|_{T_{i,n+1}} = n + 1$.

(2) the condition (3.3.4) holds up to level l_{n+1}

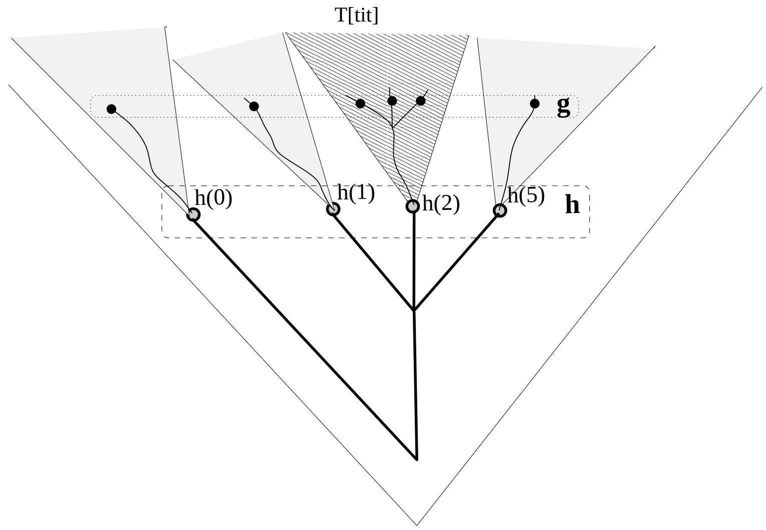

Figure 3.3: Sketch for Lemma 3.3.6 for the case $|F| = 5$

In order to construct a fusion sequence we have to argue that it is possible to fix a splitting node above a given node in each of the trees. Pick a sequence $(s_i)_{i\in|\Sigma_h|\setminus\{i_\xi\}}$ of splitting nodes such that each s_i is in $T_{i,n}$ and pick one splitting node $s_\xi \in T_{\xi,n}$ on a level above all of them such that $\|s_\xi\|_T \geq \mathbf{R}^l(F, n, 2)$. We will give the value for l in a minute. Pick a $\mathrm{IS}(s_i) \cup \mathrm{LEV}_{l_n}(T_{n,i})$-dense set $A_i \subseteq T_{n,i}$ on level $|s_\xi| + 1$ and set $T_{n+1,i} = T_{n,i}[A_i]$ for every $i \in \Sigma_h \setminus \{i_\xi\}$ and set $l_{n+1} = |s_\xi| + 1$.

Set $Ext_h = \{h' : h' \in \binom{\bigcup_{i\in\Sigma_h} A_i}{F^-}$ such that $h \prec h'\}$ and observe in the next step that the l from before has to be $|Ext_h|$. For an $f \in \binom{\mathrm{IS}(s_\xi)}{F^+}$ and a $h' \in Ext_h$ the blow up of h' by f has a color according to the original coloring c. Every $h' \in Ext_h$ thus induces a coloring on the set of embeddings $\binom{\mathrm{IS}(s_\xi)}{F^+}$. According to Theorem 3.2.3 we find a copy of $G_\omega \restriction \mathbf{R}^{l-1}(F, n, 2) + 1$ in $\mathrm{IS}(s_\xi)$ such that (3.3.4) is satisfied for this h'. We do the same thing iteratively for each of the finitely many $h' \in Ext_h$ and get a copy $G \subseteq \mathrm{IS}(s_\xi)$ of $G_\omega \restriction n + 2$. For $i = i_\xi$ we substitute the set of successors of s_i in A_i by the nodes contained in the graph G. Now we can set $T_{n+1,i} = T_{n,i}[A_i]$.

$\Box$(Claim 10)

Again, let $Ext_h = \{h' \in \binom{\bigcup_{i<\Sigma} T[t_i]}{F^-} : h \prec h'\}$ and observe that because of the definition of $\prec$ the range $\mathrm{rg}(h')$ of an $h' \in Ext_h$ is an element of $\bigotimes_{i<\Sigma_h}^{\omega} T[t_i]$. For

$h' \in Ext_h$ we pick $s' \in T[t_{i_\xi}]$ as the highest splitting point below $h'(i_\xi)$. Then because of (3.3.4) there is a color $j \in 2$ such that for any $g \in \binom{G_{\max}}{F}^\xi$ with $h \prec g$ and $g(i_\xi) \restriction \Delta^m(g) = s'$ it holds that $c(h) = j$. Assigning to every element of Ext_h the corresponding color j this defines a coloring $e : \bigotimes_{i<\Sigma_h} T[t_i] \to 2$ for which we can use Theorem 3.2.10 (compare Figure 3.3.1 for the product). This gives an infinite subset $A \subseteq \omega$ and a tuple $\vec{t^*} \in \bigotimes^\omega_{i<\Sigma_h} T[t_i]$. We increase the t_i such that they become the nodes in this tuple.

We construct a fusion sequence $(T_n)_{n\in\omega}$ starting with $T_0 = T[\{t_i : i \in \Sigma_h\}]$ and a sequence $(l_n)_{n\in\omega}$ of integers starting with $l_0 = |t_0|$. Suppose we are in stage n of the construction and we want to realize a splitting node with norm $m \in \omega$ above an $s \in \mathrm{LEV}_{l_n}(T_n)$. If $s \notin T[t_{i_\xi}]$ we pick $s' \supseteq s$ with $\|s'\|_{T_n} > m$ and extend it to a $\mathrm{LEV}_{l_n}(T_n)$-dense set on level $|s'| + 1$ which we call D and we continue with $l_{n+1} = |s'| + 1$ and $T_{n+1} = T_n[D \cup \mathrm{IS}(s')]$. In the case that $s \in T[t_{i_\xi}]$ we can find a $\mathrm{LEV}_{l_n}(T_n)$-dense set $D^* = \{s^*_0, \dots s^*_{N_n}\}$ on a level $l_{n+1} \in A$ such that e is constantly $j \in 2$ on $\bigotimes^\omega_{i<\Sigma_h}(T[t_i] \cap D^*)$. Furthermore we can find a $s^* \in D^*$ which is a successor of s. Now we pick s' as the maximal predecessor of s^* which is a splitting node. If the norm of an immediate successor of this node is too small, another D^* on a higher level in A had to be chosen. Set $T_{n+1} = T_n[D^* \cup \mathrm{IS}(s^*)]$ and let T_∞ be the fusion of this sequence. The tree $T' = T_\infty \cup T[\{t_i : i \in N_l \setminus \Sigma_h\}]$ finishes the proof of the lemma. $\square$

Lemma 3.3.7. *Let F be an ordered finite graph and let $M \subseteq \binom{G_{\max}}{F}$ be a meager set. Then there exists a $T \subseteq T_{\max}$ of infinite norm such that $\binom{[T]}{F} \cap M = \emptyset$.*

