PRINCIPAL G-BUNDLES
ON NODAL CURVES

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Introduction

Moduli spaces appear in algebraic geometry when trying to provide an algebraic variety structure to the set of equivalence classes of certain objects. Once equipped with such structure, the dimension at a point of this variety indicates how many parameters are needed to determine the equivalence class corresponding to that point, and, therefore, we partially solve the classification problem we started with. In order to make easier the study of the geometry of these spaces, we have to address the problem of compactifying them, that is, we have to find the *limiting objects*.

The aim of this work is the classification of principal G-bundles on nodal curves via the construction of a compact moduli space for such objects. Moreover, we construct this moduli space in such a way that it behaves well under degeneration of smooth curves into stable curves.

Although the classification of principal bundles has importance itself, the last 20 years much interest has emerged in some areas of theoretical physics. For instance, In the series of articles [13, 14, 15], R. Friedman, J. Morgan and E. Witten address the construction of these moduli spaces over elliptic curves and their behavior along elliptic fibrations. They realize the importance of theses moduli spaces for understanding the duality between *heteoretic string and F-theory*.

Also, in [34], an algebro-geometric version of the Gromov-Witten invariants studied in symplectic geometry through vortex equations is introduced. This leads to a gauge Gromov-Witten theory. Stable maps to a geometric quotient are given by *decorated principal bundles* on nodal (in fact stable) curves. A modular completion and the construction of a compact coarse moduli space of such objects, involves understanding compact moduli spaces of principal G-bundles on nodal curves.

Historical Introduction and Known Results

The problem of classification of bundles on curves began in the late 50's of the last century with the works of A. Grothendieck [20] and M. Atiyah [3] which were focused on the classification of vector bundles over the projective line and elliptic curves respectively. However, it was not until 1963 when D. Mumford provided, in [38], an algebraic (quasi-projective) variety structure to the set of isomorphism classes of stable vector bundles on a projective smooth curve of genus grater or equal than 2. Thus, he gave the first construction of such a moduli space. For that, he developed *geometric invariant theory* [42], started by D. Hilbert and many others in the 18th century, giving a central role to the concept of *stability*. A few years later, in 1967, C. S. Seshadri [56] gave a natural compactification of D. Mumford's moduli space by including the *semistable* vector bundles. In the next fifteen years much work was done for generalizing the above constructions for irreducible curves [44] and, more generally, for reduced curves [57]. In the first case, *semistable torsion-free sheaves* were needed and in the second case the problem was solved by including *semistable sheaves of depth one.*

R. Pandharipande gave, in 1996, a very remarkable construction. He showed that there exists a scheme, relatively projective over the moduli space of stable curves $\overline{M}_g$ such that the fiber over a point, $[C] \in \overline{M}_g$, coincide with the compact moduli space of stable torsion-free sheaves (quoted out by the group of automorphisms of C) over the curve C.

Since vector bundles of rank r can be seen as principal $\mathrm{GL}(n)$-bundles, a natural question comes up: is it possible to get similar constructions for a general algebraic group G over smooth algebraic curves? This question was answered positively by A. Ramanathan in his Ph.D thesis, which was published in 1996 [47]. A central problem in this work is the concept of *semistability for principal G-bundles*, with G a reductive algebraic group. In his construction, A. Ramanathan used the existence of the moduli space of vector bundles by considering the Lie algebra, $\mathrm{Lie}(G)$, of G. In 2002, A. Schmitt [49] realised that a principal G-bundle can always be seen as a pair formed by a vector bundle and certain morphism of sheaves of algebras, once a fully faithful representation $\rho\colon G \hookrightarrow \mathrm{SL}(n)$ is fixed. He constructed the compact moduli space of so called *δ-semistable singular principal G-bundles* over smooth projective varieties. Although this moduli space depends *a priori* on the fixed representation, he proved that the moduli space constructed by A. Ramanathan agrees with it. In 2005, he generalized this construction to the case of an irreducible nodal curve with one node [52], and in 2013, A. Langer [35], gave a construction of a compact moduli space of δ-semistable singular principal G-bundles in the case of a family of irreducible projective varieties.

This work will be focused on the construction of a compactification of the moduli space of principal G-bundles on a stable curve. Therefore, it will be given a special emphasis on the behavior of these objects along families of curves. The method of A. Schmitt, mentioned above, will be used for the construction. Let us describe it briefly.

Let X be a smooth projective curve of genus g and let G be a semisimple linear algebraic group. The heart of A. Schmitt's work consists on the following observation: given a faithful representation $\rho\colon G \hookrightarrow \mathrm{SL}(V)$ being V a complex vector space of dimension n there is a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes} \\ \text{of principal } G\text{-bundles on X} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{isomorphism classes of pairs } (\mathcal{E}, \tau) \\ \text{where } \mathcal{E} \text{ is a locally free sheaf of rank} \\ n \text{ with trivial determinant and} \\ \tau\colon X \to \underline{\mathrm{Isom}}_{\mathcal{O}_X}(V \otimes \mathcal{O}_X, \mathcal{E})/G \end{array} \right\}.$$

Here, a morphism between pairs, $(\mathcal{E}, \tau)$ and $(\mathcal{E}', \tau')$, is a morphism $f\colon \mathcal{E} \to \mathcal{E}'$ such that the diagram

$$\begin{tikzcd} \underline{\mathrm{Isom}}_{\mathcal{O}_X}(V \otimes \mathcal{O}_X, \mathcal{E})/G \arrow[rr, "f^\sharp"] & & \underline{\mathrm{Isom}}_{\mathcal{O}_X}(V \otimes \mathcal{O}_X, \mathcal{E}')/G \\ & X \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes, $f^\sharp$ being the morphism induced by f. Although the extra structure $\tau\colon X \to \underline{\mathrm{Isom}}_{\mathcal{O}_X}(V \otimes \mathcal{O}_X, \mathcal{E})/G$ is quite difficult to deal with, we can make it easier in the following way. Note that

$$\underline{\mathrm{Isom}}_{\mathcal{O}_X}(V \otimes \mathcal{O}_X, \mathcal{E}) \subset \underline{\mathrm{Hom}}_{\mathcal{O}_X}(V \otimes \mathcal{O}_X, \mathcal{E}),$$

and consider the GIT quotient $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(V \otimes \mathcal{O}_X, \mathcal{E})/\!\!/G = \underline{\mathrm{Spec}}(S^\bullet(V \otimes \mathcal{E}^\vee)^G)$, where $S^\bullet$ denotes the symmetric algebra. Since the group G can be identified with a subgroup of $\mathrm{SL}(V)$ via the representation ρ, we can show, looking at the fibers, that the determinant function defined on $\mathrm{Hom}(V, \mathcal{E}(x))$ is G-invariant, so $\mathrm{Isom}(V, \mathcal{E}(x))$ are the stable points of the above vector space. With this in hand one shows that there is an open immersion

$$\underline{\mathrm{Isom}}_{\mathcal{O}_X}(V \otimes \mathcal{O}_X, \mathcal{E})/G \subset \underline{\mathrm{Hom}}_{\mathcal{O}_X}(V \otimes \mathcal{O}_X, \mathcal{E})/\!\!/G.$$

Thus, in particular, giving a section $\tau\colon X \to \underline{\mathrm{Isom}}_{\mathcal{O}_X}(V \otimes \mathcal{O}_X, \mathcal{E})/G$ is equivalent to giving a morphism of algebras $\tau\colon \underline{\mathrm{Spec}}(V \otimes \mathcal{E}^\vee) \to \mathcal{O}_X$ such that the induced morphism $X \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(V \otimes \mathcal{O}_X, \mathcal{E})/\!\!/G$ takes values in the space of local isomorphisms. Therefore, one may give the following generalization: a *singular principal G-bundle over X* is a pair $(\mathcal{E}, \tau)$ where $\mathcal{E}$ is a locally free sheaf of rank r and $\tau\colon S^\bullet(V \otimes \mathcal{E})^G \to \mathcal{O}_X$ a non trivial morphism of $\mathcal{O}_X$-algebras. The geometric counterpart of a singular principal G-bundle is given by means of the following fibered product

$$\begin{tikzcd}
P(\mathcal{E},\tau) \arrow[r, dashed] \arrow[d, dashed] & \underline{\mathrm{Hom}}_{\mathcal{O}_X}(V \otimes \mathcal{O}_X, \mathcal{E}^\vee) \arrow[d] \\
X \arrow[r, "\tau"] & \underline{\mathrm{Hom}}_{\mathcal{O}_X}(V \otimes \mathcal{O}_X, \mathcal{E}^\vee)/\!\!/G.
\end{tikzcd}$$

If the image lies in the space of local isomorphisms, we recover the concept of a principal G-bundle. The next step is to generalize the concept of semistability for singular principal G-bundles, $(\mathcal{E}, \tau)$. The presence of the extra structure given by τ forces us to introduce a semistability parameter $\delta \in \mathbb{Q}_{>0}$. One, then, says that a singular principal G-bundle is δ-(semi)stable if for any weighted filtration $(\mathcal{E}_\bullet, \underline{m})$ the inequality

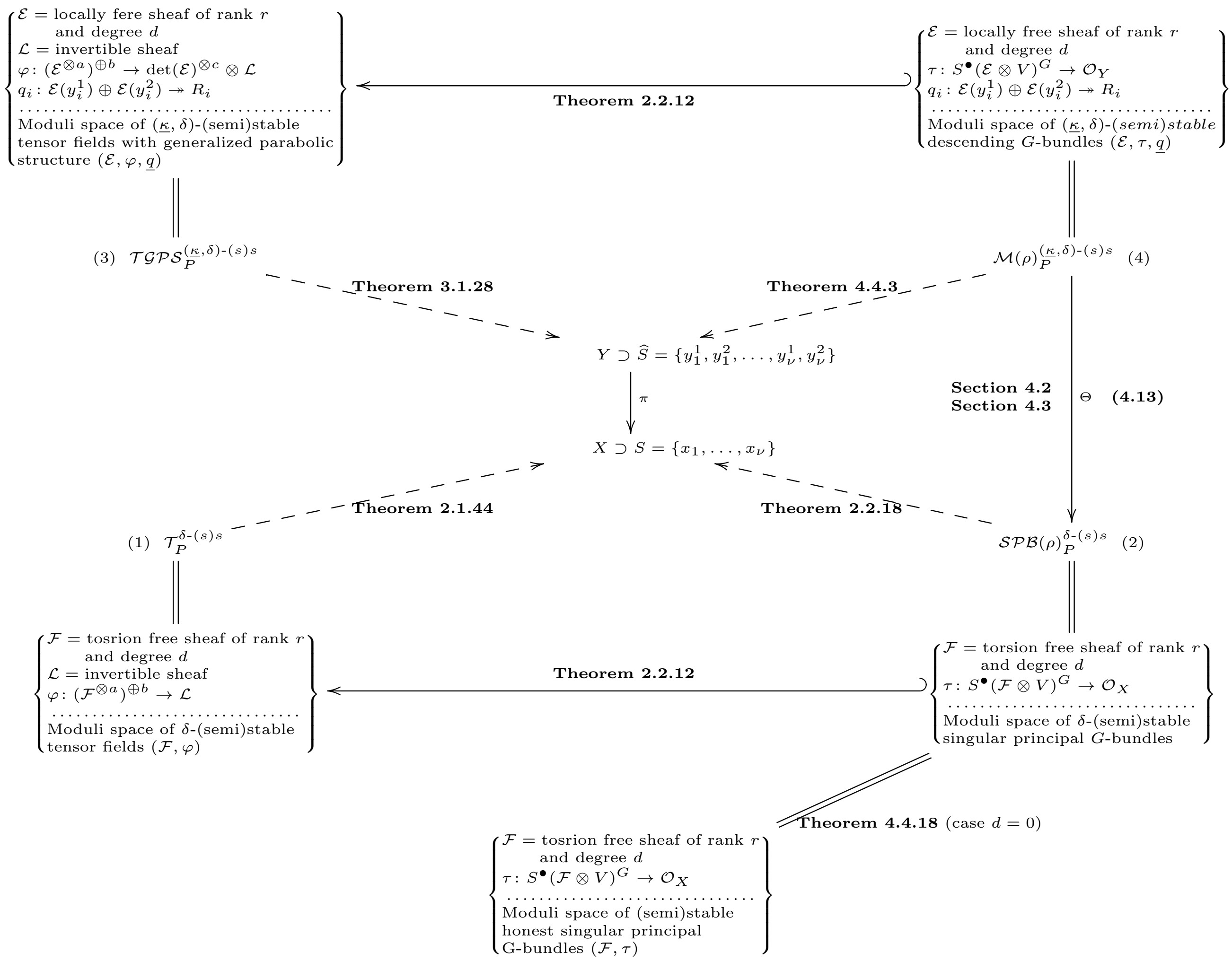
$$\sum_{i=1}^{s} m_i(P(\mathcal{E})\mathrm{rk}(\mathcal{E}_i) - P(\mathcal{E}_i)\mathrm{rk}(\mathcal{E})) + \delta\mu(\mathcal{E}_\bullet, \underline{m}, \tau)(\geq)0$$

holds true. The quantity $\mu(\mathcal{E}_\bullet, \underline{m}, \tau)$ is, essentially, the semistability function of the point τ_η with respect to the action of the group $\mathrm{SL}(\mathcal{E}_\eta)$ and the one parameter subgroup defined by the restriction of the filtration $(\mathcal{E}_\bullet, \underline{m})$, to the generic point $\eta \in X$. This let us to construct a coarse moduli space parametrizing δ-semistable singular principal G-bundles, $\mathrm{SPB}(\rho)_P^{\delta-(s)s}$, with Hilbert polynomial P. Then it is to shown that there is a closed (in fact also open) subscheme parametrizing *semistable* principal G-bundles in the sense of A. Ramanathan. Therefore, we find $\mathrm{M}(G) \subseteq \mathrm{SPB}(\rho)_P^{\delta-(s)s}$. The last step is to show $\mathrm{SPB}(\rho)_P^{\delta-(s)s} \subseteq \mathrm{M}(G)$ which follows from a global boundedness argument.

The case of an irreducible curve X with one node is more involved and uses torsion free sheaves instead of just locally free sheaves. First, the semistability notion is extended to singular principal G-bundles. Its complexity does not allow us to construct the corresponding moduli space of semistable (honest) singular principal G-bundles directly. Instead, A. Schmitt uses the theory of generalized parabolic bundles on the normalization $Y \to X$ to construct it, following the ideas of U. Bhosle for vector bundles [6]. On Y, the moduli space of *$(\underline{\kappa}, \delta)$-semistable descending principal G-bundles* is constructed. Descending principal G-bundles are those principal G-bundles with a generalized parabolic structure which descends to singular principal G-bundles on X. This construction leads to a morphism $\mathrm{M}(\rho)^{(\underline{\kappa},\delta)\text{-(s)s}} \to \mathrm{SPB}(\rho)_P^{\delta\text{-(s)s}}$ for certain values of the semistability parameters. Thus, the main moduli space is defined as the schematic image of this morphism. A. Schmitt proves that, if the representation ρ takes values in the symplectic group the last morphism is surjective and for large values of δ, $\mathrm{SPB}(\rho)_P^{\delta-(s)s}$ parametrizes semistable (honest) singular principal G-bundles.

Structure and Results of this Work

As we have said before the aim of this work is to construct a compact moduli space for principal G-bundles over a nodal curve X. The construction process of this moduli space, based on A. Schmitt' work, can be summarized in the following schema:



In Chapter 1 we give the background in GIT, coherent sheaves of depth one over reduced projective curves and principal G-bundles. In Section 1 we present some examples for the calculation of the Hilbert-Mumford semistability function which will be crucial in Chapter 3. In Section 2 we present the basic theory of coherent sheaves of depth one on reduced projective curves. We characterize them as the subsheaves of locally free sheaves with torsion cokernel (Theorem 1.2.16), therefore we complete [56, Lemma 5]. The main result in this section is Lemma 1.2.28, which will allow us to prove, among other results, Theorem 2.2.12. Finally, in Section 3, we state the basic theory of principal G-bundles following the classical work of J-P. Serre [55].

Chapter 2 is devoted to the construction of $\mathrm{SPB}(\rho)_P^{\delta\text{-(s)s}}$. In Section 1 we construct the moduli space of δ-(semi)stable tensor fields over X, $\mathcal{T}_P^{\delta\text{-(s)s}}$ (Theorem 2.1.44). We follow [8, 17] closely. Since our curve X is not irreducible we have to change ranks by multiplicities in the definition of δ-semistasbility (see Definition 2.1.9). In Section 2 we construct the moduli space of δ-semistable singular principal G-bundles, $\mathrm{SPB}(\rho)_P^{\delta\text{-(s)s}}$ (Theorem 2.2.18). We first show how to assign to any singular principal G-bundle a tensor field, for what we need to linearize the problem (Theorem 2.2.6). This is done by using some result on graded algebras (Lemma 2.2.5). After, we need to show that this assignment is injective (Theorem 2.2.12), making use of Lemma 1.2.28, which is a kind of a vanishing theorem for sections of torsion free sheaves on certain opens subsets. In this way, we built the moduli space as a closed subscheme of the moduli space of tensor fields. At this stage we make an important observation. One of the main purposes of this work is to lay the groundwork for the construction of a compactification of the universal moduli space of principal G-bundles over $\overline{M}_g$. This is the idea behind the proof of Theorem 2.2.6, since we can show that the linearization can be done uniformly along $\overline{M}_g$ (see Remark 2.2.7).

In Chapter 3 we deal with the upper level of the conceptual schema. In Section 1 we construct the moduli space of tensor fields with generalized parabolic structures over a (possibly) non connected smooth projective curve $Y = \coprod_{i=1}^{l} Y_i$. The semistability condition will depend now on $\nu + 1$ (rational) parameters, $\kappa_1, \ldots, \kappa_\nu, \delta$, due to the presence of the extra structure given by the parabolic structure. We show that the right Gieseker space in which the parameter space is embedded is not a cross product of as many Gieseker spaces (as in the irreducible case) as components we have (see Subsection 3.1.4). The right polarization is found in Subsection 3.1.5. The comparison between GIT semistability in the parameter space and $(\underline{\kappa}, \delta)$-semistability is presented in Theorem 3.1.24. The moduli space of $(\underline{\kappa}, \delta)$-semistable singular principal G-bundles with generalized parabolic structures on Y is constructed, as in the nodal case, as a closed subscheme of the moduli space of tensor fields with generalized parabolic structure. Finally, we study semistability condition for the objects, that these moduli spaces represent, for large values of the semistability parameters. The existence of several minimal points in the curve Y makes impossible to translate the results of [52]. Since each minimal point gives us a (eventually) different function field, the first step is to put all the data in the same category. For that, we restrict the tensor field to each minimal point and then we make a base change to the function field of the smooth projective variety $Y_1 \times \ldots \times Y_l$.

In Chapter 4 we describe explicitly a method for representing a given singular principal G-bundle by a descending G-bundle, and compare the semistability notion of both objects for large values of the semistability parameters. With this in hand we can define the morphism

$$\Theta\colon \mathcal{M}(\rho)_P^{(\underline{\kappa},\delta)\text{-}(s)s} \to \mathrm{SPB}(\rho)_P^{\delta\text{-}(s)s}.$$

Proposition 4.4.6 and Proposition 4.4.7 show that the schematic image of Θ satisfies the same properties as in the irreducible case (see [52]). Thus, $\mathcal{M}_X(\rho) := \mathrm{Im}(\Theta)$ consists on semistable honest singular principal G-bundles and every stable honest singular principal G-bundle lies in $\mathcal{M}_X(\rho)$. Finally Theorem 4.4.18 shows that $\mathrm{SPB}(\rho)_P^{\delta\text{-}(s)s}$ parametrizes (semi)stable honest singular principal G-bundles, which generalizes the results given in [51] to any nodal curve.

Chapter 1

Preliminaries

1.1 Geometric Invariant Theory

In this section we give an introduction to Geometric Invariant Theory, which deals with the problem of constructing quotients of algebraic group actions on schemes. We follow closely [42], [44] and [53] to develop the basic theory, and we finish this section describing some important examples which will be crucial in the construction of the moduli spaces appearing along this work.
We fix for this section an algebraically closed field k of characteristic zero. Thanks to [30], great part of the geometric invariant theory developed by Mumford (see [42]), holds true in positive characteristic. However, Hilbert-Mumford criterium fails when the characteristic is positive, and we need to use it systematically throughout all this work.

1.1.1 First Definitions and Properties

Let X be a k-scheme and G an algebraic group over k acting on X. This action is given by a morphism of k-schemes $\sigma\colon G\times X\to X$ such that $\sigma(g,\sigma(g',x))=\sigma(g\cdot g',x)$ and $\sigma(e,x)=x$, $\forall g,g'\in G$ and $\forall x\in X$.

Definition 1.1.1. A pair (Y,ϕ) consisting of a scheme Y and a morphism $\phi\colon X\to Y$ is a *categorical quotient* of X by G if

a) $\phi(\sigma(g,x))=\phi(x)$,

b) given a pair (Z,φ) as before satisfying a), there is a unique morphism $\xi\colon Y\to Z$ such that $\varphi=\xi\circ\phi$.

If a categorical quotient exists then it is unique up to a cononical isomorphism.

Definition 1.1.2. A pair (Y,ϕ) consisting of a scheme Y and a morphism $\phi\colon X\to Y$

is a *good quotient* of X by G if

a) ϕ is affine, surjective and G-invariant,
b) if U is open in Y, then the morphisms of rings $\phi^*\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(\phi^{-1}(U))$ induces an isomorphism $\mathcal{O}_Y(U) \simeq \mathcal{O}_X(\phi^{-1}(U))^G$,
c) if $Z \subset X$ is closed and G-invariant then $\phi(Z)$ is closed,
d) if $Z_1,\ Z_2 \subset X$ are closed and G-invariant such that $Z_1 \cap Z_2 = \varnothing$ then, $\phi(Z_1) \cap \phi(Z_2) = \varnothing$.

Proposition 1.1.3. ([53, Lemma 1.4.1.1]) *Every good quotient of X by G is also a categorical quotient.*

Definition 1.1.4. A pair (Y, ϕ) consisting of a scheme Y and a morphism $\phi\colon X \to Y$ is a *geometric quotient* of X by G if it is a good quotient and $\phi^{-1}(y)$ consists of a single orbit for every geometric point $y \in Y$, i.e., the map $\overline{\phi}\colon X/G \to Y$ induced by ϕ is bijective.

Definition 1.1.5. A universal categorical (resp. good, geometric) quotient is a categorical (resp. good, geometric) quotient such that for every morphism $g\colon Y' \to Y$ the projection onto the second factor $\phi' := p_2\colon X \times_Y Y' \to Y'$ is a categorical (resp. good, geometric) quotient.

Some geometric properties of a categorical quotient are inherited from the scheme X on which G is acting on.

Proposition 1.1.6. ([42, Chap. 0, §2, (2)]). *Let (Y, ϕ) be a categorical quotient. If X is reduced (resp. connected, irreducible, locally integral, locally integral and normal) then Y is reduced (resp. connected, irreducible, locally integral, locally integral and normal).*

To understand the link between constructing quotients and proving the existence of certain moduli spaces we need, first of all, to introduce some basic definitions and the concept of (local) universal families.

Let $\mathbf{Sch}_k$ be the category of k-schemes and **Sets** the category of sets. For any scheme X we denote $h_X(-) = \mathrm{Hom}_{\mathbf{Sch}_k}(-, X)$ its functor of points. Consider a contravariant functor

$$\Phi\colon \mathbf{Sch}_k \to \mathbf{Sets}.$$

Definition 1.1.7. The functor Φ is representable if there exists a pair (M, f), M being a k-scheme and $f\colon \Phi \to h_M$ a natural transformation, such that f is an isomorphism.

A *moduli problem* is determined once we fix a *class of objects*, $\mathcal{A}$, an *equivalence relation* between them, "$\sim$", and the concept of *family of objects parametrized by a k-scheme S*, $\mathbf{A}(S)$, with an *equivalence relation* $\sim_S$. We require to the pairs $(\mathbf{A}(S), \sim_S)$ to satisfy some functorial properties, and that $(\mathbf{A}(\mathrm{Spec}(k)), \sim_{\mathrm{Spec}(k)}) = (\mathcal{A}, \sim)$. With precision, $\mathcal{A}$ is a category, $p\colon \mathbf{A} \to \mathbf{Sch}_k$ is a fibered category (see [21]), $\mathbf{A}$ endowed with an equivalence relation $\sim$, compatible with pullbacks and such that $\mathbf{A}(\mathrm{Spec}(k)) = \mathcal{A}$, and the equivalence relation $\sim$ satisfies that, restricted to $\mathbf{A}(\mathrm{Spec}(k))$, is precisely given by the isomorphisms of $\mathcal{A}$.

The moduli problem is hereby presented in a categorical way as a functor

$$\begin{aligned} \Phi_{\mathcal{A}} \colon \mathbf{Sch}_k &\to \mathbf{Sets} \\ S &\mapsto Obj\ \mathbf{A}(S)/\sim \end{aligned} \tag{1.1}$$

and the main question is whether this functor is representable or not. A positive answer solves the moduli problem and we call any representative $M \in \mathbf{Sch}$ a *fine moduli space*. Suppose this is the case. Then there is a canonical family parametrized by M which is given by $\mathcal{U} := f^{-1}(\mathrm{id}_M) \in \Phi(M)$ and we call it a *universal family*.

Although most of the moduli problems we can find do not admit *fine* solutions, we can partially solve them by proving the existence of a *coarse moduli space* (schemes whose geometric points may be identified with the geometric points of the functor (1.1)).

Definition 1.1.8. A pair (M, f), as in Definition 1.1.7, is a *coarse moduli space* for the moduli functor (1.1) if $\Phi_{\mathcal{A}}(k) = h_M(\mathrm{Spec}(k))$ and for any other pair, (N, g), as before, there exists a unique natural transformation

$$\Omega \colon h_M \to h_N$$

such that $g = \Omega \circ f$.

Note that in this case we can not define the concept of universal family. Instead we define:

Definition 1.1.9. A family $\mathcal{U}$ parametrized by a scheme $M \in \mathbf{Sch}_k$ is said to be *locally universal* if for any other family $\mathcal{U}'$ parametrized by a scheme M' and any point $m' \in M'$ there exists an open neighborhood $m' \in V \subset M'$ and a morphism $t \colon V \to M$ such that $t^*\mathcal{U} \sim \mathcal{U}'|_V$.

Now we can establish the link between moduli problems and quotients by algebraic groups

Proposition 1.1.10. ([44, Proposition 2.13]) *Suppose there exists a scheme M and a local universal family $\mathcal{U}$ parametrized by M for the moduli problem (1.1). Suppose, further, that there is an algebraic group G acting on M such that for any pair of points $m, n \in M$, $\mathcal{U}_m \sim \mathcal{U}_n$ if and only if both points lie in the same orbit. Then*

- (i) *any coarse moduli space is a categorical quotient of M by G,*
- (ii) *a categorical quotient of M by G is a coarse moduli space if and only if it is an orbit space, i.e., the fibers of $\phi \colon M \to M/G$ consist of single orbits.*

1.1.2 Affine Quotients

Let G be an algebraic group. For any G-module M we denote by $M^G \subset M$ the G-submodule of invariant elements. This operation defines a left exact endofunctor $(-)^G \colon \mathbf{G\text{-}Mod} \to \mathbf{G\text{-}Mod}$. A *Reynolds operator* in G is a (functorial) retract of G-modules $R(M) \colon M \to M^G$ for every G-module M. A G-module M is *simple or irreducible* if there is no non trivial G-submodule. The algebraic group G is *reductive* if every G-module is a direct sum of simple G-submodules.

The exactness of the functor $(-)^G$ and the existence of a Reynolds operator characterize reductive groups.

Theorem 1.1.11. ([48, Theorem 7.1, Theorem 7.2]) *The following conditions are equivalent*

a) G is reductive,

b) there exists a (unique) Reynolds operator in G,

c) the functor $(-)^G$ is exact.

The problem of the existence of the quotient of an affine scheme by the action of an algebraic reductive group is solved in the following theorem.

Theorem 1.1.12. ([42, Theorem 1.1.]) *Let $X = \mathrm{Spec}(A)$ be an affine scheme and G an algebraic reductive group acting on X. Then the induced morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A^G)$ is a universal categorical quotient. Moreover, if k is a field and A is a finitely generated (resp. noetherian) k-algebra then A^G is a finitely generated (resp. noetherian) k-algebra.*

The first part of the theorem is proved by using the existence of the Reynolds operator and the second part holds also in positive characteristic. This theorem is crucial for the proof of the existence of manageable moduli spaces in algebraic geometry.

1.1.3 The General Case. Semistability

We have shown the way to construct quotients of affine schemes. Now, to understand what is going on with more general schemes, let us describe the case $X = \mathrm{Proj}(A)$, A being a graded k-algebra (see [43]).

Let A be a finitely generated graded algebra over k. The degree 0 piece is assumed to be k. Let $A_+ = \oplus_{n>0} A_i$ be the irrelevant ideal of A, and as usual we denote it by $\{0\}$ as a point in the spectrum of A. The grading always induces an action of $\mathbb{G}_m$, given by

$$\begin{aligned} \mathbb{G}_m \times A_n &\to A_n \quad , \\ (\lambda, a) &\mapsto \lambda^{-n} a \end{aligned}$$

which leaves invariant the ideal $\{0\}$. Therefore, this action induces an action on $\mathrm{Spec}(A) - \{0\}$ and the quotient by $\mathbb{G}_m$ exists, giving us the usual homogeneous spectrum of A, that is, $\mathrm{Spec}(A) - \{0\}/\mathbb{G}_m = \mathrm{Proj}(A)$.

Let A and B be finitely generated algebras over k as before and let us consider a graded morphism of rings $f^{\#} \colon B \to A$ of degree 0. This morphism induces a morphism between spectra $f \colon \mathrm{Spec}(A) \to \mathrm{Spec}(B)$. However, the formation of the homogeneous spectrum does not transform, in general, graded morphisms between graded algebras into morphisms between the homogeneous spectra, since there might be points outside from the irrelevant ideal of B lying in the irrelevant ideal of A via $f^{\#}$. Despite this obstruction, we can still do something else. Define $S'(f) := f^{-1}(\{0\})$. Then we have a morphism

$$f \colon \mathrm{Spec}(A) \setminus S'(f) \to \mathrm{Spec}(B) \setminus \{0\},$$

and taking the quotient by $\mathbb{G}_m$ we get

$$\mathrm{Proj}(f) \colon \mathrm{Proj}(A) \setminus S(f) \to \mathrm{Proj}(B),$$

$S(f)$ being the closed subset defined as the image of $S'(f) \setminus \{0\}$ by the quotient morphism $\mathrm{Spec}(A) - \{0\} \to \mathrm{Proj}(A)$. Applying this general situation to the example $A = k[x_0, \ldots, x_n]$ and $B = A^G$ with G an algebraic reductive group acting on the space of polynomials of degree one (and therefore acting on the whole ring by extending it algebraically to higher degree polynomials), we get the following morphism

$$\mathrm{Proj}(F)\colon \mathbf{P}^n \setminus S(f) \to \mathrm{Proj}(A^G). \tag{1.2}$$

Here $S(f)$ is the closed subset on whose points the G-invariant homogeneous polynomials of high degree vanish (see [23], §2, 2.8) and the complement is thus the open space to which a point belongs if it satisfies that there exists a G-invariant homogeneous polynomial not vanishing on it. We will show that the morphism (1.2) is a good quotient and we will take the above as a first approach to the definition of *semistable point*, concluding that a quotient of a projective scheme exists over the semistable locus.

Let us start with the general formalism. Let X be a scheme, G an algebraic group acting on X via $\sigma\colon G \times X \to X$ and $\mathcal{L}$ an invertible sheaf on X.

Definition 1.1.13. A G-linearization of $\mathcal{L}$ consists of an isomorphism $\phi\colon \sigma^*\mathcal{L} \to p_2^*\mathcal{L}$ on $G \times X$ satisfying that the diagram (cocycle condition)

$$\begin{tikzcd}
(\sigma \circ (1_G \times \sigma))^*\mathcal{L} \arrow[rr, "(1_G\times\sigma)^*\phi"] \arrow[dr, "(\mu\times 1_X)^*\phi"'] & & (p_2 \circ (1_G \times \sigma))^*\mathcal{L} \arrow[dl, "p_{23}^*\phi"] \\
& (p_2 \circ (\mu \times 1_X))^*\mathcal{L} &
\end{tikzcd}$$

is commutative, where $\mu\colon G \times G \to G$ is the group law, $p_2\colon G \times G \times X \to G$ and $p_{23}\colon G \times G \times X \to G \times X$ are the obvious projections.

Remark 1.1.14. ([42, Chap. 1, §3])

1) A G-linearization of G on $\mathcal{L}$ can be understood as a lifting of the action on X to an action on the fibers of the associated line bundle $L := \underline{\mathrm{Spec}}\, S^\bullet\mathcal{L}$,

$$\begin{tikzcd}
G \times L \arrow[r, "p_2\circ\phi"] \arrow[d, "1_G\times\pi"'] & L \arrow[d, "\pi"] \\
G \times X \arrow[r, "\sigma"] & X
\end{tikzcd} .$$

2) It can also be understood as a (dual) action of $H^0(G, \mathcal{O}_G)$ on $H^0(X, \mathcal{L})$, when the space of golbal section is non zero,

$$H^0(X,\mathcal{L}) \xrightarrow{\sigma^*} H^0(G \times X, \sigma^*\mathcal{L}) \xrightarrow{\phi} H^0(G \times X, p_2^*\mathcal{L}) \simeq H^0(G, \mathcal{O}_G) \otimes H^0(X, \mathcal{L}).$$

The last isomorphism is given by the Künneth formula.

3) The group of isomorphism classes of G-linearized invertible sheaves is denoted by $\mathrm{Pic}^G(X)$.

Example 1.1.15. Let $X = \operatorname{Spec}(k)$, where k is a field. Any line bundle $\mathcal{L}$ on X is isomorphic to $\mathcal{O}_X = k$, and any action of an algebraic reductive group, $\sigma\colon G\times X \to X$, is equal to the second projection. Therefore a G-linearization on $\mathcal{L}$ is just an isomorphism $\phi\colon \mathcal{O}_G \simeq \mathcal{O}_G$; that is, an invertible function on G, $G \to \mathbb{G}_m$, satisfying the cocycle condition. These morphisms form precisely the characters of G.

Definition 1.1.16. Let $\mathcal{L}$ be an invertible sheaf on X and let $\phi\colon \sigma^*\mathcal{L} \to p_2^*\mathcal{L}$ be a G-linearization on $\mathcal{L}$. Then:

a) $x \in X$ is semistable with respect to $\mathcal{L}$ and ϕ if there exists $n \gg 0$ and a section $s \in H^0(X, \mathcal{L}^n)$ such that $s(x) \neq 0$, $X_s = \{x \in X|\ s(x) \neq 0\}$ is affine and s is an invariant section.

b) $x \in X$ is polystable with respect to $\mathcal{L}$ and ϕ if there exists $n \gg 0$ and a section $s \in H^0(X, \mathcal{L}^n)$ such that $s(x) \neq 0$, X_s is affine, s is invariant, and the action on $X_s = \{x \in X|s(x) \neq 0\}$ is closed.

c) $x \in X$ is stable with respect to $\mathcal{L}$ and ϕ if there exists $n \gg 0$ and a section $s \in H^0(X, \mathcal{L}^n)$, such that $s(x) \neq 0$, $X_s = \{x \in X|\ s(x) \neq 0\}$ is affine, s is an invariant section, the action of G on X_s is closed and the isotropy group is finite.

We denote $X^{ss}(\mathcal{L})$ (resp. $X^s(\mathcal{L})$) the (open) set of semistable (resp. stable) points. Both concepts, linearizations and semistablity, allow us to solve the problem of the existence of quotients for more general schemes by the action of an algebraic reductive group. It is shown that, although we can not ensure in general the existence of a quotient, we can do so for some open G-invariant subsets, once we fix a G-linearized invertible sheaf on the scheme, as we have seen in the example of the introduction of this subsection.

Theorem 1.1.17. ([42, Theorem 1.10]) *Let X be a scheme of finite type over a field k, G a reductive algebraic group acting on it and $\mathcal{L}$ a G-linearized invertible sheaf. Then a universal categorical quotient (Y, ϕ) of $X^{\mathrm{ss}}(\mathcal{L})$ exists, Y being quasi-projective, and there is an ample invertible sheaf $\mathcal{N}$ on Y such that $\phi^*(\mathcal{N}) \simeq \mathcal{L}^n$ for some n. Moreover, there is an open subset $Y^s \subset Y$ such that $\phi^{-1}(Y^s) = X^{\mathrm{s}}(\mathcal{L})$ and such that $\phi|_{X^{\mathrm{s}}(\mathcal{L})}\colon X^{\mathrm{s}}(\mathcal{L}) \to Y^{\mathrm{s}}$ is a universal geometric quotient.*

The formation of the semistable and stable locus with respect to some linearization enjoys some remarkable functorial properties. For instance, the (semi) stability condition does not change after changing the base field and have a nice behavior under finite pullbacks to projective schemes.

Proposition 1.1.18. ([42, Proposition 1.14]) *Let X be a scheme of finite type over k, G a reductive algebraic group acting on it and $\mathcal{L} \in \operatorname{Pic}^G(X)$. Let $k \subset K$ be any field extension and denote $\overline{\mathcal{L}}$ and $\overline{X}$ the invertible sheaf and the scheme after the field base change. Then $\overline{X}^{\mathrm{s}}(\overline{\mathcal{L}}) = \overline{X^{\mathrm{s}}(\mathcal{L})}$ and $\overline{X}^{\mathrm{ss}}(\overline{\mathcal{L}}) = \overline{X^{\mathrm{ss}}(\mathcal{L})}$.*

Proposition 1.1.19. ([42, Proposition 1.18, Theorem 1.19]) *Let X, Y be schemes of finite type over a field k, G an algebraic reductive group acting on them, $f\colon X \to Y$ a G-morphism and $\mathcal{L} \in \operatorname{Pic}^G(Y)$. If f is quasi-affine then $f^{-1}(Y^s(\mathcal{L})) \subset X^{\mathrm{s}}(f^*\mathcal{L})$. Moreover, if X is proper over k, f is finite and $\mathcal{L}$ is ample then $f^{-1}(Y^{(\mathrm{s})\mathrm{s}}(\mathcal{L})) = X^{(\mathrm{s})\mathrm{s}}(f^*\mathcal{L})$.*

1.1.4 Hilbert-Mumford Criterion

In this section we will assume that X is a projective scheme over a field k of characteristic zero acted on by an algebraic reductive group G. As we have seen before, the existence of the quotient of X by G cannot be ensured in the general case, but what we can do is to take the quotient of some open G-invariant subset (the semistable locus) of X by G, once we fix a G-linearized invertible sheaf. For this reason it is quite important to find out effective tools which can help us to identify the points lying in this open subset

First of all, the following proposition let us to reduce our efforts in looking for this tool to the case of the projective space $\mathbf{P}^n$,

Proposition 1.1.20. ([42, Proposition 1.7]) *Let $\mathcal{L}$ be a G-linearized invertible sheaf on X such that its sections have no common zeroes. Then the morphism $I\colon X \to \mathbf{P}^n$ induced by the complete linear system $H^0(X, \mathcal{L})$ is G-equivariant, the action on $\mathbf{P}^n$ being the one induced by the action of G on $H^0(X, \mathcal{L})$.*

Therefore, Proposition 1.1.19 and Proposition 1.1.20 imply that it will be enough to find out the (semi)stable points of $\mathbf{P}^n$ with respect to a G-linearization of $\mathcal{O}_{\mathbf{P}^n}(1)$ to describe the (semi)stable points of X with respect to $\mathcal{L} = I^*\mathcal{O}_{\mathbf{P}^n}(1)$ with the induced linearization.

The key construction ([42, Chap. 2 §1. Def. 2.2]) is the following. We come back to the general case in which X is a projective scheme over a field k, and consider $x \in X$ a closed point and $\lambda\colon \mathbb{G}_m \to G$ a one parameter subgroup. Consider the morphism $f_\eta := \sigma_x \circ \lambda\colon \mathbb{G}_m \to X$, σ_x being the orbit map associated to x induced by the action σ of G on X. We identify $\mathbb{G}_m$ with $\operatorname{Spec}(k[t]_{(t)}) \subset \mathbb{A}^1$. Since X is projective, we can extend f_η to a morphism $f\colon \mathbb{A}^1 \to X$. The closed point $f(0)$ is

$$f(0) = \lim_{t \mapsto 0} \sigma(\lambda(t), x),$$

and is fixed by the action of $\mathbb{G}_m$ induced from that of G via λ. Let $\mathcal{L}$ be a G-linearized invertible sheaf on X and consider the induced $\mathbb{G}_m$-linearization of $\mathcal{L}|_{f(0)}$. By Example 1.1.15 we know that this linearization is given by a character $\chi(t) = t^\gamma$, $\gamma \in \mathbb{Z}$. We define

$$\mu^{\mathcal{L}}(x, \lambda) := -\gamma. \tag{1.3}$$

Proposition 1.1.21. ([42, Chap.2, §1]) *The integer $\mu^{\mathcal{L}}(x, \lambda)$ enjoys the following remarkable properties*

(*i*) $\mu^{\mathcal{L}}(\sigma(t, x), \lambda) = \mu^{\mathcal{L}}(x, t^{-1} \cdot \lambda \cdot t),\ t \in \mathbb{G}_m$.

(*ii*) *Fix x and λ, $\mu^{\mathcal{L}}(x, \lambda)$ is a morphism of groups as a function of $\mathcal{L}$.*

(*iii*) *If $f\colon X \to Y$ is G-equivariant, $\mathcal{L} \in \operatorname{Pic}^G(Y)$ and $x \in X$ is a closed point, then $\mu^{f^*\mathcal{L}}(x, \lambda) = \mu^{\mathcal{L}}(f(x), \lambda)$.*

Let us give now an interpretation of the integer defined in (1.3) in the case of the projective space and in terms of local coordinates. For that, we first need some comments about $\mathbb{G}_m$-actions on the projective space. Consider $X = \mathbf{P}^n$ acted on by $\mathbb{G}_m$ and ϕ a $\mathbb{G}_m$-linearization of $\mathcal{O}_{\mathbf{P}^n}(1)$. This induces an action on $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$

and therefore an action on the affine cone $\mathbb{A}^{n+1} = \mathrm{Spec}(S^\bullet H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)))$ of $\mathbf{P}^n$ compatible with the projection

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{A}^{n+1} - \{0\} & \xrightarrow{\sigma^*} & \mathbb{A}^{n+1} - \{0\} \\ \downarrow{\scriptstyle 1\times\pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{G}_m \times \mathbf{P}^n & \xrightarrow{\sigma} & \mathbf{P}^n \end{array} .$$

This action can be diagonalized, that is, for some basis of $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$ the action of any $t \in \mathbb{G}_m$ is given by a diagonal matrix

$$\begin{pmatrix} t^{\gamma_1} & & \\ & \ddots & \\ & & t^{\gamma_{n+1}} \end{pmatrix} . \tag{1.4}$$

With this in hand we have the following important proposition

Proposition 1.1.22. ([42, Proposition 2.3]) *Let G be an algebraic reductive group acting on $\mathbf{P}^n$. Let $x \in \mathbf{P}^n$ be a closed point, $x^* \in \mathbb{A}^{n+1}$ a closed homogeneous point with $\pi(x^*) = x$, and λ a one parameter subgroup. Fix coordinates in $\mathbb{A}^{n+1}$ which diagonalize the induced action of $\mathbb{G}_m$ as in (1.4) and write $x^* = (x_1^*, \dots, x_{n+1}^*)$. Then,*

$$\mu^{\mathcal{O}(1)}(x, \lambda) = \max\{-\gamma_i | x_i^* \neq 0\}. \tag{1.5}$$

Moreover, if we denote $f^*(0) = \lim_{t \mapsto 0} \sigma^*(\lambda(t), x^*)$, then $f^*(0)$ does exists in $\mathbb{A}^{n+1}$ and is different from 0 (resp. equal to 0, resp. does not exists) $\Leftrightarrow$ $\mu(x, \lambda) = 0$ (resp. $\mu(x, \lambda) < 0$, resp. $\mu(x, \lambda) > 0$).

