Appendix B

Mathematical Supplement

This appendix provides some mathematical results not found in the literature in the specific form necessary for this work. Their proofs are either outlined or given in detail.

B.1 Spreading of regular wave packets

In this section, a mathematical theorem regarding the spreading of free Dirac wave packets is formulated. It is applied in the existence and orthogonality proofs for the wave operators in sections 3.3 and 3.4.

A free Dirac wave packet $\phi(\boldsymbol{x})$ is a linear superposition of plane waves:

$$\phi(\boldsymbol{x}) = (2\pi)^{-\frac{3}{2}} \int \mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{p}}\, \hat{\phi}(\boldsymbol{p})\ \mathrm{d}^3 p. \tag{B.1}$$

Here $\hat{\phi}(\boldsymbol{p})$ denotes the Fourier transform of $\phi(\boldsymbol{x})$, which obviously must be four-spinors. The time-evolution of $\phi(\boldsymbol{x})$ is most easily written in terms of the Fourier transform $\hat{\phi}(\boldsymbol{p})$. It is necessary for that purpose to make a spectral decomposition of the state space with respect to the free Dirac Hamiltonian $H_0 = -\mathrm{i}\boldsymbol{\alpha}\cdot\nabla + \beta$. Orthogonal projectors $P_{\mathrm{C}\pm}$ onto the spectral subspaces of H_0 of positive and negative energy respectively,

$$P_{\mathrm{C}\pm} = \frac{1}{2}\left(1 \pm \frac{H_0}{|H_0|}\right),$$

are given in momentum space by a simple multiplication operator:

$$P_{\mathrm{C}\pm} = \frac{\mu(\boldsymbol{p}) \pm \boldsymbol{p}\cdot\boldsymbol{\alpha} \pm \beta}{2\mu(\boldsymbol{p})}$$

(see e.g. [Tha92, Sch95]). Here,

$$\mu(\boldsymbol{p}) = \sqrt{1+\boldsymbol{p}^2},$$

denotes the relativistic energy of a free electron with momentum $\boldsymbol{p}$. Due to the property,

$$P_{\mathrm{C}+} + P_{\mathrm{C}-} = 1,$$

the Fourier transform $\hat{\phi}(\boldsymbol{p})$ may be decomposed by means of these projectors $P_{\mathrm{C}\pm}$ into a sum of two functions,

$$\hat{\phi}(\boldsymbol{p}) = \hat{\phi}_+(\boldsymbol{p}) + \hat{\phi}_-(\boldsymbol{p}),$$

where

$$\hat{\phi}_\pm(\boldsymbol{p}) = P_{\mathrm{C}\pm}\hat{\phi}(\boldsymbol{p}).$$

The free time-evolution $\Phi(t, \boldsymbol{x}) = \mathrm{e}^{-\mathrm{i}tH_0}\,\phi(\boldsymbol{x})$ of the initial wave packet $\phi(\boldsymbol{x})$ is then given by,

$$\Phi(t, \boldsymbol{x}) = (2\pi)^{-\frac{3}{2}} \int \mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{p}} \left\{ \mathrm{e}^{\mathrm{i}t\mu(\boldsymbol{p})}\,\hat{\phi}_+(\boldsymbol{p}) + \mathrm{e}^{-\mathrm{i}t\mu(\boldsymbol{p})}\,\hat{\phi}_-(\boldsymbol{p}) \right\} \mathrm{d}^3 p,$$

because in the momentum representation and on the spectral subspaces of H_0, which have been introduced above, the free unitary time-evolution $\mathrm{e}^{-\mathrm{i}tH_0}$ is simply a multiplication operator (see e.g. [Tha92, Sch95]).

B.1.1 Regular wave packets. In this work, a regular wave packet is defined as a wave packet $\Phi(t, \boldsymbol{x}) = \mathrm{e}^{-\mathrm{i}tH_0}\,\phi(\boldsymbol{x})$ where each component of the Fourier transform $\hat{\phi}(\boldsymbol{x})$ has compact support and is infinitely differentiable. Making use of the notation common in the mathematical physics literature [Rud74, RS80, Kat80], a regular wave packet, therefore, satisfies by definition:

$$\hat{\phi}(\boldsymbol{x}) \in C_0^\infty(\mathbb{R}^3)^4.$$

The term ‘regular wave packet’ is taken from the corresponding definition in the case of the Klein-Gordon field in [RS79, p. 42], where the term smooth solution is used synonymously.

