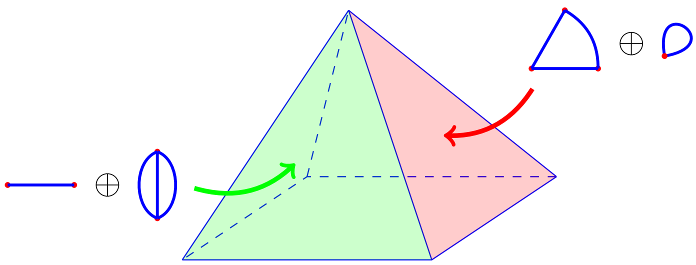
Freie Universität Berlin

On $\boldsymbol{k}$-level matroids: geometry and combinatorics



Dissertation

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Francesco Grande

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Advisor and first reviewer:
Professor Dr. Raman Sanyal

Second reviewer:
Professor Rekha Thomas, Ph.D.

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Summary

The Theta rank of a finite point configuration V is the maximal degree necessary for a sum-of-squares representation of a non-negative linear function on V. This is an important invariant for polynomial optimization that is in general hard to determine. We study the Theta rank of point configurations via levelness, that is a discrete-geometric invariant, and completely classify the 2-level (equivalently Theta-1) configurations whose convex hull is a simple or a simplicial polytope.

We consider configurations associated to the collection of bases of matroids and show that the class of matroids with bounded Theta rank or levelness is closed under taking minors. This allows us to find a characterization of matroids with bounded Theta rank or levelness in terms of forbidden minors.

We give the complete (finite) list of excluded minors for Theta-1 matroids which generalize the well-known series-parallel graphs. Moreover, we characterize the class of Theta-1 matroids in terms of the degree of generation of the vanishing ideal and in terms of the psd rank for the associated matroid base polytope.

We analyze in full detail Theta-1 matroids from a constructive perspective and discover that they are sort-closed, which allows us to determine a unimodular triangulation of every matroid base polytope and to characterize its volume by means of permutations.

A closed formula for the enumeration of Theta-1 matroids on a ground set of size n seems out of reach, but we exploit the constructive properties to provide asymptotic estimates. As a consequence, we obtain an exponential lower bound on the number of 2-level polytopes of any fixed dimension.

As for the k-level matroids with $k > 2$, we prove that the list of excluded minors is finite for every k and we describe the excluded minors for k-level graphs. We also investigate the excluded minors for graphs of Theta rank 2.

For the case of hypersimplices, that is, matroid base polytopes of uniform matroids, we present results about the non-negative rank and the Gröbner fan together with conjectures about possible generalizations to the class of Theta-1 matroids.

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Contents

Introduction

Let $V \subset \mathbb{R}^d$ be a configuration of finitely many points and $\boldsymbol{c} \in \mathbb{R}^d$ a vector. If we are asked to maximize the linear function $\langle \boldsymbol{c}, \mathbf{x} \rangle = c_1 x_1 + \ldots + c_d x_d$ on V, we are dealing with a simple task, namely, to evaluate $\langle \boldsymbol{c}, \boldsymbol{v} \rangle$ for all $\boldsymbol{v} \in V$ and record the maximum value.

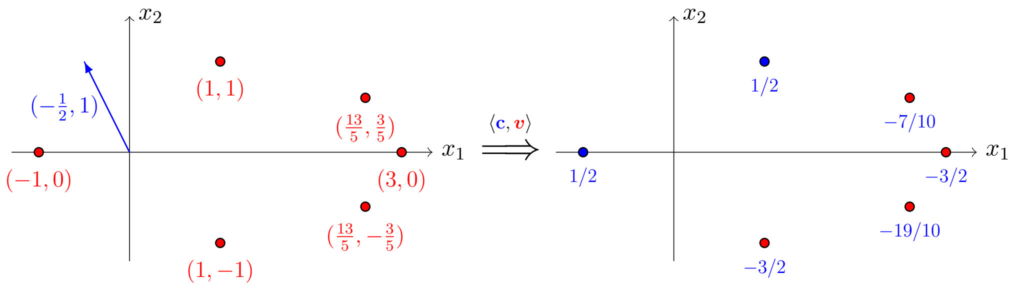

Figure 1: Linear optimization over a configuration V.

It is tempting to claim that linear optimization over a finite point configuration is computationally easy and, in particular, that the computational cost is linear in the number of points of V. These considerations rely on one key assumption, namely that V is provided as a finite list of points.

For instance, a linear optimization over the configuration V leads to the same outcome if solved on the polytope $P = \mathrm{conv}(V)$.

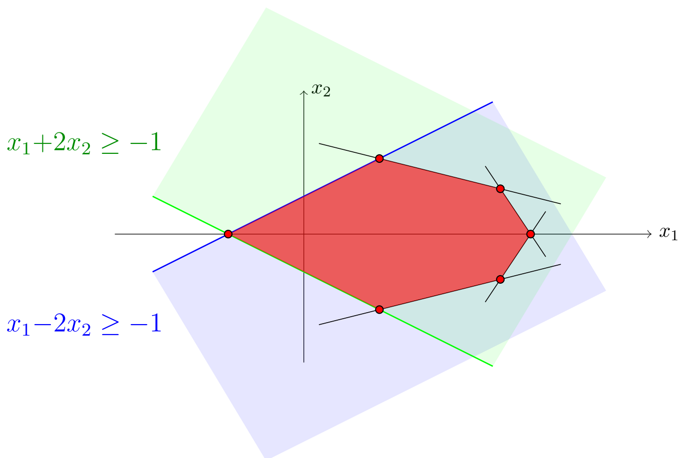

Figure 2: Linear inequalities defining $\mathrm{conv}(V)$.

Moreover, the polytope $P = \text{conv}(V)$ can be described by a system of linear inequalities $C\mathbf{x} \leq \boldsymbol{\delta}$, where $C \in \mathbb{R}^{m\times d}$ and $\boldsymbol{\delta} \in \mathbb{R}^m$ (see Figure 2). It is known that the linear programming problem

$$\max_{\boldsymbol{p}\in\mathbb{R}^d} \langle \boldsymbol{c}, \boldsymbol{p}\rangle \text{ s.t. } C\boldsymbol{p} \leq \boldsymbol{\delta}$$

can be solved in polynomial time [Sch03a, Ch. 5].

On the other hand, V could be described as the set of solutions to a system of non-linear polynomial equations (see Figure 3) in which case the direct approach to the optimization requires us to solve the system as a first step and therefore is rather unpractical. In general, performing linear optimization over a finite configuration of points defined by non-linear polynomial constraints is NP-hard [Lau09, Sect. 1].

$$\begin{cases} x_1^2+4x_2^2-2x_1-3=0 \\ (x_1-2x_2+1)(x_1-2x_2-1.4)(x_1-2x_2-3)(x_1-2x_2-3.8)=0 \\ (x_1+2x_2+1)(x_1+2x_2-1.4)(x_1+2x_2-3)(x_1+2x_2-3.8)=0 \end{cases}$$

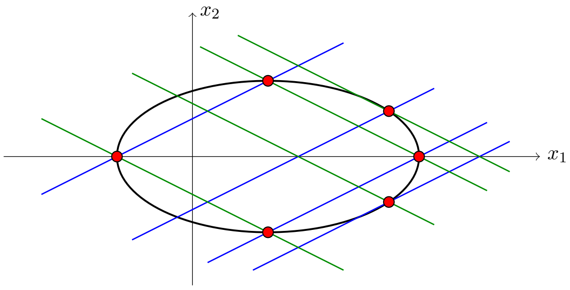

Figure 3: A polynomial description of V.

An alternative way of tackling the problem is to optimize over a relaxation of $\text{conv}(V)$, that is, a set containing $\text{conv}(V)$, and this yields an approximate solution in polynomial time. The key observation is that the polytope $\text{conv}(V)$ is determined by the set of all linear inequalities of the form $\ell(\mathbf{x}) \geq 0$, where $\ell(\mathbf{x})$ is a non-negative linear function on V.

A linear function $\ell(\mathbf{x}) = \delta - \langle \mathbf{c}, \mathbf{x}\rangle$ which is non-negative on V is called ***k*-sos** (sum of squares) with respect to V if there exist polynomials $h_1, \ldots, h_s \in \mathbb{R}[x_1, \ldots, x_d]$ such that $\deg(h_i) \leq k$ and

$$\ell(\boldsymbol{v}) \;=\; h_1^2(\boldsymbol{v}) + h_2^2(\boldsymbol{v}) + \cdots + h_s^2(\boldsymbol{v}) \qquad (1)$$

for all $\boldsymbol{v} \in V$.

Among the non-negative linear functions defining conv(V) we consider the ones that are k-sos. This set of functions cuts out a convex set $\mathsf{TB}_k(V)$, called the ***k*-Theta body** of V. Notice that the Theta bodies form a hierarchy of relaxations $\mathsf{TB}_1(V) \supseteq \mathsf{TB}_2(V) \supseteq \ldots \supseteq \mathrm{conv}(V)$.

The **Theta rank** $\mathsf{Th}(V)$ of V is the smallest $k \geq 0$ such that every non-negative linear function is k-sos with respect to V. As V is finite and $\ell(\mathbf{x})$ non-negative on V, we may interpolate $\sqrt{\ell(\mathbf{x})}$ over V by a single polynomial which shows that $\mathsf{Th}(V) \leq |V| - 1$. This, however, is a rather crude estimate as the 0/1-cube $V = \{0,1\}^n$ has Theta rank 1.

The Theta rank was introduced in [GPT10] as a measure for the 'complexity' of linear optimization over V using tools from polynomial optimization. Whenever V is given as the solutions to a system of polynomial equations, the size of a semidefinite program for the (exact) optimization of a linear function over V is of order $O(d^{\mathsf{Th}(V)})$. For this reason we are interested in the set

$$\mathcal{V}_k^{\mathsf{Th}} \;:=\; \{V \text{ point configuration} : \mathsf{Th}(V) \leq k\}.$$

For many practical applications, for instance in combinatorial optimization, an algebraic description of V is readily available and the semidefinite programming approach is the method of choice. Clearly, situations with high Theta rank make the approach impractical.

Finding the Theta rank of a configuration is a recurrent question in this thesis. For many instances it is hard to determine an exact answer, but we will exploit geometric properties for bounding the Theta rank from above. As we will provide thorough definitions in the following chapters, we try now to convey an intuition for the main objects involved in our work and how they connect to each other.

An inclusion-maximal subconfiguration $V' = \{\boldsymbol{v} \in V : \ell(\boldsymbol{v}) = 0\}$ for some linear function $\ell(\mathbf{x})$ non-negative on V is called a **facet** of V. It follows from basic convexity that $\mathsf{Th}(V)$ is the smallest k such that all facet-defining $\ell(\mathbf{x})$ are k-sos.

A point configuration V is ***k*-level** if for every facet-defining hyperplane H there are k parallel hyperplanes $H = H_1, H_2, \ldots, H_k$ with

$$V \;\subseteq\; H_1 \cup H_2 \cup \cdots \cup H_k.$$

We denote by $\mathsf{Lev}(V)$ the **levelness** of V, that is, the smallest k such that V is k-level. It is easy to see that $\mathsf{Th}(V) \leq \mathsf{Lev}(V) - 1$. Indeed any non-negative facet-defining linear function $\ell(\mathbf{x})$ attains at most $\mathsf{Lev}(V)$ values $l_1, \ldots, l_{\mathsf{Lev}(V)}$ and thus there is a polynomial $g(\mathbf{x})$ of degree $\mathsf{Lev}(V) - 1$ which interpolates the values $\sqrt{l_1}, \ldots, \sqrt{l_{\mathsf{Lev}(V)}}$ on V. Clearly $\ell(\mathbf{x}) = g^2(\mathbf{x})$ on V. Hence, the class $\mathcal{V}_k^{\mathsf{Lev}}$ of k-level point configurations is a subclass of $\mathcal{V}_{k-1}^{\mathsf{Th}}$. A main result of [GPT10] is the following characterization of $\mathcal{V}_1^{\mathsf{Th}}$.

Theorem 0.0.1 ([GPT10, Thm. 4.2])**.** *Let V be a finite point configuration. Then V has Theta rank 1 if and only if V is 2-level.*

This result encourages us to investigate the Theta rank via geometry, since it states that $\mathcal{V}_2^{\mathsf{Lev}}$ is exactly the set of configurations V for which $\mathsf{TB}_1(V) = \mathrm{conv}(V)$. With these motivations in mind, we dedicate a significant part of this work to 2-level configurations, which arise in several situations. For instance, the polytopes $P = \mathrm{conv}(V)$, $V \in \mathcal{V}_2^{\mathsf{Lev}}$, occur in the study of extremal centrally-symmetric polytopes [SWZ09] as well as in statistics under the name of *compressed* polytopes [Sul06] and include many interesting classes of polytopes (see Section 2.2.3). Furthermore, 2-level configurations are affinely equivalent to 0/1-configurations which gives them a combinatorial flavour but does not suffice to fully understand them.

We succeed in classifying the combinatorial types of simple and simplicial 2-level configurations and we identify a class of matroids (denoted by $\mathcal{M}_2^{\mathsf{Lev}}$) such that for every matroid $M \in \mathcal{M}_2^{\mathsf{Lev}}$ the matroid base configuration V_M is 2-level. The base configuration V_M contains the characteristic vectors of the bases of M. The combinatorial properties of a matroid base configuration V_M come in handy to determine the levelness $\mathsf{Lev}(V_M)$.

We try to make our presentation as complete and detailed as possible and we explore several features of the family $\mathcal{M}_2^{\mathsf{Lev}}$. The combinatorial characterization by excluded minors is one of the aspects we focus on, but we also analyze decompositions and constructive properties of $\mathcal{M}_2^{\mathsf{Lev}}$, the enumeration of this matroid family, and the structure of the vanishing ideal $I(V_M)$.

Beyond 2-levelness, we address the case of k-level matroids for $k > 2$. At this point the challenge becomes harder and no general excluded minor characterization has been found so far. In addition, geometric results about levelness only provide upper bounds to Theta rank, since $\mathcal{V}_k^{\mathsf{Lev}} \subsetneq \mathcal{V}_{k-1}^{\mathsf{Th}}$.

Chapter 1 introduces most of the basic concepts and properties that are needed in this work. We have tried to make this thesis as self-contained as possible and, for sake of clarity, we adjusted or simplified standard definitions and theorems to our needs. We identified two main areas requiring some background: point configurations and polytopes, and matroid theory. For a better readability, in some other cases we recall definitions and properties throughout the thesis, at the moment when they are needed.

Chapter 2 begins with a review of general facts about 0/1-point configurations. The second part of the chapter focuses on 2-level configurations, which can be considered as a subclass of the 0/1-configurations. We present some properties of 2-level configurations and an overview of known classes of 2-level polytopes.

Our contribution to a better understanding of 2-level configurations appears in Section 2.3: we classify the combinatorial types of simple (which easily follows from known results of [KW00]) and simplicial 2-level polytopes. More precisely, simple 2-level polytopes are Cartesian products of simplices and simplicial 2-level polytopes are direct sums of simplices of the same dimension.

Chapter 3 is the core of the thesis. We study matroids and the associated base configurations in relation to levelness. Particularly interesting to us are the classes $\mathcal{M}_k^{\mathsf{Th}}$ and $\mathcal{M}_k^{\mathsf{Lev}}$, that is, matroids whose base configurations have Theta rank $\leq k$ and levelness $\leq k$, respectively. We show that $\mathcal{M}_k^{\mathsf{Th}}$ and $\mathcal{M}_k^{\mathsf{Lev}}$ are closed under taking minors. This, in principle, allows us to find a characterization in the form of forbidden sub-structures.

In Section 3.2, we examine the class $\mathcal{M}_1^{\mathsf{Th}}$ of matroids of Theta rank 1 or, equivalently, 2-level matroids. Our main result is the excluded-minor characterization of $\mathcal{M}_1^{\mathsf{Th}}$, which in turn unlocks several doors to a deeper understanding of this family and its properties. We can summarize our findings in the following theorem.

Theorem 0.0.2. *Let $M = (E, \mathcal{B})$ be a matroid and $V_M \subset \mathbb{R}^E$ the corresponding base configuration. The following are equivalent:*

(i) *V_M has Theta rank 1 or, equivalently, is 2-level;*

(ii) *M has no minor isomorphic to $M(K_4)$, $\mathcal{W}^3$, Q_6, or P_6;*

(iii) *M can be constructed from uniform matroids by taking direct sums or 2-sums;*

(iv) *The vanishing ideal $I(V_M)$ is generated in degrees ≤ 2;*

(v) *The base polytope P_M has minimal psd rank.*

Part (ii) yields a complete and, in particular, finite list of excluded minors whereas (iii) gives a synthetic description of this class of matroids. The four excluded minors $M(K_4)$, $\mathcal{W}^3$, Q_6, and P_6 are all of rank 3 on 6 elements and we will describe them in Section 3.2. The excluded minor characterization shows that $\mathcal{M}_2^{\mathsf{Lev}}$ is the generalization to matroids of the well-known family of series-parallel graphs $\mathcal{G}_{\mathsf{SP}}$. Parts (iv) and (v) are proven in Chapter 6 (Section 6.1 and Section 6.2, respectively).

In Section 3.3 we restrict to the class of graphs. We give a complete list of excluded minors for k-level graphs (Theorem 3.3.6). The classes of 3-level and 4-level graphs appear in works of Halin (see [Die90, Ch. 6]) and Oxley [Oxl89]. In particular, we show that the wheel with 5 spokes W_5 has Theta rank 3. Combined with results of Oxley [Oxl89], this yields a list of candidates for a complete characterization of Theta-2 graphs.

Every minor-closed family of matroids has an excluded-minor characterization and the Robertson-Seymour theorem [RS04] guarantees that the list of excluded minors for minor-closed families of graphs is finite. This is not necessarily true for matroids. Part (ii) of Theorem 0.0.2 proves the finiteness of the excluded minors for $\mathcal{M}_2^{\mathsf{Lev}}$. Moreover, even though the explicit list of excluded minors for $\mathcal{M}_k^{\mathsf{Lev}}$ seems hard to provide for $k > 2$, we prove in Section 3.4 that such a list is finite.

Part (iii) of Theorem 0.0.2 hints that we could look at the family $\mathcal{M}_2^{\mathsf{Lev}}$ from the constructive side, which is indeed the topic of Chapter 4. Notice that 2-level matroids generalize series-parallel graphs and many features in the graph setting have counterparts for matroids. Every connected matroid is constructed as a sequence of 2-sums of rings, multiedges, and 3-connected matroids. This construction process can be represented by a tree-like structure whose vertices are labelled by matroids and whose edges are labelled by pairs of elements of adjacent vertex labels. The case of 2-level matroids shows some interesting features: the vertex labels are chosen among uniform matroids and we can get rid of the edge labels without losing information, as explained in Section 4.1. As a consequence, the family of connected matroids in $\mathcal{M}_2^{\mathsf{Lev}}$ is in bijection with a family of trees which we name UMR-trees.

A second result we obtain from the constructive approach is that 2-level matroids are sort-closed matroids for some ordering of the ground set. This fact implies that the corresponding matroid base polytopes are alcoved, in the sense of [LP07], and determines an explicit unimodular triangulation. Moreover, we show that 2-level matroids are contained in the class of positroids, which were introduced by Postnikov in [Pos06] and have many interesting combinatorial properties.

Every 2-level base polytope P_M has a particular inequality description which relates to the normalized volume $\mathrm{vol}(P_M)$. This allows us to compute $\mathrm{vol}(P_M)$ as the number of permutations satisfying some constraints for the position of their descents. We illustrate a way to control the evolution of the defining inequalities of P_M with respect to the tree decomposition of M and the constraints for the descents of permutations follow.

Chapter 5 is devoted to the enumeration of connected (and non-connected) 2-level matroids. As this family generalizes the family of series-parallel graphs $\mathcal{G}_{\mathsf{SP}}$, we apply methodologies of enumerative combinatorics inspired by the paper [DFK$^+$11], where a successful asymptotic enumeration of $\mathcal{G}_{\mathsf{SP}}$ is presented. We use the combinatorial class of UMR-trees as a proxy and enumerate it by a generating function $T(x) = \sum_{n\geq 1} t_n x^n$, where the coefficient t_n counts the number of connected 2-level matroids on n elements. A closed formula for the coefficients of $T(x)$ is out of reach, but we can iteratively compute its coefficients up to arbitrary degree. Moreover, we provide asymptotic estimates for the number of connected 2-level matroids on n elements. From this number we deduce a lower bound, exponential in n, on the number of combinatorially non-equivalent 2-level $(n-1)$-polytopes.

Chapter 6 is divided into two areas: it completes the missing parts (iv) and (v) of Theorem 0.0.2 and presents some results about hypersimplices followed by our conjectures about possible generalizations to 2-level matroid base polytopes.

Section 6.1 deals with the vanishing ideal of matroids $M \in \mathcal{M}_2^{\mathsf{Lev}}$, that is the vanishing ideal $I(V_M)$ of the 2-level base configuration V_M. We present the proof of Part (iv) of Theorem 0.0.2, namely that 2-level matroids are precisely those matroids M for which the base configuration V_M is cut out by quadrics (Theorem 6.1.5). This contrasts the situation for general point configurations as shown in Example 6.1.3.

Section 6.2 is dedicated to the psd rank of a base polytope P. This is the smallest "size" of a spectrahedron that linearly projects to P. The psd rank was studied in [GPT13, GRT13] and it was shown that the psd rank $\mathrm{rank}_{\mathsf{psd}}(P)$ is at least $\dim(P) + 1$. Part (v) of Theorem 0.0.2 states that the 2-level matroids are exactly those matroids for which the psd rank of the base polytope $P_M = \mathrm{conv}(V_M)$ is minimal. Again, this is in strong contrast to the psd rank of general polytopes.

The extension complexity (or non-negative rank) of a polytope P is defined in analogy to psd rank as the smallest number of facets of a polytope that linearly projects to P. The extension complexity can be seen in terms of a cone factorization for a cone of type $\mathbb{R}^m_{\geq 0}$ and is shown to be the same as the non-negative rank of the slack matrix of P ([Yan91]). This is a current topic in optimization for which some basic questions, such as the extension complexity of the Cartesian product of polytopes, remain unanswered. In Section 6.3 we study the extension complexity of the hypersimplex $\Delta_{n,k}$, that is, the matroid base polytope of the uniform matroid $U_{n,k}$, and prove that for $n \geq 6$

$$\mathrm{xc}(\Delta_{n,k}) = 2n.$$

In other words, the hypersimplex $\Delta_{n,k}$, $n \geq 6$, does not admit any extension with fewer facets than the hypersimplex itself. We conjecture this to be true for a whole class of 2-level base polytopes satisfying some additional requirements.

One last result about hypersimplices is presented in Section 6.4. More precisely, we consider the vanishing ideal of $\Delta_{n,k}$, that is the ideal $I_{n,k} := I(V_{U_{n,k}})$ for a uniform matroid $U_{n,k}$. We study the reduced Gröbner bases of $I_{n,k}$ in light of results from [HR03] and describe the Gröbner fan $\mathrm{GF}(I_{n,k})$. Finally, we interpret $\mathrm{GF}(I_{n,k})$ as the normal fan of a polyhedron, which turns out to be the Minkowski sum of a permutahedron with the cone $\mathbb{R}^n_{\leq 0}$.

Chapter 3 and the first two sections of Chapter 6 are partly joint work with Raman Sanyal ([GS14]), while Section 4.1 and Chapter 5 are joint work with Juanjo Rué ([GR15]). Section 6.3 is part of an ongoing project with Arnau Padrol and Raman Sanyal.

Chapter 1

Basics

1.1 Point configurations and polytopes

1.1.1 Basic definitions and properties

Let $V = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be a point configuration in $\mathbb{R}^d$ with no repeated points. The **affine hull** of V is the affine subspace

$$\mathrm{aff}(V) := \left\{ \sum_{i=1}^{n} \mu_i \boldsymbol{v}_i \ : \ \mu_i \in \mathbb{R}, \sum_{i=1}^{n} \mu_i = 1 \right\}$$

and the **dimension** of V is the dimension of its affine hull $\dim(V) := \dim(\mathrm{aff}(V))$.

Let $\ell(\mathbf{x}) = \delta - \langle \boldsymbol{c}, \mathbf{x} \rangle$ be a linear function for $\mathbf{0} \neq \boldsymbol{c} \in \mathbb{R}^d$, $\delta \in \mathbb{R}$. The set $H_\ell := \{\boldsymbol{p} \in \mathbb{R}^d : \ell(\boldsymbol{p}) = 0\}$ defines a **hyperplane** in $\mathbb{R}^d$ and is called **supporting** for V if $V \cap H_\ell \neq \emptyset$ and $\ell(\boldsymbol{v}) \geq 0$ for all $\boldsymbol{v} \in V$. Equivalently, H_ℓ is supporting for V if V is contained in the closed half-space $H_\ell^+ := \{\boldsymbol{p} \in \mathbb{R}^d : \ell(\boldsymbol{p}) \geq 0\}$.

For every supporting hyperplane H_ℓ, the subconfiguration $V' = \{\boldsymbol{v} \in V : \ell(\boldsymbol{v}) = 0\}$ is called a **face** of V. A face V' is **proper** if $V' \neq \emptyset$ and $V' \neq V$. The ***k*-faces** of V are the faces V' such that $\dim(V') = k$ and the $(\dim(V)-1)$-faces are called **facets**. The **face lattice** of V is the set of faces of V together with a partial order by inclusion. Two configurations V_1 and V_2 are **combinatorially equivalent** if there is a bijection $V_1 \to V_2$ which preserves the face lattice.

An **affine map** is a function $f : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ such that

$$f(\mathbf{x}) = A\mathbf{x} + \mathbf{b},$$

$A \in \mathbb{R}^{d_2 \times d_1}$ and $\mathbf{b} \in \mathbb{R}^{d_2}$. Two configurations $V_1 \subset \mathbb{R}^{d_1}$ and $V_2 \subset \mathbb{R}^{d_2}$ are **affinely equivalent** if there is an affine map $f : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ such that f restricts to a bijection $V_1 \to V_2$. Such affine map is called an **affine transformation** of V_1. Figure 1.1 shows three affinely equivalent configurations.

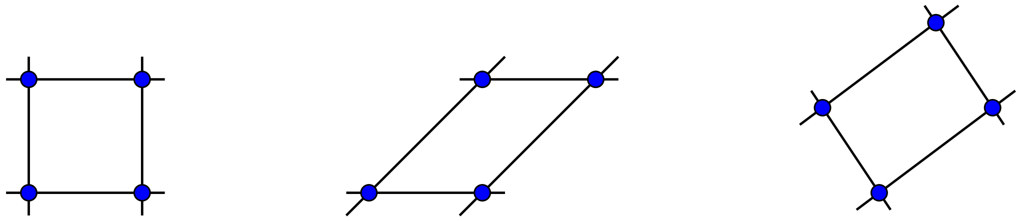

Figure 1.1: Affinely equivalent configurations of 4 points in $\mathbb{R}^2$.

Every affine transformation f of V preserves supporting hyperplanes and therefore V and $f(V)$ have the same face lattice. It follows that two affinely equivalent configurations are also combinatorially equivalent.

Given a point configuration $V = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\} \subset \mathbb{R}^d$, the convex hull

$$\operatorname{conv}(V) := \left\{ \boldsymbol{p} \in \mathbb{R}^d \ : \ \boldsymbol{p} = \sum_{i=1}^{n} \lambda_i \boldsymbol{v}_i,\ \lambda_i \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

is called a **polytope**. The set of **vertices** of $P = \operatorname{conv}(V)$ is the inclusion-minimal subconfiguration $\mathcal{V}(P) \subseteq V$ such that $P = \operatorname{conv}(\mathcal{V}(P))$. The vertices of P are the 0-faces of V, its dimension $\dim(P)$ is $\dim(\mathcal{V}(P))$ and its faces are the convex hulls of the faces of $\mathcal{V}(P)$. The **interior** of P is defined as the set $\operatorname{int}(P) := \{\boldsymbol{p} \in P : \boldsymbol{p} \notin F \text{ for every facet } F \subset P\}$. Notice that our definition of interior corresponds to the usual definition of relative interior. Two polytopes are combinatorially or affinely equivalent if and only if the configurations given by their set of vertices are.

Let us consider a point configuration V with $d{+}1$ points such that $\dim(V) = d$. Every subconfiguration of $k{+}1$ points is a k-face of V and the convex hull of V is called **d-simplex**. Among many properties, we mention that all its facets are $(d{-}1)$-simplices. A polytope such that all its facets are simplices is called **simplicial**.

In [Zie95, Thm. 2.15] several equivalent ways to represent a polytope are listed. We defined a polytope as the convex hull of a point configuration and we now emphasize that a polytope $P \subset \mathbb{R}^d$ can also be represented as the intersection of facet-defining closed half-spaces, one for each facet, with the affine hull of $\mathcal{V}(P)$. This representation for $P \subset \mathbb{R}^d$ with m facets translates into a system of inequalities given by the facet-defining half-spaces $H^+_{\ell_1}, \ldots, H^+_{\ell_m}$ and equalities given by the affine hull $\operatorname{aff}(\mathcal{V}(P))$. More precisely, if $\ell_i(\mathbf{x}) = \delta_i - \langle \boldsymbol{c}_i, \mathbf{x} \rangle$ for $i = 1, \ldots, m$, then we can write

$$P = \{\boldsymbol{p} \in \mathbb{R}^d \ : \ C\boldsymbol{p} \leq \boldsymbol{\delta}\} \cap \operatorname{aff}(\mathcal{V}(P)),$$

where $C \in \mathbb{R}^{m\times d}$, $\boldsymbol{c}_i$ is the ith row of C, and $\boldsymbol{\delta} = (\delta_1, \dots, \delta_m)$. Notice that no equalities are needed if $P \subset \mathbb{R}^d$ is d-dimensional.

Example 1.1.1. Let us consider the point configuration

$$V = \{(1,0,0),(0,1,0),(-1,-1,0),(0,0,1),(0,0,-1)\} \subset \mathbb{R}^3.$$

The configuration has 6 facets, 9 edges (1-faces), and 5 vertices (0-faces). Notice that every point of V is a 0-face, thus V is the set of vertices of $\mathrm{conv}(V)$. The supporting hyperplanes of the facets are

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 1, & -2x_1 + x_2 + x_3 &= 1, & x_1 + x_2 + x_3 &= 1,\\ x_1 + x_2 - x_3 &= 1, & x_1 - 2x_2 - x_3 &= 1, & -2x_1 + x_2 - x_3 &= 1. \end{aligned}$$

We draw the polytope $P := \mathrm{conv}(V)$ as in Figure 1.2 and observe that every facet is a 2-simplex, thus P is simplicial.

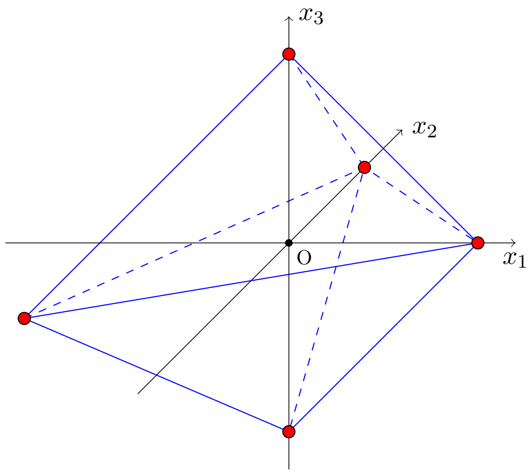

Figure 1.2: Bipyramid over a triangle.

The configuration V provides the representation of P by vertices; from the facet-defining hyperplanes we find the representation by inequalities

$$P = \{\boldsymbol{p} \in \mathbb{R}^3 \ : \ C\boldsymbol{p} \le \boldsymbol{\delta}\},$$

where $\boldsymbol{\delta} = \mathbb{1} \in \mathbb{R}^6$ and

$$C = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & -1 \\ -2 & 1 & -1 \end{pmatrix}.$$

Let $P \subset \mathbb{R}^d$ be a polytope with vertices $V = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ and m facets. Consider the representation given by the facet inequalities

$$P = \{\boldsymbol{p} \in \mathbb{R}^d \ : \ C\boldsymbol{p} \leq \boldsymbol{\delta}\} \cap \operatorname{aff}(V),$$

for $C \in \mathbb{R}^{m \times d}$ and $\boldsymbol{\delta} \in \mathbb{R}^m$. With a slight abuse of notation we consider V as a matrix in $\mathbb{R}^{n \times d}$ whose rows are represented by the vertices of P. The **slack matrix** of P with respect to the representation $(V, C, \boldsymbol{\delta})$ is the matrix $S \in \mathbb{R}^{n \times m}_{\geq 0}$ such that $S_{ij} = \delta_j - \langle \boldsymbol{c}_j, \boldsymbol{v}_i \rangle$. Equivalently,

$$S = [V, \mathbb{1}] \cdot [-C, \boldsymbol{\delta}]^T \in \mathbb{R}^{n \times m}_{\geq 0}.$$

Observe that the slack matrix is not unique: scaling columns of S by positive scalars yields several valid slack matrices for P. Namely, this operation corresponds to rescaling all coefficients in one inequality, thus it does not alter the polytope P.

We denote by $\mathbb{S}(P)$ the set of all slack matrices for P. Any $S \in \mathbb{S}(P)$ specifies an embedding of P up to affine equivalence. In particular, any slack matrix $S \in \mathbb{S}(P)$ encodes the combinatorial structure of the polytope: the **support** $\operatorname{supp}(S)$ of S is the 0/1-matrix whose zero entries are exactly the zero entries of S and is enough to reconstruct the face lattice of P ([Zie95, Ch. 2, Ex. 2.7]).

The extension complexity of P, which we discuss in Section 6.3, is another information carried by a slack matrix $S \in \mathbb{S}(P)$ as proven in [Yan91].

Example 1.1.2. We can compute the slack matrix of the polytope P defined in Example 1.1.1 with respect to the given representation $(V, C, \mathbb{1})$.

$$S = [V, \mathbb{1}] \cdot [-C, \mathbb{1}]^T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 & -1 & -1 & -1 & 2 \\ 2 & -1 & -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 3 & 0 & 0 & 0 & 3 \\ 3 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

Let $P \subset \mathbb{R}^d$ be a d-polytope (d-dimensional polytope) such that $\mathbf{0} \in \operatorname{int}(P)$. Its **polar** is defined as the set

$$P^\circ := \{\boldsymbol{y} \in \mathbb{R}^d \ : \ \boldsymbol{y}^T \boldsymbol{p} \leq 1 \text{ for all } \boldsymbol{p} \in P\}.$$

The set P° is a d-polytope and $\mathbf{0} \in \operatorname{int}(P^\circ)$. Moreover, $(P^\circ)^\circ = P$ and it easy to see that, given the vertices V of P or a representation $\{\boldsymbol{p} \in \mathbb{R}^d : C\boldsymbol{p} \leq \mathbb{1}\} \cap \operatorname{aff}(V)$, there is a straightforward description of the polar of P:

$$P^\circ = \operatorname{conv}(C) = \{\boldsymbol{y} \in \mathbb{R}^d \ : \ V\boldsymbol{y} \leq \mathbb{1}\}.$$

Polarity induces a bijection between the vertices of P and the facets of P° and vice versa. More generally, it reverses inclusion of faces and interchanges k-faces of P with $(\dim(P)-k-1)$-faces of P°. We say that the face lattice of P° is the **opposite** of the face lattice of P. We refer the reader to [Zie95, Sect 2.3] for more details.

If a d-polytope $P \subset \mathbb{R}^d$ does not contain $\mathbf{0}$ in its interior, the polar set of P is not a polytope. Nevertheless, for any translation P' of P such that $\mathbf{0} \in \operatorname{int}(P')$, the polar of P' is a polytope. The polar polytopes obtained by different translations of P, while having different metric properties, have the same combinatorial structure, since their face lattices are opposite to the face lattice of P. We say that a polytope P^Δ is a **combinatorial polar** of P if its face lattice is opposite to the face lattice of P. If $\mathbf{0} \in \operatorname{int}(P)$, then P° is also a combinatorial polar of P. Clearly, every full-dimensional polytope P has at least one combinatorial polar P^Δ.

A d-polytope $P \subset \mathbb{R}^d$ is **simple** if every vertex is adjacent to exactly d edges. Any combinatorial polar of a simple polytope P is simplicial.

Example 1.1.3. Consider again the polytope P of Example 1.1.1. We can easily find the vertices of its polar polytope. Indeed, $P = \{\boldsymbol{p} \in \mathbb{R}^3 : C\boldsymbol{p} \leq \mathbb{1}\}$ implies $P^\circ = \operatorname{conv}(C)$, where the rows of the matrix C represent the point configuration

$$\{(1,-2,1),(-2,1,1),(1,1,1),(1,1,-1),(1,-2,-1),(-2,1,-1)\}.$$

The polytope P°, shown in Figure 1.3, is simple.

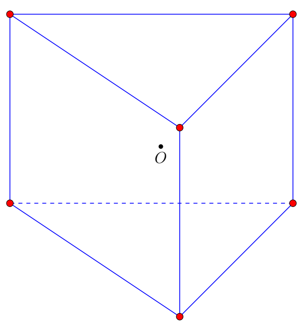

Figure 1.3: Triangular prism.

The inequality representation of the polar easily follows from V, namely $P^\circ = \{\boldsymbol{y} \in \mathbb{R}^3 : V\boldsymbol{y} \leq \mathbb{1}\}$. Observe that the slack matrix of P° with respect to the representation $(C, V, \mathbb{1})$ is S^T, where S is the slack matrix of P with respect to $(V, C, \mathbb{1})$ computed in Example 1.1.2.

Now we introduce two important constructions in the setting of finite point configurations.

Let $V_1 \subset \mathbb{R}^{d_1}$ and $V_2 \subset \mathbb{R}^{d_2}$ be point configurations. The **Cartesian product** of V_1 and V_2 is the configuration

$$V_1 \times V_2 := \left\{ \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{pmatrix} \in \mathbb{R}^{d_1+d_2} \ : \ \boldsymbol{v}_1 \in V_1 \text{ and } \boldsymbol{v}_2 \in V_2 \right\}.$$

If, in addition, $\mathbf{0} \in \mathrm{int}(\mathrm{conv}(V_1))$ and $\mathbf{0} \in \mathrm{int}(\mathrm{conv}(V_2))$, the **direct sum** (or free sum) of V_1 and V_2 is the configuration

$$V_1 \oplus V_2 := \left\{ \begin{pmatrix} \boldsymbol{v}_1 \\ \mathbf{0} \end{pmatrix} \in \mathbb{R}^{d_1+d_2} \ : \ \boldsymbol{v}_1 \in V_1 \right\} \cup \left\{ \begin{pmatrix} \mathbf{0} \\ \boldsymbol{v}_2 \end{pmatrix} \in \mathbb{R}^{d_1+d_2} \ : \ \boldsymbol{v}_2 \in V_2 \right\}.$$

The dimension of both constructions is $\dim(V_1) + \dim(V_2)$. Moreover, both of them are combinatorial, that is the face lattices of $V_1 \times V_2$ and $V_1 \oplus V_2$ can be derived from the face lattices of V_1 and V_2 [Zie95, Ch. 0].

Proposition 1.1.4. *A non-empty configuration V' is a face of $V_1 \times V_2$ if and only if there exist non-empty faces $V_1' \subseteq V_1$ and $V_2' \subseteq V_2$ such that $V' = V_1' \times V_2'$.*

Proposition 1.1.5. *The configuration V' is an i-face of $V_1 \oplus V_2$ for $i \leq \dim(V_1) + \dim(V_2) - 1$ if and only if there exist faces $V_1' \subset V_1$ and $V_2' \subset V_2$ such that $\dim(V_1') + \dim(V_2') + 1 = i$ and $V' = (V_1' \times \{\mathbf{0}\}) \cup (\{\mathbf{0}\} \times V_2')$.*

Observe that each facet V' of the direct sum $V_1 \oplus V_2$ is constructed from a facet of V_1 and a facet of V_2.

If two configurations V_1 and V_2 are sets of vertices of two polytopes $P_1 = \mathrm{conv}(V_1)$ and $P_2 = \mathrm{conv}(V_2)$, then the Cartesian product and the direct sum are the classic polytopal constructions presented in [HRGZ04, Sect. 16.1.3].

Example 1.1.6. The configuration V in Example 1.1.1 is the direct sum of the configurations $V_1 = \{(1,0), (0,1), (-1,-1)\} \subset \mathbb{R}^2$ and $V_2 = \{1, -1\} \subset \mathbb{R}^1$. The polytope $P_1 = \mathrm{conv}(V_1)$ is a 2-simplex and $P_2 = \mathrm{conv}(V_2)$ is a 1-simplex.

On the other hand, the vertices V° of the polar polytope P° described in Example 1.1.3 form a point configuration that can be obtained as the Cartesian product of $V_1^\circ = \{(1,-2), (-2,1), (1,1)\} \subset \mathbb{R}^2$ and $V_2^\circ = \{1, -1\} \subset \mathbb{R}^1$.

The last example suggests a polarity relation between direct sums and Cartesian products of polytopes. Indeed, as discussed in [HRGZ04, Sect. 16.1.3], two polytopes P_1 and P_2 containing the origin in the interior satisfy

$$(P_1 \times P_2)^\circ = P_1^\circ \oplus P_2^\circ .$$

1.1.2 Gale duality

Gale duality for vector configurations is a very useful tool which is ultimately applied to obtain results in polytope theory ([Zie95, Sect. 6.4],[Mat02, Sect. 5.6]). In particular, polytopes with few vertices (compared to the dimension) can be well understood by means of Gale duality. For instance, in Chapter 2 we will use it to study a very specific class of polytopes, that is, direct sums of simplices.

Given a point configuration $V = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\} \subset \mathbb{R}^d$ (possibly the vertices of a polytope), the columns of the matrix

$$\hom(V) := \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_n \\ 1 & 1 & \dots & 1 \end{bmatrix} \in \mathbb{R}^{(d+1)\times n}$$

define a configuration of n vectors in $\mathbb{R}^{d+1}$ which is called the **homogenization** of V.

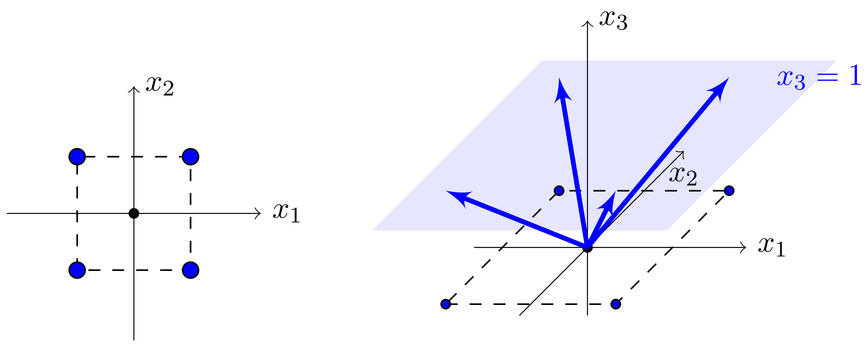

Figure 1.4: Homogenization of 4 points in $\mathbb{R}^2$.