Remark 1.1.23. We can write down the last equation in terms of vector spaces instead of varieties. For that, we recall that $\mathbb{A}^{n+1} = \mathrm{Spec}(S^\bullet(V))$, being $V := k^{n+1}$, and $\mathbf{P}^n = \mathrm{Proj}(S^\bullet(V))$. If $\{v_1, \dots, v_{n+1}\}$ is a basis of V diagonalizing λ and φ is a linear form on V representing $x \in \mathbf{P}^n$, then equality (1.5) turns into

$$\mu^{\mathcal{O}(1)}(x, \lambda) = \max\{-\gamma_i | \varphi(v_i) \neq 0\}.$$

With this, we finally have the last result of this section, which provide us, together with Proposition 1.1.22, with a very useful tool to describe the (semi)stable points.

Theorem 1.1.24 (Hilbert-Mumford criterion). ([42, Theorem 2.1]) *Let X be a projective scheme over a field k acted on by an algebraic reductive group G. Let $\mathcal{L} \in Pic^G(X)$ be an ample line bundle and $x \in X$ a closed point. Then*

$$x \in X^{ss}(\mathcal{L}) \Leftrightarrow \mu^{\mathcal{L}}(x, \lambda) \geq 0, \ \forall \textit{ one parameter subgroup } \lambda,$$
$$x \in X^{s}(\mathcal{L}) \Leftrightarrow \mu^{\mathcal{L}}(x, \lambda) > 0, \ \forall \textit{ one parameter subgroup } \lambda.$$

1.1.5 One Parameter Subgroups and Weighted Flags

Let V be a p-dimensional vector space. A weighted flag of V of length s is a pair $(V^\bullet, \underline{m})$ where

$$V^\bullet : \ (0) \subset V_1 \subset \ldots \subset V_s \subset V_{s+1} = V$$

is a flag of vector subspaces and $\underline{m} = (m_1, \dots, m_s)$ is a tuple of positive rational numbers,

$$m_i \in \frac{\mathbb{Z}}{p} := \{\frac{n}{p}, \text{ with } n \in \mathbb{Z}\}.$$

Two flags $(V_\bullet, \underline{m})$, $(V'_\bullet, \underline{m'})$ of V are isomorphic if $\underline{m} = \underline{m'}$ and there is an automorphism of V which is compatible with the flag structures. The group of flag isomorphisms of a given flag, $P \subset \mathrm{GL}(V)$, is a parabolic subgroup

Let $\lambda\colon \mathbb{G}_m \to \mathrm{SL}(V)$ be a one parameter subgroup. We know from the last section that λ is diagonalizable, i.e., there exists a basis $\underline{v} = \{v_1, \dots, v_p\}$ and integers $\gamma_1 \leq \dots \leq \gamma_p$ satisfying $\sum \gamma_i = 0$, and such that $\lambda(t) \cdot v_i = t^{\gamma_i} v_i$. Consider the vector of integers $\underline{\gamma} = (\gamma_1, \dots, \gamma_p) \in \mathbb{Z}^p$ and define

$$n_i := \frac{\gamma_{i+1} - \gamma_i}{p} \in \frac{\mathbb{Z}}{p}, \; i = 1, \dots, p-1. \tag{1.6}$$

Then, we clearly have

$$\underline{\gamma} = \sum n_i \gamma_p^{(i)}, \text{ being } \gamma_p^{(i)} = (\overbrace{i-p, \dots, i-p}^{i}, \overbrace{i, \dots, i}^{p-i}) \tag{1.7}$$

(Conversely, given rational numbers $n_1, \dots, n_{p-1} \in \frac{\mathbb{Z}}{p}$ as before, there are integers $\gamma_1, \dots, \gamma_p$ satifsying $\sum \gamma_i = 0$ such that (1.6) holds). Now, given the data $(\underline{v}, \underline{\gamma})$, we define a weighted filtration in the following way: $\Gamma_1 < \dots < \Gamma_{s+1}$ are the different integers among $\gamma_1 \leq \dots \leq \gamma_p$, the vector spaces giving the flag are defined by $V_i := \langle \{v_j\} | \Gamma_j \leq \Gamma_i \rangle$ and the weights $(m_1, \dots, m_s)$ are defined as the non zero numbers among the $n_i's$. We denote this weighted flag as $(V_\bullet(\lambda), \underline{m}(\lambda))$.

Let $(V_\bullet, \underline{m})$ be a weighted flag. To this weighted flag we can associate the vector

$$\underline{\gamma} = \sum_{i=1}^{s} m_i \gamma_p^{(\dim V_i)},$$

and its components $\gamma_1 \leq \dots \leq \gamma_p$ obviously satisfy $\sum \gamma_i = 0$. Let $\{v_1, \dots, v_{\dim V_1}\}$ be a basis of V_1, complete it to get a basis of V_2, and so on. In this way we get a basis $\underline{v}$ of V and we may define a one parameter subgroup, $\lambda = \lambda(\underline{v}, \underline{\gamma})$, as $\lambda(t) \cdot v_i = t^{\gamma_i} v_i$.

This shows that to any one parameter subgroup, we can attach a weighted flag, and any weighted flag can be obtained in this way. Moreover, if λ is a one parameter subgroup and P is the parabolic subgroup of automorphisms of the associated weighted flag $(V_\bullet(\lambda), \underline{m}(\lambda))$, then the weighted flag associated to $g^{-1} \cdot \lambda \cdot g$, for $g \in P$, is isomorphic to $(V_\bullet(\lambda), \underline{m}(\lambda))$. Because of this, we use the notation $P(\lambda)$ for the parabolic subgroup P. The importance of this relationship becomes clear after the following proposition,

Proposition 1.1.25. ([42, Proposition 2.7]) *Let G be a reductive group act on a projective scheme X over k. Then for all $x \in X$, $\mathcal{L} \in \mathrm{Pic}^G(X)$, and any one parameter subgroup, $\lambda\colon \mathbb{G}_m \to G$, we have*

$$\mu^{\mathcal{L}}(x, \lambda) = \mu^{\mathcal{L}}(x, g^{-1} \cdot \lambda \cdot g)$$

for all $g \in P(\lambda)$.

That means that the value of the semistability function $\mu^{\mathcal{L}}(x,\lambda)$, as a function of λ, depends only on the associated weighted flag $(V_\bullet(\lambda), \underline{m}(\lambda))$ when the group is isomorphic to a closed subgroup of the special linear group.

With this in hand, we can improve a bit more the equality given in Remark 1.1.23.

$$\mu^{\mathcal{O}(1)}(x,\lambda) = \max\{-\gamma_i |\ \varphi_{|_{V^i}} \neq 0\}. \tag{1.8}$$

1.1.6 Products of Groups

For this section we follow [45] and [53] closely. Let X be a projective scheme over k and let G, H be algebraic reductive groups such that $K := G \times H$ is acting on X. Let $\mathcal{L}$ be a K-linearized very ample invertible sheaf and consider the K-linear closed immersion (see Proposition 1.1.20)

$$I\colon X \to \mathbf{P}(W) = \mathrm{Proj}(S^\bullet(W)), \quad \text{being } W = H^0(X,\mathcal{L}). \tag{1.9}$$

The induced linearization in $\mathcal{O}(1)$ determines a representation $\rho\colon K = G\times H \to \mathrm{GL}(W)$ and restricting to G and H we get two more representations $\rho'\colon G \to \mathrm{GL}(W)$ and $\rho''\colon H \to \mathrm{GL}(W)$. We want to describe the semistability condition of the action of K in terms of the semistability condition of the actions of G and H. Because of the linearity of (1.9) and Proposition 1.1.19 it is enough to do this description for the projective space. We use the notation $(s)s$, $(s)s'$, $(s)s''$ for the (semi)stability with respect to ρ, ρ', ρ''.

Consider the quotient $Q_G := \mathbf{P}(W)/\!\!/_{\rho'} G$ by the induced action of G and let $\pi_G : \mathbf{P}(W)^{ss'} \to Q_G$ be the quotient map. Recall that $Q_G = \mathrm{Proj}(S^\bullet(W)^G)$. By [23, Proposition 2.4.7] we know that $Q_G \simeq \mathrm{Proj}(S^{(d)}(W)^G)$ for $d \gg 0$ (being $S^{(d)}(W)^G = \bigoplus_{d\geq 0} S^{dn}(W)^G$) and is generated in degree 1 [23, Proposition 3.1.10]. Therefore, there is a closed immersion

$$\varphi_n\colon Q_G \hookrightarrow \mathbf{P}(S^n(W)^G).$$

Since the actions ρ' and ρ'' commute with each other, we find that H acts, via ρ'', on Q_G, $S^n(W)^G$ and $\mathbf{P}(S^n(W)^G)$ in such a way that φ_n is H-linear. Consider the very ample invertible sheaf $\mathcal{L}_n := \mathcal{O}(1)|_{\mathbf{P}(S^n(W)^G)}$. The semistability notion with respect to $\mathcal{L}_n$ is independent on n, thus:

Theorem 1.1.26. ([45, Proposition 1.3.1, Proposition 1.3.2]) *We have* $\mathbf{P}(W)^{ss} = \pi_G^{-1}(Q_G^{ss''}) \subset \mathbf{P}(W)^{ss'}$. *Moreover, we have* $\mathbf{P}(W)/\!\!/_{\rho}(G\times H) \simeq Q_G/\!\!/_{\rho'} H$. *For the polystable points we have* $\mathbf{P}^{ps} = \mathbf{P}^{ps'} \cap \pi_G^{-1}(Q_G^{ps''})$.

1.1.7 Direct Sums of Representations

Let $\rho_i\colon \mathrm{GL}_{r_i}(\mathbb{C}) \longrightarrow \mathrm{GL}(V_i)$ be a finite dimensional representation of dimension d_i, which is homogeneous of degree h_i, $i = 1,\dots,s$. Assume that $h_1,\dots,h_s$ have all the same sign. Next, let m_i be a positive integer, $i = 1,\dots,s$, and

$$\iota : \mathrm{GL}_{r_i}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{r_i}(\mathbb{C}) \to \mathrm{GL}_R(\mathbb{C}), \quad R\colon = m_1\cdot r_1 + \cdots + m_s \cdot r_s,$$

the embedding which sends $(g_1,\dots,g_s)$ to the block diagonal matrix in which g_1 is first repeated m_1 times, then g_2 is repeated m_2 times, and so on.
Set

$$\overline{G} := \iota^{-1}(\mathrm{SL}_R(\mathbb{C})) \ \text{ and } \ V := V_1 \oplus \cdots \oplus V_s.$$

We obtain the representation $\rho : \overline{G} \to \mathrm{GL}(V)$. We write an element of V in the form $p = (p_1, \dots, p_s)$, p_i being the component of p in V_i, $i = 1, \dots, s$.

Lemma 1.1.27. *Let $p = (p_1, \dots, p_s) \in V$ be a linear form and $\lambda = (\lambda_1, \dots, \lambda_s)$ a one parameter subgroup of $\overline{H}$. Then*

$$\mu(p, \lambda) = \max\{\mu(p_i, \lambda_i) | i = 1, \dots, l\}$$

Proof. Note that there is a basis $\underline{w_i} = \{w_{i,1}, \dots, w_{i,d_i}\}$ of V_i such that

$$\lambda_i(z) \cdot w_{i,j} = z^{\gamma^i_j} w_{i,j},\ i = 1, \dots, s,\ j = d_1.$$

These bases $\underline{w_i}$ induces a basis $\underline{w} = \{\overline{w}_{i,1}, \dots, \overline{w}_{i,d_i} |\ i = 1, \dots s\}$ of V in the obvious way, and we have

$$\lambda(z) \cdot \overline{w}_{i,j} = \lambda_i(z) \cdot w_{i,j} = z^{\gamma^i_j} w_{i,j}. \tag{1.10}$$

Then we conclude by Remark 1.1.23. □

Proposition 1.1.28. *Let $p = (p_1, \dots, p_s) \in V$. Then, the following conditions are equivalent:*
(i) The point p is ρ-semistable.
(ii) For $i = 1, \dots, s$, the point p_i is $\overline{\rho}_i$-semistable[1]*, $\overline{\rho}_i$ being the restriction of ρ_i to $\mathrm{SL}_{r_i}(\mathbb{C})$.*

Proof. Let us first discuss the easy implication "*(ii)* ⇒ *(i)*". A one parameter subgroup $\lambda\colon \mathbb{C} \longrightarrow \overline{G}$ can be written as a tuple $(\lambda_1, \dots, \lambda_s)$ where λ_i is a one parameter subgroup of $\mathrm{GL}_{r_i}(\mathbb{C})$, $i = 1, \dots, s$. Given λ, we may find rational one parameter subgroups λ'_i of $\mathrm{SL}_{r_i}(\mathbb{C})$ and rational numbers η_i, such that

$$\lambda_i = \lambda'_i + \mathbb{E}^{\eta_i}_{r_i},\ i = 1, \dots, s.$$

Here $\mathbb{E}^{\eta_i}_{r_i}$ is the rational one parameter subgroup defined by

$$\frac{1}{b_i} \bullet (z \mapsto z^{a_i} \mathbb{E}_{r_i}),\ \text{where } \eta_i = \frac{a_i}{b_i}.$$

Since $p_i \neq 0$ and ρ_i is homogeneous of degree h_i, $i = 1, \dots, s$, we have

$$\mu_\rho(\lambda, p) = \max\{\mu_{\overline{\rho}_i}(\overline{\lambda}_i, p_i) + h_i \eta_i |\ i = 1, \dots, s\}.$$

By assumption,

$$\mu_{\overline{\rho}_i}(\overline{\lambda}_i, p_i) \geq 0,\ i = 1, \dots, s.$$

If $h_1 = \cdots = h_s = 0$, we are done. Otherwise, we use

$$m_1 \cdot r_1 \cdot \eta_1 + \cdots + m_s \cdot r_s \cdot \eta_s = 0.$$

Since the h_i are either all negative or positive, there must be an index $i_0 \in \{1, \dots, s\}$ with $h_{i_0} \cdot \eta_{i_0} \geq 0$, so that

$$\mu_\rho(\lambda, p) \geq \mu_{\overline{\rho}_{i_0}}(\overline{\lambda}_{i_0}, p_{i_0}) + h_{i_0} \cdot \eta_{i_0} \geq 0.$$

[1] This in particular means that $p_i \neq 0$ for each i.

Let us turn to the reverse implication "$(i) \Rightarrow (ii)$". We will discuss the case $h_i > 0$, $i = 1, \dots, s$. First, suppose that $p_1 = 0$. Set $\eta_2 = \dots = \eta_s := -1$ and

$$\eta_1 \colon = \frac{m_2 \cdot r_2 + \dots + m_s \cdot r_s}{m_1 \cdot r_1}.$$

Then,

$$\lambda := (\mathbb{E}_{r_1}^{\eta_1}, \dots, \mathbb{E}_{r_s}^{\eta_s})$$

is a rational one parameter subgroup of $\overline{G}$ with

$$\mu_\rho(\lambda, p) \leq \max\{-h_2, \dots, -h_s\} < 0.$$

For the rest, we may assume $p_i \neq 0$, $i = 1, \dots, s$. Let us show that p_1 is $\overline{\rho}_1$-semistable. If it were not, we would find a one parameter subgroup $\lambda_1 \colon \mathbb{G}_m(\mathbb{C}) \longrightarrow \mathrm{SL}_{r_1}(\mathbb{C})$ with

$$\mu_{\overline{\rho}_1}(\lambda_1, p_1) < 0.$$

Define $\eta_1, \dots, \eta_s$ as before, choose $\epsilon \in \mathbb{Q}_{>0}$, such that

$$\mu_{\overline{\rho}_1}(\lambda_1, p_1) + \epsilon \cdot \eta_1 \cdot h_1 < 0,$$

and set

$$\lambda := (\lambda_1 + \mathbb{E}_{r_1}^{\epsilon\eta_1}, \mathbb{E}_{r_2}^{-\epsilon}, \dots, \mathbb{E}_{r_s}^{-\epsilon})$$

This is a rational one parameter subgroup of $\overline{G}$ with

$$\mu_\rho(\lambda, p) = \max\{\mu_{\overline{\rho}_1}(\lambda_1, p_1) + \epsilon \cdot \eta_1 \cdot h_1, -\epsilon \cdot h_2, \dots, -\epsilon \cdot h_s\} < 0.$$

This proves the proposition. $\square$

1.1.8 Example 1

Let p, r be integers such that $1 \leq r \leq p - 1$. Let $\mathbf{Gr} := \mathbf{Grass}(U^{\oplus 2}, r)$ be the Grassmannian of r-dimensional quotients of $U^{\oplus 2}$, U being a p-dimensional vector space, and let N be positive integer. The Grassmannian can be embedded into the projective space through the Plücker embedding

$$\begin{gathered} \iota \colon \mathbf{Gr} \hookrightarrow \mathbf{P}(\wedge^r U^{\oplus 2}) \quad . \\ (\tau \colon U^{\oplus 2} \to R) \mapsto (\wedge^r \tau \colon \wedge^r U^{\oplus 2} \to k) \end{gathered}$$

The group $\mathrm{SL}(U)$ acts on both spaces through the diagonal $\delta : \mathrm{SL}(U) \hookrightarrow \mathrm{SL}(U^{\oplus 2})$ in the obvious way, and ι is $\mathrm{SL}(U)$-equivariant. If $\mathcal{O}(1)$ is the tautological invertible sheaf on $\mathbf{P}(\wedge^r U^{\oplus 2})$, then $\mathcal{L} := \iota^* \mathcal{O}(1)$ is a $\mathrm{SL}(U)$-linearized very ample invertible sheaf. Let us compute the semistability function of points in $\mathbf{Gr}$ with respect to $\mathcal{L}$.

Let us explicitely describe the action of $\mathrm{SL}(U)$ on $\mathbf{P}(\wedge^r U^{\oplus 2})$. Let $\{u_1, \dots, u_p\}$ be a basis of the vector space U. Then, a basis of $\wedge^r U^{\oplus 2}$ is given by the vectors

$$u_{I,J} := (u_{i_1}, 0) \wedge \dots \wedge (u_{i_l}, 0) \wedge (0, u_{j_1}) \wedge \dots \wedge (0, u_{j_{r-l}}).$$

Let $\lambda \colon \mathbb{G}_m \to \mathrm{SL}(U)$ be a one parameter subgroup. Fix a basis $\underline{u} = \{u_1, \dots, u_p\}$ and integers $\gamma_1 \leq \dots \leq \gamma_p$ such that $\lambda = \lambda(\underline{u}, \underline{\gamma})$. Then, the induced action of λ on $\wedge^r U^{\oplus 2}$ is given by

$$\lambda(t) \cdot u_{I,J} = t^{\gamma_{i_1} + \dots + \gamma_{i_l} + \gamma_{j_1} + \dots + \gamma_{j_{r-l}}} u_{I,J}.$$

Assume $\underline{\gamma} = \gamma_p^{(i)} = (\overbrace{i-p, \ldots, i-p}^{i}, \overbrace{i, \ldots, i}^{p-i})$ and let $a, c \in \mathbb{N}$ be the smallest integers such that $\gamma_{i_1} = \ldots = \gamma_{i_a} = \gamma_{j_1} = \ldots = \gamma_{j_c} = i - p$. Then

$$\gamma_{I,J} := \gamma_{i_1} + \ldots + \gamma_{i_l} + \gamma_{j_1} + \ldots + \gamma_{j_{r-l}} = ri - (a+c)p = p(2i - a - c) - i \dim(\mathrm{Ker}(\tau))$$

Now, a short calculation shows that

$$a + c = \dim \tau(L_i^1 \oplus L_i^2),$$

being $L_i^1 = \langle (u_1, 0), \ldots, (u_i, 0) \rangle$, $L_i^2 = \langle (0, u_1), \ldots, (0, u_i) \rangle$, and therefore

$$\gamma_{I,J} = p \dim(\mathrm{Ker}(\tau) \cap (L_i^1 \oplus L_i^2)) - i \dim \mathrm{Ker}(\tau).$$

Moreover, for a general weighted vector $\underline{\gamma}$ we have

$$\mu^{\mathcal{L}}(\tau, \lambda(\underline{u}, \underline{\gamma})) = \sum_{i=1}^{s} i \dim(\mathrm{Ker}(\tau)) - p \dim(\mathrm{Ker}(\tau) \cap (U_i \oplus U_i)) \tag{1.11}$$

where $(U_\bullet, \underline{m})$ is the weighted filtration associated to λ.

1.1.9 Example 2

Let $Y_1, \ldots, Y_l$ be smooth projective connected curves, and consider their disjoint union, $Y := \bigsqcup Y_i$. Let $\mathcal{N}_1, \ldots, \mathcal{N}_l$ be invertible sheaves on $Y_1, \ldots, Y_l$ respectively and denote by $\mathcal{N} := \bigoplus \mathcal{N}_i$ the corresponding invertible sheaf on Y. Let $r,\ n \in \mathbb{N}$ and let U be a k-vector space of dimension $p > r$. Consider now, for each i, the projective space given by

$$\mathbb{G}^i_{1,\mathcal{N}} := \mathbf{P}(\mathrm{Hom}(\bigwedge^r U, H^0(Y_i, \mathcal{N}_i(rn)))^\vee),$$

and define $\mathbb{G}_{1,\mathcal{N}} = \mathbb{G}^1_{1,\mathcal{N}} \times \ldots \times \mathbb{G}^l_{1,\mathcal{N}}$. Let $b_1, \ldots, b_l \in \mathbb{N}$ and consider the very ample invertible sheaf on $\mathbb{G}_1$ given by

$$\mathcal{L} := \pi_1^* \mathcal{O}_{\mathbb{G}^1_{1,\mathcal{N}}}(b_1) \otimes \ldots \otimes \pi_l^* \mathcal{O}_{\mathbb{G}^l_{1,\mathcal{N}}}(b_l)$$

with the obvious $\mathrm{SL}(U)$-linearization. For the sake of clarity we will use the symbol $\mathcal{L}_i$ to denote the invertible sheaf $\mathcal{O}_{\mathbb{G}^i_{1,\mathcal{N}}}(1)$. We want to compute the semistability function for points in the space $\mathbb{G}_{1,\mathcal{N}}$ with respect to the linearized invertible sheaf $\mathcal{L}$. By Proposition 1.1.21 (ii) and (iii) we deduce that

$$\mu^{\mathcal{L}}([h], \lambda) = \sum_{i=1}^{l} b_i \mu^{\pi_i^* \mathcal{L}_i}([h], \lambda) = \sum_{i=1}^{l} b_i \mu^{\mathcal{L}_i}([h_i], \lambda), \tag{1.12}$$

$[h_i]$ being the i-th component of $[h]$. Therefore the calculation of the semistability function of points of $\mathbb{G}_{1,\mathcal{N}}$ with respect to $\mathcal{L}$ is reduced to the calculation of the semistability function of points of $\mathbb{G}^i_{1,\mathcal{N}}$ with respect to $\mathcal{L}_i$

Let $\mathcal{E}$ be a locally free quotient sheaf of rank r

$$q \colon U \otimes \mathcal{O}_Y(-n) \to \mathcal{E} \to 0$$

whose determinant is isomorphic to $\mathcal{N}$. Restricting to the i-th component, twisting by n, taking the r-th exterior power and taking global sections we find the morphism

$$H^0(\wedge^r(q_i(n)))\colon\ \wedge^r U \to H^0(Y, \mathcal{N}_i(rn)),$$

whose equivalence class defines a point $[H^0(\wedge^r(q_i(n)))] \in \mathbb{G}^i_{1,\mathcal{N}}$.

Let us compute the semistability function for points of the form

$$([H^0(\wedge^r(q_1(n)))], \ldots, [H^0(\wedge^r(q_l(n)))]) \in \mathbb{G}_{1,\mathcal{N}}.$$

Note that the group $\mathrm{SL}(U)$ acts on $\mathbb{G}^i_{1,\mathcal{N}}$ by the rule

$$\forall g \in \mathrm{SL}(U)\ ,\ (T \bullet g)(u_{i_1}, \ldots, u_{i_r}) = T(gu_{i_1} \wedge \ldots \wedge gu_{i_r}),$$

where $\{u_1, \ldots, u_p\}$ is a basis for U. Let $\lambda\colon \mathbb{G}_m \to \mathrm{SL}(U)$ be a one parameter subgroup. Then there exists a basis $\{u_1, \ldots, u_p\}$ of U and integers $\gamma_1, \ldots, \gamma_p \in \mathbb{Z}$ with $\gamma_1 \leq \ldots \leq \gamma_p$ and $\sum_i \gamma_i = 0$ such that

$$\lambda(z)u_i = z^{\gamma_i} u_i\ ,\ \forall z \in \mathbb{G}_m.$$

For any multiindex $I = (i_1, \ldots, i_r)$ with $1 \leq i_1 < \ldots < i_r \leq p$ let $u_I = u_{i_1} \wedge \ldots \wedge u_{i_r}$ and $\gamma_I = \gamma_{i_1} + \ldots + \gamma_{i_r}$. The vectors $\{u_I\}_I$ form a basis for the vector space $\bigwedge^r U$ and $\lambda\colon \mathbb{G}_m \to \mathrm{SL}(U)$ acts on $\bigwedge^r U$ by the rule

$$\lambda(z) \bullet u_I = z^{\gamma_I} u_I\ ,\ \forall z \in \mathbb{G}_m.$$

A short calculation shows that

$$\mu([T], \lambda(\underline{u}, \underline{\gamma})) = \sum n_i \mu([T], \lambda(\underline{u}, \gamma_p^{(i)})), \tag{1.13}$$

being $\gamma_p^{(i)} = (i-p, \ldots, i-p, i, \ldots, i)$ as in (1.7), so the quantity we need to compute is $\mu([T], \lambda(\underline{u}, \gamma_p^{(i)}))$.

Consider again the locally free quotient $q\colon U \otimes \mathcal{O}_Y(-n) \to \mathcal{E}$. Let q_i be its restriction to Y_i and denote $q_i(n)$ the twisting by $\mathcal{O}_{Y_i}(n)$. Then, giving the quotient $q_i(n)$ is the same as giving p global sections,

$$e_j\colon \mathcal{O}_Y \to \mathcal{E}_{Y_i}(n),$$

one for each u_j such that, for each point $y \in Y_i$, the familly of vectors $\{e_1(y), \ldots, e_p(y)\}$ generate de fiber $\mathcal{E}_{Y_i}(n)(y)$. Now, the morphism defined by the quotient q_i,

$$H^0(\wedge^r(q_i(n)))\colon\ \wedge^r U \to H^0(Y, \mathcal{N}_i(rn)),$$

is given by

$$H^0(\wedge^r(q_i(n)))(u_I) = e_I := e_{i_1} \wedge \ldots, \wedge e_{i_r} \in H^0(Y, \mathcal{N}_i(rn)).$$

Therefore, we deduce the following

$$\begin{aligned} H^0(\wedge^r(q(n)))(u_I) \neq 0 \Leftrightarrow e_{i_1} \wedge \ldots, \wedge e_{i_r} \neq 0 \Leftrightarrow \\ \Leftrightarrow \left\{ \begin{array}{l} \{e_{i_1}(y), \ldots, e_{i_r}(y)\} \text{ is a basis} \\ \text{of } \mathcal{E}_{Y_i}(n)(y), \text{ for every point} \\ \text{in a dense open subset of } Y_i \end{array} \right\}. \end{aligned}$$

This gives to us a geometric interpretation of the multiindices that must to be taken for the calculation of the semistability function (see Remark 1.1.23).

Remark 1.1.29. Let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module and $\mathcal{F} \subset \mathcal{E}$ a locally free subsheaf. We say $\mathcal{F}$ is a *saturated subsheaf* if $\mathcal{E}/\mathcal{F}$ is again locally free. Any subsheaf $\mathcal{F} \subset \mathcal{E}$ generates a saturated subsheaf in the following way. Denote T the torsion part of $\mathcal{E}/\mathcal{F}$ and $\mathcal{G}$ the locally free part. We have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, "i"] \arrow[d, dotted, "\mathrm{id}"] & \mathcal{E} \arrow[r, "\pi"] \arrow[d, "\mathrm{id}"] & \mathcal{E}/\mathcal{F} \arrow[r] \arrow[d, "\pi'"] & 0 \\
0 \arrow[r] & Ker(\pi' \circ \pi) \arrow[r] & \mathcal{E} \arrow[r, "\pi' \circ \pi"] & \mathcal{G} \arrow[r] & 0
\end{tikzcd}$$

Because of the conmutativity of the right square, the identity morphism induces a morphism $\mathcal{F} \to Ker(\pi' \circ \pi)$. Moreover, because of Short Five Lemma, the induced morphism is injective. We say $\mathcal{F}^s := \mathrm{Ker}(\pi' \circ \pi)$ is *the saturated subsheaf generated by* $\mathcal{F}$.

Consider the locally free quotient $q \colon U \otimes \mathcal{O}_Y(-n) \to \mathcal{E}$ and denote $U_i = \langle u_1, \dots, u_i \rangle$. Then, for each index i we have the locally free (saturated) subsheaf generated by U_i:

$$\mathcal{E}_i := q(U_i \otimes \mathcal{O}_Y(-n))^s \subset \mathcal{E}$$

and, therefore, we get a filtration

$$0 \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \ldots \subseteq \mathcal{E}_{p-1} \subseteq \mathcal{E}_p = \mathcal{E}.$$

Denote k_1 the index of the first non-zero subsheaf, k_2 the first subsheaf such that $\mathcal{E}_{k_1} \subsetneq \mathcal{E}_{k_2}$, and so on. We get in this way a multiindex $\underline{K} = (k_1, \dots, k_r)$ giving us the minimum in (1.1.23). Thereby,

$$\begin{aligned}
\mu^{\mathcal{L}_j}([H^0(\wedge^r(q_j(n)))], \lambda(\underline{u}, \gamma_p^{(i)})) &= -\gamma_K = -\gamma_{k_1} - \ldots - \gamma_{k_r} = \\
&= \mathrm{rk}(\mathcal{E}_i|_{Y_j})(i-p) - (r - \mathrm{rk}(\mathcal{E}_i|_{Y_j}))i = \\
&= \mathrm{rk}(\mathcal{E}_i|_{Y_j})p - ri.
\end{aligned}$$

From (1.13) we can finally conclude

$$\mu^{\mathcal{L}_j}([H^0(\wedge^r(q_j(n)))], \lambda) = \sum_{i=1}^{s} m_i(\mathrm{rk}(\mathcal{E}_i|_{Y_j})p - r\mathrm{dim}(U_i)), \tag{1.14}$$

being (U_i, m_i) the ith term of the weighted filtration associate to λ and $\mathcal{E}_i|_{Y_j}$ the restriction to Y_j of the saturated subsheaf generated by U_i.

1.1.10 Example 3

Consider the same situation as in Example 2. Let $\mathcal{L}$ be an invertible sheaf on Y, U a p-dimensional vector space and $a, b, c, n \in \mathbb{N}$. For any other invertible sheaf $\mathcal{N}$ on Y we consider the projective space

$$\mathbb{G}_{2,N} = \mathbf{P}(\mathrm{Hom}(U_{a,b}, H^0(Y, \mathcal{N}^{\otimes c} \otimes \mathcal{L}(na)))^\vee),$$

being $U_{a,b} := (U^{\otimes a})^{\oplus b}$. Consider the pair (q, ϕ) given by a locally free quotient sheaf of rank r, $q \colon U \otimes \mathcal{O}_Y(-n) \to \mathcal{E}$, whose determinant is isomorphic to $\mathcal{N}$, and a morphism

$\phi\colon (\mathcal{E}^{\otimes a})^{\oplus b} \to \mathcal{N}^{\otimes c} \otimes \mathcal{L}$. Denote $\Delta\colon U_{a,b} \hookrightarrow U_{a,b}^{\oplus l}$ the diagonal linear map, and consider the morphism

$$H^0((q(n)^{\otimes a})^{\oplus b}) \circ \Delta\colon U_{a,b} \to H^0(Y, (\mathcal{E}^{\otimes a})^{\oplus b} \otimes \mathcal{O}_Y(na)).$$

Twisting by $\mathcal{O}_Y(na)$ the morphism ϕ and taking global sections, we find

$$H^0(\phi(na)) : H^0(Y, (\mathcal{E}^{\otimes a})^{\oplus b} \otimes \mathcal{O}_Y(na)) \to H^0(Y, \mathcal{N}^{\otimes c} \otimes \mathcal{L}(na)).$$

Composing both morphisms we get a point in $\mathbb{G}_{2,\mathcal{N}}$,

$$[H^0(\phi(na)) \circ H^0((q(n)^{\otimes a})^{\oplus b}) \circ \Delta]\colon U_{a,b} \to H^0(Y, \mathcal{N}^{\otimes c} \otimes \mathcal{L}(na))] \in \mathbb{G}_{2,\mathcal{N}}. \tag{1.15}$$

Set $p = \dim(U)$ and let $\underline{u} = (u_1, \dots, u_p)$ be a basis of U. For any multiindex $I = (i_1, \dots, i_a)$ with $i_j \in \{1, \dots, p\}$ define

$$\begin{aligned} u_I &= u_{i_1} \otimes \ldots \otimes u_{i_a}, \\ u_I^k &= (0, \ldots, 0, \overset{k)}{u_I}, 0, \ldots, 0). \end{aligned}$$

Then the elements u_I^k form a basis of $U_{a,b}$. Also, the group $\mathrm{SL}(U)$ acts on $\mathrm{Hom}(U_{a,b}, H^0(Y, \mathcal{N}^{\otimes c} \otimes \mathcal{L}(na)))$ by the rule

$$(T \bullet g)(u_I^k) = T(g u_I^k),$$

being $g u_I^k = g u_{i_1} \otimes \ldots \otimes g u_{i_a}$.

We want to compute the semistability function for points $T \in \mathbb{G}_{2,\mathcal{N}}$ of the form (1.15) with respect to the natural $SL(U)$-linearization of $\mathcal{O}_{\mathbb{G}_{2,\mathcal{N}}}(1)$.

Let $\lambda\colon \mathbb{G}_m \to \mathrm{SL}(U)$ be a one parameter subgroup. Then there exists a basis $u_1, \dots, u_p$ of U and integers $\gamma_1 \le \ldots \le \gamma_p$ with $\sum \gamma_i = 0$ such that

$$\lambda(z) u_i = z^{\gamma_i} u_i, \forall z \in \mathbb{G}_m.$$

For any multiindex $I = (i_1, \dots, i_a)$ consider u_I and define $\gamma_I = \gamma_{i_1} + \cdots + \gamma_{i_a}$. Then $\lambda\colon \mathbb{G}_m \to \mathrm{SL}(U)$ acts by

$$\lambda(z) \bullet u_I^k = z^{\gamma_I} \bullet u_I^k, \forall z \in \mathbb{G}_m.$$

By Remark 1.1.23, we know that

$$\begin{aligned} \mu([T], \lambda) =& \max\{-\gamma_I | T(u_I^k) \neq 0\} \geq 0 = \\ &= -\min\{\gamma_I | T(u_I^k) \neq 0\} \geq 0. \end{aligned}$$

Remark 1.1.30. Given a multiindex $I = (i_1, \dots, i_a)$ we want to compute $\gamma_I = \gamma_{i_1} + \cdots + \gamma_{i_a}$ for $\gamma = \gamma_p^i = (i-p, \dots, i-p, i, \dots, i)$. Denote by $\nu(I, i) = \#\{j | i_j \le i\}$. Then $i_1, \dots, i_{\nu(I,i)} \le i$ and $i_{\nu(I,i)+1}, \dots, i_a > i$. Therefore

$$\gamma_I = (i-p)\nu(I,i) + i(a - \nu(I,i)) = ia - \nu(I,i)p. \tag{1.16}$$

A short calculation, as in Example 2, shows that

$$\mu([T], \lambda) = \sum_{i=1}^{s} m_i (\nu(I, \mathrm{dim} U_i)p - \mathrm{dim} U_i a), \tag{1.17}$$

$(U_\bullet, \underline{m})$ being the associated weighted flag and $I = (i_1, \dots, i_a)$ is the multiindex giving the minimum of the semistability function.

1.2 Sheaves on Nodal Curves

In this section we develope briefly the basic theory of torsion free sheaves on nodal curves. We give some new results dealing with their characterization and their local structure.

1.2.1 Depth

Let R be a commutative ring and M a R-module. An element $x \in R$ is said to be M-regular if $xm = 0$ with $m \in M$ implies $m = 0$, that is, the endomorphism $\cdot x : M \to M$ is injective.

Definition 1.2.1. A sequence $\mathbf{x} = x_1, \ldots, x_n$ of elements of R is said to be M-regular if

1) x_i is $M/(x_1, \ldots, x_{i-1})M$-regular
2) $M/\boldsymbol{x}M \neq 0$

being $\boldsymbol{x}$ the ideal $(x_1, \ldots, x_n)$.

Definition 1.2.2. Let $I \subset R$ be an ideal. A sequence $\mathbf{x} = x_1, \ldots, x_n$ is said to be an M-sequence in I if it is an M-sequence and $x_i \in I$ for each i. An M-sequence in I, $\mathbf{x} = x_1, \ldots, x_n$, is said to be maximal if there is no $x \in I$ such that $x_1, \ldots, x_n, x$ is an M-sequence in I.

The following Theorem of Rees shows that the length of a maximal sequence in I is an invariant of the module M.

Theorem 1.2.3. ([10, Theorem 1.2.5.]). *Let R be a Noetherian ring, M a finitely generated R-module and I an ideal such that $IM \neq M$. Then, all maximal M-sequences in I have the same length given by*

$$n = \min\{i : \operatorname{Ext}^i_R(R/I, M) \neq 0\}.$$

In the above conditions, we define the *grade* of I on M as

$$\operatorname{grade}(I, M) := n = \min\{i : \operatorname{Ext}^i_R(R/I, M) \neq 0\}.$$

Now we can define the depth of a finitely generated module.

Definition 1.2.4. Let R a Noetherian local ring with maximal ideal $\mathbf{m}$ and M a finitely generated R-module. The *depth* of M is defined as

$$\operatorname{depth}(M) := \operatorname{grade}(\mathbf{m}, M).$$

Lemma 1.2.5. ([10, Proposition 1.2.10 (d)]) *Let R be a Noetherian ring, I an ideal and M a finitely generated R-module. If $\boldsymbol{x} = x_1, \ldots, x_n$ is a regular M-sequence in I, then*

$$\operatorname{grade}(I/\boldsymbol{x}, M/\boldsymbol{x}M) = \operatorname{grade}(I, M/\boldsymbol{xM}) = \operatorname{grade}(I, M) - n.$$

Let R be a Noetherian local ring. A finitely generated R-module $M \neq 0$ is a Cohen-Macaulay module if $\operatorname{depth}(M) = \dim(M)$. The ring R is Cohen-Macaulay if it itself is a Cohen-Macaulay module. If R is an arbitrary noetherian ring, then a R-module M is Cohen-Macaulay if $M_{\mathfrak{m}}$ is a Cohen-Macaulay $R_{\mathfrak{m}}$-module for all maximal ideal $\mathfrak{m} \in \operatorname{supp}(M)$. We consider the zero module to be Cohen-Macaulay.

Definition 1.2.6. (Serre's condition (S_n)). A finitely generated R-module over a Noetherian ring R satisfies Serre's condition (S_n) if

$$\mathrm{depth}(M_{\mathbf{p}}) \geq \min(n, \dim(M_{\mathbf{p}}))$$

for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

Definition 1.2.7. Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. If M is a finitely generated R-module we define its *injective dimension* as

$$\mathrm{inj.dim}(M) := \sup\{i : \mathrm{Ext}^i_R(k, M) \neq 0\}.$$

We say that M is a *Gorenstein module* if $\mathrm{inj.dim}(M) < \infty$. R is a *Gorenstein ring* if it is Gorenstein as an R-module.

Note that any Gorenstein ring is also Cohen-Macaulay. In fact, we can show that a local Noetherian ring R of Krull dimension n is Gorenstein if and only if $\mathrm{Ext}^i_R(k, R) = 0$ for all $i < n$ and $\mathrm{Ext}^n_R(k, R) \simeq k$ (see [37, Theorem 18.1]).

1.2.2 Sheaves of Depth One on Reduced Projective Curves

We assume that k is an algebraically closed field of characteristic zero. For the rest of this work, the word *curve* will mean a one dimensional noetherian scheme of finite type over k. In this case a curve X is *projective* if and only if it is proper over k. A curve is *reduced* if its local rings are reduced.

Remark 1.2.8. Every reduced Noetherian local ring of dimension one is Cohen-Macaulay (see [10] Ex.2.1.20). Therefore any *reduced projective curve* X is a Cohen-Macaulay scheme and, hence, satisfies condition (S_n) for each n.

Let X be a reduced projective curve over k and let $\mathcal{O}_X(1)$ be an ample invertible sheaf (polarization). For any coherent sheaf $\mathcal{F}$, the Euler characteristic of $\mathcal{F}$ is

$$\chi(X, \mathcal{F}) = \mathrm{dim} H^0(X, \mathcal{F}) - \mathrm{dim} H^1(X, \mathcal{F}),$$

and we denote $P_{\mathcal{F}}(n) := \chi(X, \mathcal{F}(n))$ its Hilbert function. This is a polynomial in the variable n with integral coefficients (see [55]). The natural number

$$g_X := 1 - \chi(X, \mathcal{O}_X) \tag{1.18}$$

is called the *arithmetic genus of* X. The *degree of the polarization* $\mathcal{O}_X(1)$ is defined as the leading coefficient of the Hilbert polynomial of the structure sheaf, and we denote it by h,

$$P_{\mathcal{O}_X(1)}(n) = \chi(X, \mathcal{O}_X(n)) = h \cdot n + \cdots$$

In general, if $\mathcal{F}$ is a coherent sheaf on X we define its *rank* and its *degree* with respect to the fixed polarization as the numbers $r(\mathcal{F})$, $\deg(\mathcal{F})$ such that

$$P_{\mathcal{F}}(n) = (hr(\mathcal{F}))n + \deg(\mathcal{F}) + r(\mathcal{F})\chi(X, \mathcal{O}_X) \in \mathbb{Z}[n]. \tag{1.19}$$

The *multiplicity of* $\mathcal{F}$ with respect to the given polarization is defined as the leading coefficient of its Hilbert polynomial, $\alpha(\mathcal{F}) := hr(\mathcal{F})$. Note that $r(\mathcal{F})$ and $\deg(\mathcal{F})$ are not, in general, natural numbers. The slope of $\mathcal{F}$ is defined as $\mu'(\mathcal{F}) = \deg(\mathcal{F})/\alpha(\mathcal{F})$, and the *reduced Hilbert polynomial* is

$$p_{\mathcal{F}}(n) := \frac{P_{\mathcal{F}}(n)}{\alpha(\mathcal{F})} = n + \mu'(\mathcal{F}) + \frac{\chi(X, \mathcal{O}_X)}{h}. \tag{1.20}$$

Recall that a *dualizing sheaf* for X is a coherent sheaf ω_X^0 such that $\mathrm{Hom}_X(\mathcal{F}, \omega_{X^0}) \simeq H^1(X, \mathcal{F})^\vee$, and it exists since X is projective (see [31, Proposition 7.5]). Moreover, since X is Cohen-Macaulay (see Remark 1.2.8) we already know that, for any locally free sheaf $\mathcal{F}$, the following

$$H^i(X, \mathcal{F}) \simeq H^{1-i}(X, \mathcal{F}^\vee \otimes \omega_X^0)^\vee, \; i = 0, 1$$

holds true (see [31, Corollary 7.7]).