Proof. Let M be a meager set and let $M = \bigcup_{i\in\omega} X_i$ be a representation such that the X_i form an increasing sequence of nowhere dense sets. We construct a fusion sequence starting with $T_0 = T_{\max}$ and $l_0 = 0$. Suppose we have constructed T_n and l_n. Let $\mathrm{LEV}_{l_n}(T_n) = \{t_0, \dots t_N\}$ be an enumeration of the l_nth level and suppose we want to fix a splitting node above $t_N \in \mathrm{LEV}_{l_n}(T_n)$. Pick an arbitrary splitting node $s \supseteq t_N$ with $\|s\|_{T_n} > \max\{\|t\|_{T_n} : t \subseteq t_N\}$ and pick additionally extensions $s_i \supseteq t_i$ for every $i \in N$ on the level $|s| + 1$. Now consider the tree $T'_n = \bigcup_{i\in N} T_n[s_i] \cup T_n[\mathrm{IS}(s, T_n)]$. Let $\{f^0, \dots, f^d\}$ enumerate the set of all elements of in $\binom{\mathrm{LEV}_{|s|+1} T'_n}{F}$. Choose an embedding into the branches, $h \in \binom{[T'_n[\mathrm{rg}(f_0)]]}{F}$. Hence h corresponds to a point in $\binom{G_{\max}}{F}$ and because X_{n+1} is nowhere dense, there is an open neighborhood around h. That is we can pick a $g^0 \in \binom{\mathrm{LEV}_m(T_n)}{F}$ such that $m > |s| + 1$, $f^0 \prec g^0 \prec h$ and such that the intersection $\{g \in \binom{T'_n}{F} : g \succ g^0\} \cap X_n$ is empty. Now pick a $\mathrm{LEV}_{|s|}(T'_n)$-dense set S_0 on level $|g^0(0)|$ which includes the image of g^0 and set $T_{n,0} = T'_n[S_0]$. After repeating this procedure for every f^i we set in $T_{n+1} = T_{n,d}$ and $l_{n+1} = |g^d(0)|$. $\square$

This lemma as well as its proof is very close to an argument presented in [Myc64]. Recall that a set which is Borel has the Baire property. Hence for a given Borel coloring c there exists an open set $A \subseteq \mathrm{dom}(c)$ such that the symmetrical difference $A \triangle c^{-1}(j) = (A \setminus c^{-1}(j)) \cup (c^{-1}(j) \setminus A)$ is meager. Hence based on Lemma 3.3.7 it suffices to prove the statements for continuous colorings.

Theorem 3.3.8. *Let $F \subseteq G_{\max}$ be an ordered finite graph. The partition $\binom{G_{\max}}{F} = \dot{\bigcup}_{\xi \in \mathbf{TP}^F} \binom{G_{\max}}{F}^{\xi}$ is basic $G_{\max}$-partition for Borel colorings.*

Proof. Let $\xi \in \mathbf{TP}^F$ and let $c : \binom{G_{\max}}{F} \to 2$ be a continuous coloring. There is a subtree $T \subseteq T_{\max}$ of infinite norm such that c is constant on $\binom{[T]}{F}^{\xi}$. We will prove this statement inductively on the amount of blow-ups in F, or equivalently on the amount of splitting nodes contained in the spanned tree $T_{\leq f}$ for an $f \in \binom{G_{\max}}{F}^{\xi}$. The beginning of the induction, type $\boldsymbol{\Delta}$, is exactly Lemma 3.3.4. So assume $\xi \neq \Delta$ and define a fusion sequence $(T_n)_{n \in \omega}$ and a sequence of levels $\{l_n\}_{n \in \omega}$ which starts with $T_0 = T$ and $l_0 = 0$. Assume T_n, l_n have been constructed. For each $f^- \in \binom{\mathrm{LEV}_{l_n}(T_n)}{F^-}$ we can iteratively apply Lemma 3.3.6. After finitely many steps we are left with a tree T'_n of infinite norm. Now we can pick an arbitrary splitting node s on a higher level than l_n and a $\mathrm{LEV}_{l_n}(T_n)$-dense set D on level $|s| + 1$. Then we set $T_{n+1} = T'_n[D \cup \mathrm{IS}(s)\}]$ and $l_{n+1} = |s| + 1$. Let T_∞ be the fusion of this sequence. With (3.3.3) we can assign a color j to each $f^- \in \binom{\mathrm{LEV}(T_\infty)}{F^-}$. For f^- we can apply the induction hypothesis and conclude that there is a $T_\xi \subseteq T_\infty$ of infinite norm which is c-homogeneous on ξ. $\square$

Corollary 3.3.9. Let F be an finite ordered graph and let $t = |\mathbf{TP}^F|$. For every

$r \in \omega$ the following holds:

$$G_{\max} \rightarrowtail_{\mathrm{Borel}} (G_{\max}, \leq_{\mathrm{lex}})^F_{r,t}.$$

The question, if this last Corollary can be generalized to measurable colorings remains open.

3.3.2 Variations of the Result

To conclude the section on finite graphs we want to formulate a finite version of Theorem 3.3.8. Observe that it is as well a version of Theorem 3.2.3 not directly deducible therefrom.

Theorem 3.3.10. *Given a fixed type η for an embedding in $\binom{G_{\max}}{G_{\max}}$, a finite number $k \in \omega$ and a finite graph H. There is an $N \in \omega$ such that given any initial η-embedding $f \in \binom{G_{\max}}{F^\eta_N}$ and a given continuous coloring $c : \binom{\mathrm{rg}(f)}{H} \to 2$ which is immediately decided, there is an initial η-embedding $f' \in \binom{G_{\max}}{F^\eta_k}$ such that $\mathrm{rg}(f')$ is c-homogeneous on every possible type $\xi \in \mathbf{TP}^H$.*

Proof. The theorem will be proved using Königs Lemma. Assume the statement is wrong. Then there is a finite graph H, a $j \in |\mathbf{TP}^H|$ and a $k \in \omega$ such that for every pair $N \in \omega$ and $f \in \binom{G_{\max}}{F^\eta_N}$ there is a coloring $c_f : \binom{\mathrm{rg}(f)}{H}^{\xi_j} \to 2$ which is immediately decided and such that there is no c_f-homogeneous copy of F^η_k in $\mathrm{rg}(f)$. Consider now the tree which consists of the pairs (f, c_f) such that f is an initial η-embedding and c_f is the coloring. The tree ordering is described as follows:

$$(f_0, c_{f_0}) \leq' (f_1, c_{f_1}) \iff$$
$$f_0 \prec f_1 \wedge \forall h \in \binom{\mathrm{rg}(f_1)}{H} \forall h' \in \binom{\mathrm{rg}(f_0)}{H} (h' \prec h \to c_{f_0}(h') = c_{f_1}(h))$$

This is a finitely branching but infinite tree which thus contains an infinite branch $(f_i, c_{f_i})_{i \in \omega}$. This gives rise to an $\mathfrak{f} \in \binom{G_{\max}}{G_{\max}}^\eta$ such that for all $i \in \omega$ the image $\mathrm{rg}(f_i)$ is a maximal antichain in $T_{\leq \mathfrak{f}}$. Take an $h \in \binom{\mathrm{rg}(\mathfrak{f})}{H}$ and set l as the maximum of the set $\{n \in \omega : \exists s \in \mathbf{sp}(T_{\leq h}) \wedge |s| = n\} + 1$. Pick an $i \in \omega$ such that $T_{\leq \mathfrak{f}} \restriction l+1 \subseteq T_{\leq f_i}$. Let $h \prec h'$ and $h' \in \binom{\mathrm{rg}(f_i)}{H}$ and define a coloring c as $c(h) = c_{f_i}(h')$. The resulting $\mathfrak{f}$ and the coloring c is a contradiction to Theorem 3.3.8. $\square$

The presented proof is not constructive in the sense that it does not provide an upper bound for the "Ramsey number" N. Alternatively one could prove finite versions with upper bounds for all the lemmas which appear in the proof of Theorem 3.3.8. As far as the author knows, no reasonable upper bound for the Halpern-Läuchli theorem has been established. Hence there are a lot of open questions in this direction. As mentioned in the Preface another thread of questions arises around the problem which uncountable graphs arrow themselves. The next theorem yields countably many new graphs which arrow themselves.