The columns of any matrix $W \in \mathbb{R}^{(d+1)\times n}$ give a configuration $\mathbf{w}_1, \dots, \mathbf{w}_n$ of (possibly repeated) vectors in $\mathbb{R}^{d+1}$. Given $W \in \mathbb{R}^{(d+1)\times n}$ we define the vector space of **linear dependences**

$$\mathrm{Dep}(W) := \left\{ \boldsymbol{\lambda} \in \mathbb{R}^n \, : \, \sum_{i=1}^n \lambda_i \mathbf{w}_i = \mathbf{0} \right\},$$

and the vector space of **linear evaluations**

$$\mathrm{Val}(W) := \{\mathbf{z} \in \mathbb{R}^n \ : \ z_i = \langle \boldsymbol{c}', \mathbf{w}_i \rangle, \ \boldsymbol{c}' \in \mathbb{R}^{d+1}\}.$$

Moreover, a-Dep(V) := Dep(hom(V)) is the vector space of **affine dependences** of a point configuration V and a-Val(V) := Val(hom(V)) the vector space of its **affine evaluations**.

A main feature of a-Val(V) is that it encodes a description of the faces of V: a configuration $V' \subset V$ lies in a supporting hyperplane if and only if there is an affine evaluation $\mathbf{z} \in$ a-Val(V) such that $z_i = 0$ for $\boldsymbol{v}_i \in V'$ and $z_i > 0$ otherwise. Indeed the affine evaluation $\mathbf{z}$ is given by evaluating a linear function $\ell(\mathbf{x}) = \delta - \langle \boldsymbol{c}, \mathbf{x} \rangle$ on V, where $\boldsymbol{c} \in \mathbb{R}^d$ and $\delta \in \mathbb{R}$. The corresponding evaluation on hom(V) is given by the vector $\boldsymbol{c}' = (-\boldsymbol{c}, \delta) \in \mathbb{R}^{d+1}$.

Example 1.1.7. Consider the point configuration $V \subset \mathbb{R}^3$ from Example 1.1.1. We have

$$W = \mathrm{hom}(V) = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The vector space of linear dependences of W has dimension one and we can choose the basis $(2, 2, 2, -3, -3)$.

Let us consider four different vectors in $\mathbb{R}^{3+1}$: $\boldsymbol{c}'_1 = (0, 0, 1, 0)$, $\boldsymbol{c}'_2 = (0, 1, 0, 0)$, $\boldsymbol{c}'_3 = (0, 0, 1, 1)$, $\boldsymbol{c}'_4 = (-1, -1, 1, 1)$. The corresponding linear evaluations on W are $(0, 0, 0, 1, -1)$, $(0, 1, -1, 0, 0)$, $(1, 1, 1, 2, 0)$, $(0, 0, 3, 2, 0)$. Note that the first two evaluations have both positive and negative entries, thus do not correspond to faces of V. Indeed they correspond to the hyperplanes $x_3 = 0$ and $x_2 = 0$ as shown in Figure 1.5.

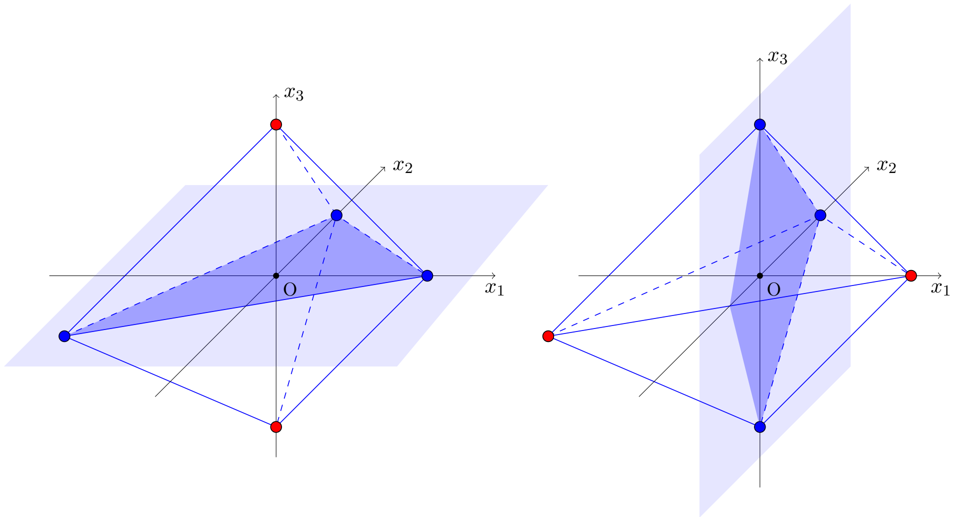

Figure 1.5: Evaluations corresponding to non-faces of V.

The third and the fourth evaluations correspond to faces of V as shown in Figure 1.6. In particular, $\boldsymbol{c}'_3$ gives the vertex $\bar{\boldsymbol{v}} = (0, 0, -1)$ and $\boldsymbol{c}'_4$ gives the facet $V' = \{(1, 0, 0), (0, 1, 0), (0, 0, -1)\}$. We see that $\bar{\boldsymbol{v}} \subset V'$ because the zeros of the evaluation of $\boldsymbol{c}'_3$ are also zeros of the evaluation of $\boldsymbol{c}'_4$. Notice that, up to multiplication by a positive scalar, the evaluation of a facet is exactly the column of the slack matrix corresponding to the facet.

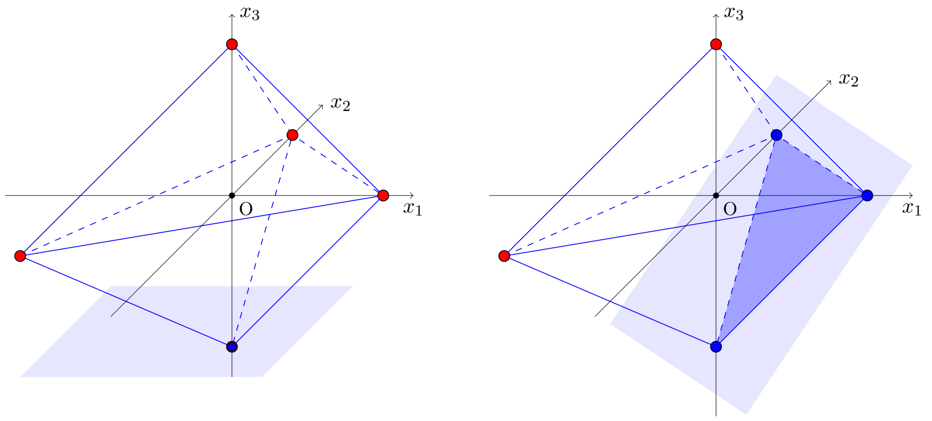

Figure 1.6: Evaluations corresponding to faces of V.

Consider a full-rank matrix $W \in \mathbb{R}^{(d+1)\times n}$, $n \geq d+1$. The kernel of W has dimension $n-d-1$. Let $G \in \mathbb{R}^{(n-d-1)\times n}$ be a full-rank matrix such that $W \cdot G^T = \mathbf{0} \in \mathbb{R}^{(d+1)\times(n-d-1)}$. The matrix G is called **Gale dual** of W and its columns form a configuration of (possibly repeated) vectors $\{\mathbf{g}_1, \dots, \mathbf{g}_n\}$

in $\mathbb{R}^{n-d-1}$. The Gale dual G is unique up to linear transformations. Notice that the ith column $\mathbf{w}_i$ of W is naturally associated to the ith column of G.

We state here the main theorem about Gale duality which suggests how to go back and forth from a vector configuration to its Gale dual in order to extract as much information as possible. The reader is referred to [Zie95, Ch. 6] for more details.

Theorem 1.1.8. *Let $W \in \mathbb{R}^{(d+1)\times n}$ be a full-rank matrix and G its Gale dual. The following holds:*

(i) $\mathrm{Val}(W) = \mathrm{Dep}(G)$;

(ii) $\mathrm{Dep}(W) = \mathrm{Val}(G)$.

The theorem can be applied directly to study a spanning point configuration $V = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\} \subset \mathbb{R}^d$. Let G be the Gale dual of $\mathrm{hom}(V)$ (shortly, the Gale dual of V): for any $I \subset [n]$, the set $\{\boldsymbol{v}_i : i \in I\}$ is a face of V if and only if there exists $\boldsymbol{\lambda} \in \mathbb{R}^n$ such that $\lambda_i = 0$ for $i \in I$, $\lambda_i > 0$ for $i \notin I$, and

$$\sum_{i \notin I} \lambda_i \mathbf{g}_i = \mathbf{0}.$$

The facets of V are the inclusion-maximal sets $I \subset [n]$ such that the subconfiguration $\{\boldsymbol{v}_i : i \in I\}$ lies on a supporting hyperplane. Therefore the facets of V are in bijection with the minimal positive dependences in G.

Since the last row of $\mathrm{hom}(V)$ contains only ones, the linear evaluation of $\mathbf{e}_{d+1} = (0, \dots, 0, 1) \in \mathbb{R}^{d+1}$ is $\mathbb{1} \in \mathbb{R}^n$. Theorem 1.1.8 implies the following proposition.

Proposition 1.1.9. *Let $V \subset \mathbb{R}^d$ be a spanning configuration of n points. Let G be its Gale dual. Then*

$$\sum_{i=1}^{n} \mathbf{g}_i = \mathbf{0}.$$

Example 1.1.10. Consider once more the configuration $V \subset \mathbb{R}^3$ from Example 1.1.1. The Gale dual of V is $G = [2, 2, 2, -3, -3] \in \mathbb{R}^{1\times 5}$.

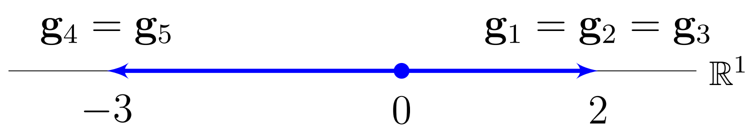

Figure 1.7: Gale dual of the bipyramid over a triangle.

It is straightforward to see that facets of V are determined by any $I \subset [5]$ such that $|I \cap \{1,2,3\}| = 2$ and $|I \cap \{4,5\}| = 1$, since they correspond to minimal dependences of the vectors in G.

1.2 Matroids

1.2.1 Basic definitions and properties

Matroids will play a central role throughout this thesis and we recall here basic definitions and properties. The combinatorial theory of matroids is a vast subject and we refer the reader to the book by Oxley [Oxl11] for further information.

Matroids have several equivalent definitions and we present the definition that fits our point of view.

Definition 1.2.1. A **matroid** of rank k is a pair $M = (E, \mathcal{B})$ consisting of a finite ground set E and a collection of bases $\emptyset \neq \mathcal{B} \subseteq \binom{E}{k}$ satisfying the basis exchange axiom: for $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, there is $y \in B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y \in \mathcal{B}$.

A set $I \subseteq E$ is **independent** if $I \subseteq B$ for some $B \in \mathcal{B}$. The **rank** of $X \subseteq E$, denoted by $\mathrm{rank}_M(X)$, is the cardinality of the largest independent subset contained in X. The **circuits** of M are the inclusion-minimal dependent subsets. Given a matroid M we denote its ground set by $E(M)$, its collection of bases by $\mathcal{B}(M)$, its collection of independent sets by $\mathcal{I}(M)$, and its collection of circuits by $\mathcal{C}(M)$.

The collection of circuits (as well as the collection of independent sets or the rank function) uniquely determines a matroid and sometimes we use this description instead of the one given by the collection of bases. For instance, we define matroid isomorphism via the collection of circuits: two matroids M_1 and M_2 are **isomorphic** if their collection of circuits are the same up to relabelling of the ground sets $E(M_1)$ and $E(M_2)$. More formally, $M_1 \cong M_2$ if there is a bijection $\varphi : E(M_1) \to E(M_2)$ such that, for all $X \subseteq E(M_1)$, $\varphi(X) \in \mathcal{C}(M_2)$ if and only if $X \in \mathcal{C}(M_1)$.

We recall here one more property of the collection of circuits.

Proposition 1.2.2 ([Oxl11, Lem. 1.1.13])**.** *Let $\mathcal{C}$ be the collection of circuits of a matroid M. Then the following hold:*

(i) $\emptyset \notin \mathcal{C}$;

(ii) *If C_1 and C_2 are in $\mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$;*

(iii) *If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus e$.*

Given a matroid $M = (E, \mathcal{B})$, an element $e \in E$ is called a **loop** if $\{e\}$ is a circuit. We say that $e, f \in E$ are **parallel** if $\{e, f\}$ is a circuit. A matroid is **simple** if it does not contain loops or parallel elements. A **parallel class** of M is a maximal subset X of E such that any two distinct elements of X are parallel and no element is a loop. The set $X \subseteq E$ is a **non-trivial** parallel class if $|X| > 1$. A **flat** of a matroid is a set $F \subseteq E$ such that $\mathrm{rank}_M(F) < \mathrm{rank}_M(F \cup e)$ for all $e \in E \setminus F$.

A particular class of matroids that we will consider are the **graphic matroids**. To a graph $G = (V, E)$ we associate the matroid $M(G) = (E, \mathcal{B})$. The bases are exactly the spanning forests of G. The running example for this section is the following.

Example 1.2.3. Let G be the graph in Figure 1.8.

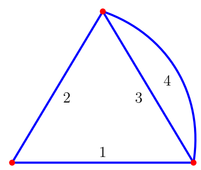

Figure 1.8: Graphic matroid with 4 elements.

The graphic matroid $M = M(G)$ has ground set $E = \{1, 2, 3, 4\}$, $\mathrm{rank}(M) = 2$, and collection of bases

$$\mathcal{B}(G) \ = \ \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}.$$

The **dual matroid** M^* of the matroid $M = (E, \mathcal{B})$ is the matroid defined by the pair $(E, \mathcal{B}^*)$ where $\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}$. A **coloop** of M is an element which is a loop of M^*. Equivalently, it is an element which appears in every basis of M.

If $e \in E$ is not a coloop, we define the **deletion** as the matroid $M \setminus e := (E \setminus e, \{B \in \mathcal{B} : e \notin B\})$. If e is a coloop, then the bases of $M \setminus e$ are $\{B \setminus e : B \in \mathcal{B}\}$. Dually, if $e \in E$ is not a loop, we define the **contraction** as the matroid $M/e := (E \setminus e, \{B \setminus e : e \in B \in \mathcal{B}\})$. These operations can be extended to subsets $X \subseteq E$ and we write $M \setminus X$ and M/X, respectively. We also define the **restriction** of M to a subset $X \subseteq E$ as $M|_X := M \setminus (E \setminus X)$.

Note that $(M \setminus X)^* = M^*/X$. A **minor** of M is a matroid obtained from M by a sequence of deletion and contraction operations. The subclass of graphic matroids is closed under taking minors but not under taking duals.

Let us introduce the following relation among elements of E: $e_1, e_2 \in E$ are related if there exists a circuit of M containing both. This is an equivalence relation and the equivalence classes are called the **connected components** of M. Let us write $c(M)$ for the number of connected components. The matroid M is **connected** if $c(M) = 1$.

Let us recall a result of Tutte about connectedness of matroid minors.

Proposition 1.2.4 ([Oxl11, Thm. 4.3.1]). *Let $M = (E, \mathcal{B})$ be a connected matroid. Then for every $e \in E$, $M \setminus e$ is connected or M/e is connected.*

Let G be a graph with at least 3 vertices which has no loops nor isolated vertices. The graphic matroid M(G) is connected if and only if G is **biconnected**, that is, the removal of any vertex leaves a connected graph.

1.2.2 Matroid base configurations

To each matroid we associate a point configuration representing the collection of bases. For a fixed ground set E let us write $\mathbf{1}_X \in \{0,1\}^E$ for the characteristic vector of $X \subseteq E$, that is $(\mathbf{1}_X)_e = 1$ if and only if $e \in X$.

Definition 1.2.5. Let $M = (E, \mathcal{B})$ be a matroid. The **base configuration** of M is the point configuration

$$V_M \;:=\; \{\mathbf{1}_B : B \in \mathcal{B}\} \;\subset\; \mathbb{R}^E.$$

The **base polytope** of M is $P_M := \operatorname{conv}(V_M)$.

Note that

$$V_{M^*} \;=\; \mathbf{1}_E - V_M. \tag{1.1}$$

In particular, V_M and V_{M^*} are related by an affine transformation.

Observe that V_M is not a full-dimensional point configuration. Indeed, V_M is contained in the hyperplane $\sum_{e \in E} x_e = \operatorname{rank}(M)$.

Let M_1 and M_2 be matroids with disjoint ground sets E_1 and E_2. The collection

$$\mathcal{B} \;:=\; \{B_1 \cup B_2 : B_1 \in \mathcal{B}(M_1), B_2 \in \mathcal{B}(M_2)\}$$

is the set of bases of a matroid on $E_1 \cup E_2$, called the **direct sum** of M_1 and M_2 and denoted by $M_1 \oplus M_2$. The corresponding base configuration is exactly the Cartesian product

$$V_{M_1 \oplus M_2} \;=\; V_{M_1} \times V_{M_2}. \tag{1.2}$$

If $E_1, \dots, E_s \subseteq E$ are the connected components of M, then $M = \bigoplus_i M|_{E_i}$. Thus, showing that $\dim V_M = |E| - 1$ if M is connected proves the following.

Proposition 1.2.6. *The smallest affine subspace containing V_M is of dimension $|E| - c(M)$.*

Moreover, since the dual M^* of a matroid M has the same ground set and V_M and V_{M^*} are affinely equivalent, we get the following corollary.

Corollary 1.2.7. *A matroid M is connected if and only if M^* is connected.*

For a subset $X \subseteq E$ let us write $\ell_X(\mathbf{x}) = \sum_{e \in X} x_e$. For $A \subseteq E$ we then have $\ell_X(\mathbf{1}_A) = |A \cap X|$. Hence, $\operatorname{rank}_M(X) = \max_{\boldsymbol{v} \in V_M} \ell_X(\boldsymbol{v})$. For $X \subseteq E$ we define the supporting hyperplane

$$H_M(X) \;:=\; \{\boldsymbol{p} \in \mathbb{R}^E : \ell_X(\boldsymbol{p}) = \operatorname{rank}_M(X)\}.$$

The corresponding faces of V_M (or equivalently of P_M) are easy to describe.

Proposition 1.2.8 ([Edm70])**.** *For a matroid $M = (E, \mathcal{B})$ and a subset $X \subset E$, we have*

$$V_M \cap H_M(X) \;=\; V_{M|_X \oplus M/X} \;=\; V_{M|_X} \times V_{M/X}.$$

Let us illustrate this on our running example.

Example 1.2.9 (continued)**.** The graph given in Example 1.2.3 yields a connected matroid on 4 elements and hence a 3-dimensional base configuration. The corresponding base polytope is shown in Figure 1.9:

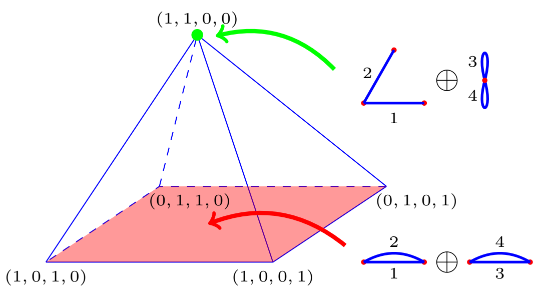

Figure 1.9: Faces of a matroid base polytope.

The 5 bases correspond to the vertices of P_M. We considered the subset $\{3,4\}$ whose associated face $\mathrm{M(G)}\,|_{\{3,4\}} \times \mathrm{M(G)}\,/\{3,4\}$ is the quadrilateral facet of the polytope, and the subset $\{1,2\}$ whose associated face $\mathrm{M(G)}\,|_{\{1,2\}} \times \mathrm{M(G)}\,/\{1,2\}$ is the vertex $(1,1,0,0)$.

In the remainder of the section we will recall the facet-defining hyperplanes of V_M which will also show that *all* faces of V_M correspond to direct sums of minors. The facial structure of V_M has been of interest originally in combinatorial optimization [Edm70] (see also [Sch03b, Ch. 40]) and later in geometric combinatorics and tropical geometry [AK06, FS05, Kim10].

Theorem 1.2.10. *Let $M = (E, \mathcal{B})$ be a connected matroid. For every facet $U \subset V_M$ there is a unique $\emptyset \neq S \subset E$ such that $U = V_M \cap H_M(S)$. Conversely, a subset $\emptyset \neq S \subset E$ gives rise to a facet if and only if*

(i) *S is a flat such that both $M|_S$ and M/S are connected;*

(ii) *$S = E \setminus e$ for some $e \in E$ such that both $M|_S$ and M/S are connected.*

In [FS05] the subsets S in (i) were called **flacets** and we stick to this name.

Example 1.2.11. The facets of the running example are four triangles and one square. The four triangles correspond to the two sets $\{1,2,4\}$, $\{1,2,3\}$ of cardinality $|E|-1$ and the two flacets $\{1\}$, $\{2\}$, while the square corresponds to the flacet $\{3,4\}$. We have already described in the previous example the square facet. In the picture we highlight two triangular facets, the first one (green) corresponding to the flacet $\{1\}$, the second one (red) to the set $\{1,2,4\}$.

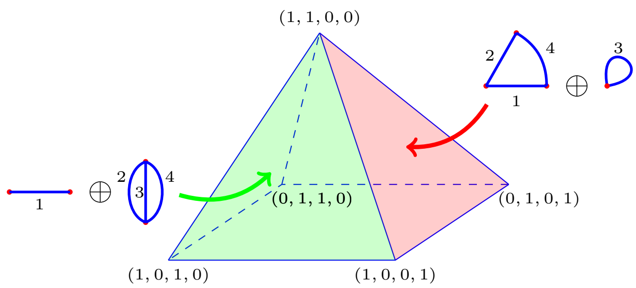

Figure 1.10: Facets of a matroid base polytope.

A useful and recurrent class of matroids is given by the **uniform matroids** $U_{n,k}$ for $0 \leq k \leq n$ with ground set $E = \{1, \dots, n\}$ and collection of bases $\mathcal{B} = \{B \subseteq E : |B| = k\}$. Uniform matroids are in general not graphic. The graphic ones are listed in Figure 1.11.

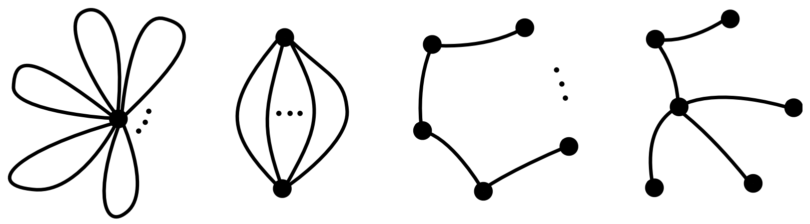

Figure 1.11: Graph representations of $U_{n,0}$, $U_{n,1}$, $U_{n,n-1}$, and $U_{n,n}$.

The **basis graph** of a matroid $M = (E, \mathcal{B})$ is the undirected graph with one vertex for each element in $\mathcal{B}$ such that two vertices B_1 and B_2 are adjacent if and only if $|B_1 \Delta B_2| = 2$, where Δ is the symmetric difference $(B_1 \setminus B_2) \cup (B_2 \setminus B_1)$. Equivalently, it is the 1-skeleton of the base polytope P_M as first characterized in [GGMS87].

Let us conclude this section with some results for base polytopes that will be used in Chapter 5 for the asymptotic enumeration of 2-level base polytopes. The first one appears as part of Exercise 4.9 in [Whi86, Ch. 4].

Proposition 1.2.12. *Let M_1 and M_2 be connected matroids. The basis graphs of M_1 and M_2 are isomorphic if and only if $M_1 \cong M_2$ or $M_1 \cong M_2^*$.*

Two polytopes are **congruent** if they are related by rigid motions and reflections. This implies that that they have the same edge lengths, volume, and 1-skeleton. The next corollary follows immediately and appears as an exercise in [BGW03, Ch. 1, Ex. 18].

Corollary 1.2.13. *Let M_1 and M_2 be connected matroids. The base polytopes P_{M_1} and P_{M_2} are congruent if and only if $M_1 \cong M_2$ or $M_1 \cong M_2^*$.*

It is known that “congruent” $\Rightarrow$ “combinatorially equivalent”. The converse is not true in general: for instance, there exist full-dimensional simplices with vertices in $\{0, 1\}^d$ and different volumes as shown in [Zie00]. Nevertheless, for the class of base polytopes, Proposition 1.2.12 yields the following corollary.

Corollary 1.2.14. *Let M_1 and M_2 be connected matroids. The polytope P_{M_1} is congruent to P_{M_2} if and only if P_{M_1} is combinatorially equivalent to P_{M_2}.*

Proof. We only need to prove one direction. If P_{M_1} is combinatorially equivalent to P_{M_2}, then the two polytopes have isomorphic face lattices and, in particular, isomorphic 1-skeletons. By Proposition 1.2.12, $M_1 \cong M_2$ or $M_1 \cong M_2^*$ and therefore P_{M_1} is congruent to P_{M_2} by Corollary 1.2.13. $\square$

1.2.3 Matroid operations

The uniform matroids turn out to be the building blocks for an interesting class of matroids we will be exploring in the next chapters. To construct matroids using these building blocks, we introduce three matroid operations that retain levelness as we will prove in Chapter 3.

Let $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$ be matroids such that $\{q\} = E_1 \cap E_2$. We call q a **base point**. If q is not a coloop of both matroids, then we define the **series connection** $\mathcal{S}(M_1, M_2)$ with respect to q as the matroid on the ground set $E_1 \cup E_2$ and with bases

$$\mathcal{B} \ = \ \{B_1 \cup B_2 : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2, B_1 \cap B_2 = \emptyset\}.$$

We also define the **parallel connection** with respect to q as the matroid $\mathcal{S}(M_1^*, M_2^*)^*$ provided that q is not a loop of both. Notice that $\mathcal{S}(M_1, M_2)$ contains both M_1 and M_2 as a minor.

The operations of series and parallel connection, introduced by Brylawski [Bry71], are inspired by the well-known series and parallel operations on graphs. The following example illustrates the construction in the graphic case.

Example 1.2.15. Let us consider the two graphic matroids $U_{3,2}$ and $M(K_4)$. Their series connection is the following graph:

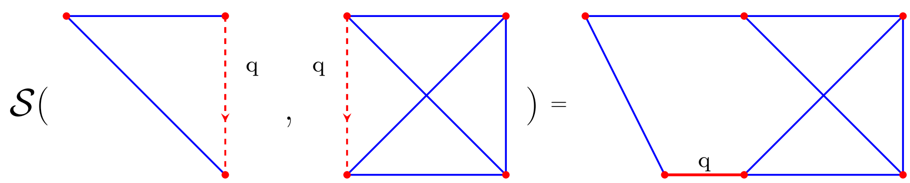

Figure 1.12: Series connection of graphic matroids.

An extensive treatment of these two operations is given in [Oxl11, Sect. 7.1].

The most important operation that we will need is derived from the series connection (or dually, by the parallel connection). Let $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$ be matroids with $E_1 \cap E_2 = \{q\}$. If neither M_1 nor M_2 have q as a loop or a coloop, then we define the **2-sum**

$$M_1 \oplus_2 M_2 \ := \ \mathcal{S}(M_1, M_2)/q.$$

This is the matroid on the ground set $E = (E_1 \cup E_2) \setminus q$ and with bases

$$\mathcal{B} \ := \ \{B_1 \cup B_2 \setminus q : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2, q \in B_1 \triangle B_2\},$$

where $B_1 \triangle B_2$ is the symmetric difference.

The 2-sum is an associative operation for matroids which defines, by analogy to the direct sum, the 3-connectedness: a connected matroid M is **3-connected** if and only if it cannot be written as a 2-sum of two matroids each with at least 3 elements and is isomorphic to a minor of M.

Example 1.2.16. Let us consider the 2-sum of a matroid $U_{3,2} \oplus_2 M(K_4)$: both matroids are graphic, therefore we can illustrate the operation for the corresponding graphs.

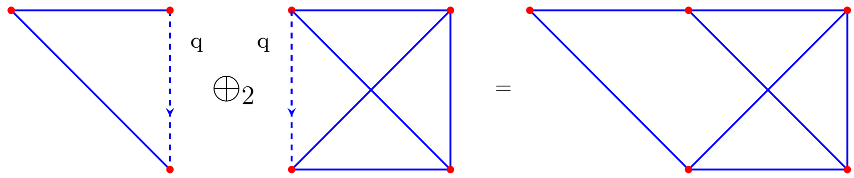

Figure 1.13: 2-sum of graphic matroids.

To perform the 2-sum, we select an element for each matroid, while in the picture it looks like we also need to orient the chosen element. This is the case only because we are drawing graph representations of graphic matroids; in fact the structure given by the vertices is forgotten when we look at the matroid from a purely combinatorial perspective. Whitney's 2-Isomorphism Theorem [Oxl11, Thm. 5.3.1] clarifies that the matroid structure does not depend on the orientation we decide for the chosen elements.

Chapter 2

Configurations and levelness

2.1 Configurations of 0/1-points

In Chapter 1 we introduced a first class of 0/1-configurations, that is, matroid base configurations. We saw that the combinatorics of a matroid M is strongly related to the geometry of the base configuration V_M and, in particular, encodes the face lattice of V_M. In the field of polyhedral combinatorics, 0/1-configurations are used to represent and analyze combinatorial objects.

In many instances, we can rephrase a combinatorial optimization problem as a maximization of a linear function over a 0/1-point configuration. We mention here the travelling salesman problem, the maximum stable set of a graph, and the maximum cut of a graph. For a comprehensive review on combinatorial optimization and a much richer collection of examples, we refer to [Sch03b]. Let us point out that whenever we write 0/1-configuration, we might as well say 0/1-polytope. Namely, any 0/1-configuration V is the set of vertices of the 0/1-polytope $\operatorname{conv}(V)$. Therefore we will use V and $\operatorname{conv}(V)$ as interchangeable objects in this context.

The class of 0/1-configurations coming from stable sets of graphs is highly inspiring for our work, because it connects to our considerations for levelness and Theta rank. We will say more about this topic in Section 2.2, while we briefly introduce now the maximum stable set problem.

A **stable set** of a graph G with vertices $[n]$ and edges E is a subset $X \subseteq [n]$ such that there is no edge $e \in E$ with endpoints in X. We associate to a stable set X its characteristic vector $\mathbf{1}_X \in \{0,1\}^n$. The **stable set configuration** of a graph G is

$$\mathrm{STAB}(G) := \{\mathbf{1}_X \ : \ X \text{ is a stable set of } G\}.$$

The maximum stable set problem consists of maximizing the linear function $\sum_{i\in[n]} x_i$ over $\mathrm{STAB}(G)$.

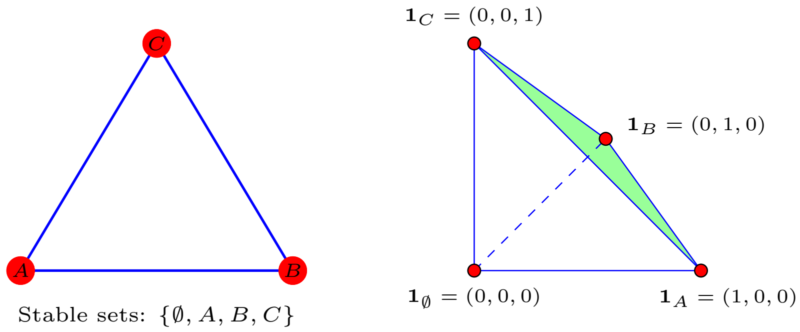

Figure 2.1: Stable set polytope of a triangle graph.

The variety of problems from combinatorial optimization which can be represented by 0/1-configurations is very broad. However, 0/1-configurations are highly complicated and many basic problems are open (see [Zie00]).

Theorem 2.1.1 ([Zie00, Sarangarajan-Ziegler])**.** *For $d \geq 6$ there is a collection of at least $2^{2^{d-2}}$ combinatorially non-equivalent d-dimensional 0/1-configurations in $\mathbb{R}^d$.*

In addition to the fact that the number of 0/1-configurations is doubly-exponential in the dimension, the following theorem shows that they can have many facets.

Theorem 2.1.2 ([FKR00],[KRGSZ97])**.** *For all large enough d, let V be the d-dimensional 0/1-configuration with the largest number of facets $\mathrm{f}(V)$. Then*

$$3.6^d < \mathrm{f}(V) < 30(d-2)!.$$

It is not a surprise that even basic questions, such as the classification of simplicial 0/1-polytopes, remain unanswered. On the other hand, simple 0/1-polytopes are fully understood.

Theorem 2.1.3 ([KW00, Thm. 1])**.** *A d-dimensional 0/1-polytope is simple if and only if it is equal to a Cartesian product of 0/1-simplices.*

Moreover, it is also shown that the following holds.

Proposition 2.1.4 ([KW00, Cor. 2])**.** *A combinatorial polar of a simple 0/1-polytope is combinatorially equivalent to a simplicial 0/1-polytope.*

The converse is not true: it is mentioned in [KW00] that there exists a 4-dimensional simplicial 0/1-polytope with 7 points and 13 facets whose combinatorial polar is not combinatorially equivalent to any 0/1-polytope. Thus, from the simple 0/1-polytopes we obtain by polarity some combinatorial types of simplicial 0/1-polytopes, but the list is incomplete.

Our contribution to the understanding of 0/1-configurations will be focused on the 2-level ones. After exploring some of their properties in Section 2.2, we will give in Section 2.3 the complete classification of combinatorial types for simple and simplicial 2-level 0/1-configurations.

2.2 An overview of 2-level configurations

This section is focused on levelness of configurations, with particular attention to the case of 2-levelness. The configurations in $\mathcal{V}_2^{\mathsf{Lev}}$ form, up to affine transformations, a subclass of 0/1-configurations and, because of Theorem 0.0.1, they have Theta rank 1 and, thus, are interesting from the point of view of optimization. The restrictive geometric condition of 2-levelness makes the configurations easier to study, but is not enough for a full understanding. In fact, $\mathcal{V}_2^{\mathsf{Lev}}$ is a broad and fascinating class of configurations, for which many questions, such as the classification of combinatorial types and the enumeration, are wide open.

2.2.1 Levelness and 2-levelness

A point configuration V is ***k*-level** if for every facet-defining hyperplane H there are k parallel hyperplanes $H = H_1, H_2, \ldots, H_k$ with

$$V \subseteq H_1 \cup H_2 \cup \cdots \cup H_k.$$

Equivalently, V is k-level if every facet-defining linear function $\ell(\mathbf{x})$ takes at most k distinct values on V. We say that a facet F is k-level if its facet-defining linear function $\ell(\mathbf{x})$ takes exactly k distinct values on V. The **levelness** $\mathsf{Lev}(V)$ of V is the smallest k such that V is k-level. A polytope is k-level if its vertices form a k-level configuration.

The levelness of the Cartesian product can be found explicitly.

Proposition 2.2.1. *Let $V_1 \subset \mathbb{R}^{d_1}$ and $V_2 \subset \mathbb{R}^{d_2}$ be point configurations. Then the levelness satisfies* $\mathsf{Lev}(V_1 \times V_2) = \max(\mathsf{Lev}(V_1), \mathsf{Lev}(V_2))$.

Proof. A linear function $\ell(\mathbf{x}, \mathbf{y})$ is facet-defining for $V_1 \times V_2$ if and only if $\ell(\mathbf{x}, \mathbf{0})$ is facet-defining for V_1 or $\ell(\mathbf{0}, \mathbf{y})$ is facet-defining for V_2. Thus a facet-defining function $\ell(\mathbf{x}, \mathbf{y})$ for $V_1 \times V_2$ takes at most $\mathsf{Lev}(V_1)$ values if it is of type $\ell(\mathbf{x}, \mathbf{0})$ or at most $\mathsf{Lev}(V_2)$ values if it is of type $\ell(\mathbf{0}, \mathbf{y})$. $\square$

The class $\mathcal{V}_2^{\mathsf{Lev}}$ has already been studied from several perspectives. For instance, we report three useful results from [GPT10, Sect. 4], where 2-level configurations appear under the name of exact sets.

Proposition 2.2.2. *Every 2-level configuration in $\mathbb{R}^d$ is affinely equivalent to a 0/1-configuration in $\{0,1\}^d$.*

Proposition 2.2.3. *Every 2-level configuration V is the set of vertices of* $\operatorname{conv}(V)$.

Proposition 2.2.4. *Every face of a 2-level configuration is 2-level.*

Proposition 2.2.3 explains why 2-level configurations can be equivalently studied as 2-level polytopes. In general, it is not true that a k-level point configuration V is equivalent to its convex hull $\operatorname{conv}(V)$, since it could happen that $\mathsf{Lev}(V) > \mathsf{Lev}(\operatorname{conv}(V))$. Figure 2.2 shows a 3-level configuration V such that the levelness of $\operatorname{conv}(V)$ is 2.

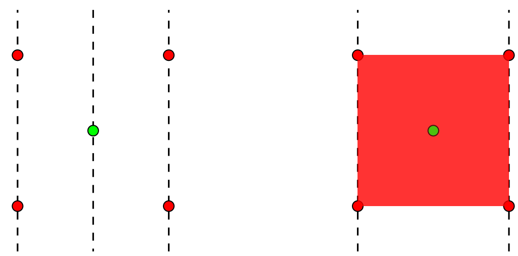

Figure 2.2: 3-level configuration, 2-level polytope.

We will present and prove Proposition 2.2.4 in a more general fashion in Section 3.1.

For a more inclusive overview about 2-level configurations we should mention [Sta80], where compressed polytopes are introduced. A full-dimensional polytope P with vertices in $\mathbb{Z}^d$ is **compressed** if every pulling triangulation of P using the points in $P \cap \mathbb{Z}^d$ is **unimodular**, that is every simplex in the triangulation has volume $1/d!$. A point configuration V is 2-level if and only if $\operatorname{conv}(V)$ is affinely equivalent to a compressed polytope. For more details about compressed polytopes we refer to [OH01], [Sul06] and [DLRS10, Ch. 9].

Affine transformations map parallel hyperplanes to parallel hyperplanes, which means that the levelness of a configuration is invariant under affine transformations.

One challenging problem is to classify 2-level configurations up to combinatorial equivalence. Since affinely equivalent configurations are also combinatorially equivalent, Proposition 2.2.2 allows us to restrict our attention to 0/1-configurations, and brute force algorithms yield results for small dimensions $d \leq 6$. Indeed the count of 2-level configurations is

dimension	2	3	4	5	6
# 2-level config.	2	5	19	106	1150

and is available at [Fis] together with several other computational results about 2-level polytopes. However, no general classification (or enumeration) is available so far.

In Chapter 3 we present and analyze a new subclass of $\mathcal{V}_2^{\mathsf{Lev}}$ which arise from matroids and is endowed with simple constructive properties, suitable for counting purposes. As a consequence, we will obtain in Chapter 5 an exponential lower bound on the number of 2-level configurations.

Section 2.3 handles the classification of combinatorial types for the case of simple and simplicial 2-level configurations and motivates the need for the following properties of the slack matrices of 2-level polytopes.

Proposition 2.2.5. *A polytope P is 2-level if and only if there exists $S \in \mathbb{S}(P)$ such that S is a 0/1-matrix.*

Proof. Let $S' \in \mathbb{S}(P)$. Since P is 2-level, every column of S' has only two different entries (0 and k) which we can rescale to form a 0/1-column. Thus we get a 0/1-matrix in $\mathbb{S}(P)$. □

A point configuration (polytope) is **combinatorially 2-level** if there exists a 2-level configuration (polytope) with the same face lattice.

Proposition 2.2.6. *A polytope P is combinatorially 2-level if and only if there exists a polytope Q such that for $S \in \mathbb{S}(P)$, $\mathrm{supp}(S) \in \mathbb{S}(Q)$.*

Proof. Consider $S \in \mathbb{S}(P)$. The support $\mathrm{supp}(S)$ determines the combinatorics of P, that is, its face lattice. If $\mathrm{supp}(S) \in \mathbb{S}(Q)$, then Q is combinatorially equivalent to P. Moreover, Q is 2-level by Proposition 2.2.5. □

2.2.2 Polytopal constructions and levelness

We briefly discuss here the relation between polytopal constructions and levelness. Proposition 2.2.1 shows that the Cartesian product of polytopes preserves 2-levelness.

We consider a second construction, that is, the **pyramid** over a polytope: for a d-polytope $P \subset \mathbb{R}^{d+1}$ and a point $a \notin \operatorname{aff}(P)$ we define

$$\operatorname{pyr}(P) := \operatorname{conv}(P \cup a).$$

The point a is called **apex** of the pyramid.

Proposition 2.2.7. *The pyramid* $\operatorname{pyr}(P)$ *over a k-level d-polytope $P \subset \mathbb{R}^{d+1}$ is a k-level polytope.*

Proof. The facets of $\operatorname{pyr}(P)$ are either P or $\operatorname{conv}(F \cup a)$, where F is a facet of P and a is the apex of the pyramid. Clearly P is a 2-level facet of $\operatorname{pyr}(P)$. If the facet F of P is k-level, there exists a sequence of at most k parallel hyperplanes $H_1, H_2, \ldots, H_k$ containing all the vertices of P; since P is not full-dimensional, we can tilt the sequence of hyperplanes in such a way that H_1 also contains a and thus the facet $(F \cup a)$ is k-level for $\operatorname{pyr}(P)$. □

The previous proposition shows that the levelness of a polytope is preserved by the pyramid construction and, in particular, 2-levelness is preserved.

Among other constructions, the join preserve levelness. On the other hand, we have already encountered a construction which fails to preserve levelness, namely the direct sum. The counterexample is provided by the polytope in Example 1.1.6: P is obtained as the direct sum of Δ_2 and Δ_1 and is not 2-level. In fact Proposition 4.4 in [GRT13] proves the even stronger fact that P is not combinatorially 2-level. The direct sum of simplices will be investigated in Section 2.3 as the key to classify simplicial 2-level polytopes.

2.2.3 A catalog of 2-level polytopes

This subsection surveys known classes of 2-level polytopes and their properties.

I) Hypersimplices

The easiest example of a 2-level polytope is certainly the simplex; more interesting is the class of *hypersimplices* which first appeared in [GGMS87].

For any positive integers n and k, $2 \leq k \leq n-2$, we define the **hypersimplex** as the convex set

$$\Delta_{n,k} := \operatorname{conv}\left(\left\{\boldsymbol{p} \in \{0,1\}^n \;:\; \sum_{i=1}^{n} p_i = k\right\}\right) = \left\{\boldsymbol{p} \in [0,1]^n \;:\; \sum_{i=1}^{n} p_i = k\right\}.$$

The hypersimplex $\Delta_{n,k}$ is not full-dimensional and the second formulation underlines that it is obtained as a section of the n-cube by the hyperplane $\sum x_i = k$. The definition provides a vertex representation of $\Delta_{n,k}$ and it is not hard to check that all facet-defining hyperplanes are of the form $x_i = 0$ and $1 - x_i = 0$ (see [Grü03, Ex. 4.8.16]).