Let X be a reduced projective curve over k. A singular point $x \in X$ is called a *node* or *ordinary double point* if $\widehat{\mathcal{O}_{X,x}} \simeq k[[x, y]]/xy$. The curve X is a *nodal curve* if all of its singularities are nodes. Note that any nodal curve is Gorenstein, and therefore it has a dualizing sheaf which is an invertible sheaf.

One of the most important objects we will deal with along this work is defined in the following:

Definition 1.2.9. Let X be a reduced projective curve over the field k. A (non-zero) coherent $\mathcal{O}_X$-module $\mathcal{F}$ on X is said to be of *depth one* if for each closed point of its support $x \in \mathrm{supp}(\mathcal{F}) \subset X$, $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module of depth one.

Definition 1.2.10. A coherent sheaf of depth one (see Definition 1.2.9) is *semistable* if for any proper subsheaf $\mathcal{G} \subset \mathcal{F}$ we have $p_{\mathcal{G}}(n) \leq p_{\mathcal{F}}(n)$, or, equivalently, $\mu'(\mathcal{G}) \leq \mu'(\mathcal{F})$.

Remark 1.2.11. Note that the semistability condition depends, in general, on the polarization $\mathcal{O}_X(1)$ we are working with.

The following definition and theorem shows the importance of the concept of semistability.

Definition 1.2.12. Let X be a reduced projective curve over k and let $\mathcal{F}$ be a non zero coherent sheaf of depth one on X. A Harder-Narasimhan filtration for $\mathcal{F}$ is an increasing filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_l = \mathcal{F}$$

such that the factors $gr_i := \mathcal{F}_i/\mathcal{F}_{i-1}$ are semistable sheaves of depth one with reduced Hilbert polynomials p_i, satisfying

$$p_{max}(\mathcal{F}) := p_1 > \cdots > p_l =: p_{min}(\mathcal{F}). \tag{1.21}$$

Theorem 1.2.13. ([33, Theorem 1.3.4]) *Every coherent sheaf of depth one on X has a unique Harder-Narasimhan filtration.*

Remark 1.2.14. i) This theorem shows, in particular, that there is no ambiguity in the notation p_{max} and p_{min}. Note that a coherent sheaf $\mathcal{F}$ of depth one is semistable if and only if $p_{max}(\mathcal{F}) = p_{min}(\mathcal{F})$.

ii) By Equation (1.20) we can rewrite Equation (1.21) in terms of slopes. In particular, the quantities $\mu'_{max}(\mathcal{F})$ and $\mu'_{min}(\mathcal{F})$ are defined as the slopes of $\mathcal{F}_1$ and $\mathcal{F}/\mathcal{F}_{l-1}$ respectively.

Let us recall the concepts of torsion free, torsionless and reflexive module, and rank(see [10, Chapter 1, §1.4]). Let R be a commutative ring, Σ its total ring of fractions and M a finitely generated R-module. We say that M is *torsion free* if the canonical map $M \to M \otimes \Sigma$ is injective. The *dual* of M is defined as $M^\vee = \mathrm{Hom}_R(M, R)$ and there is a canonical map $\varphi\colon M \to M^{\vee\vee}$ defined by $\varphi(m)(f) := f(m)$. We say that M is *torsionless* if φ is injective, and is *reflexive* if φ is an isomorphism. We say that M has *rank* r if $M \otimes \Sigma \simeq \Sigma^{\oplus r}$ as a Σ-module. Suppose now that R is noetherian and reduced. In this case R has finitely many minimal points $\eta_1, \ldots, \eta_n$ and they corresponds to the irreducible components $V(\eta_i) = \mathrm{Spec}(R/\eta_i)$ of $\mathrm{Spec}(R)$. If we denote by Σ_i the field of fractions of the ith irreducible component then $\Sigma = \prod_{i=1}^n \Sigma_i$. The multirank of a finitely generated R-module is the tuple $(r_1, \ldots, r_n)$, being r_i is the rank of M restricted to $V(\eta_i)$.

These definitions can be extended to schemes and sheaves in the obvious way. Note that, in case X is an irreducible and reduced projective curve, torsion free, torsionless and depth one describe the same concept.

Characterization on Reduced Projective Curves

Our propose is to generalize [57, Septiéme Partie, Lemme 5].

Lemma 1.2.15. *Let $\mathcal{O}$ be a noetherian local domain with maximal ideal $\mathbf{m_x}$ and M, M' free $\mathcal{O}$-modules of the same rank r. Let $f\colon M \to M'$ be a non-zero morphism of $\mathcal{O}$-modules. The following statements are equivalent*

i) $f\colon M \to M'$ is surjective

ii) $f\colon M \to M'$ is an isomorphism

iii) $f(x)\colon M/\mathbf{m_x} \to M'/\mathbf{m_x}$ is an isomorphism

Proof. 1)⇒2) Assume $f\colon M \to M'$ is surjective and denote by M'' its kernel. If Σ is the field of functions of $\mathcal{O}$ then, since both modules have the same rank, we have

$$f \otimes 1\colon M \otimes_{\mathcal{O}} \Sigma \simeq M' \otimes_{\mathcal{O}} \Sigma,$$

that is, $M'' \otimes_{\mathcal{O}} \Sigma = 0$, so that M'' is a torsion submodule of M. Since M is free, M'' must be the zero module, so $f\colon M \simeq M'$. Step 2)⇒3) is trivial. For step 3)⇒1), consider the right exact sequence

$$M \to^f M' \to \mathrm{Coker}(f) \to 0.$$

Since $f(x)$ is an isomorphism we deduce that $\mathrm{Coker}(f)/(m_x)\mathrm{Coker}(f) = 0$, that is, $\mathrm{Coker}(f) = (m_x)\mathrm{Coker}(f)$. Now by Nakayama's Lemma we finde $\mathrm{Coker}(f) = 0$, so f is surjective. □

Theorem 1.2.16. *Let X be a connected and reduced projective curve over k and $\mathcal{F}$ a coherent sheaf. The following statements are equivalent:*

i) $\mathcal{F}$ is a depth one coherent sheaf of uniform multirank r,

ii) there exists $m \gg 0$ such that there exists an injection $\mathcal{F} \hookrightarrow \bigoplus^r \mathcal{O}_X(m)$ whose cokernel is a torsion sheaf.

Proof. ii)⇒ i) is trivial. Let us see i)⇒ii). Assume $\mathcal{F}$ is a depth one coherent sheaf of uniform multirank r. Fix a non-singular point x_i one for each irreducible component X_i. Denote $D = x_1 + \cdots + x_l$ the corresponding divisor, denote with the same leter its support, and denote $i\colon D \hookrightarrow X$ the natural inclusion. Consider the sheaf

$$\mathcal{G} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \bigoplus^r \mathcal{O}_X),$$

which is also torsion free. Consider the canonical morphism

$$\phi\colon \mathcal{G} \to i_* i^* \mathcal{G} = \bigoplus^l \mathcal{G}(x_i).$$

Let $\mathcal{K}$ be the kernel. Since ϕ is surjective we have an exact sequence

$$0 \to \mathcal{K} \hookrightarrow \mathcal{G} \xrightarrow{\phi} i_* i^* \mathcal{G} \to 0.$$

Let m be an integer large enough such that $H^1(X, \mathcal{K}(m)) = 0$. Take tensor product by $\mathcal{O}_X(m)$ in the above exact sequence

$$0 \to \mathcal{K}(m) \hookrightarrow \mathcal{G}(m) \xrightarrow{\phi} i_* i^* \mathcal{G} \to 0.$$

Since $\mathcal{G}(m) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \bigoplus^r \mathcal{O}_X(m))$, taking global sections, we find a surjective linear map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \bigoplus^r \mathcal{O}_X(m)) \xrightarrow{\phi} \bigoplus^l \mathcal{G}(x_i) \to 0.$$

Since $\mathcal{F}$ is torsion free and the points x_i are non-singular we deduce that

$$\mathcal{G}(x_i) \simeq \mathrm{Hom}_k(\mathcal{F}(x_i), k^r).$$

Fix an isomorphism $g_i : \mathcal{F}(x_i) \simeq k^r$ for each point, and let $g\colon \mathcal{F} \to \bigoplus^r \mathcal{O}_X(m)$ be such that $\phi(g) = (g_1, \dots, g_l)$. In particular, because of Lemma 1.2.15, g satisfies

$$g_{x_i}\colon \mathcal{F}_{x_i} \simeq \bigoplus^r \mathcal{O}_{X,x_i}.$$

Let $Q = \mathrm{Ker}(g)$. Clearly $Q_{x_i} = 0$ for each i so that $\mathrm{supp}(Q)$ consists only on finitely many points on X. Since $\mathcal{F}$ has depth one we deduce $Q = 0$, and hence

$$g\colon \mathcal{F} \hookrightarrow \bigoplus^r \mathcal{O}_X(m)$$

□

Remark 1.2.17. Observe that, from this theorem, we infer that a coherent sheaf over a reduced projective curve is torsion free if and only if has depth one.

The non existence of torsion elements in the sheaf of rings of a reduced projective curve implies the following important property.

Proposition 1.2.18. *Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $\mathcal{F}^\vee$ is torsion free.*

Proof. It is enough to show that $\mathrm{Hom}_{\mathcal{O}_{X,x}}(k, \mathcal{F}_x^\vee) = 0$ for all $x \in X$. Assume there is a point and a non zero morphism $\phi' \colon k \to \mathcal{F}_x^\vee$. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m}_x \arrow[r, hook] & \mathcal{O}_{X,x} \arrow[r] \arrow[dr, "\phi"] & k \arrow[r] \arrow[d, "\phi'"] & 0 \\ & & & \mathcal{F}_x^\vee & \end{tikzcd}$$

Note that there exists a morphism $f \colon \mathcal{F}_x \to \mathcal{O}_{X,x}$ such that $\phi(a) = af$ for all $a \in \mathcal{O}_{X,x}$. In particular $af = 0$ for every $a \in \mathfrak{m}_x$ so there exists $z \in \mathcal{F}_x$ such that $f(z) \neq 0$ and $af(z) = 0$ for every $a \in \mathfrak{m}_x$ but this is not possible because $\mathcal{O}_{X,x}$ do not have torsion elements. $\square$

Flatness Properties on Gorenstein Curves

We show that the dual of a flat family of coherent sheaves of depth one is also flat.

Lemma 1.2.19. ([12, Chapter III, Lemma 2.5.]) *Let $\phi : A \to B$ be a local morphism of noetherian local rings, B A-flat, with maximal ideals $\mathbf{m}$ and $\mathbf{n}$ and $k = A/\mathbf{m}$. Assume $\overline{B} := B/\mathbf{m}B$ is a Gorenstein ring. Let F be a B-module of finite type, flat over A such that $\overline{F} := F \otimes k$ has $depth(\mathcal{F}) = 1$. Then $F^\vee = Hom_B(F, B)$ is flat over A and $F^\vee \otimes_A k \simeq Hom_{\overline{B}}(\overline{F}, \overline{B})$*

Corollary 1.2.20. *Let S be a noetherian scheme and $f \colon X \to S$ a flat morphism of finite type such that the geometric fibres of f are Gorenstein curves. Let $\mathcal{F}$ be an S-flat coherent $\mathcal{O}_X$-module inducing on the fibres of f Cohen-Macaulay sheaves. Then*

1) $\mathcal{F}^\vee$ is flat over S.

2) for each point $s \in S$, the canonical morphism ([22, Chapter 0, §4, 4.4.6.])

$$f_s^*(\mathcal{F}^\vee) \to (f_s^*\mathcal{F})^\vee \tag{1.22}$$

is an isomorphism, where

$$\begin{tikzcd} X_s \arrow[r, hook, "f_s"] \arrow[d] & X \arrow[d] \\ Spec(k(s)) \arrow[r, hook] & S \end{tikzcd}$$

Proof. (See [12], Chapter III, Proposition 2.7.). The problem is local on S. But if $\mathcal{F}$ is Cohen-Macaulay then $\mathcal{F}_x$ is zero or $\mathcal{F}_x$ is a depth one $\mathcal{O}_{X,x}$-module. If $\mathcal{F}_x = 0$ then $\mathcal{F}^\vee$ is trivially flat over $x \in X$ and the natural map in 2) is trivially an isomorphism in $x \in X$. Suppose that $\mathcal{F}_x \neq 0$. Then result follows applying Lemma 1.2.15 for $F := \mathcal{F}_x$, $B := \mathcal{O}_{X,x}$ and $A := \mathcal{O}_{S,f(x)}$. $\square$

A Useful Result on Torsion Free Quotients

Let X be a reduced projective curve over k and $\mathcal{E}$ a torsion free sheaf on X. Recall that a quotient sheaf of $\mathcal{E}$ is a surjection $q \colon \mathcal{E} \to \mathcal{F} \to 0$ where $\mathcal{F}$ is torsion free. An isomorphism between two quotients is an isomorphism between the two sheaves which are compatible with the surjections. Then we have:

Proposition 1.2.21. *Let $q_i\colon \mathcal{E} \to \mathcal{F} \to 0$, $i = 1, 2$, be two torsion free quotients such that*

$$\begin{tikzcd}
 & \mathcal{F}_1|_U \arrow[dd, equal, "\phi_U" , "\wr"'] \\
\mathcal{E}|_U \arrow[ur, twoheadrightarrow, "q_1|_U"] \arrow[dr, twoheadrightarrow, "q_2|_U"'] & \\
 & \mathcal{F}_2|_U
\end{tikzcd}$$

commutes over a dense open subscheme $U \subset X$. Then, there exists an isomorphism, $\phi : \mathcal{F}_1 \simeq \mathcal{F}_2$, of quotients extending ϕ_U.

Proof. Denote $\alpha_i\colon U \hookrightarrow X$ the open immersion. Since U is dense, the complement, $Z = X - U$, consists of finitely many closed points. From the exact sequence (see [29])

$$0 \to \underline{\mathcal{H}}^0_Z(\mathcal{F}_i) \hookrightarrow \mathcal{F}_i \xrightarrow{\Phi_i} \alpha_*(\mathcal{F}_i|_U) \to \underline{\mathcal{H}}^1_Z(\mathcal{F}_i) \to 0$$

and the fact that $\mathrm{depth}(\mathcal{F}_i) = 1$, we deduce that $\underline{\mathcal{H}}^0_Z(\mathcal{F}_i) = 0$. Thus, the canonical morphism,

$$\Phi_i\colon \mathcal{F}_i \hookrightarrow \alpha_*(\mathcal{F}_i|_U),$$

is injective. The same argument is applied to $\mathcal{E}$, so the canonical morphism $\Phi\colon \mathcal{E} \hookrightarrow \alpha_*(\mathcal{E}|_U)$ is injective. Therefore, we have the following commutative diagram

$$\begin{tikzcd}
 & & & \mathcal{F}_1 \arrow[dl, hook, "\Phi_1"] \\
 & & \alpha_*(\mathcal{F}_1|_U) \arrow[dd, equal, "\alpha_*(\phi_U)", "\wr"'] & \\
\mathcal{E} \arrow[r, hook, "\Phi"] \arrow[uurrr, twoheadrightarrow, bend left, "q_1"] \arrow[ddrrr, twoheadrightarrow, bend right, "q_2"'] & \alpha_*(\mathcal{E}|_U) \arrow[ur, "\alpha_*(q_1|_U)"] \arrow[dr, "\alpha_*(q_2|_U)"'] & & \\
 & & \alpha_*(\mathcal{F}_2|_U) & \\
 & & & \mathcal{F}_2 \arrow[ul, hook', "\Phi_2"']
\end{tikzcd}$$

Now we claim that the morphism $f_1 = \alpha_*(\phi_U) \circ \Phi_1$ takes values in $\mathcal{F}_2$ and the morphism $f_2 = \alpha_*(\phi_u)^{-1} \circ \Phi_2$ takes values in $\mathcal{F}_1$. It is enough to show this at the level of stalks. By the surjectivity of q_1 and the injectivity of Φ_2, it is obvious that for any element $m \in \mathcal{F}_1$ there is a unique element $m' \in \mathcal{F}_2$ such that $f_1(m) = \Phi_2(m')$ (the same argument holds for f_2). This, together with the fact that the diagram commutes, implies that f_1 and f_2 are inverses to each other, so they determine an isomorphism between both quotients. $\square$

Remark 1.2.22. Note, that this proposition says that if $\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{E}$ are saturated subsheaves (i.e. with torsion free quotients) which are equal on some dense open subscheme, then they are globally equal.

1.2.3 Local Structure of Torsion Free Sheaves on Nodal Curves

Let us recall the local structure of a torsion free sheaf $\mathcal{F}$ on the nodal points of the nodal curve X.

Proposition 1.2.23. ([57, Chap. 8, Proposition 2]) *Let x be a nodal point of X lying in only one irreducible component X_i of X. Then a finitely generated $\mathcal{O}_x$-module M has depth one if and only if there exist $a_i, b_i \in \mathbb{N}$ with*

$$M \simeq \mathcal{O}_x^{a_i} \oplus \mathbf{m}_x^{b_i}. \tag{1.23}$$

Remark 1.2.24. The integers a_i and b_j are unique because

$$\begin{aligned} \mathrm{rk}(M \otimes_{\mathcal{O}_x} k) &= a_i + 2b_i, \\ \mathrm{rk}(M) &= a_i + b_i. \end{aligned}$$

Proposition 1.2.25. ([57, Chap. 8 Proposition 3]) *Let x be a nodal point of X lying in two irreducible components X_i and X_j of X. Then a finitely generated $\mathcal{O}_x$-module M has depth one if and only if there exist $a_{ij}, b_{ij}, c_{ij} \in \mathbb{N}$ with*

$$M \simeq \mathcal{O}_x^{a_i} \oplus \mathcal{O}_{x_i}^{b_{ij}} \oplus \mathcal{O}_{x_j}^{c_{ij}}. \tag{1.24}$$

Remark 1.2.26. Again, the integers a_{ij}, b_{ij} and c_{ij} are unique because

$$\begin{aligned} \mathrm{rk}(M \otimes_{\mathcal{O}_x} \mathcal{O}_{x_i}) &= a_{ij} + b_{ij} \\ \mathrm{rk}(M \otimes_{\mathcal{O}_x} \mathcal{O}_{x_j}) &= a_{ij} + c_{ij} \\ \mathrm{rk}(M \otimes_{\mathcal{O}_x} k) &= a_{ij} + b_{ij} + c_{ij} \end{aligned}$$

Suppose that $\mathcal{F}$ has uniform multirank r. If x is a nodal point lying in only one irreducible component X_i, then the local structure of $\mathcal{F}$ at x is determined by the number a_i because $r = a_i + b_i$. In the same way, if x is a nodal point lying in two irreducible components X_i and X_j the the local structure of $\mathcal{F}$ at x is determined by the number a_{ij} because $r = a_{ij} + b_{ij}$ and $r = a_{ij} + c_{ij}$ (so $b_{ij} = c_{ij} = r - a_{ij}$). Since for a nodal point lying in two irreducible components we have $\mathbf{m}_x \simeq \mathcal{O}_{x_i} \oplus \mathcal{O}_{x_j}$ (see [57]) we also find, in this particular case, that $M \simeq \mathcal{O}_x^{a_{ij}} \oplus \mathcal{O}_{x_i}^{r-a_{ij}} \oplus \mathcal{O}_{x_j}^{r-a_{ij}} \simeq \mathcal{O}_{x_i}^{a_{ij}} \oplus \mathbf{m}_x^{r-a_{ij}}$.

Definition 1.2.27. A torsion free sheaf of rank r is *of type a_i at a nodal point x_i* (lying in one or two irreducible components) if $\mathcal{F}_x \simeq \mathcal{O}_{X,x}^{a_i} \oplus \mathfrak{m}_x^{r-a_i}$.

1.2.4 Extending the Local Structure

Let X be a projective connected nodal curve over k of genus g and let us denote $x_1, \ldots, x_\nu$ its nodes.

Lemma 1.2.28. *If $\mathcal{F}$ is a torsion free sheaf on X and x_i a node lying in only one irreducible component X' (resp. lying in two ireducible components X', X'') then for each open subset U such that $x_i \in U$, $x_j \notin U$ for $j \neq i$ and contained in X' (resp. contained in $X' \cup X''$), we have*

$$\mathcal{F}_U \subset \mathcal{F}_{x_i}.$$

Proof. Let U be an open subset satisfying the above conditions. Consider the natural morphism

$$\begin{aligned}\Phi\colon \mathcal{F}_U &\to \mathcal{F}_{x_i},\\ s &\mapsto \frac{s}{1}\end{aligned}$$

and let $s \in \mathcal{F}_U$ be an element of the kernel and assume that $s \neq 0$. Since $\dfrac{s}{1} = 0$ there exists $0 \neq f \in \mathcal{O}_U \setminus \overline{\mathfrak{m}}_{x_i}$ such that $fs = 0$, $\overline{\mathfrak{m}}_{x_i} \subset \mathcal{O}_U$ being the maximal ideal of x_i. Consider the submodule $M := \langle s \rangle \subset \mathcal{F}_U$. Since $\Phi(s) = 0$ we have $M_{x_i} = 0$. Therefore there is a finite set of points $C = \{p_1, \ldots, p_l\} \subset U$ suth that $M_V = 0$ being $V = U \setminus C$, that is, M is supported on C (this is because of the properties that U satisfies). Then on each point of C we find that $M_{p_i} \simeq k^{n_i}$, and therefore for each i we have an inclusion $M_{p_i} \simeq k^{n_i} \hookrightarrow \mathcal{F}_{p_i}$ which can not be possible, so $s = 0$ and Φ is injective. □

Theorem 1.2.29. *Let $\mathcal{F}$ be a torsion free sheaf on X of uniform multirank r. Let $x \in X$ be a node and suppose that $\mathcal{F}$ is of type a on x, i.e. $\mathcal{F}_x \simeq \mathfrak{m}_x^a \oplus \mathcal{O}_x^{r-a}$. Then there is an (affine) open neighborhood, U, of x not containing more nodes and an isomorphism*

$$\Phi_U : \mathcal{F}_U \simeq \overline{\mathfrak{m}}_x^a \oplus \mathcal{O}_U^{r-a}$$

satisfying $\Phi_{U,x} = \Phi_x$.

Proof. Let V be an open subset as in the Lemma 1.2.28. Then we have

$$\begin{tikzcd}
\mathcal{F}_x \arrow[r, "\Phi_x"] & \mathfrak{m}_x^a \oplus \mathcal{O}_x^{r-a} \\
\mathcal{F}_V \arrow[u, hook] &
\end{tikzcd}$$

Let $\{m_1, \ldots, m_l\}$ be generators of $\mathcal{F}_V$. Then

$$\Phi_x(m_i) = \frac{s_i}{t_i} + \frac{f_i}{g_i},\ s_i \in \mathfrak{m}_x^a,\ f_i \in \mathcal{O}_x^{r-a},\ g_i, t_i \in \mathcal{O}_V \setminus \overline{\mathfrak{m}}_x.$$

Consider the ideal $I = \langle \{g_i\}, \{t_i\} \rangle$ and let $U' = V \setminus V(I)$. Then we have a conmutative diagram

$$\begin{tikzcd}
\mathcal{F}_x \arrow[r, "\Phi_x"] & \mathfrak{m}_x^a \oplus \mathcal{O}_x^{r-a} \\
\mathcal{F}_{U'} \arrow[u, hook] \arrow[r, hook, dashed, "\Phi_{U'}"] & \overline{\mathfrak{m}}_x^a \oplus \mathcal{O}_{U'}^{r-a} \arrow[u, hook]
\end{tikzcd}$$

Since $\Phi_{U',x}$ is an isomorphism there exists an open subset $U' \supseteq U \ni x$ such that Φ_U is an isomorphism. □

1.3 Sheaves on Non Connected Smooth Projective Curves

Let $Y = \amalg_{i=1}^{l} Y_i$ be a curve where Y_i is an irreducible smooth projective curve for all i, and let $\mathcal{O}_Y(1)$ be a polarization. For each i we denote by $u_i\colon Y_i \hookrightarrow Y$ the natural inclusion. We assume along this section that $l > 1$. A coherent sheaf $\mathcal{E}$ of depth one on Y a locally free sheaf. Moreover, for any locally free sheaf $\mathcal{E}$ on Y we have

$$\mathcal{E} = \bigoplus_{i=1}^{l} u_{i*}\mathcal{E}_i, \text{ where } \mathcal{E}_i = \mathcal{E}|_{Y_i}. \tag{1.25}$$

Recall from Remark 1.2.11 that the semistability condition depends on the polarization we fix on the curve. The aim of this section is to show that, for some special polarizations, the tensor product of two semistable locally free sheaves, $\mathcal{E} \otimes \mathcal{F}$, is semistable.

Let us denote $h = \deg(\mathcal{O}_Y(1))$ and $h_i = \deg(\mathcal{O}_{Y_i}(1))$. First of all note that, by Equation (1.25), we have

$$\begin{aligned}\chi(\mathcal{E}) &= \sum_{i=1}^{l} \chi(\mathcal{E}_i),\\ P_{\mathcal{E}}(n) &= \sum_{i=1}^{l} \{h_i \mathrm{rk}(\mathcal{E}_i) n + \deg(\mathcal{E}_i) + \mathrm{rk}(\mathcal{E}_i)\chi(Y_i, \mathcal{O}_{Y_i})\}.\end{aligned}$$

Since $P_{\mathcal{E}}(n) = \alpha(\mathcal{E})n + \deg(\mathcal{E}) + \mathrm{rk}(\mathcal{E})\chi(Y, \mathcal{O}_Y)$, we can easily show that

$$\begin{aligned}\mathrm{rk}(\mathcal{E}) &= \sum_{i=1}^{l} e_i \mathrm{rk}(\mathcal{E}_i), \text{ where } e_i = \frac{h_i}{h},\\ \deg(\mathcal{E}) &= -\mathrm{rk}(\mathcal{E})\chi(Y, \mathcal{O}_Y) + \sum_{i=1}^{l} \Big\{\mathrm{rk}(\mathcal{E}_i)\chi(Y_i, \mathcal{O}_{Y_i}) + \deg(\mathcal{E}_i)\Big\}.\end{aligned} \tag{1.26}$$

Recall that the slope of a locally free sheaf $\mathcal{E}$ is defined as $\mu(\mathcal{E}) = \deg(\mathcal{E})/\alpha(\mathcal{E})$. We say that $\mathcal{E}$ is semistable if for any subsheaf $\mathcal{F} \subset \mathcal{E}$, we have $\mu(\mathcal{F}) \subset \mu(\mathcal{E})$. For any tuple of locally free sheaves, $\mathcal{E}^1, \dots, \mathcal{E}^n$ and for any polarization $\mathcal{O}_Y(1)$, we define

$$\begin{aligned}A^i(\mathcal{E}^1, \dots, \mathcal{E}^n) :=& \frac{h_i \mathrm{rk}(\mathcal{E}_i^1) \dots \mathrm{rk}(\mathcal{E}_i^n)}{\sum_{j=1}^{l} h_j \mathrm{rk}(\mathcal{E}_j^1) \dots \mathrm{rk}(\mathcal{E}_j^n)} = \frac{\alpha(\mathcal{E}_i)}{\alpha(\mathcal{E})},\\ B^i(\mathcal{O}_Y(1)) :=& \frac{\chi(Y, \mathcal{O}_{Y_i})}{h_i} - \frac{\chi(Y, \mathcal{O}_Y)}{h},\end{aligned} \tag{1.27}$$

being $\mathcal{E} = \mathcal{E}^1 \otimes \cdots \otimes \mathcal{E}^n$ and $\mathcal{E}_i = \mathcal{E}|_{Y_i}$, the restriction to the ith component.

Lemma 1.3.1. *For any tuple of locally free sheaves $\mathcal{E}^1, \dots, \mathcal{E}^n$ we have*

$$\mu(\mathcal{E}^1 \otimes \dots \otimes \mathcal{E}^n) = \sum_{i=1}^{l} A^i(\mathcal{E}^1, \dots \mathcal{E}^n)\Big\{B^i(\mathcal{O}_Y(1)) + \sum_{j=1}^{n} \mu(\mathcal{E}_i^j)\Big\}.$$

Proof. By equality (1.26) we know that

$$\begin{aligned}\deg(\mathcal{E}^1 \otimes \cdots \otimes \mathcal{E}^n) =& -\mathrm{rk}(\mathcal{E}^1 \otimes \dots \otimes \mathcal{E}^n)\chi(Y, \mathcal{O}_Y) + \\ &+ \sum_{i=1}^{l} \Big\{\mathrm{rk}(\mathcal{E}_i^1 \otimes \dots \otimes \mathcal{E}_i^n)\chi(Y, \mathcal{O}_{Y_i}) + \deg(\mathcal{E}_i^1 \otimes \dots \otimes \mathcal{E}_i^n)\Big\}.\end{aligned}$$

Since

$$\mathrm{rk}(\mathcal{E}_i^1 \otimes \ldots \otimes \mathcal{E}_i^n) = \prod_{j=1}^n \mathrm{rk}(\mathcal{E}_i^j),$$
$$\mathrm{rk}(\mathcal{E}^1 \otimes \ldots \otimes \mathcal{E}^n) = \sum_{i=1}^l e_i (\prod_{j=1}^n \mathrm{rk}(\mathcal{E}_i^j)),$$
$$\deg(\mathcal{E}_i^1 \otimes \ldots \otimes \mathcal{E}_i^n) = \sum_{k=1}^n \deg(\mathcal{E}_i^k)(\prod_{j \neq k} \mathrm{rk}(\mathcal{E}_i^j)),$$

we deduce that

$$\begin{aligned}\deg(\mathcal{E}^1 \otimes \ldots \otimes \mathcal{E}^n) = &-(\sum_{i=1}^l e_i(\prod_{j=1}^n \mathrm{rk}(\mathcal{E}_i^j)))\chi(Y, \mathcal{O}_Y)+ \\ &+ \sum_{i=1}^l \Big\{ \prod_{j=1}^n \mathrm{rk}(\mathcal{E}_i^j)\chi(Y, \mathcal{O}_{Y_i}) + \sum_{k=1}^n \deg(\mathcal{E}_i^k)(\prod_{j\neq k} \mathrm{rk}(\mathcal{E}_i^j))\Big\}.\end{aligned}$$

Since the slope is

$$\mu(\mathcal{E}^1 \otimes \ldots \otimes \mathcal{E}^n) = \frac{\deg(\mathcal{E}^1 \otimes \ldots \otimes \mathcal{E}^n)}{h\mathrm{rk}(\mathcal{E}^1 \otimes \ldots \otimes \mathcal{E}^n)},$$

we get

$$\begin{aligned}\mu(\mathcal{E}^1 \otimes \ldots \otimes \mathcal{E}^n) &= \frac{-\chi(\mathcal{O}_Y)}{h} + \frac{\sum_{i=1}^l \Big\{ \prod_{j=1}^n \mathrm{rk}(\mathcal{E}_i^j)\chi(\mathcal{O}_{Y_i}) + \sum_{k=1}^n \deg(\mathcal{E}_i^k)(\prod_{j\neq k}\mathrm{rk}(\mathcal{E}_i^j)))\Big\}}{h(\sum_{i=1}^l e_i \prod_{j=1}^n \mathrm{rk}(\mathcal{E}_i^j))} = \\ &= \frac{-\chi(\mathcal{O}_Y)}{h} + \frac{\sum_{i=1}^l \Big\{ \prod_{j=1}^n \mathrm{rk}(\mathcal{E}_i^j)\chi(\mathcal{O}_{Y_i}) + h_i \prod_{j=1}^n \mathrm{rk}(\mathcal{E}_i^j)\sum_{k=1}^n \mu(\mathcal{E}_i^k)\Big\}}{h(\sum_{i=1}^l e_i \prod_{j=1}^n \mathrm{rk}(\mathcal{E}_i^j))} = \\ &= \frac{\sum_{i=1}^l \prod_{j=1}^n \mathrm{rk}(\mathcal{E}_i^j)(\chi(\mathcal{O}_{Y_i}) - e_i\chi(\mathcal{O}_Y))}{h(\sum_{i=1}^l e_i \prod_{j=1}^n \mathrm{rk}(\mathcal{E}_i^j))} + \frac{\sum_{i=1}^l (h_i \prod_{j=1}^n \mathrm{rk}(\mathcal{E}_i^j)\sum_{k=1}^n \mu(\mathcal{E}_i^k))}{h(\sum_{i=1}^l e_i \prod_{j=1}^n \mathrm{rk}(\mathcal{E}_i^j))} = \\ &= \sum_{i=1}^l \Big\{ \frac{h_i \prod_{j=1}^n \mathrm{rk}(\mathcal{E}_i^j)}{\sum_{i=1}^l h_i \prod_{j=1}^n \mathrm{rk}(\mathcal{E}_i^j)} \{ (\frac{\chi(\mathcal{O}_{Y_i})}{h_i} - \frac{\chi(\mathcal{O}_Y)}{h}) + \sum_{k=1}^n \mu(\mathcal{E}_i^k)\}\Big\} = \\ &= \sum_{i=1}^l A^i(\mathcal{E}^1, \ldots, \mathcal{E}^n)\Big\{B^i(\mathcal{O}_Y(1)) + \sum_{k=1}^n \mu(\mathcal{E}_i^k)\Big\}.\end{aligned}$$

□

Remark 1.3.2. From the last lemma we find, in particular, that

$$\mu(\mathcal{E}) = \sum_{i=1}^l A^i(\mathcal{E})\Big\{B^i(\mathcal{O}_Y(1)) + \mu(\mathcal{E}_i)\Big\}.$$

Note that

i) If there is just one component $l = 1$, then we obviously have $A(\mathcal{E}) = 1$ and $B(\mathcal{O}_Y(1)) = 0$.

ii) if $B^i(\mathcal{O}_Y) = 0$ for all i, we find $\mu(\mathcal{E}) = \dfrac{\sum_{i=1}^l \deg(\mathcal{E}_i)}{\alpha(\mathcal{E})}$, that is, $\deg(\mathcal{E}) = \sum_{i=1}^l \deg(\mathcal{E}_i)$.

iii) let $\mathcal{F}$ be a locally free sheaf on the ith component Y_i, and consider the sheaf $\mathcal{E} = u_{i*}\mathcal{F}$ on Y. Then $\mu(\mathcal{E}) = \mu(\mathcal{F}) + B^i(\mathcal{O}_Y(1))$.

Lemma 1.3.3. *If $\mathcal{E}$ is a semistable locally free sheaf on Y, then $\mu(\mathcal{E}) = B^i(\mathcal{O}_Y(1)) + \mu(\mathcal{E}_i)$, for all i with $\mathcal{E}_i \neq 0$.*

Proof. We can reduce the lemma to the case $\mathcal{E}_i \neq 0$ for all i. For the sake of notation, we set $D^i := B^i(\mathcal{O}_Y(1)) + \mu(\mathcal{E}_i)$ for each i. Then we know that $\mu(u_{i*}\mathcal{E}_i) \leq \mu(\mathcal{E})$ since $\mathcal{E}$ is semistable. Therefore

$$\begin{aligned}
0 \leq \mu(\mathcal{E}) - \mu(u_{i*}\mathcal{E}_i) &= \big(\sum_{j=1}^l A^j(\mathcal{E})\{B^j(\mathcal{O}_Y(1)) + \mu(\mathcal{E}_j)\}\big) - (B^i(\mathcal{O}_Y(1)) + \mu(\mathcal{E}_i)) = \\
&= (\sum_{j=1}^l A^j(\mathcal{E})D^j) - D^i = \\
&= A^1(\mathcal{E})D^1 + \cdots + (A^i(\mathcal{E}) - 1)D^i + \cdots + A^l(\mathcal{E})D^l.
\end{aligned}$$

Recall that $A^j(\mathcal{E}) = \dfrac{\alpha(\mathcal{E}_j)}{\alpha(\mathcal{E})}$ for all j. Thus $A^i(\mathcal{E}) - 1 = \dfrac{\alpha(\mathcal{E}_i) - \alpha(\mathcal{E})}{\alpha(\mathcal{E})}$. Since $\alpha(\mathcal{E}) = \sum_{i=1}^l \alpha(\mathcal{E}_i)$, we get,

$$\begin{aligned}
0 \leq \mu(\mathcal{E}) - \mu(u_{i*}\mathcal{E}_i) =& A^1(\mathcal{E})(D^1 - D^i) + \cdots + A^{i-1}(\mathcal{E})(D^{i-1} - D^i) + \\
&+ A^{i+1}(\mathcal{E})(D^{i+1} - D^i) + \cdots + A^l(\mathcal{E})(D^l - D^i).
\end{aligned}$$

Since $A^j(\mathcal{E}) > 0$ for all j, we deduce that there must be at least one superindex j such that $(D^j - D^i) \geq 0$, that is, D^i can not be the maximum of the set of rational numbers $\{D^1, \ldots, D^l\}$. Since this is true for all i, we deduce that the set $\{D^1, \ldots, D^l\}$ has no maximum. Thus $D^1 = \ldots = D^l$. In particular, we find that $0 = \mu(\mathcal{E}) - \mu(u_{i*}\mathcal{E}_i)$ for all i, so

$$\mu(\mathcal{E}) = \mu(u_{i*}\mathcal{E}_i) = B^i(\mathcal{O}_Y(1)) + \mu(\mathcal{E}_i).$$

□

Lemma 1.3.4. *If $\mathcal{E}$ is a semistable locally free sheaf on Y, then $\mathcal{E}_i$ is semistable for all i with $\mathcal{E}_i \neq 0$.*

Proof. We can reduce the lemma to the case $\mathcal{E}_i \neq 0$ for all i. Let $\mathcal{F}' \subset \mathcal{E}_i$ be a subsheaf on Y_i and let $\mathcal{F} := u_{i*}\mathcal{F}' \subset \mathcal{E}$. Since $\mathcal{E}$ is semistable, we know that $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$. By Lemma 1.3.3, we deduce that

$$\mu(\mathcal{F}) \leq B^i(\mathcal{O}_Y(1)) + \mu(\mathcal{E}_i).$$

Finally, by Remark 1.3.2 (iii), we know that $\mu(\mathcal{F}) = B^i(\mathcal{O}_Y(1)) + \mu(\mathcal{F}_i)$ and therefore $\mu(\mathcal{F}_i) \leq \mu(\mathcal{E}_i)$, that is, $\mathcal{E}_i$ is semistable. □

Theorem 1.3.5. *A locally free sheaf $\mathcal{E}$ on Y is semistable if and only if all non zero components $\mathcal{E}_i$ are semistable and $B^i(\mathcal{O}_Y(1)) + \mu(\mathcal{E}_i) = \mu(\mathcal{E})$.*

Proof. We can reduce the theorem to the case $\mathcal{E}_i \neq 0$ for all i. The direct implication follows from Lemma 1.3.3 and Lemma 1.3.4. Let us see the inverse. Let $\mathcal{F} \subset \mathcal{E}$ be a subsheaf and let $\mathcal{F}_i$ be a non zero component. Since $\mathcal{E}_i$ is semistable, we know that

$$\mu(\mathcal{F}_i) \leq \mu(\mathcal{E}_i) = \mu(\mathcal{E}) - B^i(\mathcal{O}_Y(1)).$$

Therefore,

$$\mu(\mathcal{F}) = \sum_{\mathcal{F}_i \neq 0} A^i(\mathcal{F})\Big\{B^i(\mathcal{O}_Y(1)) + \mu(\mathcal{F}_i)\Big\} \leq \sum_{\mathcal{F}_i \neq 0} A^i(\mathcal{F})\mu(\mathcal{E}) = \mu(\mathcal{E}),$$

so $\mathcal{E}$ is semistable. □

Remark 1.3.6. This in particular shows that any semistable sheaf $\mathcal{E}_i$ on Y_i is semistable as a sheaf on Y independently on the polarization we are working with.

Definition 1.3.7. We say that the polarized curve $(Y, \mathcal{O}_Y(1))$ has the property $\mathbf{P}$ if the following holds true on Y

$$\mathbf{P} \equiv \left\{ \begin{array}{c} \text{if } \mathcal{E} \text{ and } \mathcal{F} \text{ are semistable locally free} \\ \text{sheaves with support equal to } Y \\ \text{then } \mathcal{E} \otimes \mathcal{F} \text{ is also semistable} \end{array} \right\}.$$

Theorem 1.3.8. *The polarized curve $(Y, \mathcal{O}_Y(1))$ has the property $\boldsymbol{P}$ if and only if $B^i(X) = 0$ for all i.*

Proof. We first prove that there exists a constant Δ such that $B^i(\mathcal{O}_Y(1)) = \Delta$ and then we show that Δ must be zero. Suppose that $(Y, \mathcal{O}_Y(1))$ satisfies property $\mathbf{P}$. Now, let $\mathcal{E}$, $\mathcal{F}$ be semistable locally free sheaves of uniform rank. By Theorem 1.3.5, we know that

i) all components $\mathcal{E}_i$, $\mathcal{F}_i$ are semistable
ii) $B^i(\mathcal{O}_Y(1)) + \mu(\mathcal{E}_i) = \mu(\mathcal{E})$ for all i
iii) $B^i(\mathcal{O}_Y(1)) + \mu(\mathcal{F}_i) = \mu(\mathcal{F})$ for all i

By Lemma 1.3.1 and Theorem 1.3.5, $\mathcal{E} \otimes \mathcal{F}$ is semistable if and only if $\mathcal{E}_i \otimes \mathcal{F}_i$ is semistable and $B^i(\mathcal{O}_Y(1)) + \mu(\mathcal{E}_i) + \mu(\mathcal{F}_i) = \mu(\mathcal{E} \otimes \mathcal{F})$. Since it is semistable because $(Y, \mathcal{O}_Y(1))$ satisfies property $\mathbf{P}$, the last equality holds. Therefore

$$\begin{aligned} \mu(\mathcal{E} \otimes \mathcal{F}) &= B^i(\mathcal{O}_Y(1)) + (-B^i(\mathcal{O}_Y(1)) + \mu(\mathcal{E})) + (-B^i(\mathcal{O}_Y(1)) + \mu(\mathcal{F})) = \\ &= -B^i(\mathcal{O}_Y(1)) + \mu(\mathcal{E}) + \mu(\mathcal{F}). \end{aligned}$$

thus $B^i(\mathcal{O}_Y(1))$ is constant. The reciprocal follws by the same argument. Let us show that the constant must be zero. Let Δ be such that $B^i(\mathcal{O}_Y(1)) = \Delta$ for all i. Then

$$h_i \Delta = \chi(\mathcal{O}_{Y_i}) - h_i \frac{\chi(\mathcal{O}_Y)}{h},$$

and taking the sum over all the components we get

$$h\Delta = \chi(\mathcal{O}_Y) - \chi(\mathcal{O}_Y) = 0.$$

Since $h \neq 0$, we get $\Delta = 0$. □

Corollary 1.3.9. *If $B^i(\mathcal{O}_Y(1)) = 0$ for all i, then for every couple of semistable locally free sheaves,*

i) the tensor product, $\mathcal{E} \otimes \mathcal{F}$, is semistable.

ii) we have $\mu(\mathcal{E} \otimes \mathcal{F}) = \mu(\mathcal{E}) + \mu(\mathcal{F})$.