The importance of regular wave packets of the free Dirac equation in this work is manifested in following property, which is useful in scattering theory. A regular free Dirac wave packet (in three spatial dimensions) satisfies for any time t and coordinate $\boldsymbol{x}$ the following inequality:

$$\|\Phi(t, \boldsymbol{x})\|_2 \leq \frac{\text{const.}}{(1 + |t|)^{3/2}}. \tag{B.2}$$

Since the L^2-norm $\|\Phi(t)\|$ is time-independent this inequality describes the spatial spreading of the wave packet for large times t.

A mathematical proof of this statement is possible with the aid of the method of stationary phase [Hör76, RS79]. Noting that $\hat{\phi}_+(\boldsymbol{x}), \hat{\phi}_-(\boldsymbol{x}) \in C_0^\infty(\mathbb{R}^3)^4$ for regular wave packets, mainly the Corollary to Theorem XI.15 in [RS79] has to be be applied. Although the explicit proof in the case of the Dirac equation was not found in the literature, its details will not be presented here. A similar result for regular wave packets of the Klein-Gordon equation constitutes Theorem XI.17(b) in [RS79].

B.2 Lorentz invariance of the scalar product

In this section, it is proved that the scalar product $(\Psi_1(t), \Psi_2(t))$ between two wave functions $\Psi_1(t, \boldsymbol{x})$ and $\Psi_2(t, \boldsymbol{x})$ is invariant under Lorentz-boosts, if they are both solutions of the same Dirac equation. Lorentz-invariance means that the scalar product $(\Psi_1'(t'), \Psi_2'(t'))'$ in a Lorentz-transformed frame between the Lorentz-transformed wave functions $\Psi_1'(t', \boldsymbol{x}')$ and $\Psi_2'(t', \boldsymbol{x}')$ satisfies,

$$(\Psi_1'(t'), \Psi_2'(t'))' = (\Psi_1(t), \Psi_2(t))\,, \tag{B.3}$$

for arbitrary t and t'. The assumptions necessary to prove this result will be stated in the subsequent presentation. Only Lorentz boosts will be considered since rotations and translations do not transform the time axis and the time-independence of the scalar product is equivalent to the unitarity of the time evolution. The existence of such a unitary time evolution will be assumed here.

Consider two Dirac wave functions $\Psi_1(t, \boldsymbol{x})$ and $\Psi_2(t, \boldsymbol{x})$ which are solutions of the same Dirac equation,

$$[H_0 + W(t, \boldsymbol{x}) - \mathrm{i}\partial_t]\, \Psi_i(t, \boldsymbol{x}) = 0, \qquad i = 1, 2.$$

The external field $W(t, \boldsymbol{x})$ is required to be a hermitian matrix, i.e.

$$W(t, \boldsymbol{x})^\dagger = W(t, \boldsymbol{x}),$$

which is true in particular for external electromagnetic fields $(A^0, \boldsymbol{A})$, where $W(t, \boldsymbol{x}) = q(A^0 - \boldsymbol{\alpha} \cdot \boldsymbol{A})$. The hermitian conjugate Dirac spinors $\Psi_i^\dagger(t, \boldsymbol{x})$ then solve the following hermitian conjugate equation,

$$\mathrm{i}\nabla \cdot (\Psi_i^\dagger \boldsymbol{\alpha}) + \Psi_i^\dagger \gamma^0 + \mathrm{i}\partial_t \Psi_i^\dagger + \Psi_i^\dagger W = 0.$$

By taking the difference between the hermitian conjugate equation for Ψ_1 multiplied from the right by Ψ_2 and the Dirac equation for Ψ_2 multiplied from the left by $\Psi_1^\dagger$ one obtains:

$$\nabla \cdot (\Psi_1^\dagger \boldsymbol{\alpha} \Psi_2) + \partial_t(\Psi_1^\dagger \Psi_2) = 0.$$

Recalling the definition of the adjoint spinor, $\overline{\Psi} = \Psi^\dagger \gamma^0$, this equation may be rewritten as the four-divergence of a complex Lorentz four-vector,

$$\partial_\mu(\overline{\Psi}_1 \gamma^\mu \Psi_2) = 0. \tag{B.4}$$

The familiar continuity equation of the four-current density $j^\mu = q\overline{\Psi}\gamma^\mu\Psi$ is an implication of this result.

