Introduction

The reliable simulation of contact problems is of great importance in many applications. These include the production of gear boxes or tires, the design of cars, metal forming processes or implants in biomechanics. For example, when replacing a damaged joint by an endoprosthesis, both surfaces of the joint are replaced by artificial materials. This gives rise to a contact problem. Nowadays, the components of prostheses are made of stainless steel, chrome cobalt molybdenum alloy, titanium and high density polyethylene. For the applied loads, these materials can be assumed to behave linearly elastic.

The mathematical modeling of a contact problem gives rise to partial differential equations. More precisely, the actual contact zone is unknown and as a consequence free boundary value problems are obtained. Unfortunately, an analytical solution for the arising systems of nonlinear partial differential equations is generally not known. We refer the reader to [Her82] for a special case where the solution can be computed analytically. Thus, reliable and efficient numerical solution techniques are extremely important. The efficiency of the simulation depends strongly on the use of special numerical algorithms. In combination with the possibilities of modern high performance computing, this has led to an increased research activity in the field of numerical contact mechanics.

In 1933, Signorini [Sig33] considered the frictionless contact of an elastic body with a rigid foundation. Since then, Signorini's problem with and without friction has been discussed by many authors, e.g., [DL72, EJ98, Has92, Sch88, Kin82a, KO88]. Although constituting a significant simplification over an elastic contact problem with friction, the construction of fast and reliable solvers for Signorini's problem is even nowadays a challenging task. This is due to the intrinsic non-differentiable nonlinearity of the problem, which is common to all contact problems. A priori, the subset of the body's boundary which actually comes into contact with the foundation is not known. In particular, the body's displacement has to satisfy a non-penetration condition, which causes the nonlinearity of the contact problem.

Very often, penalty methods or global active set strategies are used to solve the arising discrete system. In case of a penalty method, the accuracy of the solution depends strongly on the penalty parameter. To obtain an accurate solution, the penalty parameter has to be very small. This gives rise to a bad condition number of the resulting linear system. On the other hand, global active set strategies are easy to realize, but require a sequence of linear problems to be solved. In general, this leads to a high computational cost.

For linear elliptic problems, standard multigrid methods give rise to efficient iterative solvers of optimal complexity. In this work, we introduce and analyze a new nonlinear multigrid method for contact problems with friction. We point out that our method does not use any regularization. Moreover, nonlinear contact problems can be solved within the same optimal complexity as linear elliptic problems. This is of particular importance for threedimensional contact problems with friction.

In Chapter 1, we introduce the basic concepts of linear elasticity. We introduce the concept of stress axiomatically and discuss the linearized quantities. As a result, we obtain the equations of linear elasticity which describe the displacements of an elastic body subjected to volume and traction forces. In addition to the boundary value problem of linear elasticity in its strong formulation, we also give it's weak formulation and its formulation as a minimization problem. Minimization is done with respect to the energy functional, which is closely associated to the principle of virtual work introduced in Section 1.2. The energy functional will be of particular importance for constructing our nonlinear solver in Chapter 3.

Chapter 2 is devoted to the description of Signorini's problem. It is is given in its strong as well as in its weak formulation. Here, the weak formulation is given by a variational inequality. The special energy functional associated with Signorini's problem is introduced. Unfortunately, it turns out to be nonlinear and nondifferentiable at the contact boundary. This constitutes the main difficulty in constructing a solver for the arising discrete system. Furthermore, we give some results from convex analysis, which are necessary for the construction and analysis of our method.

In Chapter 3, we introduce and analyze our method. We define our monotone multigrid method as an extended relaxation using sophisticated truncated functions for the coarse grid corrections. The global convergence of the method is shown. Moreover, the discrete contact zone is shown to be detected after a finite number of iteration steps. Special emphasis is put on the case of varying normals at the contact boundary. As it turns out, for a curvilinear boundary the standard multigrid methods cannot be applied without losing their optimal complexity. The remedy is to use our truncated coarse grid functions. They give rise to a globally convergent nonlinear solver as well as a linear multigrid method. Global control of the nonlinear iteration process is provided by successive minimization of the energy.

In Chapter 4, we explain the concept of our implementation of the method. Due to an object oriented implementation, we can solve scalar obstacle problems and obstacle problems involving systems of equations using the same code. We show where the structural aspects of the developed method are reflected by the structure of the developed code. The resulting code can be regarded as an abstract tool for turning linear subspace solvers into nonlinear ones. This is demonstrated in Section 4.6, where we apply our code to an algebraic multigrid method.

Numerical examples in Chapter 5 in two and three space dimensions illustrate the performance of our new algorithm. Here, the performance of the method within the nonlinear and the linear phase of the solution process is discussed. A first test concerns the accuracy of the computed discrete boundary stresses. In a second test, we consider the influence of highly varying normals at the contact boundary. Finally, we consider the behavior of the method in case of a bad start iterate. For each example, we consider the convergence rate of our method with respect to the asymptotic linear problem, i.e., the problem to be solved when the discrete contact zone has been identified.

Contact problems with Coulomb friction are investigated in Chapter 6. Unfortunately, the functional associated with frictional contact problems is nonconvex, nonquadratic and nondifferentiable. To overcome this difficulty, we use a fixed point iteration in the normal stresses. In each step of this fixed point iteration, we have to solve a contact problem with Tresca friction, which leads to a nondifferentiable but convex functional. The necessary modifications needed to solve contact problems with Tresca friction are explained. Moreover, we propose a Gauß–Seidel type algorithm. Using this special Gauß–Seidel variant, no outer iteration is necessary. Numerical results for both algorithms are given.

In our last Chapter, we present a nonlinear Dirichlet–Neumann algorithm for the elastic contact of two bodies. The information transfer at the interface between the bodies is realized in terms of dual Lagrange multiplier spaces known from nonconforming domain decomposition. The resulting nonlinear Dirichlet–Neumann algorithm can be used for solving elastic contact problems with and without friction. Numerical examples in two and three space dimensions illustrate the performance of our algorithm.

Acknowledgments: It is a great pleasure for me to thank Prof. Dr. R. Kornhuber, who introduced me to this field. In particular, I would like to thank him for his support throughout this work. I wish to thank Prof. Dr. Dr. h.c. Deuflhard, Prof. Dr. Wendland and Prof. Dr. Wittum for their interest in my work. This work was supported in part by the Deutsche Forschungsgemeinschaft. I would also like to thank my colleagues in Berlin and Stuttgart for the friendly atmosphere. I wish to thank Priv.-Doz. Christian Wieners, who gave me an introduction to UG. My thanks are also to my friends and my family for their support. In particular, I would like to thank Barbara for her encouragement, criticism and many fruitful discussions.

Berlin, July 2001 *Rolf Hermann Krause*