Proof. It is implicitly proved in the last theorem. □

1.4 Principal G-bundles on Curves

In this section we will introduce the theory of principal G-bundles following [55]. Reductions and extensions of the structure group will become quite important and they will allow us to construct our compact moduli spaces of principal G-bundles considering locally free sheaves with an extra structure and their degenerations. This point is in the heart of all the constructions of the moduli spaces of principal G-bundles we have so far.

Let k be an algebraically closed field of characteristic 0. Any scheme considered in this section will be a separated noetherian k-scheme.

1.4.1 Coverings

Definition 1.4.1. Let Y be a scheme. A pair (X, f) consisting of a scheme X and a morphism of schemes $f\colon X \to Y$ is said to be a *covering* if f is finite (i.e. there exists a covering of affine open subschemes $U_i = \mathrm{Spec}(A_i) \subset Y$ such that $V_i := f^{-1}(U_i)$ is affine, say $V_i = \mathrm{Spec}(B_i)$, and B_i is a A_i-module of finite type) and surjective.

Remark 1.4.2. Observe that given a finitely generated B_i-module we automatically get a finitely generated A_i-module, simply by considering M with its induced A_i-module structure. Therefore, the functor $f_*(-)$ tranforms coherent $\mathcal{O}_X$-modules into coherent $\mathcal{O}_Y$-modules.

We will find different types of coverings depending of the characteristics of the A_i-algebras B_i. Those who will be considered here are the unramified, étalé and Galois coverings.

Definition 1.4.3. Let X and Y be schemes, $f\colon X \to Y$ a morphism locally of finite type, $x \in X$ a point of X and $y = f(x) \in Y$. Then f is said to be unramified at x if $\mathfrak{m}_x = \mathfrak{m}_y \mathcal{O}_X$ and $k(x)$ is a finite separable extension of $k(y)$. The morphism f is said to be unramified if it is unramified at every point $x \in X$.

The next proposition allow us to characterize geometrically the unramified condition,

Proposition 1.4.4. ([1, Proposition 3.3]) *Let X and Y be schemes, $f\colon X \to Y$ a morphism locally of finite type, $x \in X$ a point of X and $y = f(x) \in Y$. Then the following conditions are equivalent:*

(i) *$\Omega^1_{X/Y}$ is zero at x.*

(ii) *the diagonal $\Delta_{X/Y}$ is an open immersion in a neighborhood of x.*

(iii) *f is unramified at x.*

Remark 1.4.5. The last proposition says that the morphism $f\colon X \to Y$ is unramified if and only if $\Omega^1_{X/Y} = 0$, but this condition must to be checked only on closed (thereby, rational in our condiitions) points. Thus, f is unramified if and only if $\Omega^1_{X/Y,x} = 0$ at every rational point x.

Definition 1.4.6. Let X and Y be schemes, $f\colon X \to Y$ a morphism locally of finite type, $x \in X$ a point of X and $y = f(x) \in Y$. Then f is said to be étalé at x if it is flat and unramified at x.

Proposition 1.4.7. ([1, Corollary 4.5]) *Let X and Y be schemes, $f\colon X \to Y$ a morphism locally of finite type, $x \in X$ a point of X and $y = f(x) \in Y$. If $\widehat{f_x}\colon \widehat{\mathcal{O}}_y \to \widehat{\mathcal{O}}_x$ is an isomorphism, then f is étalé at x. Conversely, suppose that $k(x) = k(y)$ or that $k(y)$ is algebraically closed. If f is étalé then $\widehat{f_x}$ is an isomorphism.*

Remark 1.4.8. Since we are dealing with noetherian k-schemes, we have the equivalence: f is étalé at a closed point x if and only if $\widehat{f_x}$ is an isomorphism. As before, the morphism f is étalé if it is étalé at every point, thus, if and only if $f_*\mathcal{O}_X$ is a flat $\mathcal{O}_Y$-module and $\Omega^1_{X/Y} = 0$.

We say that a morphism locally of finite type $f\colon X \to Y$ is flat (resp. unramified, resp. étalé) at a point $y \in Y$ if it is flat (resp. unramified, resp. étalé) at every poitn $x \in X$ such that $f(x) = y$. With this in hand, we have, as a trivial consequence, the following

Corollary 1.4.9. *Let $f\colon X \to Y$ be a covering (finite, surjective), étalé at a point $y \in Y$. Then $(f_*\mathcal{O}_X)_y$ is free of rank $n = \sharp\{x \in X | f(x) = y\}$. Moreover, if f is an étalé cover then $f_*\mathcal{O}_X$ is a locally free sheaf on Y with constant rank on each connected component. Therefore, $f_*(-)$ transforms locally free sheaves into locally free sheaves.*

In the particular case in which we have that $f_*\mathcal{O}_Y$ is locally free of rank r, we say that the étalé covering f has *degree n*.

A morphism between coverings, $f\colon X \to Y$ and $g\colon X' \to Y$, is a morphism between X and X' as Y-schemes. The group of automorphism of a given covering $f\colon X \to Y$, is denoted by G_f. If the covering is étalé connected and of degree n we can bound the order of the group of automorphisms by n.

Proposition 1.4.10. *Let $f\colon X \to Y$ be an étalé covering of degree n. Let X' be a connected scheme and $\varphi\colon X' \to Y$ be a morphism. Then*

$$\mathrm{Hom}_Y(X', X) = \mathrm{Hom}_{X'}(X', X \times_Y X') = \left\{ \begin{array}{c} \textit{connected components of} \\ X \times_Y X' \textit{ isomorphic to } X' \end{array} \right\}.$$

In particular, $\#\mathrm{Hom}_Y(X', Y) \leq n$ and the inequality becomes into an equality if and only if $X \times_Y X' \to X'$ is a trivial covering.

Proof. The two equalities are clear. Let us show the inequality. We define

$$r := \#\mathrm{Hom}_X(X', Y) = \#\mathrm{Hom}_{X'}(X', Y \times_X X').$$

Then, there is an injective morphism $X' \amalg \overset{r}{\ldots} \amalg X' \hookrightarrow Y \times_X X'$ of coverings over X'. Therefore, $r \leq n$. Note that the equality holds if and only if $r = n$, i.e. the above injection is a bijection. $\square$

The next theorem let us to introduce the concept of Galois covering.

Theorem 1.4.11. *Let $f\colon X \to Y$ be an étalé covering with Y connected. Then there exists an étalé covering $g\colon X' \to Y$ which trivializes f,*

$$X' \sqcup \ldots \sqcup X' \simeq X \times_Y X' \to X'.$$

Proof. We proceed by induction on the degree. For degree equal to one is obvious. Let n be the degree. Since the identity of Y is an automorphism of coverings, by Proposition 1.4.10, we conclude that $X \times_Y X = X \amalg X_1 \amalg \ldots \amalg X_r$. Here, each X_i is a connected étalé covering of degree strictly smaller that n. Now, we conclude by induction. $\square$

We arrive, therefore, naturally to the following definition

Definition 1.4.12. A connected étalé covering $f\colon X \to Y$ is principal or Galois if it trivializes itself.

Example 1.4.13. The Galois coverings of the point $\mathrm{Spec}(k)$ are precisely the Galois extensions of k.

A link between geometric invariant theory for finite groups and Galois coverings is stablished in the following theorem of Artin

Theorem 1.4.14. **(Artin)** *Let $f\colon X \to Y$ be an étalé covering and $G \subseteq \mathrm{Aut}_Y(X)$ a subgroup. Then the geometric quotient X/G is equal to Y if and only if f is a Galois covering and $G = \mathrm{Aut}_Y(X)$.*

1.4.2 Fiber Spaces and Principal Bundles

Definition 1.4.15. Let X be a scheme and G an algebraic group. A fibered system over X with group G is a pair (P, π) where P is a scheme acted on by G (on the right) and $\pi\colon P \to X$ is G-invariant, i.e., $\pi(g \cdot p) = \pi(p)$. A morphism between fibered spaces over X with group G is a morphism of X-schemes, $f\colon P \to P'$, which is G-equivariant, i.e. $f(g \cdot p) = g \cdot f(p)$.

We denote by $\mathrm{Fib}_X(G)$ the category of fibered spaces over X with group G. Given a fibered space over X with group G and an X-scheme $f\colon X' \to X$, the fibered product $P \times_X X'$ with the induced morphism $\pi'\colon P \times_X X' \to X'$ is a fibered space. It is usually denoted by $f^*(P)$. Therefore, any X-scheme induces a functor

$$f^*\colon \mathrm{Fib}_X(G) \to \mathrm{Fib}_{X'}(G).$$

For any scheme X we can consider the first projection $\pi\colon X \times G \to X$ and the standard action of G on $X \times G$. Then $(X \times G, \pi)$ is a fibered space and we say that a fibered space is trivial if it is isomorphic to $(X \times G, \pi)$.

Definition 1.4.16. A principal G-bundle over X with structure group G is a fibered space (P, π) with group G which is *isotrivial*: for any point $x \in X$ there exists an open neighborhood $U \subset X$ of x and an unramified covering $f\colon X' \to U$ such that the induced fibered space $f^*P|_U \to X'$ is trivial.

Morphisms between principal G-bundles over X are the morphisms as fibered spaces. We will denote by $\mathrm{Bun}_X(G)$ the category of principal G-bundles over X. This category is also stable under pullbacks, and for any X-scheme we have an induced functor

$$f^*\colon \mathrm{Bun}_X(G) \to \mathrm{Bun}_{X'}(G)$$

as before.

Example 1.4.17. 1) Trivial fibered spaces $(X \times G, \pi)$ are principal bundles.

2) Given a finite group G, any Galois cover of a scheme X is a principal bundle over X.

3) If G is an affine algebraic group over k and $H \subset G$ is a closed subgroup of G, then G/H is a smooth quasi-projective scheme and G is a principal G-bundle over G/H with structure group H (see [5, Theorem 6.8]).

A basic property of the category $\mathrm{Bun}_X(G)$ is stated in the following proposition,

Proposition 1.4.18. ([55, §3.1]) *Let X be a scheme and $\pi\colon P \to X$, $\pi'\colon P' \to X$ principal G-bundles. Then* $\mathrm{Mor}_{\mathrm{Bun}_X(G)}(P, P') = \mathrm{Isom}_{\mathrm{Bun}_X(G)}(P, P')$.

1.4.3 Isotriviality Criterion

Let X be a scheme, G an algebraic group, and (P, π) a fibered system over X with group G. Let us denote by $\sigma\colon P \times G \to P$ the group action on P and $p_2 : P \times G \to P$ the second projection. Consider the following properties for fibered spaces

P1) The morphism $\Phi = (\sigma, p_2)\colon P \times G \to P \times_X P$ over X is an isomorphism of X-schemes.

P2) For any point $x \in X$, there is an étalé covering $f\colon U' \to U \subset X$ over an open neighborhood of x and a morphism $s\colon U' \to P$ such that the diagram

$$\begin{tikzcd} & P \arrow[d, "\pi"] \\ U' \arrow[ur, "s"] \arrow[r, "f"'] & U \end{tikzcd}$$

is commutative.

Then we have,

Proposition 1.4.19. ([55, Proposition 2]) *The fibered system (P, π) is locally isotrivial (so, a principal G-bundle) if and only if P1) and P2) are satified.*

1.4.4 Associated Fibered Spaces. Extensions and Reductions of the Structure Group

Let X be a scheme and G an algebraic group. Let $\pi\colon P \to X$ be a principal G-bundle and F a quasi-projective scheme (left) acted on by G. Consider the product $P \times F$ and let G act on $P \times F$ by the rule

$$(p, f) \cdot g := (p \cdot g, g^{-1} \cdot f).$$

Then, the result is

Proposition 1.4.20. ([55, Proposition 4]) *There is a unique separated scheme of finite type, Q, such that $P \times F$ is a principal G-bundle over Q.*

Remark 1.4.21. Note that Q is just the categorical quotient $(P \times F)/G$. This scheme is usually denoted by $P \times^G F$, and it is called the *associated fibered space with tyPical fiber F.*

Example 1.4.22. ([53, Example 2.1.1.8.]) Let X be a scheme and V a k-vector space of finite dimension r. Consider the group $G := \mathrm{GL}(V)$ and let P be a principal G-bundle. The group G acts on V by matrix multiplication and, therefore, we can consider the associated fibered space $Q := (P \times V)/\mathrm{GL}(V)$. One can easily see that Q is, in fact, a vector bundle with tyPical fiber V over X. The map $P \rightsquigarrow Q$ is functorial and it establishes an equivalence between isomorphism classes of principal $\mathrm{GL}(V)$-bundles and isomorphism classes of vector bundles with typical fiber V. The converse of this map is, precisely, given by the construction of the frame bundle associated to the vector bundle Q, that is, $\mathrm{Isom}(V \otimes \mathcal{O}_X, Q)$.

There are another two important applications of this proposition: extensions and reductions of the structure group.

Extensions of the Structure Group

Let $\alpha \colon G \to G'$ be a morphism of groups. Then G acts on G' via α in an obvious way, $g \cdot g' := \alpha(g)g'$. Therefore, applying the above proposition to any principal G-bundle, we get the associated principal G-bundle $P \times G' \to P \times^G G'$. Moreover

Proposition 1.4.23. ([55, Proposition 5]) *The associated fiber space $P \times^G G'$ is a principal G'-bundle.*

Thus, any morphism of groups $\alpha \colon G \to G'$ defines a functor

$$\alpha_* \colon \mathrm{Bun}_X(G) \to \mathrm{Bun}_X(G'),\ \alpha_*(P) \colon = P \times^G G'.$$

Reductions of the Structure Group

Let G be an algebraic group, $H \subset G$ a subgroup and $P \to X$ a principal G-bundle. Note that the natural action of H on G induces an action of H on P. Consider the scheme $F := G/H$ and let $P \times (G/H) \to P \times^G (G/H)$ the principal G-bundle deduced from Proposition 1.4.20. Then

Proposition 1.4.24. ([55, Proposition 8]) *The induced morphism $P \to P \times^G (G/H)$ is a principal H-bundle. Therefore, $P \times^G (G/H) = P/H$.*

Thereby, we find the following equivalence,

Proposition 1.4.25. ([55, Proposition 9]) *Let G be an algebraic group and $\alpha \colon H \hookrightarrow G$ a subgroup. Giving a principal H-bundle over X is equivalent to giving a principal G-bundle over X and a global section of the associated fibered space $s \colon X \to P \times^G (G/H) = P/H$.*

Proof. Let Q be a principal H-bundle and let $\alpha_*(Q) = Q \times^H G$ be a associated principal G-bundle. Consider now the associated fibered space $\alpha_*(Q) \times^G (G/H)$, which is isomorphic to $Q \times^H (G/H)$. Since the action of H on G/H leaves the neutral element fixed, there is a canonical section $s\colon X \to \alpha_*(Q) \times^G (G/H)$.

Conversely, let $P \to X$ be a principal G-bundle and $s\colon X \to P \times^G (G/H)$ a global section. Then we define the principal H-bundle Q by means of the pullback

$$\begin{tikzcd} s^*(P) =: Q \arrow[r, dashed] \arrow[d, dashed] & P \arrow[d, "H-\text{bundle}"] \\ X \arrow[r, "s"] & P \times^G (G/H) \end{tikzcd}$$

Finaly, we can show that reduction and extension of structure group are operations inverse to each other (up to canonical isomorphism). $\square$

Chapter 2

Singular Principal G-Bundles on Nodal Curves

The goal of this chapter is to construct the moduli space of singular principal G-bundles over a nodal projective curve over the complex numbers. Following [49] and [8], we first construct the moduli space of tensor fields and then we construct our moduli space by associating a tensor field to any singular principal G-bundle (*linearizing the moduli problem*). The construction of the moduli space of tensor fields is done following the same steps as in [17], and adapting the calculations to our situation. The hardest part of the construction is to show that linearizing the moduli problem is an injective map, as in the irreducible case.

Along this chapter, X will be a nodal projective curve over an algebraically closed $k = \mathbb{C}$ and $\mathcal{O}_X(1)$ an ample invertible sheaf whose degree will be denoted by h (see Chapter 1, Section 1.2.2).

2.1 Moduli Space of Tensor Fields

The main result of this Section is Theorem 2.1.44, which claims the existence of a projective moduli scheme for δ-semistable tensor fields on the nodal curve X. We extend the δ-semistability notion given in [17] substituting ranks by multiplicities. The multiplicity of a coherent $\mathcal{O}_X$-module, $\mathcal{F}$, with respect to the polarization $\mathcal{O}_X(1)$ is defined as the leading coefficient of its Hilbert polynomial $P_{\mathcal{F}}(n) := \chi(\mathcal{F}(n))$, $n \in \mathbb{N}$. Since the rank of a locally free sheaf on a smooth projective curve is defined in the same way, this permits us to follow the same argument as in [17] to solve the problem.

The calculations in [17, Lemma 2.6.] are adapted to our case in Lemma 2.1.23, which is crucial in proving Lemma 2.1.26 and, hence, in giving the equivalence between δ-semistability and sectional semistability. This, and an adaptation of the polarization of the parameter space (2.24), allow us to compare δ-semistability and GIT semistability in the parameter space (Subsection 2.1.5)

2.1.1 Tensor Fields and δ-Semistability

Let P be a polynomial with integral coefficients of degree one, and let D be a locally free sheaf on X. We also fix natural numbers $a, b \in \mathbb{N}$.

Definition 2.1.1. A tensor field over X is a pair $(\mathcal{F},\phi)$ where $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module of uniform multirank r, with Hilbert polynomial P and a non-zero morphism of $\mathcal{O}_X$-modules,

$$\phi\colon (\mathcal{F}^{\otimes a})^{\oplus b} \to D.$$

From now on we will say that P is the Hilbert polynomial of the tensor field $(\mathcal{F},\phi)$ and r is its rank.

Definition 2.1.2. Let $(\mathcal{F},\phi)$ and $(\mathcal{G},\gamma)$ be two tensor fields with the same Hilbert polynomial P. A morphism between them is a pair (f,α) where $\alpha \in k$ and $f\colon \mathcal{F} \to \mathcal{G}$ is a morphism such that the following square

$$\begin{tikzcd}
(\mathcal{F}^{\otimes a})^{\oplus b} \arrow[r, "(f^{\otimes a})^{\oplus b}"] \arrow[d, "\phi"] & (\mathcal{G}^{\otimes a})^{\oplus b} \arrow[d, "\gamma"] \\
D \arrow[r, "\alpha \mathrm{Id}"] & D
\end{tikzcd}$$

commutes.

Definition 2.1.3. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module on X. A weighted filtration, $(\mathcal{F}_\bullet, \underline{m})$, of $\mathcal{F}$ is a filtration

$$\mathcal{F}_\bullet \equiv (0) \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_t \subset \mathcal{F}_{t+1} = \mathcal{F},$$

equipped with positive numbers $m_1 \ldots, m_t \in \mathbb{Q}_{>0}$. We adapt the following convention: the one step filtration is always equipped with $m = 1$. A filtration is called saturated if the quotients $\mathcal{F}/\mathcal{F}_i$ are torsion free sheaves.

Definition 2.1.4. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module over X. Two weighted filtrations, $(\mathcal{F}_\bullet, \underline{m})$ and $(\mathcal{F}'_\bullet, \underline{m})$, are isomorphic if there is an isomorphism $f\colon \mathcal{F} \simeq \mathcal{F}$ such that $f(\mathcal{F}_i) = \mathcal{F}'_i$, that is, if there is a commutative diagram

$$\begin{tikzcd}
(0) \arrow[r, hook] & \mathcal{F}_1 \arrow[r, hook] \arrow[d, "f|_{\mathcal{F}_1}", "\wr"'] & \ldots \arrow[r, hook] & \mathcal{F}_t \arrow[r, hook] \arrow[d, "f|_{\mathcal{F}_t}", "\wr"'] & \mathcal{F} \arrow[d, "f", "\wr"'] \\
(0) \arrow[r, hook] & \mathcal{F}'_1 \arrow[r, hook] & \ldots \arrow[r, hook] & \mathcal{F}'_t \arrow[r, hook] & \mathcal{F}'
\end{tikzcd}$$

Definition 2.1.5. Let $\phi\colon (\mathcal{F}^{\otimes a})^{\oplus b} \to D$ be a tensor field on X, and $(\mathcal{F}_\bullet, \underline{m})$, $(\mathcal{F}'_\bullet, \underline{m})$, weighted filtrations. Suppose that there is a flag isomorphism $f\colon (\mathcal{F}_\bullet, \underline{m}) \simeq (\mathcal{F}'_\bullet, \underline{m})$. We say that f is compatible with the tensor structure if $(f,1)$ is a morphism of tensor fields.

Let $\phi\colon (\mathcal{F}^{\otimes a})^{\oplus b} \to D$ be a tensor field on X and let $(\mathcal{F}_\bullet, \underline{m})$ be a weighted filtration. For each $\mathcal{F}_i$ denote by α_i its multiplicity and just α the multiplicity of $\mathcal{F}$. Define the vector

$$\Gamma = \sum_{1}^{t} m_i \Gamma^{(\alpha_i)},$$

where $\Gamma^{(l)} = (\overbrace{l-\alpha, \ldots, l-\alpha}^{l}, \overbrace{l, \ldots, l}^{\alpha - l})$. Let us denote by J the set

$$J = \{ \text{ multi-indices } I = (i_1, \ldots, i_a) | I_j \in \{1, \ldots, t+1\}\}.$$

Define

$$\mu(\mathcal{F}_\bullet, \underline{m}, \phi) = \min_{I\in J}\{\Gamma_{\alpha_{i_1}} + \ldots + \Gamma_{\alpha_{i_a}} | \phi|_{(\mathcal{F}_{i_1}\otimes\ldots\otimes\mathcal{F}_{i_a})^{\oplus b}} \neq 0\}. \tag{2.1}$$

Lemma 2.1.6. *Let* $\phi\colon (\mathcal{F}^{\otimes a})^{\oplus b} \to D$ *be a tensor field on* X *and let* $(\mathcal{F}_\bullet, \underline{m})$ *be a weighted filtration as above. Denote* $\epsilon_i(\mathcal{F}_\bullet)$ *the number*

$$\epsilon_i(\mathcal{F}_\bullet) = \#\{k \in (i_1, \ldots, i_a) | \alpha_k \leqslant \alpha_i\},$$

$(i_1, \ldots, i_a)$ *being one multi-index giving the minimum in* $\mu(\mathcal{F}_\bullet, \underline{m}, \phi)$*. Then, the following holds,*

$$\mu(\mathcal{F}_\bullet, \underline{m}, \phi) = \sum_{i=1}^{t} m_i(\alpha_i a - \epsilon_i(\mathcal{F}_\bullet)\alpha).$$

Proof. We know that

$$\Gamma^{(\alpha_i)} = (\overbrace{\alpha_i - \alpha, \ldots, \alpha_i - \alpha}^{\alpha_i}, \overbrace{\alpha_i, \ldots, \alpha_i}^{\alpha-\alpha_i}). \tag{2.2}$$

Then Γ_k is the k-th component of the vector $\Gamma = \sum_1^t m_i \Gamma^{\alpha_i}$. Therefore

$$\begin{aligned}\Gamma_{\alpha_{i_j}} &= m_1\alpha_1 + \ldots + m_{i_j-1}\alpha_{i_j-1} + \\ &+ m_{i_j}(\alpha_{i_j} - \alpha) + m_{i_j+1}(\alpha_{i_j+1} - \alpha) + \ldots + m_t(\alpha_t - \alpha) = \\ &= \sum_1^t m_i\alpha_i - \alpha\sum_{k=i_j}^{t} m_k,\end{aligned}$$

so

$$\begin{aligned}\Gamma_{\alpha_{i_1}} + \ldots + \Gamma_{\alpha_{i_a}} &= a\sum_1^t m_i\alpha_i - \alpha\sum_{k=i_1}^{t} m_k - \ldots - \alpha\sum_{k=i_a}^{t} m_k = \\ &= a\sum_1^t m_i\alpha_i - \alpha\sum_1^t m_i\nu_i(I) = \\ &= \sum_1^t m_i(a\alpha_i - \alpha\nu_i(I)),\end{aligned}$$

where $\nu_i(I) = \sharp\{k \in I = (i_1, \ldots, i_a) | \alpha_k \leqslant \alpha_i\}$. Note that if I is a multi-index giving the minimum, then $\nu_i(I) = \epsilon_i(\mathcal{F}_\bullet)$, and we are done. □

Lemma 2.1.7. *Let* $(\mathcal{F}_\bullet, \underline{m})$ *be a weighted filtration, with*

$$\mathcal{F}_\bullet \equiv (0) \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_s \subset \mathcal{F}_{s+1} = \mathcal{F}.$$

Consider a partition of the multitindex $(1, 2, \ldots, s)$

$$I := (1, 2, \ldots, s) = I_1 \sqcup I_2,$$

let us say $I_1 = (i_1, \ldots, i_t)$ *and* $I_2 = (k_1, \ldots, k_{s-t})$*. Then*

$$1)\ (\sum_{i=1}^{s} m_i)a(\alpha - 1) \geq \mu(\mathcal{F}_\bullet, \underline{m}, \phi) \geq -(\sum_{i=1}^{s} m_i)a(\alpha - 1),$$

$$2)\ \mu(\mathcal{F}_\bullet, \underline{m}, \phi) \leq \mu(\mathcal{F}^1_\bullet, \underline{m_1}, \phi) + (\sum_{i=1}^{s-t} m_{2,i})a(\alpha - 1),$$

being $\mathcal{F}^1_j = \mathcal{F}_{i_j}$ *and* $\mathcal{F}^2_j = \mathcal{F}_{k_j}$*.*

Proof. For the sake of clarity, we first introduce some notation that will be used later. We denote $I_1' = (1, \dots, t)$ and $I_2' = (1, \dots, s-t)$, and by ϕ_1 (resp. ϕ_2) the bijection between I_1 and I_1' (resp. I_2 and I_2') given by $\phi_1(i_j) = j$ (resp. $\phi_2(k_j) = j$).
1) Being $\nu_i(J) \leq a$, we have

$$\begin{aligned}\mu(\mathcal{F}_\bullet, \underline{m}, \phi) = &\sum_{i=1}^{s} m_i(\nu_i(J)\alpha - a\alpha_i) \leq \\ \leq &\sum_{i=1}^{s} am_i(\alpha - \alpha_i) \leq \\ \leq &\sum_{i=1}^{s} am_i(\alpha - 1) = \\ = &(\sum_{i=1}^{s} m_i)a(\alpha - 1).\end{aligned}$$

On the other hand, being $\nu_i(J) \geq 0$ and $\alpha_i + 1 \leq \alpha$ we deduce

$$\begin{aligned}\mu(\mathcal{F}_\bullet, \underline{m}, \phi) = &\sum_{i=1}^{s} m_i(\nu_i(J)\alpha - a\alpha_i) \geq \\ \geq &\sum_{i=1}^{s} m_i(a(1-\alpha)) = -a(\sum_{i=1}^{s} m_i)(\alpha - 1).\end{aligned}$$

2) Let J be a multiindex giving the minimum in $\mu(\mathcal{F}^1_\bullet, \underline{m}_1, \phi)$. Then we have

$$\begin{aligned}\mu(\mathcal{F}_\bullet, \underline{m}, \phi) \leq &\sum_{i=1}^{s} m_i(a\alpha_i - \nu_i(J)\alpha) = \\ = &\sum_{j=1}^{t} m_{1,j}(a\alpha_{i_j} - \nu_{i_j}(J)\alpha) + \sum_{j=1}^{s-t} m_{2,j}(a\alpha_{k_j} - \nu_{k_j}(J)\alpha) \leq \\ \leq &\mu(\mathcal{F}^1_\bullet, \underline{m}_1, \phi) + (\sum_{i=1}^{s-t} m_{2,i})a(\alpha - 1)\end{aligned}$$

□

Lemma 2.1.8. *Let $\phi\colon (\mathcal{F}^{\otimes a})^{\oplus b} \to D$ be a tensor field on X, and let $(\mathcal{F}_\bullet, \underline{m})$ and $(\mathcal{F}'_\bullet, \underline{m})$ be two weighted filtrations for whom there is an isomoprhism compatible with the tensor structure over a dense open subscheme $U \subset X$. Then*

$$\mu(\mathcal{F}_\bullet, \underline{m}, \tau) = \mu(\mathcal{F}'_\bullet, \underline{m}, \tau).$$

Proof. Follows trivially by Proposition 1.2.21, Remark 1.2.22, Definition 2.1.5 and de construction of $\mu(-,-,-)$. □

Definition 2.1.9. Let δ be a positive rational number. A tensor field $(\mathcal{F}, \phi)$ is δ-(semi)stable if for each weighted filtration $(\mathcal{F}_\bullet, \underline{m})$ the following holds

$$\sum_{1}^{t} m_i(\alpha P_{\mathcal{F}_i} - \alpha_i P) + \delta\mu(\mathcal{F}_\bullet, \underline{m}, \phi)(\leq)0. \tag{2.3}$$

The following lemma will be quite important in the analysis of the δ-semistability condition and, therefore, in the construction of the moduli space.

Lemma 2.1.10. *There is a positive integer A, depending only on the numerical input data (P, a, b and D), such that it is enough to check the δ-semistability condition (2.3) for weighted filtrations with $m_i < A$.*

Proof. Note that a tensor field is δ-(semi)stable if and only if Equation (2.3) holds for every integral weighted filtration, i.e., filtrations with integral weights. Now, the result follows from [17, Lemma 1.4] changing ranks by multiplicities. $\square$

2.1.2 Boundedness

In order to construct our moduli space we have to be sure that the set of torsion free sheaves appearing in our problem is small enough. We develop the basic definitions and results about bounded families, and we prove, following closely [17], the boundedness of the family of torsion free sheaves appearing in δ-(semi)stable tensor fields.

Let X be a k-scheme of finite type. For any field extension $k \hookrightarrow K$, we denote by $X_K := X \times_k K$ the base change. Let $k \hookrightarrow K_i,\ i = 1, 2$ be two field extensions of k, and let $\mathcal{F}_1$, $\mathcal{F}_2$ be coherent sheaves over X_{K_1} and X_{K_2} respectively. We say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent, $\mathcal{F}_1 \sim \mathcal{F}_2$, if there exists a common field extension

$$\begin{tikzcd} & K_1 \arrow[dr, hook] & \\ k \arrow[ur, hook] \arrow[dr, hook] & & K \\ & K_2 \arrow[ur, hook] & \end{tikzcd}$$

such that $\mathcal{F}_1 \otimes_{K_1} K \simeq \mathcal{F}_2 \otimes_{K_2} K$ over X_K.

Definition 2.1.11. Let E be a set of equivalence classes of coherent sheaves on X. We say that E is bounded if there exists a k-scheme of finite type and a coherent sheaf $\mathcal{F}$ over $X \times S \to S$ such that for any member $\mathcal{E}$ of E on X_K, there is a point $s \in S$ such that $\mathcal{E} \sim \mathcal{F}_s$, being $\mathcal{F}_s = \mathcal{F} \otimes_S k(s)$.

This can be generalized to the relative case, in which X is a S-scheme for some k-scheme S, in the obvious way. In case X is projective, there is a very important characterization,

Theorem 2.1.12. ([20, n^o 221, Theorem 2.1]) *Let X be a noetherian projective scheme over k and $\mathcal{O}_X(1)$ a very ample invertible sheaf. Let E be a set of equivalence classes of coherent sheaves over X. E is bounded if and only if the following holds:*

i) there exist natural numbers $n, N \in \mathbb{N}$ such that E is contained in the set of equivalence classes of coherent sheaves which are quotients of $\mathcal{O}_X(-n)^{\oplus N}$.

ii) the set of Hilbert polynomials $P_{\mathcal{F}}$ of sheaves $\mathcal{F} \in E$ is finite.

Remark 2.1.13. Note that Theorem 2.1.12 implies that the union of a finite number of bounded sets of equivalence classes of coherent sheaves is also bounded.

We can also characterize boundedness of the set E looking at the *regularity* of the sheaves living inside it.

Definition 2.1.14. Let m be an integer and $\mathcal{F}$ a coherent sheaf on X. We say that $\mathcal{F}$ is m-regular if

$$H^i(X, \mathcal{F}(m-i)) = 0, \text{ for all } i > 0.$$

Clearly, if $\mathcal{F}$ is m-regular and $m' > m$ then $\mathcal{F}$ is also m'-regular. From Serre's vanishing theorem, it follows that there is always an integer m such that $\mathcal{F}$ is m-regular. Therefore, we can define,

Definition 2.1.15. Let $\mathcal{F}$ a coherent sheaf on X. The regularity of $\mathcal{F}$ is defined by

$$\operatorname{reg}(\mathcal{F}) = \inf\{m \in \mathbb{Z} : \mathcal{F} \text{ is } m\text{-regular}\}.$$

Therefore, we have

Theorem 2.1.16. ([33, Lemma 1.7.6]) *Let X be a noetherian projective scheme over k and $\mathcal{O}_X(1)$ a very ample invertible sheaf. Let E be a set of equivalence classes of coherent sheaves over X. E is bounded if and only if the following holds:*
i) there is a uniform bound $\operatorname{reg}(\mathcal{F}) \leq \rho$ for all $\mathcal{F} \in E$.
ii) the set of Hilbert polynomials $P_{\mathcal{F}}$ of sheaves $\mathcal{F} \in E$ is finite.

Corollary 2.1.17. ([20, n^o 221, Lemma 2.5]) *Suppose that X has dimension l. Let $\mathcal{G}$ be a coherent sheaf on X and E a set of equivalence classes of sheaves $\mathcal{F}$ which are quotients of $\mathcal{G}$. The Hilbert polynomial of the sheaves in E are of the form*

$$P_{\mathcal{F}}(n) = an^l/l! + bn^{l-1}/(l-1)! + \ldots$$

Then, a is bounded from above and b is bounded from below. If b is bounded from above, then the family of quotients $\mathcal{F}$ is bounded.

Let X be a reduced projective curve of genus g and $\mathcal{O}_X(1)$ a very ample invertible sheaf. Let us denote by h its degree and let H be an ample divisor determining its class. Given a coherent sheaf, $\mathcal{F}$, we have defined its *slope* as

$$\mu'(\mathcal{F}) = \frac{\deg(\mathcal{F})}{\alpha(\mathcal{F})}.$$

Nevertheless, we will also use, in this chapter, the quantity

$$\mu(\mathcal{F}) := \frac{r(1-g) + \deg(\mathcal{F})}{\alpha},$$

which is also defined as the slope of $\mathcal{F}$ by C. Simpson (see [58]). As always, α its the multiplicity of $\mathcal{F}$, $r = \alpha/h$ its rank and $\deg(\mathcal{F})$ its degree (see Chapter 1, Section 1.2.2). Recall that a torsion free sheaf, $\mathcal{F}$, is semistable if for any subsheaf $\mathcal{F}' \subset \mathcal{F}$ (see Chapter 1, Section 1.2.2),

$$\mu'(\mathcal{F}') \leq \mu'(\mathcal{F}).$$

Note that this is the same as saying that $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$. Recall also that for any torsion free sheaf, $\mathcal{F}$, there is a unique filtration (Harder-Narasimhan filtration)

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_k = \mathcal{F}$$

such that the quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ are semistable torsion free sheaves with decreasing slopes (Chapter 1, Section 1.2.2). Note that this is true independently on the definition of the slope, μ or μ', that we use.

Lemma 2.1.18. ([58, Corollary 1.7],[32, Lemma 2.2]) *Let $\alpha \in \mathbb{N}$. There exists a positive integer B, depending only on α, such that if X is a reduced projective curve and $\mathcal{F}$ is a semistable torsion free sheaf of multiplicity less or equal than α, then*

$$\frac{h^0(X, \mathcal{F}(m))}{\alpha} \leq [\mu(\mathcal{F}) + m + B]_+, \text{ with } n \in \mathbb{N}.$$

Lemma 2.1.19. *Let X be a reduced projective curve over k. Let $\alpha > 0$ be an integer. Then there exists a positive integer B such that for every torsion free sheaf $\mathcal{F}$, with multiplicity $0 < \alpha' < \alpha$, we have,*

$$h^0(X, \mathcal{F}(m)) \leq ((\alpha' - 1)[\mu_{max}(\mathcal{F}) + m + B]_+ + [\mu_{min}(\mathcal{F}) + m + B]_+).$$

Proof. We follow [32]. Consider the Harder-Narassimhan filtration of $\mathcal{F}$

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_k = \mathcal{F},$$

where $\mathcal{F}_i/\mathcal{F}_{i-1}$ are semistable torsion free, and we denote their multiplicities by $\beta_i = \alpha(\mathcal{F}_i/\mathcal{F}_{i-1})$. For every $i = 1, \ldots, k$, we have exact sequences

$$0 \to \mathcal{F}_{i-1} \hookrightarrow \mathcal{F}_i \to \mathcal{F}_i/\mathcal{F}_{i-1} \to 0.$$

Taking global sections and applying Lemma 2.1.18 we get

$$h^0(X, \mathcal{F}(m)) \leq \sum_{i=1}^{k} \beta_i[\mu_i + m + B]_+.$$

Moreover,

$$\begin{aligned}
&\sum_{i=1}^{k} \beta_i[\mu_i + m + B]_+ \leq \beta_{i_0}[\mu_{\min}(\mathcal{F}) + m + B] + \sum_{i \neq i_0}^{k} \beta_i[\mu_{\max}(\mathcal{F}) + m + B]_+ \leq \\
\leq &(\alpha_{i_0} - \alpha_{i_0 - 1})[\mu_{\min}(\mathcal{F}) + m + B]_+ + (\alpha_{i_0-1} - \alpha_{i_0} + \alpha')[\mu_{max}(\mathcal{F}) + m + B]_+ \\
\leq &[\mu_{\min}(\mathcal{F}) + m + B]_+ + (\alpha' - 1)[\mu_{\max}(\mathcal{F}) + m + B]_+.
\end{aligned}$$

□

Lemma 2.1.20. *Let C'' be a constant, and let E be a bounded set of equivalence classes of coherent sheaves on X. The set of torsion free quotients $\mathcal{F} \twoheadrightarrow \mathcal{F}''$ of sheaves $\mathcal{F}$ in E with $\deg(\mathcal{F}'') \leq C''$, is bounded.*

Proof. Follows from Theorem 2.1.12 and Corollary 2.1.17. □

Theorem 2.1.21. *Let $P(n) \in \mathbb{Z}[n]$ be a polynomial of degree one and $C \in \mathbb{R}$ a constant. The family of sheaves $\mathcal{F}$ with Hilbert polynomial P and such that $\mu_{max}(\mathcal{F}) \leq C$ is bounded.*

Proof. This is a particular case of [58, Theorem 1.1]. □

Corollary 2.1.22. *Let X be a reduced projective curve over k, let $\delta \in \mathbb{Q}_{>0}$ and $\alpha \in \mathbb{N}$. The set of torsion free sheaves of multiplicity α and degree d occurring in δ-semistable tensor fields $(\mathcal{F}, \phi)$ is bounded.*

Proof. We follow [17]. Consider the one-step flag $0 \subset \mathcal{F}_1 \subseteq \mathcal{F}_2 = \mathcal{F}$ with $\alpha(\mathcal{F}_1) = k$. The associated one parameter subgroup is determined by

$$\Gamma^{(k)} = (\overbrace{k-\alpha, \ldots, k-\alpha}^{k}, \overbrace{k, \ldots, k}^{\alpha-k}).$$

For each multiindex $I = (i_1, \ldots, i_a)$ denote

$$g_I = \Gamma_{\alpha_{i_1}} + \ldots + \Gamma_{\alpha_{i_a}},$$

and s_1 the number of indices $i_j = 1$ and s_2 the number of indices $i_j = 2$. Then we have

$$g_I = s_1(k-\alpha) + s_2 k.$$

Since $s_1 \geq 0$ and $s_2 \leq a$ we find, for every multiindex I

$$a(k-\alpha) \leq g_I \leq ak.$$

Hence, applying Lemma 2.1.7, we get,

$$a(k-\alpha) \leq \mu(\mathcal{F}_\bullet, \phi, 1) \leq ak.$$

Now, the semistability condition means that $\alpha P_{\mathcal{F}_1} - kP \leq \delta a(\alpha - k)$. If we denote by C the constant

$$\mu(\mathcal{F}) + \frac{a(\alpha-1)}{\alpha}\delta$$

(note that $\mu(\mathcal{F})$ is constant since the degree and the multiplicity are fixed) then we conclude that $\mu(\mathcal{F}_1) \leq C$. Thus, by the Theorem 2.1.21 the family is bounded. □

2.1.3 Characterizing δ-Semistability

We want to prove Theorem 2.1.26. The proof will be done in several steps, following closely [17]. Just minor changes must be done to adapt their proofs to the case of reduced projective curves.

Let us fix some notation. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, and suppose we have a filtration, $\mathcal{F}_\bullet$, of $\mathcal{F}$. We will denote by α^i the multiplicity of $\mathcal{F}/\mathcal{F}_i$ and by α_i the multiplicity of $\mathcal{F}_i$ (thus, $\alpha(\mathcal{F}) = \alpha_i + \alpha^i$). Let now $P(x) \in \mathbb{Z}[x]$ be a polynomial, α, d rational numbers such that $P(x) = \alpha x + \frac{\alpha}{h}(1-g) + d$, and m a natural number. Then, we define:

1) S^s is the set of δ-semistable tensor fields $(\mathcal{F}, \phi)$ with a torsion free sheaf with Hilbert polynomial P.

2) S'_m is the set of tensor fields $(\mathcal{F}, \phi)$ with $\mathcal{F}$ a torsion free sheaf with Hilbert polynomial P, and such that

$$\sum_{i=1}^{t} m_i(\alpha h^0(X, \mathcal{F}_i(m)) - \alpha_i P(m))) + \delta\mu(\mathcal{F}_\bullet, \underline{m}, \phi) \leq 0$$

for every weighted filtration $(\mathcal{F}_\bullet, \underline{m})$.

3) S''_m is the set of tensor fields $(\mathcal{F}, \phi)$ with $\mathcal{F}$ a torsion free sheaf with Hilbert polynomial P, and such that

$$\sum_{i=1}^{t} m_i(\alpha^i P(m) - \alpha h^0(X, \mathcal{F}^i(m))) + \delta\mu(\mathcal{F}_\bullet, \underline{m}, \phi) \leq 0,$$

for every weighted filtration $(\mathcal{F}_\bullet, \underline{m})$.