Definition 3.3.11. For an $n \in \omega$ a graph G is called *K_n-free* if it does not contain an induced copy of K_n. The *K_n-free random graph* $G_\omega^{K_n-free}$ is the minimal graph which satisfies that for every finite K_n-free Graph $G = (G, E)$, a node $a \in G$ and every embedding $f : G \setminus \{a\} \to G_\omega^{K_n-free}$ there is an embedding $f^* : G \to G_\omega^{K_n-free}$ such that $f^* \restriction (G \setminus \{a\}) = f$. Now we can construct a graph on $\mathbb{R}_C$ just as before, taking the K_n-free random Graph instead of the random Graph. The resulting graph is called $G_{\max}^{K_n-free}$.

Lemma 3.3.12. *[NesRoed76, NesRoed] Let H be a finite ordered graph and $m, n, r \in \omega$ three natural numbers. Then there exists $\mathbf{R}(H, m, n, r) \in \omega$ such that*

$$G_\omega^{K_n-free} \restriction (\mathbf{R}(H, m, n, r) + 1) \rightarrowtail (G_\omega^{K_n-free} \restriction (m+1), \leq)_r^H.$$

One can easily check that the proof for Theorem 3.3.8 works as well with the K_n-free graph and we can thus conclude the following:

Theorem 3.3.13. *Let F be an ordered finite Graph and let $k \in \omega$ be a natural number. Then there is a finite number t such that for all $r \in \omega$ the following holds:*

$$G_{\max}^{K_n-free} \rightarrowtail_{\mathrm{cont}} (G_{\max}^{K_n-free}, \leq_{\mathrm{lex}})_{r,t}^F.$$

Observe that the graph $G_{\max}^{K_n-free}$ contains copies of K_n. These uncountable graphs are nevertheless different for different values n. Consider for the case K_3 the following graph $\{0,1,2,3,4\}$ with the following edge relation:
$H = \{\{0,2\},\{1,2\},\{2,3\},\{3,4\},\{1,3\}\}$. It can be easily checked, that H can be embedded in $G_{\max}$ but not in $G_{\max}^{K_3-free}$. For every $n \in \omega$ an example is given by a finite graph H which satisfies the following requirements:

(1) H contains K_n as an induced subgraph.

(2) For every two elements in H there is a $z \in H$ such that z is connected with an edge to exactly one of them.

3.4 Coloring Closed Graphs

We want to generalize our results to colorings of arbitrary closed subgraphs of $G_{\max}$. It will turn out that the type partition of the set of infinite closed induced subgraphs, as introduced earlier, is a basic $G_{\max}$ partition for Borel colorings. As mentioned before, it suffices to consider continuous colorings. We will present two theorems and two proofs. The reader who is only interested in the results might skip Theorem 3.4.2 since it follows directly from Theorem 3.4.14. This less general proof is of independent value because it only uses very basic tools whereas the later proof requires advanced technology such as Forcing and the non-trivial Halpern-Läeuchli theorem.

3.4.1 Copies of $G_{\max}$

First we will restrict our attention to colorings of graphs which are themselves copies of $G_{\max}$. The here presented proof is inspired by the almost classical theorem of Galvin and Prikry [GaPr73] which states that every Borel subset of the natural numbers is Ramsey. In our terminology this means that for every Borel coloring $c : [\omega]^\omega \to 2$ of infinite sets of natural numbers, there exists an infinite c-homogeneous set. We have already observed that even in the case of finite subsets of perfect trees this sort of result can only be obtained modulo the type partition. The here presented result is a generalization of the theorem of Galvin and Prikry. For related results in a slightly different direction compare [Mil76] and [Mil81], where strongly embedded trees are studied.

We fix a type $\eta \in \mathbf{TP}^{G_{\max}}$ for this proof and consider only embeddings of this type. Let T be a tree of infinite norm, let $l_0 \leq l_1$ be natural numbers and let $f_0 : F^\eta_{k_0} \to \mathrm{LEV}_{l_0}(T)$, $f_1 : F^\eta_{k_1} \to \mathrm{LEV}_{l_1}(T)$ be two initial η-embeddings for integers $k_0 < k_1$. Recall that $f_0 \prec f_1$ means that $\mathrm{rg}(f_0) = \mathrm{LEV}_{l_0}(T_{\leq f_1})$ and we say that f_1 extends f_0. If the images of both embeddings are in some tree T, we say that f_1 extends f_0 in T and we write

$$f_0 \prec_T f_1.$$

Let $n \in \omega$ be a natural number $(F^\eta_{k_i})_{i\in\omega}$ and let $f : F^\eta_{k_0} \to \mathrm{LEV}_n(T)$ be an initial η-embedding of $G_{\max}$. We define accordingly

$$f \prec T \iff \mathrm{rg}(f) = \mathrm{LEV}_n(T).$$

Given a sequence of initial η-embeddings f_i such that $f_i \prec_T f_{i+1}$ for i in ω. There is an embedding $\mathfrak{f} \in \binom{[T]}{G_{\max}}^\eta$, the *limit* of this sequence such that $T_{\leq \mathfrak{f}} = \bigcup_{i\in\omega} T_{\leq f_i}$. Let $\mathfrak{f}$ be an element of $\binom{[T]}{G_{\max}}^\eta$ and let $f \prec T$ be an initial η-embedding. We set $f \prec_T \mathfrak{f}$ if there exists a $\prec_T$-increasing sequence such that its limit is $\mathfrak{f}$. Let $T \subseteq T_{\max}$ be a tree of infinite norm and let f_0, f_1 be both initial η -embeddings. We define

$$f_0 \subseteq^* f_1 \text{ iff } \mathrm{rg}(f_0) \subseteq T_{\leq f_1}.$$

$$f_0 \subseteq^* T \text{ iff } \mathrm{rg}(f_0) \subseteq T.$$

In this situation we say that f_0 is a *subset embedding of* f_1, respectively a *subset embedding of* T. If f and g are initial η-embeddings such that $g \subseteq^* f$, then we define the *spread of* g *into* f as:

$$\mathrm{spd}_f(g) = \min\{\mathrm{hght}(f') : g \subseteq^* f' \wedge f' \prec_T f\}.$$

Observation 3.4.1. A decreasing sequence of trimmed trees $(T_i)_{i\in\omega}$ and a sequence of initial ξ-embeddings $(f_i)_{i\in\omega}$ such that $f_i \prec_{T_i} f_{i+1}$ corresponds to a fusion sequence: set $l_i \in \omega$ such that $\mathrm{rg}(f_i) \subseteq \mathrm{LEV}_{l_i}(T_i)$. The tree $\bigcup_{i\in\omega} T_{\le f_i}$ has infinite norm.

Theorem 3.4.2. *Let η be a possible type for $G_{\max}$ and let $c : \binom{G_{\max}}{G_{\max}}^{\eta} \to 2$ be a continuous coloring of induced copies of this type. Then $G_{\max}$ contains a copy of $G_{\max}$ which is c-homogeneous on η.*

$$G_{\max} \to_{Borel} (G_{\max}, \le)^{G_{\max},\eta}_{r,1}.$$

For this proof we agree on the following conventions: all Ts are subtrees of $T_{\max}$, which have infinite norm, all small letters f, g are initial η-embeddings, and the letters $\mathfrak{f}, \mathfrak{g}$ are always elements of $\binom{G_{\max}}{G_{\max}}^{\eta}$. The small letter h usually refers to an embedding of an initial segment of the random graph $G_\omega \restriction m$ for a natural number $m \in \omega$.