Proposition 2.2.8. *Hypersimplices are 2-level polytopes.*

Proof. Every hypersimplex $\Delta_{n,k}$ has vertices in $\{0,1\}^n$, thus every facet-defining linear function of type $\ell(\mathbf{x}) = x_i$ or $\ell(\mathbf{x}) = 1 - x_i$ can only attain the values 0 or 1. □

Up to permutations of the columns, the slack matrix associated to the facet representation of $\Delta_{n,k}$ is a matrix of the form $[A|\mathbf{1}-A] \in \{0,1\}^{\binom{n}{k}\times 2n}$, where the rows of A are all possible vectors in $\{0,1\}^n$ with exactly k ones and $\mathbf{1}$ is the all-ones matrix. Because of Proposition 2.2.5 we already expected to find a 0/1-matrix in $\mathbb{S}(\Delta_{n,k})$.

One last observation about hypersimplices is that they represent the matroid base polytopes for the non-graphic uniform matroids and they will play a main role throughout the thesis.

II) Hanner polytopes

Let us now mention a class of 2-level polytopes that is entirely constructed using Cartesian products and direct sums, namely *Hanner polytopes*. We report here the recursive definition and refer to [Han56] for more details. A polytope P is a **Hanner polytope** if it satisfies one of the following:

(i) P is a centrally symmetric line segment;

(ii) P is the Cartesian product of two Hanner polytopes;

(iii) P is the direct sum of two Hanner polytopes.

A polytope P with vertices V is **centrally symmetric** if $-\boldsymbol{v} \in V$ for every $\boldsymbol{v} \in V$. Notice that all Hanner polytopes are centrally symmetric, since both the Cartesian product and the direct sum preserve central symmetry.

We have shown that the direct sum does not always preserve 2-levelness. Nevertheless this is true if we apply this construction to two Hanner polytopes (the word Hanner configuration refers to the vertices of a Hanner polytope).

To show this fact, we observe that, by central symmetry, every facet V' of a Hanner configuration V has a corresponding parallel facet $V'' := \{-\boldsymbol{v} : \boldsymbol{v} \in V'\}$. Given two Hanner 2-level configurations V_1 and V_2, every facet V' of $V_1 \oplus V_2$ is of the form $(V_1' \times \{\mathbf{0}\}) \cup (\{\mathbf{0}\} \times V_2')$ where V_i' is a facet of V_i, $i = 1, 2$. Let V_i'' be the opposite facet of V_i': $V'' = (V_1'' \times \{\mathbf{0}\}) \cup (\{\mathbf{0}\} \times V_2'')$ is a facet of $V_1 \oplus V_2$ and, more precisely, V'' is opposite to V'. Moreover, since facet-defining hyperplanes of opposite facets are parallel and

$$\begin{aligned} V' \cup V'' = ((V_1' \times \{\mathbf{0}\}) \cup (\{\mathbf{0}\} \times V_2')) \cup ((V_1'' \times \{\mathbf{0}\}) \cup (\{\mathbf{0}\} \times V_2'')) = \\ = (V_1 \times \{\mathbf{0}\}) \cup (\{\mathbf{0}\} \times V_2) = V, \end{aligned}$$

we have that $V_1 \oplus V_2$ is 2-level and therefore, by inductive reasoning, we conclude that all Hanner polytopes are 2-level.

III) Stable set polytopes of perfect graphs

A beautiful class of 2-level polytopes arises by considering a subclass of the 2-level stable set polytopes for graphs. This subclass has been studied by Lovász in [GLS93, Ch. 9] and has a surprising combinatorial characterization based on the properties of the graph.

A graph is **perfect** if and only if the clique number coincides with the chromatic number for all its induced subgraphs. The class of perfect graphs contains many interesting graphs such as bipartite graphs, chordal graphs, line graphs of bipartite graphs, and comparability graphs. For more details we refer to [BM08, Sect. 14.4].

Theorem 2.2.9 ([GPT10, Thm. 3.1])**.** *The stable set polytope of a graph G is 2-level if and only if G is a perfect graph.*

The beauty of Lovász's theorem lies in the ability to link a geometric property such as levelness to a purely combinatorial property of graphs. This remarkable characterization encourages the combinatorial investigation of other classes of 2-level polytopes: in Chapter 3 we will unveil an analogous link between the geometry of matroid base polytopes and the combinatorics of matroids. This will yield a new class of 2-level polytopes.

IV) Hansen polytopes

The class of **Hansen polytopes**, that is twisted prisms over stable set polytopes of perfect graphs, first appeared in [Han77] and has been analyzed in the context of Kalai's 3d conjecture in [SWZ09]. Every polytope in this class is centrally symmetric and 2-level. Notice that the twisted prism over a 2-level polytope is not necessarily a 2-level polytope as we can check for the hypersimplex $\Delta_{6,2}$. It is not known which 2-level polytopes retain 2-levelness after the twisted prism construction.

V) Order polytopes

The last class of 2-level polytopes, namely order polytopes, was first introduced in [Sta86]. An **order polytope** $\mathcal{O}_A$ is defined from a partially ordered set $A = \{a_1, \dots, a_n\}$ as the convex hull of all the 0/1-points $\boldsymbol{v} \in \{0,1\}^n$ satisfying that if $v_i = 1$ and $a_j \leq a_i$ in A, then $v_j = 1$.

2.3 Simple and simplicial

We classify first the combinatorial types of simple 2-level polytopes. This is a straightforward consequence of Proposition 2.2.2 and Theorem 2.1.3.

Proposition 2.3.1. *A 2-level polytope is simple if and only if it is affinely equivalent to a Cartesian product of 0/1-simplices.*

The goal of this section is to understand 2-level simplicial polytopes by means of polarity. The main obstacle we have to overcome is that the combinatorial polar of a 2-level polytope is not necessarily combinatorially 2-level.

We have already encountered an instance of this fact in Example 1.1.6. Indeed the 2-level polytope $\Delta_2 \times \Delta_1$ is affinely equivalent to a 2-level 0/1-polytope but its polar is not combinatorially 2-level. As a first step towards the understanding of simplicial 2-level polytopes, we look at the polar of Cartesian products of simplices and determine which ones are combinatorially 2-level. To answer this question, we use to the Gale duality introduced in Chapter 1.

Proposition 2.3.2. *Let $V = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\} \subset \mathbb{R}^d$ be a spanning point configuration and $G = [\mathbf{g}_1, \dots, \mathbf{g}_n] \in \mathbb{R}^{(n-d-1)\times n}$ its Gale dual. The configuration V is 2-level if and only if every facet $V' = \{\boldsymbol{v}_i : i \in I \subset [n]\}$ of V is such that*

$$\sum_{i \notin I} \mathbf{g}_i = \mathbf{0}.$$

Proof. If V is 2-level, for any facet V' of V with supporting hyperplane H_ℓ, the linear function $\ell(\mathbf{x})$ takes two values on V. The evaluation of $\ell(\mathbf{x})$ on V

is a vector in $\mathbb{R}^n$ with ith component equal to

$$\ell(\boldsymbol{v}_i) = \begin{cases} 0\ , & \text{if } i \in I \\ a\ , & \text{if } i \notin I. \end{cases}$$

Since affine evaluations on V correspond to linear dependences of G (Theorem 1.1.8), we conclude that

$$\mathbf{0} = \sum_{i=1}^{n} \ell(\boldsymbol{v}_i)\mathbf{g}_i = \sum_{i \notin I} a\mathbf{g}_i \quad \Longrightarrow \quad \mathbf{0} = \sum_{i \notin I} \mathbf{g}_i.$$

Conversely, suppose that the condition holds for all facets. The linear dependence in G corresponds to an affine evaluation on V that takes only values 0 and 1, hence V is 2-level. □

Theorem 2.3.3. *The direct sum $P = \Delta_{k_1} \oplus \Delta_{k_2} \oplus \ldots \oplus \Delta_{k_l}$, $k_i > 0$ for $i = 1, \ldots, l$, is a combinatorially 2-level polytope if and only if*

$$k_1 = k_2 = \ldots = k_l.$$

Proof. If P is combinatorially 2-level, there exists a 2-level polytope P' with the same face lattice. Let $V = \{\boldsymbol{v}_{1,1}, \ldots, \boldsymbol{v}_{1,k_1+1}, \ldots, \boldsymbol{v}_{l,1}, \ldots, \boldsymbol{v}_{l,k_l+1}\}$ be the configuration of vertices of P' where the first index i of a point indicates that it comes from the vertices of Δ_{k_i}. The Gale dual of V is of the form

$$G = [\underbrace{\mathbf{g}_{1,1}, \ldots, \mathbf{g}_{1,k_1+1}}_{\Delta_{k_1}}, \ldots, \underbrace{\mathbf{g}_{l,1} \ldots \mathbf{g}_{l,k_l+1}}_{\Delta_{k_l}}] \in \mathbb{R}^{(l-1)\times(\sum k_i + l)}.$$

Let us consider the facet obtained by excluding the first vertex of each simplex. Since P' is 2-level, Proposition 2.3.2 yields the equation

$$\sum_{i=1}^{l} \mathbf{g}_{i,1} = \mathbf{0}.$$

By considering the facet obtained by excluding the first vertex of each simplex and a vertex j of Δ_{k_l}, $j = 2, \ldots, k_l + 1$, we get

$$\mathbf{0} = \sum_{i=1}^{l-1} \mathbf{g}_{i,1} + \mathbf{g}_{l,j} = \sum_{i=1}^{l} \mathbf{g}_{i,1} \ \Rightarrow\ \mathbf{g}_{l,1} = \mathbf{g}_{l,j} \ \text{ for all } j = 2, \ldots, k_{l+1}.$$

By the same argument, we show that all vectors associated to same simplex are equal. It follows that

$$G = [\underbrace{\mathbf{g}_1, \ldots, \mathbf{g}_1}_{(k_1+1)\text{-times}}, \ldots, \underbrace{\mathbf{g}_l, \ldots \mathbf{g}_l}_{(k_l+1)\text{-times}}],$$

where $\mathbf{g}_i := \mathbf{g}_{i,1} = \ldots = \mathbf{g}_{i,k_i+1}$. We saw that

$$\sum_{i=1}^{l} \mathbf{g}_i = \mathbf{0}.$$

Furthermore, by Proposition 1.1.9 we obtain

$$\mathbf{0} = \sum_{i=1}^{l} (k_i + 1)\mathbf{g}_i.$$

Without loss of generality we may assume $k_1 \leq k_2 \leq \ldots \leq k_l$. Suppose that at least one inequality is strict and consider the equalities

$$\mathbf{0} = \sum_{i=1}^{l} (k_i + 1)\mathbf{g}_i - (k_1 + 1) \sum_{i=1}^{l} \mathbf{g}_i = \sum_{i=2}^{l} (k_i - k_1)\mathbf{g}_i.$$

This linear dependence among at most $l-1$ vectors of G yields a contradiction because it corresponds to a proper face strictly containing all the vertices of a facet. We conclude that $k_1 = k_2 = \ldots = k_l$.

Conversely, suppose that all simplices have dimension k. We consider a set of vectors $\{\mathbf{g}_1, \ldots, \mathbf{g}_l\} \subset \mathbb{R}^{l-1}$ such that any $l-1$ vectors are independent and

$$\mathbf{0} = \sum_{i=1}^{l} \mathbf{g}_i.$$

For instance, the set of vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_{l-1}, -\mathbb{1}\}$ satisfies the requirements. From $\{\mathbf{g}_1, \ldots, \mathbf{g}_l\}$ we create the matrix

$$G = [\underbrace{\mathbf{g}_1, \ldots, \mathbf{g}_1}_{(k+1)\text{-times}}, \ldots, \underbrace{\mathbf{g}_l, \ldots \mathbf{g}_l}_{(k+1)\text{-times}}] \in \mathbb{R}^{(l-1)\times(kl+l)}.$$

The matrix G is the Gale dual of a point configuration which is 2-level and combinatorially equivalent to $\mathcal{V}(\Delta_{k_1} \oplus \Delta_{k_2} \oplus \ldots \oplus \Delta_{k_l})$. □

The last theorem describes a family of simplicial combinatorially 2-level polytopes that we can obtain by polarity from the simple 2-level polytopes. We investigate whether there can be any other combinatorial types of simplicial 2-level polytopes.

Our considerations concern slack matrices of polytopes. In particular, let us recall a proposition that follows from [GGK+13, Thm. 6].

Proposition 2.3.4. *If a matrix $S \in \mathbb{R}^{n\times m}_{\geq 0}$ with $\operatorname{rank}(S) \geq 2$ is a slack matrix of a polytope, then the vector $\mathbb{1} \in \mathbb{R}^n$ is in the column span of S. Moreover, if $\mathbb{1} \in \mathbb{R}^m$ is in the row span of S, then S^T is also a slack matrix of some polytope.*

Example 2.3.5. In Example 1.1.3 we found a slack matrix S of the polytope $P := \Delta_2 \times \Delta_1 \subset \mathbb{R}^3$. We get another possible $S' \in \mathbb{S}(P)$ by rescaling the columns of S.

$$S' = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

This is a 0/1-matrix and thus P is 2-level. Notice that S^T is the slack matrix of $\Delta_2 \oplus \Delta_1$, while $(S')^T$ is not a slack matrix since it does not contain $\mathbb{1}$ in the column span.

Nevertheless it is true that for every polytope P, there exists $S \in \mathbb{S}(P)$ such that S^T is a slack matrix of some polytope. Furthermore, as one could expect, there is a connection between the transpose of a slack matrix and polytope polarity.

Proposition 2.3.6 ([GGK$^+$13, Prop. 18])**.** *Suppose $S \in \mathbb{R}^{n\times m}_{\geq 0}$ such that S and S^T are both slack matrices. Then there exists a polytope P, with $\mathbf{0} \in \operatorname{int}(P)$, such that $S \in \mathbb{S}(P)$ and $S^T \in \mathbb{S}(P^\circ)$.*

The properties of the slack matrices of a polytope allow us to provide an alternative proof for one direction of Theorem 2.3.3.

Proof. (Alternative proof for $\Rightarrow$ of Thm. 2.3.3) Let $S \in \mathbb{R}^{n\times m}$ be a slack matrix of $P = \Delta_{k_1} \oplus \Delta_{k_2} \oplus \ldots \oplus \Delta_{k_l}$ and $S' = \operatorname{supp}(S)$. By Proposition 2.2.6 if P is combinatorially 2-level, then there exists a 2-level polytope Q such that $S' \in \mathbb{S}(Q)$. In particular, S' is an admissible slack matrix and by Proposition 2.3.4 the vector $\mathbb{1}$ is in the column span of S'. Equivalently, there exists a column vector $\mathbf{a} \in \mathbb{R}^m$ such that $S'\mathbf{a} = \mathbb{1}$.

The structure of the facets of P implies that the matrix S' has exactly one entry 1 for each set of vertices associated to the same simplex Δ_{k_i}. Therefore the left kernel of S' contains all elements of the form

$$\mathbf{z}^{ij} := (0, \ldots, 0, \underbrace{1, \ldots, 1}_{\mathcal{V}(\Delta_{k_i})}, 0, \ldots, 0, \ldots, 0, \underbrace{-1, \ldots, -1}_{\mathcal{V}(\Delta_{k_j})}, 0, \ldots, 0),$$

that is, for any $i, j \in [l]$, $i < j$, $\mathbf{z}^{ij} S' = \mathbf{0}$. We obtain

$$0 = \mathbf{z}^{ij} S' \mathbf{a} = \mathbf{z}^{ij} \, \mathbb{1} = k_i + 1 - (k_j + 1) = k_i - k_j.$$

Since $k_i = k_j$ for all $i, j \in [l]$, we conclude that $k_1 = k_2 = \ldots = k_l$. □

Theorem 2.3.7. *Let P be a simplicial polytope and P^{Δ} a combinatorial polar of P. If P is 2-level, then P^{Δ} is combinatorially 2-level.*

Proof. Consider $S \in \mathbb{S}(P)$. Since P is 2-level, $\text{supp}(S) \in \mathbb{S}(P)$. Moreover, P is simplicial and therefore all facets contain the same number of vertices. It follows that the columns of $\text{supp}(S)$ have constant sum, thus the vector $\mathbb{1}$ is in the row span of $\text{supp}(S)$. This implies that $\text{supp}(S)^T$ is a slack matrix of some polytope Q, which is a 2-level polytope and is combinatorially equivalent to P^{Δ}. □

We conclude the section with a corollary, that provides a complete answer to the classification of combinatorial types for simplicial 2-level polytopes.

Corollary 2.3.8. *A 2-level polytope is simplicial if and only if it is combinatorially equivalent to a direct sum $\Delta_k \oplus \ldots \oplus \Delta_k$ of simplices with identical dimension k.*

Proof. If P is a simplicial 2-level polytope, Theorem 2.3.7 shows that any combinatorial polar P^{Δ} of P is combinatorially equivalent to a 2-level simple polytope. Proposition 2.3.1 states that every 2-level simple polytope is affinely equivalent to a product of simplices $\Delta_{k_1} \times \Delta_{k_2} \times \ldots \times \Delta_{k_l}$, thus, P must be combinatorially equivalent to $\Delta_{k_1} \oplus \Delta_{k_2} \oplus \ldots \oplus \Delta_{k_l}$. Since P is 2-level, by Theorem 2.3.3 we conclude that $k_1 = \cdots = k_l$. □

Chapter 3

Matroid base configurations

This chapter is entirely dedicated to matroids and graphs. We characterize the class of 2-level matroids and generalize the characterization to k-level graphs. In addition, we find upper bounds on the size of excluded minors for k-level matroids. This shows that there are finitely many of them.

We start with a section about face-hereditary properties of point configurations. This motivates our interest in matroid base configurations, as faces of base configurations are base configurations of matroid minors.

3.1 Face-hereditary properties

The definitions of levelness and Theta rank make only reference to the affine hull of the configuration V and thus neither depend on the embedding nor on a choice of coordinates. To have it on record we note the following basic property.

Proposition 3.1.1. *The levelness and the Theta rank of a point configuration are invariant under affine transformations.*

That this does not hold for (admissible) projective transformations is clear for the levelness and for the Theta rank follows from Theorem 0.0.1. Moreover, a proposition analogous to Proposition 2.2.1 holds for Theta rank.

Proposition 3.1.2. *Let* $V_1 \subset \mathbb{R}^{d_1}$ *and* $V_2 \subset \mathbb{R}^{d_2}$ *be point configurations. Then the Theta rank satisfies* $\mathsf{Th}(V_1 \times V_2) = \max(\mathsf{Th}(V_1), \mathsf{Th}(V_2))$.

Proof. A linear function $\ell(\mathbf{x}, \mathbf{y})$ is facet-defining for $V_1 \times V_2$ if and only if $\ell(\mathbf{x}, \mathbf{0})$ is facet-defining for V_1 or $\ell(\mathbf{0}, \mathbf{y})$ is facet-defining for V_2. Thus any representation (1) lifts to $\mathbb{R}[\mathbf{x}, \mathbf{y}]$. □

The Theta rank as well as the levelness of a point configuration are not monotone with respect to taking subconfigurations as can be seen by removing a single point from $\{0,1\}^d$. However, it turns out that monotonicity holds for subconfigurations induced by supporting hyperplanes. Let us call a collection $\mathcal{P}$ of point configurations **face-hereditary** if it is closed under taking faces. That is, $V \cap H \in \mathcal{P}$ for any $V \in \mathcal{P}$ and supporting hyperplane H for V.

As promised, the following lemma generalizes Proposition 2.2.4.

Lemma 3.1.3. *The classes $\mathcal{V}_k^{\mathsf{Th}}$ and $\mathcal{V}_k^{\mathsf{Lev}}$ are face-hereditary.*

Proof. Let $V \subset \mathbb{R}^d$ be a full-dimensional point configuration and $H = \{\boldsymbol{p} \in \mathbb{R}^d : g(\boldsymbol{p}) = 0\}$ a supporting hyperplane such that the affine hull of $V' := V \cap H$ has codimension 1. Let $\ell(\mathbf{x})$ be facet-defining for V'. Observe that $\ell(\mathbf{x})$ and $\ell_\delta(\mathbf{x}) := \ell(\mathbf{x}) + \delta g(\mathbf{x})$ give the same linear function on V' for all δ. For

$$\delta \;=\; \max\{\tfrac{-\ell(\boldsymbol{v})}{g(\boldsymbol{v})} : \boldsymbol{v} \in V \setminus V'\}$$

$\ell_\delta(\mathbf{x})$ is non-negative on V. Hence any representation (1) of ℓ_δ over V yields a representation for ℓ over V'. Moreover, the levelness of ℓ_δ gives an upper bound on the levelness of ℓ. □

It is interesting to note that these properties are not hereditary with respect to arbitrary hyperplanes. Indeed, consider the point configuration

$$V \;=\; (\{0,1\}^n \times \{-1,0,1\}) \setminus \{\mathbf{0}\}$$

It can be easily seen that $\mathsf{Th}(V) = \mathsf{Lev}(V) - 1 = 2$. The hyperplane $H = \{\boldsymbol{p} \in \mathbb{R}^{n+1} : p_{n+1} = 0\}$ is not supporting and $V' = V \cap H = \{0,1\}^n \setminus \{\mathbf{0}\}$. The linear function $\ell(x) = x_1 + \cdots + x_n - 1$ is facet-defining for V' with n levels. As for the Theta rank, any representation (1) yields a polynomial $f(\mathbf{x}) = \ell(\mathbf{x}) - \sum_i h_i^2(\mathbf{x})$ of degree $2k$ that vanishes on V' and $f(\mathbf{0}) = -1 - \sum_i h_i^2(\mathbf{0}) < 0$. For $n > 4$, the following proposition assures that $\mathsf{Th}(V') \geq 3$.

Proposition 3.1.4. *Let $V' = \{0,1\}^n \setminus \{\mathbf{0}\}$ and $f(\mathbf{x})$ a polynomial vanishing on V' and $f(\mathbf{0}) \neq 0$. Then $\deg f \geq n$.*

Proof. For a monomial $\mathbf{x}^\alpha$, let $\tau = \{i : \alpha_i > 0\}$ be its support. Over the set of 0/1-points it follows that $\mathbf{x}^\alpha$ and $\mathbf{x}^\tau := \prod_{i\in\tau} x_i$ represent the same function. Hence, we can assume that f is of the form $f(\mathbf{x}) = \sum_{\tau\subseteq[n]} c_\tau \mathbf{x}^\tau$ for some $c_\tau \in \mathbb{R}$, $\tau \subseteq [n]$. Moreover $c_\emptyset = f(\mathbf{0}) \neq 0$ and without loss of generality we can assume $c_\emptyset = 1$. Any point $\boldsymbol{v} \in V'$ is of the form $\boldsymbol{v} = \mathbf{1}_\sigma$ for some $\emptyset \neq \sigma \subseteq [n]$ and we calculate

$$0 \;=\; f(\boldsymbol{v}) \;=\; \sum_{\emptyset\subseteq\tau\subseteq\sigma} c_\tau .$$

It follows that c_τ satisfies the defining conditions of the *Möbius function* of the Boolean lattice and hence equals $c_\tau = (-1)^{|\tau|}$ for all $\tau \subseteq [n]$. In particular $c_{[n]} \neq 0$, which finishes the proof. □

3.2 2-level matroids

We define the following families of matroids:

$$\begin{aligned} \mathcal{M}_k^{\mathsf{Lev}} &:= \{M \text{ matroid} : \mathsf{Lev}(V_M) \leq k\}, \quad \text{and} \\ \mathcal{M}_k^{\mathsf{Th}} &:= \{M \text{ matroid} : \mathsf{Th}(V_M) \leq k\}. \end{aligned}$$

We will say that a matroid M is of Theta rank k or k-level if the corresponding base configuration V_M is. Now combining Proposition 1.2.8 with Lemma 3.1.3 proves the main theorem of this section.

Theorem 3.2.1. *The classes $\mathcal{M}_k^{\mathsf{Th}}$ and $\mathcal{M}_k^{\mathsf{Lev}}$ are closed under taking minors.*

Proof. Every minor N of M can be written in the form $M/Y\backslash X$. By Proposition 1.2.8, the configuration $V_{M/Y}$ is a face of $V_M \cap H_M(Y)$. Moreover, we can apply Proposition 1.2.8 again to show that $V_{M/Y\backslash X}$ is a face of $V_{M/Y}$. Thus there is a supporting hyperplane H such that $V_M \cap H$ is affinely isomorphic to V_N. Lemma 3.1.3 assures us that $\mathsf{Th}(V_N) \leq \mathsf{Th}(V_M)$. □

Let us analogously define the classes $\mathcal{G}_k^{\mathsf{Th}}$ and $\mathcal{G}_k^{\mathsf{Lev}}$ of graphic matroids of Theta rank and levelness bounded by k. These are also closed under taking minors and the Robertson-Seymour theorem ([RS04]) asserts that there is a finite list of excluded minors characterizing each class.

In our study of the Theta rank and the levelness of base configurations, the following asserts that we will only need to consider flacets. For brevity, a k-level flacet refers to a flacet whose corresponding facet is k-level.

Proposition 3.2.2. *Let M be a connected matroid and $S = E(M) \setminus e$. Then $\ell_S(\mathbf{x})$ takes 2 values on V_M and hence is 1-sos.*

Proof. Let r be the rank of M. Restricted to the affine hull of V_M, we have that $\ell_S(\mathbf{x})$ and $r - \mathbf{x}_e$ induce the same linear function. As V_M is a 0/1-configuration, it follows that $\ell_S(\mathbf{x})$ takes the 2 values r and $r-1$ on V_M. □

Proposition 3.2.3. *Uniform matroids are in $\mathcal{M}_2^{\mathsf{Lev}}$ and hence in $\mathcal{M}_1^{\mathsf{Th}}$.*

Proof. The base polytope of $U_{n,k}$ is given by

$$P_{U_{n,k}} \;=\; \mathrm{conv}\{\mathbf{1}_B : B \subseteq E, |E| = k\} \;= \Delta_{n,k}.$$

Proposition 2.2.8 concludes the proof. $\square$

In this section we investigate the excluded minors for the matroids in $\mathcal{M}_2^{\mathsf{Lev}}$ (shortly, 2-level matroids) and, by Theorem 0.0.1, the matroids of Theta rank 1. In this case we can give the complete and in particular finite list of excluded minors. We start by showing that matroids with few elements and of small rank cannot be excluded minors for $\mathcal{M}_2^{\mathsf{Lev}}$.

Proposition 3.2.4. *Let $M = (E, \mathcal{B})$ be a matroid. If $\mathrm{rank}(M) \leq 2$ or $|E| \leq 5$, then M is 2-level.*

Proof. The case $\mathrm{rank}(M) = 1$ is trivial since there is no proper flacet. On the other hand, if $\mathrm{rank}(M) = 2$ the proper flacets are necessarily flacets of rank 1. The linear function $\ell_F(\mathbf{x})$ for any such flacet F only takes values in $\{0, 1\}$ and thus is 2-level. By (1.1) and Proposition 3.1.1, M and M^* have the same Theta rank and levelness. If $|E| \leq 5$, then either M or M^* is of rank ≤ 2. $\square$

A first example of a matroid of levelness ≥ 3 is given by the graphic matroid associated to the complete graph K_4.

Figure 3.1: K_4 and its geometric representation.

Proposition 3.2.5. *The graphic matroid $M(K_4)$ is 3-level.*

Proof. Let F be the flat $\{1, 2, 3\}$ corresponding to the labelled example shown in Figure 3.1. Both the contraction of F and the restriction to F are connected (or biconnected on the level of graphs) and thus F is a flacet with $\ell_F(\mathbf{x}) = x_1 + x_2 + x_3$. The spanning trees $B_1 = \{1, 5, 6\}$ and $B_2 = \{4, 5, 6\}$ satisfy $|F \cap B_2| < |F \cap B_1| < \mathrm{rank}(F)$ which shows that $M(K_4)$ is at least 3-level. To see that $M(K_4)$ is at most 3-level we notice that every proper flacet F has rank smaller or equal than $\mathrm{rank}(M(K_4)) - 1 = 2$ and hence $\ell_F(\mathbf{x})$ can take at most three different values. $\square$

Before analyzing other matroids we quickly recall a **geometric representation** of certain matroids of rank 3: the idea is to draw a diagram in the plane whose points correspond to the elements of the ground set. Subsets of 3 elements constitute a basis unless they are contained in a depicted line.

Example 3.2.6. Let us consider the graph K_4 and its geometric representation as a matroid as shown in Figure 3.1. The geometric representation consists only of the four lines associated to the 3-circuits of K_4.

Starting from the geometric representation of $M(K_4)$ we define three new matroids by removing one, two or three lines of the representation and we call them respectively $\mathcal{W}^3$, Q_6, and P_6. None of these matroids is graphic, but we can easily draw their geometric representations:

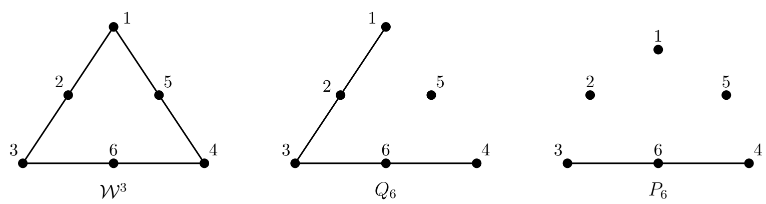

Figure 3.2: Geometric representations of $\mathcal{W}^3$, Q_6, and P_6.

It is also interesting to observe that the four matroids $M(K_4)$, $\mathcal{W}^3$, Q_6, and P_6 are self-dual matroids.

Proposition 3.2.7. *The matroids $\mathcal{W}^3$, Q_6, and P_6 are 3-level.*

Proof. Let M be any of the three given matroids and consider $F = \{3, 4, 6\}$. It is easy to check that $M|_F \cong U_{3,2}$ and $M/F \cong U_{3,1}$ which marks F as a flacet. The vertices of the matroid polytope associated to the bases $\{1, 2, 5\}, \{1, 2, 4\}, \{1, 3, 4\}$ lie on distinct hyperplanes parallel to $H_M(F) = \{\boldsymbol{p} \in \mathbb{R}^6 : \ell_F(\boldsymbol{p}) = \text{rank}_M(F)\}$. Therefore the matroids are at least 3-level. Since $\text{rank}(M) = 3$, we can use the same argument as in the proof of Proposition 3.2.5. $\square$

The list of excluded minors for $\mathcal{M}_2^{\mathsf{Lev}}$ so far includes $M(K_4)$, $\mathcal{W}^3$, Q_6, and P_6. To show that this list is complete, we will approach the problem from the constructive side and consider how to synthesize 2-level matroids. We already saw that $\mathcal{M}_2^{\mathsf{Lev}}$ is closed under taking direct sums. In Chapter 1 we introduced three more operations, namely series and parallel connection and 2-sum. Now we see that these operations retain levelness.

For the following result, we write $E_1 \uplus E_2 = (E_1 \cup E_2 \cup \{q_1, q_2\}) \setminus \{q\}$ for the *disjoint union* of E_1 and E_2.

Lemma 3.2.8. *Let $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$ be matroids with $\{q\} = E_1 \cap E_2$ not a coloop of both. Then the base polytope $P_{\mathcal{S}}$ of the series connection $\mathcal{S} = \mathcal{S}(M_1, M_2)$ is linearly isomorphic to*

$$(P_{M_1} \times P_{M_2}) \cap \{\boldsymbol{p} \in \mathbb{R}^{E_1 \uplus E_2} : p_{q_1} + p_{q_2} \leq 1\}.$$

Proof. It is clear that the base configuration $V_{\mathcal{S}}$ is isomorphic to

$$V' = (V_{M_1} \times V_{M_2}) \cap \{\boldsymbol{p} \in \mathbb{R}^{E_1 \uplus E_2} : p_{q_1} + p_{q_2} \leq 1\}$$

under the linear map $\pi : \mathbb{R}^{E_1 \uplus E_2} \to \mathbb{R}^{E_1 \cup E_2}$ given by $\pi(\mathbf{1}_{q_1}) = \pi(\mathbf{1}_{q_2}) = \mathbf{1}_q$ and $\pi(\mathbf{1}_e) = \mathbf{1}_e$ otherwise. Indeed, let $r_i = \operatorname{rank}(M_i)$, then a linear inverse is given by $s : \mathbb{R}^{E_1 \cup E_2} \to \mathbb{R}^{E_1 \uplus E_2}$ with $s(\mathbf{x})_{q_i} = r_i - \ell_{E_i}(\mathbf{x})$ for $i = 1, 2$ and the identity otherwise.

It is therefore sufficient to show that the vertices of

$$P' = (P_{M_1} \times P_{M_2}) \cap \{\boldsymbol{p} \in \mathbb{R}^{E_1 \uplus E_2} : p_{q_1} + p_{q_2} \leq 1\}.$$

are exactly the points in V'. Clearly V' is a subset of the vertices and any additional vertex of P' would be the intersection of the relative interior of an edge of $P_{M_1} \times P_{M_2}$ with the hyperplane $H = \{\boldsymbol{p} \in \mathbb{R}^{E_1 \uplus E_2} : p_{q_1} + p_{q_2} = 1\}$. However, every edge of $P_{M_1} \times P_{M_2}$ is parallel to some $\mathbf{1}_e - \mathbf{1}_f$ for $e, f \in E_1$ or $e, f \in E_2$. Thus every edge of $P_{M_1} \times P_{M_2}$ can meet H only in one of its endpoints which proves the claim. $\square$

It is interesting to note that the operation that related P_{M_1} and P_{M_2} to $P_{\mathcal{S}(M_1,M_2)}$ is exactly a *subdirect product* in the sense of McMullen [McM76]. From the description of $P_{\mathcal{S}(M_1,M_2)}$ we instantly get information about the Theta rank and levelness of the series and parallel connection.

Corollary 3.2.9. *Let $\mathcal{S} = \mathcal{S}(M_1, M_2)$ be the series connection of matroids M_1 and M_2. Then*

$$\mathsf{Th}(\mathcal{S}) = \max(\mathsf{Th}(M_1), \mathsf{Th}(M_2)).$$

The same holds true for the parallel connection as well as the levelness.

Proof. Lemma 3.2.8 shows that the facet-defining linear functions of $P_{\mathcal{S}}$ are among those of $P_{M_1} \times P_{M_2}$ and $\ell(\mathbf{x}) = 1 - x_{q_1} - x_{q_2}$. However, by the characterization of the bases of $\mathcal{S}$, $\ell(\mathbf{x})$ can take only values in $\{0, 1\}$. Hence, $\mathsf{Th}(V_{\mathcal{S}}) = \mathsf{Th}(V_{M_1} \times V_{M_2})$ and Proposition 3.1.2 finishes the proof. $\square$

Corollary 3.2.10. *The classes $\mathcal{M}_k^{\mathsf{Th}}$ and $\mathcal{M}_k^{\mathsf{Lev}}$ are closed under taking series and parallel connections.*

We will need the following two properties.

Lemma 3.2.11 ([CO03, Lem. 2.3]). *Let M be a 3-connected matroid having no minor isomorphic to any of $M(K_4)$, $\mathcal{W}^3$, Q_6, P_6. Then M is uniform.*

Lemma 3.2.12 ([Oxl11, Thm. 8.3.1]). *Every matroid that is not 3-connected can be constructed from 3-connected proper minors of itself by a sequence of direct sums and 2-sums.*

We can finally give a complete characterization of the class $\mathcal{M}_2^{\mathsf{Lev}} = \mathcal{M}_1^{\mathsf{Th}}$.

Theorem 3.2.13. *For a matroid M the following are equivalent:*

(i) *M has Theta rank 1;*

(ii) *M is 2-level;*

(iii) *M has no minor isomorphic to $M(K_4)$, $\mathcal{W}^3$, Q_6, or P_6;*

(iv) *M can be constructed from uniform matroids by taking direct or 2-sums.*

Proof. (i) $\Rightarrow$ (ii) is just Theorem 0.0.1. (ii) $\Rightarrow$ (iii) follows from Theorem 3.2.1 and Proposition 3.2.7. Let M be a matroid satisfying (iii). If M is 3-connected, then M is uniform by Lemma 3.2.11. If M is not 3-connected, then Lemma 3.2.12 shows that it satisfies (iv). Finally, uniform matroids have Theta rank 1 by Proposition 3.2.3. Theta rank $\leq k$ is retained by series connection (Corollary 3.2.9) and, by definition, also by the 2-sum. □

Example 3.2.14. If we look at the family of 2-level graphic matroids, the only excluded minor is the graph K_4. The class of graphs which do not contain K_4 as a minor is the well-known class of **series-parallel graphs** $\mathcal{G}_{\mathsf{SP}}$. The theorem implies $\mathcal{G}_2^{\mathsf{Lev}} = \mathcal{G}_{\mathsf{SP}}$.

Notice that the class of base polytopes of 2-level matroids is not contained in any of the families of 2-level polytopes listed in Section 2.2.3. For instance, it strictly contains all hypersimplices and no hypersimplex can be obtained as an order polytope. Moreover, there are 2-level base polytopes (like $U_{7,2}$) with an odd number of vertices which implies that they are not centrally symmetric, thus neither Hanner nor Hansen polytopes. Finally, every 2-level polytope has a simple vertex if and only if it is isomorphic to a stable set polytope of a perfect graph (see [FFF$^+$]), and the hypersimplex $\Delta_{4,2}$ is 3-dimensional with vertices of degree 4.

3.3 k-level graphs

In this section we study the class $\mathcal{G}_k^{\mathsf{Lev}}$ of k-level graphs for arbitrary k. The Robertson-Seymour theorem assures that the list of excluded minors characterizing $\mathcal{G}_k^{\mathsf{Lev}}$ is finite and we give an explicit description in the next subsection. In Section 3.3.2, we focus on the class of 3-level graphs which is characterized by exactly one excluded minor, the wheel W_4 with 4 spokes. The class of W_4-minor-free graphs was studied by Halin and we recover its building blocks from levelness considerations. In Section 3.3.3 we focus on the class of graphs with Theta rank 2. Excluded minors for this class can be obtained from the structure of 4-level graphs.

3.3.1 Excluded minors for k-level graphs

A consequence of Theorem 3.2.13 is that a graph G is 2-level if and only if G does not have K_4 as a minor. In order to give a characterization of k-level graphs in terms of excluded minors, we first need to view K_4 from a different angle.

Definition 3.3.1. The **cone** over a graph $G = (V, E)$ with apex $w \notin V$ is the graph

$$\operatorname{cone}(G) = (V \cup \{w\}, E \cup \{wv : v \in V\}).$$

Let us denote by C_n the n-**cycle**. Thus, we can view K_4 as the cone over C_3. As in the previous section, we only need to consider graphic matroids $M(G)$ which are connected, that is we restrict to biconnected graphs. For a flacet F let us denote by $V_F \subseteq V$ the vertices covered by F.

Proposition 3.3.2. *Let $G = (V, E)$ be a biconnected graph and $F \subset E$ a flacet with $|E \backslash F| \geq 2$. Then $G|_F$ is a vertex-induced subgraph.*

Proof. By contradiction, suppose that $e \in E \backslash F$ is an edge with both endpoints in V_F. Since F is a flacet, G/F is a biconnected graph with loop e. This contradicts $|E \backslash F| \geq 2$. □

The definition of flacets requires the graph G/F to be biconnected. This, in turn, implies that $G|_{E \backslash F}$ is connected. Let us write $C(F) := \{uv \in E : u \in V_F, v \notin V_F\}$ for the induced **cut**. Moreover, let us write $\overline{F} := E \backslash (F \cup C(F))$. The next result allows us to find minors G' of G with $\mathsf{Lev}(G') = \mathsf{Lev}(G)$.

Lemma 3.3.3. *Let G be a biconnected graph and F a k-level flacet. Then F is a k-level flacet of the graph $G/\overline{F}$.*

Proof. Let $H = G/\overline{F}$. It follows from the definition of flacets, that $G|_{\overline{F}}$ is connected and thus $H/F = G/(F \cup \overline{F}) = U_{|C(F)|,1}$ is biconnected. Moreover, $H|_F = G|_F$ is biconnected and therefore F is a flacet of H.

The levelness of the flacet F cannot be bigger than k. Let $T_1 \subset E$ be a spanning tree such that the restriction to the connected graph $G|_{E \setminus F}$ is also a spanning tree. In particular, $|T_1 \cap F|$ is minimal among all spanning trees. It now suffices to show that there is a sequence of spanning trees $T_1, T_2, \ldots, T_k \subset E$ with $|T_i \cap F| = |T_1 \cap F| + i - 1$ for all $i = 1, \ldots, k$ and such that $T_i \cap \overline{F} = T_j \cap \overline{F}$ for all i, j. The contractions $T_i/\overline{F}$ then show that F is at least k-level for H.

If $T_i \cap F$ is not a spanning tree for $G|_F$, then pick $e \in F \setminus T_i$ such that e connects two connected components of $(V_F, T_i \cap F)$. Since T_i is a spanning tree, there is a cycle in $T_i \cup e$ that uses at least one cut edge $f \in C(F) \cap T_i$. Hence $T_{i+1} = (T_i \setminus e) \cup f$ is the new spanning tree with the desired properties. □

The contraction of $\overline{F}$ in G gives a graph with vertices $V_F \cup \{w\}$, where w results from the contraction of $\overline{F}$.

Proposition 3.3.4. *Let $G = (V, E)$ be a simple, biconnected graph and let w be a vertex such that the set of edges F of $G - w$ is a flacet. Then F is k-level if and only if* $\deg(w) = k$.

Proof. Let E_w be the edges incident to w. For a spanning tree $T \subseteq E$, we have $\ell_F(\mathbf{1}_T) = |F \cap T| = |T| - |E_w \cap T|$. Hence, F is k-level if and only if there are at most k spanning trees $T_1, \ldots, T_k$ such that every T_i uses a different number of edges from E_w. Since $|E_w| = \deg(w)$ and every spanning tree contains at least one edge of E_w, there are at most $\deg(w)$ spanning trees with different size of the intersection with F, thus $k \leq \deg(w)$. Moreover, G is simple, thus there exists a spanning tree T_1 such that $E_w \subseteq T_1$. Applying the same reasoning of the proof of Lemma 3.3.3, we obtain the sequence of spanning trees with the desired properties. Finally, we observe that $T_1 \cap F$ has $\deg(w) - 1$ connected components, thus the sequence is made of at least $\deg(w)$ trees, proving that $\deg(w) \leq k$. □

It follows from Proposition 3.3.4 that the cone over a biconnected graph on k vertices has a k-level flacet. The next result gives a strong converse to this observation. A graph G is called **minimally biconnected** if $G \setminus e$ is not biconnected for all $e \in E$. For more background on this class of graphs we refer to [Plu68] and [Dir67].