4) $S_N = (\cup_{m \geq N} S''_m) \cup S^s$, $N \in \mathbb{N}$.

Lemma 2.1.23. *There exist integers, N_1 and C, such that, if $(\mathcal{F}, \phi) \in S_{N_1}$ then, for all saturated weighted filtrations, the following holds $\forall$ i:*

$$\deg(\mathcal{F}_i) - \alpha_i \mu_s \leq C, \text{ where } \mu_s = \frac{d - a\delta}{\alpha},$$

and either 1) $-C \leq \deg(\mathcal{F}_i) - \alpha_i \mu_s$, *or*
2.a) $h^0(X, \mathcal{F}_i(m)) < \alpha_i(P(m) - a\delta)$, *if* $(\mathcal{F}, \phi) \in S^s$ *and* $m \geq N_1$
2.b) $\alpha^i(P - a\delta) < \alpha(P_{\mathcal{F}^i} - a\delta)$ *if* $(\mathcal{F}, \phi) \in \cup_{m \geq N_1} S''_m$.

Proof. Let B be as in Lemma 2.1.18 and $B' = B + \dfrac{(1-g)}{h}$ being $h = \deg(\mathcal{O}_X(1))$. Choose $C \gg a\delta$ and such that the leading coefficient of the polynomial $G - (P - a\delta)/\alpha$ is negative, where

$$G(m) = (1 - \frac{1}{\alpha})(\mu_s + s\delta + m + B') + \frac{1}{\alpha}(\mu_s - \frac{1}{\alpha}C + m + B')$$

(Note that $G - (P - a\delta)/\alpha$ is, in fact, a constant polynomial, so the above condition means that $G(m) - (P(m) - a\delta)/\alpha < 0$ for each m). Choose $N_1 \gg 0$ such that

$$\mu_s - \frac{C}{\alpha} + m + B' > 0, \quad \forall m \geq N_1.$$

Then the proof will be done considering the two possible cases.

Case 1: $(\mathcal{F}, \phi) \in S^s$. Let $(\mathcal{F}_\bullet, \underline{m})$ be a saturated weighted filtration. For each i consider now the one-step filtration $\mathcal{F}_i \subsetneq \mathcal{F}$. Since $(\mathcal{F}, \phi)$ is δ-semistable, we have

$$(\alpha P_{\mathcal{F}_i} - \alpha_i P) + \delta\mu(\mathcal{F}_\bullet, \underline{m}, \phi) \leq 0$$

or, equivalently

$$\alpha \deg(\mathcal{F}_i) - \alpha_i d + \delta\mu(\mathcal{F}_\bullet, \underline{m}, \phi) \leq 0.$$

Then

$$\deg(\mathcal{F}_i) - \frac{\alpha_i d}{\alpha} + \frac{\delta\mu(\mathcal{F}_\bullet, \underline{m}, \phi)}{\alpha} \leq 0.$$

If we sum and subtract $\alpha_i a\delta/\alpha$ in the last inequality, we find

$$\deg(\mathcal{F}_i) - \alpha_i \mu_s - \frac{\alpha_i a\delta}{\alpha} + \frac{\mu(\mathcal{F}_\bullet, \underline{m}, \phi)\delta}{\alpha} \leq 0$$

or, equivalently

$$\deg(\mathcal{F}_i) - \alpha_i \mu_s \leq \frac{(\alpha_i a - \mu(\mathcal{F}_\bullet, \underline{m}, \phi))\delta}{\alpha}.$$

Since $\mu(\mathcal{F}_\bullet, \underline{m}, \phi) = \alpha_i a - \epsilon(\mathcal{F}_i \subsetneq \mathcal{F})\alpha$ and $\epsilon(\mathcal{F}_i \subsetneq \mathcal{F}) \leq a$ we get

$$\frac{(\alpha_i a - \mu(\mathcal{F}_\bullet, \underline{m}, \phi))\delta}{\alpha} \leq a\delta,$$

so

$$\deg(\mathcal{F}_i) - \alpha_i \mu_s \leq a\delta, \tag{2.4}$$

but we know that $a\delta < C$, so

$$\deg(\mathcal{F}_i) - \alpha_i \mu_s < C.$$

Now, assume that 2a) do not holds, that is $-C > \deg(\mathcal{F}_i) - \alpha_i \mu_s$. Let $\mathcal{F}_{i,\max} \subsetneq \mathcal{F}_i$ be the term in the Harder-Narasimham filtration of $\mathcal{F}_i$ with maximal slope. Because of Equation (2.4), we have

$$\mu'_{\max}(\mathcal{F}_i) = \mu'(\mathcal{F}_{i,\max}) = \frac{\deg(\mathcal{F}_{i,\max})}{\alpha(\mathcal{F}_{i,\max})} \leq \mu_s + \frac{a\delta}{\alpha(\mathcal{F}_{i,\max})} \leq \mu_s + a\delta. \tag{2.5}$$

Since $-C > \deg(\mathcal{F}_i) - \alpha_i \mu_s$, we find

$$\mu'_{\min}(\mathcal{F}_i) \leq \mu'(\mathcal{F}_i) = \frac{\deg(\mathcal{F}_i)}{\alpha_i} < \mu_s - \frac{C}{\alpha}. \tag{2.6}$$

By Lemma 2.1.19 we know that:

$$\begin{aligned} h^0(X, \mathcal{F}_i(m)) &\leq ((\alpha_i - 1)[\mu_{\max}(\mathcal{F}_i) + m + B]_+ + [\mu_{\min}(\mathcal{F}_i) + m + B]) = \\ &= ((\alpha_i - 1)[\mu'_{\max}(\mathcal{F}_i) + m + B']_+ + [\mu'_{\min}(\mathcal{F}_i) + m + B']). \end{aligned} \tag{2.7}$$

Using equations (2.5), (2.6) and (2.5), we find

$$h^0(X, \mathcal{F}_i(m)) < ((\alpha_i - 1)[\mu_s + a\delta + m + B']_+ + [\mu_s - \frac{C}{\alpha} + m + B'])$$

and therefore

$$\frac{h^0(X, \mathcal{F}_i(m))}{\alpha_i} < (1 - \frac{1}{\alpha_i})[\mu_s + a\delta + m + B']_+ + \frac{1}{\alpha_i}[\mu_s - \frac{C}{\alpha} + m + B']_+.$$

If we denote by $G_i(m)$ the right hand side then

$$G_i(m) - G(m) = \frac{\alpha - \alpha_i}{\alpha\alpha_i}(-\frac{C}{\alpha} - a\delta) < 0,$$

so

$$\frac{h^0(X, \mathcal{F}_i(m))}{\alpha_i} < G_i(m) < G(m) < \frac{P(m) - a\delta}{\alpha}$$

hence, we have the result.

Case 2: $(\mathcal{F}, \phi) \in S''_m$ with $m \geq N_1$. Let $(\mathcal{F}_\bullet, \underline{m})$ be a saturated weighted filtration, and for each i consider the quotient $\mathcal{F}^i = \mathcal{F}/\mathcal{F}_i$. Let $\mathcal{F}^i_{\min}$ the last factor of the Harder-Narasimham filtration of $\mathcal{F}^i$ (so $\mu(\mathcal{F}^i_{\min}) = \mu_{\min}(\mathcal{F}^i)$). Denote by $\mathcal{F}'$ the kernel, so we have

$$0 \to \mathcal{F}' \hookrightarrow \mathcal{F} \to \mathcal{F}^i_{\min} \to 0,$$

and let $\mathcal{F}' \subsetneq \mathcal{F}$ be the one step filtration. Since

$$\mu_s - \frac{C}{\alpha} + m + B' > 0$$

we deduce that, for the fixed m, $G(m) > 0$. Since $(\mathcal{F}, \phi) \in S''_m$, we know also that $\alpha^i_{\min} P(m) - \alpha h^0(X, \mathcal{F}^i_{\min}) + \mu(\mathcal{F}' \subsetneq \mathcal{F}, 1, \phi)\delta \leq 0$. Hence we have,

$$\begin{aligned} G(m) &< \frac{1}{\alpha}(P(m) - a\delta) \leq \\ &\leq \frac{1}{\alpha}(\frac{\alpha}{\alpha^i_{\min}} h^0(X, \mathcal{F}^i_{\min}(m)) - \frac{\mu(\mathcal{F}' \subsetneq \mathcal{F}, 1, \phi)}{\alpha^i_{\min}}\delta - a\delta) = \\ &= \frac{h^0(X, \mathcal{F}^i_{\min}(m))}{\alpha^i_{\min}} - \delta\frac{\mu(\mathcal{F}' \subsetneq \mathcal{F}, 1, \phi) + a\alpha^i_{\min}}{\alpha\alpha^i_{\min}}. \end{aligned}$$

Now, we know that

$$\begin{aligned} \gamma^{(\alpha')} &= (\alpha' - \alpha, \dots, \alpha' - \alpha, \alpha', \dots, \alpha') = \\ &= (-\alpha^i_{\min}, \dots, -\alpha^i_{\min}, \alpha - \alpha^i_{\min}, \dots, \alpha - \alpha^i_{\min}), \end{aligned}$$

so there is an integer $l \leq a$, such that

$$\mu(\mathcal{F}' \subsetneq \mathcal{F}, 1, \phi) = l(-\alpha^i_{\min}) + (a - l)(\alpha - \alpha^i_{\min}).$$

Finally we get that

$$\mu(\mathcal{F}' \subsetneq \mathcal{F}, 1, \phi) + a\alpha^i_{\min} = (a - l)\alpha \geq 0.$$

Since $\delta > 0$ we find,

$$G(m) < \frac{h^0(X, \mathcal{F}^i_{\min}(m))}{\alpha^i_{\min}},$$

and because of Lemma 2.1.18, we have

$$G(m) < \frac{h^0(X, \mathcal{F}^i_{\min}(m))}{\alpha^i_{\min}} \leq \mu_{\min}(\mathcal{F}^i) + m + B = \mu'_{\min}(\mathcal{F}^i) + m + B'.$$

Show that the above inequality implies the inequality for the constant coefficients

$$\mu'_{\min}(\mathcal{F}^i) \geq \mu_s + (1 - \frac{1}{\alpha})a\delta - \frac{C}{\alpha^2}.$$

From the above equation and the fact that $\mu'_{\min}(\mathcal{F}^i) \leq \mu'(\mathcal{F}^i)$, we get

$$\frac{\deg(\mathcal{F}^i)}{\alpha - \alpha_i} = \frac{\deg(\mathcal{F}^i)}{\alpha^i} = \mu'(\mathcal{F}^i) \geq \mu_s + (1 - \frac{1}{\alpha})a\delta - \frac{C}{\alpha^2}.$$

Therefore,

$$d - \deg(\mathcal{F}_i) = \deg(\mathcal{F}^i) \geq \alpha\mu_s - \alpha_i\mu_s + ((1 - \frac{1}{\alpha})a\delta - \frac{C}{\alpha^2})(\alpha - \alpha_i)$$

Reordering and using the fact that $\alpha \geq \alpha_i > 0$ (and assuming $C + \alpha(1-\alpha) > 0$) we find,

$$\begin{aligned}\deg(\mathcal{F}_i) - \alpha_i\mu_s \leq d - \alpha\mu_s + ((\frac{1}{\alpha} - 1)a\delta + \frac{C}{\alpha^2})(\alpha - \alpha_i) = \\ = a\delta + ((\frac{1}{\alpha} - 1) + \frac{C}{\alpha^2})(\alpha - \alpha_i) \leq \\ \leq a\delta + ((\frac{1}{\alpha} - 1) + \frac{C}{\alpha^2})\alpha = \\ = a\delta(2-\alpha) + \frac{C}{\alpha} \leq C.\end{aligned}$$

Assume now that the first alternative does not hold, i.e. $-C > \deg(\mathcal{F}_i) - \alpha_i\mu_s$. Then,

$$\begin{aligned}\alpha^i\mu_s = (\alpha - \alpha_i)\mu_s = \alpha\mu_s - \alpha_i\mu_s < \\ < \alpha\mu_s - \deg(\mathcal{F}_i) - C = \\ = d - a\delta - \deg(\mathcal{F}_i) - C = \\ = d - a\delta - d + \deg(\mathcal{F}^i) - C = \\ = \deg(\mathcal{F}^i) - a\delta - C < \deg(\mathcal{F}^i) - a\delta.\end{aligned}$$

So, we finally deduce

$$\alpha^i(P - a\delta) < \alpha(P_{\mathcal{F}^i} - a\delta).$$

□

Lemma 2.1.24. *The set S_N is bounded.*

Proof. Let $(\mathcal{F}, \phi) \in S_N$. Let $\mathcal{F}'$ be a subsheaf of $\mathcal{F}$ and $\mathcal{F}''$ the saturated subsheaf of $\mathcal{F}$ generated by $\mathcal{F}'$. We always have

$$\mu(\mathcal{F}') \leq \mu(\mathcal{F}'').$$

Therefore, by Lemma 2.1.23

$$\mu(\mathcal{F}') \leq \mu(\mathcal{F}'') \leq \mu_s + \frac{C}{\alpha(\mathcal{F}'')} + \frac{1-g}{h} \leq \mu_s + C + \frac{1-g}{h}.$$

Then, by Theorem 2.1.21, the set S_N is bounded. □

Lemma 2.1.25. *Let S_0 be the set of saturated subsheaves, $\mathcal{F}' \subset \mathcal{F}$, of coherent sheaves, $\mathcal{F}$, appearing in tensor fields $(\mathcal{F}, \phi) \in S_N$, and satisfying*

$$|\deg(\mathcal{F}') - \alpha'\mu_s| \leq C.$$

Then S_0 is bounded.

Proof. Consider $\mathcal{F}' \in S_0$. The sheaf $\mathcal{F}'' = \mathcal{F}/\mathcal{F}'$ is torsion free. Since S_N is bounded, there are just finitely many Hilbert polynomials in S_N so there is a maximun $l = \max_{\mathcal{F} \in S_N} |\deg(\mathcal{F})|$. Then

$$\begin{aligned}|\deg(\mathcal{F}'')| =& |\deg(\mathcal{F}) - \deg(\mathcal{F}')| \leq |\deg(\mathcal{F})| + |\deg(\mathcal{F}')| \leq \\ =& \max_{\mathcal{F} \in S_N} |\deg(\mathcal{F})| + C + \alpha|\mu_s| = \\ =& l + C + \alpha|\mu_s|.\end{aligned}$$

That is, $\deg(\mathcal{F}'')$ is bounded. Then by Lemma 2.1.20 the set of quotients $\mathcal{F}''$ as above is bounded and therefore S_0 is bounded too. □

Then we have

Theorem 2.1.26. *There is an integer N_0 such that if $m \geq N_0$, the following properties of tensors $(\mathcal{F}, \phi)$, with $\mathcal{F}$ torsion free and $P_{\mathcal{F}} = P$, are equivalent:*

1) $(\mathcal{F}, \phi)$ is δ-(semi)stable.

2) $\forall$ $(\mathcal{F}_\bullet, \underline{m})$ we have $\sum_1^t m_i(\alpha h^0(\mathcal{F}_i(m)) - \alpha_i P(m)) + \delta\mu(\mathcal{F}_\bullet, \underline{m}, \phi)(\leq)0$.

3) $\forall$ $(\mathcal{F}_\bullet, \underline{m})$ we have $\sum_1^t m_i(\alpha^i P(m) - \alpha h^0(\mathcal{F}^i(m))) + \delta\mu(\mathcal{F}_\bullet, \underline{m}, \phi)(\leq)0$.
Furthermore, for any tensor field satisfying these conditions, we have $h^1(X, \mathcal{F}(m)) = 0$.

Proof. By Theorem 2.1.16, a family E is bounded if and only if $\{P_{\mathcal{F}_t}\}_{t\in\Sigma}$ is finite and $\mathrm{reg}(\mathcal{F}_t) \leq \rho$ for all t. Since S_N and S_0 are bounded (for all N), we can fix $N_0 > N_1$ such that sheaves $\mathcal{F}$ in S and S_0 are N_0-regular and $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_a$ is N_0-regular for all $\mathcal{F}_1, \ldots, \mathcal{F}_a$ is S_0 (see Remark 2.1.13).

2) $\Rightarrow$ 3) Fix $m > N_0$. Let $(\mathcal{F}, \phi) \in S'_m$ and consider a weighted filtration $(\mathcal{F}_\bullet, \underline{m})$. Then

$$\begin{aligned}
&\sum_{i=1}^t m_i(\alpha^i P(m) - \alpha h^0(X, \mathcal{F}^i(m))) + \delta\mu(\mathcal{F}_\bullet, \underline{m}, \phi) \leq \\
\leq &\sum_{i=1}^t m_i(\alpha h^0(X, \mathcal{F}_i(m)) - \alpha_i P(m))) + \delta\mu(\mathcal{F}_\bullet, \underline{m}, \phi)(\leq)0.
\end{aligned}$$

The above inequality is because

$$\begin{aligned}
&\sum_{i=1}^t m_i(\alpha^i P(m) - \alpha h^0(X, \mathcal{F}^i(m))) - \sum_{i=1}^t m_i(\alpha h^0(X, \mathcal{F}_i(m)) - \alpha_i P(m))) = \\
= &\sum_{i=1}^t m_i\{(\alpha^i + \alpha_i)P(m) - \alpha(h^0(X, \mathcal{F}^i(m)) + h^0(X, \mathcal{F}_i(m)))\} \leq \\
\leq &\sum_{i=1}^t m_i\{\alpha P(m) - \alpha h^0(X, \mathcal{F}(m))\} \leq 0.
\end{aligned}$$

1) $\Rightarrow$ 2) Let $(\mathcal{F}, \phi) \in S^s$ and consider a saturated weighted filtration $(\mathcal{F}_\bullet, \underline{m})$. Since $\mathcal{F}$ is N_0-regular, $P(m) = h^0(X, \mathcal{F}(m))$. If $\mathcal{F}_i \in S_0$, then $P_{\mathcal{F}_i}(m) = h^0(X, \mathcal{F}_i(m))$. If $\mathcal{F}_i$ do not belongs to S_0, then the second alternative of Lemma 2.1.23 holds, so

$$\alpha h^0(\mathcal{F}_i(m)) < \alpha_i(P(m) - a\delta). \tag{2.8}$$

Let $T' \subset T = \{1, \ldots, t\}$ be the subset of those i for which $\mathcal{F}_i \in S_0$. Let $(\mathcal{F}'_\bullet, \underline{m})$ the corresponding subfiltration. Then

$$
\begin{aligned}
&(\sum_{i=1}^{t} m_i(\alpha h^0(X,\mathcal{F}_i(m)) - \alpha_i P(m))) + \delta\mu(\mathcal{F}_\bullet,\underline{m},\phi) \leq \\
\leq&(\sum_{i=1}^{t} m_i(\alpha h^0(X,\mathcal{F}_i(m)) - \alpha_i P(m))) + \delta\mu(\phi,\mathcal{F}'_\bullet,m'_\bullet) + \delta(\sum_{i\in T-T'} m_i a\alpha_i) = \\
=&(\sum_{i\in T'} m_i(\alpha h^0(X,\mathcal{F}_i(m)) - \alpha_i P(m))) + \delta\mu(\phi,\mathcal{F}'_\bullet,m'_\bullet) + \\
+&(\sum_{i\in T-T'} m_i(\alpha h^0(X,\mathcal{F}_i(m)) - \alpha_i P(m)) + a\alpha_i\delta)) \leq \\
\leq&(\sum_{i\in T'} m_i(\alpha P_{\mathcal{F}_i}(m) - \alpha_i P(m))) + \delta\mu(\phi,\mathcal{F}'_\bullet,m'_\bullet)(\leq)0.
\end{aligned}
\tag{2.9}
$$

The first inequality follows by Lemma 2.1.7 and the last inequality comes form the fact that $h^0(X,\mathcal{F}_i(m)) = P_{\mathcal{F}_i}(m)$ if $i \in T'$, and from the fact that

$$\alpha h^0(X,\mathcal{F}_i(m)) - \alpha_i P(m) + a\alpha_i\delta < 0$$

because of equation (2.8). We can remove the condition that $\mathcal{F}_i$ is saturated because $h^0(X,\mathcal{F}_i(m)) \leq h^0(X,\overline{\mathcal{F}}_i(m))$ and $\mu(\phi,\overline{\mathcal{F}}_\bullet,\underline{m}) = \mu(\mathcal{F}_\bullet,\underline{m},\phi)$, where $\overline{\mathcal{F}}_i$ is the saturated subsheaf generated by $\mathcal{F}_i$ in $\mathcal{F}$.

3) $\Rightarrow$ 1) Let $(\mathcal{F},\phi) \in S''_m$. Thus, $\mathcal{F}$ is N_0-regular and $P(m) = h^0(X,\mathcal{F}(m))$. Consider a saturated weighted filtration $(\mathcal{F}_\bullet,\underline{m})$. If $\mathcal{F}_i \in S_0$, then $P_{\mathcal{F}_i}(m) = h^0(X,\mathcal{F}_i(m))$. Let $(\mathcal{F}'_\bullet,\underline{m'})$ be the subfiltration formed by those terms $\mathcal{F}_i$ lying in S_0. Then, by hypothesis

$$(\sum_{\mathcal{F}_i\in S_0} m_i(\alpha^i P(m) - \alpha P_{\mathcal{F}_i}(m))) + \delta\mu(\phi,\mathcal{F}'_\bullet,\underline{m'})(\leq)0.$$

Since $\alpha = \alpha_i + \alpha^i$ and $P(m) = P_{\mathcal{F}_i}(m) + P_{\mathcal{F}^i}(m)$, the above is equivalent to

$$(\sum_{\mathcal{F}_i\in S_0} m_i(\alpha P_{\mathcal{F}_i}(m) - \alpha_i P(m))) + \delta\mu(\phi,\mathcal{F}'_\bullet,\underline{m'})(\leq)0$$

and, therefore, equivalent to

$$(\sum_{\mathcal{F}_i\in S_0} m_i(\alpha P_{\mathcal{F}_i} - \alpha_i P)) + \delta\mu(\phi,\mathcal{F}'_\bullet,\underline{m'})(\leq)0. \tag{2.10}$$

If $\mathcal{F}_i$ do not belongs to S_0, then the second alternative in Lemma 2.1.23 holds so

$$\alpha P_{\mathcal{F}_i} - \alpha_i P + s\alpha_i\delta < 0. \tag{2.11}$$

Using Lemma 2.1.7 and equations (2.10) and (2.11), we get

$$(\sum_{i=1}^{t} m_i(\alpha P_{\mathcal{F}_i} - \alpha_i P)) + \delta\mu(\mathcal{F}_\bullet,\underline{m},\phi)(\leq)0.$$

We can remove the condition $\mathcal{F}_i$ is saturated as we did before. □

Corollary 2.1.27. *Let $(\mathcal{F},\phi)$ be a δ-semistable, $m \geq N_0$ and assume that there is a weighted filtration $(\mathcal{F}_\bullet, \underline{m})$ such that*

$$(\sum_{i=1}^{t} m_i(\alpha h^0(\mathcal{F}_i(m)) - \alpha_i P(m))) + \delta\mu(\mathcal{F}_\bullet, \underline{m}, \phi) = 0. \tag{2.12}$$

Then $\mathcal{F}_i \in S_0$ and $h^0(X, \mathcal{F}_i(m)) = P_{\mathcal{F}_i}(m)$ for all i.

Proof. Equality (2.12) implies that the inequalities in (2.9) become equalities, so $T = T'$, $\mathcal{F}_i \in S_0$ for all i, and we are done. $\square$

Let us recall the result [35, Lemma 2.3.]. Let k be an algebraically closed field and X a reduced projective curve over k . Let C be a smooth curve, and fix a point $0 \in C$ and denote $Z := X \times C$. Consider the projections $p_X\colon Z \to X$ and $p_C\colon Z \to C$. Let Y be a non-empty proper closed subscheme of $X \times \{0\}$ of dimension 0 and denote $i\colon Y \hookrightarrow Z$ the closed embbeding. Denote also $U = Z - Y$ and let $j\colon U \hookrightarrow Z$ be the open immersion.

Lemma 2.1.28. [35, Lemma 2.3] *If $\mathcal{F}$ is a torsion free sheaf, then we have a canonical isomorphism $p_X^*\mathcal{F} \simeq j_*j^*(p_X^*\mathcal{F})$. In particular, $\mathcal{O}_Z \simeq j_*\mathcal{O}_U$ and for any locally free sheaf $\mathcal{E}$ on Z we have $\mathcal{E} \simeq j_*j^*\mathcal{E}$.*

Proof. Denote $\mathcal{G} = p_X^*\mathcal{F}$ and let $\underline{H}^0_Y(\mathcal{G})$, $\underline{H}^0_Y(\mathcal{G})$ be the cohomology sheaves with support in Y (see [29, Chapter 1, Section 2]). From the canonical morphism $\mathcal{G} \to j_*j^*\mathcal{G}$ we get an exact sequence (see [29, Corollaire 2.11])

$$0 \to \underline{H}^0_Y(\mathcal{G}) \to \mathcal{G} \to j_*j^*\mathcal{G} \to \underline{H}^1_Y(\mathcal{G}) \to 0. \tag{2.13}$$

This shows that it will be sufficient to prove that $\underline{H}^i_Y(\mathcal{G}) = 0$ for $i = 0, 1$. Because of [29, Proposition 3.3.], it will be sufficient to prove that $\operatorname{depth}(\mathcal{G}_y) \geq 2$ for each point $y \in Y$. Let s be a parameter of $\mathcal{O}_{C,0}$. We have the diagram

$$\begin{tikzcd}
X \simeq X \times \{0\} \arrow[r, hook, "u"] \arrow[d] & X \times C \arrow[r, "p_X"] \arrow[d, "p_X"] & X \\
\operatorname{Spec}(\mathcal{O}_{C,0}/s\mathcal{O}_{C,0}) = \{0\} \arrow[r, hook] & C &
\end{tikzcd}$$

Then, for each $y \in Y$,

$$\mathcal{G}_y/s\mathcal{G}_y = (u^*\mathcal{G})_y = ((p_X \circ u)^*\mathcal{F})_y \simeq \mathcal{F}_y. \tag{2.14}$$

Since $\mathcal{F}$ is torsion free, we know that $\operatorname{depth}(\mathcal{F}_y) \geq 1$ (in fact, is equal to one because $\operatorname{depth}(\mathcal{F}_y)$ is bounded by $\dim(X) = 1$). Consider the projection $p_C\colon X \times C \to C$. This induces a local morphism of local rings

$$\Phi_y\colon A := \mathcal{O}_{C,0} \to \mathcal{O}_{Z,y} =: B$$

(that is $\Phi_y^{-1}(\mathbf{m}_B) = \mathbf{m}_A$). Show also that the sheaf $\mathcal{G}_y/s\mathcal{G}_y$ is, with precision, the sheaf $\mathcal{G}_y/(\Phi(s))\mathcal{G}_y$. Now, applying Lemma 1.2.5 to this situation ($\Phi(s)$ is regular in $\mathbf{m}_B$) we get

$$\operatorname{depth}(\mathcal{G}_y) = \operatorname{depth}(\mathcal{G}_y/s\mathcal{G}_y) + 1 \geq 2, \tag{2.15}$$

and the result follows. For the second part, show that if $\mathcal{E}$ is locally free on Z then $u^*\mathcal{E}$ is locally free on X so we can repeat the proof. $\square$

Lemma 2.1.29. *Let C be a smooth curve, $(0) \in C$ a fixed point and let $\mathcal{G}$ be a family of coherent sheaves parametrized by C with Hilbert polynomial P (that is, a coherent $\mathcal{O}_{X\times C}$-module flat over C) such that $\mathcal{G}_t$ is torsion free for each $t \in C \setminus \{(0)\}$. Denote $\mathcal{F} = \mathcal{G}_{(0)}$. Then, there exists a torsion free sheaf $\mathcal{F}'$ with Hilbert polynomial P and an inclusion*

$$0 \to \mathcal{F}/T(\mathcal{F}) \hookrightarrow \mathcal{F}', \tag{2.16}$$

where $T(\mathcal{F})$ is the torsion subsheaf of $\mathcal{F}$.

Proof. Let $X \hookrightarrow \mathbf{P}^N$ be the closed immersion given by the polarization $\mathcal{O}_X(1)$ and consider the diagram

$$\begin{tikzcd}
& C & \\
X \times C \arrow[ur] \arrow[rr, hook, "i"] & & \mathbf{P}^N \times C \arrow[ul] \\
X \arrow[u, hook, "j"] \arrow[rr, hook, "i_{(0)}"] \arrow[dr] & & \mathbf{P}^N \arrow[u, hook, "j"'] \arrow[dl] \\
& \{0\} \arrow[uuu, bend left=80] &
\end{tikzcd}$$

Denote by $\overline{\mathcal{G}}$ the pushforward of $\mathcal{G}$ to $\mathbf{P}^N \times C$ and $\overline{\mathcal{F}} = \overline{\mathcal{G}}_{(0)}$. Applying [58, Lemma 1.17] we find a coherent sheaf of depth 1, $\mathcal{F}''$, and an injection

$$\overline{\mathcal{F}}/T(\overline{\mathcal{F}}) \hookrightarrow \mathcal{F}''.$$

Since $\overline{\mathcal{F}}/T(\overline{\mathcal{F}})$ is torsion free, the restriction of this injection to the curve is still injective

$$i^*_{(0)}\overline{\mathcal{F}}/T(\overline{\mathcal{F}}) \hookrightarrow i^*_{(0)}\mathcal{F}''.$$

Since $i^*_{(0)}\overline{\mathcal{F}} = j^*\mathcal{G} = \mathcal{G}_{(0)} = \mathcal{F}$ we finaly get the desired injection

$$\mathcal{F}/T(\mathcal{F}) \hookrightarrow \mathcal{F}' := \mathcal{F}''|_X.$$

□

Then, we finally have,

Proposition 2.1.30. *Let C be a smooth curve, $0 \in C$ a fixed point and $(\mathcal{G}, \Phi)$ a family of tensor fields parametrized by C with Hilbert polynomial P such that $\mathcal{G}_t$ is of pure dimension one for each $t \in C \setminus 0$. Denote by $(\mathcal{F}, \phi)$ the tensor field on the fibre corresponding to 0. Then there exists a tensor field $(\mathcal{F}', \phi')$ with $\mathcal{F}'$ of pure dimension one and Hilbert polynomial P and a morphism*

$$(\beta, \alpha)\colon (\mathcal{F}, \phi) \to (\mathcal{F}', \phi')$$

such that

$$\mathrm{Ker}(\beta) = T(\mathcal{F}).$$

Proof. Follows as in [8, Proposition 2.12], using Lemma 2.1.28 and Lemma 2.1.29 □

2.1.4 The Parameter Space

Let D be a locally free sheaf on X, let us fix a polynomial P of degree one with integral coefficients and $a, b \in \mathbb{N}$. Given $m \in \mathbb{N}$, let $\mathcal{H}$ be the subscheme of the quot-scheme parametrizing torsion free quotients $V \otimes \mathcal{O}_X(-m) \to \mathcal{F}$ with Hilbert polynomial P (m is fixed later). Let N_0 be as in Theorem 2.1.26. Since the set of sheaves which can appear in δ-(semi)stable tensor fields $(\mathcal{F}, \phi)$ is bounded, we can find $N > N_0$ such that for each $m > N$ and for each δ-(semi)stable tensor field, $(\mathcal{F}, \phi)$, $\mathcal{F}(m)$ is generated by its global sections, $h^1(X, \mathcal{F}(m)) = 0$, and also such that $D(m)$ is generated by global sections and $h^1(X, D(m)) = 0$.

Fix such a natural number $m > N$ and let V be a vector space of dimension $p = P(m)$. For any integer $l > m$ denote $V' = H^0(X, \mathcal{O}_X(l-m))$. For l large enough, there is a projective embedding (Grothendieck embedding)

$$\mathcal{H} \hookrightarrow \mathbf{P}\Big(\bigwedge^{P(l)} (V^\vee \otimes V'^\vee)\Big) \quad ,$$
$$q \mapsto \bigwedge^{P(l)} H^0(q \otimes 1)$$

where

$$q \otimes 1 \colon V \otimes \mathcal{O}_X(l-m) \longrightarrow \mathcal{F}(l) \quad .$$
$$\bigwedge^{P(l)} H^0(q \otimes 1) : \bigwedge^{P(l)} (V \otimes V') \longrightarrow \bigwedge^{P(l)} H^0(X, \mathcal{F}(l)) \simeq k$$

Let $\mathcal{P}$ be the projective space

$$\mathcal{P} = \mathbf{P}(((V^{\otimes a})^{\oplus b})^\vee \otimes H^0(D(sm))). \tag{2.17}$$

Its functor of points is given by

$$\mathcal{P}^\bullet(T) = \left\{ \begin{array}{c} \text{equivalence classes of invertible quotients} \\ ((V^{\otimes a})^{\oplus b} \otimes H^0(X, D(am))^\vee) \otimes \mathcal{O}_T \to \mathcal{L} \\ \text{over } T \end{array} \right\}. \tag{2.18}$$

For any scheme T we define a family of δ-(semi)sable tensor fields with Hilbert polynomial P and uniform multirank r parametrized by T as a tuple $(\mathcal{F}_T, \phi_T, N)$, where $\mathcal{F}_T$ is a relatively torsion free sheaf of uniform multirank r on $X \times T$ flat over T, with Hilbert polynomial P on each fiber, N is an invertible sheaf on T and ϕ_T is a morphism

$$\phi_T \colon (\mathcal{F}_T^{\otimes a})^{\oplus b} \to \pi_X^* D \otimes \pi_T^* N \tag{2.19}$$

such that for each point $t \in T$ the pair $(\mathcal{F}_{T,t}, \phi_{T,t})$ is δ-(semi)stable.

We want to solve the moduli problem defined by the functor

$$\mathbf{Tensors}^{\delta\text{-(s)s}}_{P,\mathcal{D},a,b}(T) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \delta\text{-(semi)stable torsion free tensor fields} \\ (\mathcal{F}_T, \phi_T, N) \text{ of uniform multi rank } r \\ \text{and with Hilbert polynomial } P \end{array} \right\}. \tag{2.20}$$

The strategy which we will follow for giving a coarse solution for the above moduli problem consists of, first rigidify the problem and give a fine solution and, finally, quot out that solution by the automorphisms of the rigidifying datum. The rigidifying datum will consist only in giving an isomorphism

$$g_T\colon V\otimes \mathcal{O}_T \simeq \pi_{T*}\mathcal{F}_T(m).$$

First we need to represent the functor (for m fixed as in the introduction)

$$ {}^{\mathrm{rig}}\mathbf{Tensors}^m_{P,\mathcal{D},a,b}(T) = \left\{ \begin{array}{c} \text{isomorphism classes of tuples } (\mathcal{F}_T,\phi_T,N,g_T) \\ \text{where } (\mathcal{F}_T,\phi_T,N) \text{ is a tensor field with} \\ \text{Hilbert polynomial } P \text{ and } g_T \text{ is a morphism} \\ g_T\colon V\otimes\mathcal{O}_T\to\pi_{T*}\mathcal{F}_T(m) \text{ such that} \\ \text{the induced morphism } V\otimes\mathcal{O}_{X\times T}\to\mathcal{F}_T(m) \\ \text{is surjective} \end{array} \right\}. \tag{2.21}$$

where two tuples, $(\mathcal{F}_T,\phi_T,N,g_T)$ and $(\mathcal{F}'_T,\phi'_T,N',g'_T)$, are isomorphic if there exists an isomorphism (f,α) (see Definition 2.1.2) between $(\mathcal{F}_T,\phi_T,N)$ and $(\mathcal{F}'_T,\phi'_T,N')$ such that $\pi_{T*}(f(m))\circ g_T = g'_T$.

Lemma 2.1.31. *Let X be a reduced connected projective curve, T a scheme and $\mathcal{G}$ a coherent $\mathcal{O}_{X\times T}$-module such that $h^1(X_t,\mathcal{G}_t)=0$ for all $t\in T$. Assume there is a non zero global section $s\in\mathcal{G}$. Then, for any point $i_t\colon t\hookrightarrow T$, $i_t^*\pi_{T*}s\neq 0$ if and only if $\pi_{t*}i_t^*s\neq 0$., being π_T and π_t the projections onto the second factor,*

$$\begin{tikzcd} X_t \arrow[r, hook, "i_t"] \arrow[d, "\pi_t"'] & X\times T \arrow[d, "\pi_T"] \\ t \arrow[r, hook, "i_t"'] & T \end{tikzcd}$$

Proof. By base change theorem, condition $h^1(X_t,\mathcal{G}_t)=0$ implies that $i_t^*\pi_{T*}\mathcal{G}\simeq\pi_{t*}i_t^*\mathcal{G}$. Then we just have to show that $i_t^*\pi_{T*}s\in i_t^*\pi_{T*}\mathcal{G}$ (resp. $\pi_{t*}i_t^*s\in\pi_{t*}i_t^*\mathcal{G}$), which follows from the fact that X is connected. □

Proposition 2.1.32. *There is a natural transformation of functors*

$$u\colon\ {}^{\mathrm{rig}}\mathbf{Tensors}^m_{P,\mathcal{D},a,b}\to\mathcal{H}^\bullet\times\mathcal{P}^\bullet.$$

Proof. The trasformation u is defined as follows. For each scheme T, let $(\mathcal{F}_T,\phi_T,N,g_T)\in {}^{\mathrm{rig}}\mathbf{Tensors}^m_{P,\mathcal{D},a,b}(T)$ and let

$$\pi_T^*\pi_*\mathcal{F}_T(m)\to\mathcal{F}_T(m)$$

be the canonical map. Composing with g_T we get a surjection $q_T\colon V\otimes\mathcal{O}_{X\times T}(-m)\to\mathcal{F}_T$, by definition of ${}^{\mathrm{rig}}\mathbf{Tensors}^m_{P,\mathcal{D},a,b}$. This gives an element in $\mathcal{H}^\bullet(T)$. Now, composing the morphisms

$$\begin{aligned} (q_T^{\otimes a})^{\oplus b} &: (V^{\otimes a})^{\oplus b}\otimes\mathcal{O}_{X\times T}\to(\mathcal{F}_T(m)^{\otimes a})^{\oplus b} \\ \phi^{\otimes am} &: (\mathcal{F}_T(m)^{\otimes a})^{\oplus b}\to\pi_X^*D(am)\otimes\pi_T^*N \end{aligned}$$

we get $\Phi\colon (V^{\otimes a})^{\oplus b} \otimes \mathcal{O}_{X\times T} \to \pi_X^* D(am) \otimes \pi_T^* N$. Since $\pi_{T*}\mathcal{O}_{X\times T} \simeq \mathcal{O}_T$, because X is connected, and $\pi_{T*}\pi_X^* D(sm) \simeq H^0(X, D(sm)) \otimes \mathcal{O}_T$ we find

$$\pi_{T*}(\Phi)\colon (V^{\otimes a})^{\oplus b} \otimes \mathcal{O}_T \to H^0(X, D(am)) \otimes N,$$

and therefore a morphism, which we denote with the same letter,

$$\pi_{T*}(\Phi)\colon ((V^{\otimes a})^{\oplus b} \otimes H^0(X, D(am))^\vee) \otimes \mathcal{O}_T \to N.$$

Now we claim that $\pi_{T*}(\Phi)$ is surjective. Since a morphism is surjective if and only if it is residually surjective we just have to show that $i_t^*\pi_{T*}(\Phi) \neq 0$ for each point $t \in T$. Observe that the fact that $h^1(X, D(m)) = 0$ implies that

$$h^1(X_t, i_t^* \mathcal{H}om_{\mathcal{O}_{X\times T}}((V^{\otimes a})^{\oplus b} \otimes \mathcal{O}_{X\times T}, \pi_X^* D(am) \otimes \pi_T^* N)) = 0 \text{ for all } t \in T$$

Now the claim is proved following [8, Section 2.2] and using Lemma 2.1.31. So the transformation is given, at the level of objects, by

$$(\mathcal{F}_T, \phi_T, N, g_T) \mapsto (q_T, \pi_{T*}(\Phi)).$$

The definition at the level of morphisms is the obvious one. $\square$

Consider a pair $(q_T, \varphi_T) \in \mathcal{H}^\bullet(T) \times \mathcal{P}^\bullet(T)$. Then we can construct

$$\begin{tikzcd}
(V^{\otimes a})^{\oplus b} \otimes \mathcal{O}_{X\times T} \arrow[r, "(q_T^{\otimes a})^{\oplus b}"] \arrow[d, "\pi_T^*(\varphi_T)"] & (\mathcal{F}(m)^{\otimes a})^{\oplus b} \arrow[r] & 0 \\
H^0(X, D(am)) \otimes \pi_T^*(N) \arrow[d, "\pi_X^*(f)\otimes 1"] & & \\
\pi_X^*(D(am)) \otimes \pi_T^*(N) & &
\end{tikzcd} \tag{2.22}$$

where $f\colon H^0(X, D(am)) \otimes \mathcal{O}_X \to D(am)$ is the natural surjection.

We need the following Lemma,

Lemma 2.1.33. *Let X be a connected k-scheme, $\mathcal{G}$ a coherent sheaf generated by its global sections and W a finite dimensional k-vector space. Let $d\colon W \otimes \mathcal{O}_X \to \mathcal{G}$ be a morphism, $\Psi_d\colon = \pi^*\pi_*(d)\colon W \otimes \mathcal{O}_X \to H^0(X, \mathcal{G}) \otimes \mathcal{O}_X$ the associated morphism, and $f\colon H^0(X, \mathcal{G}) \otimes \mathcal{O}_X \to \mathcal{G}$ the canonical morphism. Then $d = f \circ \Psi_d$. Furthermore, if $d\colon W \otimes \mathcal{O}_X \to \mathcal{G}$ and $\Psi\colon W \otimes \mathcal{O}_X \to H^0(X, \mathcal{G}) \otimes \mathcal{O}_X$ are morphisms such that $d = f \circ \Psi$, then $\Psi = \Psi_d$.*

Proof. The first part is trivial. Let us see the second part. Suppose we have a commutative diagram

$$\begin{tikzcd}
W \otimes \mathcal{O}_X \arrow[r, "d"] \arrow[dr, "\Psi"'] & \mathcal{G} \\
& H^0(X, \mathcal{G}) \otimes \mathcal{O}_X \arrow[u, "f"']
\end{tikzcd}$$

Then we can construct

$$
\begin{tikzcd}
 & H^0(\mathcal{G})\otimes\mathcal{O}_X \arrow[d, "f"] \\
W\otimes\mathcal{O}_X \arrow[ur, "\Psi_d"] \arrow[r, "d"] \arrow[dr, "\Psi"'] & \mathcal{G} \\
 & H^0(\mathcal{G})\otimes\mathcal{O}_X \arrow[u, "f"']
\end{tikzcd}
$$

from which we deduce that $f\circ\Psi_d = f\circ\Psi$, that is, $\Psi_d-\Psi$ factorizes through the kernel of the surjection $f\colon H^0(\mathcal{G})\otimes\mathcal{O}_X\to\mathcal{G}$. Since $H^0(\mathrm{Ker}(f))=(0)$, it follows that $\Psi_d=\Psi$. $\square$

Proposition 2.1.34. *Let T be a k-scheme. A point (q_T,φ_T) belongs to $\mathrm{Im}(u(T))$ if and only if there is a morphism*

$$(\mathcal{F}_T(m)^{\otimes a})^{\oplus b}\to\pi_X^*(D(am))\otimes\pi_T^*N$$

closing diagram (2.22). Accordingly, the natural transformation u is injective.