Definition 3.4.3. Given a coloring $c : \binom{[T_{\max}]}{[T_{\max}]} \to 2$, a tree $T \subseteq T_{\max}$ and an initial η-embedding $f \subseteq^* T$ we define

$$T \text{ accepts } f \iff \forall \mathfrak{f}(f \prec_T \mathfrak{f}(c(\mathfrak{f}) = 0).$$

$$T \text{ rejects } f \iff \forall T' \subseteq T(f \subseteq^* T' \to T' \text{ does not accept } f).$$

The tree T decides f if $f \subseteq^* T$ and T either rejects or accepts f. T is said to *reject(accept, decide) all its initial η-embeddings* if for every subset embedding $g \subseteq^* T$ it is the case that T rejects (accepts, decides) g.

Observation 3.4.4. If T rejects (accepts) f then every $T' \subseteq T$ which contains the range of f rejects (accepts) f.

Observation 3.4.5. Let $f \subseteq^* T$ be an initial η-embedding into a tree T of infinite norm. Then there is a trimmed $T' \subseteq T$ such that $f \prec T'$.

Observation 3.4.6. Let $T \subseteq T_{\max}$ be a tree of infinite norm and let f be a subset embedding of T. Then the following equivalence holds:

$$T \text{ rejects } f \iff \forall T' \subseteq T(f \prec T' \to T' \text{ does not accept } f).$$

Proof. $\Rightarrow$ is trivial. To see the other direction suppose towards a contradiction that there is a $T^* \subseteq T$ such that $f \subseteq^* T^*$ and T^* accepts f. That is all $\mathfrak{f}$ with $f \prec_{T^*} \mathfrak{f}$ are colored with 0. Set $T' = T^*[\mathrm{rg}(f)] \subseteq T$ and trim it if necessary such that $f \prec T'$. For any $\mathfrak{f}$, which satisfies $f \prec_{T'} \mathfrak{f}$, it is true that $f \prec_{T^*} \mathfrak{f}$ and hence that $c(\mathfrak{f}) = 0$. Hence T' accepts f. $\square$

Lemma 3.4.7. *There is a $T \subseteq T_{\max}$ of infinite norm which decides all its initial η-embeddings.*

Proof. We define a decreasing sequence of trimmed trees $(T_i)_{i\in\omega}$ and a sequence $f_i : F_i^\eta \to \mathrm{LEV}(T_i)$ such that $f_i \prec_{T_i} f_{i+1}$, $f_i \prec T_i$ and T_i decides all initial η-embeddings g with $\mathrm{spd}_{f_i}(g) \leq i$:
If $T_{\max}$ rejects $\emptyset$ we set $T_0 = T_{\max}$ and $f_0 = \emptyset$. If $T_{\max}$ does not reject $\emptyset$ we choose $T_0 \subseteq T_{\max}$ such that T_0 accepts $\emptyset$. Suppose f_i, T_i is already defined. We take an arbitrary $f_{i+1} \in \binom{T_i}{F_{i+1}^\eta}$ such that $f_i \prec_{T_i} f_{i+1}$. Let $g \subseteq^* f_{i+1}$ be an initial η-embedding and let $T' \subseteq T_i$ be a tree of infinite norm such that $f_{i+1} \prec T'$.

Claim 11.
$$\exists T'' \subseteq T'(g \prec T'' \wedge T'' \text{ decides } g).$$

Proof of Claim 11. Suppose there are $g \subseteq^* f_{i+1} \prec T'$ such that no $T'' \subseteq T'$ with $g \prec T''$ decides g. Hence in particular none of them accepts g. Thus in this case $T'[\mathrm{rg}(g)]$ rejects g.

$\square$(Claim 11)

For each $g \subseteq^* f_{i+1}$ such that $\mathrm{spd}_{f_{i+1}}(g) = i + 1$ we iteratively do the following procedure:
According to Claim 11 there is a $T^g \subseteq T_i$ of infinite norm such that $g \prec T^g$. Now we set
$$T_i^g = (T_i[\mathrm{rg}(f_{i+1}) \setminus \mathrm{rg}(g)]) \cup T^g.$$

That is, we "thin out above $\mathrm{rg}(g)$". Observe that this tree is still of infinite norm but that trimming might be necessary. Set T_i as T_i^g and repeat the procedure iteratively for all initial η-embeddings $g \subseteq^* f_{i+1}$ which are of spread $i+1$. This results in a new tree T_{i+1} and we set $T = \bigcup_{i\in\omega} T_{\leq f_i}$. $\square$

Lemma 3.4.8. *There is a $T \subseteq T_{\max}$ such that for all initial η-embeddings $g', g'' \subseteq^* T$ the following statement holds:*

$$\mathbf{sp}(T_{\leq g'}) = \mathbf{sp}(T_{\leq g''}) \rightarrow (T \text{ accepts } g', g'' \text{ or } T \text{ rejects both}).$$

Proof. We assume that T decides every initial η-embedding. Again we define sequences f_i, T_i, l_i with $i \in \omega$ starting with $f_0 = \emptyset$ and $T_0 = T$. Assume we have defined $f_i : F_i^\eta \rightarrow \mathrm{LEV}_{l_i}(T_i)$ for some l_i. Consider $g \subseteq^* f_i$ such that $\mathrm{hght}(g) = k$. If there are extensions $f_i' : F_{i+1}^\eta \rightarrow \mathrm{LEV}(T_i)$ and $g' : F_{k+1}^\eta \rightarrow \mathrm{LEV}(T_i)$ such that $f_i \prec_{T_i} f_i'$, $g \prec_{T_i} g'$ and $\mathrm{hght}(f_i') = \mathrm{spd}_{f_i'}(g') = i + 1$ we will thin out the tree T_i as follows. Otherwise do nothing:
Recall the definition of the type function η. Let $\eta(i) = (j_i, H_i)$ and $\eta(k) = (j_g, H_g)$. We will thin out above the node $f_i(j_i)$. Set S_i as the tree $T_i[f_i(j_i)])$ and define a coloring $d_s : \binom{\mathrm{IS}(s,S)}{H_g} \rightarrow 2$ for every $s \in \mathbf{sp}(S_i) \setminus T_{\leq f_i}$: Let $g' : F_{k+1} \rightarrow T_i$ be an extension of g in T_i such that $s \in \mathbf{sp}(T_{\leq g'})$. There is a unique graph embedding $h : H_g \rightarrow T_{\leq g'} \cap \mathrm{IS}(s)$. If g' is accepted by T_i we set $d_s(h) = 1$ otherwise we set $d_s(h) = 0$. Exactly as in Claim 10 of Section 3.3 we can construct a subtree of infinite norm $S_i' \subseteq S_i$ such that for each $s \in \mathbf{sp}(S')$ the coloring d_s is constant on $\binom{\mathrm{IS}(s,S')}{H_g}$. We substitute T_i by a trimmed subtree of $(T_i \setminus S_i) \cup S_i'$ such that $f_i \prec T_i$ still holds.