In the following $\Lambda = \Lambda(\boldsymbol{v})$ shall denote a pure Lorentz boost from an unprimed Lorentz frame to a primed frame moving with velocity $\boldsymbol{v}$ with respect to the unprimed frame:

$$\begin{aligned} (\Lambda^\mu{}_\nu) &= \begin{pmatrix} \gamma & -\gamma \boldsymbol{v}^{\mathrm{T}} \\ -\gamma \boldsymbol{v} & \left(1 + (\gamma - 1)\hat{\boldsymbol{v}}\hat{\boldsymbol{v}}^T\right) \end{pmatrix}, \\ x'^\mu &= \Lambda^\mu{}_\nu x^\nu. \end{aligned} \tag{B.5}$$

Again, γ is the Lorentz factor corresponding to the velocity $\boldsymbol{v}$. In order to show the invariance of the scalar product of Dirac spinors under Lorentz boosts we note that the scalar product at time $t = a$ in the unprimed frame is an integral over a three-dimensional flat hypersurface of Minkowski space,

$$\int_{t=a} \Psi_1^\dagger(t, \boldsymbol{x}) \Psi_2(t, \boldsymbol{x})\, \mathrm{d}^3x. \tag{B.6}$$

The same comment applies to the scalar product computed at time $t' = b$ in the primed frame,

$$\int_{t'=b} \Psi_1'^\dagger(t', \boldsymbol{x}') \Psi_2'(t', \boldsymbol{x}')\, \mathrm{d}^3x'. \tag{B.7}$$

The hyperplanes $t = a$ and $t' = b$ may be characterised using the unit timelike normal vectors n^μ and m^μ pointing forward in time:

$$\begin{aligned} \partial D^{\mathrm{a}} &= \{x : n_\mu x^\mu - a = 0, n^\mu = (1, \mathbf{0})\} \\ \partial D^{\mathrm{b}} &= \{x : m_\mu x^\mu - b = 0, m^\mu = \gamma(1, \boldsymbol{v})\}. \end{aligned}$$

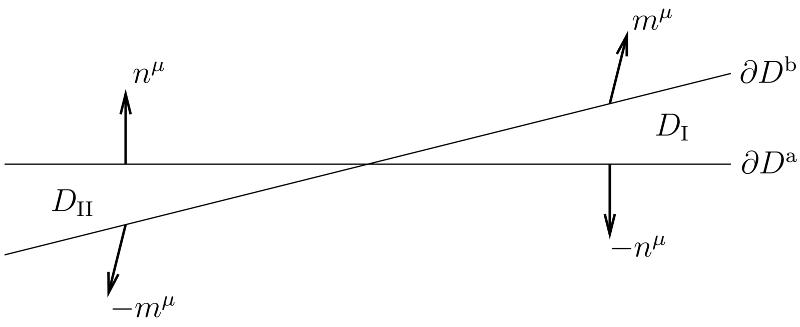

FIGURE B.1. Sketch of four-wedges.

The hyperplanes ∂D^{a} and ∂D^{b} are depicted in figure B.1. The Minkowski space is cut into four subsets by these hyperplanes. Two of them, necessary in the subsequent calculation, are given by the following definitions:

$$D_{\mathrm{I}} = \{x : n_\mu x^\mu \geq a \text{ and } m_\mu x^\mu \leq b\} = \Big\{x : a \leq x^0 \leq \gamma^{-1} b + \boldsymbol{v} \cdot \boldsymbol{x}\Big\},$$
$$D_{\mathrm{II}} = \{x : n_\mu x^\mu \leq a \text{ and } m_\mu x^\mu \geq b\} = \Big\{x : \gamma^{-1} b + \boldsymbol{v} \cdot \boldsymbol{x} \leq x^0 \leq a\Big\}.$$