Proof. It is enough to show that $(q_T,\varphi_T)=u(T)(\mathcal{F}_T,\phi_T,N,g_T)\in\mathcal{H}^\bullet(T)\times\mathcal{P}^\bullet(T)$ if and only if the morphism

$$\phi_T^{\otimes am}\colon(\mathcal{F}_T(m)^{\otimes a})^{\oplus b}\to\pi_X^*(D(am))\otimes\pi_T^*N$$

close the diagram (2.22). We show it for k-points. The general case follows by the same argument. Assume there exists a tensor field $(\mathcal{F},\phi)$ such that $u(k)(\mathcal{F},\phi)=(q,\varphi)$. Then, the above diagram is

$$
\begin{tikzcd}
(V^{\otimes a})^{\oplus b}\otimes\mathcal{O}_X \arrow[r, "(q^{\otimes a})^{\oplus b}"] \arrow[d, "\pi^*\pi_*\Phi"] & (\mathcal{F}(m)^{\otimes a})^{\oplus b} \arrow[r] \arrow[ddl, dashed, "\phi^{\otimes am}"] & 0 \\
H^0(X,D(am))\otimes\mathcal{O}_X \arrow[d, "f"] & & \\
D(sm) & &
\end{tikzcd}
$$

and the equality $\phi^{am}\circ(q^{\otimes a})^{\oplus b}=f\circ\pi^*\pi_*(\phi^{\otimes am}\circ(q^{\otimes a})^{\oplus b})$ follows from Lemma 2.1.33. The converse follows also trivially from the above lemma. $\square$

Consider now the relative version of the above diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \arrow[r, "i"] \arrow[ddr, bend right, "h':=i\circ h"'] & (V^{\otimes a})^{\oplus b}\otimes\mathcal{O}_{X\times\mathcal{H}\times\mathcal{P}} \arrow[r, "(q_T^{\otimes a})^{\oplus b}"] \arrow[d, "\pi_T^*(\Phi)"] \arrow[dd, bend left=60, "h:=(\pi_X^*(f)\otimes 1)\circ(\pi_T^*(\Phi))"] & (\pi^*_{X\times\mathcal{H}}\mathcal{F}_{\mathcal{H}}(m)^{\otimes a})^{\oplus b} \arrow[r] & 0 \\
 & & H^0(X,D(am))\otimes\pi^*_{\mathcal{H}\times\mathcal{P}}(\upsilon^*\mathcal{N}) \arrow[d, "\pi_X^*(f)\otimes 1"] & & \\
 & & \pi_X^*(D(am))\otimes\pi^*_{\mathcal{H}\times\mathcal{P}}(\upsilon^*\mathcal{N})=:\mathcal{A} & &
\end{tikzcd}
$$

where $\mathcal{N}$ is the universal invertible sheaf of $\mathcal{P}$. Note that h factorizes if and only if $h' = 0$. Recall the following lemma,

Lemma 2.1.35. ([17, Lemma 3.1]) *Let Y be a scheme, and let $f\colon \mathcal{G} \to \mathcal{F}$ be a morphism of coherent sheaves on $X \times Y$. Assume $\mathcal{F}$ is Y-flat. Then there is a unique closed subscheme $Z \subset Y$ satisfying the following universal property: given a Cartesian diagram*

$$\begin{tikzcd} X \times S \arrow[r, "h'=\mathrm{id}_X \times h"] \arrow[d] & X \times Y \arrow[d] \\ S \arrow[r, "h"] & Y \end{tikzcd}$$

$h'^* f = 0$ if and only if h factors through Z.

If we apply Lemma 2.1.35 to $Y = \mathcal{H} \times \mathcal{P}$ and $h'\colon \mathcal{K} \to \mathcal{A}$ we get a closed subscheme $Z'_{m,\mathcal{D}} \subset \mathcal{H} \times \mathcal{P}$ whose points parametrizes tenor fields. If we denote by i the inclusion of $Z'_{m,\mathcal{D}}$, then $\overline{i}^* h' = 0$ and $\overline{i}^* h$ factorizes giving us a universal family of tensor fields.

Theorem 2.1.36. *The functor* ${}^{\mathrm{rig}}\mathbf{Tensors}^{m}_{P,\mathcal{D},a,b}$ *is represented by the closed subscheme $Z'_{m,\mathcal{D}}$.*

Proof. Follows trivially from the last results. $\square$

2.1.5 Semistability in the Parameter Space

We want to compare δ-semistability for tensor fields and GIT semistability in the parameter space with respect to the action of $\mathrm{SL}(V)$. The main result is Theorem 2.1.41. To prove it we will follow [17]. The polarization, and its linearization, is the one given there, adapted to our case.

Let $Z_{m,D} \subset Z'_{m,\mathcal{D}}$ be the closure of the locus representing δ-semistable tensor fields. Consider the projections

$$\begin{aligned} p_{\mathcal{H}}&\colon Z_{m,D} \to \mathcal{H} \\ p_{\mathcal{P}}&\colon Z_{m,D} \to \mathcal{P} \end{aligned}$$

and define a polarization on $Z_{m,D}$ by

$$\mathcal{O}_{Z_{m,D}}(n_1, n_2) := p_{\mathcal{H}}^* \mathcal{O}_{\mathcal{H}}(n_1) \otimes p_{\mathcal{P}}^* \mathcal{O}_{\mathcal{P}}(n_2), \tag{2.23}$$

n_1 and n_2 being positive integers such that

$$\frac{n_1}{n_2} = \frac{P(l) - \dim(V)}{\dim(V) - s\delta}\delta. \tag{2.24}$$

The natural action of $\mathrm{SL}(V)$ on $\mathcal{H} \times \mathcal{P}$ preserves the projective scheme Z and the linearizations on $\mathcal{O}_{\mathcal{H}}(1)$ and $\mathcal{O}_{\mathcal{P}}(1)$ induces a linearization on $\mathcal{O}_Z(n_1, n_2)$.

The objective of this section is to analyze the semistable points of the projective scheme $Z_{m,D}$ with respect to the linearized polarization $\mathcal{O}_{Z_{m,D}}(n_1, n_2)$.

Given a subspace $V' \subset V$ and a quotient $q\colon V \otimes \mathcal{O}_X(-m) \to \mathcal{F}$, we define the subsheaf $\mathcal{F}_{V'}$ of $\mathcal{F}$ as $\mathrm{Im}(q|_{V'\otimes \mathcal{O}_X(-m)})$, and we have

$$\begin{tikzcd}
V \otimes \mathcal{O}_X(-m) \arrow[r] & F \\
V' \otimes \mathcal{O}_X(-m) \arrow[r] \arrow[u, hook] & F_{V'}. \arrow[u, hook]
\end{tikzcd}$$

Hence, the subsheaf $\mathcal{F}_{V'}(m)$ is always generated by its global sections.

On the other hand, for any surjection $q\colon V \otimes \mathcal{O}_X(-m) \to \mathcal{F}$ and any subsheaf $\mathcal{F}' \subset \mathcal{F}$ we define $V_{\mathcal{F}'} := q^{-1}(H^0(X, \mathcal{F}'(m)))$, where we denote with the same letter q the induced linear map $V \to H^0(X, \mathcal{F}(m))$ (observe that $H^0(X, \mathcal{F}'(m)) \subset H^0(X, \mathcal{F}(m))$).

Lemmas [17, Lemma 3.2, Lemma 3.3] are easily adapted from the smooth case to our case.

Lemma 2.1.37. *Given a rational point $(q, [\phi]) \in Z$ (which corresponds to a tensor $(\mathcal{F}, \phi)$) such that q induces an injection $V \hookrightarrow H^0(X, \mathcal{F}(m))$, and a weighted filtration $(\mathcal{F}_\bullet, \underline{m})$ of $\mathcal{F}$, we have*

1) $\mathcal{F}_{V_{\mathcal{F}_i}} \subset \mathcal{F}_i$

2)If $\phi|_{(\mathcal{F}_{i_1}\otimes \ldots \otimes \mathcal{F}_{i_a})^{\oplus b}} = 0$ then $\Phi|_{(V_{\mathcal{F}_{i_1}}\otimes \ldots \otimes V_{\mathcal{F}_{i_a}})^{\oplus b}} = 0$

3)$\sum_1^n -m_i \epsilon(\phi, \mathcal{F}_\bullet, \underline{m}) \le \sum_1^t -m_i \epsilon(\Phi, V_{\mathcal{F}_\bullet}, \underline{m})$

Furthermore, if q induces a linear isomorphism $V \simeq H^0(X, \mathcal{F}(m))$, all $\mathcal{F}_i$ are m-regular and all $\mathcal{F}_{i_1} \otimes \cdots \otimes \mathcal{F}_{i_a}$ are am-regular, then 1) becomes an equality, 2) becomes an if and only if, and 3) an equality.

Proof. 1) Let $\mathcal{F}' \subset \mathcal{F}$ be a subsheaf. Then we have $H^0(X, \mathcal{F}'(m)) \subset H^0(X, \mathcal{F}(m))$. From the injectivity of the map $V \hookrightarrow H^0(X, \mathcal{F}(m))$ we get $V_{\mathcal{F}'} = V \cap H^0(X, \mathcal{F}'(m))$, thus the commutative diagram

$$\begin{tikzcd}
V \otimes \mathcal{O}_X(-m) \arrow[rr] & & \mathcal{F} \\
& \mathcal{F}' \arrow[ur, hook] & \\
V_{\mathcal{F}'} \otimes \mathcal{O}_X(-m) \arrow[uu, hook] \arrow[ur] \arrow[rr, two heads] & & \mathcal{F}_{V_{\mathcal{F}'}} \arrow[uu, hook]
\end{tikzcd}$$

from which follows that $\mathcal{F}_{V_{\mathcal{F}'}} \subset \mathcal{F}$.

2) Let $0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \ldots \subset \mathcal{F}_t \subset \mathcal{F}_{t+1} = \mathcal{F}$ be a filtration. Assume that, for $I = (i_1, \ldots, i_s)$, the restriction $\phi|_I\colon (\mathcal{F}_{i_1} \otimes \ldots, \otimes \mathcal{F}_{i_a})^{\oplus b} \to D$ is the zero morphism, that is, $\phi|_{(\mathcal{F}_{i_1}\otimes \ldots \otimes \mathcal{F}_{i_a})} = 0$. Hence, ϕ induces a zero morphism

$$\phi|_I^{\otimes am}\colon (\mathcal{F}_{i_1}(m) \otimes \ldots, \otimes \mathcal{F}_{i_a}(m))^{\oplus b} \to D(am))$$

and, therefore, the zero linear map

$$\Phi_1^I\colon H^0(X, \bigotimes \mathcal{F}_{i_j}(m))^{\oplus b} \to H^0(X, D(am)).$$

In the other hand, $q\colon V\otimes\mathcal{O}_X\to\mathcal{F}(m)$ induces

$$\Phi_2^I\colon = H^0(q_I^{\otimes a})^{\oplus b} : (V_{\mathcal{F}_{i_1}}\otimes\ldots\otimes V_{\mathcal{F}_{i_a}})^{\oplus b}\to H^0(X,\bigotimes\mathcal{F}_{i_j}(m))^{\oplus b}.$$

Composing we get the zero morphism

$$\Phi|_{(V_{\mathcal{F}_{i_1}}\otimes\ldots\otimes V_{\mathcal{F}_{i_a}})^{\oplus b}} := \Phi_1^I\circ\Phi_2^I : (V_{\mathcal{F}_{i_1}}\otimes\ldots\otimes V_{\mathcal{F}_{i_a}})^{\oplus b}\to H^0(X,D(am)).$$

3) Follows trivially from 2).
Suppose now that q induces an isomorphism. Then, clearly, 1) becomes an equality. Furthermore, if $\mathcal{F}_i$ are m-regular and $\mathcal{F}_{i_1}\otimes\cdots\otimes\mathcal{F}_{i_a}$ are am-regular too, then Φ_2^I is surjective. Therefore, $\Phi|_{(V_{\mathcal{F}_{i_1}}\otimes\ldots\otimes V_{\mathcal{F}_{i_a}})^{\oplus b}}=0$ implies that $\Phi_1^I=0$. Since $\mathcal{F}_{i_1}(m)\otimes\cdots\otimes\mathcal{F}_{i_a}(m)$ and $D(am)$ are globally generated, it implies that $\phi|_{(\mathcal{F}_{i_1}\otimes\cdots\otimes\mathcal{F}_{i_a})^{\oplus b}}=0$.

$\square$

Lemma 2.1.38. *Given a rational point $(q,[\Phi])\in Z$ (which corresponds to a tensor $(\mathcal{F},\phi)$) such that q induces an injection $V\hookrightarrow H^0(X,\mathcal{F}(m))$, and a weighted filtration $(V_\bullet,\underline{m})$ of V, we have*

1) $V_i\subset V_{\mathcal{F}_{V_i}}$

2)$\phi|_{(\mathcal{F}_{V_{i_1}}\otimes\ldots\otimes\mathcal{F}_{V_{i_a}})^{\oplus b}}=0$ if and only if $\Phi|_{(V_{i_1}\otimes\ldots\otimes V_{i_a})^{\oplus b}}=0$

3)$\sum_1^n -m_i\epsilon(\phi,\mathcal{F}_{V_\bullet},\underline{m})=\sum_1^t -m_i\epsilon(\Phi,V_\bullet,\underline{m})$

Proof. 1)Let $V'\subset V$ be a vector subspace. Then we have $\mathcal{F}_{V'}\subset\mathcal{F}$, hence $V_{\mathcal{F}_{V'}}\subset V$. Since the square

$$\begin{tikzcd} V\otimes\mathcal{O}_X(-m) \arrow[r] & F \\ V'\otimes\mathcal{O}_X(-m) \arrow[r] \arrow[u, hook] & F_{V'} \arrow[u, hook] \end{tikzcd}$$

commutes, we get commutative diagram

$$\begin{tikzcd} & & V \arrow[r, hook] & H^0(X,\mathcal{F}(m)) \\ & & V' \arrow[u, hook] \arrow[r, hook] & H^0(X,\mathcal{F}_{V'}(m)) \arrow[u, hook] \\ V\cap H^0(X,\mathcal{F}_{V'}(m)) := V_{\mathcal{F}_{V'}} \arrow[urru, hook] \arrow[urrr, hook] & & & \end{tikzcd}$$

By the commutativity follows the existence of an injection $V'\subset V_{\mathcal{F}_{V'}}$.

2) Consider a filtration

$$0\subset V_1\subset V_2\ldots\subset V_t\subset V_{t+1}=V. \tag{2.25}$$

By 1) of this lemma and 2) of Lemma 2.1.37, we deduce the direct implication. Let us see the inverse. Assume that

$$\Phi|_I\colon (V_{i_1}\otimes\ldots\otimes V_{i_a})^{\oplus b}\to H^0(X,D(am)) \tag{2.26}$$

is the zero morphism. Each $V_{i_j} \subset V$ defines a subsheaf $\mathcal{F}_{V_{i_j}}$ and we have a commutative diagram

$$\begin{tikzcd} V \otimes \mathcal{O}_X(-m) \arrow[r] & \mathcal{F} \\ V_{i_j} \otimes \mathcal{O}_X(-m) \arrow[r] \arrow[u, hook] & \mathcal{F}_{V_{i_j}}. \arrow[u, hook] \end{tikzcd}$$

Consider the restriction $\phi|_I \colon (\mathcal{F}_{V_{i_1}} \otimes \ldots \oplus \mathcal{F}_{V_{i_a}})^{\oplus b} \to D$. Taking tensor product am times by $\mathcal{O}_X(1)$, we get

$$\phi|_I \colon (\mathcal{F}_{V_{i_1}}(m) \otimes \ldots \oplus \mathcal{F}_{V_{i_a}}(m))^{\oplus b} \to D(am).$$

Now, from $q_{i_j} \colon V_{i_j} \otimes \mathcal{O}_X(-m) \to \mathcal{F}_{V_{i_j}}$ we get the surjective map

$$q_I^{\otimes a} \colon (V_{i_1} \otimes \ldots \otimes V_{i_a})^{\oplus b} \otimes \mathcal{O}_X \to (\mathcal{F}_{V_{i_1}}(m) \otimes \ldots \otimes \mathcal{F}_{V_{i_a}}(m))^{\oplus b} \to 0.$$

Composing these two morphisms we find the morphism

$$q|_I^{\otimes a} \circ \phi|_I \colon (V_{i_1} \otimes \ldots \otimes V_{i_a})^{\oplus b} \otimes \mathcal{O}_X \to D(am),$$

from which taking global sections we get the initial (zero) morphism. But then we deduce that

$$\phi|_I \colon (\mathcal{F}_{V_{i_1}}(m) \otimes \ldots \oplus \mathcal{F}_{V_{i_a}}(m))^{\oplus b} \to D(am)$$

is the zero morphism, thus $\phi|_{(\mathcal{F}_{V_{i_1}} \otimes \ldots \mathcal{F}_{V_{i_a}})^{\oplus b}} = 0$.

3) Follows trivially from 2). □

Proposition 2.1.39. *For sufficiently large l, the point $(q, \phi) \in Z$ is GIT-(semi)stable with respect to $\mathcal{O}_X(n_1, n_2)$ if and only if for every weighted filtration $(V_\bullet, \underline{m})$ of V*

$$n_1(\sum_1^t m_i(\dim V_i P(l) - \dim V P_{\mathcal{F}_{V_i}}(l))) + n_2 \delta \mu(\phi, V_\bullet, \underline{m})(\leq)0.$$

Furthermore, there is an integer A_2 (depending on m, P, s, b, c and D) such that it is enough to consider weighted filtrations with $m_i \leq A_2$

Proof. It follows exactly as in [17, Proposition 3.4] by applying the same argument given in Lemma 2.1.10. □

Proposition 2.1.40. *A point (q, ϕ) is GIT-(semi)stable if and only if for all weighted filtrations $(\mathcal{F}_\bullet, \underline{m})$ of $\mathcal{F}$,*

$$\sum_1^t m_i((\dim V_{\mathcal{F}_i} - \epsilon_i(V_\bullet)\delta)(P - a\delta) - (P_{\mathcal{F}_{V_i}} - \epsilon_i(V_\bullet)\delta)(\dim V - a\delta))(\preceq)0.$$

Furthermore, if $(q, [\phi])$ is GIT-semistable, then the induced map $f_q \colon V \to H^0(X, \mathcal{F}(m))$ is injective.

Proof. We follow [19] closely. First, let us show the second part. Let $(q, [\phi])$ be a GIT semistable point. The quotient q induces a linear map

$$f_q \colon V \to H^0(X, \mathcal{F}(m)).$$

Let $V' \subseteq V$ be ist kernel. Obviously, $\mathcal{F}_{V'} = 0$ and $\mu(\phi, V' \subseteq V) = a\mathrm{dim}(V')$. Proposition 2.1.39 give us

$$n_1 \mathrm{dim}(V')P(l) + n_2 \mathrm{dim}(V') \leq 0$$

hence $V' = 0$.

Using the polarization given in (2.24), the inequality of Proposition 2.1.39 becomes

$$\sum_{i=1}^{t} m_i((\mathrm{dim}(V_i) - \epsilon(V_\bullet)\delta)(P(l) - a\delta) - (P_{\mathcal{F}_{V_i}}(l) - \epsilon_i(V_\bullet)\delta)(\mathrm{dim}(V) - a\delta))(\leq)0.$$

Since the family $\{\mathcal{F}_{V'}\}_{V' \subseteq V}$ is bounded there are just finitely many polynomials $P_{\mathcal{F}_{V'}}$. By Proposition 2.1.39 there is an A_2 such that we just need to choose $m_i < A_2$ Then we can take l large enough (depending on m, s, b, c, P, D and δ) so that the inequality holds for l if and only if it holds as an equality of polynomials. Now, the proposition follows as in [17, Proposition 3.5] using Lemma 2.1.37 and Lemma 2.1.38. □

Theorem 2.1.41. *Assume $m > N$. For l large enough, a point $(q, [\phi])$ in Z is GIT-(semi)stable if and only if the corresponding tensor field $(\mathcal{F}, \phi)$ is δ-(semi)stable and the linear map $f_q \colon V \to H^0(X, \mathcal{F}(m))$ is an isomorphism.*

Proof. 1) We will see that if $(q, [\phi])$ is GIT-(semi)stable then $(\mathcal{F}, \phi)$ is δ-(semis)stable and q induces the isomorphism. The leading coefficient of (2.1.40) gives the inequality

$$\sum_{1}^{t} m_i((\mathrm{dim}(V_{\mathcal{F}_i}) - \epsilon_i(V_\bullet)\delta)\alpha - \alpha_i(\mathrm{dim}(V) - a\delta)) \leq 0,$$

or, equivalently

$$\sum_{i=1}^{t} m_i(\mathrm{dim}(V_{\mathcal{F}_i})\alpha - \alpha_i \mathrm{dim}(V)) + \delta\mu(\mathcal{F}_\bullet, \underline{m}, \phi) \leq 0. \tag{2.27}$$

Since $\mathrm{dim}(V) = P(m)$ and $P(m) \leq h^0(\mathcal{F}_i(m)) + h^0(\mathcal{F}^i(m))$, inequality (2.27) becomes

$$(\sum_{i=1}^{t} m_i(\alpha^i P(m) - \alpha h^0(X, \mathcal{F}^i(m)))) + \delta\mu(\mathcal{F}_\bullet, \underline{m}, \phi) \leq 0. \tag{2.28}$$

To be able to apply Theorem 2.1.26 we need to show that $\mathcal{F}$ is torsion free. Applying Proposition 2.1.30, we know that there exists a tensor field $(\mathcal{G}, \psi)$ with $\mathcal{G}$ torsion free and Hilbert polynomial P, and an exact sequence

$$0 \to T(\mathcal{F}) \to \mathcal{F} \to \mathcal{G}. \tag{2.29}$$

Consider a weighted filtration $(\mathcal{G}_\bullet, \underline{m})$ of $\mathcal{G}$. Let $\mathcal{G}^i = \mathcal{G}/\mathcal{G}_i$, $\mathcal{F}^i$ be the image of $\mathcal{F}$ in $\mathcal{G}^i$ and $\mathcal{F}_i$ the kernel of $\mathcal{F} \to \mathcal{F}^i$. We can construct the diagram

$$\begin{tikzcd}
0 \arrow[r] & Q(\mathcal{G}_i) \arrow[r, hook] & Q(\mathcal{G}) & & \\
0 \arrow[r] & \mathcal{G}_i \arrow[r, hook] \arrow[u] & \mathcal{G} \arrow[r] \arrow[u] & \mathcal{G}^i \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{F}_i \arrow[r, hook] \arrow[u, dotted] & \mathcal{F} \arrow[r] \arrow[u] & \mathcal{F}^i \arrow[r] \arrow[u, hook] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[u] & T(\mathcal{F}) \arrow[u] & &
\end{tikzcd}$$

By the Short Five Lemma we get $K \simeq T(\mathcal{F})$, so K is a torsion sheaf. From the fact that $P = P_{\mathcal{F}} = P_{\mathcal{G}}$, we get $P_{T(\mathcal{F})} = P_{Q(\mathcal{G})} \in k$, so $Q(\mathcal{G})$ and, hence, $Q(\mathcal{G}_i)$ are torsion sheaves. Since the Hilbert polynomials of K and $Q(\mathcal{G}_i)$ are scalars we deduce that the leading coefficients of the Hilbert polynomials of $\mathcal{F}_i$ and $\mathcal{G}_i$ are the same so $\alpha(\mathcal{F}_i) = \alpha(\mathcal{G}_i)$. Also, because of the right vertical injection, we know that $h^0(X, \mathcal{G}i(m)) \geq h^0(X, \mathcal{F}^i(m))$. Let us see now that $\mu(\mathcal{G}_\bullet, \underline{m}, \psi) = \mu(\mathcal{F}_\bullet, \underline{m}, \phi)$. Recall that for each multi-index $I = (i_1, \ldots, i_s)$ we have (note that we have seen that $\alpha_i := \alpha(\mathcal{F}_i) = \alpha(\mathcal{G}_i)$),

$$\gamma_{\alpha_{i_1}} + \ldots + \gamma_{\alpha_{i_a}} = \sum_{i=1}^{t} m_i(a\alpha_i - \nu_i(I)\alpha)$$

where $\nu_i(I)$ is the number of elements k of the multi-index I such that $\alpha_k \leq \alpha_i$. Since $\psi|_{(\mathcal{G}_{i_1} \otimes \ldots \otimes \mathcal{G}_{i_a})^{\oplus b}} \neq 0$ if and only if $\phi|_{(\mathcal{F}_{i_1} \otimes \ldots \otimes \mathcal{F}_{i_a})^{\oplus b}}$ we, finally, get

$$\mu(\mathcal{G}_\bullet, \underline{m}, \psi) = \mu(\mathcal{F}_\bullet, \underline{m}, \phi).$$

Using this and applying (2.28) to $\mathcal{G}_i$ we find

$$\begin{aligned}
&(\sum_{i=1}^{t} m_i(\alpha^i P(m) - \alpha h^0(X, \mathcal{G}^i(m)))) + \delta\mu(\mathcal{G}_\bullet, \underline{m}, \phi) \leq \\
&(\sum_{i=1}^{t} m_i(\alpha^i P(m) - \alpha h^0(X, \mathcal{F}^i(m)))) + \delta\mu(\mathcal{G}_\bullet, \underline{m}, \phi) = \\
&(\sum_{i=1}^{t} m_i(\alpha^i P(m) - \alpha h^0(X, \mathcal{F}^i(m)))) + \delta\mu(\mathcal{F}_\bullet, \underline{m}, \phi) \leq 0.
\end{aligned}$$

Now, applying Theorem 2.1.26 we deduce that $(\mathcal{G}, \psi)$ is δ-semistable. If we show that $T(\mathcal{F}) = 0$, we will deduce that $(\mathcal{F}, \phi) \simeq (\mathcal{G}, \psi)$ because the Hilbert polynomials are the same. Define $\mathcal{F}''$ as the image of $\mathcal{F}$ in $\mathcal{G}$. Since $(\mathcal{G}, \psi)$ is δ-semistable, $\mathcal{G}$ is m-regular so $P(m) = h^0(X, \mathcal{G}(m))$. Therefore,

$$P(m) - a\delta = h^0(X, \mathcal{G}(m)) - a\delta \geq h^0(X, \mathcal{F}''(m)) - a\delta \geq P(m) - a\delta$$

where the last inequality follows from the third equation applied to the one-step filtration $T(\mathcal{F}) \subset \mathcal{F}$. Then, instead of inequalities we have equalities so $h^0(X, \mathcal{G}(m)) =$

$h^0(X, \mathcal{F}''(m))$. Since $\mathcal{G}$ is globally generated (see [39, Lecture 14, Proposition]), $\mathcal{G} = \mathcal{F}''$ so $T(\mathcal{F})$=0. Finally we have seen that $f_q\colon V \to H^0(X, \mathcal{F}(m))$ is injective (Proposition 2.1.40) and since $(\mathcal{F}, \phi)$ is δ-semistable, $\dim(V) = h^0(X, \mathcal{F}(m))$ so f_q is, in fact, an isomorphism.

2)Assume $(\mathcal{F}, \phi)$ is δ-(semi)stable and that q induces an isomorphism $f_q : V \simeq H^0(X, \mathcal{F}(m))$. Since f_q is an isomorphism then $V_{\mathcal{F}'} = H^0(X, \mathcal{F}'(m))$ for any subsheaf $\mathcal{F}' \subset \mathcal{F}$. Thus, by Theorem 2.1.26 we have

$$\sum_{i=1}^{t} m_i(\alpha \mathrm{dim} V_{\mathcal{F}_i} - \alpha_i P(m)) + \delta\mu(\mathcal{F}_\bullet, \underline{m}, \phi)(\leq)0 \tag{2.30}$$

for all weighted filtrations. Observe that the left hand side of Equation (2.30) is precisely the leading coefficient of the polynomial

$$\sum_{i=1}^{t} m_i((\mathrm{dim} V_{\mathcal{F}_i} - \epsilon_i(\mathcal{F}_\bullet)\delta)(P - a\delta) - (P_{\mathcal{F}_i} - \epsilon_i(\mathcal{F}_\bullet)\delta)(\mathrm{dim} V - a\delta)).$$

We deduce that if we have a strict inequality in Equation (2.30) then,

$$\sum_{i=1}^{t} m_i((\mathrm{dim} V_{\mathcal{F}_i} - \epsilon_i(\mathcal{F}_\bullet)\delta)(P - s\delta) - (P_{\mathcal{F}_i} - \epsilon_i(\mathcal{F}_\bullet)\delta)(\mathrm{dim} V - s\delta)) \prec 0.$$

If $(\mathcal{F}, \phi)$ is strictly δ-semistable, by Theorem 2.1.26 there is a filtration $(\mathcal{F}_\bullet, \underline{m})$ giving an equality in (2.30)

$$\sum_{i=1}^{t} m_i(\alpha \mathrm{dim}(V_{\mathcal{F}_i}) - \alpha_i P(m)) + \delta\mu(\mathcal{F}_\bullet, \underline{m}, \phi) = 0. \tag{2.31}$$

Note that

$$\sum_{i=1}^{t} m_i \left\{(\mathrm{dim}(V_{\mathcal{F}_i}) - \epsilon_i\delta)(P - a\delta) - (P_{\mathcal{F}_i} - \epsilon_i\delta)(\mathrm{dim}(V) - a\delta)\right\} =$$
$$= \sum_{i=1}^{t} m_i \left\{(\mathrm{dim}(V_{\mathcal{F}_i})P - \mathrm{dim}(V)P_{\mathcal{F}_i}) + \delta(P_{\mathcal{F}_i}a - \epsilon_i P) - \right.$$
$$\left. -\delta(\mathrm{dim}(V_{\mathcal{F}_i})a - \epsilon_i \mathrm{dim}(V))\right\}.$$

The degree one coefficient of this polynomial is given by

$$\sum_{i=1}^{t} m_i((\mathrm{dim}(V_{\mathcal{F}_i})\alpha - \mathrm{dim}(V)\alpha_i) + \delta(\alpha_i a - \epsilon_i\alpha)) =$$
$$= \sum_{i=1}^{t} m_i(\alpha \mathrm{dim}(V_{\mathcal{F}_i}) - \alpha_i P(m)) + \delta\mu(\mathcal{F}_\bullet, \underline{m}, \phi) = 0,$$

which is equal to 0 because (2.31) holds. Using the equalities $P(n) = (\mathrm{dim}(V) - \alpha m) + \alpha n$, $P_{\mathcal{F}_i}(n) = (\mathrm{dim}(V_{\mathcal{F}_i}) - \alpha_i m) + \alpha_i n$ (this last equality follows from Corollary 2.1.27) and again (2.31), it follows that the constant coefficient of this polynomial is also 0. Finally the result follows by Proposition 2.1.40 $\square$

2.1.6 Construction of the Moduli Space

We fix a polynomial $P \in \mathbb{Z}[n]$ of degree one, natural numbers a and b, a rational number $\delta \in \mathbb{Q}_{>0}$ and a locally free sheaf D on X. For $m \in \mathbb{N}$, let $Z_{m,D}$ be the scheme constructed in Section 2.1.5, and let $Z^0_{m,D}$ be the open subscheme parametrizing points $(\mathcal{F}, \phi, g)$ with $\mathcal{F}$ a torsion free sheaf and g an isomorphism.

Proposition 2.1.42. (Glueing Property) *Let S be a scheme of finite type over $\mathbb{C}$ and $s_1, s_2\colon S \to Z^0_{m,D}$ two morphisms such that the pullbacks of $(\mathcal{F}_{Z^0_{m,D}}, \phi_{Z^0_{m,D}})$ via $s_1 \times \mathrm{id}_X$ and $s_2 \times \mathrm{id}_X$ are isomorphic. Then there exists an étalé covering $c\colon T \to S$ and a morphism $g\colon T \to \mathrm{SL}(W)$ such the triangle*

$$\begin{tikzcd}
\mathrm{SL}(W) \times Z^0_{m,D} \arrow[rr, "\Gamma"] & & Z^0_{m,D} \\
& T \arrow[ul, "g\times(s_1\circ c)"] \arrow[ur, "s_2\circ c"'] &
\end{tikzcd}$$

is commutative.

Proof. Morphisms $s_i \times \mathrm{id}_X\colon S \times X \to Z^0_{m,D} \times X$ provide us with two families, $(\mathcal{F}^1_S, \phi^1_S, N^1_S, g^1_s)$ and $(\mathcal{F}^2_S, \phi^2_S, N^2_S, g^2_S)$, via pullback of the universal family, such that there are isomorphisms $\Phi : \mathcal{F}^1_S \simeq \mathcal{F}^2_S$ and $\psi : N_S \simeq N'_S$, making the diagram commutative

$$\begin{tikzcd}
((\mathcal{F}^1_S)^{\otimes a})^{\oplus b} \arrow[r, "(\Phi^{\otimes a})^{\oplus b}", "\sim"'] \arrow[d, "\phi^1_S"] & ((\mathcal{F}^2_S)^{\otimes a})^{\oplus b} \arrow[d, "\phi^2_S"] \\
\pi_X^* D(am) \otimes \pi_S^* N^1_S \arrow[r, "id\otimes\pi_S^*\psi", "\sim"'] & \pi_X^* D(am) \otimes \pi_S^* N^2_S
\end{tikzcd}$$

Note also that there is an isomorphism,

$$V \otimes \mathcal{O}_S \xrightarrow{\quad g^1_S \quad} \pi_{S*}(\mathcal{F}^1_S \otimes \pi_X^* \mathcal{O}_X(n)) \longrightarrow$$

$$\xrightarrow{\pi_{S*}(\Phi \otimes \mathrm{id}_{\pi_X^*\mathcal{O}_X(n)})} \pi_{S*}(\mathcal{F}^2_S \otimes \pi_X^* \mathcal{O}_X(n)) \xrightarrow{\quad g^2_S \quad} V \otimes \mathcal{O}_S$$

which determines a morphism $h'\colon S \to \mathrm{GL}(V)$. Let $\det(h') = det \circ h'\colon S \to \mathbb{G}_m$ be the determinant morphism, Now we define T by means of the following cartesian product

$$\begin{tikzcd}
T := S \times_{\mathbb{G}_m} \mathbb{G}_m \arrow[r] \arrow[d, "c"] \arrow[dr, "\Delta"] & \mathbb{G}_m \arrow[d, "\chi_p"] & z \arrow[d, mapsto] \\
S \arrow[r, "\det(h')"] & \mathbb{G}_m & z^p
\end{tikzcd}$$

Obviously $c\colon T \to S$ is a Galois covering (therefore étalé) of degree p. Denote $\Delta_e\colon T \to \mathbb{G}_m$ the morphism $\chi_e \circ \Delta$. Now, the T-point of $\mathrm{SL}(V)$ is obtained from g' by composing with c and dividing by the determinant, i.e

$$h : T \xrightarrow{\Delta_{-1}\times(h'\circ c)} \mathbb{G}_m \times \mathrm{GL}(V) \xrightarrow{\quad\cdot\quad} \mathrm{SL}(V).$$

All of this together determines a T-point of $Z^0_{m,D}$

$$T \xrightarrow{h\times(s_1\circ c)} \mathrm{SL}(V)\times Z^0_{m,D} \xrightarrow{\Gamma} Z^0_{m,D},$$

which corresponds to the family of tensor fields on T obtained by pulling back to T (via c) the family on S given by $h \bullet (c\times \mathrm{id}_X)^*(\mathcal{F}^1_S, \phi^1_S, N^1_S, g^1_S) =: (\mathcal{F}'^1_T, \phi'^1_T, N'^1_T, g'^1_T)$. Here, $\mathcal{F}'^1_S = (c\times\mathrm{id}_X)^*\mathcal{F}^1_S$, $N'^1_T = c^*N^1_S$, g'^1_T is given by the composition

$$V\otimes \pi_X^*\mathcal{O}_X(-n) \xrightarrow{h^{-1}\otimes \mathrm{id}_{\pi_X^*\mathcal{O}_X(-n)}} V\otimes \pi_X^*\mathcal{O}_X(-n) \longrightarrow \pi_{T*}((c\times\mathrm{id}_X)^*\mathcal{F}^1_S)$$

and ϕ'^1_T is the morphism obtained by pulling ϕ^1_S back to $T\times X$ via $(c\times\mathrm{id})$. Finally, the isomorphism

$$\widehat{\Phi} := \Delta\cdot((c\times\mathrm{id}_X)^*\Phi^{-1}) : (c\times\mathrm{id}_X)^*\mathcal{F}^2_S \longrightarrow (c\times\mathrm{id}_X)^*\mathcal{F}^1_S$$

gives an equivalence with the family $(c\times\mathrm{id}_X)^*(\mathcal{F}^2_S, \phi^2_S, N^2_S, g^2_S)$. □

Proposition 2.1.43. (Local Universal Property) *Let S be a scheme of finite type over $\mathbb{C}$ and $(\mathcal{F}_S, \phi_S)$ a family of δ-(semi)stable tensor fields parametrized by S. Then there exists an open covering S_i, $i\in I$ of S and morphisms $\beta_i\colon S_i \to Z^0_{m,D}$, $i\in I$ such that the restriction of the family $(\mathcal{F}_S, \tau_S)$ to $S_i\times X$ is equivalent to the pullback of $(\mathcal{F}_{Z^0_{m,D}}, \phi_{Z^0_{m,D}})$ via $\beta_i\times\mathrm{id}_X$ for all $i\in I$.*

Proof. Since n is large enough so that $h^1(X_s, \mathcal{F}_{S,s}\otimes\mathcal{O}_{X_s}) = 0$ for all $s\in S$ we deduce that $\pi_{S*}(\mathcal{F}_{S,s}\otimes\pi_X^*\mathcal{O}_X(n))$ is locally free. Then, any finite covering $\{S_i\}$ of S trivializing it satisfies the statement of the proposition (see [50, Proposition 2.8]). □

Finally we have,

Theorem 2.1.44. *Fix a polynomial P, natural numbers a and b, a rational number $\delta\in\mathbb{Q}_{>0}$ and a locally free sheaf D on X. There is a projective scheme $\mathcal{T}^{\delta\text{-}ss}_P$ and an open subscheme $\mathcal{T}^{\delta\text{-}s}_P \subset \mathcal{T}^{\delta\text{-}ss}_P$ together with a natural tranformation*

$$\alpha^{(s)s}\colon \mathbf{Tensors}^{\delta\text{-}(s)s}_{P,D,a,b} \to h_{\mathcal{T}^{\delta\text{-}(s)s}_P}$$

with the following propoerties:

1) For every scheme $\mathcal{N}$ and every natural transformation $\alpha'\colon \mathbf{Tensors}^{\delta\text{-}(s)s}_{P,D,a,b} \to h_{\mathcal{N}}$, there exists a unique morphism $\varphi\colon \mathcal{T}^{\delta\text{-}(s)s}_P \to \mathcal{N}$ with $\alpha' = h(\varphi)\circ\alpha^{(s)s}$.

2) The scheme $\mathcal{T}^{\delta\text{-}s}_P$ is a coarse moduli space for the functor $\mathbf{Tensors}^{\delta\text{-}s}_{P,D,a,b}$

Proof. Consider the closed immersion $Z_{m,D} \hookrightarrow \mathcal{H}\times\mathcal{P}$. Let m and l be large enough so that Theorem 2.1.41 holds. Then, the GIT construction ensures that the categorical quotient$Z^{ss}_{m,D}/\!\!/\mathrm{SL}(V)$ exists and is a projective scheme. The categorical quotient $Z^s_{m,D}/\mathrm{SL}(V) \subset Z^{ss}_{m,D}/\!\!/\mathrm{SL}(V)$ is an open subset and a geometric quotient. Now the theorem follows by the same argument as in [17, Theorem 1.8], using Proposition 2.1.42, Proposition 2.1.43 and Proposition 1.1.10. □

2.2 Moduli Space of Singular Principal G-Bundles

In this section, the moduli problem for singular principal G-bundles on a nodal curve and a semisimple algebraic group G is solved. Although a semistability notion for this objects is given (Definition 2.2.4), we will attack the problem by considering the notion of δ-semistabillity.

We follow [8, 49] for the construction of the moduli space $\mathrm{SPB}(\rho)_P^{\delta\text{-}(s)s}$. The main difficulty will be to give the notion of δ-semistability. We first attach to any singular principal G-bundle, $(\mathcal{F},\tau)$, a tensor field, $(\mathcal{F},\phi)$, of certain type (s,b), which depends only on the numerical input data, and then we define the function $\mu(\mathcal{F}_\bullet,\underline{m},\tau)$ as the semistability function of the corresponding tensor field. The key results in this process are Theorem 2.2.6 and Theorem 2.2.12, which make use of the structure of torsion free sheaves around nodal points (Lemma 1.2.28 and Lemma 2.2.5).

2.2.1 Singular Principal G-Bundles on Nodal Curves and Semistability

Let G be a semisimple linear algebraic group and $\rho\colon G\to \mathrm{SL}(V)\subset \mathrm{GL}(V)$ a faithful representation, V being a k-vector space of dimension n.

Let X be a nodal projective curve with ν nodes. We denote by $x_1,\dots,x_\nu$ the nodes of X. If we need to refer to the nodal points lying in more than one component we will use the notation z_i for the ith of those nodal points. Let $X_1,\dots,X_l$ be the irreducible components of X and let $p_i\colon X_i\hookrightarrow X$ be the closed immersion of the i-th irreducible component. Let $U:=X-\mathrm{Sing}(X)$ be the regular part, $U_i=U\cap X_i$ and $j_i\colon U_i\hookrightarrow X_i$ the ith open embedding. We fix an ample invertible sheaf $\mathcal{O}_X(1)$ and we denote by h its degree.

Definition 2.2.1. A singular principal G-bundle over the nodal curve X is a pair $(\mathcal{F},\tau)$ where $\mathcal{F}$ is a torsion free sheaf on X and $\tau\colon S^\bullet(\mathcal{F}\otimes V)^G\to\mathcal{O}_X$ a non trivial surjective morphism (is not the projection onto the degree zero component) of $\mathcal{O}_X$-algebras. Let $r\in\mathbb{N}$, $P\in\mathbb{Z}[n]$ a polynomial of degree one. Then, $(\mathcal{F},\tau)$ is of type r,P if $\mathcal{F}$ has uniform multirank r and Hilbert polynomial P.