Proposition 3.3.5. *Let G be a simple, biconnected graph with a vertex w such that the set of edges F not incident to w is a flacet. If F is k-level, then G has a minor* $\text{cone}(H)$ *with apex w where H is a minimally biconnected graph on k vertices.*

Proof. Let $m = |V_F|$. By Proposition 3.3.4, $\deg(w) = k$ and thus $m \geq k$. By removing edges if necessary, we can assume that $G|_F$ is minimally biconnected. By Proposition 1.2.4 the contraction of any edge of $G|_F$ leaves a biconnected graph. Contract an edge such that at most one endpoint is connected to w. The new edge set F' is still a k-level flacet. By iterating these deletion-contraction steps, we obtain a cone over F' with apex w. □

Theorem 3.3.6. *A graph G is k-level if and only if G has no minor* $\text{cone}(H)$ *where H is a minimally biconnected graph on $k+1$ vertices.*

Proof. Let $G = (V, E)$ be a graph and $F \subset E$ a m-level flacet such that $m > k$. By Lemma 3.3.3, we may assume that F is the set of edges not incident to some $w \in V$. By Proposition 3.3.5, we may also assume that $G|_F$ is minimally biconnected on m vertices. Now, $G|_F$ contains a minor H that is minimally biconnected on $k+1$ vertices and hence G contains $\text{cone}(H)$ as a minor. □

3.3.2 The class of 3-level graphs

According to Theorem 3.3.6, the excluded minors for $\mathcal{G}_3^{\mathsf{Lev}}$ are cones over minimally biconnected graphs on 4 vertices. The only minimally biconnected graph on 4 vertices is the 4-cycle and hence the excluded minor is $W_4 = \text{cone}(C_4)$, the wheel with 4 spokes. In general, let us write $W_n = \text{cone}(C_n)$ for the **n-wheel**, which is a graph of levelness n. The family of W_4-minor-free graphs was considered by R. Halin (see [Die90, Ch. 6]). In this section, we will rediscover the *building blocks* for this class.

We start with the observation that by Lemma 3.2.12 and Corollary 3.2.9, we may restrict to 3-connected, simple graphs. Recall that a graph G (and its matroid) is **k-connected** if the removal of any $k-1$ vertices leaves G connected. Also, a graph is **k-regular** if every vertex is incident to exactly k edges.

Proposition 3.3.7. *A 3-level, 3-connected simple graph is 3-regular.*

Proof. A graph G with a vertex of degree at most 2 cannot be 3-connected. If there is a vertex w of degree at least 4, then $G - w$ is biconnected. It follows that the set of edges F not incident to w form a flacet and Proposition 3.3.4 yields the claim. $\square$

The following well-known result (see [Oxl11, Thm 8.8.4]) puts strong restrictions on minimally 3-connected matroids. A n**-whirl** is the matroid of the n-wheel $W_n = \text{cone}(C_n)$ with the additional basis being the rim of the wheel $B = E(C_n)$.

Theorem 3.3.8 (Tutte's wheels and whirl theorem)**.** *Let $M = (E, \mathcal{B})$ be a 3-connected matroid. Then the following are equivalent:*

(i) *For all $e \in E$ neither $M \backslash e$ nor M/e is 3-connected;*

(ii) *M is a n-whirl or n-wheel, for some n.*

We will come back to whirls in the next section. For now, we note that the only minimally 3-connected graphs are the wheels. Moreover note that every 3-regular simple graph must have an even number of vertices ($3|V(G)| = 2|E(G)|$).

Lemma 3.3.9. *Let G be a 3-connected 3-regular simple graph with at least 6 vertices. Then G is at least 4-level.*

Proof. By assumption G cannot be a wheel. By Theorem 3.3.8, there must be an edge e such that $G \setminus e$ or G/e is 3-connected. Now, $G \setminus e$ has a degree-2 vertex for all $e \in E$ and hence is not 3-connected. On the other hand, G/e is 3-connected and the removal of multiple edges does not alter 3-connectivity. This rules out all the cases where G/e has multiple edges, because there would be a vertex of degree 2 (not counting multiple edges). The only possibility is that G/e is a simple 3-connected graph with a vertex of degree 4. By Proposition 3.3.4, we conclude that $G \setminus e$ (and consequently G) is at least 4-level . $\square$

Corollary 3.3.10. *K_4 is the only 3-level, 3-connected simple graph.*

The following gives a complete characterization of 3-level graphs.

Theorem 3.3.11. *For a graph G the following are equivalent.*

(i) *G has no minor isomorphic to W_4;*

(ii) *G is 3-level;*

(iii) *G can be constructed from the cycles C_2, C_3, the dual C_3^*, and K_4 by taking direct or 2-sums.*

Proof. (i) $\Leftrightarrow$ (ii) is Theorem 3.3.6 together with the fact that C_4 is the unique minimally biconnected graph on 4 vertices. (ii) $\Rightarrow$ (iii) follows from Corollary 3.3.10. (iii) $\Rightarrow$ (ii) follows from Corollary 3.2.9 and (1.2). $\square$

By inspecting the building blocks for 2-level (Example 3.2.14) and 3-level graphs, it is tempting to think that the building blocks of k-level graphs are given by the building blocks and the excluded minors of $\mathcal{G}_{k-1}^{\mathsf{Lev}}$. This turns out to be false even for $\mathcal{G}_4^{\mathsf{Lev}}$. Indeed $\mathsf{Lev}(K_5) = 4$ and we cannot obtain it as a sequence of direct sums and 2-sums of C_2, C_3, C_3^*, $K_4 = W_3$, and W_4.

3.3.3 4-level and Theta-2 graphs

A further hope one could nourish is that 3-level graphs coincide with the graphs of Theta rank 2. This would be the case if and only if $\mathsf{Th}(W_4) = 3$. The only k-level flacet F of W_n with $k > 3$ is given by the rim of the wheel $F = E(C_n)$. To find a sum-of-squares representation of $\ell_F(\mathbf{x})$ for the basis configuration $V_{M(W_n)}$ of W_n, we may project onto the coordinates of F which coincides with the configuration of *forests* of C_n. Now, every subset of $E(C_n)$ is independent except for the complete cycle $I = E(C_n)$. Hence the configuration of forests is given by $\{0,1\}^n \setminus \{\mathbb{1}\}$ and the linear function in question is $\ell(\mathbf{x}) = n - 1 - \sum_i x_i$. For $n = 4$ and for all $\boldsymbol{v} \in \{0,1\}^4 \setminus \{\mathbb{1}\}$,

$$18\ell(\boldsymbol{v}) \ = \ 2(\ell(\boldsymbol{v})(\ell(\boldsymbol{v}) - 4))^2 + (\ell(\boldsymbol{v})(\ell(\boldsymbol{v}) - 1))^2,$$

and this gives a sum-of-squares representation (1) of degree ≤ 2. We may now pullback the 2-sos representation to $\ell_F(\mathbf{x})$ which shows that W_4 is Theta-2.

Towards a list of excluded minors for $\mathcal{G}_2^{\mathsf{Th}}$, we focus on the class of 4-level graphs. Using Theorem 3.3.6 we easily find the two excluded minors for $\mathcal{G}_4^{\mathsf{Lev}}$:

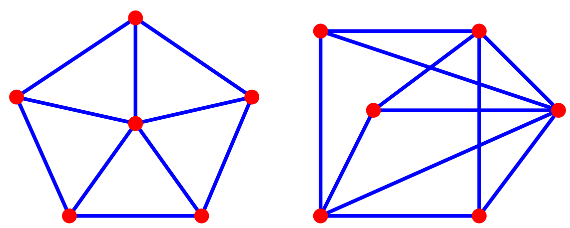

Figure 3.3: Excluded minors of $\mathcal{G}_4^{\mathsf{Lev}}$.

The first graph is the 5-wheel W_5, the second graph is the cone over $K_{2,3}$ and is called $A_3\backslash x$ in [Oxl89]. The next result states that this is the right class to study.

Proposition 3.3.12. *The wheel W_5 has Theta rank 3.*

Proof. Let $F = E(C_5)$ be the edges of the rim of the wheel which is a flat of rank 4. This is the unique 5-level flacet and it is sufficient to show that $4 - \ell_F(\mathbf{x})$ is not 2-sos with respect to the spanning trees $V = V_{M(W_5)}$ of W_5. Arguing by contradiction, let us suppose that there are polynomials $h_1(\mathbf{x}), \dots, h_m(\mathbf{x})$ of degree ≤ 2 such that

$$f(\mathbf{x}) \ := \ 4 - \ell_F(\mathbf{x}) - h_1(\mathbf{x})^2 - \cdots - h_1(\mathbf{x})^2$$

is identically zero on V.

Consider the point $\boldsymbol{p} = \mathbf{1}_F$. This is not a basis of $M(W_5)$ and a polynomial separating $\boldsymbol{p}$ from V is given by $f(\mathbf{x})$. That is, by construction $f(\mathbf{x})$ is a polynomial that vanishes on V and $f(\boldsymbol{p}) \leq -1 \neq 0$. Now we may compute a degree-compatible Gröbner basis of the vanishing ideal $I = I(V)$ (see Chapter 6 for more) using *Macaulay2* [GS]. Evaluating the elements of the Gröbner basis at $\boldsymbol{p}$ shows that the only polynomials not vanishing on $\boldsymbol{p}$ are of degree 5. As $\deg(f) \leq 4$ by construction, this yields a contradiction. □

The proof suggests an interesting connection to Tutte's wheels and whirls theorem (Theorem 3.3.8): for $n = 4$ it states that the vanishing ideal of the n-wheel $I(W_n)$ is generated by $I(\mathcal{W}^n)$ and a unique polynomial of degree n. This should be viewed in relation to Proposition 3.1.4: projecting $V_{\mathcal{W}^n}$ and V_{W_n} onto the coordinates of $F = E(C_n)$ yields $\{0,1\}^n$ and $\{0,1\}^n \setminus \{\mathbb{1}\}$, respectively.

Oxley [Oxl89] determined that the class of 3-connected graphs not having W_5 as a minor consists of 17 individual graphs and 4 infinite families. The graph $A_3\backslash x$ is 5-level and belongs to this set of graphs. In addition, it is a minor of all the elements of the 4 infinite families and it is minor of three further graphs. This proves the following result.

Theorem 3.3.13. *Every 4-level graph is obtained by direct and 2-sums of C_2, C_3, C_3^*, and the 14 graphs represented in Figure 3.4.*

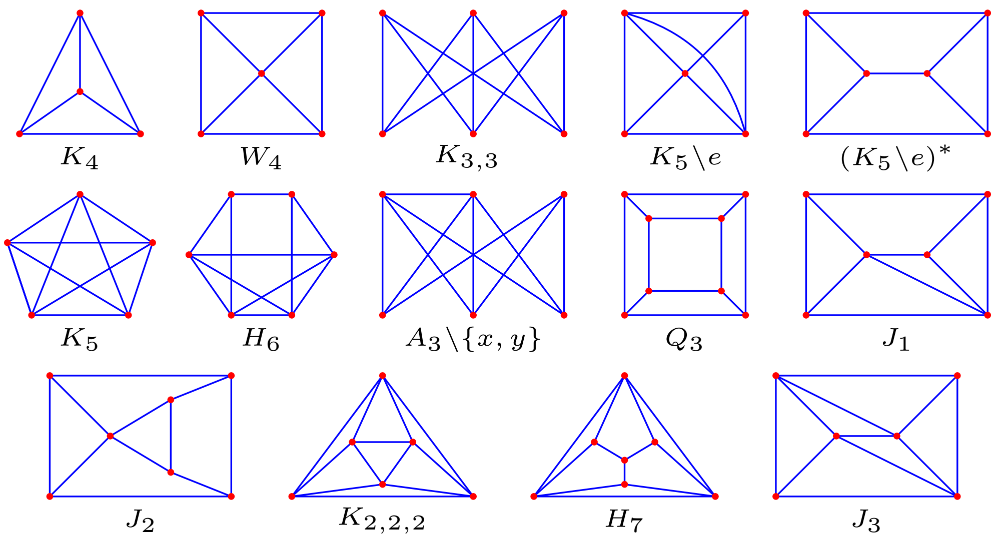

Figure 3.4: List of 3-connected graphs of $\mathcal{G}_4^{\mathsf{Lev}}$.

As $A_3\backslash x$ is Theta-2, a complete list of excluded minor has to be extracted from the 17 graphs plus 4 families in [Oxl89]. As a last remark, we note that the Theta-1 graphs are given by series-parallel graphs. On the contrary, the property of being Theta-2 is unrelated to graph planarity. A computational approach to this list of graphs has been tried: as a result we checked that some of the graphs in the list are Theta-2. However, whenever we do not obtain a Theta-2 decomposition of a facet-defining linear function (in a reasonable amount of computational time), we are missing a certificate to decide if the graph is Theta-3.

Proposition 3.3.14. *The graphs K_5 and $K_{3,3}$ have Theta rank* 2.

Proof. For both cases we use the idea that for a given flacet $F \subseteq E$, we may project the basis configuration V onto the coordinates given by F and find a 2-sos representation of the linear function $\mathrm{rank}(F) - \sum_i x_i$.

For the graph $K_{3,3}$, the only flacets of levelness > 3 are given by 4-cycles. Projecting onto these coordinates yields $\{0,1\}^4 \setminus \{\mathbb{1}\}$ which is a point configuration of Theta rank 2.

For the complete graph K_5, we note that any flacet F of levelness > 3 is given by the edges of an embedded K_4. For such a flacet, we might equivalently consider $\ell_{E\backslash F}(\mathbf{x}) - 1 \geq 0$. Projecting onto $E\backslash F$ again yields $\{0,1\}^4 \backslash \{\mathbf{0}\}$. $\square$

3.4 Excluded minors for k-level matroids

Theorem 3.2.1 implies that the class $\mathcal{M}_k^{\mathsf{Lev}}$ and $\mathcal{M}_k^{\mathsf{Th}}$ have an excluded minor characterization. Theorem 3.2.13 provides the list of excluded minors for $\mathcal{M}_2^{\mathsf{Lev}} = \mathcal{M}_1^{\mathsf{Th}}$, and in particular, shows that the list is finite. In general, the list of excluded minors is not expected to be finite for all minor-closed matroid families.

In the previous section we described the excluded minors of $\mathcal{G}_k^{\mathsf{Lev}}$, for every k. Since we are dealing with a minor-closed class of graphs, they are finitely many as implied by Robertson-Seymour theorem ([RS04]). On the other hand, the problem of characterizing the finite excluded minors of $\mathcal{G}_k^{\mathsf{Th}}$ remains open, even for $k = 2$.

This section is devoted to the excluded minors of $\mathcal{M}_k^{\mathsf{Lev}}$ for $k > 2$. A complete characterization, analogous to the graph setting, seems out of reach. For instance, the cone construction introduced for graphs (3.3.1) is not a matroid operation, since it relates to the vertex structure. In this section we prove that the list of excluded minors of $\mathcal{M}_k^{\mathsf{Lev}}$ is finite for every k.

Let us define the following property of matroids, which relates to their levelness.

Definition 3.4.1. A matroid M is called **minimally k-level** if $\mathsf{Lev}(M) = k$ and for every minor N, $\mathsf{Lev}(N) < \mathsf{Lev}(M)$.

Since $M_1 \oplus_2 M_2$ contains both M_1 and M_2 as minors ([Oxl11, Thm. 8.3.1]), it follows from Lemma 3.2.12 and Corollary 3.2.9 that every minimally k-level matroid is 3-connected.

The minimally k-level matroids relate to the excluded minors for levelness. In particular, we immediately realize that every excluded minor of $\mathcal{M}_k^{\mathsf{Lev}}$ is a minimally $(k+i)$-level matroid for some $i \geq 1$. Moreover, all minimally $(k+1)$-level matroids are excluded minors of $\mathcal{M}_k^{\mathsf{Lev}}$. However, it is not clear a priori whether there exists an excluded minor of $\mathcal{M}_k^{\mathsf{Lev}}$ which is minimally $(k+i)$-level for $i > 1$. We answer this question in Proposition 3.4.14.

We state now the main theorem of this section.

Theorem 3.4.2. *For every k, there are finitely many minimally k-level matroids. In particular, the list of excluded minors for $\mathcal{M}_{k-1}^{\mathsf{Lev}}$ is finite.*

To prove the theorem, we show that for a fixed k, the size of the ground set of a minimally k-level matroid is bounded, which implies that we can construct finitely many of such matroids. Moreover, Proposition 3.4.14 gives that the excluded minors for $\mathcal{M}^{\mathsf{Lev}}_{k-1}$ are exactly the minimally k-level matroids and the second claim of the theorem follows.

To bound the size of the ground set of a minimally k-level matroid M, we choose one of its k-level flacets F and bound separately the size of F and the size of its complement $\overline{F} = E(M) \setminus F$. Both tasks require us to analyze the structure of minimally k-level matroids and their properties.

We already saw in the proof of Lemma 3.3.3 the approach we will often use in this section to determine the levelness of a matroid. More precisely, in the graph setting we constructed a particular sequence of spanning trees of a graph with respect to a flacet in order to deduce the levelness. We now generalize this idea to any matroid M. A **k-sequence of bases** for a flacet F is a collection of bases $B_1, \ldots, B_k \in \mathcal{B}(M)$ such that:

(i) $|F \cap B_1|$ is minimal among all bases;

(ii) $|F \cap B_i| = |F \cap B_1| + i - 1$, for $1 \leq i \leq k$;

(iii) $F \cap B_i \subset F \cap B_{i+1}$, for $1 \leq i \leq k-1$;

(iv) $|F \cap B_k| = \mathrm{rank}_M(F)$.

Lemma 3.4.3. *A flacet F of a matroid M is k-level if and only if there is a k-sequence of bases for F.*

Proof. If F is a k-level flacet, we start from $B_1 \in \mathcal{B}(M)$ such that $|F \cap B_1|$ is minimal among all bases. We construct the sequence in the following way: if $F \cap B_i$ is not a basis of $M|_F$, we pick $f \in F$ such that $\mathrm{rank}_M((F \cap B_i) \cup f) = \mathrm{rank}_M(F \cap B_i) + 1$. The set $B_i \cup f$ contains a circuit with at least one element $e \in \overline{F}$. Let us set $B_{i+1} = B_i \setminus e \cup f$. The sequence can be extended until $|F \cap B_i| = \mathrm{rank}_M(F)$. If F is k-level, then B_k must be the first basis in the sequence such that $F \cap B_k$ is a basis of $M|_F$.

Conversely, given a k-sequence of bases for a flacet F, F is clearly k-level. □

Before we start to investigate the family of minimally k-level matroids, we introduce some technical lemmas. The following two lemmas follow from the properties of the collection of circuits described in Proposition 1.2.2.

Lemma 3.4.4. *Let M be a connected matroid and B any basis. For any $e \in E(M)$, the set $B \cup e$ contains a circuit C with at least 2 elements such that $e \in C$. Moreover, for any $e' \in C$ the set $B \setminus e' \cup e$ is a basis of M.*

Proof. Since B is a basis, it is immediate that $B \cup e$ contains a circuit C and $e \in C$. Moreover, M is connected, thus e is not a loop and C contains at least another element. For any $e' \in C$, consider $B' := B \setminus e' \cup e$.

Suppose by contradiction that B' is not a basis. The set B' necessarily contains a circuit C' and $e \in C'$. Since $e \in C \cap C'$, by Proposition 1.2.2 there exists a circuit C'' such that $C'' \subseteq (C \cup C') \setminus e \subseteq B$. This yields the contradiction and concludes the proof. □

Lemma 3.4.5. *Let M be a matroid such that the set $\{e, e'\} \subseteq E(M)$ is a circuit. Then for every circuit $C \in \mathcal{C}(M)$ such that $e \in C$ and $e' \notin C$, the set $C \setminus e \cup e'$ is also a circuit.*

Proof. Let us consider the two circuits $\{e, e'\}$ and C: since $e \in C \cap \{e, e'\}$, by Proposition 1.2.2 the set $(C \cup \{e, e'\}) \setminus e$ contains a circuit C'. By contradiction, suppose that $C' \neq C \setminus e \cup e'$: since C is a circuit, $e' \in C'$. Moreover, $e' \in C' \cap \{e, e'\}$ allows us to apply again Proposition 1.2.2 to claim that there exists a circuit C'' such that $C'' \subseteq (C' \cup \{e, e'\}) \setminus e' \subset C$. Since C is a circuit, this yields a contradiction and proves that $C' = C \setminus e \cup e' \in \mathcal{C}(M)$. □

The third lemma is specifically related to levelness.

Lemma 3.4.6. *Let M be a connected matroid and F a k-level flacet. For any $e \in \overline{F}$, there exists a k-sequence of bases $B_1, \ldots, B_k$ such that $e \in B_i$ for $i = 1, \ldots, k$.*

Proof. Consider any basis B such that $|F \cap B|$ is minimal among the bases. If $e \in B$, we set $B_1 = B$. Otherwise, by Lemma 3.4.4, $B \cup e$ contains a circuit C (with at least 2 elements) such that $e \in C$ and for any $e' \in C$, the set $B \setminus e' \cup e$ is a basis. We can choose an element $e' \in \overline{F} \cap C$. Indeed suppose that $\overline{F} \cap C = \{e\}$; since F is a flat, this would imply $e \in F$ which is a contradiction. Thus we choose $e' \in \overline{F} \cap C$ and set $B_1 = B \setminus e' \cup e$. Now we construct the sequence starting from B_1 as in the proof of Proposition 3.4.3, with the additional requirement that we never remove e when passing from B_i to B_{i+1}. This is possible because whenever e belongs to the circuit C in $B_i \cup f$ for $f \in F$, C contains at least another element in $\overline{F}$. □

A matroid M is **minimally connected** if M is connected and there is no element $e \in E(M)$ such that $M \setminus e$ is connected.

We prove now four propositions which eventually will provide us the tools to obtain the upper bounds on the size of minimally k-level matroids.

Proposition 3.4.7. *Let M be a minimally k-level matroid and F a k-level flacet of M. Then $(M/F)^*$ is a minimally connected matroid.*

Proof. Suppose that $(M/F)^*$ is not minimally connected. There exists an element $e \in \overline{F}$ such that the deletion $(M/F)^* \setminus e$ is a connected matroid. Dually, Corollary 1.2.7 implies that $(M/F)/e$ is connected.

Since M is minimally k-level, it is 3-connected. By Lemma 3.4.6 we can construct a k-sequence of bases for F such that all bases contain e. We have that $B_1 \setminus e, \ldots, B_k \setminus e$ is a k-sequence of bases of the matroid M/e for F. We only need to check that F is a flacet of M/e.

If C is a circuit containing e and some elements of F, it must contain at least a second element $e' \in \overline{F}$ because F is a flat. In addition, there must be at least a third element $e'' \in \overline{F}$, otherwise e' would be a loop of $(M/F)/e$, which is connected by hypothesis. This shows that F is a flat of M/e. Moreover, $(M/e)/F \cong (M/F)/e$ and $(M/e)|_F \cong M|_F$ are connected. Thus F is a k-level flacet of M/e, contradicting the k-level minimality of M. □

Proposition 3.4.8. *Let M be a minimally k-level matroid and F a k-level flacet of M. Then* $\mathrm{rank}_M(\overline{F}) = |\overline{F}|$.

Proof. By contradiction, suppose $\mathrm{rank}_M(\overline{F}) < |\overline{F}|$. Consider a k-sequence of bases $B_1, \ldots, B_k$ for F. Because of the assumption $\mathrm{rank}_M(\overline{F}) < |\overline{F}|$, we can pick an element $e \in \overline{F} \setminus B_1$. By Proposition 3.4.7, $(M/F)/e$ is not connected. Since F is a flacet, M/F is connected and Proposition 1.2.4 implies that $(M/F) \setminus e$ is connected.

The set F is a flacet of the matroid $M \setminus e$. Indeed F is a flat of $M \setminus e$ and both $(M \setminus e)|_F \cong M|_F$ and $(M \setminus e)/F \cong (M/F) \setminus e$ are connected.

The bases $B_1, \ldots, B_k$ are also bases for $M \setminus e$ and form a k-sequence for the flacet F. Thus $M \setminus e$ is a k-level minor of M, contradicting the k-level minimality of M. □

Proposition 3.4.9. *Let M be a minimally k-level matroid and F a k-level flacet of M. Then $M|_F$ is a minimally connected matroid.*

Proof. Suppose that $M|_F$ is not minimally connected. Consider $e \in F$ such that $(M|_F) \setminus e$ is connected. $\hat{F} = F \setminus e$ is a flat of $M \setminus e$. We show that $\hat{F}$ is a k-level flacet of $M \setminus e$.

The matroid $(M \setminus e)|_{\hat{F}} \cong (M|_F) \setminus e$ is connected by hypothesis and the matroid $(M \setminus e)/\hat{F}$ is also connected. To prove the second claim, we show that $(M \setminus e)/\hat{F} \cong M/F$. Since $M|_F$ is connected, we can consider one circuit

C of $M|_F$ containing e such that it has the least possible number of elements. Let us choose another element $e' \in C$ and define the set $C^- := C \setminus \{e, e'\}$. The set $\{e, e'\}$ is a circuit of the matroid $M' := M/C^-$. By Lemma 3.4.5 $e \in C' \in \mathcal{C}(M')$ implies $C' \setminus e \cup e' \in \mathcal{C}(M')$. Thus, $(M' \setminus e)/e' \cong (M'/e)/e'$ and the claim follows.

At last, we show that there is a k-sequence of bases of $M \setminus e$ for $\hat{F}$. Let us consider a k-sequence $B_1, \dots, B_k \in \mathcal{B}(M)$ for F such that $e \notin B_k$. To find this sequence we exploit the fact that $M|_F$ is connected and therefore there exists a basis of $M|_F$ avoiding any chosen element e. We can complete this basis to a basis B_k of M. Then, we construct the sequence $B_1, \dots, B_k$ backwards: given B_i, consider an element $f \in \overline{F}$ such that $f \notin B_i$. The set $B_i \cup f$ contains a circuit with at least one element $f' \in F$ because M is minimally k-level and by Proposition 3.4.8 there is no circuit entirely contained in $\overline{F}$. Thus, we set $B_{i-1} = B_i \setminus f' \cup f$.

The k-sequence $B_1, \dots, B_k$ is also a k-sequence of bases of $M \setminus e$ for the flacet $\hat{F}$, contradicting the k-level minimality of M. $\square$

Proposition 3.4.10. *Let M be a minimally k-level matroid and F a k-level flacet of M. Then* $\operatorname{rank}_M(F) = k-1$.

Proof. Suppose that $\operatorname{rank}_M(F) > k-1$. Consider a k-sequence $B_1, \dots, B_k$ for F: by definition $|F \cap B_k| = \operatorname{rank}_M(F) > k-1$ and thus $|F \cap B_1| > 0$. Equivalently, there is an element $e \in F$ such that $e \in B_i$ for $i = 1, \dots, k$. We prove that the matroid M/e is k-level with respect to the flacet $\hat{F} = F \setminus e$.

The matroid $(M/e)|_{\hat{F}} \cong (M|_F)/e$ is connected because $M|_F$ is minimally connected by Proposition 3.4.9 and $(M/e)/\hat{F} \cong M/F$ is connected because F is a flacet of M. Finally, $B_1 \setminus e, \dots, B_k \setminus e$ are bases of M/e and form a k-sequence of the flacet $\hat{F}$, contradicting the k-level minimality of M. $\square$

We report two useful propositions whose proofs are available in Oxley ([Oxl11]).

Proposition 3.4.11 ([Oxl11, Prop. 4.3.11])**.** *Let M be a minimally connected matroid of rank r where $r \geq 3$. Then $|E(M)| \leq 2r-2$. Moreover, equality holds if and only if $M \cong M(K_{2,r-1})$.*

Proposition 3.4.12 ([Oxl11, Ch. 4, Ex. 10 (d)])**.** *Let M be a matroid for which M^* is minimally connected. Then either $M \cong U_{n,1}$ for some $n \geq 3$ or M has at least* $\operatorname{rank}(M)+1$ *non-trivial parallel classes.*

Notice that there is only one minimally connected matroid of rank 1, namely $U_{1,1}$. Also the rank 2 case is easy to understand.

Corollary 3.4.13. *The uniform matroid $U_{3,2}$ is the only minimally connected matroid of rank* 2.

Proof. Let M be a minimally connected matroid of rank 2 on $n \geq 3$ elements. The dual matroid M^* is a rank $n-2$ matroid. By Proposition 3.4.12, either $M^* \cong U_{n,1}$ or M^* has at least $\text{rank}(M^*)+1 = n-2+1 = n-1$ non-trivial parallel classes. The first possibility implies $n-2 = 1$, hence $M = U_{3,2}$; the second possibility would require at least $2n-2 > n$ elements, which is clearly not possible. $\square$

The following proposition implies the equivalence of excluded minors for $\mathcal{M}_{k-1}^{\mathsf{Lev}}$ and minimally k-level matroids.

Proposition 3.4.14. *Every minimally k-level matroid M has a minor N such that* $\mathsf{Lev}(N) = k-1$.

Proof. Consider a k-level flacet F of a minimally k-level matroid M and choose a k-sequence $B_1, \ldots, B_k$ for F. Pick the element $e \in F$ such that $e \notin B_1$ and $e \in B_i$ for $i = 2, \ldots, k$.

Since M is minimally k-level, $\mathsf{Lev}(M/e) \leq k-1$. Moreover, $\hat{F} := F \setminus e$ is a flacet of M/e (we apply here the same argument used in the proof of Proposition 3.4.10) and it is $(k-1)$-level since we can exhibit the $(k-1)$-sequence of bases $B_2 \setminus e, \ldots, B_k \setminus e$. $\square$

We now present the proof of the main theorem of this section.

Proof of Theorem 3.4.2. Let us consider a minimally k-level matroid M for $k \geq 4$: any k-level flacet F of M is of rank $k-1$ by Proposition 3.4.10. In particular, $\text{rank}(F) \geq 3$ and therefore, by Proposition 3.4.9 and Proposition 3.4.11, F has at most $2(k-1) - 2 = 2k-4$ elements.

We try to find an upper bound for the number of elements in $\overline{F}$. First, we partition $\overline{F}$ into two sets T and $S = \overline{F} \setminus T$ such that

$$T = \{e \in \overline{F} \ : \ \exists\, C \text{ circuit of } M \text{ with } e \in C \text{ and } |C \cap \overline{F}| = 2\}.$$

Every element $e \in T$ is in a non-trivial parallel class of M/F: by definition, $e \in T$ if there is a circuit C of M containing elements of F and only another element $e' \in \overline{F}$ which implies that $\{e, e'\}$ is a circuit of M/F. On the other hand, every element $e \in S$ is in a circuit containing at least 3 elements of $\overline{F}$ and therefore e is a trivial parallel class of M/F. Since all the non-trivial parallel classes originate from elements in T and every class has at least 2 elements, their number is bounded from above by $|T|/2$.

Let us set $h := \text{rank}(M) - \text{rank}(F) - 1$, so that $\text{rank}(M/F) = h + 1$. By Proposition 3.4.7, $(M/F)^*$ is a minimally connected matroid on at least 3 elements (since the levelness requirements imply $|\overline{F}| \geq 4$) and by Proposition 3.4.12 there are two possibilities:

(i) $M/F \cong U_{|\overline{F}|,1}$. This means $\text{rank}(M) = k$ and $|\overline{F}| \leq k$ because of Proposition 3.4.8. It follows that $|E(M)| = |F|+|\overline{F}| \leq 2k-4+k = 3k-4$;

(ii) $\text{rank}(M/F) = h+1 > 1$. By proposition 3.4.12 M/F has at least $h+2$ non trivial parallel classes, hence we obtain the inequality

$$\frac{|T|}{2} \geq h + 2 \Rightarrow |T| \geq 2h+4. \tag{3.1}$$

Moreover, $\overline{F}$ has exactly $\text{rank}(F)+h+1 = k+h$ elements and this fact yields a second inequality, namely

$$|T| \leq k+h. \tag{3.2}$$

Combining 3.1 and 3.2 we get

$$2h+4 \leq k+h \quad \Longrightarrow \quad h \leq k-4.$$

It is immediate that $|\overline{F}| = k+h \leq 2k-4$ and finally

$$|E(M)| = |F|+|\overline{F}| \leq 2k-4+2k-4 = 4k-8.$$

We have shown that for every minimally k-level matroid M, $k \geq 4$, $|E(M)|$ is bounded.

The case $k = 2$ is trivial since $M|_F = U_{1,1}$ and the case $k = 3$ is such that $M|_F = U_{3,2}$ and $|\overline{F}| \geq 3$. We can find an upper bound $|\overline{F}|$ using the same argument as before: the only difference is that the first case yields the upper bound $|\overline{F}| \leq 3$ and the second case $|\overline{F}| \leq 2k - 4 = 2$ which is not realizable. Thus $|\overline{F}| = 3$.

For any k, the number of elements of a minimally k-level matroid is bounded, therefore there exist finitely many minimally k-level matroids.

If we consider an excluded minor M of $\mathcal{M}_{k-1}^{\mathsf{Lev}}$, every minor of M is at most $(k-1)$-level. From Proposition 3.4.14 it follows that every excluded minor M of $\mathcal{M}_{k-1}^{\mathsf{Lev}}$ is such that $\mathsf{Lev}(M) = k$. We conclude that the excluded minors of $\mathcal{M}_{k-1}^{\mathsf{Lev}}$ are the minimal k-level matroids, which are finitely many. $\square$

Chapter 4

The constructive approach

4.1 Structural properties

Part (iv) of Theorem 3.2.13 suggests us to try a constructive approach to 2-level matroids. This point of view can be framed in the general theory of matroid decompositions and yields some useful results for the case we are interested in. More precisely, 2-level matroids turn out to be in bijection with a particular class of trees.

4.1.1 Tree decompositions of matroids

To understand matroid properties from the constructive perspective, it is convenient to adapt the definition of 2-sum for matroids with disjoint ground sets. Let M_1 and M_2 be matroids such that $E(M_1) \cap E(M_2) = \emptyset$. For any pair $e_1 \in E(M_1)$ and $e_2 \in E(M_2)$ that are not loops nor coloops, we define the 2-sum $(M_1, e_1) \oplus_2 (M_2, e_2)$ as the matroid $(E(M_1) \cup E(M_2) \setminus \{e_1, e_2\}, \mathcal{B})$, where

$$\mathcal{B} := \{B_1 \cup B_2 \setminus \{e_1, e_2\} : B_1 \in \mathcal{B}(M_1), B_2 \in \mathcal{B}(M_2), |(B_1 \cup B_2) \cap \{e_1, e_2\}| = 1\}.$$

The elements e_1 and e_2 are called the **base points** of the 2-sum.

As we already mentioned, there is a theory of matroid decompositions and we refer to [Oxl11, Sect. 8.3] for a complete overview on this topic. The decomposition relies heavily on the 2-sum operation and proves itself to be a valuable tool for the understanding of 2-level matroids. In this section we summarize a few definitions and results that are relevant for our purposes.

Definition 4.1.1. A **matroid-labelled tree** is a tree T with vertex set $\{N_1, \ldots, N_s\}$ for some positive integer s such that

(i) the N_i's are matroids with pairwise disjoint ground sets;

(ii) an edge joining N_i and N_j is labelled by a set $\{e_i, e_j\}$ such that $e_i \in E(N_i)$, $e_j \in E(N_j)$, and e_i, e_j are neither loops nor coloops;

(iii) every element $e_i \in E(N_i)$ is used in at most one edge label.

We call $N_1, \ldots, N_s$ the **vertex labels** of T.

From now on we will call $U_{n,1}$ a **multiedge** and denote it by M_n and we will call $U_{n,n-1}$ a **ring** and denote it by R_n. As already observed in Chapter 1, rings and multiedges are graphic matroids.

Given a matroid-labelled tree T, let us consider an edge t with label $\{e_i, e_j\}$, which connects two vertex labels N_i and N_j. The contraction of t yields a matroid-labelled tree T/t with the same vertex labels except N_i and N_j which have been merged in a vertex label $(N_i, e_i) \oplus_2 (N_j, e_j)$. The adjacencies of T/t are given by the contraction of the edge t in T.

For each vertex label N_i of a matroid-labelled tree, we partition the ground set $E(N_i)$ into two sets: the set $W(N_i)$ of elements which appear in some edge label and the set $F(N_i) = E(N_i) \setminus W(N_i)$. We call $W(N_i)$ the set of **ideal elements** (generalizing the notion of ideal edge in [Tut01, Sect. IV.3]) and $F(N_i)$ the set of **free elements**.

Example 4.1.2. Figure 4.1 displays, on the left, 5 graphs related by 2-sums, whose base points are specified by pairs of equal colors. Performing all operations yields the graph on the right which is series-parallel, since obtained as a sequence of 2-sums of rings and multiedges.

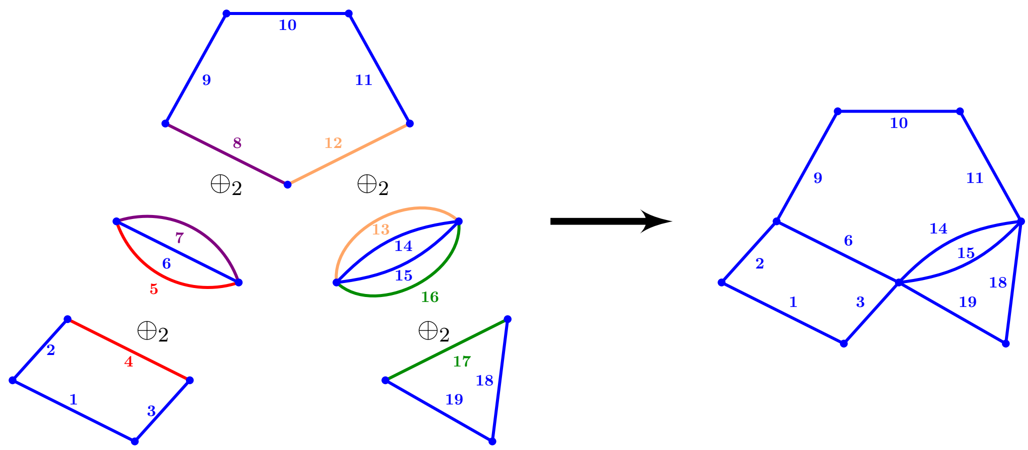

Figure 4.1: Sequence of 2-sums of graphs.

Each graph is labelled in order to describe its ground set and its collection of bases. The free elements are colored blue. We can represent the structure of the 2-sums by the tree shown in Figure 4.2.

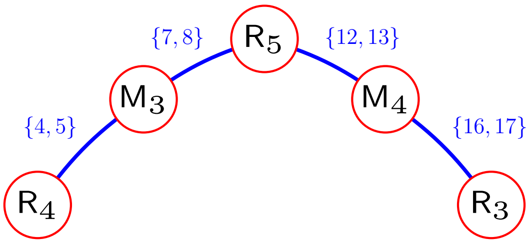

Figure 4.2: Example of matroid-labelled tree.

The tree T is a matroid-labelled tree. The contraction of all edges in T yields a matroid-labelled tree with a unique vertex label corresponding to the graph shown on the right in Figure 4.1.

Definition 4.1.3. A **tree decomposition** of a connected matroid M is a matroid-labelled tree T such that if $V(T) = \{N_1, \ldots, N_s\}$ and $E(T) = \{t_1, \ldots, t_{s-1}\}$, then

(i) $E(M) = \bigcup_{i=1}^{s} F(N_i)$;

(ii) $|E(N_i)| \geq 3$ for all i, unless $|E(M)| < 3$, in which case $s = 1$ and $N_1 = M$;

(iii) M is the matroid that labels the single vertex of $T/\{t_1, t_2, \ldots, t_{s-1}\}$.

The condition (ii) ensures that every tree decomposition of a matroid M has a finite number of vertex labels. Otherwise, for instance, we could add a chain of arbitrarily many vertex labels $U_{2,1}$ and still obtain a tree decomposition of M.

Notice that the matroid-labelled tree of Example 4.1.2 satisfies all the requirements for being a tree decomposition of the series-parallel graph drawn in Figure 4.1.

The following theorem is taken from [Oxl11, Thm. 8.3.10] and originally appeared in [CE80]. According to our definitions, we replace the words "circuit" and "cocircuit" with "ring" and "multiedge", respectively.

Theorem 4.1.4. *Let M be a connected matroid. Then M has a tree decomposition T_M in which every vertex label is a 3-connected matroid, a ring, or a multiedge, and there are no two adjacent vertices that are both labelled by rings or by multiedges. Moreover, T_M is unique up to relabelling of its edges.*

The requirement of no adjacent rings or multiedges is strictly necessary for uniqueness as Figure 4.3 shows.

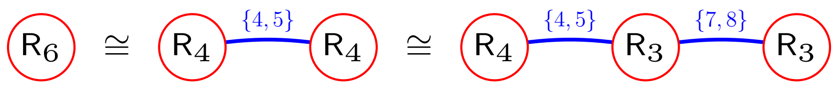

Figure 4.3: Tree decompositions of R_6.

Every matroid-labelled tree T which satisfies the condition (ii) of Definition 4.1.3 is a tree decomposition of the matroid obtained by contracting all edges of T. We want to "reverse" the decomposition process and understand which non-isomorphic (in terms of the corresponding matroids) tree decompositions can be constructed when we restrict the possible choices for vertex labels. The constructive process requires to choose the tree structure (the vertex labels and their adjacencies) and an appropriate labelling for the edges.

Theorem 4.1.4 explains how to prevent the construction of different tree structures representing isomorphic matroids: we only use vertex labels that are 3-connected matroids, rings, or multiedges. In addition, two vertex labels N_i and N_j can be adjacent only if they are not both rings or both multiedges.

Our focus is on 2-level matroids, hence the set of possible vertex labels restricts to uniform matroids by Part (iv) of Theorem 3.2.13. We divide them into three categories:

(i) M-vertices: correspond to multiedges of size at least 3;

(ii) R-vertices: correspond to rings of size at least 3;

(iii) U-vertices: correspond to uniform matroids $U_{n,k}$ such that $n \geq 4$ and $2 \leq k \leq n-2$.

Once we decided the tree structure, we have to choose an appropriate edge labelling. For any pair of vertex labels N_i and N_j connected by an edge, we choose two base points $e_i \in F(N_i)$ and $e_j \in F(N_j)$ for the 2-sum. After we add the edge label $\{e_i, e_j\}$, the elements e_i and e_j are no longer free elements.

Since every free element can be used at most once to label an edge, the **degree** $\deg(N_i)$ of a vertex label N_i, that is, the number of edges incident to N_i, can be at most $|E(N_i)|$. This condition must be satisfied by the tree structure. Corollary 3.2.10 implies that every tree constructed with labels of type M, R, and U yields a 2-level matroid if it satisfies the condition that $\deg(N_i) \leq E(N_i)$ for all vertex labels.

In general, different choices of the base points yield non-isomorphic matroids, despite the fact that the tree structure is the same. Example 4.1.5 shows this possibility for a tree with only two vertex labels.

At the beginning of this chapter we described the collection of bases for the 2-sum; the collection of circuits of $(M_1, e_1) \oplus_2 (M_2, e_2)$ is

$$\begin{aligned} &\mathcal{C}(M_1 \setminus e_1) \cup \mathcal{C}(M_2 \setminus e_2) \cup \\ &\cup \{(C_1 \setminus e_1) \cup (C_2 \setminus e_2) \ : \ e_1 \in C_1 \in \mathcal{C}(M_1) \text{ and } e_2 \in C_2 \in \mathcal{C}(M_2)\} . \end{aligned} \tag{4.1}$$

Example 4.1.5. Consider two copies of the 3-connected matroid P_6 whose geometric representation is given in Figure 3.2. P_6 has one circuit with 3 elements and all other circuits with 4 elements. Let us label the ground set of the first copy by $[6] := \{1, \dots, 6\}$ in such a way that the only circuit with 3-elements is $\{1, 2, 3\}$; analogously the second copy has ground set $[7, 12] := \{7, \dots, 12\}$ and circuit with 3 elements $\{7, 8, 9\}$. Figure 4.4 shows two non-isomorphic 2-sums obtained by different choices of the base points.