After we did this procedure for each $g \subseteq^* f_i$ the final tree can be taken as T_{i+1}. Set l_{i+1} as the minimal level such that there is splitting point on a level in the interval $[l_i, l_{i+1})$ and pick $f_{i+1} : F_{i+1}^\eta \rightarrow \mathrm{LEV}_{l+1}(T_{i+1})$. In the end we take the unique $\mathfrak{g} \in \binom{G_{\max}}{G_{\max}}$ such that $f_i \prec_{T_i} \mathfrak{g}$ and $T = T_{\leq \mathfrak{g}}$ satisfies the requirements formulated in the lemma. $\square$

Lemma 3.4.9. *Let $T \subseteq T_{\max}$ be a tree of infinite norm and let $f \prec T$ be an initial η-embedding such that T rejects all $g \subseteq^* f$. Let $i = \mathrm{hght}(f)$ then there is a subtree $T' \subseteq T$ and an immediate extension $f' \in \binom{T'}{F_{i+1}^\eta}$ of the embedding f such that T' rejects all subset embeddings, $g \subseteq^* f'$, of f.*

Proof. We assume that T decides all its initial η-embeddings and satisfies the statement of Lemma 3.4.8. Let $\{g_0, \ldots g_{l-1}\}$ be an enumeration of the initial η-embeddings

$g \subseteq^* f$ and let $\eta(i)$ be the pair (j_i, H_i). We define a coloring $d : \mathbf{sp}(T[f(j_i)]) \to l$. For a node $s \in \mathbf{sp}(T[f(j_i)])$ we pick an embedding $f' \in \binom{T}{F^\eta_{i+1}}$ such that $s \in \mathbf{sp}(T_{\leq f'})$. If there is a $T' \subseteq T$ such that $f' \prec T'$ and T' rejects all $g \subseteq^* f'$ this choice would prove the lemma. Hence we assume that this does not happen for any choice of s. Thus there is a $g' \subseteq^* f'$ which is not rejected by any of the trees T' with $f' \prec T' \subseteq T$. This implies that $s \in T_{\leq g'}$ and there is an index $m \in l$ such that g' extends g_m. Set $d(s) = m$ where $m \in l$ is the minimal index such that the extensions of g_m, which contain s, are not rejected. It should be observed that in two places Lemma 3.4.8 ensures that this coloring is well-defined. There is a tree $T_0 \subseteq T[f(j_i)]$ of infinite norm in which all remaining splitting nodes are colored with the same color. Suppose this color is some $m_0 \in l$ and consider $g_{m_0} \subseteq^* f \prec T_1$ where $T_1[f(j_i)] \subseteq T_0$ and T_1 is skew.

Claim 12. T_1 accepts g_{m_0}.

Proof of Claim 12. Let $\mathfrak{g} \in \binom{[T_1]}{G_{\max}}$ be an extension of g_{m_0} into T_1. Then there is a $g' \prec_{T_1} \mathfrak{g}$ and a $f' \in \binom{T_1}{F^\eta_{i+1}}$ such that $g' \subseteq^* f'$. Because T_1 is d-homogeneous it follows that T_1 does not reject g'. Hence g' is accepted and hence $c(\mathfrak{g}) = 0$.

□(Claim 12)

This is a contradiction to the premises of the lemma and hence we can find an extension f' as described. □

Proof of Theorem 3.4.2. Consider a trimmed tree $T \subseteq T_{\max}$ which satisfies the statement in Lemma 3.4.8. If T accepts $\emptyset$ then $[T]$ is a c-homogeneous copy of $G_{\max}$ of color 0. So suppose that T rejects $\emptyset$. We construct sequences f_i, T_i for $i \in \omega$ such that $f_i \prec_{T_i} f_{i+1}$ and such that T_i rejects all $g \subseteq^* f_i$. Suppose we are in step i: according to Lemma 3.4.9 we can find an $f_{i+1} \in \binom{T_i}{F^\eta_{i+1}}$ and a tree $T_{i+1} \subseteq T_i$ such that all subset embeddings of f_{i+1} are rejected by T_{i+1}. After ω many steps there is an $\mathfrak{f} \in \binom{G_{\max}}{G_{\max}}$ such that $f_i \prec_{T_i} \mathfrak{f}$ and $\mathrm{rg}(\mathfrak{f})$ is homogeneous for color 1. Set $T' = T_{\leq \mathfrak{f}}$, the homogeneity can be seen as follows: Suppose there is a $\mathfrak{g} \in \binom{[T']}{G_{\max}}$ such that $c(\mathfrak{g}) = 0$. Because the color 0 is open, there is an initial η-embedding $g \prec_{T'} \mathfrak{g}$ such that $c(\mathfrak{g}') = 0$ for all $g \prec_{T'} \mathfrak{g}'$. On the other hand there is a stage $i \in \omega$ such that $g \subseteq^* f_i$ and T_i rejects g which is a contradiction.

$\square$(Theorem 3.4.2)

Looking closer at the proof of Theorem 3.4.2 it appears that the induced copy of $G_{\max}$ thus constructed spans a tree of type η.

We want to close this section with a generalization of the Nash-Williams theorem. We follow here very closely the ideas appearing in [Mil81] in the 3rd section.

Definition 3.4.10. A family of finite sets $\mathcal{C} \subseteq [\omega]^{<\omega}$ is called *thin* if there are no two distinct sets $A, B \in \mathcal{C}$ such that A is a subset of B and every $a \in A$ is smaller than all $b \in B \setminus A$. That means that for a set $B \in \mathcal{C}$ no end-extension of B can be contained in $\mathcal{C}$.

Theorem 3.4.11 (Nash-Williams)**.** *Suppose that $\mathcal{C} \subseteq [\omega]^{<\omega}$ is thin, and $\mathcal{C}$ is contained in $C_0 \dot\cup C_1$. Then there is a set $X \in [\omega]^\omega$ and a $j \in 2$ such that $\mathcal{C} \cap [X]^{<\omega} \subseteq C_j$.*

With the main result of this chapter we can easily prove a generalization of this theorem:

Definition 3.4.12. Let η be a possible type for $G_{\max}$ and let $\mathcal{F}$ be a family of initial η-embeddings into $T_{\max}$. The set $\mathcal{F}$ will be called *thin* if there are no two elements $f_0, f_1 \in \mathcal{F}$ such that $f_0 \prec_{T_{\max}} f_1$.

Theorem 3.4.13. *Let $\mathcal{F}$ be a thin set of initial η-embeddings into $G_{\max}$ and let $c : \mathcal{F} \to 2$ be a coloring of this set. Then there is $T \subseteq T_{\max}$ which spans $G_{\max}$ and a $j \in 2$, such that $c(f) = j$ for each $f \in \mathcal{F}$ such that $f \subseteq^* T$.*

Proof. We define a coloring d on $\{\mathfrak{f} \in \binom{G_{\max}}{G_{\max}}^\eta : \exists f \in \mathcal{F}(f \prec_T \mathfrak{F})\}$ such that $d(\mathfrak{f}) = c(f)$ for an $f \in \mathcal{F}$ such that $f \prec_{T_{\max}} \mathfrak{f}$. Hence d is open in one color and Theorem 3.4.2 proves the lemma. $\square$

3.4.2 Arbitrary Closed Subgraphs

In this section we will prove, that the type partition is a basic $G_{\max}$-partition for Borel colorings of arbitrary closed subgraphs of $G_{\max}$. This is our most general result and in the proof we will use Forcing as well as the theorem of Halpern-Läuchli. For a closed set $X \subseteq \omega^\omega$ the set $\mathrm{INC}([X]^\omega)$ contains all countable sets which can be enumerated as an omega sequence in increasing lexicographical order. In [LSV93] a type partition of $\mathrm{INC}([X]^\omega)$ into countably many types was given and then it was proven that this is a basic perfect partition. We generalize this result in two respects. First of all we are looking for copies of $G_{\max}$, which are homogeneous on a given set of the partition. Secondly, we will color arbitrary closed subgraphs instead of countable well-orderable graphs.