Let $(\mathcal{F},\tau)$ be a singular principal G-bundle. Note that if $\mathcal{F}$ is locally free then $S^\bullet(V\otimes\mathcal{F})^G=\underline{\mathrm{Hom}}_{\mathcal{O}_X}(V\otimes\mathcal{O}_X,\mathcal{F}^\vee)/\!\!/G$. For the sake of notation we will denote $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(V\otimes\mathcal{O}_X,\mathcal{F}^\vee):=S^\bullet(V\otimes\mathcal{F})$ for any torsion free sheaf $\mathcal{F}$. Giving τ is, therefore, the same as giving a section $\tau\colon X\to\underline{\mathrm{Hom}}_{\mathcal{O}_X}(V\otimes\mathcal{O}_X,\mathcal{F}^\vee)/\!\!/G$. The pair $(\mathcal{F},\tau)$ provides us with a commutative diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{F},\tau) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{\mathcal{O}_X}(V\otimes\mathcal{O}_X,\mathcal{F}^\vee) \arrow[d] \\
X \arrow[r, "\tau"] & \underline{\mathrm{Hom}}_{\mathcal{O}_X}(V\otimes\mathcal{O}_X,\mathcal{F}^\vee)/\!\!/G
\end{tikzcd}$$

being $\mathcal{P}(\mathcal{F},\tau)$ the fiber product. Note that over $U:=X-\mathrm{Sing}(X)$, $\mathcal{F}$ is always locally free, so if $\tau(U)\subset\underline{\mathrm{Isom}}_{\mathcal{O}_U}(V\otimes\mathcal{O}_U,\mathcal{F}|_U^\vee)/\!\!/G$, then $\mathcal{P}(\mathcal{F},\tau)|_U$ determines a principal G-bundle over U.

Definition 2.2.2. We say that a singular principal G-bundle $(\mathcal{F},\tau)$ on X is honest if the image of $\tau|_U$ is contained in the open subscheme $\underline{\mathrm{Isom}}_X(\mathcal{O}_X|_U\otimes V,\mathcal{F}|_U^\vee)/G$. We say that it is quasi-honest if it is honest over some subcurve $X'\subset X$.

Definition 2.2.3. Let $(\mathcal{F},\tau)$ and $(\mathcal{G},\lambda)$ be singular principal G-bundles on X. A morphism between them is a morphism of $\mathcal{O}_X$-modules $f\colon\mathcal{F}\to\mathcal{G}$ such that the triangle

$$\begin{tikzcd} S^\bullet(\mathcal{F}\otimes V)^G \arrow[rr, "\overline{f}"] \arrow[dr, "\tau"'] & & S^\bullet(\mathcal{G}\otimes V)^G \arrow[dl, "\lambda"] \\ & \mathcal{O}_X & \end{tikzcd}$$

is commutative, being $\overline{f}$ the morphism of $\mathcal{O}_X$-algebras induced by f. Isomorphisms are the obvious ones.

Now, let us define the *semistability* condition for honest singular principal G-bundles. Let $\lambda\colon\mathbb{G}_m\to G\times\ldots\times G$ be a one parameter subgroup of the product of l copies of G. Then, $\lambda=(\lambda_1,\ldots,\lambda_l)$, $\lambda_i\colon\mathbb{G}_m\to G$ being a one parameter subgroup for all i. For any i, the one parameter subgroup λ_i defines a weighted flag $(V_\bullet(\lambda_i),\underline{m}(\lambda_i))$ in V and a parabolic subgroup, $Q_G(\lambda)\subset G$, defined as the G-stabilizer of the flag. Define $U_i:=U\cap X_i$. *A reduction to the one parameter subgroup* λ is a tuple of sections $\beta:=(\beta_1,\ldots,\beta_l)$, being

$$\beta_i\colon U_i\to\mathcal{P}(\mathcal{F}|_{X_i},\lambda_i)/Q_G(\lambda_i).$$

This defines a *weighted flag of* $\mathcal{F}$ as follows. Any section

$$\beta_i\colon U_i\to\mathcal{P}(\mathcal{F}|_{X_i},\lambda_i)/Q_G(\lambda_i)\hookrightarrow\underline{\mathrm{Isom}}(V\otimes\mathcal{O}_{U_i},\mathcal{F}|_{U_i}^\vee)/Q_{Gl(V)}(\lambda_i)$$

induces a weighted filtration of locally free sheaves

$$\mathcal{F}|_{U_i\bullet}^\vee\equiv(0)\subset\mathcal{F}_1^i\subset\ldots\subset\mathcal{F}_{s(i)}^i\subset\mathcal{F}|_{U_i}^\vee$$
$$\underline{m}_i=(m_1^i,\ldots,m_{s(i)}^i):=\underline{m}(\lambda_i)$$

such that $\mathrm{rk}(\mathcal{F}_j^i)=\dim(V_j(\lambda_i))$, since the bundle $\underline{\mathrm{Isom}}(V\otimes\mathcal{O}_{U_i},\mathcal{F}|_{U_i}^\vee)/Q_{Gl(V)}(\lambda_i)$ is the bundle of flags of the same type as $(V_\bullet(\lambda_i),\underline{m}(\lambda_i))$. For any $t\in\{1,\ldots,s(i)\}$, consider the surjection

$$\mathcal{F}|_{U_i}\to\mathcal{F}_t^{i\vee}\to 0,$$

and denote by $\mathcal{G}_t^i$ its kernel. Note that there is a canonical isomorphism $\mathcal{F}|_{U_i}\simeq(p_i^*\mathcal{F}/T_i)|_{U_i}$ being T_i the torsion part of $p_i^*\mathcal{F}$. Therefore, we can consider $\mathcal{G}_t^i$ as a subsheaf of $(p_i^*\mathcal{F}/T_i)|_{U_i}=j_i^*(p_i^*\mathcal{F}/T_i)$. Pushing it forward to X_i we get

$$j_{i,*}\mathcal{G}_t^i\hookrightarrow j_{i*}j_i^*(p_i^*\mathcal{F}/T_i)\to p_i^*\mathcal{F}/T_i$$

and denote by $\mathcal{R}_t^i$ the image of this morphism. Then, we get weighted filtrations of torsion free sheaves

$$(p_i^*\mathcal{F}/T_i)_\bullet\equiv(0)\subset\mathcal{R}_1^i\subset\ldots\subset\mathcal{R}_{s(i)}^i\ .$$
$$\underline{m}_i=(m_{s(i)},\ldots,m_1^i)$$

These filtrations lead to a weighted filtration of

$$\bigoplus_{i=1}^{l} p_{i*}(p_i^*\mathcal{F}/T_i)_\bullet \equiv (0) \subset \mathcal{R}_1 \subset \ldots \subset \mathcal{R}_\theta \subset \bigoplus_{i=1}^{l} p_{i*}(p_i^*\mathcal{F}/T_i)$$
$$\underline{m} = (m_1, \ldots, m_\theta)$$

by the procedure described in Chapter 3, Section 3.3.1. From the exact sequence (see [57, Septième Partie, §1]),

$$0 \to \mathcal{F} \hookrightarrow \bigoplus_{i=1}^{l} p_{i*}(p_i^*\mathcal{F}/T_i) \to T \to 0$$

we finally get a weighted filtration of $\mathcal{F}$,

$$\mathcal{F}_{\beta\bullet} \equiv (0) \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_\theta \subset \mathcal{F}\ .$$
$$\underline{m}_\beta = (m_1, \ldots, m_\theta)$$

Thus, we define,

Definition 2.2.4. An honest singular principal G-bundle, $(\mathcal{F}, \tau)$, is *(semi)stable* if for every reduction $\beta = (\beta_1, \ldots, \beta_l)$ of $(\mathcal{F}, \tau)$ to a one parameter subgroup $\lambda = (\lambda_1, \ldots, \lambda_l)$ the inequality

$$L(\mathcal{F}_{\beta\bullet}, \underline{m}_\beta) := \sum_{i=1}^{\theta} m_i(\alpha P_{\mathcal{F}_i} - \alpha_i P_{\mathcal{F}})(\leq)0$$

holds true, being α the multiplicity of $\mathcal{F}$ and α_i the multiplicity of $\mathcal{F}_i$.

2.2.2 Some Results on Graded Algebras

We are going to prove that given a submodule of a commutative R-algebra generating it at a point $\mathfrak{p}_x \subset R$, then the submodule generates it over some open neighborhood of that point. This result will be crucial in order to get a satisfactory linearization of our moduli problem for principal bundles, and will allow us to construct this space going by the moduli space of tensor fields.

Lemma 2.2.5. *Let R be a commutative ring and $X = \mathrm{Spec}(R)$. Let B be a finitely generated commutative R-algebra and $A \subset B$ a sub-R-algebra. Let $x \in X$ be a point such that $A_x = B_x$, then there exists an affine open neighborhood $U \subset X$ of x such that $A_U = B_U$.*

Proof. Consider such a point x. From the equality $A_x = B_x$ it follows that $\dfrac{b}{s} \in A_x$ for any $b \in B$ and any $s \in R \setminus \mathfrak{p}_x$. Thus, we conclude that for all $b \in B$ there exists $s \in R \setminus \mathfrak{p}_x$ such that $sb \in A$. Since B is finitely generated we can consider $\{b_1, \ldots, b_l\}$ a finite set of generators. For any $i = 1, \ldots, l$ there exists $s_i \in R \setminus \mathfrak{p}_x$ such that $b_i s_i \in A$. Define $f := s_1 \cdot \ldots \cdot s_l$.

Let us show now that for any $b \in B$ there exists a natural number $n \in \mathbb{N}$ such that $f^n b \in A$. Let $b \in B$. Since B is finitely generated there there exists a polynomial $P \in R[x_1, \ldots, x_l]$ such that

$$b = P(b_1, \ldots, b_l) = \sum a_{j_1 \ldots j_l} b_1^{k_1} \ldots b_l^{k_l}.$$

Let $n := \deg(P)$. Then

$$f^n b = \sum f^{d_{i,\dots,l}} a_{j_1\dots j_l}(fb_1)^{k_1}\dots(fb_l)^{k_l}, \text{ where } d_{1,\dots,l} = n - k_1 - \dots - k_l.$$

Since all the terms in this sum belong to A it follows that $f^n b \in A$.

Consider now the inclusion $A_f \subset B_f$ and an element $\frac{b}{f^e} \in B_f$. From the last argument there is a natural number $n \in \mathbb{N}$ such that

$$\frac{b}{f^e} = \frac{bf^n}{f^{e+n}} \in A_f$$

from what we deduce that $A_f = B_f$. □

Theorem 2.2.6. *Let X be a nodal projective curve over k with nodes $z_1, \dots, z_\nu$, G a reductive group and $\rho\colon G \hookrightarrow \mathrm{SL}(V)$ a faithfull representation, V being a k-vector space of dimension p. Fix $r \in \mathbb{N}$. Then there exists a natural number $s = s(p, r, \nu)$ such that for any torsion free sheaf $\mathcal{F}$ of rank r on X, the graded $\mathcal{O}_X$-algebra $S^\bullet(V \otimes \mathcal{F})^G$ is generated by the submodule $\bigoplus_{i=0}^{s} S^i(V \otimes \mathcal{F})^G$.*

Proof. Let us denote $\mathcal{B} = S^\bullet(V \otimes \mathcal{F})^G$. We will find this natural number in three steps.

Step 1:

a) Let $\mathcal{F}$ be a torsion free sheaf on X of rank r and structure constants $a_1, \dots, a_\nu$ (see Definition 1.2.27). Let $x \in X$ be a point and consider the $\mathcal{O}_{X,x}$-algebra $\mathcal{B}_x = S^\bullet(V \otimes \mathcal{F}_x)^G$. Let s_x be the minimal natural number such that $\bigoplus_{i=1}^{s_x} S^i(V \otimes \mathcal{F}_x)^G$ contains a set of generators of $\mathcal{B}_x$.

b) For any point $x \in X$ we construct an affine open neighborhood in the following way. Fix $x \in X$ and consider the sub-$\mathcal{O}_X$-algebra $\mathcal{A} \subset \mathcal{B}$ generated by the sub-$\mathcal{O}_X$-module $\bigoplus_{i=0}^{s_x}(V \otimes \mathcal{F})^G$. Then obviously $\mathcal{A}_x = \mathcal{B}_x$. By Theorem 2.2.5 we deduce that there exists an affine open neighborhood U_x such that $\mathcal{A}|_{U_x} = \mathcal{B}|_{U_x}$.

c) We get in this way a covering of X, $\{U_x\}_{x \in X}$ by affine open subschemes and we can choose finitely many regular points $x_1, \dots, x_l \in X$ such that $X = U_{z_1} \cup \dots \cup U_{z_\nu} \cup U_{x_1} \cup \dots \cup U_{x_l}$. Let $s_{z_1}, \dots, s_{z_\nu}, s_{x_1}, \dots, s_{x_l}$ be the corresponding natural numbers defined in a) and define $s' = max(s_{z_1}, \dots, s_{z_\nu}, s_{x_1}, \dots, s_{x_l})$.

d) Note that the natural number s' constructed in c) does not depends on the finite open cover we have chosen, since $s_{x_1} = \dots = s_{x_l}$ for all regular points.

Step 2:

The natural number constructed in Step 1 depends apparently on $\mathcal{F}$, but it does not. Actually, it just depends on the structure constants $a_1, \dots, a_\nu$, the rank r and p. Suppose we have two torsion free sheaves $\mathcal{F}$, $\mathcal{G}$ of rank r and the same structure constants. Observe that the natural numbers defined in a) depends just on $\mathcal{F}_x$ and $\mathcal{G}_x$. But with the above assumption, $\mathcal{F}_x \simeq \mathcal{G}_x$ for all $x \in X$, and therefore they have the same structure constants, so $s' = s'(r, p, a_1, \dots, a_\nu)$.

Step 3:

Since there are only finitely many possibilities for the structure constants (once we fix the rank) we get in this way finitely many natural numbers s'. Consider the maximum of all of them and denote it by s. Then, s depends just on p, r and ν, and satisfies the properties of the statement.

□

Remark 2.2.7. 1) The last theorem also holds for non-connected cuurves.

2) Let $\pi\colon Y \to X$ be the normalization of X. If X has ν nodal points, then we have an exact sequence

$$0 \to \mathcal{O}_X \hookrightarrow \pi_*\mathcal{O}_Y \to \bigoplus_{i=1}^{\nu} k \to 0,$$

and therefore $g_X = g_Y + \nu$. Since both, g_Y and ν, are natural numbers, we deduce that for a fixed genus $g = g_X$ there are finitely many possibilities for the number of nodes ν. Fixing the genus $g = g_X$ and taking the maximum of the numbers $s = s(p, r, \nu)$ varying ν we get a number $s = s(p, r, g)$ which does not depend on the curve X. This will be the main result in solving the problem of the compactification of the universal moduli space of principal G-bundles over $\overline{\mathcal{M}_g}$.

2.2.3 Associated Tensor Field and δ-Semistability of Singular Principal G-Bundles

Let X be a projective possibly non connected nodal curve. Consider a singular principal G-bundle on X, $\tau\colon S^\bullet(V \otimes \mathcal{F})^G \to \mathcal{O}_X$. Let $s \in \mathbb{N}$ be as in the last section. Then $S^\bullet(V \otimes \mathcal{F})^G$ is generated by the submodule $\bigoplus_{i=0}^{s} S^i(V \otimes \mathcal{F})^G$. Let $\underline{d} \in \mathbb{N}^s$ be such that $\sum id_i = s!$. Then we have:

$$\bigotimes_{i=1}^{s} (V \otimes \mathcal{F})^{\otimes id_i} \to \bigotimes_{i=1}^{s} S^{d_i}(S^i(V \otimes \mathcal{F})) \to \bigotimes_{i=1}^{s} S^{d_i}(S^i(V \otimes \mathcal{F}))^G \to \mathcal{O}_X$$

Adding up these morphisms as $\underline{d} \in \mathbb{N}$ varies we find a tensor field

$$\phi_\tau\colon ((V \otimes \mathcal{F})^{\otimes s!})^{\oplus N} \to \mathcal{O}_X \tag{2.32}$$

We want to prove that the assignment $\tau \mapsto \phi_\tau$ is injective on isomorphism classes. We start with the following proposition,

Proposition 2.2.8. *Let $x \in X$ be a node, $A = \mathcal{O}_{X,x}$ the local ring and $\tau, \tau'\colon A^m \to A$ two non zero morphisms such that $S^d(\tau) = S^d(\tau')$ for some natural number $d \in \mathbb{N}$. Then there exists a d-th root of unity, ξ, such that $\tau' = \xi\tau'$.*

Proof. We have to consider two cases

a) x is the intersecting point of two components:
Let $\mathfrak{p}_{\eta_1}, \mathfrak{p}_{\eta_2}$ be the minimal prime ideals of A. Let $\{e_1, \ldots, e_m\}$ be the canonical basis of A^m. Then $S^d(\tau) = S^d(\tau')$ means that

$$\tau(e_{i_1}) \ldots \tau(e_{i_d}) = \tau'(e_{i_1}) \ldots \tau'(e_{i_d})\ \forall (i_1, \ldots, i_d) \text{ with } 1 \le i_j \le m. \tag{2.33}$$

In particular $\tau(e_j)^d = \tau'(e_j)^d\ \forall 1 \le j \le m$. Let j be such that $\tau(e_j)$ and $\tau'(e_j)$ are non zero. Since A is reduced this means that

$$\tau(e_j), \tau'(e_j) \notin \mathfrak{p}_{\eta_1} \cap \mathfrak{p}_{\eta_2} = (0)$$

so, in particular, there exists $l = 1, 2$ such that

$$\tau(e_j), \tau'(e_j) \notin \mathfrak{p}_{\eta_l},$$

and therefore $\tau(e_j), \tau'(e_j)$ are invertible in $A_{\mathfrak{p}_{\eta_l}} = \Sigma_l$. Consider the equality $\tau(e_j)^d = \tau'(e_j)^d$ in Σ_l. This exactly means that

$$(\frac{\tau(e_j)}{\tau'(e_j)})^d = 1 \text{ in } \Sigma_l,$$

so there exists a d-th root of unit ξ_j such that

$$\tau'(e_j) = \xi_j \tau(e_j) \text{ for all } j|\tau(e_j), \tau'(e_j) \neq 0.$$

From Equation (2.33), we deduce that ξ_j does not depend on j, so $\tau = \xi\tau'$.

b) x is not an intersecting point of two components:
This case follow from the last part of the above argument. □

Corollary 2.2.9. *Let $x \in X$ be a node, $A = \mathcal{O}_{X,x}$ the local ring, M be a finitely generated A-module and $\tau, \tau' \colon M \to A$ two non zero morphisms such that $S^d(\tau) = S^d(\tau')$ for some natural number $d \in \mathbb{N}$. Then there exists a d-th root of unity ξ such that $\tau' = \xi\tau'$.*

Proof. Let $\{m_1, \ldots, m_t\} \in M$ be such that $\{\overline{m}_1, \ldots, \overline{m}_t\} \in M \otimes_A k$ is a basis. Consider the canonical surjection

$$\begin{aligned} \pi \colon A^t &\to M \ . \\ e_J &\mapsto m_j \end{aligned}$$

Composing with τ and τ' we find

$$\tau \circ \pi, \tau' \circ \pi \colon A^t \to A.$$

By Theorem 2.2.11 there exists a d-th root of unity ξ such that $\tau' \circ \pi = \xi \tau \circ \pi$, that is

$$\tau'(m_j) = \xi \tau(m_j) \text{ for all } 1 \leq j \leq t$$

so $\tau' = \xi\tau$. □

Theorem 2.2.10. *Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module on X and*

$$\tau, \tau' \colon \mathcal{F} \to \mathcal{O}_X$$

non zero morphisms such that $S^d(\tau) = S^d(\tau')$ for some $d \in \mathbb{N}$. Then, there exists a dth root of unity, ξ_i, one for each connected component, such that

$$\tau'|_{X_i} = \xi^i \tau|_{X_i}.$$

Proof. For all $x \in X$ we have $\tau_x, \tau'_x \colon \mathcal{F}_x \to \mathcal{O}_{X,x}$ with $S^d(\tau_x) = S^d(\tau'_x)$. By Corollary 2.2.9 there is a d-th root of unit ξ_x such that $\tau'_x = \xi_x \tau_x$. We know that for any point x there is an open subset $x \in U \subset X$ such that $\mathcal{F}_U^\vee \subset \mathcal{F}_x^\vee$ (see Lemma 1.2.28) so for any point $x \in X$ there is an open subset U such that

$$\tau'_U = \xi_x \tau$$

from which we deduce that ξ_x do not depends on x and that $\tau'|_{X_i} = \xi^i \tau|_{X_i}$. □

Lemma 2.2.11. *Let $s \in \mathbb{N}$ and ξ_i a $\frac{s!}{i}$-th root of unity for $i = 1, \ldots s$ such that for any partition of $s!$, $d_1 + 2d_2 + \ldots sd_s = s!$, the following holds:*

$$1 = \prod_{i=1}^{s} \xi_i^{d_i}. \tag{2.34}$$

Then $\xi_1^j = \xi_j$ for all $j \in \{1, \ldots, s\}$.

Proof. Consider the complex representation for each ξ

$$\xi_j = \exp(2\pi i \frac{jk_j}{s!}),\ k_j \in \mathbb{N}.$$

Equations (2.34) are equivalent to

$$k_1 d_1 + 2k_2 d_2 + \ldots + sk_s d_s = 0 \bmod(s!) \text{ for all partitions } (d_1, \ldots, d_s), \tag{2.35}$$

which writen in matrix form become to

$$\begin{pmatrix} d_1^1 & 2d_2^1 & \ldots & sd_s^1 \\ d_1^2 & 2d_2^2 & \ldots & sd_s^2 \\ \vdots & \vdots & \vdots & \vdots \\ d_1^m & 2d_2^m & \ldots & sd_s^m \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \bmod(s!),$$

m being the total number of partitions. Note that we can easily find s linearly independent solutions for this linear system

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{s!}{2} \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \frac{s!}{s} \end{pmatrix}.$$

Therefore the general solution for the system is given by

$$\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_s \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ \frac{s!}{2} \\ \vdots \\ 0 \end{pmatrix} + \ldots + \lambda_s \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \frac{s!}{s} \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_1 + \lambda_2 \frac{s!}{2} \\ \vdots \\ \lambda_1 + \lambda_s \frac{s!}{s} \end{pmatrix}. \tag{2.36}$$

with $\lambda_1, \ldots, \lambda_s \in \mathbb{Z}/s!$. Note also that $\xi_1^j = \xi_j$ if and only if $jk_1 - jk_j = 0 \bmod(s!)$. But by equation (2.36), we know that

$$jk_1 - jk_j = jk_1 - j(\lambda_1 + \lambda_j \frac{s!}{j}) = -\lambda_j s! = 0 \bmod(s!).$$

$\square$

Theorem 2.2.12. *The assignment*

$$\left\{\begin{array}{c}\textit{isomorphism classes}\\ \textit{of singular principal}\\ G\textit{-bundles}\end{array}\right\} \to \left\{\begin{array}{c}\textit{isomorphism classes}\\ \textit{of tensor fields}\\ \textit{of type } (s!, N, \mathcal{O}_X)\end{array}\right\}$$

is injective.

Proof. Following [53] page 187, we have to show that, if $\phi_\tau = \phi_{\tau'}$ then $(\mathcal{F}, \tau) \simeq (\mathcal{F}, \tau')$. For $i > 0$ consider the degree i components of τ and τ', τ_i, $\tau'_i \colon S^i(V \otimes \mathcal{F})^G \to \mathcal{O}_X$. Observe that τ (resp. τ') is completely determined by $\oplus_{i=1}^s \tau_i$ (resp. $\oplus_{i=1}^s \tau_i$). Consider now:

$$\widehat{\tau_s} : \bigoplus_{\substack{(d_1,\dots,d_s)\\ \sum i d_i = s!}} (S^{d_1}((V\otimes\mathcal{F})^G) \otimes \ldots \otimes S^{d_s}(S^s(V\otimes\mathcal{F})^G)) \to \mathcal{O}_X,$$

the morphism induced by $\tau_1, \ldots, \tau_s$. Define in the same way $\widehat{\tau'_s}$. Note that from the surjectivity of the first two morphisms defining (2.32) it follows that if $\varphi_\tau = \varphi_{\tau'}$ then $\widehat{\tau_s} = \widehat{\tau'_s}$. This implies, in particular, that

$$S^{s!/i}(\tau_i) = S^{s!/i}(\tau'_i), \ \forall 0 < i \le s.$$

From Theorem 2.2.10 and Lemma 2.2.11 we deduce that for any connected component X_j $(j = 1, \ldots, t)$ there exists an $s!$-root of unity ξ_j such that:

$$\tau'_i|_{X_j} = (\xi_j)^i \tau_i|_{X_j}, \ i = 1, \ldots, s.$$

Denote by

$$u = \begin{pmatrix} \xi_1 & & \\ & \ddots & \\ & & \xi_t \end{pmatrix} : \mathcal{F} \simeq \mathcal{F},$$

the induced automorphism on $\mathcal{F}$. If we apply u to the singular principal G-bundle $(\mathcal{F}, \tau)$ we get a singular principal G-bundle $(\mathcal{F}, \tau'')$

$$\begin{tikzcd} S^\bullet(V\otimes\mathcal{F})^G \arrow[r, "S^\bullet(1\times u)"] \arrow[dr, dashed, "\tau''"] & S^\bullet(V\otimes\mathcal{F})^G \arrow[d, "\tau"] \\ & \mathcal{O}_X \end{tikzcd}$$

Clearly, $\tau''_i = \tau'_i$, $\forall 0 < i \le s$ on each connected component, and therefore on the whole curve. Since s is large enough we deduce that $\tau' = \tau''$ and hence $(\mathcal{F}, \tau) \simeq (\mathcal{F}, \tau'')$. □

Let $\tau \colon S^\bullet(V \otimes \mathcal{F})^G \to \mathcal{O}_X$ be a singular principal G-bundle. Let $s \in \mathbb{N}$ be as in Theorem 2.2.6, so $\bigoplus_{i=0}^s S^i(V \otimes \mathcal{F})^G$ contains a set of generators, and let $\phi_\tau \colon ((V \otimes \mathcal{F})^{\otimes s!})^{\oplus N} \to \mathcal{O}_X$ be the associated tensor field. Then,

Definition 2.2.13. Let $\delta \in \mathbb{Q}_{>0}$. A singular principal G-bundle $(\mathcal{F}, \tau)$ is δ-(semi)stable if its associated tensor field $(\mathcal{F}, \phi_\tau)$ is δ-(semi)stable.

2.2.4 The Parameter Space

The aim of this section is to prove the existence of a coarse projective moduli space for the moduli functor given by

$$\mathbf{SPB}(\rho)_P^{\delta\text{-(s)s}}(S) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{families of } \delta\text{-(semi)stable singular} \\ \text{principal } G\text{-bundles on } X \text{parametrized} \\ \text{by S with Hilbert polynomial } P \end{array} \right\}.$$

We will use the same strategy as in Section 2.1.4 for the construction of this moduli space. Therefore, we need to rigidify the moduli problem. Let $n \in \mathbb{N}$ and W a vector space of dimension $P(m)$. Consider the functor

$$^{\text{rig}}\mathbf{SPB}(\rho)_P^n(S) = \left\{ \begin{array}{c} \text{isomorphism classes of tuples } (\mathcal{F}_S, \tau_S, g_S) \text{ where} \\ (\mathcal{F}_S, \tau_S) \text{ is a family of singular principal } G\text{-bundles} \\ \text{parametrized by } S \text{ with Hilbert polynomial P and} \\ g_S \colon W \otimes \mathcal{O}_S \to \pi_{S*}\mathcal{F}_S(n) \text{ is a morphism such that the} \\ \text{induced morphism } W \otimes \pi^*\mathcal{O}_X \to \mathcal{F}_S(m) \text{ is surjective} \end{array} \right\}. \tag{2.37}$$

and let us show that there is a representative for it. Let $\mathcal{Q}$ be the Quot scheme of quotients

$$W \otimes \mathcal{O}_X(-n) \to \mathcal{F}$$

with Hilbert polynomial P. Consider the following morphism on $\mathcal{Q} \times X$

$$h \colon S^\bullet(V \otimes W \otimes \pi_X^*\mathcal{O}_X(-n)) \twoheadrightarrow S^\bullet(V \otimes \mathcal{F}_\mathcal{Q}) \twoheadrightarrow S^\bullet(V \otimes \mathcal{F}_\mathcal{Q})^G,$$

induced by the universal quotient and the Reynolds operator. Let $s \in \mathbb{N}$ be as in Theorem 2.2.6. Then

$$h(\bigoplus_{i=1}^{s} S^i(V \otimes W \otimes \pi_X \mathcal{O}_X(-n)))$$

contains a set of generators of $S^\bullet(V \otimes \mathcal{F}_\mathcal{Q})^G$. A morphism $k \colon \bigoplus_{i=1}^s S^i(V \otimes W \otimes \mathcal{O}_X(-n)) \to \mathcal{O}_X$ breaks into a family of morphisms

$$k^i \colon S^i(V \otimes W) \otimes \mathcal{O}_X(-in) \simeq S^i(V \otimes W \otimes \mathcal{O}_X(-n)) \to \mathcal{O}_X,$$

obtaining, therefore, a family of linear maps

$$k^i \colon S^i(V \otimes W) \to H^0(\mathcal{O}_X(in)).$$

Thus, any singular principal G-bundle $\tau \colon S^\bullet(V \otimes \mathcal{F})^G \to \mathcal{O}_X$ is completely determined by a point in the space

$$\mathcal{Q}^* := \mathcal{Q} \times \bigoplus_{i=1}^{s} \underline{\mathrm{Hom}}(S^i(V \otimes W), H^0(\mathcal{O}_X(in))).$$

We want to put a scheme structure on the locus given by the points $([q],[k])$ that comes from a morphism of algebras

$$S^\bullet(V \otimes \mathcal{F}_{\mathcal{Q}^0|_{[q]} \times X})^G \to \mathcal{O}_X.$$

On $\mathcal{Q}^* \times X$ there are universal morphisms

$$\varphi'^i \colon S^i(V \otimes W) \otimes \mathcal{O}_{\mathcal{Q}^* \times X} \to H^0(\mathcal{O}_X(in)) \otimes \mathcal{O}_{\mathcal{Q}^* \times X}$$

Consider the pullbacks of the evaluation maps to $\mathcal{Q}^* \times X$

$$H^0(\mathcal{O}_X(in)) \otimes \mathcal{O}_{\mathcal{Q}^* \times X} \to \pi_X^* \mathcal{O}_X(in).$$

Composing we get

$$\varphi^i \colon S^i(V \otimes W) \otimes \mathcal{O}_{\mathcal{Q}^* \times X} \to \pi_X^* \mathcal{O}_X(in).$$

Summing up we get

$$\varphi \colon V_{\mathcal{Q}^*} : = \bigoplus_{i=1}^{s} S^i(V \otimes W \otimes \pi_X^* \mathcal{O}_X(-n)) \to \mathcal{O}_{\mathcal{Q}^* \times X}.$$

Now φ gives a morphism

$$\tau'_{\mathcal{Q}^*} \colon S^\bullet(V_{\mathcal{Q}^*}) \to \mathcal{O}_{\mathcal{Q}^* \times X}.$$

Consider again the universal quotient $q_{\mathcal{Q}}$ and the following chain of surjections

$$\begin{tikzcd}
S^\bullet(V \otimes W \otimes \otimes \pi_X^* \mathcal{O}_X(-n)) \arrow[r, two heads, "S^\bullet(1 \otimes q_{\mathcal{Q}})"] & S^\bullet(V \otimes \pi_{\mathcal{Q}^* \times X}^* \mathcal{F}_{\mathcal{Q}^*}) \arrow[d, two heads, "\text{Reynolds}"] \\
S^\bullet V_{\mathcal{Q}^*} \arrow[u, two heads] & S^\bullet(V \otimes \pi_{\mathcal{Q}^* \times X}^* \mathcal{F}_{\mathcal{Q}^*})^G
\end{tikzcd}$$

Let us denote by β the composition of these morphisms and consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & Ker(\beta) \arrow[r, hook] \arrow[dr, dashed, "\tau'_{\mathcal{Q}^*}"] & S^\bullet V_{\mathcal{Q}^*} \arrow[r, "\beta"] \arrow[d, "\tau'_{\mathcal{Q}^*}"] & S^\bullet(V \otimes \pi_{\mathcal{Q}^* \times X}^* \mathcal{F}_{\mathcal{Q}^*})^G \arrow[r] & 0 \\
& & \mathcal{O}_{\mathcal{Q}^* \times X} & &
\end{tikzcd}$$

Define $\mathbb{D} = \{c = ([q],[h]) | \tau'_{\mathcal{Q}^*}|_c = 0\}$. This is a closed subscheme of $\mathcal{Q}^*$ over which $\tau'_{\mathcal{Q}^*}$ lifts to

$$\tau_{\mathbb{D}} \colon S^\bullet(V \otimes \pi_{\mathcal{Q}^* \times X}^* \mathcal{F}_{\mathcal{Q}^*})^G \to \mathcal{O}_{\mathcal{Q}^* \times X}.$$

To see this, note that $\mathbb{D} = \cap_{d \geq 0} \mathbb{D}^d$ with

$$\mathbb{D}^{d \geq 0} := \{c = ([q],[h]) | \tau'^d_{\mathcal{Q}^*}|_c : \mathrm{Ker}(\beta^d)|_c \to \mathcal{O}_X \text{ is trivial}\}$$

which are closed in $\mathcal{Q}^*$. Then, we have,

Theorem 2.2.14. *The functor* $^{rig}\mathbf{SPB}(\rho)_P^n$ *is represented by the scheme* $\mathbb{D}$.

Proof. Follows trivially from the construction of $\mathbb{D}$. □

2.2.5 Construction of the Moduli Space

Recall from Corollary 2.1.22 that the family of torsion free sheaves $\mathcal{F}$ which appears in a δ-(semi)stable tensor field is bounded. As a consequence, there is a natural number $n \in \mathbb{N}$ such that for $n \geq n_0$, $\mathcal{F}(n)$ is globally generated and $h^1(X, \mathcal{F}(n)) = 0$. Fix such a natural number n and consider the functors

$$^{\mathrm{rig}}\mathbf{Tensors}^n_{P,\mathcal{O}_X}(S) = \left\{ \begin{array}{c} \text{isomorphism classes of tuples } (\mathcal{F}_S, \phi_S, N, g_S) \\ \text{where } \mathcal{F}_S \text{ is a coherent sheaf with Hilbert} \\ \text{polynomial } P \text{ and } g_S \text{ is a morphism} \\ g_S \colon W \otimes \mathcal{O}_S \to \pi_{S*}\mathcal{F}_S(n) \text{ such that} \\ \text{its image genrates } \mathcal{F}_S \text{ and } \phi \text{ is a} \\ \text{morphism } \phi\colon ((V \otimes \mathcal{F}_S)^{\otimes s!})^{\oplus N} \to \pi_S^* N \end{array} \right\},$$

$$^{\mathrm{rig}}\mathbf{SPB}(\rho)^n_P(S) = \left\{ \begin{array}{c} \text{isomorphism classes of tuples } (\mathcal{F}_S, \tau_S, g_S) \text{ where} \\ (\mathcal{F}_S, \tau_S) \text{ is a family of singular principal } G\text{-bundles} \\ \text{parametrized by } S \text{ with Hilbert polynomial P and} \\ g_S \colon W \otimes \mathcal{O}_S \to \pi_{S*}\mathcal{F}_S(n) \text{ is a morphism such that the} \\ \text{induced morphism } W \otimes \pi^*\mathcal{O}_X \to \mathcal{F}_S(m) \text{ is surjective} \end{array} \right\}.$$

Note that the natural $\mathrm{GL}(W)$-action on the universal quotient $W \otimes \pi_X^*\mathcal{O}_X(-n) \twoheadrightarrow \mathcal{F}_{\mathcal{Q}^*}$ determines an action on the space $\mathbb{D}$,

$$\Gamma\colon \mathrm{GL}(W) \times \mathbb{D} \to \mathbb{D}.$$

We can view the $\mathrm{GL}(W)$-action as a $(\mathbb{C}^* \times \mathrm{SL}(W))$-action. Thus, we will construct the quotient of $\mathbb{D}$ by $\mathrm{GL}(W)$ in two steps, considering the actions of $\mathbb{C}^*$ and $\mathrm{SL}(W)$ separately. Consider the action of $\mathbb{C}^*$ on $^{\mathrm{rig}}\mathbf{SPB}(\rho)^n_P$. By Theorem 2.2.12 and Definition 2.1.2, there is a $\mathbb{C}^*$-invariant natural transformation

$$^{\mathrm{rig}}\mathbf{SPB}(\rho)^n_P \hookrightarrow {}^{\mathrm{rig}}\mathbf{Tensors}^n_{P,\mathcal{O}_X}.$$

Moreover, the morphism induced between the representatives is a $\mathrm{SL}(W)$-equivariant injective and proper morphism

$$\beta\colon \mathbb{D}/\!\!/\mathbb{C}^* \hookrightarrow Z'_{n,\mathcal{O}_X,P}.$$

Let $\mathbb{D}_0 \subset \mathbb{D}$ be the open subscheme consisting of points such that $W \to H^0(\mathcal{F}(n))$ is an isomorphism and $\mathcal{F}$ is torsion free. Then we have,

Proposition 2.2.15. (Glueing Property) *Let S be a scheme of finite type over $\mathbb{C}$ and $s_1, s_2 : S \to \mathbb{D}_0$ two morphisms such that the pullbacks of $(\mathcal{F}_{\mathbb{D}}, \tau_{\mathbb{D}})$ via $s_1 \times \mathrm{id}_X$ and $s_2 \times \mathrm{id}_X$ are isomorphic. Then there exists an étalé covering $c : T \to S$ and a morphism $h : T \to \mathrm{SL}(W)$ such the triangle*

$$\begin{tikzcd} \mathrm{SL}(W) \times \mathbb{D}_0 \arrow[rr, "\Gamma"] & & \mathbb{D}_0 \\ & T \arrow[ul, "h\times(s_1\circ c)"] \arrow[ur, "s_2\circ c"'] & \end{tikzcd}$$

is commutative.

Proof. Morphisms $s_i \times \mathrm{id}_X : S \times X \to \mathbb{D} \times X$ provide us with two families, $(\mathcal{F}^1_S, \tau^1_S, g^1_S)$ and $(\mathcal{F}^2_S, \tau^2_S, g^2_S)$, via pullback of the universal family, such that there is an isomorphism $\Phi : \mathcal{F}^1_S \simeq \mathcal{F}^2_S$ making the diagram commutative

$$\begin{tikzcd} S^\bullet(V \otimes \pi^*_{S\times X}\mathcal{F}^1_S)^G \arrow[rr, "\mathrm{id}\otimes\pi^*_{S\times X}\Phi"] \arrow[dr, "\tau^1_S"'] & & S^\bullet(V \otimes \pi^*_{S\times X}\mathcal{F}^2_S)^G \arrow[dl, "\tau^2_S"] \\ & \mathcal{O}_{S\times X} & \end{tikzcd}$$

Note also that there is an isomorphism,

$$V \otimes \mathcal{O}_S \xrightarrow{g^1_S} \pi_{S*}(\mathcal{F}^1_S \otimes \pi^*_X \mathcal{O}_X(n)) \longrightarrow$$

$$\xrightarrow{\pi_{S*}(\Phi\otimes\mathrm{id}_{\pi^*_X\mathcal{O}_X(n)})} \pi_{S*}(\mathcal{F}^2_S \otimes \pi^*_X \mathcal{O}_X(n)) \xrightarrow{g^2_S} V \otimes \mathcal{O}_S.$$

which determines a morphism $h' : S \to \mathrm{GL}(V)$. Let $\det(h') = \det \circ h' : S \to \mathbb{G}_m$ be the determinant morphism, Now we define T by means of the following cartesian product

$$\begin{tikzcd} T := S \times_{\mathbb{G}_m} \mathbb{G}_m \arrow[r] \arrow[d, "c"] \arrow[dr, "\Delta"] & \mathbb{G}_m \arrow[d, "\chi_p"] & z \arrow[d, mapsto] \\ S \arrow[r, "\det(h')"] & \mathbb{G}_m & z^p \end{tikzcd}$$

Obviously $c : T \to S$ is a Galois covering (therefore étalé) of degree p. Denote $\Delta_e : T \to \mathbb{G}_m$ the morphism $\chi_e \circ \Delta$. Now, the T-point of $\mathrm{SL}(V)$ is obtained from h' by composing with c and dividing by the determinant, i.e

$$h : T \xrightarrow{\Delta_{-1}\times(h'\circ c)} \mathbb{G}_m \times \mathrm{GL}(V) \xrightarrow{\cdot} \mathrm{SL}(V).$$

All of this together determines a T-point of $\mathbb{D}$

$$T \xrightarrow{h\times(s_1\circ c)} \mathrm{SL}(V) \times \mathbb{D} \xrightarrow{\Gamma} \mathbb{D},$$

which corresponds to the family of singular principal G-bundles on T obtained by pulling back to T (via c) the family on S given by $g \bullet (\mathcal{F}^1_S, \tau^1_S) = (\mathcal{F}'^1_S, \tau'^1_S)$. Here, $\mathcal{F}'^1_S$ is the quotient

$$\widehat{q}^1_S : V \otimes \pi^*_X \mathcal{O}_X(-n) \xrightarrow{g^{-1}\otimes\mathrm{id}_{\pi^*_X\mathcal{O}_X(-n)}} V \otimes \pi^*_X \mathcal{O}_X(-n) \longrightarrow (c \times \mathrm{id}_X)^*\mathcal{F}^1_S,$$

and τ'^1_S is the morphism of algebras obtained by composing. Finally, the isomorphism

$$\widehat{\Phi} := \Delta \cdot ((c \times \mathrm{id}_X)^*\Phi^{-1}) : (c \times \mathrm{id}_X)^*\mathcal{F}^2_S \longrightarrow (c \times \mathrm{id}_X)^*\mathcal{F}^1_S$$

gives an equivalence with the family $(c \times \mathrm{id}_X)^*(\mathcal{F}^2_S, \tau^2_S)$. $\square$

Proposition 2.2.16. (Local Universal Property) *Let S be a scheme of finite type over $\mathbb{C}$ and $(\mathcal{F}_S,\tau_S)$ a family of δ-(semi)stable singular principal G-bundles parametrized by S. Then there exists an open covering S_i, $i\in I$ of S and morphisms $\beta_i\colon S_i\to\mathbb{D}$, $i\in I$ such that the restriction of the family $(\mathcal{F}_S,\tau_S)$ to $S_i\times X$ is equivalent to the pullback of $(\mathcal{F}_{\mathbb{D}},\tau_{\mathbb{D}})$ via $\beta_i\times\mathrm{id}_X$ for all $i\in I$.*

Proof. Since n is large enough so that $h^1(X_s,\mathcal{F}_{S,s}\otimes\mathcal{O}_{X_s})=0$ for all $s\in S$ we deduce that $\pi_{S*}(\mathcal{F}_{S,s}\otimes\pi_X^*\mathcal{O}_X(n))$ is locally free. Then, any finite covering $\{S_i\}$ of S trivializing it satisfies the statement of the proposition (see [50, Proposition 2.8]). $\square$

Consider the linearized invertible sheaf $\mathcal{O}_{Z'_{n,\mathcal{O}_X,P}}(n_1,n_2)$ given in Section 2.1.5 and let $\mathcal{L}:=\beta^*\mathcal{O}_{Z'_{n,\mathcal{O}_X,P}}(n_1,n_2)$.

Proposition 2.2.17. *In the above situation, we have:*

1) All $\mathcal{L}$-semistable points lie in $\mathbb{D}_0$.

2) A point $y\in\mathbb{D}_0$ is $\mathcal{L}$-(semi)stable if and only if the restriction of the universal singular principal G-bundle to $\{y\}\times X$ is δ-(semi)stable.