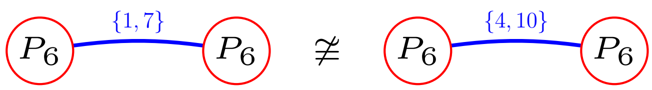

Figure 4.4: Non-isomorphic matroids from 2-sum of two copies of P_6.

From the description of circuits (4.1), it is immediate that the first matroid has circuits of size 4, 5, and 6 and the second one has circuits of size 3, 4, and 6. Thus, the two matroids are not isomorphic.

4.1.2 The family of UMR-trees

We define a class of trees whose vertices are labelled by uniform matroids and show that it is in bijection with the family of connected 2-level matroids.

Definition 4.1.6. Let T be a tree whose vertex labels are of type U, M, and R and such that no two M-vertices and no two R-vertices are adjacent. The tree T is a **UMR-tree** if $\deg(N_i) \leq |E(N_i)|$ for every vertex label N_i.

Notice that the edges of a UMR-tree T are unlabelled. Every matroid-labelled tree that we obtain by an appropriate edge-labelling of T is the unique tree decomposition of some 2-level matroid. Example 4.1.5 shows that, a priori, different edge-labellings of T can yield tree decompositions of non-isomorphic matroids. We prove in this section that this does not happen: every UMR-tree corresponds to exactly one connected 2-level matroid.

Let M be a matroid and $\pi : E(M) \to E(M)$ a permutation of its ground set. Given the collection of bases $\mathcal{B}$ of M, we define $\pi(\mathcal{B}) := \{\pi(B) : B \in \mathcal{B}\}$. A matroid M is called **permutation invariant** if $\pi(\mathcal{B}) = \mathcal{B}$ for every permutation π of the ground set. We say that M is **transposition invariant** with respect to $e_1 \in E(M)$ and $e_2 \in E(M)$ if $\pi(\mathcal{B}) = \mathcal{B}$ for the transposition $\pi = (e_1, e_2)$. Notice that the previous definitions apply if we consider the collection of circuits $\mathcal{C} = \mathcal{C}(M)$ instead of the collection of bases $\mathcal{B}$. Indeed $\pi(\mathcal{B}) = \mathcal{B}$ if and only if $\pi(\mathcal{C}) = \mathcal{C}$.

Every uniform matroid is permutation invariant and it is easy to see that every permutation invariant matroid is uniform.

The definition of permutation invariant matroid is too restrictive, therefore we introduce a weaker definition of invariance. Let M be a matroid and T_M its unique tree decomposition. We say that M is N_i-**transposition invariant** for a vertex label N_i of T_M if it is transposition invariant with respect to every pair of elements in $F(N_i)$. We say that M is **node-invariant** if it is N_i-transposition invariant for every vertex label N_i of T_M.

In the rest of the section, whenever we refer to a uniform matroid U, we assume that $E(U) \geq 3$.

Lemma 4.1.7. *Let M be a N_i-transposition invariant connected matroid and U a uniform matroid. For every choice of $f \in F(N_i)$ and $u \in E(U)$, the* 2*-sum $(M, f) \oplus_2 (U, u)$ yields the same matroid up to isomorphism.*

Proof. The uniform matroid U is permutation invariant and thus the choice of $u \in E(U)$ does not affect the result of the 2-sum. Consider any two elements $f_1, f_2 \in F(N_i)$. We want to show that

$$S_{f_1} := (M, f_1) \oplus_2 (U, u) \cong (M, f_2) \oplus_2 (U, u) =: S_{f_2}.$$

Notice that $E(S_{f_2}) = E(S_{f_1}) \setminus \{f_2\} \cup \{f_1\}$. We claim that the bijection $\varphi : E(S_{f_1}) \to E(S_{f_2})$ such that

$$\varphi(e) = \begin{cases} f_1 & \text{, if } e = f_2 \\ e & \text{, otherwise.} \end{cases}$$

yields the matroid isomorphism. We need to show that for every $X \subset E(S_{f_1})$, $X \in \mathcal{C}(S_{f_1})$ if and only if $\varphi(X) \in \mathcal{C}(S_{f_2})$.

As we mentioned in (4.1), a circuit C of S_{f_1} can be of 3 different types:

(i) $C \in \mathcal{C}(U \setminus u)$. Clearly $\varphi(C) = C$ and $C \in \mathcal{C}(S_{f_2})$;

(ii) $C \in \mathcal{C}(M \setminus f_1)$. This implies that C is a circuit of M not containing f_1. Since M is N_i-transposition invariant, we have that $\pi(C) \in \mathcal{C}(M)$ for $\pi = (f_1, f_2)$. Moreover, $f_1 \notin C$ implies $f_2 \notin \pi(C)$, that is $\pi(C) \in \mathcal{C}(M \setminus f_2)$. Finally, $\varphi(C) = \pi(C) \in \mathcal{C}(M \setminus f_2)$ and thus $\varphi(C) \in \mathcal{C}(S_{f_2})$;

(iii) $C = (C_1 \setminus f_1) \cup (C_2 \setminus u)$, $f_1 \in C_1 \in \mathcal{C}(M)$ and $u \in C_2 \in \mathcal{C}(U)$. Since M is N_i-transposition invariant, for $\pi = (f_1, f_2)$ we have $\pi(C_1) \in \mathcal{C}(M)$ and $f_2 \in \pi(C_1)$. Moreover, $\varphi(C) = (\pi(C_1) \setminus f_2) \cup (C_2 \setminus u)$, $f_2 \in \pi(C_1) \in \mathcal{C}(M)$ and $u \in C_2 \in \mathcal{C}(U)$ and thus $\varphi(C) \in \mathcal{C}(S_{f_2})$.

We proved that the image under φ of any circuit of S_{f_1} is a circuit of S_{f_2}. The same argument applies to show that all circuits of S_{f_2} are circuits of S_{f_1} under the map $\varphi^{-1} : E(S_{f_2}) \to E(S_{f_1})$. □

Lemma 4.1.8. *Let M be a node-invariant connected matroid and U a uniform matroid. The 2-sum $(M, f) \oplus_2 (U, u)$ is a node-invariant matroid for any choice of $f \in E(M)$ and $u \in E(U)$.*

Proof. Let N_i be a vertex label of the unique tree decomposition T_M of the matroid M: without loss of generality, we assume $f \in E(N_i)$. In particular, we choose $f \in F(N_i)$. To prove that $S_f := (M, f) \oplus_2 (U, u)$ is node-invariant, we check that S_f is N_j-transposition invariant for every vertex label N_j. Let N_j be a vertex label and $f_1, f_2 \in F(N_j)$, $f_1, f_2 \neq f$. The set $\mathcal{C}(S_f)$ is transposition invariant for $\pi = (f_1, f_2)$. Indeed $\mathcal{C}(M \setminus f)$ and $\mathcal{C}(U \setminus u)$ are transposition invariant for π because M is node-invariant and π does not act on $E(U)$. The same holds true for the circuits of the third type, since

$$\pi((C_1 \setminus f) \cup (C_2 \setminus u)) = (\pi(C_1) \setminus f) \cup (C_2 \setminus u)$$

and $f \in \pi(C_1) \in \mathcal{C}(M)$ by node-invariance of M.

The tree decomposition of S_f has one new vertex, labelled by the uniform matroid U. We still have to check that S_f is U-transposition invariant. The same argument used above applies, since U is permutation invariant. □

Now we are ready to state the main theorem of this section.

Theorem 4.1.9. *The class of connected 2-level matroids is in bijection with the class of* UMR*-trees.*

Proof. By definition, every UMR-tree T admits an edge-labelling that turns T into the unique tree decomposition T_M of a matroid M. Now we show by induction on the number of vertex labels n that all edge-labellings for T yield the same matroid M up to isomorphism and that M is node-invariant.

The base case $n = 1$ is trivial because the tree has no edges and a uniform matroid is clearly node-invariant. Suppose that the tree T has $n > 1$ vertex labels. Let U (a uniform matroid) be a leaf of T connected to a vertex label N_i. By induction hypothesis, if we remove U from T, we obtain a UMR-tree T' for which all edge-labellings determine the same matroid M' up to isomorphism. Moreover, M' is node-invariant. Now we consider the 2-sum $(M', f) \oplus_2 (U, u)$, where $f \in F(M_i)$ and $u \in E(U)$. By Lemma 4.1.7 every choice of $f \in F(N_i)$ and $u \in E(U)$ yields the same matroid M (up to isomorphism). In addition, M is node-invariant by Lemma 4.1.8. Therefore every edge-labelling of T yields a tree decomposition of the same matroid M and the bijection follows.

□

4.2 Base polytopes and 2-sum

In this section we prove a property of the collection of bases of 2-level matroids which allow us to explicitly determine a unimodular triangulation for the associated base polytopes. Moreover, we show that these base polytopes are alcoved, which implies the possibility of studying their volumes by means of permutations.

Most proofs exploit the fact that 2-level matroids are constructed by sequential 2-sums of uniform matroids.

4.2.1 Sort-closed matroids and triangulations

Let $M \in \mathcal{M}_2^{\mathsf{Lev}}$ be a matroid of rank k on n elements. The base polytope P_M is a subpolytope of the hypersimplex $\Delta_{n,k}$. We will show that M is *sort-closed* which implies that a well-known triangulation of $\Delta_{n,k}$ due to Stanley induces a unimodular triangulation of P_M as proven in [LP07].

Let $\mathcal{B}$ be a collection of k-subsets of $[n]$. We consider an ordering $\mathfrak{o}$ of the elements $[n]$. For B_1 and B_2 in $\mathcal{B}$ we define

$$\mathrm{sort}_{\mathfrak{o}}(B_1, B_2) := \{b_1 \leq_{\mathfrak{o}} b_2 \leq_{\mathfrak{o}} \ldots \leq_{\mathfrak{o}} b_{2k}\}$$

as the ordered sequence of elements of $B_1 \sqcup B_2$, that is, the multiset merging the elements of B_1 and B_2. We also define

$$\begin{aligned}\mathrm{Odd}_{\mathfrak{o}}(B_1, B_2) &:= \{b_1, b_3, \ldots, b_{2k-1}\}, \text{ and}\\ \mathrm{Even}_{\mathfrak{o}}(B_1, B_2) &:= \{b_2, b_4, \ldots, b_{2k}\}.\end{aligned}$$

Notice that $\mathrm{Odd}_{\mathfrak{o}}(B_1, B_2)$ and $\mathrm{Even}_{\mathfrak{o}}(B_1, B_2)$ are k-subsets of $[n]$ and depend on the chosen ordering $\mathfrak{o}$.

The collection $\mathcal{B}$ is called **sort-closed** if for every pair B_1 and B_2 in $\mathcal{B}$, both sets $\mathrm{Odd}_{\mathfrak{o}}(B_1, B_2)$ and $\mathrm{Even}_{\mathfrak{o}}(B_1, B_2)$ are in $\mathcal{B}$.

Definition 4.2.1. A matroid $M = ([n], \mathcal{B})$ with an ordering $\mathfrak{o}$ of the ground set is **sort-closed** if the collection of bases $\mathcal{B}$ is sort-closed for $\mathfrak{o}$.

Sort-closed matroids appeared first in [Blu01] with the name of base-sortable matroids. The property of being sort-closed depends on the ordering of the ground set; up to relabelling of the ground set $[n]$, we can assume that a matroid is sort-closed for the natural ordering $1 < 2 < \ldots < n$ of $[n]$.

Example 4.2.2. Consider the matroid M in Example 1.2.3 with the natural ordering $\mathfrak{o}$ of the ground set $1 < 2 < 3 < 4$. Its collection of bases $\mathcal{B}$ contains all 2-subsets of $[4]$ except $\{3, 4\}$. For every pair B_1 and B_2 in $\mathcal{B}$, the non-decreasing sequence is $\mathrm{sort}_{\mathfrak{o}}(B_1, B_2) = \{b_1 \leq_{\mathfrak{o}} b_2 \leq_{\mathfrak{o}} b_3 \leq_{\mathfrak{o}} b_4\}$. Suppose that $\mathrm{Even}_{\mathfrak{o}}(B_1, B_2) = \{b_2, b_4\} = \{3, 4\}$: clearly $b_3 = 3$ or $b_3 = 4$ and in both cases $B_2 = \{3, 4\}$. Since this is not possible, M is sort-closed.

If we consider the isomorphic matroid M' obtained by exchanging labels 1 and 4 as shown in Figure 4.5 (this is equivalent to consider the ordering $4 < 2 < 3 < 1$ of $E(M)$), we have the collection of bases

$$\mathcal{B}' = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

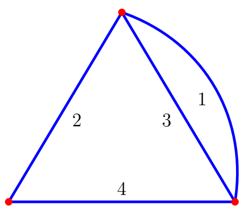

Figure 4.5: Relabelling of the triangle with one double edge.

For $B_1 = \{1, 2\}$ and $B_2 = \{3, 4\}$ in $\mathcal{B}'$, we have $\mathrm{Odd}_{\mathfrak{o}}(B_1, B_2) = \{1, 3\} \notin \mathcal{B}'$. Thus M' is not sort-closed for the natural ordering of its ground set.

Given a matroid on n elements and an ordering $\mathfrak{o}$ of the ground set, we may assume, up to relabelling of the ground set, that the ordering $\mathfrak{o}$ is $1 < 2 < \ldots < n$. A **cyclic shift** of this ordering is an ordering $i < i+1 < \ldots < n < 1 < \ldots < i-1$ for some $i \in [n]$.

Proposition 4.2.3. *A sort-closed matroid $M = ([n], \mathcal{B})$ with the ordering $\mathfrak{o}$ of the ground set is sort-closed for any cyclic shift $\mathfrak{o}^{cyc}$ of $\mathfrak{o}$.*

Proof. Since M with $\mathfrak{o}$ is sort-closed, for any pair B_1 and B_2 in $\mathcal{B}$, the sequence $\mathrm{sort}_{\mathfrak{o}}(B_1, B_2)$ yields $\mathrm{Odd}_{\mathfrak{o}}(B_1, B_2) \in \mathcal{B}$ and $\mathrm{Even}_{\mathfrak{o}}(B_1, B_2) \in \mathcal{B}$. The sequence $\mathrm{sort}_{\mathfrak{o}^{cyc}}(B_1, B_2)$ is a cyclic shift of the sequence $\mathrm{sort}_{\mathfrak{o}}(B_1, B_2)$ which implies that either $\mathrm{Odd}_{\mathfrak{o}^{cyc}}(B_1, B_2) = \mathrm{Odd}_{\mathfrak{o}}(B_1, B_2)$ and $\mathrm{Even}_{\mathfrak{o}^{cyc}}(B_1, B_2) = \mathrm{Even}_{\mathfrak{o}}(B_1, B_2)$, or $\mathrm{Odd}_{\mathfrak{o}^{cyc}}(B_1, B_2) = \mathrm{Even}_{\mathfrak{o}}(B_1, B_2)$ and $\mathrm{Even}_{\mathfrak{o}^{cyc}}(B_1, B_2) = \mathrm{Odd}_{\mathfrak{o}}(B_1, B_2)$. □

Next we show that the 2-sum of matroids preserves the property of being sort-closed.

Theorem 4.2.4. *Let M_1 and M_2 be two connected sort-closed matroids. Then the 2-sum $M = (M_1, e_1) \oplus_2 (M_2, e_2)$ is a sort-closed matroid for some ordering of the ground set $E(M)$.*

Proof. Assume that $E(M_1) = [n]$ and $E(M_2) = [n+1, n+m] := \{n+1, n+2, \ldots, n+m\}$ and they are sort-closed for the natural ordering $\mathfrak{o}$. Moreover, by Proposition 4.2.3 we can assume without loss of generality that $e_1 = n$ and $e_2 = n+1$. By definition, $E(M) = \{1, 2, \ldots, n-1, n+2, \ldots, n+m\}$ and we consider it together with its natural ordering $\mathfrak{o}$.

By definition of the 2-sum, every basis $B \in \mathcal{B}(M)$ is obtained from a pair of bases $B' \in \mathcal{B}(M_1)$ and $B'' \in \mathcal{B}(M_2)$ such that either $n \in B'$ or $n+1 \in B''$. From now on we denote by B' and B'' the unique pair of bases such that $B = (B' \cup B'') \setminus \{n, n+1\}$.

For any B_1 and B_2 in $\mathcal{B}(M)$, we show that $\mathrm{Odd}_{\mathfrak{o}}(B_1, B_2) \in \mathcal{B}(M)$. Consider B_1' and B_2' in $\mathcal{B}(M_1)$ and B_1'' and B_2'' in $\mathcal{B}(M_2)$: since both M_1 and M_2 are sort-closed, then $\mathrm{Odd}_{\mathfrak{o}}(B_1', B_2') \in \mathcal{B}(M_1)$ and $\mathrm{Odd}_{\mathfrak{o}}(B_1'', B_2'') \in \mathcal{B}(M_2)$.

In addition, if $n \in \mathrm{Odd}_{\mathfrak{o}}(B_1', B_2')$, then $n+1 \notin \mathrm{Odd}_{\mathfrak{o}}(B_1'', B_2'')$: since n is the biggest element in $E(M_1)$ and $\mathrm{Odd}_{\mathfrak{o}}(B_1', B_2')$ does not contain the biggest element of the sequence $\mathrm{sort}_{\mathfrak{o}}(B_1', B_2')$, there must be two elements n in the sequence, which means $n \in B_1'$ and $n \in B_2'$, thus $n+1 \notin B_1''$ and $n+1 \notin B_2''$. Vice versa, if $n+1 \in \mathrm{Odd}_{\mathfrak{o}}(B_1'', B_2'')$ then $n \notin \mathrm{Odd}_{\mathfrak{o}}(B_1', B_2')$. It follows that $B^U := (\mathrm{Odd}_{\mathfrak{o}}(B_1', B_2') \cup \mathrm{Odd}_{\mathfrak{o}}(B_1'', B_2'')) \setminus \{n, n+1\}$ is in $\mathcal{B}(M)$.

It is left to prove that $B^U = \mathrm{Odd}_{\mathfrak{o}}(B_1, B_2)$. We have that

$$\mathrm{sort}_{\mathfrak{o}}(B_1, B_2) = \mathrm{sort}_{\mathfrak{o}}\left(((B_1' \sqcup B_2') \sqcup (B_1'' \sqcup B_2'')) \setminus \{n, n{+}1\}\right) ,$$

where the operation $\setminus$ removes all the occurrences of n and $n{+}1$ from the sequence. By the properties of the 2-sum, the operation $\setminus$ leaves out exactly two elements that, in addition, are consecutive in the sequence ordered by $\mathfrak{o}$. It follows that

$$\mathrm{Odd}(B_1, B_2) = (\mathrm{Odd}_{\mathfrak{o}}(B_1', B_2') \cup \mathrm{Odd}_{\mathfrak{o}}(B_1'', B_2'')) \setminus \{n, n{+}1\} = B^U .$$

The same argument applies to show that $\mathrm{Even}_{\mathfrak{o}}(B_1, B_2) \in \mathcal{B}(M)$. □

For the class of 2-level matroids, we have the following.

Corollary 4.2.5. *Let M be a connected 2-level matroid. Then M is sort-closed.*

Proof. By Part (iv) of Theorem 3.2.13, M can be written as a sequence of 2-sums of uniform matroids. Uniform matroids are trivially sort-closed and Theorem 4.2.4 implies that all 2-sums we compute to construct M yield sort-closed matroids. □

The corollary is also true for non-connected 2-level matroids, since the direct sum of sort-closed matroids yields a sort-closed matroid. Before we state one of the main properties of sort-closed matroids, it is necessary to introduce a particular triangulation of the hypersimplex due to Stanley.

In his paper [Sta77] Stanley defined the map

$$\begin{aligned} \psi : [0,1]^{n-1} &\to [0,1]^{n-1} \\ (x_1, \ldots, x_{n-1}) &\mapsto (y_1, \ldots, y_{n-1}) \end{aligned}$$

where $y_i = x_1 + \ldots + x_i - \lfloor x_1 + \ldots + x_i \rfloor$. He observed that ψ is piecewise-linear, volume-preserving, and it fails to be bijective on a measure 0 set. There exists a unimodular triangulation of the hypercube into open simplices ∇_σ, labelled by the permutations $\sigma \in \mathfrak{S}_{n-1}$ and given by

$$\nabla_\sigma := \{(y_1, \ldots, y_{n-1}) \in [0,1]^{n-1} \ : \ 0 < y_{\sigma(1)} < y_{\sigma(2)} < \ldots < y_{\sigma(n-1)} < 1\}.$$

Applying the map ψ^{-1} to the simplices ∇_σ, we obtain another unimodular triangulation of the hypercube which is compatible with the subdivision of the hypercube into full-dimensional hypersimplices (the usual hypersimplices from which we remove the last coordinate, see Example 4.2.6). This is called **Stanley's triangulation** of the hypersimplex.

Example 4.2.6. The hypersimplex $\Delta_{4,2}$ can be represented inside $[0,1]^3$ as a full-dimensional polytope by removing the last coordinate and is a combinatorial octahedron as shown in Figure 4.6. Stanley's triangulation of the hypersimplex is made of the four simplices

$$\begin{aligned}
\psi^{-1}(\nabla_{132}) &= \mathrm{conv}\{(0,0,1),(0,1,0),(0,1,1),(1,0,1)\}\,,\\
\psi^{-1}(\nabla_{231}) &= \mathrm{conv}\{(0,0,1),(1,0,0),(1,0,1),(0,1,0)\}\,,\\
\psi^{-1}(\nabla_{312}) &= \mathrm{conv}\{(1,1,0),(0,1,0),(1,0,0),(1,0,1)\}\,,\\
\psi^{-1}(\nabla_{213}) &= \mathrm{conv}\{(1,1,0),(0,1,0),(0,1,1),(1,0,1)\}.
\end{aligned}$$

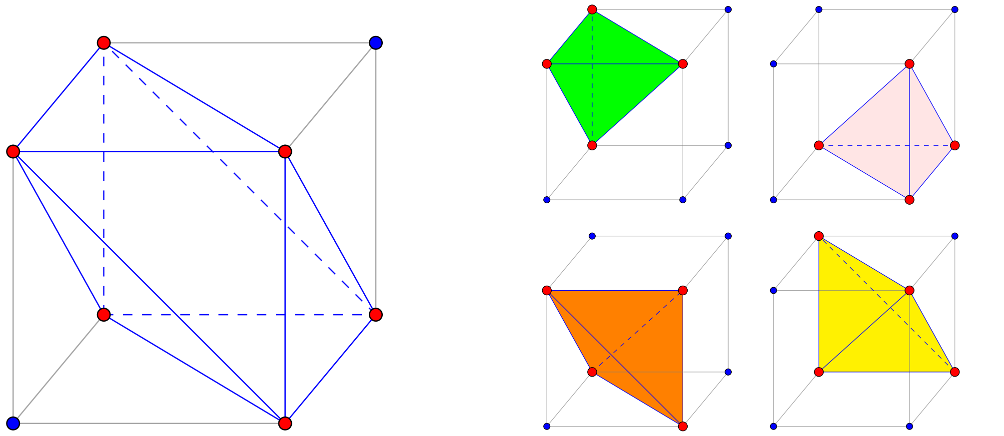

Figure 4.6: Stanley's triangulation of $\Delta_{4,2}$.

It is shown in [LP07] that the base polytope of every sort-closed matroid has a unimodular triangulation which is compatible with Stanley's triangulation of the hypersimplex.

Theorem 4.2.7 ([LP07, Thm. 4.2])**.** *Let M be a matroid. Stanley's triangulation of the hypersimplex $\Delta_{n,k}$ induces a triangulation of the base polytope P_M if and only if M is sort-closed.*

By Corollary 4.2.5, the theorem shows how to find a unimodular triangulation for base polytopes of 2-level matroids, namely the one induced by Stanley's triangulation. The vertices of the simplices in the triangulation are characterized from the collection of bases of a sort-closed matroid.

Let $M = ([n],\mathcal{B})$ be a sort-closed matroid with respect to the natural ordering $\mathfrak{o}$ of the ground set. A **sorted subset** of M is a collection $\{B_1, B_2, \ldots, B_r\} \subseteq \mathcal{B}$ such that $\mathrm{Odd}_{\mathfrak{o}}(B_i, B_j) = B_i$ and $\mathrm{Even}_{\mathfrak{o}}(B_i, B_j) = B_j$ for $1 \le i < j \le r$. For a sorted subset of size $r = \dim(P_M)+1$, the points $\mathbf{1}_{B_1}, \mathbf{1}_{B_2}, \ldots, \mathbf{1}_{B_{\dim(P_M)+1}}$ form the vertices of a simplex in the unimodular triangulation.

Example 4.2.8. Consider the graphic matroid in Example 1.2.3. Its base polytope is a subpolytope of the hypersimplex $\Delta_{4,2}$ and we see that it has a unimodular triangulation induced by Stanley's triangulation. The sorted subsets of M are $\{\{1,2\},\{1,3\},\{1,4\},\{2,4\}\}$ and $\{\{1,2\},\{1,3\},\{2,3\},\{2,4\}\}$, which respectively correspond to the yellow and pink simplices of the triangulation presented in Example 4.2.6.

4.2.2 Alcoved polytopes and volumes

Alcoved polytopes appeared in [LP07] and include many interesting classes of polytopes such as hypersimplices, order polytopes and base polytopes of sort-closed matroids. In particular, the base polytope P_M of every 2-level matroid M is alcoved. We will show how the tree decomposition T_M determines an inequality description of P_M, which in turn relates to the volume of P_M.

A polytope $P \subset \mathbb{R}^n$ is **alcoved** if P is given by inequalities of the form $b_{ij} \leq x_j - x_i \leq c_{ij}$, with $b_{ij}, c_{ij} \in \mathbb{Z}$. For more details, we refer to [LP07]. We already mentioned that base polytopes of sort-closed matroids are alcoved and Theorem 4.2.9 implies that all sort-closed matroids are *positroids*.

A **positroid** is a matroid on some ordered ground set which can be represented by the columns of a full-rank matrix such that all its maximal minors are non-negative. Positroids were first introduced in the context of the totally non-negative Grassmannian ([Pos06]) and are in bijection with several interesting classes of combinatorial objects. The class of positroids is closed under restriction, contraction, duality and cyclic shifts. For more about properties of positroids we refer to [ARW15]. Moreover, in [Oh09] an excluded minor characterization of this matroid family has been provided.

Our dedication to 2-level matroids draws the attention to the following theorem, whose proof will appear in [LP].

Theorem 4.2.9. *Let M be a matroid such that P_M is alcoved. Then M is a positroid.*

The theorem has a straightforward application to sort-closed matroids and consequently to 2-level matroids.

Corollary 4.2.10. *Every 2-level matroid is a positroid.*

Besides the relation to positroids, the geometric nature of an alcoved polytope P has a deep connection to the volume of P. Indeed every alcoved polytope comes with a natural unimodular triangulation; since all unimodular simplices have normalized volume 1, the normalized volume of P is equal to the number of simplices in the triangulation.

For instance, the normalized volume of the hypersimplex $\Delta_{n,k}$ is the Eulerian number $A_{n-1,k-1}$, that is, the number of permutations in $\mathfrak{S}_{n-1}$ with exactly $k-1$ descents. We can write any permutation $\sigma \in \mathfrak{S}_{n-1}$ as $\sigma(1)\sigma(2)\cdots\sigma(n-1)$ or, shortly, $\sigma_1\sigma_2\cdots\sigma_{n-1}$. A **descent** of σ is a position $1 \le i < n-1$ such that $\sigma(i) > \sigma(i+1)$.

In [LP07] it is shown that an alcoved polytope P which lies in the hypersimplex $\Delta_{n,k}$ has an inequality description

$$\begin{cases} \sum_{i=1}^{n} x_i = k \\ 0 \le x_i \le 1 & , \text{ for } i \in [n] \\ b_{ij} \le x_{i+1} + \ldots + x_j \le c_{ij} & , \text{ for some } (i,j)\ ,\ 0 \le i < j \le n-1, \end{cases} \tag{4.2}$$

for b_{ij} and c_{ij} non-negative integer parameters and $0 \le i < j \le n-1$. Note that the last coordinate is only used in the inequality $0 \le x_n \le 1$. We denote by $D_P \subseteq [0, n-2] \times [0, n-1]$ the set of pairs (i,j) defining an inequality of type $b_{ij} \le x_{i+i} + \ldots + x_j \le c_{ij}$.

Let us define $\mathfrak{S}_P \subset \mathfrak{S}_{n-1}$ as the set of permutations $\sigma = \sigma_1\sigma_2\cdots\sigma_{n-1} \in \mathfrak{S}_{n-1}$ satisfying the following conditions:

(i) σ has $k-1$ descents;

(ii) for every $(i,j) \in D_P$ the sequence $\sigma_i \cdots \sigma_j$ has at least b_{ij} descents. Furthermore, if $\sigma_i \cdots \sigma_j$ has exactly b_{ij} descents, then $\sigma_i < \sigma_j$;

(iii) for every $(i,j) \in D_P$ the sequence $\sigma_i \cdots \sigma_j$ has at most c_{ij} descents. Furthermore, if $\sigma_i \cdots \sigma_j$ has exactly c_{ij} descents, then $\sigma_i > \sigma_j$.

Note that we assume $\sigma_0 = 0$.

Proposition 4.2.11 ([LP07, Prop. 6.1])**.** *The normalized volume of P is equal to $|\mathfrak{S}_P|$.*

Given a sort-closed matroid M, the normalized volume of P_M is equal to the number of sorted subsets of M of size $\dim(P_M)+1$ ([LP07, Thm. 4.2]). Proposition 4.2.11 provides a different way of counting the number of unimodular simplices in the triangulation of P_M. This last point of view motivates our interest for the inequality description of base polytopes of 2-level matroids.

The fact that every 2-level matroid M is a positroid (Corollary 4.2.10) tells us something about the base polytope: P_M has a simple description with few inequalities which are explicitly presented in [ARW15, Prop. 5.5].

In what follows we analyze the inequalities of 2-level base polytopes and relate them to the tree decomposition. Once again, the constructive approach works well. In fact, for any 2-level matroid M we prove that the inequality description of P_M determines the description of P_N, where N is the 2-sum of M with a uniform matroid.

Let M be a 2-level matroid on $[n]$ with the natural ordering. The base polytope P_M is alcoved and has an inequality description of the form (4.2).

Let us consider the 2-sum $N := (M, e_1) \oplus_2 (U, e_2)$, where U is a uniform matroid of rank l on the ground set $[n{+}1, n{+}m] := \{n{+}1, n{+}2 \dots, n{+}m\}$. The matroid N is 2-level, thus P_N is alcoved with a description of type (4.2).

We assume without loss of generality that $e_1 = n$ and $e_2 = n+1$. Thus, N is defined on the ground set $E(N)=\{1, \dots, n{-}1, n{+}2, \dots, n{+}m\}$ and we will use the variables $x_1, \dots, x_{n-1}, x_{n+2}, \dots, x_{n+m}$ for the corresponding coordinates.

Theorem 4.2.12. *With the above assumptions, the base polytope P_N has an inequality description of the form*

$$\begin{cases} \text{(i)} & x_1 + \ldots + x_{n-1} + x_{n+2} + \ldots + x_{n+m} = k + l - 1 & \\ \text{(ii)} & 0 \le x_i \le 1 & , \textit{ for } i \in E(N) \\ \text{(iii)} & k - 1 \le x_1 + \ldots + x_{n-1} \le k & \\ \text{(iv)} & b_{ij} \le x_{i+1} + \ldots + x_j \le c_{ij} & , \textit{ for } (i,j) \in D_{P_M}, \end{cases}$$

where D_{P_M}, the b_{ij}'s and the c_{ij}'s are determined by the inequalities of P_M.

Proof. First we prove that these inequalities describe a 0/1-polytope P. We consider the transformation $z_i = x_1 + \ldots + x_i$ for $1 \le i \le n{-}1$ and $z_i = x_1 + \ldots + x_{n-1} + x_{n+2} + \ldots + x_{i+2}$ for $n \le i \le n{+}m{-}2$. All inequalities are of the form $z_i - z_j \le a_{ij}$ for integers a_{ij}. Since the matrix whose row vectors are $\mathbf{e}_i - \mathbf{e}_j$ is totally unimodular, the vertices of P have integer z-coordinates ([Sch86, Thm. 19.3]), and hence also integer x-coordinates. Since $0 \le x_i \le 1$, the vertices of P are 0/1-points.

To prove the equality $P_N = P$, we check that the two polytopes have the same vertices. Choose any vertex of P_N, that is, a point $\mathbf{1}_B$ for $B \in \mathcal{B}(N)$. Let $B' \in \mathcal{B}(M)$ and $B'' \in \mathcal{B}(U)$ be the two bases associated with B.

Equality (i) is satisfied by definition of 2-sum, since $|B| = \mathrm{rank}(N) = k{+}l{-}1$. The inequalities (ii) are satisfied because $\mathbf{1}_B$ is a 0/1-point. The inequality

(iii) follows from the fact that $|B \cap [1, n-1]| \in \{k-1, k\}$ by definition of 2-sum.

Consider any pair $(i,j) \in D_{P_M}$ and the corresponding inequality of type (iv): since $B' \in \mathcal{B}(M)$, we have that $\mathbf{1}_{B'}$ satisfies the inequality $b_{ij} \leq x_{i+1} + \ldots + x_j \leq c_{ij}$ describing the polytope P_M. The element x_n is not involved in the inequality and $B \cap [n-1] = B' \cap [n-1]$, thus $\mathbf{1}_B$ satisfies the inequality $b_{ij} \leq x_{i+1} + \ldots + x_j \leq c_{ij}$ of P.

We have that $\mathcal{V}(P_N) \subseteq \mathcal{V}(P)$. Let us consider any 0/1-vertex $\boldsymbol{p} \in \mathcal{V}(P)$. We construct the set

$$B'_p := \begin{cases} \{i \in [n-1] \ : \ p_i = 1\} & \text{, if } p_1 + \ldots + p_{n-1} = k \\ \{i \in [n-1] \ : \ p_i = 1\} \cup \{n\} & \text{, if } p_1 + \ldots + p_{n-1} = k - 1 \end{cases}$$

and check that it forms a basis of the matroid M, because $\mathbf{1}_{B'_p}$ satisfies all inequalities of P_M.

Combining (i) and (iii) we see that $l-1 \leq p_n + \ldots + p_{n+m-2} \leq l$. If we construct the set

$$B''_p := \begin{cases} \{i \ : \ i{\in}[n{+}2, n{+}m] \text{ and } p_i{=}1\} & \text{, if } p_{n+2}{+}\ldots{+}p_{n+m}{=}l \\ \{i \ : \ i{\in}[n{+}2, n{+}m] \text{ and } p_i{=}1\} \cup \{n{+}1\} & \text{, if } p_{n+2}{+}\ldots{+}p_{n+m}{=}l{-}1, \end{cases}$$

and we easily see that $|B''_p| = l$ and thus it forms a basis of U. Moreover $n{+}1 \in B''_p$ if and only if $n \notin B'_p$. Therefore there is a basis B of N associated to B'_p and B''_p such that $\mathbf{1}_B = \boldsymbol{p}$. □

Chapter 5

Enumeration of 2-level matroids

5.1 The generating function $T(x)$

In this section we apply the results of Section 4.1 to get formulas for the number of connected 2-level matroids of fixed size. It is noteworthy that this matroid family generalizes the family of series-parallel graphs, which appears in various areas and has several interesting properties. In particular, series-parallel graphs have been already successfully studied from an enumerative point of view in [BGKN07] and [DFK$^+$11]. By means of Theorem 4.1.9, the enumeration of connected 2-level matroids is equivalent to a tree enumeration problem, namely the enumeration of UMR-trees.

5.1.1 Preliminaries

In this subsection we introduce the tools from enumerative combinatorics that we need to count special families of trees. For the sake of brevity, we do not include all the details and refer the reader to [FS09, Ch. 1] for a thorough treatment.

A **combinatorial class** is a set $\mathcal{A}$ of combinatorial objects endowed with a size function $|\cdot|$ such that the number of elements in $\mathcal{A}$ of any given size is finite. The **generating function** (GF for short) associated to $\mathcal{A}$ is the formal power series $A(x) = \sum_{a \in \mathcal{A}} x^{|a|} = \sum_{n \geq 0} a_n x^n$. In particular, a_n is the number of elements in $\mathcal{A}$ of size n and we write $[x^n]A(x) := a_n$. We assume that every combinatorial class contains no object of size 0, thus $a_0 = 0$. Given two generating functions $A(x)$ and $B(x)$, we write $A(x) \leq B(x)$ if for each n, $[x^n]A(x) \leq [x^n]B(x)$.

The **symbolic method** in enumerative combinatorics (see [FS09, Ch. 1]) gives a direct way to translate combinatorial operations among combinatorial classes into operations involving their generating functions. Besides the disjoint union and the Cartesian product of combinatorial families, which translate into sums and products of GFs, respectively, we introduce the multiset construction: given a combinatorial class $(\mathcal{A}, |\cdot|)$ with generating function $A(x)$, the **multiset** construction of $\mathcal{A}$ is the combinatorial family obtained by taking all multisets of elements in $\mathcal{A}$. The corresponding GF is equal to

$$\begin{aligned}\mathrm{Mul}(A(x)) &= \prod_{\alpha\in\mathcal{A}}(1-x^{|\alpha|})^{-1} = \prod_{n=1}^{\infty}(1-x^n)^{-a_n}\\ &= \exp\left(\sum_{n=1}^{\infty} a_n \log(1-x^n)^{-1}\right) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}A(x^n)\right).\end{aligned}$$

Notice that we obtain a formal power series with coefficients in $\mathbb{Q}$. We also define the **restricted multiset construction** for a set of positive integers Λ as the combinatorial family obtained by taking multisets of elements in $\mathcal{A}$ with the restriction that the number of components lies in Λ. We denote this by $\mathrm{Mul}_\Lambda(A(x))$. In particular,

$$\mathrm{Mul}_0(A(x)) = 1, \quad \mathrm{Mul}_1(A(x)) = A(x), \quad \mathrm{Mul}_2(A(x)) = \frac{1}{2}\left(A(x)^2 + A(x^2)\right).$$

The notation $\mathrm{Mul}_{\geq k}$ refers to the multiset operator restricted to $\Lambda{=}\{j : j{\geq}k\}$.

5.1.2 The combinatorial class of UMR-trees

Let us consider a UMR-tree T with vertex labels $\{N_1, \dots, N_s\}$. For technical reasons we introduce an additional type of vertex that we call **leg**. Legs always have degree 1 and we represent them graphically by small red disks as in Figure 5.1. For each free element of a vertex label N_i, we draw a leg connected to N_i. Hence legs are exactly the leaves of the tree.

With the addition of legs, the degree of every vertex N_i is equal to the number of elements in $E(N_i)$. Let M be the matroid associated to T. From part (i) of Definition 4.1.3, we see that the number of elements of $E(M)$ is exactly the number of legs of T.

The generating function $T(x) = \sum_{n\geq 1} a_n x^n$, where a_n is the number of UMR-trees with n legs, leads directly to the enumeration of 2-level matroids.

Before starting with the enumeration of UMR-trees, we recall the combinatorial restrictions derived from the matroid setting:

(i) The edges are unlabelled;

(ii) No two R-vertices and no two M-vertices are adjacent;

(iii) The degree of the R-vertices and M-vertices is greater or equal than 3, and the degree of the U-vertices is greater or equal than 4.

Instead of trying to obtain enumerative formulas for UMR-trees via a direct approach, we tackle the problem with the help of the *Dissymmetry Theorem* for trees (Subsection 5.1.4) and turn our attention to rooted families.

In order to encode families of rooted UMR-trees, we will use an auxiliary type of tree: a UMR-tree is **pointed** at a vertex if it has a special leaf that we call the **virtual leg**. The virtual leg does not contribute to the total number of legs and basically pinpoints its adjacent vertex, which we call the **pointed** vertex, and distinguishes it from the other vertices.

A red triangle graphically represents the virtual leg. See Figure 5.1 for an example of a UMR-tree pointed at a R-vertex. It is important to avoid confusion between the pointed UMR-trees and the vertex-rooted UMR-trees: both kinds of trees have a distinguished vertex but only pointed UMR-trees have the virtual leg.

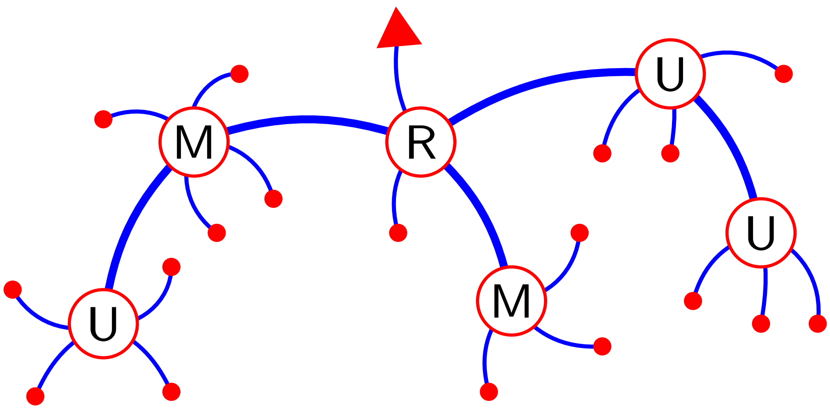

Figure 5.1: A pointed UMR-tree with 18 legs and 1 virtual leg.

Notice that the degree m of any R-vertex (or M-vertex) uniquely determines the vertex label R_m (or M_m). On the contrary, a U-vertex of degree m does not specify the rank of the corresponding vertex label. Therefore the U-vertex can represent any uniform matroid $U_{m,k}$ such that $k \in \{2, 3, \ldots, m-2\}$ and this fact must be encoded in the enumeration.

5.1.3 Counting pointed UMR-trees

In this subsection we analyze pointed UMR-trees and their generating functions.

We denote by $A_{\mathsf{R}}(x)$, $A_{\mathsf{M}}(x)$ and $A_{\mathsf{U}}(x)$ the generating functions for UMR-trees pointed at a vertex such that the virtual leg is adjacent to a R-vertex, a M-vertex, and a U-vertex, respectively. Additionally, we write $A_l(x)$ for the generating function of the tree pointed at a leg (see Figure 5.2).

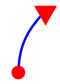

Figure 5.2: The tree pointed at a leg.

Clearly, $A_l(x) = x$. Notice that the first non-zero coefficients of the generating functions of pointed UMR-trees are $[x^2]A_{\mathsf{R}}(x) = [x^2]A_{\mathsf{M}}(x) = 1$, and $[x^3]A_{\mathsf{U}}(x) = 1$.