Their distinctive feature is that the intersections of D_{I} and D_{II} respectively with the four-dimensional cylinder of radius R, defined through $\boldsymbol{x}^2 \leq R^2$, have finite volume in Minkowski space. The boundaries ∂D_{I} and ∂D_{II} of D_{I} and D_{II} respectively may be decomposed uniquely into flat bounded hypersurfaces,

$$\partial D_{\mathrm{I}} = \partial D_{\mathrm{I}}^{\mathrm{a}} \cup \partial D_{\mathrm{I}}^{\mathrm{b}} \quad \text{and} \quad \partial D_{\mathrm{II}} = \partial D_{\mathrm{II}}^{\mathrm{a}} \cup \partial D_{\mathrm{II}}^{\mathrm{b}},$$

such that the following decomposition is valid at the same time:

$$\partial D^{\mathrm{a}} = \partial D_{\mathrm{I}}^{\mathrm{a}} \cup \partial D_{\mathrm{II}}^{\mathrm{a}} \quad \text{and} \quad \partial D^{\mathrm{b}} = \partial D_{\mathrm{I}}^{\mathrm{b}} \cup \partial D_{\mathrm{II}}^{\mathrm{b}}.$$

See figure B.1 in order to understand this quite formal definitions easily.

To complete the proof, the scalar product (B.7) in the primed frame is rewritten as follows:

$$\begin{aligned}
&\int_{t'=b} {\Psi'_1}^\dagger(t', \boldsymbol{x}')\Psi'_2(t', \boldsymbol{x}')\,\mathrm{d}^3x' \\
=\ &\int_{t'=b} \overline{\Psi}_1(\Lambda^{-1}(t', \boldsymbol{x}'))\,\gamma(\gamma^0 - \boldsymbol{v}\cdot\boldsymbol{\gamma})\,\Psi_2(\Lambda^{-1}(t', \boldsymbol{x}'))\,\mathrm{d}^3x' \\
=\ &\int_{\partial D^{\mathrm{b}}} \overline{\Psi}_1(t, \boldsymbol{x})\gamma^\mu\Psi_2(t, \boldsymbol{x})\,g_{\mu\nu}\,m^\nu\,\mathrm{d}S(x) \\
=\ &\int_{\partial D_{\mathrm{I}}^{\mathrm{b}}} \overline{\Psi}_1(x)\gamma^\mu\Psi_2(x)\,m_\nu\,\mathrm{d}S(x)\ - \int_{\partial D_{\mathrm{II}}^{\mathrm{b}}} \overline{\Psi}_1(x)\gamma^\mu\Psi_2(x)\,(-m_\nu)\,\mathrm{d}S(x)
\end{aligned}$$

Here $\mathrm{d}S(x)$ denotes the hypersurface element at the space-time point x (following the notation of [For84]). The scalar product in the unprimed frame may be written in a similar form:

$$\begin{aligned}
&\int_{t=a} {\Psi_1}^\dagger(t, \boldsymbol{x})\Psi_2(t, \boldsymbol{x})\,\mathrm{d}^3x \\
=\ &-\int_{\partial D_{\mathrm{I}}^{\mathrm{a}}} \overline{\Psi}_1(x)\gamma^\mu\Psi_2(x)\,(-n_\nu)\,\mathrm{d}S(x)\ + \int_{\partial D_{\mathrm{II}}^{\mathrm{a}}} \overline{\Psi}_1(x)\gamma^\mu\Psi_2(x)\,n_\nu\,\mathrm{d}S(x)
\end{aligned}$$

The unit four-vectors appearing in the integrands above are the outer normal vectors on the boundaries of the four-volumes D_{I} and D_{II} respectively. If the solutions $\Psi_1(x)$

and $\Psi_2(x)$ decay sufficiently rapidly at *spatial* infinity for all t then the integral theorem of Gauß in four dimensions [For84, Das93] may be used to conclude that the difference between the scalar products (B.6) and (B.7) is given in terms of four-dimensional volume integrals over the 'wedges' D_{I} and D_{II}:

$$\begin{aligned} &\int_{t'=b} {\Psi'_1}^\dagger(t',\boldsymbol{x}')\Psi'_2(t',\boldsymbol{x}')\,\mathrm{d}^3x' - \int_{t=a} {\Psi_1}^\dagger(t,\boldsymbol{x})\Psi_2(t,\boldsymbol{x})\,\mathrm{d}^3x \\ = &\int_{D_{\mathrm{I}}} \partial_\mu(\overline{\Psi}_1\gamma^\mu\Psi_2)\,\mathrm{d}^4x - \int_{D_{\mathrm{II}}} \partial_\mu(\overline{\Psi}_1\gamma^\mu\Psi_2)\,\mathrm{d}^4x \end{aligned}$$