Theorem 3.4.14. *Let $F \subseteq G_{\max}$ be a closed subgraph of $G_{\max}$ and let ξ be a possible type for F. For every continuous coloring $c : \binom{G_{\max}}{F}^\xi \to 2$ there exists a $G' \subseteq G_{\max}$, an induced copy of $G_{\max}$ which is c-homogeneous on ξ. Using the arrow notation:*

$$G_{\max} \rightarrowtail_{cont} (G_{\max}, \leq_{\mathrm{lex}})^{F^{\xi}_{r,1}}.$$

We define two Forcing notions. The first notion $\mathbb{GH}$ will add a tree of infinite norm and thus a copy of $G_{\max}$ and the second $\mathbb{X}$ will add a ξ-embedding of F into $G_{\max}$. Since there are different graph embeddings involved, we agree to use f only for initial ξ-embeddings of F and the letter g ranges over initial embeddings of other graphs including $G_{\max}$ itself. Accordingly, $\mathfrak{f}$ refers to embeddings of F into $G_{\max}$.

Definition 3.4.15. $[G_{\max}]$-Forcing $\mathbb{GH}$:
A condition consists of a pair (T, g) where $T \subseteq T_{\max}$ is a tree of infinite norm and $g \prec T$ is an initial embedding of some graph onto a maximal antichain in T. For two conditions $(T_0, g_0), (T_1, g_1) \in \mathbb{GH}$ we define

$$(T_0, g_0) \leq (T_1, g_1) \text{ iff } g_1 \prec_{T_1} g_0 \wedge T_0 \subseteq T_1.$$

That is (T_0, g_0) carries more information or is stronger than (T_1, g_1). If furthermore g_0 equals g_1 we call (T_0, g_0) a *pure extension of* (T_1, g_1). The height of a condition $\mathrm{hght}(T, g)$ is the height of the embedding g. If $f \prec_T f'$ and $\mathrm{hght}(f) + 1 = \mathrm{hght}(f')$ we call f' *an immediate successor of* f.

Fact 3.4.16. Let $H_{\mathbb{GH}}$ be a $\mathbb{GH}$-generic filter. Then the *generic tree* $T_H = \bigcup\{T_{\leq g} :$

$(T, g) \in H_{\mathbb{GH}}\}$ has infinite norm and hence there exist embeddings $\mathfrak{g} \in \binom{[T_H]}{G_{\max}}$. We mean those embeddings if we talk about *new copies of* $G_{\max}$

That the resulting tree has infinite norm follows from the fact that all appearing trees have infinite norm and that for every $m \in \omega$ the following set is dense:

$$\{(T, g) : (T, g) \in \mathbb{GH} \wedge \forall t \in T_{\leq g} \exists s \in T_{\leq g}(\|s\|_T \geq m)\}.$$

Definition 3.4.17. [ξ-Forcing $\mathbb{X}$] Let $F \subseteq G_{\max}$ be a closed induced subgraph and let ξ be a possible type for F. We define the ξ-Forcing $\mathbb{X} \subseteq \mathbb{GH}$:

$$(T, f) \in \mathbb{X} \iff f \prec T \wedge f \text{ is an initial } \xi\text{-embedding of } F.$$

The set $\mathbb{X}$ is ordered by the ordering $\leq$ from $\mathbb{GH}$.

Fact 3.4.18. Let $(T, f_0), (T, f_1) \in \mathbb{X}$ be two conditions of equal height. Then the two conditions force exactly the same statements.

Since $\mathrm{rg}(f_0)$ and $\mathrm{rg}(f_1)$ are both maximal antichains in T, the two conditions have exactly the same extensions.

Fact 3.4.19. Let $H_{\mathbb{X}}$ be an $\mathbb{X}$-generic filter over a model M and let $T_{\mathbb{X}} = \bigcup\{T : (T, f) \in H_{\mathbb{X}}\}$ be the generated tree. Then there is a surjective embedding $\mathfrak{f}_H : F \to [T_{\mathbb{X}}]$ of type ξ and furthermore $f \prec_{T_{\mathbb{X}}} \mathfrak{f}_H$ for every f which appears in the filter. We will call this embedding the $\mathbb{X}$-*generic* embedding.

This can be easily seen because for every $d \in \omega$ the set $D_d = \{(T, f) \in H_{\mathbb{X}} : \mathrm{hght}(T, f) = d\}$ is a dense subset of $\mathbb{X}$.

Lemma 3.4.20. *Let $(T, f) \in \mathbb{X}$ and let ϕ be a statement in the language of* ZFC*. Then there is a pure extension $(T', f) \leq (T, f)$ which forces either ϕ or $\neg\phi$.*

Lemma 3.4.21. *Let T_H be the $\mathbb{GH}$-generic tree. Then every $f \in \binom{[T_H]}{F}^{\xi}$ is $\mathbb{X}$-generic.*

Before we start with the proof of Lemma 3.4.20 we need another auxiliary lemma:

Lemma 3.4.22. *Let $(T_0, f_0) \in \mathbb{X}$ be a condition of height d_0. Let ϕ be a statement in the language of set theory and assume that no pure extension of (T_0, f_0) forces ϕ, then there is a pure extension $(T_1, f_0) \leq (T_0, f_0)$ such that no extension of height $d_0 + 1$ forces ϕ.*

Proof. Since the height of f_0 is d_0 the value $\xi(d_0)$ describes a node m such that any immediate successor of f_0 is constructed by a blow up of the node $f_0(m)$ by a graph H. We define a coloring on the extensions f_1 of height d_0 which satisfy $f_0 \prec_{T_0} f_1$ and which are on a sufficiently high level:
Take the highest splitting node $s_1 \in T_0$ which is below $f_1(m)$ and consider the initial ξ-embedding f_1' of height $d_0 + 1$ such that s_1 is a splitting node in $T_{\leq f_1'}$.

$$e(f_1) = \begin{cases} 1, & \text{if } (T_0[\mathrm{rg}(f_1')], f_1') \Vdash \phi \\ 2, & \text{if } (C_0[\mathrm{rg}(f_1')], f_1') \Vdash \neg\phi \\ 0, & \text{otherwise} \end{cases}$$

By an easy fusion argument we may assume that if a pure extension of such a $(T_0[\mathrm{rg}(f_1')], f_1')$ decides ϕ, then already $(T_0[\mathrm{rg}(f_1')], f_1')$ decides ϕ. Furthermore this coloring is well-defined after applying Claim 10 of Section 3.3 as in the proof of Lemma 3.3.6. Observe that there is a bijection between the embeddings like f_1 and the tree product $\bigotimes_{t \in \mathrm{rg}(f_0)} T_0[t]$. Hence we can apply Theorem 3.2.10 and thus obtain a set $D \subseteq T_0$ such that $T_1 = T_0[D]$ is of infinite norm and such that $f_0 \prec T_1$ is exactly as in the proof of Lemma 3.3.6. The coloring e has constant value on the tree T_1 where defined and we can conclude that if there is an extension of depth $d_0 + 1$ which forces ϕ, then all extensions of depth $d_0 + 1$ force ϕ. This implies that any extension of (T_1, f_0) forces ϕ and hence that $(T_1, f_1) \Vdash \phi$. Since this contradicts the initial assumption and it also finishes the proof. □

Proof of Lemma 3.4.20. For $(T, f) \in \mathbb{X}$ let $\psi(T, f)$ refer to the following statement:

"no pure extension of (T, f) forces ϕ",

and $\psi^*(T, f)$ refers to "no pure extension of (T, f) forces $\neg\phi$". Starting with the assumption $\psi(T,f) \wedge \psi^*(T,f)$ we will construct a fusion sequence $(T_i)_{i\in\omega}$ together with a sequence of integers $(l_i)_{i\in\omega}$ which results in a tree $T_\infty = \bigcap_{i\in\omega} T_i$ with infinite norm such that (T_∞, f) is a pure extension of (T,f) and no extension of (T_∞, f) forces ϕ or $\neg\phi$ which is in contradiction to the Main Forcing lemma. The fusion sequence $(T_i, l_i)_{i\in\omega}$ shall satisfy the following requirements:

(1) $T_0 = T$ and l_0 such that $\mathrm{rg}(f) \subseteq \mathrm{LEV}_{l_0}(T_0)$.