We obtain some relations among $A_{\mathsf{R}}(x)$, $A_{\mathsf{M}}(x)$, $A_{\mathsf{U}}(x)$, and $A_l(x)$ by decomposing the trees at the pointed vertex. Let us consider $A_{\mathsf{R}}(x)$: a tree pointed at a R-vertex can be described as a R-vertex (adjacent to the virtual leg) followed by a multiset of size greater or equal than 2 of (the disjoint union of) trees pointed at a leg, at a M-vertex or at a U-vertex (see Figure 5.3). Applying the symbolic method introduced in Subsection 5.1.1, the combinatorial description is translated into the equation $A_{\mathsf{R}}(x) = \mathrm{Mul}_{\geq 2}(A_{\mathsf{M}}(x) + A_{\mathsf{U}}(x) + A_l(x))$ which can also be written in the form

$$\begin{aligned} A_{\mathsf{R}}(x) = \exp &\left(\sum_{r=1}^{\infty} \frac{1}{r} \left(A_{\mathsf{M}}(x^r) + A_{\mathsf{U}}(x^r) + A_l(x^r) \right) \right) \\ &- 1 - \left(A_{\mathsf{M}}(x) + A_{\mathsf{U}}(x) + A_l(x) \right). \end{aligned} \tag{5.1}$$

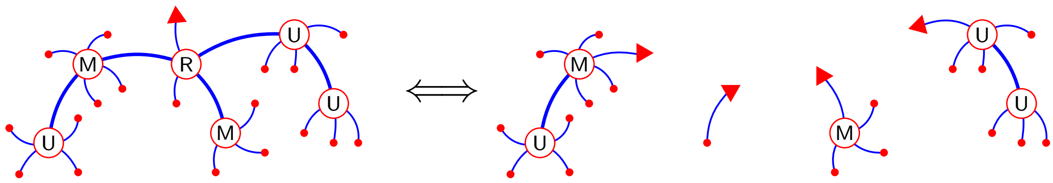

Figure 5.3: Decomposition of a pointed UMR-tree.

The same reasoning applies if the pointed vertex is of type M and gives an analogous equation for $A_{\mathsf{M}}(x)$.

$$\begin{aligned} A_{\mathsf{M}}(x) = \exp &\left(\sum_{r=1}^{\infty} \frac{1}{r} \left(A_{\mathsf{R}}(x^r) + A_{\mathsf{U}}(x^r) + A_l(x^r) \right) \right) \\ &- 1 - \left(A_{\mathsf{R}}(x) + A_{\mathsf{U}}(x) + A_l(x) \right). \end{aligned} \tag{5.2}$$

Subtracting Equation (5.1) from Equation (5.2), we get

$$\sum_{r\geq 1}\frac{1}{r}A_{\mathsf{R}}(x^r)=\sum_{r\geq 1}\frac{1}{r}A_{\mathsf{M}}(x^r)$$

and thus $[x^n]\sum_{r\geq 1}\frac{1}{r}A_{\mathsf{R}}(x^r)=[x^n]\sum_{r\geq 1}\frac{1}{r}A_{\mathsf{M}}(x^r)$. In addition, notice that the nth coefficient is determined as follows:

$$[x^n]\sum_{r\geq 1}\frac{1}{r}A_{\mathsf{R}}(x^r)=[x^n]\sum_{r\leq n}\frac{1}{r}A_{\mathsf{R}}(x^r)=\sum_{r|n}[x^{\frac{n}{r}}]\frac{A_{\mathsf{R}}(x)}{r},$$

where $r|n$ refers to all positive integers r dividing n. The same holds true if we replace $A_{\mathsf{R}}(x)$ by $A_{\mathsf{M}}(x)$.

Since the first non-zero coefficient $[x^2]A_{\mathsf{R}}(x)=[x^2]A_{\mathsf{M}}(x)=1$, we can inductively see that for every n, we have $[x^n]A_{\mathsf{R}}(x)=[x^n]A_{\mathsf{M}}(x)$, thus $A_{\mathsf{M}}(x)=A_{\mathsf{R}}(x)$.

Getting formulas for $A_{\mathsf{U}}(x)$ is slightly more involved: the multiplicity $m-3$ of a U-vertex of degree m must be encoded in the generating function.

If a vertex is adjacent to the virtual leg, we define its **restricted degree** as the degree m minus 1. Let us use an auxiliary variable u which marks the restricted degree of the pointed U-vertex. We emphasize that we do not consider the contribution of the virtual leg to the total number of legs n. However, the multiplicity of the U-vertex must be considered with respect to the degree of the vertex and not with respect to the restricted degree. Let us denote by $d=m-1$ the restricted degree of a tree pointed at a U-vertex of degree m. The multiplicity of the U-vertex is $m-3=d-2$.

We write $a_{n,d}$ for the number of pointed trees with n non-virtual legs whose virtual leg is attached to a U-vertex of restricted degree r counted with multiplicity 1. The notation $a_{\mathsf{U}}(x,u):=\sum_{n,\,d\geq 3}a_{n,d}x^nu^d$ refers to the corresponding generating function. Then we have

$$A_{\mathsf{U}}(x)=\sum_{n,d\geq 3}(d-2)a_{n,d}x^nu^d\Bigg|_{u=1}=\frac{\partial}{\partial u}a_{\mathsf{U}}(x,u)\Bigg|_{u=1}-2a_{\mathsf{U}}(x,1). \tag{5.3}$$

Observe that $a_{\mathsf{U}}(x,u)$ satisfies the equation $a_{\mathsf{U}}(x,u)=\mathrm{Mul}_{\geq 3}(u(A_{\mathsf{M}}(x)+A_{\mathsf{R}}(x)+A_{\mathsf{U}}(x)+A_l(x)))$, which arises from the fact that the pointed U-vertex has restricted degree ≥ 3 (or equivalently, degree ≥ 4). Hence

$$\begin{aligned} a_{\mathsf{U}}(x,u) = &\exp\left(\sum_{r=1}^{\infty} \frac{u^r}{r}\left(A_{\mathsf{R}}(x^r) + A_{\mathsf{M}}(x^r) + A_{\mathsf{U}}(x^r) + A_l(x^r)\right)\right) \\ &- 1 - u\left(A_{\mathsf{R}}(x) + A_{\mathsf{M}}(x) + A_{\mathsf{U}}(x) + A_l(x)\right) \\ &- \mathrm{Mul}_2(u(A_{\mathsf{M}}(x) + A_{\mathsf{R}}(x) + A_{\mathsf{U}}(x) + A_l(x))). \end{aligned}$$

Using Equation (5.3) we can write $A_{\mathsf{U}}(x)$ in terms of $a_{\mathsf{U}}(x,1)$ and its derivative at $u = 1$. We set $\psi := \sum_{r=1}^{\infty} \frac{1}{r}\left(A_{\mathsf{R}}(x^r) + A_{\mathsf{M}}(x^r) + A_{\mathsf{U}}(x^r) + A_l(x^r)\right)$ and obtain

$$\begin{aligned} A_{\mathsf{U}}(x) = &\psi \exp(\psi) - \left(A_{\mathsf{R}}(x) + A_{\mathsf{M}}(x) + A_{\mathsf{U}}(x) + A_l(x)\right) \\ &- 2\,\mathrm{Mul}_2(A_{\mathsf{M}}(x) + A_{\mathsf{R}}(x) + A_{\mathsf{U}}(x) + A_l(x)) \\ &- 2\exp(\psi) + 2 + 2\left(A_{\mathsf{R}}(x) + A_{\mathsf{M}}(x) + A_{\mathsf{U}}(x) + A_l(x)\right) \\ &+ 2\,\mathrm{Mul}_2(A_{\mathsf{M}}(x) + A_{\mathsf{R}}(x) + A_{\mathsf{U}}(x) + A_l(x)). \end{aligned} \tag{5.4}$$

Hence, we have three equations relating $A_{\mathsf{R}}(x)$, $A_{\mathsf{M}}(x)$ and $A_{\mathsf{U}}(x)$. Moreover, we know the first non-zero coefficient of each generating function and thus we can iteratively compute the coefficients of $A_{\mathsf{R}}(x)$, $A_{\mathsf{M}}(x)$ and $A_{\mathsf{U}}(x)$ up to an arbitrary degree.

5.1.4 The Dissymmetry Theorem

The Dissymmetry Theorem for trees (see [BLLR97]) provides a general methodology to relate a combinatorial class of trees with given properties to the corresponding classes of rooted trees. A (vertex-)**rooted** tree is a tree with a distinguished vertex. Analogously, we can root a tree at an edge for which we can in addition choose an orientation. Figure 5.4 shows three possible ways of rooting a tree.

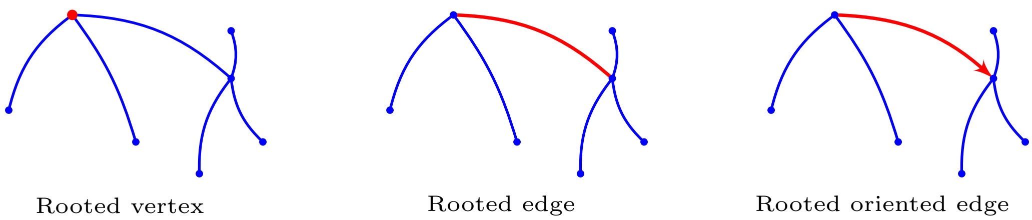

Figure 5.4: Examples of rooted trees.

Notice that different choices of the root of a tree yield different objects in the corresponding family of rooted trees.

Let $\mathcal{T}$ be a class of unrooted trees with a size function $|\cdot|$ (in our case it counts legs) and the corresponding generating function $T(x)$. We define three families of rooted trees: $\mathcal{T}_\circ$ is built from $\mathcal{T}$ by rooting vertices, $\mathcal{T}_{\circ-\circ}$ by rooting edges and $\mathcal{T}_{\circ\to\circ}$ by rooting and orienting edges. Let $T_\circ(x)$, $T_{\circ-\circ}(x)$, and $T_{\circ\to\circ}(x)$ be the corresponding generating functions.

The **Dissymmetry Theorem for trees** asserts that there is a bijection between $\mathcal{T}\cup\mathcal{T}_{\circ\to\circ}$ and $\mathcal{T}_{\circ-\circ}\cup\mathcal{T}_\circ$. This fact translates directly into the equality among generating functions

$$T(x) + T_{\circ\to\circ}(x) = T_{\circ-\circ}(x) + T_v(x).$$

We use the Dissymmetry Theorem to express the UMR-trees in terms of rooted UMR-trees. Let $T(x)$ be the generating function of UMR-trees and denote by $T_\circ(x)$, $T_{\circ-\circ}(x)$, and $T_{\circ\to\circ}(x)$ the generating functions associated to families of UMR-trees with a rooted vertex, a rooted edge, and a rooted oriented edge, respectively. The Dissymmetry Theorem relates these generating functions by the equation

$$T(x) = T_\circ(x) + T_{\circ-\circ}(x) - T_{\circ\to\circ}(x). \tag{5.5}$$

Let us compute each generating function in terms of the pointed families obtained in Subsection 5.1.3. First we see that $T_{\circ-\circ}(x)$ can be written as the sum of generating functions

$$\begin{aligned} T_{\circ-\circ}(x) = T_{\mathsf{M-R}}(x) + T_{\mathsf{M-U}}(x) + T_{\mathsf{M-\bullet}}(x) \\ + T_{\mathsf{R-U}}(x) + T_{\mathsf{R-\bullet}}(x) \\ + T_{\mathsf{U-U}}(x) + T_{\mathsf{U-\bullet}}(x)\,, \end{aligned} \tag{5.6}$$

where the index of each term specifies the type of the end vertices of the rooted edge (for instance, the first term $T_{\mathsf{M-R}}(x)$ counts trees whose rooted edge connects a M-vertex and a R-vertex). If we cut the rooted edge and paste two virtual legs as shown in Figure 5.5, we obtain two pointed UMR-trees. Each term in the sum (5.6) with the exception of $T_{\mathsf{U-U}}(x)$ is the product of the corresponding generating functions of the pointed families.

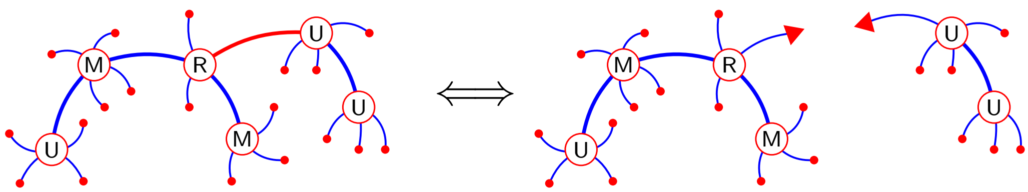

Figure 5.5: A UMR-tree rooted at an edge (colored in red).

To deal with the case of $T_{\mathsf{U}-\mathsf{U}}(x)$, observe that cutting the rooted edge results in a multiset of size 2 of U-pointed trees. We conclude that

$$\begin{aligned} T_{\circ-\circ}(x) &= A_{\mathsf{M}}(x)(A_{\mathsf{R}}(x) + A_{\mathsf{U}}(x) + A_l(x)) + A_{\mathsf{R}}(x)(A_{\mathsf{U}}(x) + A_l(x)) \\ &\quad + \mathrm{Mul}_2(A_{\mathsf{U}}(x)) + A_{\mathsf{U}}(x)A_l(x). \end{aligned} \tag{5.7}$$

A decomposition analogous to Equation (5.6) applies for $T_{\circ\to\circ}(x)$. This generating function can be written as

$$\begin{aligned} T_{\circ\to\circ}(x) &= T_{\mathsf{M}\to\mathsf{R}}(x) + T_{\mathsf{M}\to\mathsf{U}}(x) + T_{\mathsf{M}\to\bullet}(x) \\ &\quad + T_{\mathsf{R}\to\mathsf{M}}(x) + T_{\mathsf{R}\to\mathsf{U}}(x) + T_{\mathsf{R}\to\bullet}(x) \\ &\quad + T_{\mathsf{U}\to\mathsf{M}}(x) + T_{\mathsf{U}\to\mathsf{R}}(x) + T_{\mathsf{U}\to\mathsf{U}}(x) + T_{\mathsf{U}\to\bullet}(x) \\ &\quad + T_{\bullet\to\mathsf{M}}(x) + T_{\bullet\to\mathsf{R}}(x) + T_{\bullet\to\mathsf{U}}(x) \end{aligned}$$

where the index of each term shows the type of the end vertices for the rooted oriented edge. The computations are analogous to the ones for $T_{\circ-\circ}(x)$ with the difference that orienting the edge destroys all symmetries. We get

$$\begin{aligned} T_{\circ\to\circ}(x) &= A_M(x)(A_R(x) + A_U(x) + A_l(x)) \\ &\quad + A_R(x)(A_M(x) + A_U(x) + A_l(x)) \\ &\quad + A_U(x)(A_M(x) + A_R(x) + A_U(x) + A_l(x)) \\ &\quad + A_l(x)(A_R(x) + A_M(x) + A_U(x)). \end{aligned} \tag{5.8}$$

The last generating function to study is $T_\circ(x)$. Observe that $T_\circ(x)$ differs from the sum $A_{\mathsf{R}}(x) + A_{\mathsf{M}}(x) + A_{\mathsf{U}}(x) + A_l(x)$ because, as we already mentioned, pointed trees are different from trees rooted at a vertex. We write

$$T_\circ(x) = T_{\mathsf{R}}(x) + T_{\mathsf{M}}(x) + T_{\mathsf{U}}(x) + T_\bullet(x) \tag{5.9}$$

where the index of each term indicates the type of the rooted vertex. We want to express each term by means of pointed families. It is clear that

$$T_\bullet(x) = A_l(x)(A_{\mathsf{R}}(x) + A_{\mathsf{M}}(x) + A_{\mathsf{U}}(x)) \tag{5.10}$$

because a rooted leg induces canonically a rooted edge. For the other cases observe that $T_{\mathsf{R}}(x) = \mathrm{Mul}_{\geq 3}(A_{\mathsf{M}}(x) + A_{\mathsf{U}}(x) + A_l(x))$, which is obtained by cutting the edges incident with the rooted R-vertex and pasting a virtual leg to the resulting subtrees (this reasoning is analogous to the vertex decomposition represented in Figure 5.3). We conclude that

$$\begin{aligned} T_{\mathsf{R}}(x) &= \mathrm{Mul}_{\geq 3}(A_{\mathsf{M}}(x) + A_{\mathsf{U}}(x) + A_l(x)) \\ &= \mathrm{Mul}_{\geq 2}(A_{\mathsf{M}}(x) + A_{\mathsf{U}}(x) + A_l(x)) - \mathrm{Mul}_2(A_{\mathsf{M}}(x) + A_{\mathsf{U}}(x) + A_l(x)) \\ &= A_{\mathsf{R}}(x) - \mathrm{Mul}_2(A_{\mathsf{M}}(x) + A_{\mathsf{U}}(x) + A_l(x)) \end{aligned} \tag{5.11}$$

and an analogous expression holds for $T_{\mathsf{M}}(x)$. At last, we study $T_{\mathsf{U}}(x)$: let $t_{\mathsf{U}}(x,u)$ be the generating function of trees with a rooted U-vertex, whose multiplicity is not yet encoded and whose degree is tracked by the variable u. Then, $t_{\mathsf{U}}(x,u) = \mathrm{Mul}_{\geq 4}(u(A_{\mathsf{R}}(x) + A_{\mathsf{M}}(x) + A_{\mathsf{U}}(x) + A_l(x)))$ and

$$\begin{aligned} t_{\mathsf{U}}(x,u) = \sum_{n,m\geq 4} t_{n,m}x^n u^m \quad &\Longrightarrow \quad T_{\mathsf{U}}(x) = \sum_{n,m\geq 4} (m-3)t_{n,m}x^n u^m \Big|_{u=1} \\ &= \frac{\partial}{\partial u} t_{\mathsf{U}}(x,u)\Big|_{u=1} - 3t_{\mathsf{U}}(x,1). \end{aligned}$$

Applying the same trick we used for $a_{\mathsf{U}}(x,u)$ in Subsection 5.1.3, we get that

$$\begin{aligned} T_{\mathsf{U}}(x) &= \frac{\partial}{\partial u} t_{\mathsf{U}}(x,u)\Big|_{u=1} - 3t_{\mathsf{U}}(x,1) \qquad (5.12) \\ &= \left(\frac{\partial}{\partial u} - 3\right)\left(a_{\mathsf{U}}(x,u) - u^3\,\mathrm{Mul}_3(A_{\mathsf{R}}(x) + A_{\mathsf{M}}(x) + A_{\mathsf{U}}(x) + A_l(x))\right)\Big|_{u=1} \\ &= A_{\mathsf{U}}(x) - a_{\mathsf{U}}(x,1) + (3-3)\,\mathrm{Mul}_3(A_{\mathsf{R}}(x) + A_{\mathsf{M}}(x) + A_{\mathsf{U}}(x) + A_l(x))) \\ &= A_{\mathsf{U}}(x) - \mathrm{Mul}_{\geq 3}(A_{\mathsf{R}}(x) + A_{\mathsf{M}}(x) + A_{\mathsf{U}}(x) + A_l(x))). \end{aligned}$$

Substituting Equations (5.10),(5.11) and (5.12) in (5.9), we get an expression for $T_\circ(x)$. Finally, we replace (5.7), (5.8) and the expression of $T_\circ(x)$ into Equation (5.5). Once we have an expression for the generating function $T(x)$ in terms of the generating functions $A_R(x)$, $A_M(x)$, and $A_U(x)$, we can compute its coefficients up to an arbitrary degree using `Maple`:

$$2x^3+4x^4+10x^5+27x^6+78x^7+246x^8+818x^9+2871x^{10}+10446x^{11}+39358x^{12}+\ldots$$

The scripts used for computations are available at [Rué].

5.2 Asymptotic analysis of $T(x)$

5.2.1 Preliminaries

In Section 5.1 we considered the generating function $A(x)$ of a combinatorial class $\mathcal{A}$ as a *formal* power series. We now interpret $A(x)$ in the context of complex analysis as an *analytic* power series and study its singularities.

An analytic power series $A(x)$ with disc of convergence D has no singularity inside D and at least one singularity on the boundary of D ([FS09, Thm. IV.5]). The singularities located on the boundary of D are called **dominant**. In addition, if $A(x)$ has non-negative coefficients, one of its dominant singularities is a positive real number as stated by Pringsheim's Theorem ([FS09, Thm. IV.6]).

Our interest in the singularities of $A(x)$ is motivated by the fact that the position and the nature of the dominant singularities provide information regarding the coefficients of $A(x)$ and, in particular, their asymptotic growth rate. We refer to [FS09, Sect. VI.4] for a detailed description of the singularity analysis process.

When $A(x)$ is not provided explicitly, but satisfies a functional equation, we first have to determine the dominant singularities. The case we are interested in requires to find the singularities of functions which satisfy a system of functional equations.

Let us explain how to get an asymptotic estimate of the coefficients for a function $A(x)$, when there is a unique dominant singularity ρ, and, in addition, we have an expansion around this singularity. The expansion around ρ is defined on a dented domain. A **dented domain** $\Delta(\phi, R)$ at $\rho \in \mathbb{C}$ is the set

$$\{x \in \mathbb{C} : x \neq \rho,\ |x| < R,\ |\mathrm{Arg}(x - \rho)| > \phi\},$$

for $|\rho| < R \in \mathbb{R}$ and $0 < \phi < \pi/2$. The shape of a dented domain is that of an indented disk as shown in Figure 5.6.

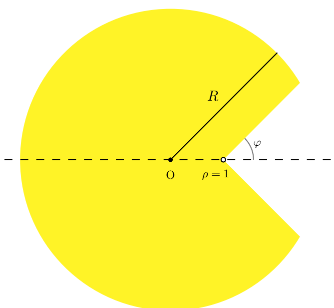

Figure 5.6: Dented domain at $\rho = 1$.

We state here a simplified version of the **Transfer Theorem** for singularity analysis ([Drm09, Cor. 2.16]).

Theorem 5.2.1 (Transfer Theorem)**.** *Let $A(x)$ be a function with a unique dominant singularity ρ. Assume that $A(x)$ is analytic in a dented domain*

$\Delta(\phi, R)$ at ρ and that as $x \to \rho$ in $\Delta(\phi, R)$, $A(x)$ admits an expansion

$$A(x) = C\left(1 - \frac{x}{\rho}\right)^{-\alpha} + O\left(\left(1 - \frac{x}{\rho}\right)^{-\alpha+1}\right),$$

where $\alpha \in \mathbb{Q}$ and $\alpha \notin \{0, -1, -2, \dots\}$. Then, as $n \to \infty$

$$[x^n]A(x) \approx C\frac{1}{\Gamma(\alpha)} \cdot n^{\alpha-1} \cdot \rho^{-n},$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$ denotes the classical Gamma function.

As mentioned before, in some cases there is a way to find the real dominant singularities of a set of functions which are implicitly described by a system of functional equations.

Let $\mathbf{F}(x, \boldsymbol{y}) = (F_1(x, \boldsymbol{y}), \dots, F_k(x, \boldsymbol{y}))$ be a vector of functions $F_i(x, \boldsymbol{y})$ with complex variables x and $\boldsymbol{y} = (y_1, \dots, y_k)$ which are analytic around $(0, \mathbf{0})$ for $i = 1, \dots, k$. In addition, we require all functions $F_i(x, \boldsymbol{y})$ to have non-negative Taylor coefficients around $(0, \mathbf{0})$.

Let $\mathbf{A}(x) = (A_1(x), \dots, A_k(x))$ be a vector of functions defined in a neighborhood U_0 of the origin and such that $\mathbf{A}(0) = \mathbf{0}$. Suppose that $\mathbf{A}(x)$ satisfies the system of functional equations $\mathbf{A}(x) = \mathbf{F}(x, \mathbf{A}(x))$, that is, for all $x \in U_0$

$$\begin{aligned} A_1(x) &= F_1(x, \mathbf{A}(x)) \\ A_2(x) &= F_2(x, \mathbf{A}(x)) \\ &\vdots \\ A_k(x) &= F_k(x, \mathbf{A}(x)). \end{aligned}$$

The **dependency graph** $G = (V, \mathcal{D})$ associated to the system $\mathbf{A}(x) = \mathbf{F}(x; \mathbf{A}(x))$ is the directed graph with vertex set $\{A_1, \dots, A_k\}$ and $\overrightarrow{A_iA_j} \in \mathcal{D}$ if and only if $\frac{\partial F_i(x, \mathbf{A}(x))}{\partial A_j(x)} \neq 0$ (the condition means that F_i depends on the function $A_j(x)$). A dependency graph is called **strongly connected** if every pair of vertices is connected by a directed path.

Theorem 5.2.2. *Let $\boldsymbol{A}(x) = \boldsymbol{F}(x; \boldsymbol{A}(x))$ be a system of functional equations satisfying the conditions described above. Additionally, assume that the dependency graph associated to $\boldsymbol{A}(x)$ is strongly connected. Denote by $\boldsymbol{I}_k$ the $k \times k$ identity matrix and by $\mathrm{Jac}(\boldsymbol{F})$ the $k \times k$-matrix where $\mathrm{Jac}(\boldsymbol{F})_{ij} = \frac{\partial F_i(x, \boldsymbol{y})}{\partial y_j}$. If the system*

$$\begin{cases} \boldsymbol{y} &= \boldsymbol{F}(x; \boldsymbol{y}) \\ 0 &= \det\left(\boldsymbol{I}_k - \mathrm{Jac}(\boldsymbol{F})\right) \end{cases} \tag{5.13}$$

has a unique positive real solution $(x_0, \boldsymbol{y}_0) \in \mathbb{R}^{k+1}_{>0}$*, then the functions* $A_i(x)$ *have non-negative coefficients and a unique dominant singularity at* x_0*. Moreover, they have a square-root expansion in a domain dented at* x_0*.*

The statement in its full generality is available in [Drm09, Sect. 2.2.5]). For an application of this theorem in a setting similar to the one we are going to approach, we refer to [DFK+11].

The square-root expansion of $A_i(x)$ is of the form

$$\overline{A_i}(X) = a_0 + a_1 X + a_2 X^2 + \dots$$

in a domain dented at x_0, where $X = \sqrt{1 - \frac{x}{x_0}}$. In other words, the theorem says that every function $A_i(x)$ can be written in the form

$$g_i(x) + h_i(x)\sqrt{1 - \frac{x}{x_0}},$$

where the functions $g_i(x)$ and $h_i(x)$ are analytic at x_0. From Theorem 5.2.1 we see that only the part related to $h_i(x)$ contributes to the asymptotic estimate of the coefficients.

5.2.2 Asymptotic analysis

In this section we will get asymptotic estimates for $[x^n]T(x)$. More precisely, since we proved in Subsection 5.1.4 that the generating function $T(x)$ can be expressed in terms of the generating functions $A_{\mathsf{R}}(x)$, $A_{\mathsf{M}}(x)$ and $A_{\mathsf{U}}(x)$, we study the system of functional equations based on (5.1), (5.2), and (5.4) and check computationally that it is of type (5.13). From the unique positive real solution $(x_0, \boldsymbol{y}_0)$ of the system, we obtain the unique dominant singularity $\rho = x_0$ for the functions $A_{\mathsf{R}}(x)$, $A_{\mathsf{M}}(x)$ and $A_{\mathsf{U}}(x)$ and by Theorem 5.2.2 we know that they have a square-root expansions in a domain dented at ρ.

The generating function $T(x)$ has its dominant singularity at ρ and its singular expansion follows easily. Moreover, the Transfer Theorem 5.2.1 can be applied to $T(x)$ to obtain asymptotic estimates for its coefficients. In particular, we get the growth constant $\rho^{-1} \approx 4.88052854$.

It is shown in [DFK+11] that the number of unlabelled biconnected series-parallel graphs on n vertices grows exponentially in n and the growth constant is ≈ 8.05153567. Since series-parallel graphs are also 2-level matroids, we try to generalize the methodologies from graphs to matroids.

Despite several similarities, there are few caveats that we have to take into account:

(i) matroids do not have a vertex structure; instead we count them by the number of elements in the ground set, which will also pay off when relating our results to the enumeration of 2-level base polytopes;

(ii) the tree decompositions of series-parallel graphs have only R-vertices and M-vertices. General 2-level matroids are constructed using also the U-vertices, that is, a much wider variety of building blocks.

(iii) series-parallel graphs with different graph realizations can correspond to isomorphic matroids (see Figure 5.7) and must be counted only once in the matroid setting.

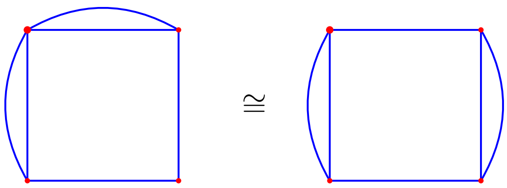

Figure 5.7: Different graphs, isomorphic matroids.

It is clear that even though we use analogous tools, the comparison between the asymptotic estimates that we get for 2-level matroids and the results presented in [DFK+11] is not meaningful.

The first step for getting the asymptotic estimate of the coefficients of $T(x)$ is to determine the dominant singularities and the singular expansions of $A_{\mathsf{R}}(x), A_{\mathsf{M}}(x)$ and $A_{\mathsf{U}}(x)$ up to the desired precision. In particular, it is enough to consider the expansion up to X^3 to avoid a cancellation of all relevant terms when computing $T(x)$.

Lemma 5.2.3. *The generating functions $A_{\mathsf{R}}(x), A_{\mathsf{M}}(x)$ and $A_{\mathsf{U}}(x)$ have a unique dominant singularity at $\rho \approx 0.20489584$. The singular expansions of $A_{\mathsf{R}}(x)$, $A_{\mathsf{M}}(x)$ and $A_{\mathsf{U}}(x)$ in a domain dented at ρ are of the form*

$$\begin{aligned} A_{\mathsf{R}}(X) = A_{\mathsf{M}}(X) &= A_0 + A_1X + A_2X^2 + A_3X^3 + O(X^4), \\ A_{\mathsf{U}}(X) &= U_0 + U_1X + U_2X^2 + U_3X^3 + O(X^4), \end{aligned}$$

where $X = \sqrt{1 - x/\rho}$, $A_0{\approx}0.13529174$, $A_1{\approx}{-0.23137622}$, $A_2{\approx}0.04653888$, $A_3{\approx}0.06281332$, $U_0{\approx}0.06921673$, $U_1{\approx}{-0.19340420}$, $U_2{\approx}0.15045323$ and $U_3{\approx}0.01018058$.

Proof. Since $A_{\mathsf{R}}(x) = A_{\mathsf{M}}(x)$, we can replace $A_{\mathsf{M}}(x)$ by $A_{\mathsf{R}}(x)$ in (5.1) and (5.4). For $\mathbf{A}(x) = (A_{\mathsf{R}}(x), A_{\mathsf{U}}(x))$, we can obtain from these two equations a system of the form

$$\begin{cases} A_R(x) = F_1(x, \mathbf{A}(x)) \\ A_U(x) = F_2(x, \mathbf{A}(x)) \end{cases} \tag{5.14}$$

satisfying the conditions of Theorem 5.2.2. In fact, if $A_{\mathsf{R}}(x)$ and $A_{\mathsf{U}}(x)$ have a dominant singularity ρ, then the functions $A_{\mathsf{M}}(x^r)$ and $A_{\mathsf{U}}(x^r)$ for $r \geq 2$ have dominant singularity $\rho^{\frac{1}{r}} > \rho$ (since $\rho < 1$) and thus, are analytic at ρ. With the same reasoning the exponential functions such as

$$\exp\left(\sum_{r=2}^{\infty} \frac{1}{r}\left(A_{\mathsf{R}}(x^r) + A_{\mathsf{U}}(x^r) + A_l(x^r)\right)\right)$$

are analytic at ρ. Hence, we can approximate all these functions up to arbitrary precision by their truncated Taylor series at ρ, which can be computed by means of an iterative algorithm.

After replacing the truncated Taylor series in (5.1) and (5.4), we obtain two equations depending only on x, $A_{\mathsf{R}}(x)$, and $A_{\mathsf{U}}(x)$. This yields the system (5.14) to which we apply Theorem 5.2.2: we add the equation $0 = \det\left(\mathbf{I}_k - \mathrm{Jac}(\mathbf{F})\right)$ and solve the three equations in three variables by means of `Maple` computations. We find out that there is a unique solution

$$(x_0, \widehat{A_{\mathsf{R}}}, \widehat{A_{\mathsf{U}}}) \approx (0.20489584, 0.13529174, 0.06921673).$$

Thus, the singularity of both $A_{\mathsf{R}}(x)$ and $A_{\mathsf{U}}(x)$ is located at $\rho = x_0 \approx 0.20489584$, and the truncated singular square-root expansions in a domain dented at ρ are of the form

$$\begin{aligned} \overline{A_{\mathsf{R}}}(X) = \overline{A_{\mathsf{M}}}(X) &= A_0 + A_1 X + A_2 X^2 + A_3 X^3, \\ \overline{A_{\mathsf{U}}}(X) &= U_0 + U_1 X + U_2 X^2 + U_3 X^3, \end{aligned}$$

where $X = \sqrt{1 - x/\rho}$ and A_i and U_i are computable constants for $i = 0, 1, 2, 3$. In particular, we have $A_0 = \widehat{A_{\mathsf{R}}}$ and $U_0 = \widehat{A_{\mathsf{U}}}$. Let $\overline{\mathbf{A}}(X) = (\overline{A_R}(X), \overline{A_U}(X))$. By substituting the truncated singular expansions as follows

$$\begin{cases} \overline{A_R}(X) = F_1\left(\rho(1 - X^2), \overline{\mathbf{A}}(X)\right) \\ \overline{A_U}(X) = F_2\left(\rho(1 - X^2), \overline{\mathbf{A}}(X)\right), \end{cases}$$

and equating the coefficient of same degree, we obtain a system of equations in the A_i's and the U_i's, whose solution gives the approximate values reported in the statement of the theorem. □

Once we have computed the (truncated) singular square-root expansions at ρ for $A_{\mathsf{R}}(x)$, $A_{\mathsf{M}}(x)$, and $A_{\mathsf{U}}(x)$, we focus on $T(x)$ and get asymptotic estimates for its coefficients $[x^n]T(x)$.

Theorem 5.2.4. *Let $T(x)$ be the generating function of* UMR*-trees. We have the asymptotic estimate*

$$[x^n]T(x) \approx C \cdot n^{-5/2} \cdot \rho^{-n} \, (1 + o(1)) ,$$

where $C \approx 0.07583455$ and $\rho \approx 0.20489584$.

Proof. We use the square-root expansions at ρ for $A_{\mathsf{R}}(x)$, $A_{\mathsf{M}}(x)$ and $A_{\mathsf{U}}(x)$ obtained in Lemma 5.2.3 together with the expressions in equations (5.5)-(5.12) and we compute the square-root singular expansion

$$T(X) = T_0 + T_2X^2 + T_3X^3 + O(X^4),$$

where $X = \sqrt{1 - x/\rho}$ and $T_0 \approx 0.03457946$, $T_2 \approx -0.18596384$ and $T_3 \approx 0.17921766$. Observe that the constant multiplying X in the singular expansion is equal to zero: the cancellation of degree one terms is a consequence of the application of the Dissymmetry Theorem for trees. The first term of type $(1 - x/\rho)^{-\alpha}$ such that $\alpha \notin \{0, -1, -2, \ldots\}$ comes from T_3X^3. We apply the Transfer Theorem 5.2.1 to $T(X) - T_0 - T_2X^2$ (whose coefficients are asymptotically the same as the coefficients of $T(x)$) and obtain the asymptotic estimate for the coefficients of $T(x)$. $\square$

The coefficients of $T(x)$ encode the number of connected 2-level matroids. We consider now the generating function $T^{nc}(x)$ which counts the number of 2-level matroids, including the non-connected ones. For this enumeration, we need to consider UMR-forests instead of UMR-trees. Equivalently, we apply the multiset construction over UMR-trees and thus

$$T^{nc}(x) = \mathrm{Mul}(T(x)) = \exp\left(\sum_{r=1}^{\infty} \frac{1}{r}(T(x^r))\right).$$

Observe that

$$\exp\left(\sum_{r=1}^{\infty} \frac{1}{r}(T(x^r))\right) = \exp(T(x)) \exp\left(\sum_{r=2}^{\infty} \frac{1}{r}(T(x^r))\right),$$

and the second factor is analytic at $x = \rho$. Hence, in a domain dented at $x = \rho$ the singular expansion of $\mathrm{Mul}(T(x))$ is

$$\mathrm{Mul}(T(x)) = \exp(T_0 + T_2X^2 + T_3X^3 + O(X^4)) \exp\left(g(x)\right),$$

where we replace $T(x)$ with its square-root expansion at ρ described in Theorem 5.2.4 and the second exponent with its truncated Taylor series $g(x)$ at ρ. We compute the square-root expansion

$$T^{nc}(x) = T_0^{nc} + T_2^{nc} X^2 + T_3^{nc} X^3 + O(X^4),$$

and we obtain $T_0^{nc} \approx 1.03526853$, $T_2^{nc} \approx -0.19252251$, $T_3^{nc} \approx 0.18553841$. Applying Theorem 5.2.1, we conclude that

$$[x^n]\,\mathrm{Mul}(T(x)) \approx C' \cdot n^{-5/2} \cdot \rho^{-n},$$

with $C' \approx 0.07850913$.

It is interesting to observe that

$$\lim_{n\to\infty} \frac{[x^n]T(x)}{[x^n]T^{nc}(x)} \approx \frac{C}{C'} = 0.9659329 \neq 1.$$

A conjecture of Mayhew, Newman, Welsh, Whittle [MNWW11] claims that as $n \to \infty$, the ratio between the number of connected matroids on n elements and the number of matroids on n elements is 1. On the other hand, Corollary 4.2.10 states that all 2-level matroids are positroids. In [ARW15, Thm. 10.7] it is shown that this ratio for positroids is $\frac{1}{e^2} \approx 0.1353$.

Even though the class $\mathcal{M}_2^{\mathsf{Lev}}$ is contained in the class of positroids and, as $n \to \infty$, the ratio between the number of connected 2-level matroids on n elements and the number of 2-level matroids on n elements is different from 1, this result appears more reassuring with respect to the conjecture that most matroids are connected.

5.2.3 Self-duality

This part is devoted to show that the number of self-dual connected 2-level matroids is asymptotically negligible compared to the number of connected 2-level matroids. This fact will be relevant when dealing with the enumeration of 2-level matroid base polytopes.

Let us begin with a proposition from [Oxl11, Prop. 7.1.22].

Proposition 5.2.5. *Let M_1 and M_2 be two matroids and $e_i \in E(M_i)$. Then*

$$((M_1, e_1) \oplus_2 (M_2, e_2))^* = (M_1^*, e_1) \oplus_2 (M_2^*, e_2).$$

Let M be a matroid and T_M its unique tree decomposition, then Proposition 5.2.5 implies that the tree decomposition of M^* has the same tree structure of T_M. Moreover, we replace each vertex label N_i with its dual matroid N_i^*.

We now consider the self-dual connected 2-level matroids. The vertex labels are chosen among uniform matroids, and the operation of duality converts M-vertices into R-vertices and vice versa, and U-vertices into U-vertices. Since $U^*_{n,k} = U_{n,n-k}$, the self-dual U-vertices are necessarily labelled by $U_{2k,k}$. Moreover, for technical reasons, we consider virtual legs and legs to be self-dual.

Provided that matroid duality extends naturally to UMR-trees, we estimate the contribution of the family of self-dual UMR-trees to the total number of UMR-trees. We start analyzing the pointed situation: we write $A_{\mathsf{R}}(x) = S_{\mathsf{R}}(x) + N_{\mathsf{R}}(x)$, $A_{\mathsf{M}}(x) = S_{\mathsf{M}}(x) + N_{\mathsf{M}}(x)$ and $A_{\mathsf{U}}(x) = S_{\mathsf{U}}(x) + N_{\mathsf{U}}(x)$, where the generating functions $S_{\mathsf{R}}(x)$, $S_{\mathsf{M}}(x)$ and $S_{\mathsf{U}}(x)$ encode self-dual trees whose pointed vertex is of type R, M, and U, respectively. The generating functions $N_{\mathsf{R}}(x)$, $N_{\mathsf{M}}(x)$ and $N_{\mathsf{U}}(x)$ encode trees pointed at a vertex which are not self-dual. Observe that in particular $S_{\mathsf{R}}(x) = S_{\mathsf{M}}(x) = 0$, because the dual of each R-vertex is an M-vertex. Thus there are no self-dual UMR-trees pointed at either a R-vertex or a M-vertex.

We also use a similar notation for unrooted UMR-trees. We write $T(x) = S(x) + N(x)$, where $S(x)$ is the generating function associated to self-dual (unrooted) trees.

The next lemma tells us that the contribution of self-dual pointed trees to the number of pointed trees is exponentially small.

Lemma 5.2.6. *The following estimate holds:*

$$[x^n]S_{\mathsf{U}}(x) = o([x^n]A_{\mathsf{U}}(x)).$$

Proof. We want to analyze $S_{\mathsf{U}}(x)$. In this situation, the pointed vertex is a U-vertex associated to a uniform matroid of type $U_{2k,k}$. Hence, we notice that the degree of the pointed vertex determines the rank of the vertex label, which implies that the multiplicity in the counting is 1. Moreover, the possible restricted degrees of the vertex belong to the set $\Lambda = \{3, 5, 7, \dots\}$.

The collection of pointed subtrees glued to the U-vertex is a multiset of pairs of pointed trees such that one is the dual of the other, followed by a multiset of odd size of self-dual pointed trees. Hence,

$$\begin{aligned} S_{\mathsf{U}}(x) &= \mathrm{Mul}_{\{3,5,7,\dots\}}(S_{\mathsf{U}}(x) + A_l(x)) \qquad (5.15)\\ &\quad + \mathrm{Mul}_{\geq 1}(N_{\mathsf{R}}(x^2) + N_{\mathsf{M}}(x^2) + N_{\mathsf{U}}(x^2))\mathrm{Mul}_{\{1,3,5,7,\dots\}}(S_{\mathsf{U}}(x) + A_l(x)). \end{aligned}$$

Let η be the dominant singularity of $S_{\mathsf{U}}(x)$. It is obvious that $\eta \geq \rho$, because the family of self-dual U-pointed trees is contained in the family of U-pointed trees. We need to show that $\eta > \rho$.