In order to understand the reasoning, remember that the finite volume integral over the intersection of D_{I} with the four-dimensional cylinder of radius R converges to the integral over D_{I} itself as the radius R goes to infinity. The Gaussian theorem may be used to rewrite this finite volume integral as a sum over hypersurface integrals over the boundary of the volume of the intersection. It is then noted that the hypersurface integral over that part of the four-dimensional cylinder which appears in this sum vanishes as $R \to \infty$. We conclude that both four-volume integrals vanish as a result of equation (B.4) and hence the scalar products (B.6) and (B.7) are equal and equation (B.3) is verified.

Obviously, the invariance property proved here implies the orthonormality of orthonormal stationary Dirac eigenstates after their Lorentz transformation to a moving frame.

The idea of using the integral theorem of Gauß was taken from the discussion of the free Dirac equation in [Sch95]. Thirring gives a similar proof for the Lorentz boost invariance of the total charge in classical electrodynamics [Thi90, (1.3.18,2)]. Note also that for vanishing external field $W(t,\boldsymbol{x})$ the Lorentz invariance (B.3) proved in this section is a consequence of the fact that Lorentz boosts, only in this particular situation, are represented by a time-independent unitary operator $\exp(-\mathrm{i}\boldsymbol{v}\cdot\boldsymbol{N})$. The self-adjoint generator of this unitary transform is given by $\boldsymbol{N} = \frac{1}{2}(H_0\boldsymbol{x} + \boldsymbol{x}H_0)$ where H_0 is the free Dirac-Hamiltonian [Tha92]. Such a generator does not exist if the Dirac field is subject to a time-dependent external field.

B.3 Transformations of eigenstates

Consider an eigenfunction $\psi(\boldsymbol{x})$ of a time-independent Dirac-Hamiltonian $H_0 + W(\boldsymbol{x})$ with eigenvalue ϵ,

$$[H_0 + W(\boldsymbol{x})]\,\psi(\boldsymbol{x}) = \epsilon\,\psi(\boldsymbol{x}). \tag{B.8}$$

Indeed, the external field $W(\boldsymbol{x})$ does not necessarily need to originate in a minimally coupled external electromagnetic field. Other kinds of covariant external fields, like scalar potentials or non-minimally coupled electromagnetic fields (Pauli term, etc.), are not explicitly excluded in this section. (See, for example, [Tha92] for a complete classification of covariant external fields.) The time-dependent wave function,

$$\Psi(t,\boldsymbol{x}) = \exp(-\mathrm{i}t\epsilon)\,\psi(\boldsymbol{x}),$$

solves the corresponding time-dependent Dirac equation,

$$[H_0 + W(\boldsymbol{x}) - \mathrm{i}\partial_t]\,\Psi(t,\boldsymbol{x}) = 0. \tag{B.9}$$

Clearly, $\Psi(t, \boldsymbol{x})$ is also an eigenfunction of $H_0 + W(\boldsymbol{x})$ for any time t.

B.3.1 Lorentz boosts. In the unprimed coordinate system of equations (B.9) and (B.8) the hermitian matrix $W(\boldsymbol{x})$ shall not have an explicit time-dependence, as emphasised above. Now consider primed coordinates $(t', \boldsymbol{x}')$ obtained by a Lorentz boost with velocity $\boldsymbol{v}$ from the unprimed coordinates $(t, \boldsymbol{x})$, as in equation (B.5). As usual, let $S(\Lambda)$ denote the four-spinor representation matrix of the Lorentz boost $\Lambda^{\mu}{}_{\nu}$, corresponding to the representation of the γ-matrices employed in the definition of the free Dirac Hamiltonian H_0. The Lorentz transform of the time-dependent Dirac equation (B.9) in the primed frame is given by,

$$[H_0' + W'(t', \boldsymbol{x}') - \mathrm{i}\partial_{t'}]\, \Psi'(t', \boldsymbol{x}') = 0, \tag{B.10}$$