(2) T_{i+1} contains exactly one splitting node between the levels l_i and $l_{i+1}-1$.

(3) Every ξ-condition $(T', f') \leq (T_{i+1}, f)$ such that $\mathrm{rg}(f') \subseteq T' \restriction l_{i+1}$ satisfies ψ and ψ^*.

In order to construct a fusion sequence as defined in (3.2.2) we have to show that we can pick a splitting node t of any norm above any given node such that a thinning procedure above level $|t|$ establishes (3). If there are splitting nodes below the level of $|t|$ which are incomparable to t, we tacitely replace them by non splitting nodes in order not to violate (2). We assume that (3) holds for the step i. Let $\{f_j : j \in d\}$ be an enumeration of those initial ξ-embeddings of F which satisfy $\mathrm{rg}(f_j) \subseteq \mathrm{LEV}_{l_i}(T_i)$ for each $j \in d$. We apply Lemma 3.4.22 to the condition $(T_i[\mathrm{rg}(f_0)], f_0)$ first with the formula ψ and then for the formula ψ^*. This yields a pure extension and hence $|\mathrm{rg}(f_0)|$ many trees of infinite norm which we substitute for the trees $T_i[y]$ for $y \in \mathrm{rg}(f_0)$. Then we do the same for the embedding f_1 and so on. After d iterations we can pick an arbitrary splitting node s above l_i. It is no problem to drop nodes in order to satisfy (2) in the resulting tree. Let T_{i+1} be the resulting tree. The satisfaction of (3) can be seen as follows: Take $(T', f') \leq (T_{i+1}, f)$ as in (3). If $\mathrm{hght}(f') = \mathrm{hght}(f)$ the condition (3) follows from Fact 3.4.18 together with the inductive hypothesis. In the other case we can conclude with (2) that $\mathrm{hght}((T', f')) = \mathrm{hght}((T_{i+1}, f)) + 1$ and the statement follows due to the construction.

Claim 13. No condition in $\mathbb{X}$ which is stronger than $(T_\infty = \bigcup_{i\in\omega} T_i \restriction l_i, n)$ decides ϕ.

Proof of Claim 13. Let $(T^*, f^*) \leq (C_\omega, n)$ be a stronger condition in $\mathbb{X}$. Then there is an $i \in \omega$ such that the highest splitting point in $T_{\leq(}f^*)$ is between l_i, l_{i+1}. Hence because of Fact 3.4.18 there is an extension (T^*, f^{**}) of the condition (T^*, f^*) with the same height, which forces exactly the same statements. Furthermore f^{**} has appeared in the construction but (T^*, f^{**}) is a pure extension of $(T_i[\mathrm{rg}(f^{**})], f^{**}$ and hence it forces neither ϕ nor $\neg\phi$.

$\square$(Claim 13)

$\square$(Lemma 3.4.20)

Lemma 3.4.23. *Let $D \subseteq \mathbb{X}$ be dense open. Then the set of $\mathbb{GH}$-conditions (T, g) which satisfy*

$$\forall \mathfrak{f} \exists f (f \prec_T \mathfrak{f} \subseteq^* T \wedge (T[\mathrm{rg}(f)], f) \in D) \tag{3.4.1}$$

is a dense subset of $\mathbb{GH}$

We will need the following abbreviation in the proof: for an initial ξ-embedding $f \subseteq^* T$ we define the set of "admitted extensions"
$\mathcal{F}_\xi(T, f) = \{\mathfrak{f} \in \binom{[T]}{F}^\xi : f \prec_T \mathfrak{f}\}$

Proof. Let $(T_0, g_0) \in \mathbb{GH}$ we show that there is a stronger (even pure) condition which satisfies (3.4.1). We define the set of potential generic ξ-embeddings in T_0:

$$U(T_0) = \left\{\mathfrak{f} \in \binom{[T_0]}{F}^\xi : \exists f \prec_{T_0} \mathfrak{f} (T_0[\mathrm{rg}(f)], f) \in D)\right\}.$$

We show that there is a stronger condition (T_∞, g_0) which forces the statement "$\mathfrak{F}_\xi(T', f) \subseteq U$". As in the proof of Lemma 3.4.20 we construct a fusion sequence T_i for a set of levels l_i for $i \in \omega$ such that $\mathrm{rg}(g_0) \subseteq \mathrm{LEV}_{l_0}(T_0)$ and for each $i \in \omega$ there is only one splitting node between l_i and $l_{i+1} - 1$ in the tree T_{i+1}. Let Θ_i be the set of initial ξ-embeddings f such that $\mathrm{rg}(f) \subseteq \mathrm{LEV}_{l_i}(T_i)$. In each stage $i \in \omega$ of the fusion sequence we consider each $f \in \Theta_i$. If there is a pure extension $(T', f) \leq (T_i[\mathrm{rg}(f)], f)$ such that $\mathfrak{F}_\xi(T', f) \subseteq U$, then we thin out above the nodes contained in $\mathrm{rg}(f)$. That is, we substitute T_i above $\mathrm{rg}(f)$ with the trees coming from this pure extension just as in the previous proof. Let T_∞ be the fusion of this sequence.

Claim 14. For each $f \subseteq^* T_\infty$ the following statement is true: $\mathfrak{F}_\xi(T_\infty, f) \subseteq U$.

Proof of Claim 14. Assume there is an f_0 such that $(T_\infty[\mathrm{rg}(f_0)], f_0)$ is a condition in $\mathbb{X}$ and such that $\mathrm{hght}(f_0)$ equals d. Assume furthermore that $\mathfrak{F}_\xi(T_\infty, f_0)$ is not contained in U. If the following is unclear, one might look again at the proof and the statement of Lemma 3.4.22. Take an initial ξ-embedding f_1 such that $f_0 \prec_T f_1$ and $\mathrm{hght}(f_1) = d$. Consider the highest f_2 with height $d+1$ such that $f_0 \prec_{T_\infty} f_2$ but $\mathrm{rg}(f_2)$ is below the level of $\mathrm{rg}(f_1)$. If $\mathfrak{F}_\xi(T_\infty[\mathrm{rg}(f_2)], f_2) \subseteq U$ we set $e(f_1) = 0$ otherwise we set $e(f_1) = 1$. With the theorem of Halpern-Läuchli there is T'' on which the coloring e is constant with a color $j \in 2$ and $(T'', f_0) \in \mathbb{X}$. If j equals 1, then for any initial ξ-embedding f'' with height $d+1$ it holds that $\mathcal{F}_\xi(T'', f'') \not\subseteq U$. Inductively on d we can build a tree T'_∞ such that $(T'_\infty, f_0) \in \mathbb{X}$ but D contains no extension of this condition thus contradicting the density of D.

$\square$(Claim 14)

Hence the condition (T_∞, g_0) is an extension of (T_0, g_0) which proves the lemma.