Equation (5.15) can be analyzed with the same methods used in the proof of Lemma 5.2.3. However, for our purposes it is enough to bound the coefficients of $S_{\mathsf{U}}(x)$ by means of rough estimates. Observe that $\mathrm{Mul}_{\geq 3}(S_{\mathsf{U}}(x)+A_l(x)) \geq \mathrm{Mul}_{\{3,5,7,\dots\}}(S_{\mathsf{U}}(x)+A_l(x))$ and

$$\begin{aligned}&\mathrm{Mul}_{\geq 1}(N_{\mathsf{R}}(x^2)+N_{\mathsf{M}}(x^2)+N_{\mathsf{U}}(x^2))\mathrm{Mul}_{\geq 1}(S_{\mathsf{U}}(x)+A_l(x)) \geq \\ &\mathrm{Mul}_{\geq 1}(N_{\mathsf{R}}(x^2)+N_{\mathsf{M}}(x^2)+N_{\mathsf{U}}(x^2))\mathrm{Mul}_{\{1,3,5,7,\dots\}}(S_{\mathsf{U}}(x)+A_l(x)).\end{aligned}$$

Hence, if $s(x)$ satisfies the equation

$$\begin{aligned}s(x) = {} &\mathrm{Mul}_{\geq 3}(s(x)+A_l(x)) \\ &+ \mathrm{Mul}_{\geq 1}(N_{\mathsf{R}}(x^2)+N_{\mathsf{M}}(x^2)+N_{\mathsf{U}}(x^2))\mathrm{Mul}_{\geq 1}(s(x)+A_l(x)),\end{aligned} \tag{5.16}$$

then $S_{\mathsf{U}}(x) \leq s(x)$. Let γ be the real dominant singularity of $s(x)$. Observe that this singularity arises either from the square-root singularity of the terms $A_{\mathsf{R}}(x^2) = N_{\mathsf{R}}(x^2)$, $A_{\mathsf{M}}(x^2) = N_{\mathsf{M}}(x^2)$ and $A_{\mathsf{U}}(x^2) - S_{\mathsf{U}}(x^2) = N_{\mathsf{U}}(x^2)$ at $\sqrt{\rho} \approx 0.45265421$ or from a branch point of Equation (5.16).

A branch point $x = \beta$ can be obtained as a solution to the pair of equations $s(x) = F(s(x), x)$, $1 = F_{s(x)}(s(x), x)$, where $F(s(x), x)$ is the right-hand side of 5.16. Computations performed in `Maple` show that there is no branch point $\beta \leq \sqrt{\rho}$. It follows that $\sqrt{\rho}$ is the dominant singularity of $s(x)$.

To conclude, $[x^n]s(x) \geq [x^n]S_{\mathsf{U}}(x)$ and the coefficients of $s(x)$ have exponential growth of order $\rho^{-n/2}$, thus exponentially small compared with ρ^{-n}. $\square$

Since the number of pointed self-dual UMR-trees is exponentially small compared to the total number of pointed trees, we can prove that the number of self-dual unrooted UMR-trees is also exponentially small compared to the total number of unrooted UMR-trees.

Proposition 5.2.7. *The following estimate hold:*

$$[x^n]S(x) = o([x^n]T(x)).$$

Proof. Let us split the class of self-dual UMR-trees by the type of the center. The **center** of a connected graph is the set of vertices that minimize the maximal path-distance from other vertices in the graph. The center of a tree consists of a single vertex or two adjacent vertices (thus an edge). Accordingly, we split the generating function $S(x) = S_{\circ}(x) + S_{\circ-\circ}(x)$. For each summand we find an upper bound.

Consider the self-dual trees whose center is a vertex: the center is necessarily a U-vertex labelled by a matroid of type $U_{2k,k}$. In this case the degree of the pointed vertex determines the rank of the U-vertex, which counts with multiplicity one. Hence, we have the rough upper bound $S_\circ(x) \leq \mathrm{Mul}(A_{\mathsf{R}}(x^2)+A_{\mathsf{M}}(x^2)+N_U(x^2))\mathrm{Mul}(S_{\mathsf{U}}(x)+A_l(x))$, whose dominant singularity is strictly bigger than ρ, because of previous considerations on $A_{\mathsf{R}}(x^2)$, $A_{\mathsf{M}}(x^2)$, $A_{\mathsf{U}}(x^2) \geq N_{\mathsf{U}}(x^2)$, and $S_{\mathsf{U}}(x)$ (Lemma 5.2.6).

As for the self-dual trees whose center is an edge, we consider the pair of pointed UMR-trees that arise when cutting the center of the tree and pasting a virtual leg for each side of the cut. Two situations may happen:

(i) Each tree is self-dual;

(ii) Each tree is not self-dual, but one is the dual of the other.

In both cases (i) and (ii) we can easily find an upper bound: $S_{\mathsf{U}}(x)^2$ and $A_{\mathsf{R}}(x^2)+A_{\mathsf{M}}(x^2)+N_U(x^2)$, respectively. It follows that $S_{\circ-\circ}(x) \leq S_{\mathsf{U}}(x)^2 + A_{\mathsf{R}}(x^2)+A_{\mathsf{M}}(x^2)+N_U(x^2)$. Again, the dominant singularity of $S_{\circ-\circ}(x)$ is strictly bigger than ρ. Hence the result follows. □

5.2.4 Many 2-level polytopes from matroids

In Section 2.2 we highlighted the difficulties in enumerating 2-level polytopes. Notice that connected 2-level matroids yield a class of 2-level polytopes and in Section 3.2 we show that this class is not contained in any of the known classes of Subsection 2.2.3. The bijection with UMR-trees eases the approach to the enumeration of 2-level matroids. An additional effort results in an asymptotic estimate for the number of combinatorially non-equivalent 2-level base polytopes. Some properties of base polytopes presented in Section 1.2.2 become useful in this context: in particular, Corollary 1.2.14 reduces our investigation to the number of non-congruent 2-level base polytopes, instead of the combinatorially non-equivalent ones.

Theorem 5.2.8. *The asymptotic number of combinatorially non-equivalent $(n-1)$-dimensional 2-level base polytopes of connected matroids is*

$$c \cdot n^{-5/2} \cdot \rho^{-n},$$

where $c \approx 0.03791727$ and $\rho^{-1} \approx 4.88052854$.

Proof. Every connected 2-level matroid M on n elements is, by definition, associated with a 2-level base polytope P_M. The connectedness of M implies that $\dim(P_M) = n-1$ and by Theorem 1.2.13, there is only another matroid whose base polytope is congruent to P_M, namely M^*.

Let us denote the number of connected 2-level matroids on n elements by $L_2(n)$ and the number of self-dual ones by $S_2(n)$. The number of non-congruent $(n-1)$-dimensional 2-level base polytopes is $\frac{L_2(n)+S_2(n)}{2}$.

Applying the structural results of Section 4.1 and using the notation of Subsection 5.1.4, we easily see that $L_2(n) = [x^n]T(x)$ and $S_2(n) = [x^n]S(x)$. We do not have closed formulas for the coefficients of the generating functions, but we are able to provide asymptotic estimates: by Theorem 5.2.4, the number of UMR-trees is asymptotically equal to $C \cdot n^{-5/2} \cdot \rho^{-n}$, where $C \approx 0.07583455$ and $\rho \approx 0.20489584$. Due to Proposition 5.2.7, the contribution of self-dual UMR-trees to this asymptotic is exponentially small. Hence, the number of non-self-dual UMR-trees is asymptotically equal to the number of UMR-trees. Finally, we want to count a matroid and its dual as one, since they yield congruent base polytopes. Thus we divide by 2 and obtain that the asymptotic estimate for the number of 2-level base polytopes equals the asymptotic estimate for the number of UMR-trees except for the constant, which is $c = \frac{C}{2}$. □

The number of combinatorially non-equivalent 0/1-polytopes in dimension n is bounded from below by the number $2^{2^{n-2}}$ as stated in Theorem 2.1.1. An asymptotic exponential lower bound on the number of 2-level polytopes follows immediately from Theorem 5.2.8.

Observe that the constant C' for the asymptotic estimate of the coefficients of $T^{nc}(x)$ is slightly bigger than the constant for $T(x)$. However, this asymptotic estimate is not suitable to get a better lower bound on the number of 2-level polytopes in fixed dimension, because the dimension of a 2-level base polytope depends on the number of connected components of the matroid (Proposition 1.2.6).

Chapter 6

Matroid ideals and cone-ranks

The first two sections of the chapter study the degree of the vanishing ideal and the psd rank for matroids in $\mathcal{M}_2^{\mathsf{Lev}}$, respectively. Results obtained in these sections complete the proof of Theorem 0.0.2. The last two sections contain results about hypersimplices that we believe can be generalized to 2-level matroids. More specifically, the third section is concerned with the extension complexity of hypersimplices, while the fourth one deals with their vanishing ideal.

6.1 Degree of base configurations

For a point configuration $V \subset \mathbb{R}^d$, the **vanishing ideal** of V is

$$I(V) \ := \ \{f(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_d] : f(\boldsymbol{v}) = 0 \text{ for all } \boldsymbol{v} \in V\}.$$

We say that V is of **degree** $\leq k$ if there is a minimal generating set of $I(V)$ of maximal degree k. We write $\mathsf{Gen}(V) = k$ if k is the smallest positive integer such that $I(V)$ is of degree $\leq k$. We define

$$\mathcal{V}_k^{\mathsf{Gen}} \ := \ \{V \text{ point configuration} : \mathsf{Gen}(V) \leq k\}$$

It is clear that $\mathsf{Gen}(V)$ is an affine invariant and, since all point configurations are finite, we get the following.

Proposition 6.1.1. *The class* $\mathcal{V}_k^{\mathsf{Gen}}$ *is face-hereditary.*

Proof. Let $H = \{\boldsymbol{p} \in \mathbb{R}^d : \ell(\boldsymbol{p}) = 0\}$ be a supporting hyperplane for V. The vanishing ideal of $V' = V \cap H$ is the ideal generated by $I(V)$ and $\ell(\mathbf{x})$. Since $\ell(\mathbf{x})$ is linear, this then shows that $\mathsf{Gen}(V') \leq \mathsf{Gen}(V)$. $\square$

The relation to point configurations of Theta rank 1 is given by the following proposition which is implicit in [GPT10].

Proposition 6.1.2. *If $V \subset \mathbb{R}^d$ is a point configuration of Theta rank 1, then* $\mathsf{Gen}(V) \leq 2$.

Proof. From Theorem 0.0.1 we infer that V is 2-level. We may assume that the configuration is spanning and hence up to affine equivalence, the 2-level polytope $P = \mathrm{conv}(V)$ is given by

$$P = \left\{ \boldsymbol{p} \in \mathbb{R}^d : \begin{array}{rcll} 0 \leq & p_i & \leq 1 & \text{for } i = 1, \ldots, d \\ \delta_j^- \leq & \ell_j(\boldsymbol{p}) & \leq \delta_j^+ & \text{for } j = 1, \ldots, m \end{array} \right\}$$

for unique facet-defining linear functions $\ell_j(\mathbf{x})$ and $\delta_j^- < \delta_j^+$. In particular, $V \subseteq \{0,1\}^d$. We claim that $I(V)$ is generated by the quadrics

$$x_i(x_i - 1) \quad \text{for } 1 \leq i \leq d, \quad \text{and} \quad (\ell_j(\mathbf{x}) - \delta_j^-)(\ell_j(\mathbf{x}) - \delta_j^+) \quad \text{for } 1 \leq j \leq m.$$

The vanishing locus U defined by the quadrics is a smooth subset of $\{0,1\}^d$. Thus, the polynomials span a real radical ideal. Now, every vertex $\boldsymbol{v} \in V \subseteq \{0,1\}^d$ satisfies $\ell_j(\boldsymbol{v}) = \delta_j^{\pm}$. Hence $V \subseteq U$. Conversely, every $\boldsymbol{u} \in U$ is a vertex of P and hence $U \subseteq V$. □

The following example illustrates the fact that the degree of generation is invariant under projective transformations while Theta rank is not.

Example 6.1.3. To see that generation in degrees ≤ 2 is necessary for Theta rank 1 but not sufficient, consider the planar point configuration $V = \{(1,0),(0,1),(2,0),(0,2)\}$. The configuration is clearly not 2-level and hence not Theta 1, however the vanishing ideal $I(V)$ is generated by x_1x_2 and $(x_1 + x_2 - 1)(x_1 + x_2 - 2)$ which implies $\mathsf{Gen}(V) \leq 2$.

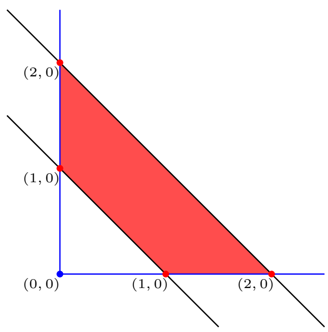

Figure 6.1: 3-level configuration in $\mathcal{V}_2^{\mathsf{Gen}}$.

The vanishing ideals of base configurations are easy to write down explicitly.

Proposition 6.1.4. *Let $M = (E, \mathcal{B})$ be a matroid of rank r. The vanishing ideal for V_M is generated by*

$$x_e^2 - x_e \text{ for all } e \in E, \quad \ell_E(\mathbf{x}) - r, \quad \text{and} \quad \mathbf{x}^C = \prod_{e \in C} x_e \text{ for all circuits } C \subset E.$$

Proof. Any solution to the first two sets of equations is of the form $\mathbf{1}_B$ for some $B \subseteq E$ with $|B| = r$. For the last set of equations, we note that $(\mathbf{1}_B)^C = 0$ for all circuits C if and only if B does not contain a circuit. This is equivalent to $B \in \mathcal{B}$. Arguments similar to those used in the proof of Proposition 6.1.2 show that the polynomials generate a real radical ideal. □

Let us write $\mathcal{M}_k^{\mathsf{Gen}}$ for the class of matroids M with $\mathsf{Gen}(V_M) \leq k$. The previous proposition is a little deceiving in the sense that it suggests a direct connection between the size of circuits and the degree of generation. This is not quite true. Indeed, let us consider the 2-level matroid $M = U_{n,n-1} \oplus U_{n,1}$. Then both M and M^* have a circuit of cardinality n but $M \in \mathcal{M}_2^{\mathsf{Lev}} \subseteq \mathcal{M}_2^{\mathsf{Gen}}$ by Theorem 3.2.13 and Proposition 6.1.2. The main result of this section is that for base configurations the condition of Proposition 6.1.2 is also sufficient.

Theorem 6.1.5. *Let M be a matroid. Then V_M is Theta 1 if and only if $\mathsf{Gen}(V_M) \leq 2$.*

Proof. In light of Proposition 6.1.2, we already know that $\mathcal{M}_1^{\mathsf{Th}} \subseteq \mathcal{M}_2^{\mathsf{Gen}}$. Now, if $M \in \mathcal{M}_2^{\mathsf{Gen}} \setminus \mathcal{M}_1^{\mathsf{Th}}$, then M has a minor isomorphic to $M(K_4)$, P_6, Q_6, or $\mathcal{W}^3$. Since $\mathcal{M}_2^{\mathsf{Gen}}$ is closed under taking minors, the following proposition yields a contradiction. □

Proposition 6.1.6. *The matroids $M(K_4)$, $\mathcal{W}^3$, Q_6, and P_6 are not in $\mathcal{M}_2^{\mathsf{Gen}}$.*

Proof. For a point configuration $V \subset \mathbb{R}^n$, let $I \subset \mathbb{R}[x_1, \ldots, x_n]$ be its vanishing ideal. Let us consider the (unique) reduced Gröbner basis $\mathcal{G}$ of $I(V)$ with respect to a degree-compatible term order. If $I(V)$ is generated in degrees $\leq k$, then the set of all polynomials of degree $\leq k$ of $\mathcal{G}$ forms a set of generators of $I(V)$. By using a software like `Macaulay2` [GS], we can verify that, for $M \in \{M(K_4), \mathcal{W}^3, Q_6, P_6\}$, the set of quadratic polynomials of a degree-compatible Gröbner basis does not generate the vanishing ideal $I(V_M)$. □

6.2 Psd rank

Let $\mathcal{S}^m \subset \mathbb{R}^{m\times m}$ be the vector space of symmetric $m \times m$ matrices. The **psd cone** is the closed convex cone $\mathcal{S}^m_{\geq 0} = \{A \in \mathcal{S}^m : A \text{ positive semidefinite}\}$.

Definition 6.2.1. A polytope $P \subset \mathbb{R}^d$ has a **psd-lift** of size m if there is an affine subspace $L \subset \mathcal{S}^m$ and a linear projection $\pi : \mathcal{S}^m \to \mathbb{R}^d$ such that $P = \pi(\mathcal{S}^m_{\geq 0} \cap L)$. The **psd rank** $\operatorname{rank}_{\mathsf{psd}}(P)$ is the size of a smallest psd-lift.

Psd-lifts together with lifts for more general cones were introduced by Gouveia, Parrilo, and Thomas [GPT13] as natural generalizations of *polyhedral lifts* or *extended formulations*. Let us define $\mathcal{V}^{\mathsf{Psd}}_k$ as the class of point configurations V in convex position such that $\operatorname{conv}(V)$ has a psd-lift of size $\leq k$. In [GRT13] it was shown that for a d-dimensional polytope P the psd rank is always $\geq d+1$. A polytope P is called **psd-minimal** if $\operatorname{rank}_{\mathsf{psd}}(P) = \dim(P) + 1$. We write $\mathcal{V}^{\mathsf{Psd}}_{\mathsf{min}}$ for the class of psd-minimal (convex position) point configurations.

Proposition 6.2.2. *The classes $\mathcal{V}^{\mathsf{Psd}}_k$ and $\mathcal{V}^{\mathsf{Psd}}_{\mathsf{min}}$ are face-hereditary.*

Proof. Let $V \in \mathcal{V}^{\mathsf{Psd}}_k$ and let (L, π) be a psd-lift of $P = \operatorname{conv}(V)$. For a supporting hyperplane H we observe that $(L \cap \pi^{-1}(H), \pi)$ is a psd-lift of $P \cap H$ of size k.

Let P be psd-minimal and let $F = P \cap H$ be a face of dimension $\dim(F) = \dim(P) - 1$. If F is not psd-minimal, then by [GRT13, Prop. 3.8], $\operatorname{rank}_{\mathsf{psd}}(P) \geq \operatorname{rank}_{\mathsf{psd}}(F) + 1 > \dim(F) + 2 = \dim(P) + 1$. □

A characterization of psd-minimal polytopes in small dimensions was obtained in [GRT13] and, in particular, the following relation was shown.

Proposition 6.2.3. *Let V be a point configuration in convex position. If $\mathsf{Th}(V) = 1$, then $P = \operatorname{conv}(V)$ is psd-minimal.*

In [GRT13] the polytope of Example 1.1.6 is proposed as an instance of psd-minimal polytope which is not (combinatorially) 2-level. This shows that the condition above is sufficient but not necessary. The main result of this section is that the situation is much better for base configurations.

Theorem 6.2.4. *Let M be a matroid. The base polytope $P_M = \operatorname{conv}(V_M)$ is psd-minimal if and only if $\mathsf{Th}(M) = 1$.*

In light of Proposition 6.2.3 it remains to show that there is no psd-minimal matroid M with $\mathsf{Th}(M) > 1$. Since $\mathcal{V}_{\mathsf{min}}^{\mathsf{Psd}}$ is face-hereditary, it is sufficient to show that the excluded minors $M(K_4)$, $\mathcal{W}^3$, Q_6, and P_6 are not psd-minimal.

In order to do so, we need to recall the connection to slack matrices and Hadamard square roots developed in [GRT13]. For a more coherent picture of the relations in particular to cone factorizations we refer to the papers [GPT13, GRT13]. Let P be a polytope with v vertices and f facets. A **Hadamard square root** of a slack matrix $S \in \mathbb{S}(P)$ is a matrix $H \in \mathbb{R}^{v\times f}$ such that $S_{ij} = H_{ij}^2$ for all i, j. Moreover, we define $\operatorname{rank}_{\sqrt{}} S$ as the smallest rank among all Hadamard square roots. The following is the main connection between Hadamard square roots and the psd-rank.

Theorem 6.2.5 ([GRT13, Thm. 3.5])**.** *A polytope P is psd-minimal if and only if* $\operatorname{rank}_{\sqrt{}}(S) = \dim P + 1$ *for* $S \in \mathbb{S}(P)$.

Thus, we will complete the proof of Theorem 6.2.4 by showing that the slack matrices for the excluded minors of $\mathcal{M}_1^{\mathsf{Th}}$ have Hadamard square roots of rank ≥ 7. We start with a technical result.

Proposition 6.2.6. *The matrix*

$$A_0 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

has $\operatorname{rank}_{\sqrt{}} A_0 = 4$.

Proof. Every Hadamard square root of A_0 is of the form

$$H = \begin{pmatrix} 0 & y_1 & y_2 & y_3 \\ y_4 & 0 & y_5 & y_6 \\ y_7 & y_8 & 0 & y_9 \\ y_{10} & y_{11} & y_{12} & 0 \end{pmatrix}$$

with $y_i^2 = 1$, $i = 1, .., 12$. Claiming that $\operatorname{rank}_{\sqrt{}} A_0 = 4$ is equivalent to the claim that every Hadamard square root H is non-singular. Using the computer algebra software `Macaulay2` [GS] it can be checked that the ideal

$$I = \langle y_1^2 - 1, ..., y_{12}^2 - 1, \det H\rangle \subseteq \mathbb{C}[y_1, \dots, y_{12}]$$

contains 1, which excludes the existence of a rank-deficient Hadamard square root. □

Proposition 6.2.7. *Let $P = P_M$ be the base polytope for a matroid $M \in \{M(K_4), \mathcal{W}^3, Q_6, P_6\}$ and $S \in \mathbb{S}(P)$. Then* $\operatorname{rank}_{\sqrt{}}(S) \geq 7$.

Proof. We explicitly give the argument for $M = M(K_4)$ and $P = P_M$. This proof works also for the other matroids for the same choice of the subcollection of bases and flacets. It is sufficient to find a 7×7-submatrix A of S with $\operatorname{rank}_{\sqrt{}}(N) \geq 7$. Consider the following subcollection of bases B_i's and flacets F_i's of M:

$$\begin{array}{rcl@{\qquad}rcl}
B_1 &=& \{1,2,4\} & F_1 &=& \{1\} \\
B_2 &=& \{1,2,5\} & F_2 &=& \{2\} \\
B_3 &=& \{1,2,6\} & F_3 &=& \{3\} \\
B_4 &=& \{1,3,6\} & F_4 &=& \{4\} \\
B_5 &=& \{1,4,6\} & F_5 &=& \{5\} \\
B_6 &=& \{1,5,6\} & F_6 &=& \{6\} \\
B_7 &=& \{2,4,6\} & F_7 &=& \{3,4,6\}
\end{array}$$

and the induced submatrix of S

$$A = \begin{array}{c} \\ \scriptstyle\{1,2,4\} \\ \scriptstyle\{1,2,5\} \\ \scriptstyle\{1,2,6\} \\ \scriptstyle\{1,3,6\} \\ \scriptstyle\{1,4,6\} \\ \scriptstyle\{1,5,6\} \\ \scriptstyle\{2,4,6\} \end{array}
\begin{array}{c}
\begin{array}{ccccccc} \scriptstyle\{1\} & \scriptstyle\{2\} & \scriptstyle\{3\} & \scriptstyle\{4\} & \scriptstyle\{5\} & \scriptstyle\{6\} & \scriptstyle\{3,4,6\} \end{array} \\
\begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 2 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 0
\end{pmatrix}
\end{array}.$$

Then $\operatorname{rank}_{\sqrt{}}(A) = 7$ if and only if the determinant

$$\begin{vmatrix}
0 & 0 & \pm1 & 0 & \pm1 & \pm1 & \pm1 \\
0 & 0 & \pm1 & \pm1 & 0 & \pm1 & \pm\sqrt{2} \\
0 & 0 & \pm1 & \pm1 & \pm1 & 0 & \pm1 \\
0 & \pm1 & 0 & \pm1 & \pm1 & 0 & 0 \\
0 & \pm1 & \pm1 & 0 & \pm1 & 0 & 0 \\
0 & \pm1 & \pm1 & \pm1 & 0 & 0 & \pm1 \\
\boxed{\pm1} & 0 & \pm1 & 0 & \pm1 & 0 & 0
\end{vmatrix} = \pm
\begin{vmatrix}
0 & \pm1 & 0 & \pm1 & \pm1 & \pm1 \\
0 & \pm1 & \pm1 & 0 & \pm1 & \boxed{\pm\sqrt{2}} \\
0 & \pm1 & \pm1 & \pm1 & 0 & \pm1 \\
\pm1 & 0 & \pm1 & \pm1 & 0 & 0 \\
\pm1 & \pm1 & 0 & \pm1 & 0 & 0 \\
\pm1 & \pm1 & \pm1 & 0 & 0 & \pm1
\end{vmatrix}$$

is non-zero. Since it is of the form $a + \sqrt{2} \cdot b$ for some integers a, b, we can check that b is non-zero. By Laplace expansion, this is the case if

$$\begin{vmatrix} 0 & \pm 1 & 0 & \pm 1 & \boxed{\pm 1} \\ 0 & \pm 1 & \pm 1 & \pm 1 & 0 \\ \pm 1 & 0 & \pm 1 & \pm 1 & 0 \\ \pm 1 & \pm 1 & 0 & \pm 1 & 0 \\ \pm 1 & \pm 1 & \pm 1 & 0 & 0 \end{vmatrix} = \pm \begin{vmatrix} 0 & \pm 1 & \pm 1 & \pm 1 \\ \pm 1 & 0 & \pm 1 & \pm 1 \\ \pm 1 & \pm 1 & 0 & \pm 1 \\ \pm 1 & \pm 1 & \pm 1 & 0 \end{vmatrix} \neq 0.$$

The latter is exactly the claim that the matrix A_0 of Proposition 6.2.6 has $\operatorname{rank}_{\sqrt{}}(A_0) = 4$. □

6.3 Extension complexity of hypersimplices

Given a polytope P, we denote by $\mathrm{v}(P)$ its number of vertices and by $\mathrm{f}(P)$ its number of facets. A polytope $\widehat{P}$ is called an **extension** of P if $\pi(\widehat{P}) = P$ for some linear projection $\pi : \mathbb{R}^e \to \mathbb{R}^d$. Notice that the image of a face of $\widehat{P}$ under π is not necessarily a face of P; on the other hand, for any face F of P the set $\pi^{-1}(F) := \{\boldsymbol{p} \in \widehat{P} \ : \ \pi(\boldsymbol{p}) \in F\}$ is a face of $\widehat{P}$.

The **extension complexity** $\mathrm{xc}(P)$ is the smallest number of facets of an extension $\widehat{P}$ of P. An extension $\widehat{P}$ of P is **minimal** if it has $\mathrm{xc}(P)$ facets.

Equivalently, it is possible to define the extension complexity of a polytope in analogy to Definition 6.2.1, where we use the non-negative cone $\mathbb{R}^m_{\geq 0}$ instead of the psd cone $\mathcal{S}^m_{\geq 0}$. The extension complexity of P is then the size of the smallest non-negative lift.

The polytope P is an extension of itself, therefore $\mathrm{xc}(P) \leq \mathrm{f}(P)$. Moreover, $\mathrm{xc}(P) \leq \mathrm{v}(P)$ since every polytope is the projection of a simplex.

Let us state the following lemma in analogy to [GRT13, Prop. 3.8].

Lemma 6.3.1. *Let P be a polytope and $F \subset P$ a face. Then*

$$\mathrm{xc}(P) \ \geq \ \mathrm{xc}(F) + 1.$$

Proof. Let $(\widehat{P}, \pi)$ be a minimal extension of P. Let F be a face of P: the set $\widehat{F} = \pi^{-1}(F)$ is a face of $\widehat{P}$ and is an extension of F by definition. Since $\widehat{F}$ is a face of $\widehat{P}$, it is contained in a facet of $\widehat{P}$. Furthermore, every facet of $\widehat{F}$ is obtained as an intersection $\widehat{F} \cap \widehat{G}$, with $\widehat{G}$ facet of $\widehat{P}$. Since $\mathrm{f}(\widehat{F}) \geq \mathrm{xc}(F)$, $\widehat{P}$ has necessarily at least $\mathrm{xc}(F)$ facets and, in addition, at least one facet containing $\widehat{F}$. Thus $\mathrm{xc}(P) \geq \mathrm{xc}(F) + 1$. □

Two corollaries follow immediately from the lemma.

Corollary 6.3.2. *Let $P \subset \mathbb{R}^d$ be a d-polytope and $F \subseteq P$ a k-face. Then*

$$\operatorname{xc}(P) \;\geq\; \operatorname{xc}(F) + d - k.$$

Corollary 6.3.3. *If $P \subset \mathbb{R}^d$ is a d-dimensional polytope,* $\operatorname{xc}(P) \geq d + 1$.

We can strengthen the incremental bound of Lemma 6.3.1 whenever there are two disjoint facets with the same extension complexity.

Lemma 6.3.4. *Let P be a polytope and let F_1 and F_2 be disjoint facets of P such that* $\operatorname{xc}(F_1) = \operatorname{xc}(F_2) = r$. *Then*

$$\operatorname{xc}(P) \;\geq\; r + 2.$$

Proof. Let $(\widehat{P}, \pi)$ be a minimal extension of P. For $i = 1, 2$, the face $\widehat{F_i} = \pi^{-1}(F_i)$ is contained in $c_i \geq 1$ facets of $\widehat{P}$ and there are at least $r_i \geq r$ facets of $\widehat{P}$ meeting $\widehat{F_i}$ in facets. We only need to prove the lemma for the case $c_i = 1$ and $r_i = r$, since in all other cases $c_i + r_i \geq r + 2$ and this yields enough facets of $\widehat{P}$.

If $c_i = 1$, $\widehat{F_i}$ is a facet of P. Moreover, $\widehat{F_1} \cap \widehat{F_2} = \emptyset$ because their projections F_1 and F_2 are disjoint by hypothesis. Therefore, $\widehat{F_2}$ is not counted among the r facets of $\widehat{P}$ intersecting $\widehat{F_1}$ in a facet. Thus $\widehat{P}$ has at least $r + 2$ facets. $\square$

This simple lemma trivially implies the extension complexity for the cube (see [FKPT13a, Prop. 5.9]).

Corollary 6.3.5. *Let P be a polytope combinatorially equivalent to the n-dimensional cube C_n. Then* $\operatorname{xc}(P) = 2n$.

Proof. We know that $\operatorname{xc}(P) \leq \mathrm{f}(P) = 2n$ and we now show that $\operatorname{xc}(P) \geq 2n$. For $n = 1$, P is a 1-simplex and $\operatorname{xc}(P) = \operatorname{xc}(C_1) = 2$. For $n > 1$ any facet F of P is a combinatorial $(n-1)$-cube and there is a unique facet of P disjoint from F, which is again a combinatorial cube. Hence, by Lemma 6.3.4 and induction, we get

$$\operatorname{xc}(P) \;\geq\; \operatorname{xc}(C_{n-1}) + 2 \;=\; 2(n-1) + 2 = 2n. \qquad \square$$

In this section we determine the extension complexity of hypersimplices. Notice that the hypersimplex $\Delta_{n,k}$ is congruent to the hypersimplex $\Delta_{n,n-k}$ and therefore we only study the case $k \leq \lfloor \frac{n}{2} \rfloor$. We already mentioned that the facets of $\Delta_{n,k}$ are defined by the inequalities of type $x_i \geq 0$ and $1 - x_i \geq 0$, for $1 \leq i \leq n$. Thus $\mathrm{f}(\Delta_{n,k}) = 2n \geq \mathrm{xc}(\Delta_{n,k})$.

We start to investigate the hypersimplices with small dimension: the first instance is the hypersimplex $\Delta_{4,2}$ where $\mathrm{v}(\Delta_{4,2}) = \binom{4}{2} = 6$ and $\mathrm{f}(\Delta_{4,2}) = 8$. Hence, we have the trivial bound $\mathrm{xc}(\Delta_{4,2}) \leq 6$. The canonical projection $\Delta_5 \to \Delta_{4,2}$ yields an extension with 6 facets. Observe that $\Delta_{4,2}$ is affinely equivalent to a 3-dimensional octahedron. The simplex Δ_5 is a minimal extension since the polar P° of $\Delta_{4,2}$ (more precisely, a translation of it containing the origin in the interior) is a combinatorial 3-cube (thus $\mathrm{xc}(P^\circ) = 6$ by Corollary 6.3.5) and the extension complexity is preserved under polarity.

A second instance is the 4-dimensional polytope $P = \Delta_{5,2}$. We consider this polytope in light of the tools provided by [OVW14]: if we remove the vertices $(1,1,0,0,0)$ and $(0,0,0,1,1)$ from P, we obtain a subpolytope Q with 8 vertices and 7 facets. Since $\mathrm{xc}(Q) \leq 7$ and $P = Q \cup \{(1,1,0,0,0)\} \cup \{(0,0,0,1,1\}$, it follows by [OVW14, Thm. 2.2] that $\mathrm{xc}(\Delta_{5,2}) \leq 7 + 2 = 9$.

Before approaching the general problem, we study the slack matrices of the hypersimplices $\Delta_{6,2}$ and $\Delta_{6,3}$. More precisely, we try to find lower bounds on their *rectangle covering number*, which in turn yield lower bounds on the extension complexity. First we introduce some definitions that will help us to handle the slack matrices under consideration.

Let $S \in \mathbb{R}^{\mathrm{v} \times \mathrm{f}}$ be a non-negative matrix. A **rectangle** of S is a set $R = I \times J$ such that $I \subseteq [\mathrm{v}]$, $J \subseteq [\mathrm{f}]$, and $S_{ij} > 0$ for every $(i,j) \in I \times J$. A **rectangle covering** of S is a collection of rectangles $R_1, \ldots, R_m$ such that $S_{ij} > 0$ if and only if $(i,j) \in R_l$ for some $l = 1, \ldots, m$.

The **rectangle covering number** $\mathrm{rc}(S)$ of S is the minimal number of rectangles among all possible rectangle coverings of S. This number represents a lower bound on the non-negative rank of S (see [FMP$^+$15, Thm. 4]) and whenever S is the slack matrix of some polytope P, it follows that $\mathrm{xc}(P) \geq \mathrm{rc}(S)$. For further details about non-negative rank and rectangles coverings we refer to [Yan91] and [FKPT13b].

The rectangle covering number for both slack matrices of $\Delta_{6,2}$ and $\Delta_{6,3}$ is 12 and has been computed by Stefan Weltge. The straightforward consequence is that $\mathrm{xc}(\Delta_{6,2}) = \mathrm{xc}(\Delta_{6,3}) = 12$ and this last piece of information is essential to prove the following theorem.

Theorem 6.3.6. *For $n \geq 6$ and $2 < k < n-2$,*

$$\mathrm{xc}(\Delta_{n,k}) \;=\; 2n.$$

Proof. As already mentioned we can assume $k \leq \lfloor \frac{n}{2} \rfloor$. The proof is by induction on n with base case $n = 6$ for which we know $\mathrm{xc}(\Delta_{6,2}) = \mathrm{xc}(\Delta_{6,3}) = 12$.

Observe that $\Delta_{n,k} \cap \{\boldsymbol{p} \in \mathbb{R}^n : p_i = 0\} \cong \Delta_{n-1,k}$ and $\Delta_{n,k} \cap \{\boldsymbol{p} \in \mathbb{R}^n : 1 - p_i = 0\} \cong \Delta_{n-1,k-1}$ for all $i = 1, \dots, n$. The symbol $\cong$ is used here for congruent polytopes. If $k > 2$ and $n \geq 7$, then $\mathrm{xc}(\Delta_{n-1,k}) = \mathrm{xc}(\Delta_{n-1,k-1}) = 2n - 2$ by induction hypothesis. We apply Lemma 6.3.4 and obtain

$$\mathrm{xc}(\Delta_{n,k}) \;\geq\; \mathrm{xc}(\Delta_{n-1,k}) + 2 \;=\; 2(n-1) + 2.$$

It only remains to prove the case $k = 2$. The hypersimplex $\Delta_{n,2}$ has n facets of type $\Delta_{n,2} \cap \{\boldsymbol{p} \in \mathbb{R}^n : p_i = 0\} \cong \Delta_{n-1,2}$ and n facets of type $\Delta_{n,2} \cap \{\boldsymbol{p} \in \mathbb{R}^n : 1 - p_i = 0\} \cong \Delta_{n-1}$.

Consider $P = \Delta_{n,2}$ and a minimal extension $(\widehat{P}, \pi)$. For any facet F of type $\Delta_{n-1,2}$, the face $\widehat{F} = \pi^{-1}(F)$ is contained in $c \geq 1$ facets of $\widehat{P}$ and there are at least $r \geq \mathrm{xc}(F)$ facets of $\widehat{P}$ meeting $\widehat{F}$ in facets. By induction hypothesis $\mathrm{xc}(F) = 2n - 2$ and therefore, as in the proof of Lemma 6.3.4, the only case to consider is $c = 1$ and $r = \mathrm{xc}(F)$.

In consequence, we assume that for every facet F_i of P of type $\Delta_{n-1,2}$, $\widehat{F_i} = \pi^{-1}(F_i)$ is a facet of $\widehat{P}$.

The remaining facets G_i, $1 \leq i \leq n$ of P are simplices and each $\widehat{G_i} = \pi^{-1}(G_i)$ is the intersection of some facets R_j, $1 \leq j \leq m$, of $\widehat{P}$. None of these facets R_j can be one of the $\widehat{F_i}$'s, since $\widehat{G_i} \subseteq \widehat{F_i}$ would imply that $G_i \subseteq F_i$.

Now consider the polyhedron $\widehat{P}'$ obtained from $\widehat{P}$ by removing the inequalities corresponding to the facets $\widehat{F_i}$'s. That is, $\widehat{P}'$ is the intersection of all half-spaces that define facets of $\widehat{P}$ different from $\widehat{F_i}$, $1 \leq i \leq n$. Similarly, by removing the inequalities corresponding to the facets $F_1, \dots, F_n$ from the inequality description of P, we obtain

$$P' = \left\{ \boldsymbol{p} \in \mathbb{R}^n \;:\; \sum_{i=1}^{n} p_i = 2 \text{ and } 1 - p_i \geq 0, \text{ for } i \in 1, \dots, n \right\},$$

which is a $(n-1)$-simplex.

Since none of the facet-defining hyperplanes containing the faces $\widehat{G_i}$'s is removed, $\pi(\widehat{P}') \subseteq P'$. Furthermore, $\widehat{P}'$ is bounded. By contradiction, if $\widehat{P}'$ is unbounded, then it can only be unbounded in directions that are parallel to the kernel $\ker(\pi)$ of the projection π. Since $\widehat{F_i} = \pi^{-1}(F_i)$, it follows that for all $i = 1, \dots, n$ the facet-defining hyperplanes $\widehat{H_i} := \mathrm{aff}(\widehat{F_i})$ are parallel to

$\ker(\pi)$ and therefore

$$\widehat{P} = \widehat{P}' \cap \left(\bigcap_{i=1}^{n} \widehat{H_i}^{+} \right)$$

is also unbounded, which yields the contradiction.

Now, $\widehat{P}'$ is an extended formulation of the $(n-1)$-dimensional polytope $\pi(\widehat{P}')$. Therefore, by Corollary 6.3.3, $\widehat{P}'$ has at least n facets. These facets, together with the n facets of type $\widehat{F}_i$, prove that $\mathrm{xc}(P) \geq 2n$. □

The theorem implies that every hypersimplex $\Delta_{n,k}$ with $n \geq 6$ is a minimal extension of itself. It is legitimate to ask whether there are other minimal extensions, which is quickly answered with the help of `Matlab` scripts made available by the authors of [VGGT14]. For instance, the hypersimplex $\Delta_{6,2}$ admits a minimal extension different from $\Delta_{6,2}$.

Conjecture 6.3.7. *Let P_M be the base polytope of a connected 2-level matroid M whose tree decomposition T_M has only* `U`*-vertices of degree ≥ 6. Then $\mathrm{xc}(P) = \mathrm{f}(P)$.*

This last conjecture is tied to our understanding of the extension complexity in relation to the subdirect product and other polytopal operations. It was proven by Hans Raj Tiwary ([Tiw]) that the the join $\star$ of two polytopes P_1 and P_2 is such that $\mathrm{xc}(P_1 \star P_2) = \mathrm{xc}(P_1) + \mathrm{xc}(P_2)$. The understanding of the Cartesian product would represent a relevant step towards the understanding of the subdirect product. Unfortunately, the statement $\mathrm{xc}(P_1 \times P_2) = \mathrm{xc}(P_1) + \mathrm{xc}(P_2)$, even though supported by experimental data, still remains a conjecture. Notice that if this last statement was true, our conjecture would readily extend to non-connected 2-level matroids.

6.4 Vanishing ideal of uniform matroids

6.4.1 Gröbner bases of $I_{n,k}$

Consider the uniform matroid $U_{n,k}$ and its base configuration $\Delta_{n,k}$ (with a slight abuse of notation this refers to the set of vertices and includes the cases $k = 1$ and $k = n-1$). Since simplices and hypersimplices are 2-level polytopes, Proposition 6.1.2 shows that the vanishing ideal $I_{n,k} := I(\Delta_{n,k})$ has a set of generators of degree ≤ 2. In particular, $I_{n,k}$ admits the system of quadratic generators

$$x_i^2 - x_i \text{ for all } i \in [n], \quad \text{and} \quad \sum x_i - k\,.$$

In this section we study the ideal $I_{n,k}$. More precisely, using the combinatorial description of its *Gröbner bases* provided in [HR03], we analyze the *Gröbner fan* and characterize the *state polyhedron* of $I_{n,k}$.

We shortly recap the basic definitions from Gröbner bases theory that are instrumental for our purposes. For more details we point to [Stu96, Ch. 1] and [EH12, Ch. 2].

Let $\mathbb{R}[\mathbf{x}]$ be the polynomial ring in n indeterminates. A total order $\prec$ on $\mathbb{N}^n$ induces an order on the monomials $\mathbf{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} \dots x_n^{\alpha_n}$, that is $\mathbf{x}^{\boldsymbol{\alpha}} \prec \mathbf{x}^{\boldsymbol{\beta}} \Leftrightarrow \boldsymbol{\alpha} \prec \boldsymbol{\beta}$. If $\mathbf{0}$ is the unique minimal element, and $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$ implies $\boldsymbol{\alpha}+\boldsymbol{\gamma} \prec \boldsymbol{\beta}+\boldsymbol{\gamma}$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{N}^n$, then $\prec$ is called a **term order**. Given a polynomial $p \in \mathbb{R}[\mathbf{x}]$ and the term order $\prec$, the **leading monomial** $\mathrm{LM}_\prec(p)$ is the maximal monomial with respect to $\prec$ appearing in p.

Given an ideal $I \subset \mathbb{R}[\mathbf{x}]$, the **initial ideal** $\mathrm{in}_\prec(I)$ is the monomial ideal $\{\mathrm{LM}_\prec(p) : p \in I\}$. A set of generators $\{g_1, \dots, g_m\}$ of I is a **Gröbner basis** of I with respect to the order $\prec$ if and only if $\mathrm{in}_\prec(I) = \langle \mathrm{LM}_\prec(g_i) : i = 1, \dots, m\rangle$. A Gröbner basis $\mathcal{G}$ is called **reduced** if for every element in the basis the coefficient of the leading monomial is 1 and for any $g, g' \in \mathcal{G}$ $\mathrm{LM}_\prec(g)$ does not divide any of the monomials of g'.