with $\Psi'(t', \boldsymbol{x}') = S(\Lambda)\Psi(t, \boldsymbol{x})$ and $(t, \boldsymbol{x}) = \Lambda^{-1}(t', \boldsymbol{x}')$. Due to the Lorentz boost to the unprimed frame, the transformed external field $W'(t', \boldsymbol{x}')$, generally given by [Tha92],

$$W'(t', \boldsymbol{x}') = S(\Lambda)^{-1^{\dagger}}\, W\big(\Lambda^{-1}(t', \boldsymbol{x}')\big)\, S(\Lambda)^{-1}, \tag{B.11}$$

picks up an explicit (though trivial) time-dependence. Here $W(t, \boldsymbol{x}) = W(\boldsymbol{x})$ has been introduced only to simplify the notation. Since, by construction, $\Psi'(t', \boldsymbol{x}')$ solves equation (B.10), the following holds:

$$\begin{aligned} H'(t')\, \Psi'(t', x') &= \mathrm{i}\partial_{t'}\, \Psi'(t', x') \\ &= S(\Lambda)\, \mathrm{i}\partial_{t'}\, \{\exp(-\mathrm{i}t\epsilon)\psi(\boldsymbol{x})\} \\ &= \epsilon\, S(\Lambda) \exp(-\mathrm{i}t\epsilon)\psi(\boldsymbol{x}) \frac{\partial t}{\partial t'} + S(\Lambda) \exp(-\mathrm{i}t\epsilon)\, (\mathrm{i}\partial_i \psi(\boldsymbol{x}))\, \frac{\partial x^i}{\partial t'} \\ &= \gamma\epsilon\, \Psi'(t', x') - \gamma\, S(\Lambda) \exp(-\mathrm{i}t\epsilon)\, \boldsymbol{v} \cdot (-\mathrm{i}\nabla\psi(\boldsymbol{x})), \end{aligned}$$

with $\gamma = (1 - \boldsymbol{v}^2)^{-1/2}$. This means that $\Psi'(t', x')$ is an eigenstate of $H_0' + W'(t', \boldsymbol{x}')$, if and only if the eigenfunction $\psi(\boldsymbol{x})$ of $H_0 + W(\boldsymbol{x})$ in the unprimed reference frame, is also an eigenfunction of the momentum operator $\boldsymbol{P} = -\mathrm{i}\nabla$. In fact the latter condition is equivalent to the property, that the external potential $W(\boldsymbol{x})$ of equation (B.8) is a constant, i.e. does not depend on the unprimed spatial coordinate $\boldsymbol{x}$. Clearly, this is precisely the case of free motion, and furthermore the only case, where the Lorentz-boosted Hamilton operator $H'(t')$ is time-independent.

Hence, it is meaningful only in that circumstance to Lorentz-transform the energy eigenvalue ϵ to a moving frame. The usual transformation law of the energy-momentum four-vector is then retained from the preceding calculation:

$$\begin{aligned} H'(t')\, \Psi'_{p'}(t', x') &= \gamma\epsilon\, \Psi'_{p'}(t', x') - \gamma S(\Lambda) \exp(-\mathrm{i}t\epsilon)\, \boldsymbol{v} \cdot (\boldsymbol{p}\Psi_p(\boldsymbol{x})) \\ &= \gamma(\epsilon - \boldsymbol{v} \cdot \boldsymbol{p})\Psi'_{p'}(t', x') \;=\; \epsilon' \Psi'_{p'}(t', x'). \end{aligned}$$

Here the usual notation for energy-momentum four-vectors p and p' is employed, with $p = (\epsilon, \boldsymbol{p})$ and $p' = \Lambda p = (\epsilon', \boldsymbol{p}')$.