$\square$

Proof of Lemma 3.4.21. Let M be a countable elementary submodel of a sufficiently large part of V which contains the Forcing notions $\mathbb{X}$ and $\mathbb{GH}$. Let H be an $\mathbb{GH}$-generic filter over M and $T_H \in M[H]$ is the corresponding generic tree as defined before. Let $\mathfrak{f} \subseteq [T_H]$ be an embedding of F of type ξ. Define the filter $\mathcal{G}(\mathfrak{f})$ as the filter which is generated by the set $\{(T, f) \in \mathbb{X} : f \prec_T \mathfrak{f}\} \cap M$. This is a filter because for every $(T_0, f_0), (T_1, f_1) \in \mathcal{G}(\mathfrak{f}) \cap M$ the intersection $T_0 \cap T_1$ contains a perfect set in M which contains $\mathfrak{f}$ as a subset. That is true because none of the infinite branches through T_H is an element of M. Observe that $\mathfrak{f}$ is the generic ξ-embedding which stems from this filter, that is $\mathfrak{f} \in M[\mathcal{G}(\mathfrak{f})]$. We show that $\mathcal{G}(\mathfrak{f})$ is M-generic:
Suppose not. Then there is a dense open set $D \subseteq \mathbb{X}$ in M such that $D \cap \mathcal{G}(\mathfrak{f}) \cap M$ is empty. Because of Lemma 3.4.23 there is a $(T^*, g^*) \in H \cap M$ which satisfies (3.4.1) for the set D. Because M is an elementary submodel we can conclude (3.4.1) in V and get a contradiction. $\square$(Lemma 3.4.21)

Now we can finally prove the theorem:

Proof of Theorem 3.4.14. Let M be a countable elementary submodel of a sufficiently big initial segment of V such that M contains the code for the Borel coloring $c : \binom{G_{\max}}{F}^{\xi} \to 2$, the graph F as well as the type function ξ and the Forcing $\mathbb{X}$ as well as the Forcing $\mathbb{GH}$. Let $\dot{\mathfrak{f}} \in V$ be a $\mathbb{X}$-name for the embedding defined via the generic filter as described in Fact 3.4.19 and let ϕ be the statement '$\dot{\mathfrak{f}}$ is of color 0 '. Then there is a condition $(T_0, \emptyset) \in \mathbb{X}$ which decides this statement. Let us assume the condition decides that $\dot{\mathfrak{f}}$ is colored by 0. Now consider the Forcing extension $M[H_{\mathbb{GH}}]$ where $H_{\mathbb{GH}}$ is a $\mathbb{GH}$-generic filter over M which contains $(T_0, \emptyset)$. Let $T_H \in M[H_{\mathbb{X}}]$ be the tree defined as in Fact 3.4.16 which generates a copy of $G_{\max}$. It is important that we argue entirely in V and prove in V that every copy of $G_{\max}$ in T_H is c-homogeneous on ξ:
Since every $\mathfrak{f} \in \binom{T_H}{F}^{\xi}$ is M-generic for the Forcing $\mathbb{X}$ and since $\mathrm{rg}(\mathfrak{f}) \subseteq T_H$ we can conclude that $(T_0, \emptyset)$ is an element of the filter $\mathcal{G}(\mathfrak{f})$ and hence $M[\mathcal{G}(\mathfrak{f})] \models c(\dot{\mathfrak{f}}) = 0$. This statement is absolute because the following statement is absolute: "the collection of initial comb-segments for which the color is not yet decided is wellfounded. " Hence there is a segment which is an initial segment of $\mathfrak{f}$ on which the color is decided and hence we can conclude $c(\mathfrak{f}) = 0$ in our world V as well.

$\square$(Theorem 3.4.14)

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Appendix

Zusammenfassung der Ergebnisse:

Der Gegenstand der vorliegenden Arbeit sind die unendliche Kombinatorik und die Ramsey-Theorie, wobei unter anderem einige Konsistenzresultate mit Hilfe von Forcing erzielt werden. Die Arbeit besteht aus zwei Teilen mit jeweils gesonderter aber verwandter Fragestellung.

Für eine natürliche Zahl m und eine Menge X verstehen wir unter einer m-Färbung eine Abbildung, die jeder m-elementigen Teilmenge von X eine Farbe, also einen Wert aus $\{0,1\}$ zuordnet. Das Theorem von Ramsey besagt, dass zu jeder m-Färbung einer unendlichen Menge X eine unendliche Teilmenge $Y \subseteq X$ existiert, auf der nur eine Farbe vorkommt–eine "homogene " Teilmenge. Um überabzählbare homogene Mengen zu erhalten, kann zu stetigen Färbungen übergegangen und der Satz von Galvin gezeigt werden: "Für jede stetige 2-Färbung einer perfekten Teilmenge der reellen Zahlen gibt es eine perfekte homogene Teilmenge". Die Anzahl homogener Mengen, die benötigt werden, um die reellen Zahlen zu überdecken, variiert abhängig von der Färbung. Die obere Schranke $\hom^2$ dieser Anzahlen nennen wir die 2-dimensionale Homogenitätszahl. Der Wert dieser Zahl ist jedoch abhängig vom betrachteten Modell der Mengenlehre. So wird in [GKKS02] ein Modell konstruiert, in dem $\hom^2$ (echt) kleiner als die Mächtigkeit der reellen Zahlen 2^ω ist. Der Satz von Galvin kann auch auf 3-Färbungen verallgemeinert werden, wobei die Eigenschaft "homogen" zu "schwach homogen"abgeschwächt werden muss: zuerst wird die Menge der 3-elementigen Teilmengen in möglichst wenige disjunkte Klassen zerlegt, so dass jedes Tripel genau zu einer Klasse gehört (eine basic partition). Relativ zu einer gegebenen Färbung ist eine Menge X genau dann schwach homogen, wenn die Farbe eines Tripels nur von seiner Klasse abhängt. Das bringt uns zur Formulierung unserer Ausgangsfrage: ist es konsistent, dass sich die reellen Zahlen immer mit weniger als 2^ω vielen schwach homogenen Mengen überdecken lassen? Diese Frage wird in Theorem 2.2.3 positiv beantwortet. Bezeichnen wir mit $\hom^3$ die 3-dimensionale Homogenitätszahl, so wird anschließend in Theorem 2.3.3 die Konsistenz von "$\hom^2 < \hom^3 = 2^\omega$" gezeigt.

Der zweite Teil widmet sich dem Studium des Graphen $G_{\max}$. Dies ist ein stetiger abgeschlossener Graph, dessen Knotenmenge ein kompakter Unterraum der reellen Zahlen und damit ein toplogischer Hausdorffscher Raum ist. In jeder offenen Teilmenge von $G_{\max}$ finden sich alle endlichen Graphen als induzierte Kopien wieder. Anstelle m-elementiger Teilmengen werden induzierte Kopien eines endlichen Graphen F in $G_{\max}$ betrachtet, wobei F aus m vielen Knoten besteht. Für diese Menge der induzierten Kopien von F schreiben wir $\binom{G_{\max}}{F}$. Der Begriff schwach homogen wird weiter verallgemeinert, und wir zeigen in Theorem 3.3.8, dass $G_{\max}$ für jede stetige Färbung von $\binom{G_{\max}}{F}$ eine schwach homogene Kopie von sich selbst enthält. Im letzten Abschnitt werden beliebige abgeschlossene Graphen H betrachtet und die induzierten Kopien $\binom{G_{\max}}{H}$ gefärbt. Durch eine weitere Verallgemeinerung der schwachen Homogenität lassen sich analoge Resultate (vgl.Theorem 3.4.2 sowie Theorem 3.4.14) für diese Färbungen unendlicher Graphen beweisen.

Lebenslauf

Der Lebenslauf ist in der Online-Version
aus Gründen des Datenschutzes nicht enthalten

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Eidesstattliche Erklärung:

Hiermit erkläre ich an Eides statt, die vorliegende Arbeit eigenhändig angefertigt zu haben. Alle verwendeten Hilfsmittel sind aufgeführt. Des weitern versichere ich, dass diese Arbeit nicht in dieser oder ähnlicher Form an einer weiteren Universität im Rahmen eines Prüfungsverfahrens eingereicht wurde. Stefanie Frick