Following the notation of [HR03], for given positive integers n and t, such that $1 \le t \le n/2$ we define the sets

$$\mathcal{D}_t := \{\{d_1 < \dots < d_{t+1}\} \subset [n] \,:\, d_i \ge 2i, \text{ for } i = 1, \dots, t\}, \text{ and}$$
$$\mathcal{H}_t := \{\{h_1 < \dots < h_t\} \subset [n] \,:\, h_i \ge 2i \text{ for } i = 1, \dots, t{-}1 \text{ and } h_t < 2t\}.$$

For $J \subseteq [n]$ and $0 \le i \le |J|$, let $\sigma_{J,i}$ denote the ith elementary symmetric polynomial supported on $\{x_j : j \in J\}$

$$\sigma_{J,i} := \sum_{T \subseteq J, |T|=i} \mathbf{x}^T \in \mathbb{R}[\mathbf{x}].$$

We define the polynomials

$$f_{H,k} := \sum_{j=0}^{t} (-1)^{t-j} \binom{k-j}{t-j} \sigma_{H',j} \ ,$$

where $H \in \mathcal{H}_t$ and $H' = H \cup \{2t, 2t{+}1, \dots, n\}$. We now state the main result from [HR03].

Theorem 6.4.1 ([HR03, Cor. 1.4])**.** *Let k and n be integers such that $0 \le k \le n/2$ and $\prec$ an arbitrary term order such that $x_n \prec x_{n-1} \prec \dots \prec x_1$. The set of polynomials*

$$\mathcal{G} = \{x_2^2{-}x_2, \dots, x_n^2{-}x_n\} \cup \{\mathbf{x}^D : D \in \mathcal{D}_k\} \cup \{f_{H,k} : H \in \mathcal{H}_t \text{ for some } 0{<}t{\le}k\}$$

is the reduced Gröbner basis of $I_{n,k}$ with respect to $\prec$.

If $k > n/2$, then the reduced Gröbner basis of $I_{n,k}$ is the set

$$\{x_2^2-x_2,\dots,x_n^2-x_n\}\cup\{\mathbf{x}^D : D\in\mathcal{D}_{n-k}\}\cup\{f_{H,k} : H\in\mathcal{H}_t \text{ for some } 0<t\leq n-k\},$$

which only differs from the case $I_{n,n-k}$ because of the different values of the coefficients of the polynomials $f_{H,k}$. In what follows we always assume that $k \leq n/2$. The case $k > n/2$ works similarly and leads to the same results.

Since the Gröbner basis depends only on the order of the indeterminates induced by a term order $\prec$, there can be at most $n!$ distinct Gröbner bases of $I_{n,k}$. From the combinatorial description of the Gröbner basis provided in Theorem 6.4.1 we see that different orders of the indeterminates yield the same reduced Gröebner basis. In this section we find the number of different reduced Gröbner bases.

Let us fix a term order $\prec$ such that $x_n \prec x_{n-1} \prec ... \prec x_1$. Since the order of the indeterminates encodes the information necessary to determine the Gröbner basis, we study which orders of the indeterminates yield the same Gröbner basis as $\prec$. The permutation $\pi \in \mathfrak{S}_n$ corresponds to the order $x_{\pi(n)} \prec \dots \prec x_{\pi(1)}$.

We say that a permutation $\pi \in \mathfrak{S}_n$ is **Gröbner invariant** for $I_{n,k}$ if the order $x_{\pi(n)} \prec x_{\pi(n-1)} \prec ... \prec x_{\pi(1)}$ and the order $x_n \prec x_{n-1} \prec ... \prec x_1$ yield the same reduced Gröbner basis of $I_{n,k}$. We denote by $\mathfrak{S}^{\mathrm{gb}}(n,k) \subset \mathfrak{S}_n$ the set of all Gröbner invariant permutations for $I_{n,k}$.

For instance, if $\pi \in \mathfrak{S}^{\mathrm{gb}}(n,k)$, then $\pi(1) = 1$. Indeed, by Theorem 6.4.1, the polynomial $x_i^2 - x_i$ is not in the reduced Gröbner basis of $I_{n,k}$ if and only if x_i is the biggest in the order of the indeterminates and therefore $\pi(1) = 1$. There are also other conditions that are satisfied by π and have the following concise combinatorial description.

Proposition 6.4.2. *A permutation $\pi \in \mathfrak{S}_n$ is Gröbner invariant for $I_{n,k}$ if and only if $\pi(1) = 1$, $\pi(\mathcal{D}_k) = \mathcal{D}_k$, and $\pi(\mathcal{H}_t) = \mathcal{H}_t$ for $0 < t \leq k$.*

Proof. If $\pi \in \mathfrak{S}^{\mathrm{gb}}(n,k)$, then the Gröbner basis $\mathcal{G}$ of $I_{n,k}$ with respect to the order $x_n \prec x_{n-1} \prec ... \prec x_1$ is the same as the one obtained from the order $x_{\pi(n)} \prec x_{\pi(n-1)} \prec ... \prec x_{\pi(1)}$. Equivalently, for every $g(x_1,\dots,x_n) \in \mathcal{G}$, the polynomial $\pi(g(x_1,\dots,x_n)) := g(x_{\pi(1)},\dots,x_{\pi(n)}) \in \mathcal{G}$.

From Theorem 6.4.1, it is clear that every monomial in $\mathcal{G}$ is of the form $\mathbf{x}^D$ for some $D \in \mathcal{D}_k$, which implies $\pi(\mathbf{x}^D) \in \mathcal{G}$ if and only if $\pi(D) \in \mathcal{D}_k$. Therefore $\pi(\mathbf{x}^D) \in \mathcal{G}$ for all $D \in \mathcal{D}_k$ if and only if $\pi(\mathcal{D}_k) = \mathcal{D}_k$. Moreover, any polynomial $f_{H,k}$ for $H \in \mathcal{H}_t$ has degree t and therefore $\pi(f_{H,k})$ has degree t. It follows that $\pi(f_{H,k}) \in \mathcal{G}$ for every $H \in \mathcal{H}_t$ if and only if $\pi(\mathcal{H}_t) = \mathcal{H}_t$. Finally, no polynomial $f_{H,k}$ is of the form $x_i^2 - x_i$, thus $\pi(1) = 1$. $\square$

Lemma 6.4.3. *Let H be in $\mathcal{H}_t$. Then the maximal element of H is at most $2t-1$.*

Proof. The set $H = \{h_1, \dots, h_t\} \in \mathcal{H}_t$ is such that $h_1 < \dots < h_t$ and $h_t < 2t$. $\square$

Lemma 6.4.4. *Let π be a permutation in $\mathfrak{S}_n$. If $\pi \in \mathfrak{S}^{\mathrm{gb}}(n,k)$, then $\pi(2i) \leq 2i+1$ and $\pi(2i+1) \leq 2i+1$ for $1 \leq i \leq k-1$.*

Proof. For every i, $1 \leq i \leq k-1$, consider the set

$$H = \{2, 4, 6, ..., 2i, 2i+1\} \in \mathcal{H}_{i+1}.$$

By Proposition 6.4.2, $\pi(H) \in \mathcal{H}_{i+1}$. If $\pi(2i) > 2i+1$ or $\pi(2i+1) > 2i+1$, $\pi(H)$ would contain an element bigger than $2i+1$, contradicting Lemma 6.4.3 $\square$

Theorem 6.4.5. *Let π be a permutation in $\mathfrak{S}_n$. Then $\pi \in \mathfrak{S}^{\mathrm{gb}}(n,k)$ if and only if π is a product of transpositions of type $(2i, 2i+1)$ for $1 \leq i \leq k-1$ and a permutation on the elements $\{2k, 2k+1, \dots, n\}$.*

Proof. If $\pi \in \mathfrak{S}^{\mathrm{gb}}(n,k)$, then we have that $\pi(1) = 1$. Moreover, by Lemma 6.4.4, either $\pi(2) = 2$ and $\pi(3) = 3$ or $\pi(2) = 3$ and $\pi(3) = 2$. Inductively, it follows that either $\pi(2i) = 2i$ and $\pi(2i+1) = 2i+1$ or $\pi(2i) = 2i+1$ and $\pi(2i+1) = 2i$ for $1 \leq i \leq k-1$.

In order to prove that the permutations described in the statement of the theorem are Gröbner invariant for $I_{n,k}$, it is enough to consider every transposition and the permutations of the elements $\{2k, 2k+1, \dots, n\}$, separately. Let us apply a transposition of type $\pi = (2i, 2i+1)$ to $H \in \mathcal{H}_t$. If $2i \in H$ and $2i+1 \notin H$, the set $H \setminus \{2i\} \cup \{2i+1\} \in \mathcal{H}_t$. In fact, the element we remove and the element we add occupy the same position j in the sequence $h_1 < \dots < h_t$ and the condition $2i \geq 2j$ (or $2i < 2j$ if j is the last position) implies the condition $2i+1 \geq 2j$ (or $2i+1 < 2j$).

Analogously, if $2i+1 \in H$ and $2i \notin H$, the set $H \setminus \{2i+1\} \cup \{2i\} \in \mathcal{H}_t$, since $2i+1 \geq 2j$ (or $2i+1 < 2j$) implies $2i \geq 2j$ (or $2i < 2j$). An arbitrary permutation on the elements $\{2k, 2k+1, \dots, n\}$ involves no elements in any set $H \in \mathcal{H}_t$, $1 \leq t \leq k$ (it follows from Lemma 6.4.3).

The same kind of argument applies to show that $\pi(\mathcal{D}_k) = \mathcal{D}_k$, since a transposition does not alter the order and the inequalities conditions. Furthermore, any element in $\{2k, 2k+1, \ldots, n\}$ is bigger than $2j$ for every position $1 \leq j \leq k$ and can be replaced with any other element in $\{2k, 2k+1, \ldots, n\}$, still remaining in $\mathcal{D}_k$. □

Corollary 6.4.6. *Fix an order $\prec$ on the indeterminates and let $\mathcal{G}$ be the corresponding reduced Gröbner basis. There are exactly*

$$2^{k-1}(n-2k+1)!$$

different orders of the indeterminates yielding the reduced Gröbner basis $\mathcal{G}$.

Proof. It is enough to count the number of permutations in $\mathfrak{S}^{\mathrm{gb}}(n,k)$: by Theorem 6.4.5 this set of permutation is generated by $k-1$ transpositions and a permutation on $n-2k+1$ elements. The count follows from the fact that all generators act on disjoint sets. □

Another straightforward corollary is the following.

Corollary 6.4.7. *Let $\mathrm{gb}(I_{n,k})$ be the set of all reduced Gröbner bases with respect to all possible term orders. We have that*

$$|\,\mathrm{gb}(I_{n,k})| = \frac{n!}{2^{k-1}(n-2k+1)!}.$$

We get a similar result if we consider the ideal $I_{n,k}$ such that $k > n/2$. More precisely,

$$|\,\mathrm{gb}(I_{n,k})| = \frac{n!}{2^{n-k-1}(n-2(n-k)+1)!}.$$

6.4.2 Gröbner fan and state polyhedron

Before discussing the *Gröbner fan* of $I_{n,k}$ and the corresponding *state polyhedron*, we recall few definitions, following [Stu96, Ch. 1].

A set $C \subset \mathbb{R}^n$ is a **polyhedral cone** if there exists a finite set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\} \subset \mathbb{R}^n$ such that

$$C = \{\boldsymbol{p} \in \mathbb{R}^n\ ,\ \boldsymbol{p} = \lambda_1 \boldsymbol{v}_1 + \ldots + \lambda_m \boldsymbol{v}_m \text{ for some } \lambda_i \geq 0,\ i = 1, \ldots, m\}.$$

A **fan** is a finite collection $\mathcal{F}$ of polyhedral cones such that

(i) if $F \in \mathcal{F}$ and F' is a face of F, then $F' \in \mathcal{F}$;

(ii) if F_1, $F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$.

In the rest of the section we use the compact notation $\boldsymbol{a} \cdot \boldsymbol{b}$ to denote the scalar product $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \sum_{i=1}^{n} a_i b_i$ of two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$.

Any vector $\boldsymbol{\omega} \in \mathbb{R}^n$ induces a monomial ordering as follows: $\mathbf{x}^{\boldsymbol{\alpha}} \prec \mathbf{x}^{\boldsymbol{\beta}}$ if and only if $\boldsymbol{\omega} \cdot \boldsymbol{\alpha} < \boldsymbol{\omega} \cdot \boldsymbol{\beta}$. Notice that there exist monomials such that $\boldsymbol{\omega} \cdot \boldsymbol{\alpha} = \boldsymbol{\omega} \cdot \boldsymbol{\beta}$. We define the **initial form** $\mathrm{in}_{\boldsymbol{\omega}}(p)$ of a polynomial $p = \sum c_i \mathbf{x}^{\boldsymbol{\alpha}_i} \in \mathbb{R}[\mathbf{x}]$ as the sum of all terms $c_i \mathbf{x}^{\boldsymbol{\alpha}_i}$ such that $\boldsymbol{\omega} \cdot \boldsymbol{\alpha}_i$ is maximal. Moreover, we define the **initial ideal**

$$\mathrm{in}_{\boldsymbol{\omega}}(I) := \langle \mathrm{in}_{\boldsymbol{\omega}}(p) \ : \ p \in I \rangle,$$

which is not necessarily a monomial ideal. If we fix an arbitrary term order $\prec$, then $\mathrm{in}_{\prec}(\mathrm{in}_{\boldsymbol{\omega}}(I))$ is a monomial ideal and, in particular, it is the initial ideal for the term order $\prec_{\boldsymbol{\omega}}$ such that $\mathbf{x}^{\boldsymbol{\alpha}} \prec_{\boldsymbol{\omega}} \mathbf{x}^{\boldsymbol{\beta}}$ if and only if $\boldsymbol{\omega} \cdot \boldsymbol{\alpha} < \boldsymbol{\omega} \cdot \boldsymbol{\beta}$ or $\boldsymbol{\omega} \cdot \boldsymbol{\alpha} = \boldsymbol{\omega} \cdot \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$.

The **Gröbner region** $\mathrm{GR}(I)$ is the set of $\boldsymbol{\omega} \in \mathbb{R}^n$ such that $\mathrm{in}_{\boldsymbol{\omega}}(I) = \mathrm{in}_{\boldsymbol{\omega}^+}(I)$ for some $\boldsymbol{\omega}^+ \in \mathbb{R}^n_{\geq 0}$. We set an equivalence relation among weight vectors of the Gröbner region: the equivalence class of a vector $\boldsymbol{\omega}$ is defined as

$$[\boldsymbol{\omega}] := \{\boldsymbol{\omega}' \in \mathrm{GR}(I) \ : \ \mathrm{in}_{\boldsymbol{\omega}'}(I) = \mathrm{in}_{\boldsymbol{\omega}}(I)\}.$$

Each equivalence class turns out to be a relatively open convex polyhedral cone ([Stu96, Prop. 2.3]) and the set of closed cones $\overline{[\boldsymbol{\omega}]}$ forms a fan, namely the so-called **Gröbner fan** $\mathrm{GF}(I)$ of the ideal I.

The reader is referred to [Stu96, Ch. 1-2] for more details about Gröbner bases, Gröbner regions, and Gröbner fans of ideals.

For a homogeneous ideal $I \subseteq \mathbb{R}[\mathbf{x}]$ it is known that $\mathrm{GR}(I) = \mathbb{R}^n$ (see [Stu96, Prop. 1.12]) and for I non-homogeneous $\mathrm{GR}(I) \supseteq \mathbb{R}^n_{\geq 0}$. We prove that the Gröbner region of the ideal $I_{n,k}$ is exactly $\mathbb{R}^n_{\geq 0}$.

To see why, consider any $\boldsymbol{\omega}^+ \in \mathbb{R}^n_{\geq 0}$ and $\boldsymbol{\omega} \in \mathbb{R}^n$ such that $\mathrm{in}_{\boldsymbol{\omega}}(I_{n,k}) = \mathrm{in}_{\boldsymbol{\omega}^+}(I_{n,k})$. Notice that whenever $\omega_i < 0$, $\mathrm{in}_{\boldsymbol{\omega}}(x_i^2 - x_i) = x_i$ and thus $x_i \in \mathrm{in}_{\boldsymbol{\omega}}(I_{n,k})$. If $\boldsymbol{\omega}$ has at least two negative components ω_i and ω_j, then the monomials x_i and x_j belong to $\mathrm{in}_{\boldsymbol{\omega}}(I_{n,k})$ and therefore to $\mathrm{in}_{\prec_{\boldsymbol{\omega}}}(I_{n,k})$, with $\prec$ any term order. The description of the Gröbner basis provided in Theorem 6.4.1 shows that this is impossible. If $\boldsymbol{\omega}$ has only one negative component ω_i, we have $\mathrm{in}_{\boldsymbol{\omega}}(x_i^2 - x_i) = x_i$ and $\mathrm{in}_{\boldsymbol{\omega}'}(x_1 + \ldots + x_n - k) = x_j \neq x_i$ and the same argument applies. We conclude that $\boldsymbol{\omega} \in \mathbb{R}^n_{\geq 0}$.

Let $\boldsymbol{\omega}$ be a vector in $\mathbb{R}^n_{\geq 0}$ and $\pi \in \mathfrak{S}_n$ a permutation such that $\omega_{\pi(1)} \geq \omega_{\pi(2)} \geq \ldots \geq \omega_{\pi(n)}$. We call π an **ordering permutation** of $\boldsymbol{\omega}$. If $\boldsymbol{\omega}$ has all distinct components, there exists a unique ordering permutation.

The reduced Gröbner basis of $I_{n,k}$ only depends on the order of the indeterminates induced by a term order (Theorem 6.4.1). Thus, given two vectors $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ with the same unique ordering permutation, we have $[\boldsymbol{\omega}] = [\boldsymbol{\omega}']$ because they induce the same order of the indeterminates. Moreover, this shows that $\mathrm{GF}(I_{n,k})$ is a coarsening of the braid arrangement fan described in [PRW08, Sect. 3.2] restricted to the positive octant.

Each maximal cone of $\mathrm{GF}(I_{n,k})$ is the union of maximal cones of the braid arrangement fan. Every maximal cone of the braid arrangement fan is associated with the ordering permutations of its element. Our goal is to characterize the Gröbner fan of $I_{n,k}$ as the *normal fan* of a suitable polyhedron.

Let us consider a polyhedron $P \subset \mathbb{R}^n$. For any vector $\boldsymbol{\omega} \in \mathbb{R}^n$ such that

$$\max_{\boldsymbol{p} \in P} \ \boldsymbol{\omega} \cdot \boldsymbol{p} = p_{\boldsymbol{\omega}} < \infty,$$

we define the face of P maximizing in the direction $\boldsymbol{\omega}$ as the set

$$P^{\boldsymbol{\omega}} := \{\boldsymbol{p} \in P \ : \ \boldsymbol{\omega} \cdot \boldsymbol{p} = p_{\boldsymbol{\omega}}\}.$$

We denote by $\mathrm{R}(P)$ the set of vectors which define a face and partition it into equivalence classes defined as

$$[\boldsymbol{\omega}]_{\max} := \{\boldsymbol{\omega}' \in \mathrm{R}(P) \ : \ P^{\boldsymbol{\omega}'} = P^{\boldsymbol{\omega}}\}.$$

The equivalence classes are relatively open cones and the set of closed cones $\overline{[\boldsymbol{\omega}]}_{\max}$ forms a fan called the **normal fan** $\mathrm{NF}(P)$ of P.

To construct the polyhedron we are looking for, we define the **Minkowski sum** of two polyhedra P_1 and P_2 in $\mathbb{R}^n$ as the polyhedron

$$P_1 + P_2 := \{\boldsymbol{p}_1 + \boldsymbol{p}_2 \in \mathbb{R}^n \ : \ \boldsymbol{p}_1 \in P_1 \text{ and } \boldsymbol{p}_2 \in P_2\}.$$

Notice that the normal fan of the Minkowski sum $Q := P + \mathbb{R}^n_{\leq 0}$, where P is a polytope in $\mathbb{R}^n$ and $\mathbb{R}^n_{\leq 0} := \{\boldsymbol{p} \in \mathbb{R}^n : p_i \leq 0 \text{ for all } i\}$, is defined on $\mathrm{R}(Q) = \mathbb{R}^n_{\geq 0}$.

Given the point $(a_1, \ldots, a_n) \in \mathbb{R}^n$, the **permutahedron** $\Pi(a_1, \ldots, a_n)$ is the polytope obtained as the convex hull of all possible permutations of the coordinates $(a_1, \ldots, a_n)$. For $k \leq n/2$, we define the polyhedron

$$\Pi_{n,k} := \Pi(k, k-1, k-1, k-2, k-2, \ldots, 1, 1, \underbrace{0, 0, \ldots, 0, 0}_{(n-2k+1)-\text{times}}) + \mathbb{R}^n_{\leq 0}.$$

We analyze the normal fan of $\Pi_{n,k}$. Let $\boldsymbol{\omega} \in \mathbb{R}^n_{\geq 0}$ be a vector with unique ordering permutation $\pi \in \mathfrak{S}_n$ (thus $\omega_{\pi(1)} > \omega_{\pi(2)} > \ldots > \omega_{\pi(n)}$). The only vertex $\boldsymbol{v} = (v_1, \ldots, v_n)$ of $\Pi_{n,k}$ that maximize $\boldsymbol{\omega} \cdot \boldsymbol{v}$ is $v_{\pi(1)} = k$, $v_{\pi(2)} = k-1, \ldots, v_{\pi(n)} = 0$ (this follows from the rearrangement inequality). Thus, $Q^{\boldsymbol{\omega}}$ is a vertex and all vectors that can be strictly ordered by the same permutation π necessarily belong to the same maximal cone. We conclude that $\mathrm{NF}(\Pi_{n,k})$ is a coarsening of the braid arrangement fan restricted to $\mathbb{R}^n_{\geq 0}$.

Example 6.4.8. Let us consider the polyhedron $\Pi_{3,1}$. Its normal fan has three maximal cones. In Figure 6.2 we illustrate the equivalence classes $[(1,1,1)]_{\max}$, $[(0,1,1)]_{\max}$, and $[(0,0,1)]_{\max}$. The maximizing faces are respectively a 2-simplex, an edge, and a vertex. The closure of $[(0,0,1)]$ is a maximal region of the normal fan and contains the vectors with ordering permutations (312) or (321).

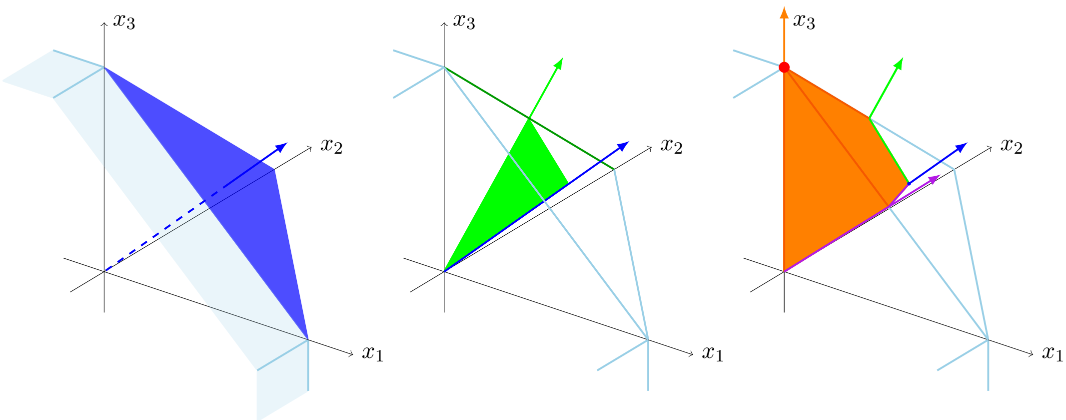

Figure 6.2: Normal fan of $\Pi_{3,1}$.

Given an ideal $I \subset \mathbb{R}[\mathbf{x}]$, its **state polyhedron** $\mathrm{SP}(I)$ is the polyhedron such that $\mathrm{NF}(\mathrm{SP}(I)) = \mathrm{GF}(I)$.

Theorem 6.4.9. *The polyhedron $\Pi_{n,k}$ is the state polyhedron of $I_{n,k}$.*

Proof. Let us consider the Gröbner fan $\mathrm{GF}(I_{n,k})$ and the normal fan $\mathrm{NF}(\Pi_{n,k})$, both defined on $\mathbb{R}^n_{\geq 0}$. To show that they are equal, we check that the maximal regions of the two fans are the same. More precisely, we determine which strictly ordered vectors are equivalent. Both fans are coarsenings of the braid arrangement fan B, therefore they are equal if their maximal regions gather together the same maximal regions of B (which we represent by permutations).

Theorem 6.4.5 implies that two strictly ordered weight vectors $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ are in the same maximal cone of $\mathrm{GF}(I_{n,k})$ if and only if their ordering permutations π and π' are such that $\pi = \pi' \circ \sigma$, where $\sigma \in \mathfrak{S}^{\mathrm{gb}}(n,k)$.

A maximal region of $\mathrm{NF}(\Pi_{n,k})$ is the set of vectors that are maximized at a same vertex of $\Pi_{n,k}$ (including vectors that give faces strictly containing the vertex). Two strictly ordered vectors $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ with ordering permutations π and π' are maximized by the same vertex if and only if $\pi = \pi' \circ \sigma$, where σ can exchange the pairs of type $(2i, 2i+1)$ for $i = 1, \dots k-1$ and permute all the elements $(2k, 2k+1, \dots, n)$ because all these operations do not alter the values in the rearrangement inequality.

This set of permutation is clearly $\mathfrak{S}^{\mathrm{gb}}(n,k)$, therefore the two fans coincide. $\square$

If we consider the Gröbner fan of $I_{n,k}$ for $k > n/2$, we easily find out that $\mathrm{GF}(I_{n,k}) = \mathrm{GF}(I_{n,n-k})$. This also implies that $\mathrm{SP}(I_{n,k}) = \mathrm{SP}(I_{n,n-k})$, thus the state polyhedron for $k > n/2$ is defined as

$$\Pi(n-k, n-k-1, n-k-1, \dots, 1, 1, \underbrace{0, 0, \dots, 0, 0}_{(n-2(n-k)+1)-\text{times}}) + \mathbb{R}^n_{\leq 0}.$$

The description of $\Pi_{n,k}$ follows quite naturally from the structure of the ideal $I_{n,k}$. In general, the vanishing ideal of a base configuration V_M is the ideal $I_{n,k}$ together with additional monomials corresponding to the circuits of the matroid M. As soon as we run few computational experiments, we notice that some classes of matroids appear to have a "well-behaved" Gröbner fan.

Conjecture 6.4.10. *The Gröbner fan of the vanishing ideal of a configuration V_M, for $M \in \mathcal{M}_2^{\mathsf{Lev}}$, is a coarsening of the braid arrangement fan.*

The conjecture is supported by several experiments we computed in `Gfan`. Unfortunately, all the symmetries of the maximal cones that we observe for the uniform matroids are destroyed as soon as we add the monomials of the circuits of M to the ideal. To conclude, we want to mention two other classes of matroids whose Gröbner fans appear to be coarsenings of the braid arrangement fan, namely lattice path matroids ([BdMN03]) and laminar matroids ([CCPV07]).

Bibliography

[AK06] Federico Ardila and Caroline J. Klivans, *The Bergman complex of a matroid and phylogenetic trees*, J. Combin. Theory Ser. B **96** (2006), no. 1, 38–49.

[ARW15] Federico Ardila, Felipe Rincón, and Lauren Williams, *Positroids and non-crossing partitions*, *To appear in* Trans. Amer. Math. Soc. (2015), Available at http://arxiv.org/abs/1308.2698.

[BLLR97] François Bergeron, Gilbert Labelle, Pierre Leroux, and Margaret Readdy, *Combinatorial species and tree-like structures*, Cambridge University Press, 1997.

[Blu01] Stefan Blum, *Base-sortable matroids and Koszulness of semigroup rings*, European J. Combin. **22** (2001), no. 7, 937–951.

[BGKN07] Manuel Bodirsky, Omer Giménez, Mihyun Kang, and Marc Noy, *Enumeration and limit laws for series-parallel graphs*, European J. Combin. **28** (2007), no. 8, 2091–2105.

[BM08] John A. Bondy and Uppaluri S. R. Murty, *Graph theory*, Graduate Texts in Mathematics, vol. 244, Springer, New York, 2008.

[BdMN03] Joseph Bonin, Anna de Mier, and Marc Noy, *Lattice path matroids: enumerative aspects and Tutte polynomials*, J. Combin. Theory Ser. A **104** (2003), no. 1, 63–94.

[BGW03] Alexandre V. Borovik, Israel M. Gel′fand, and Neil White, *Coxeter matroids*, Progress in Mathematics, vol. 216, Birkhäuser Boston, Inc., Boston, MA, 2003.

[Bry71] Thomas H. Brylawski, *A combinatorial model for series-parallel networks*, Trans. Amer. Math. Soc. **154** (1971), 1–22.

[CCPV07] Gruia Calinescu, Chandra Chekuri, Martin Pál, and Jan Vondrák, *Maximizing a submodular set function subject to a matroid constraint (extended abstract)*, Integer programming and combinatorial optimization, Lecture Notes in Comput. Sci., vol. 4513, Springer, Berlin, 2007, pp. 182–196.

[CO03] Brahim Chaourar and James Oxley, *On series-parallel extensions of uniform matroids*, European J. Combin. **24** (2003), no. 7, 877–879.

[CE80] William H. Cunningham and Jack Edmonds, *A combinatorial decomposition theory*, Canad. J. Math. **32** (1980), no. 3, 734–765.

[DLRS10] Jesús A. De Loera, Jörg Rambau, and Francisco Santos, *Triangulations*, Algorithms and Computation in Mathematics, vol. 25, Springer-Verlag, Berlin, 2010.

[Die90] Reinhard Diestel, *Graph decompositions. A study in infinite graph theory*, Oxford Science Publications, Oxford University Press, New York, 1990.

[Dir67] Gabriel A. Dirac, *Minimally 2-connected graphs*, J. Reine Angew. Math. **228** (1967), 204–216.

[Drm09] Michael Drmota, *Random trees*, Springer Wien New York, Vienna, 2009.

[DFK+11] Michael Drmota, Éric Fusy, Mihyun Kang, Veronika Kraus, and Juanjo Rué, *Asymptotic study of subcritical graph classes*, SIAM J. Discrete Math. **25** (2011), no. 4, 1615–1651.

[Edm70] Jack Edmonds, *Submodular functions, matroids, and certain polyhedra*, Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969), Gordon and Breach, New York, 1970, pp. 69–87.

[EH12] Viviana Ene and Jürgen Herzog, *Gröbner bases in commutative algebra*, Graduate Studies in Mathematics, vol. 130, American Mathematical Society, Providence, RI, 2012.

[FFF+] Yuri Faenza, Samuel Fiorini, Vissarion Fisikopoulos, Marco Macchia, and Kanstantsin Pashkovich, *Survey on 2-level polytopes*, In preparation.

[FS05] Eva Maria Feichtner and Bernd Sturmfels, *Matroid polytopes, nested sets and Bergman fans*, Port. Math. (N.S.) **62** (2005), no. 4, 437–468.

[FKPT13a] Samuel Fiorini, Volker Kaibel, Kanstantsin Pashkovich, and Dirk Oliver Theis, *Combinatorial bounds on nonnegative rank and extended formulations*, Discrete Math. **313** (2013), no. 1, 67–83.

[FKPT13b] ______, *Combinatorial bounds on nonnegative rank and extended formulations*, Discrete Math. **313** (2013), no. 1, 67–83.

[FMP+15] Samuel Fiorini, Serge Massar, Sebastian Pokutta, Hans Raj Tiwary, and Ronald de Wolf, *Exponential lower bounds for polytopes in combinatorial optimization*, arXiv:1111.0837v5 (2015).

[Fis] Vissarion Fisikopoulos, *Computational results on 2-level polytopes*, `http://homepages.ulb.ac.be/~vfisikop/data/2-level.html`.

[FS09] Philippe Flajolet and Robert Sedgewick, *Analytic combinatorics*, Cambridge University Press, Cambridge, 2009.

[FKR00] Tamás Fleiner, Volker Kaibel, and Günter Rote, *Upper bounds on the maximal number of facets of 0/1-polytopes*, European J. Combin. **21** (2000), no. 1, 121–130.

[GGMS87] Izrail′ M. Gel′fand, Mark Goresky, Robert D. MacPherson, and Vera V. Serganova, *Combinatorial geometries, convex polyhedra, and Schubert cells*, Adv. Math. **63** (1987), no. 3, 301–316.

[GGK+13] João Gouveia, Roland Grappe, Volker Kaibel, Kanstantsin Pashkovich, Richard Z. Robinson, and Rekha R. Thomas, *Which nonnegative matrices are slack matrices?*, Linear Algebra Appl. **439** (2013), no. 10, 2921–2933.

[GPT10] João Gouveia, Pablo A. Parrilo, and Rekha R. Thomas, *Theta bodies for polynomial ideals*, SIAM J. Optim. **20** (2010), no. 4, 2097–2118.

[GPT13] ______, *Lifts of convex sets and cone factorizations*, Math. Oper. Res. **38** (2013), no. 2, 248–264.

[GRT13] João Gouveia, Richard Z. Robinson, and Rekha R. Thomas, *Polytopes of minimum positive semidefinite rank*, Discrete Comput. Geom. **50** (2013), no. 3, 679–699.

[GR15] Francesco Grande and Juanjo Rué, *Many 2-level polytopes from matroids*, *To appear in* Discrete Comput. Geom. (2015), Available at arXiv:1409.2233.

[GS14] Francesco Grande and Raman Sanyal, *Theta rank, levelness, and matroids minors*, arXiv:1408.1262 (2014).

[GS] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at `http://www.math.uiuc.edu/Macaulay2`.

[GLS93] Martin Grötschel, László Lovász, and Alexander Schrijver, *Geometric algorithms and combinatorial optimization*, second ed., Algorithms and Combinatorics, vol. 2, Springer-Verlag, Berlin, 1993.

[Grü03] Branko Grünbaum, *Convex polytopes*, second ed., Graduate Texts in Mathematics, vol. 221, Springer-Verlag, New York, 2003.

[Han56] Olof Hanner, *Intersections of translates of convex bodies*, Math. Scand. **4** (1956), 65–87.

[Han77] Allan B. Hansen, *On a certain class of polytopes associated with independence systems*, Math. Scand. **41** (1977), no. 2, 225–241.

[HR03] Gábor Hegedűs and Lajos Rónyai, *Gröbner bases for complete uniform families*, J. Algebraic Combin. **17** (2003), no. 2, 171–180.

[HRGZ04] Martin Henk, Jürgen Richter-Gebert, and Günter M. Ziegler, *Basis properties of convex polytopes*, Handbook of discrete and computational geometry, 2nd ed., 355-382, Chapman & Hall/CRC Press, Boca Raton, 2004.

[KW00] Volker Kaibel and Martin Wolff, *Simple 0/1-polytopes*, European J. Combin. **21** (2000), no. 1, 139–144.

[Kim10] Sangwook Kim, *Flag enumerations of matroid base polytopes*, J. Combin. Theory Ser. A **117** (2010), no. 7, 928–942.

[KRGSZ97] Ulrich H. Kortenkamp, Jürgen Richter-Gebert, Aravamuthan. Sarangarajan, and Günter M. Ziegler, *Extremal properties of* 0/1-*polytopes*, Discrete Comput. Geom. **17** (1997), no. 4, 439–448.

[LP] Thomas Lam and Alexander Postnikov, *Polypositroids*, In preparation.

[LP07] ———, *Alcoved polytopes* I, Discrete Comput. Geom. **38** (2007), no. 3, 453–478.

[Lau09] Monique Laurent, *Sums of squares, moment matrices and optimization over polynomials*, Emerging applications of algebraic geometry, IMA Vol. Math. Appl., vol. 149, Springer, New York, 2009, pp. 157–270.

[Mat02] Jiří Matoušek, *Lectures on Discrete Geometry*, Graduate Texts in Mathematics, vol. 212, Springer-Verlag, New York, 2002.

[MNWW11] Dillon Mayhew, Mike Newman, Dominic Welsh, and Geoff Whittle, *On the asymptotic proportion of connected matroids*, European J. Combin. **32** (2011), no. 6, 882–890.

[McM76] Peter McMullen, *Constructions for projectively unique polytopes*, Discrete Math. **14** (1976), no. 4, 347–358.

[OVW14] Michael Oelze, Arnaud Vandaele, and Stefan Weltge, *Computing the extension complexities of all 4-dimensional 0/1-polytopes*, arXiv:1406.4895 (2014).

[Oh09] Suho Oh, *Combinatorics of positroids*, 21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009), Discrete Math. Theor. Comput. Sci. Proc., AK, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2009, pp. 721–732.

[OH01] Hidefumi Ohsugi and Takayuki Hibi, *Convex polytopes all of whose reverse lexicographic initial ideals are squarefree*, Proc. Amer. Math. Soc. **129** (2001), no. 9, 2541–2546 (electronic).

[Oxl89] James G. Oxley, *The regular matroids with no* 5*-wheel minor*, J. Combin. Theory Ser. B **46** (1989), no. 3, 292–305.

[Oxl11] ———, *Matroid theory*, second ed., Oxford Graduate Texts in Mathematics, vol. 21, Oxford University Press, Oxford, 2011.

[Plu68] Michael D. Plummer, *On minimal blocks*, Trans. Amer. Math. Soc. **134** (1968), 85–94.

[PRW08] Alex Postnikov, Victor Reiner, and Lauren Williams, *Faces of generalized permutohedra*, Doc. Math. **13** (2008), 207–273.

[Pos06] Alexander Postnikov, *Total positivity, grassmannians, and networks*, arXiv:math/0609764v1 (2006).

[RS04] Neil Robertson and Paul D. Seymour, *Graph minors. XX. Wagner's conjecture*, J. Combin. Theory Ser. B **92** (2004), no. 2, 325–357.

[Rué] Juanjo Rué, *Many 2-level polytopes from matroids (maple file)*, `http://www-ma2.upc.edu/jrue/research.html`.

[SWZ09] Raman Sanyal, Axel Werner, and Günter M. Ziegler, *On Kalai's conjectures concerning centrally symmetric polytopes*, Discrete Comput. Geom. **41** (2009), no. 2, 183–198.

[Sch86] Alexander Schrijver, *Theory of linear and integer programming*, Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons, Ltd., Chichester, 1986.

[Sch03a] ______, *Combinatorial optimization. Polyhedra and efficiency. Vol. A*, Algorithms and Combinatorics, vol. 24, Springer-Verlag, Berlin, 2003, Paths, flows, matchings, Chapters 1–38.

[Sch03b] ______, *Combinatorial optimization. Polyhedra and efficiency. Vol. B*, Algorithms and Combinatorics, vol. 24, Springer-Verlag, Berlin, 2003.

[Sta77] Richard P. Stanley, *Eulerian partitions of a unit hypercube*, Higher Combinatorics (M. Aigner, ed.) (1977), 49.

[Sta80] ______, *Decompositions of rational convex polytopes*, Ann. Discrete Math. **6** (1980), 333–342, Combinatorial mathematics, optimal designs and their applications (Proc. Sympos. Combin. Math. and Optimal Design, Colorado State Univ., Fort Collins, Colo., 1978).

[Sta86] ______, *Two poset polytopes*, Discrete Comput. Geom. **1** (1986), no. 1, 9–23.

[Stu96] Bernd Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996.

[Sul06] Seth Sullivant, *Compressed polytopes and statistical disclosure limitation*, Tohoku Math. J. (2) **58** (2006), no. 3, 433–445.

[Tiw] Hans R. Tiwary, *Introduction to extended formulations*, `http://kam.mff.cuni.cz/~tancer/Rynartice/hans_raj_tiwary.pdf`.

[Tut01] William T. Tutte, *Graph theory*, Encyclopedia of Mathematics and its Applications, vol. 21, Cambridge University Press, Cambridge, 2001.

[VGGT14] Arnaud Vandaele, Nicolas Gillis, François Glineur, and Daniel Tuyttens, *Heuristics for exact nonnegative matrix factorization*, arXiv:1411.7245v1 (2014).

[Whi86] Neil White (ed.), *Theory of matroids*, Encyclopedia of Mathematics and its Applications, vol. 26, Cambridge University Press, Cambridge, 1986.

[Yan91] Mihalis Yannakakis, *Expressing combinatorial optimization problems by linear programs*, J. Comput. System Sci. **43** (1991), no. 3, 441–466.

[Zie95] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995.

[Zie00] ———, *Lectures on* 0/1-*polytopes*, Polytopes—combinatorics and computation (Oberwolfach, 1997), DMV Sem., vol. 29, Birkhäuser, Basel, 2000, pp. 1–41.

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Zusammenfassung

Der Theta-Rang einer endlichen Punktkonfiguration V ist der maximal benötigte Grad, um eine beliebige, nicht-negative, lineare Funktion auf V als Summe von Quadraten darzustellen. Diese Zahl ist eine wichtige Invariante für Probleme der polynomiellen Optimierung und ist im Allgemeinen schwer zu bestimmen. Wir untersuchen den Theta-Rang einer Punktkonfiguration mittels *levelness*, einer Invariante aus der diskreten Geometrie, und klassifizieren die 2-level (d.h. Theta-1) Konfigurationen deren konvexe Hülle ein simples oder simpliziales Polytop ist.

Wir betrachten Konfigurationen, die man aus der Familie von Basen eines Matroids erhält und zeigen, dass die Klasse von Matroiden mit beschränktem Theta-Rang beziehungsweise levelness abgeschlossen bezüglich Minoren ist. Dies gestattet es, Matroide mit beschränktem Theta-Rang oder beschränkter levelness durch verbotene Minoren zu charakterisieren.

Die vollständige (endliche) Liste ausgeschlossener Minoren wird für Theta-1 Matroide angegeben, die den Fall von *series-parallel graphs* verallgemeinern. Zudem lässt sich die Klasse der Theta-1 Matroide über den *degree of generation* des Verschwindungs-Ideals sowie über den psd-Rang des assozierten Matroidenbasispolytops bestimmen.

Theta-1 Matroide sind *sort-closed*. Dies gestattet es, unimodulare Triangulierungen des Matroidpolytops zu finden und sein Volumen mittels Permutationen zu charakterisieren.

Asymptotische Schranken für die Anzahl an Theta-1 Matroiden auf einer Grundmenge mit fester Größe werden gefunden. Somit gelingt es auch, eine exponentielle untere Schranke an die Anzahl von 2-*level polytopes* einer beliebigen aber festen Dimension anzugeben.

Es wird bewiesen, dass k-level Matroide für $k > 2$ sich durch nur endlich viele ausgeschlossene Minoren beschreiben lassen. Zudem wird eine Charakterisierung von k-*level graphs* durch verbotene Minoren angegeben und die verbotenen Minoren für Graphen von Theta-Rang 2 untersucht.

Der nicht-negative Rang und Gröbnerfächer von Hypersimplizes – also Matroidbasispolytopen von uniformen Matroiden – werden vollständig beschrieben. Vermutungen über mögliche Verallgemeinerungen auf Theta-1 Matroide werden präsentiert.

Eidesstattliche Erklärung

Gemäß §7 (4) der Promotionsordnung des Fachbereichs Mathematik und Informatik der Freien Universität Berlin versichere ich hiermit, dass ich alle Hilfsmittel und Hilfen angegeben und auf dieser Grundlage die Arbeit selbständig verfasst habe. Des Weiteren versichere ich, dass ich diese Arbeit nicht schon einmal zu einem früheren Promotionsverfahren eingereicht habe.

Berlin, den 16. Juli

Francesco Grande