B.3.2 Galilean boosts in nonrelativistic quantum theory. The same problem may be addressed in nonrelativistic quantum mechanics. In order to discuss this similarity briefly, consider a solution $\Phi(t,\boldsymbol{x})$ of the time-dependent Schrödinger equation,

$$\Big[-\frac{1}{2}\nabla^2 + V(\boldsymbol{x}) - \mathrm{i}\partial_t\Big]\,\Phi(t,\boldsymbol{x}) = 0,$$

which is of the form $\Phi(t,\boldsymbol{x}) = \exp(-\mathrm{i}t\epsilon)\,\phi(\boldsymbol{x})$ and, therefore, an eigenstate of the time-independent Hamiltonian $H = -\frac{1}{2}\nabla^2 + V(\boldsymbol{x})$. The Galilean-boosted Schrödinger wave function,

$$\Phi'(t,\boldsymbol{x}') = \exp\!\Big(-\tfrac{\mathrm{i}}{2}t\boldsymbol{v}^2\Big)\exp(\mathrm{i}\boldsymbol{v}\cdot\boldsymbol{x}')\,\exp(-\mathrm{i}t\epsilon)\,\phi(\boldsymbol{x}'+t\boldsymbol{v}),$$

corresponding to the passive Galilean boost, $\boldsymbol{x}' = \boldsymbol{x} - t\boldsymbol{v}$, with boost velocity $\boldsymbol{v}$, solves the Galilean-transformed Schrödinger equation,

$$\Big[-\frac{1}{2}\nabla'^2 + V(\boldsymbol{x}'+t\boldsymbol{v}) - \mathrm{i}\partial_t\Big]\,\Phi'(t,\boldsymbol{x}') = 0.$$

In the nonrelativistic Schrödinger theory, the Galilean-boosted wave function $\Phi'(t,\boldsymbol{x}')$ is likewise *not* an eigenfunction of the Galilean-boosted time-dependent Hamiltonian,

$$H'(t) = -\frac{1}{2}\nabla'^2 + V(\boldsymbol{x}'+t\boldsymbol{v}),$$

at any time t, except the external potential $V(\boldsymbol{x})$ is a constant. This is easily verified as in the previous subsection. However, contrary to the relativistic case, supposing that $\phi(\boldsymbol{x})$ is a bound state, the energy expectation value $\bar{\epsilon}'$ in the primed Galilean frame is given by a simple expression:

$$\bar{\epsilon}' = \Big(\Phi'(t), H'(t)\,\Phi'(t)\Big) = \epsilon + \frac{\boldsymbol{v}^2}{2}.$$

Moreover, the energy uncertainty is time-independent and grows at most linearly with the modulus of the boost velocity $\boldsymbol{v}$, since the following estimate holds:

$$0 \le \Big(\Phi'(t), H'(t)^2\,\Phi'(t)\Big) - \Big(\Phi'(t), H'(t)\,\Phi'(t)\Big)^2 \le \boldsymbol{v}^2\int \phi(\boldsymbol{x})^*(-\nabla^2\phi)(\boldsymbol{x})\,\mathrm{d}^3x.$$

Therefore, $\Phi'(t,\boldsymbol{x})$ is an approximate eigenstate of the Galilean-boosted Schrödinger operator $H'(t)$ for small boost velocities $\boldsymbol{v}$.

B.3.3 Local gauge transformations. The discussion of local gauge transformations can be carried out along the lines of the discussion of Lorentz boosts. We sketch it briefly. It is well-known already in nonrelativistic and relativistic classical mechanics that the Hamiltonian is not a gauge-invariant observable [Thi88]. The same is true in (non-)relativistic quantum theory. The locally gauge-transformed Dirac spinor,

$$\breve{\Psi}(t,\boldsymbol{x}) = \exp(-\mathrm{i}g(t,\boldsymbol{x}))\,\exp(-\mathrm{i}t\epsilon)\,\psi(\boldsymbol{x}),$$

solves the gauge-transformed Dirac equation, $\Big[H_0 + \breve{W}(t,\boldsymbol{x}) - \mathrm{i}\partial_t\Big]\,\breve{\Psi}(t,\boldsymbol{x}) = 0$, with the transformed external field,

$$\breve{W}(t,\boldsymbol{x}) = W(\boldsymbol{x}) + \{(\partial_t + \boldsymbol{\alpha}\cdot\nabla)g(t,\boldsymbol{x})\}\,.$$

The gauge-transformed wave function $\breve{\Psi}(t, \boldsymbol{x})$ is an eigenfunction of the transformed Dirac operator $\breve{H}(t) = H_0 + \breve{W}(t, \boldsymbol{x})$, if and only if the gauge function $g(t, \boldsymbol{x})$is time-independent. Then, of course, $\breve{W}$ is time-independent as well.