Proof. Follows from the construction of the morphism β, Definition 2.2.13 and Theorem 2.1.41. $\square$

We finally have,

Theorem 2.2.18. *There is a projective scheme $\mathrm{SPB}(\rho)_P^{\delta\text{-}(s)s}$ and an open subscheme $\mathrm{SPB}(\rho)_P^{\delta\text{-}s}\subset\mathrm{SPB}(\rho)_P^{\delta\text{-}(s)s}$ together with a natural tranformation*

$$\alpha^{(s)s}\colon\mathbf{SPB}(\rho)_P^{\delta\text{-}(s)s}\to h_{\mathrm{SPB}(\rho)_P^{\delta\text{-}(s)s}}$$

with the following propoerties:

1) For every scheme $\mathcal{N}$ and every natural transformation $\alpha'\colon\mathbf{SPB}(\rho)_P^{\delta\text{-}(s)s}\to h_{\mathcal{N}}$, there exists a unique morphism $\varphi\colon\mathrm{SPB}(\rho)_P^{\delta\text{-}(s)s}\to\mathcal{N}$ with $\alpha'=h(\varphi)\circ\alpha^{(s)s}$.

2) The scheme $\mathrm{SPB}(\rho)_P^{\delta\text{-}s}$ is a coarse moduli space for the functor $\mathbf{SPB}_P(\rho)^{\delta\text{-}s}$

Proof. By Proposition 2.2.15, Proposition 2.2.16 and Proposition 2.2.17, the quotients $\mathrm{SPB}(\rho)_P^{\delta\text{-(s)s}}:=\mathbb{D}^{(s)s}/\!\!/\mathrm{GL}(V)$ exist, $\mathbb{D}^{ss}/\!\!/\mathrm{GL}(V)$ is a projective scheme and satisfies 1) and 2). $\square$

Chapter 3

Generalised Parabolic Structures on Smooth Curves

The goal of this chapter is to construct the moduli space of singular principal G-bundles with generalized parabolic structure over a smooth projective (possibly) non-connected curve over an algebraically closed field k of characteristic zero. Following [52], we first construct the moduli space of tensor fields with generalized parabolic structure and then we construct our moduli space by associating a tensor field to any singular principal G-bundle. This is the same strategy we have followed in Chapter 2. Since Theorem 2.2.12 holds in this case, this part will not imply extra difficulties. The construction of the moduli space of tensor fields is done following the same steps as in [52], and adapting the calculations to our situation. We put a special emphasis in the properties for very large values of the semistability parameters. Since the semistability function does not split as a sum of the different semistability functions on each connected component, we need a different geometric interpretation of such semistability function from that of [52].

Let X be a nodal curve with nodes $x_1, \ldots, x_\nu$ and $\pi\colon Y \to X$ its normalization. We fix an ample line bundle $\mathcal{O}_X(1)$ on X and we denote $\mathcal{O}_Y(1)$ the ample line bundle obtained by pulling back $\mathcal{O}_X(1)$ to $Y = \coprod Y_i$. We denote $h = \deg(\mathcal{O}_Y(1))$. We denote by y_1^i, y_2^i the points in the preimage of the ith nodal point x_i. We denote also by $D_i = y_1^i + y_2^i$ the corresponding divisor on Y and by $D = \sum D_i$ the total divisor.

3.1 Moduli Space of Tensor Fields with Generalised Parabolic Structures

The main result of this section is Theorem 3.1.28. In Subsection 3.1.5 and Subsection 3.1.6, we find the right Gieseker space and its polarization, respectively, that allow us to compare $(\underline{\kappa}, \delta)$-semistability and GIT semistability in Theorem 3.1.24. Given a coherent sheaf, $\mathcal{E}$, on Y we denote by $\alpha := \alpha(\mathcal{E})$ its multiplicity (see Chapter 1) and given a subsheaf, $\mathcal{E}' \subset \mathcal{E}$, we use the notation α' for $\alpha(\mathcal{E}')$.

3.1.1 Generalized Parabolic Structures on Tensor Fields

Definition 3.1.1. A generalized parabolic bundle of rank r on the smooth curve Y is a tuple $(\mathcal{E}, q_1, \dots, q_\nu)$ where $\mathcal{E}$ is a locally free sheaf of rank r (that is, uniform multirank equal to r) and q_i is a quotient of dimension r

$$q_i \colon \Gamma(D_i, \mathcal{E}|_{D_i}) = \mathcal{E}(y_1^i) \oplus \mathcal{E}(y_2^i) \to R_i \to 0,$$

$\mathcal{E}(y_j^i)$ being the fibre of $\mathcal{E}$ over y_j^i.

Denote by $R := \oplus R_i$ the total vector space. Since the supports of the divisors D_i are disjoint we get the equality $\Gamma(D, \mathcal{E}|_D) = \bigoplus \Gamma(D_i, \mathcal{E}|_{D_i})$. From this, we can form the quotient

$$q \colon = \oplus q_i \colon \Gamma(D, \mathcal{E}|_D) \to R \to 0.$$

Definition 3.1.2. Let $(\mathcal{E}, q_i, \dots, q_\nu)$ and $(\mathcal{E}', q_1', \dots, q_\nu')$ be generalized parabolic bundles on Y. A homomorphism between them is a tuple $(f, u_1, \dots, u_\nu)$ where $f \colon \mathcal{E} \to \mathcal{E}'$ is a homomorphism of $\mathcal{O}_X$-modules and $u_i \colon R_i \to R_i'$ is a homomorphism of vector spaces such that the diagram commutes

$$\begin{array}{ccc} \mathcal{E}(y_1^i) \oplus \mathcal{E}(y_2^i) & \xrightarrow{f(y_1^i) \oplus f(y_2^i)} & \mathcal{E}'(y_1^i) \oplus \mathcal{E}'(y_2^i) \\ \downarrow{\scriptstyle q_i} & & \downarrow{\scriptstyle q_i'} \\ R_i & \xrightarrow{u_i} & R_i' \end{array} \quad .$$

Notation. In order to abreviate the notation we will use the symbol $\underline{q}$ to refer to the tuple $(q_1, \dots, q_\nu)$.

Definition 3.1.3. Let $\underline{\kappa} = (\kappa_1, \dots, \kappa_\nu)$ be a vector of rational numbers with $\kappa_i \in (0,1) \cap \mathbb{Q}$. Let $(\mathcal{E}, \underline{q})$ be a generalized parabolic bundle. We define the $\underline{\kappa}$-parabolic degree for any subsheaf $\mathcal{F} \subseteq \mathcal{E}$ as

$$\underline{\kappa}\text{-pardeg}(\mathcal{F}) = \deg(\mathcal{F}) - \sum_{i=1}^{\nu} \kappa_i \dim q_i(\mathcal{F}(y_1^i) \oplus \mathcal{F}(y_2^i)). \tag{3.1}$$

Now we will introduce the notion of a family of these objects. Let us introduce some notation. If S is a scheme we will denote by S_{D_i} the subscheme $S \times \{y_1^i, y_2^i\}$ of $S \times Y$ and by S_{x_i} the subscheme $S \times \{x_i\}$ of $S \times X$.

Definition 3.1.4. Let S be a scheme. A family of generalized parabolic bundles parametrized by S is a tuple $(\mathcal{E}_S, q_{S1}, \dots, q_{S\nu})$ which consists of a family of locally free sheaves $\mathcal{E}_S$ on Y parametrized by S of rank r and locally free quotients of rank r on S_{x_i}

$$q_{Si} \colon \pi_{Si*}(\mathcal{E}_S|_{S_{D_i}}) \to R_i \to 0, \tag{3.2}$$

$\pi_{Si} \colon S_{D_i} \to S_{x_i}$ being the natural projection.

Definition 3.1.5. Let S be a scheme and $(\mathcal{E}_S, \underline{q}_S)$, $(\mathcal{E}_S', \underline{q}'_S)$ generalized parabolic vector bundles. A morphism between them is a pair $(f, \underline{u})$ whith $f \colon \mathcal{E}_S \to \mathcal{E}_S'$ and

$\underline{u} = (u_1, \ldots, u_\nu)$, where u_i is a morphism $u_i \colon R_i \to R'_i$ making the diagram commute,

$$\begin{tikzcd}
\pi_{Si*}(\mathcal{E}_S|_{S_{D_i}}) \arrow[r, "\pi_{Si*}(f))"] \arrow[d, "q_{Si}"] & \pi_{Si*}(\mathcal{E}'_S|S_{D_i}) \arrow[d, "q_{Si}"] \\
R_i \arrow[r, "u_i"] & R'_i \; .
\end{tikzcd}$$

Definition 3.1.6. Fix non negative integers a, b, c and an invertible sheaf $\mathcal{L}$ on Y. A generalized parabolic tensor field is a triple $(\mathcal{E}, \underline{q}, \phi)$ such that $(\mathcal{E}, \underline{q})$ is a generalized parabolic vector bundle and $\phi \colon (\mathcal{E}^{\otimes a})^{\oplus b} \to \det(\mathcal{E})^{\otimes c} \otimes \mathcal{L}$ is a non-zero morphism.

Definition 3.1.7. We fix numbers $\delta \in \mathbb{Q}_{>0}$ and $\kappa_i \in (0,1) \cap \mathbb{Q}$, $i = 1, \ldots, \nu$. Denote $\underline{\kappa} = (\kappa_1, \ldots, \kappa_\nu)$. A generalized parabolic tensor field $(\mathcal{E}, \underline{q}, \phi)$ is $(\underline{\kappa}, \delta)$-(semi)stable if for every weigted filtration $(\mathcal{E}_\bullet, \underline{m})$ of $\mathcal{E}$, the inequality

$$P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m}) + \delta\mu(\mathcal{E}_\bullet, \underline{m}, \phi)(\geq)0 \tag{3.3}$$

holds, being $P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m}) = \sum_{i=1}^{s} m_i(\underline{\kappa}\text{-pardeg}(\mathcal{E})\alpha_i - \underline{\kappa}\text{-pardeg}(\mathcal{E}_i)\alpha)$ and $\alpha,\ \alpha_i$ the multiplicities of $\mathcal{E}$ and $\mathcal{E}_i$ respectively. Here, $\mu(\mathcal{E}_\bullet, \underline{m}, \phi)$ is defined as in Chapter 2: for each $\mathcal{E}_i$ denote by α_i its multiplicity and just by α the multiplicity of $\mathcal{E}$. Define the vector

$$\Gamma = \sum_{1}^{t} m_i \Gamma^{(\alpha_i)},$$

where $\Gamma^{(l)} = (\overbrace{l-\alpha, \ldots, l-\alpha}^{l}, \overbrace{l, \ldots, l}^{\alpha - l})$. Let us denote by J the set

$$J = \{ \text{ multi-indices } I = (i_1, \ldots, i_a) | I_j \in \{1, \ldots, t+1\}\}.$$

Then we define

$$\mu(\phi, \mathcal{E}_\bullet, \underline{m}) = -\min_{I \in J}\{\Gamma_{\alpha_{i_1}} + \ldots + \Gamma_{\alpha_{i_a}} \mid \phi|_{(\mathcal{E}_{i_1} \otimes \ldots \otimes \mathcal{E}_{i_a})^{\oplus b}} \neq 0\}. \tag{3.4}$$

Lemma 3.1.8. *There is an integer A, depending only on the input data (P, a, b, c and $\mathcal{L}$), such that it is enough to check the semistability condition (3.3) for weighted filtrations with $m_i < A$.*

Proof. The same observation we have made in the proof of Lemma 2.1.10 is valid in this case. Now the lemma follows from [17, Lemma 1.4] changing ranks by multiplicities, since the function $P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m})$ is linear on the cone $\mathcal{C} = \{(\gamma_1, \ldots, \gamma_\alpha) : \gamma_1 \leq \ldots \leq \gamma_\alpha\} \in \mathbb{Z}^\alpha$, being α the leading coefficient of the polynomial P. □

Definition 3.1.9. Let S be a scheme and fix non negative integers a, b, c and $\mathcal{L}$ an invertible sheaf on Y. A family of generalized parabolic tensor fields is a quadruple $(\mathcal{E}_S, \underline{q}_S, \mathcal{N}_S, \phi_S)$ where $(\mathcal{E}_S, \underline{q}_S)$ is a family of generalized parabolic bundles, $\mathcal{N}_S$ is an invertible sheaf on S, and $\phi_S \colon (\mathcal{E}_S^{\otimes a})^{\oplus b} \to \det(\mathcal{E}_S)^{\otimes c} \otimes \pi_Y^* \mathcal{L} \otimes \pi_S^* \mathcal{N}_S$ is a morphism of vector bundles on $S \times Y$ such that $\phi_S|_{\{s\} \times Y}$ is non-zero for all $s \in S$.

Homomorphisms (hence isomorphisms) between quadruples are defined in the obvious way and $(\underline{\kappa},\delta)$-(semi)stable families are families which are $(\underline{\kappa},\delta)$-(semi)stable fiberwise.

The aim of this section is to solve the moduli problem given by the following functor

$$\mathbf{ParTensors}^{(\underline{k},\delta)-(s)s}_{P,\mathcal{L},\{D_i\}}(S) = \left\{ \begin{array}{c} \text{isomorphism classes of families} \\ \text{of } (\underline{\kappa},\delta)\text{-(semi)stable generalized parabolic} \\ \text{tensor fields } (\mathcal{E}_S, \underline{q}_S, \mathcal{N}_S, \phi_S) \text{ parametrized} \\ \text{by } S \text{ with Hilbert polynomial } P \end{array} \right\}. \tag{3.5}$$

3.1.2 Boundedness

Let $r,d\in\mathbb{N}$ and let us denote by $E_{d,r}$ the family of locally free sheaves on Y of uniform multirank r and degree d, and let $h=\deg(\mathcal{O}_Y(1))$. Recall that a family of sheaves $E\subset E_{d,r}$ on Y is bounded if and only if there is a natural number n_0 such that for all $n\geq n_0$ and all locally free sheaves $\mathcal{E}\in E$, $h^1(Y,\mathcal{E}(n))=0$ and $\mathcal{E}(n)$ is globally generated (see Theorem 2.1.12).

Lemma 3.1.10. *A family $E\subset E_{d,r}$ of sheaves on Y is bounded if and only if there exists a constant C such that for each $\mathcal{E}\in E$ the inequality*

$$\begin{aligned}\mu_{\min}(\mathcal{E}) &= \min\Big\{\mu(\mathcal{Q})=\frac{\deg(\mathcal{Q})}{\alpha(\mathcal{Q})}\mid\ \mathcal{E}\twoheadrightarrow\mathcal{Q}\ \textit{locally free}\ \Big\}\\ &\geq \mu(\mathcal{E})+C\end{aligned} \tag{3.6}$$

holds true.

Remark 3.1.11. Before proving the lemma let us remark that the above condition is equivalent to de condition: there exists a constant C' such that for each $\mathcal{E}\in E$ the inequality

$$\begin{aligned}\mu_{\max}(\mathcal{E}) &= \max\Big\{\mu(\mathcal{F})=\frac{\deg(\mathcal{F})}{\alpha(\mathcal{F})}|\mathcal{F}\subset\mathcal{E}\ \text{locally free}\Big\}\\ &\leq \mu(\mathcal{E})+C'.\end{aligned}$$

holds true.

Proof. We follow [53, Proposition 2.2.3.7] closely. Assume E is bounded and suppose the lemma is false. Let $n_0\in\mathbb{N}$ be such that $h^1(Y,\mathcal{E}(n_0))=0$ for every $\mathcal{E}\in E$. Let $l_0:=\#\{i|g_{Y_i}=0\}$, the number of rational components, and let Y' be the subcurve consisting on the components of genus equal or greater than 2. Then, for the constant

$$C:=-\frac{d}{\alpha}-n_0-\frac{r}{h}l_0-\frac{1}{\alpha}\chi(Y',\mathcal{O}_{Y'}),$$

there is a locally free sheaf $\mathcal{E}\in E$ and a locally free quotient, $q\colon\mathcal{E}\twoheadrightarrow\mathcal{Q}$, with $\mathrm{rk}(\mathcal{Q})=r'$ and degree d', such that $h\mu(\mathcal{Q})=\dfrac{d'}{r'}<h(\mu(\mathcal{E})+C)=-hn_0-rl_0-\dfrac{1}{r}\chi(Y',\mathcal{O}_{Y'})$. Denote

by $r'_i = \mathrm{rk}(\mathcal{Q}_i)$. Then we have

$$\begin{aligned}
0 = \frac{h^1(Y,\mathcal{E}(n_0))}{r'} = \frac{h^0(\mathcal{E}^\vee(-n_0)\otimes\omega_Y)}{r'} &\geq \frac{h^0(\mathcal{Q}^\vee(-n_0)\otimes\omega_Y)}{r'} \geq \\
&\geq \frac{\chi(\mathcal{Q}^\vee(-n_0)\otimes\omega_Y)}{r'} = \frac{\sum_{i=1}^l \chi(\mathcal{Q}_i^\vee(-n_0)\otimes\omega_{Y_i})}{r'} = \\
&= \frac{\sum_{i=1}^l -r'_i\chi(Y_i,\mathcal{O}_{Y_i}) - \deg(\mathcal{Q}_i) - r'_i h_i n_0}{r'} \geq \\
&\geq -\frac{\sum_{i=1}^l r'_i\chi(Y_i,\mathcal{O}_{Y_i})}{r'} - h\mu(\mathcal{Q}) - hn_0 \geq \\
&\geq -\sum_{i=1}^l \frac{r'_i}{r'}\chi(Y_i,\mathcal{O}_{Y_i}) - h\mu(\mathcal{Q}) - hn_0.
\end{aligned}$$

We have to consider different cases to analyze the term $-\sum_{i=1}^l \frac{r'_i}{r'}\chi(Y_i,\mathcal{O}_{Y_i})$,

$$\begin{aligned}
&g_{Y_i} = 0 \Rightarrow \chi(Y_i,\mathcal{O}_{Y_i}) = 1 \Rightarrow -\frac{r'_i}{r'}\chi(Y_i,\mathcal{O}_{Y_i}) = -\frac{r'_i}{r'} \geq -\frac{\max_j\{r'_j\}}{\min_j\{r'_j\}} \geq -r, \\
&g_{Y_i} = 1 \Rightarrow \chi(Y_i,\mathcal{O}_{Y_i}) = 0 \Rightarrow -\frac{r'_i}{r'}\chi(Y_i,\mathcal{O}_{Y_i}) = 0, \\
&g_{Y_i} \geq 2 \Rightarrow \chi(Y_i,\mathcal{O}_{Y_i}) < 0 \Rightarrow -\frac{r'_i}{r'}\chi(Y_i,\mathcal{O}_{Y_i}) \geq -\frac{1}{r}\chi(Y_i,\mathcal{O}_{Y_i}),
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
0 = \frac{h^1(Y,\mathcal{E}(n_0))}{r'} &\geq -\sum_{i=1}^l \frac{r'_i}{r'}\chi(Y_i,\mathcal{O}_{Y_i}) - h\mu(\mathcal{Q}) - hn_0 \geq \\
&\geq -rl_0 - \frac{1}{r}\chi(Y',\mathcal{O}_{Y'}) - h\mu(Q) - hn_0 > 0,
\end{aligned}$$

which is a contradiction.
Assume now that there exists a constant C such that (3.6) holds true. Let $\mathcal{E} \in E$, and let n be a natural number such that $h^1(Y,\mathcal{E}(n)) \neq 0$. From Serre duality theorem we know that $h^1(Y,\mathcal{E}(n)) = \dim_k \mathrm{Hom}_Y(\mathcal{E}(n),\omega_Y)$. Therefore, there is a non trivial morphism $\varphi\colon \mathcal{E}(n) \to \omega_Y$, whose image is denoted by $\mathcal{L}$. Let us denote by $\mathcal{F} \subset \mathcal{E}$ the kernel of the projection $\varphi\colon \mathcal{E} \twoheadrightarrow \mathcal{L}$, which is a locally free sheaf. Then we have

$$\begin{aligned}
\alpha n + d = \deg(\mathcal{E}(n)) &= \deg(\mathcal{F}) + \deg(\mathcal{L}) = \alpha(\mathcal{F})\mu(\mathcal{F}) + \deg(\mathcal{L}) \leq \\
&\leq \alpha(\mathcal{F})\frac{d}{\alpha} + \alpha(\mathcal{F})C + \deg(\omega_Y) \leq \\
&\leq (\alpha-1)\frac{d}{\alpha} + (\alpha-1)n + (\alpha-1)C - 2\chi(Y,\mathcal{O}_Y).
\end{aligned}$$

Therefore, for $n > (-\frac{d}{\alpha} + (\alpha-1)C - 2\chi(Y,\mathcal{O}_Y))$, the conclusion $h^1(Y,\mathcal{E}(n)) = 0$ holds true. This, in particular, implies that for any $n > (-\frac{d}{\alpha} + (\alpha-1)C - 2\chi(Y,\mathcal{O}_Y)) + 1$, every $\mathcal{E} \in E$ verifies that $\mathcal{E}(n)$ is globally generated. This is because, given $n \in \mathbb{N}$ as above, $h^1(Y, \mathcal{E}(n)(-y)) = 0$, and the long exact sequence in cohomology deduced from the short exact sequence

$$0 \to \mathcal{E}(n)(-y) \hookrightarrow \mathcal{E}(n) \to \mathcal{E}(n)|_y \to 0$$

is truncated in $H^1(Y, \mathcal{E}(n)(-y))$, that is, there is a surjection

$$H^0(Y, \mathcal{E}(n)) \to \mathcal{E}(n)|_y \to 0$$

Therefore, E is bounded. □

Lemma 3.1.12. *Let* $(\mathcal{E}_\bullet, \underline{m})$ *be a weighted filtration, with*

$$\mathcal{E}_\bullet \equiv (0) \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_s \subset \mathcal{E}_{s+1} = \mathcal{E}.$$

Consider a partition of the multitindex $(1, 2, \ldots, s)$

$$I := (1, 2, \ldots, s) = I_1 \sqcup I_2,$$

let us say $I_1 = (i_1, \ldots, i_t)$ *and* $I_2 = (k_1, \ldots, k_{s-t})$*. Then*

$$1)\ (\sum_{i=1}^{s} m_i) a(\alpha - 1) \geq \mu(\mathcal{E}_\bullet, \underline{m}, \phi) \geq -(\sum_{i=1}^{s} m_i) a(\alpha - 1),$$

$$2)\ \mu(\mathcal{E}_\bullet, \underline{m}, \phi) \geq \mu(\mathcal{E}^1_\bullet, \underline{m_1}, \phi) - (\sum_{i=1}^{s-t} m_{2,i}) a(\alpha - 1),$$

being $\mathcal{E}^1_j = \mathcal{E}_{i_j}$ *and* $\mathcal{E}^2_j = \mathcal{E}_{k_j}$*.*

Proof. For the sake of clarity, we first introduce some notation that will be used later. We denote $I'_1 = (1, \ldots, t)$ and $I'_2 = (1, \ldots, s-t)$, and by ϕ_1 (resp. ϕ_2) the bijection between I_1 and I'_1 (resp. I_2 and I'_2) given by $\phi_1(i_j) = j$ (resp. $\phi_2(k_j) = j$).
1) Being $\nu_i(J) \leq a$, we have

$$\begin{aligned}
\mu(\mathcal{E}_\bullet, \underline{m}, \phi) &= \sum_{i=1}^{s} m_i(\nu_i(J)\alpha - a\alpha_i) \leq \\
&\leq \sum_{i=1}^{s} a m_i(\alpha - \alpha_i) \leq \\
&\leq \sum_{i=1}^{s} a m_i(\alpha - 1) = \\
&= (\sum_{i=1}^{s} m_i) a(\alpha - 1).
\end{aligned}$$

On the other hand, being $\nu_i(J) \geq 0$ and $\alpha_i + 1 \leq \alpha$ we deduce

$$\begin{aligned}
\mu(\mathcal{E}_\bullet, \underline{m}, \phi) &= \sum_{i=1}^{s} m_i(\nu_i(J)\alpha - a\alpha_i) \geq \\
&\geq \sum_{i=1}^{s} m_i(a(1 - \alpha)) = -a(\sum_{i=1}^{s} m_i)(\alpha - 1).
\end{aligned}$$

2) Let J be a multiindex giving the minimum in $\mu(\mathcal{E}^1_\bullet, \underline{m}_1, \phi)$. Then we have

$$\begin{aligned}\mu(\mathcal{E}_\bullet, \underline{m}, \phi) &\geq -\sum_{i=1}^{s} m_i(a\alpha_i - \nu_i(J)\alpha) = \\ &= -\sum_{j=1}^{t} m_{1,j}(a\alpha_{i_j} - \nu_{i_j}(J)\alpha) - \sum_{j=1}^{s-t} m_{2,j}(a\alpha_{k_j} - \nu_{k_j}(J)\alpha) \geq \\ &\geq \mu(\mathcal{E}^1_\bullet, \underline{m}_1, \phi) - (\sum_{i=1}^{s-t} m_{2,i})a(\alpha-1)\end{aligned}$$

□

Proposition 3.1.13. *There is a non negative constant C_1 depending only on d, a, r, h, ν and δ, such that for every $(\underline{\kappa}, \delta)$-(semi)stable tensor field with generalized parabolic structure, $(\mathcal{E}, \underline{q}, \phi)$, of degree d and rank r, and every non trivial proper subsheaf $\mathcal{E}' \subset \mathcal{E}$, $\mu(\mathcal{E}') = \frac{\deg(\mathcal{E}')}{\alpha'} \leq \frac{\deg(\mathcal{E})}{\alpha} + C_1 = \mu(\mathcal{E}) + C_1$.*

Proof. Let $\mathcal{E}' \subset \mathcal{E}$ be a proper subsheaf. By Lemma 3.1.12, $\mu(\mathcal{E}', 1, \tau) \leq a(\alpha - 1)$. Then we have

$$\begin{aligned}&\underline{\kappa}\text{-pardeg}(\mathcal{E})\alpha' - \underline{\kappa}\text{-pardeg}(\mathcal{E}')\alpha + \delta a(\alpha-1) \geq \\ \geq &\underline{\kappa}\text{-pardeg}(\mathcal{E})\alpha' - \underline{\kappa}\text{-pardeg}(\mathcal{E}')\alpha + \delta\mu(\mathcal{E}', 1, \tau) \geq 0,\end{aligned}$$

from which we deduce

$$\frac{\underline{\kappa}\text{-pardeg}(\mathcal{E})}{\alpha} - \frac{\underline{\kappa}\text{-pardeg}(\mathcal{E}')}{\alpha'} + \frac{\delta a(\alpha-1)}{\alpha\alpha'} \geq 0.$$

Since $\underline{\kappa}\text{-pardeg}(\mathcal{E}') = \deg(\mathcal{E}') - \sum_{i=1}^{\nu} \kappa_i \dim(q(\mathcal{E}'_u(y_1^i) \oplus \mathcal{E}'_u(y_2^i))$ and $\underline{\kappa}\text{-pardeg}(\mathcal{E}) = \deg(\mathcal{E}) - r(\sum_{i=1}^{\nu} \kappa_i)$, we find

$$\begin{aligned}\frac{\deg(\mathcal{E}')}{\alpha'} &\leq \frac{\deg(\mathcal{E})}{\alpha} - \frac{r(\sum_{i=1}^{\nu}\kappa_i)}{\alpha} + \frac{\sum_{i=1}^{\nu}\kappa_i \dim(q(\mathcal{E}'_u(y_1^i) \oplus \mathcal{E}'_u(y_2^i))}{\alpha'} + \frac{\delta a(\alpha-1)}{\alpha\alpha'} \\ &\leq \frac{\deg(\mathcal{E})}{\alpha} + \frac{\delta a(\alpha-1)}{\alpha} + r\nu \leq \frac{\deg(\mathcal{E})}{\alpha} + a\delta + r\nu.\end{aligned}$$

Then, defining $C_1 = a\delta + r\nu$ and applying Lemma 3.1.10 we get the result. □

Remark 3.1.14. Let $C_1' = \alpha C_1$. Note that if $\deg(\mathcal{E}) \leq 0$ then $\deg(\mathcal{E}') \leq \dfrac{\deg(\mathcal{E})}{\alpha} + C_1'$, and if $\deg(\mathcal{E}) > 0$ then $\deg(\mathcal{E}') \leq \deg(\mathcal{E}) + C_1'$. In both cases the degree of any subsheaf $\mathcal{E}' \subset \mathcal{E}$ is bounded by a constant depending only on $a, \delta, \alpha, \nu, d$. This in particular means that for any locally free sheaf $\mathcal{E}$ of rank r and degree d appearing in a $(\underline{\kappa}, \delta)$-semistable tensor field with generalized parabolic structure we have that $\deg(\mathcal{E}|_{Y_i})$ is bounded from below and above by constants depending only on $a, \delta, \alpha, \nu, d$ which we denote by $A_-(r, d, \delta)$ and $A_+(r, d, \delta)$, or just by A_- and A_+ if there is no confusion.

As a trivial consequence we find,

Proposition 3.1.15. *Fix $\kappa_i \in (0,1) \cap \mathbb{Q}$, $i = 1, \dots \nu$ and $\delta \in \mathbb{Q}_{>0}$. Then the family of locally free sheaves of degree d and rank r appearing in $(\underline{\kappa}, \delta)$-(semi)stable generalized parabolic tensor fields of type $(a, b, c, \mathcal{L})$ is bounded.*

Proof. Follows from Lemma 3.1.10 and Proposition 3.1.13. □

3.1.3 Sectional Semistability

Fix the Hilbert Polynomial P, $a, b, c \in \mathbb{N}$ and $\kappa_1, \dots, \kappa_\nu \in (0,1) \cap \mathbb{Q}_{>0}$. Given a tensor field with generalized parabolic structure, $(\mathcal{E}, \underline{q}, \phi)$, we will use the notation

$$\mathrm{par}\chi(\mathcal{E}(n)) := \chi(\mathcal{E}(n)) - \sum_{j=1}^{\nu} \kappa_j \mathrm{dim}(q_j(\mathcal{E}(y_1^j) \oplus \mathcal{E}(y_2^j))),$$

$$\mathrm{par}h^0(\mathcal{E}(n)) := h^0(\mathcal{E}(n)) - \sum_{j=1}^{\nu} \kappa_j \mathrm{dim}(q_j(\mathcal{E}(y_1^j) \oplus \mathcal{E}(y_2^j))).$$

In the next theorem, we adapt [50, Theorem 2.12] to our case.

Theorem 3.1.16. *There exists $n_1 \in \mathbb{N}$ such that for any $n > n_1$ and every $(\underline{\kappa}, \delta)$-(semi)stable $(\mathcal{E}, \underline{q}, \phi)$, the following inequality*

$$\sum_{i=1}^{s} m_i(\mathrm{par}\chi(\mathcal{E}(n))\alpha_i - \mathrm{par}h^0(\mathcal{E}_i(n))\alpha) + \delta\mu(\mathcal{E}_\bullet, \underline{m}, \phi)(\geq)0$$

holds true for every weighted filtration $(\mathcal{E}_\bullet, \underline{m})$.

Proof. Let $(\mathcal{E}_\bullet, \underline{m})$ be a weighted filtration. Assume that each $\mathcal{E}_i$ satifies that $\mathcal{E}_i(n)$ is globally generated and $h^1(Y, \mathcal{E}_i(n)) = 0$ for each $i = 1, \dots, s$. Then, for each i we have

$$\begin{aligned}
&\mathrm{par}\chi(\mathcal{E}(n))\alpha_i - \mathrm{par}h^0(\mathcal{E}_i(n))\alpha = \\
=&(\chi(\mathcal{E}(n))\alpha_i - h^0(\mathcal{E}_i(n))\alpha) - \\
&-((r(\sum_{j=1}^{\nu}\kappa_j))\alpha_i - (\sum_{j=1}^{\nu}\kappa_j \mathrm{dim}(q_j(\mathcal{E}_i(y_1^j) \oplus \mathcal{E}_i(y_2^j)))\alpha) = \\
=&P_{\mathcal{E}}(n)\alpha_i - P_{\mathcal{E}_i}(n)\alpha) - \\
&-((r(\sum_{j=1}^{\nu}\kappa_j))\alpha_i - (\sum_{j=1}^{\nu}\kappa_j \mathrm{dim}(q_j(\mathcal{E}_i(y_1^j) \oplus \mathcal{E}_i(y_2^j))) = \\
=&d(\mathcal{E})\alpha_i - d(\mathcal{E}_i)\alpha - \\
&-((r(\sum_{j=1}^{\nu}\kappa_j))\alpha_i - (\sum_{j=1}^{\nu}\kappa_j \mathrm{dim}(q_j(\mathcal{E}_i(y_1^j) \oplus \mathcal{E}_i(y_2^j))) = \\
=&\mathrm{pardeg}(\mathcal{E})\alpha_i - \mathrm{pardeg}(\mathcal{E}_i)\alpha,
\end{aligned}$$

and we are done.

By Proposition 3.1.13, there exist a positive constant C_1 such that for all $(\underline{\kappa}, \delta)$-(semi)stable tensor fields with generalized parabolic structure, $(\mathcal{E}, \underline{q}, \phi)$, and for all $\mathcal{E}' \subset \mathcal{E}$,

$$\mu(\mathcal{E}') \leq \frac{d}{\alpha} + C_1.$$

Fix another positive constant C_2. Consider the set of isomorphism classes of locally free sheaves $\mathcal{E}'$ such that

$$\begin{aligned} &\text{a) } \mu(\mathcal{E}') \geq \frac{d}{\alpha} - C_2, \\ &\text{b) } 1 \leq \alpha' \leq \alpha - 1, \\ &\text{c) } \mu_{\max}(\mathcal{E}') \leq \frac{d}{\alpha} + C_1. \end{aligned} \tag{3.7}$$

Note that $a), b), c)$ in Equation (3.7) implies that there are finitely many different Hilbert polynomials in the family. This fact and $c)$ again implies that the family is bounded (see Theorem 2.1.21). For any $(\underline{\kappa}, \delta)$-(semi)stable tensor field $(\mathcal{E}, \underline{q}, \phi)$ and every $(0) \subset \mathcal{E}' \subset \mathcal{E}$, either $\mu(\mathcal{E}') < \frac{d}{\alpha} - C_2$ or $\mu(\mathcal{E}') \geq \frac{d}{\alpha} - C_2$. In the second case (boundedness) we know that there exists $n_2 \in \mathbb{N}$ such that $\mathcal{E}'(n)$ is globally generated and $h^1(\mathcal{E}'(n)) = 0$. We deduce that there is a natural number $n_1 \in \mathbb{N}$ such that for all $n > n_1$, for all $(\underline{\kappa}, \delta)$-(semi)stable $(\mathcal{E}, \underline{q}, \phi)$ and for all $\mathcal{E}' \subset \mathcal{E}$ either $\mu(\mathcal{E}') < \frac{d}{\alpha} - C_2$ or $\mathcal{E}'(n)$ is globally generated and $h^1(\mathcal{E}'(n)) = 0$. From Lemma 2.1.19 we know that

$$h^0(\mathcal{E}'(n)) \leq \alpha' \left(\frac{\alpha' - 1}{\alpha'} \left[\frac{d}{\alpha} + C_1 + n + B' \right]_+ + \frac{1}{\alpha'} \left[\frac{d}{\alpha} - C_2 + n + B' \right]_+ \right).$$

Assume n is large enough so that $\frac{d}{\alpha} + C_1 + n + B'$ and $\frac{d}{\alpha} - C_2 + n + B'$ are positive . Then,

$$\begin{aligned} h^0(\mathcal{E}'(n)) &\leq ((\alpha' - 1)(\frac{d}{\alpha} + C_1 + n + B') + (\frac{d}{\alpha} - C_2 + n + B')) = \\ &= \alpha'(\frac{d}{\alpha} + C_1 + n + B') - C_1 - C_2 = \\ &= \alpha'(\frac{d}{\alpha} + n + B' - \frac{C_2}{\alpha'}) + C_1(\alpha' - 1) \leq \\ &\leq \alpha'(\frac{d}{\alpha} + n + B' - \frac{C_2}{\alpha}) + C_1(\alpha' - 1) \leq \\ &\leq \alpha'(\frac{d}{\alpha} + n + B' - \frac{C_2}{\alpha} + C_1(\alpha - 1)). \end{aligned}$$

Thus, we deduce that

$$\begin{aligned} &\chi(\mathcal{E}(n))\alpha' - h^0(\mathcal{E}'(n))\alpha \geq (\frac{\alpha}{h}(1 - g) + d + \alpha n)\alpha' - \\ &-\alpha(\frac{d}{\alpha} + n + B' - \frac{C_2}{\alpha} + C_1(\alpha - 1))\alpha' = \\ &= \alpha\alpha'(\frac{1}{h} - B) - \frac{\alpha\alpha'}{h} g + \alpha' C_2 - C_1 \alpha\alpha'(\alpha - 1) \geq \\ &\geq \alpha\alpha'(-B) + C_2 - C_1 \alpha(\alpha - 1)^2 \geq \\ &\geq \alpha^2(-B) + C_2 - C_1 \alpha(\alpha - 1)^2. \end{aligned}$$

where the last inequality follows from the fact that B is positive. Then,

$$\begin{aligned}
&\operatorname{par}\chi(\mathcal{E}(n))\alpha_i - \operatorname{par}h^0(\mathcal{E}_i(n))\alpha = \\
=&(\chi(\mathcal{E}(n))\alpha_i - h^0(\mathcal{E}_i(n))\alpha) - \\
-&((r(\sum_{j=1}^{\nu}\kappa_j))\alpha_i - (\sum_{j=1}^{\nu}\kappa_j \dim(q_j(\mathcal{E}_i(y_1^j)\oplus\mathcal{E}_i(y_2^j)))\alpha) \geq \\
\geq&\alpha^2(-B) + C_2 - C_1\alpha(\alpha-1)^2 - \\
-&((r(\sum_{j=1}^{\nu}\kappa_j))\alpha_i - (\sum_{j=1}^{\nu}\kappa_j \dim(q_j(\mathcal{E}_i(y_1^j)\oplus\mathcal{E}_i(y_2^j)))\alpha) \geq \\
\geq&\alpha^2(-B) + C_2 - C_1\alpha(\alpha-1)^2 - r\alpha(\sum_{j=1}^{\nu}\kappa_j).
\end{aligned}$$

Since B depends only on α, we can define

$$\begin{aligned}
K = K(C_1, C_2, \alpha, l, \{k'_j\}, d') :=& \\
= \alpha^2(-B) + C_2 - C_1\alpha(\alpha-1)^2 - r\alpha(\sum_{j=1}^{\nu}\kappa_j).&
\end{aligned}$$

Let C_2 be lange enough so that $K > \delta a(\alpha - 1)$ and let n_1 be as before. Let $(\mathcal{E}_\bullet, \underline{m})$ be a weighted filtration with $\mathcal{E}_\bullet \equiv (0) \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_s \subset \mathcal{E}$ and $\underline{m} = (m_1, \ldots, m_s)$. We make a partition of this filtration as follows. Let $j_1, \ldots, j_t$ the indices such that

$$\begin{aligned}
&a)\ \mu(\mathcal{E}_{j_i}) \geq \frac{d}{hr} - C_2 \\
&b)\ \mathcal{E}_{j_i} \text{ is globally generated} \\
&c)\ h^1(Y, \mathcal{E}_{j_i}) = 0
\end{aligned}$$

for $i = 1, \ldots, t$. Let $l_1, \ldots, l_{s-t}$ the set of indices $\{1, 2, \ldots, s\} - \{j_1, \ldots, j_t\}$ in increasing order. Define the weighted filtrations $(\mathcal{E}_{1,\bullet}, \underline{m}_1)$ and $(\mathcal{E}_{2,\bullet}, \underline{m}_2)$ as

$$\begin{aligned}
\mathcal{E}_{\bullet,1} \equiv\ & (0) \subset \mathcal{E}_{j_1} \subset \ldots \subset \mathcal{E}_{j_t} \subset \mathcal{E},\ \ \underline{m}_1 = (m_{j_1}, \ldots, m_{j_t}), \\
\mathcal{E}_{\bullet,2} \equiv\ & (0) \subset \mathcal{E}_{l_1} \subset \ldots \subset \mathcal{E}_{l_{s-t}} \subset \mathcal{E},\ \underline{m}_2 = (m_{l_1}, \ldots, m_{l_{s-t}}).
\end{aligned}$$

From Lemma 3.1.12 we find that

$$\mu(\mathcal{E}_\bullet, \underline{m}, \phi) \geq \mu(\mathcal{E}_{\bullet,2}, \underline{m}_2, \phi) - (\sum_{q=1}^{t} m_{j_q}) a(\alpha - 1).$$

Thus

$$
\begin{aligned}
&\sum_{i=1}^{s} m_i(\mathrm{par}\chi(\mathcal{E}(n))\alpha_i - \mathrm{par}h^0(\mathcal{E}_i(n))\alpha) + \delta\mu(\mathcal{E}_\bullet, \underline{m}, \phi) \geq \\
&\geq \sum_{q=1}^{t} m_{j_q}(\mathrm{par}\chi(\mathcal{E}(n))\alpha_{j_q} - \mathrm{par}h^0(\mathcal{E}_{j_q}(n))\alpha) + \delta\mu(\mathcal{E}_{\bullet,1}, \underline{m}_1, \phi) + \\
&+ \sum_{q=1}^{s-t} m_{l_q}(\mathrm{par}\chi(\mathcal{E}(n))\alpha_{l_q} - \mathrm{par}h^0(\mathcal{E}_{l_q}(n))\alpha) - \delta(\sum_{q=1}^{s-t} m_{l_q})a(\alpha-1) \geq \\
&\geq \sum_{q=1}^{t} m_{j_q}(\mathrm{par}\chi(\mathcal{E}(n))\alpha_{j_q} - \mathrm{par}h^0(\mathcal{E}_{j_q}(n))\alpha) + \delta\mu(\mathcal{E}_{\bullet,1}, \underline{m}_1, \phi) + \\
&+ (\sum_{q=1}^{s-t} m_{l_q})K - \delta(\sum_{q=1}^{s-t} m_{l_q})a(\alpha-1) \geq 0,
\end{aligned}
$$

and the result is proved. □

3.1.4 The Parameter Space

Let H be an effective divisor of degree h in Y such that $\mathcal{O}_Y(H) \simeq \mathcal{O}_Y(1)$. By Proposition 3.1.15 we know that there exists a natural number $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every $(\underline{\kappa}, \delta)$-(semi)stable generalized parabolic tensor field of type $(a, b, c, \mathcal{L})$ with rank r and degree d the following holds

$H^1(Y, \mathcal{E}(n)) = 0$ and $\mathcal{E}(n)$ is golbally generated,

$H^1(\det(\mathcal{E}(rn))) = 0$ and $\det(\mathcal{E}(rn))$ is globally generated,

$H^1(\det(\mathcal{E})^{\otimes c} \otimes \mathcal{L} \otimes \mathcal{O}_Y(an)) = 0$ and $\det(\mathcal{E})^{\otimes c} \otimes \mathcal{L} \otimes \mathcal{O}_Y(an)$ is globally .generated

Let $n_1 \in \mathbb{N}$ be as in Theorem 3.1.16, and fix $n > \max\{n_0, n_1\}$ and $\underline{d} = (d_1, \ldots, d_l) \in \mathbb{N}^l$ with $d = \sum_{i=1}^{l} d_i$, and let $p = r\chi(\mathcal{O}_Y) + d + \alpha n$ (recall $\alpha = hr$). Let U be the vector space $k^{\oplus p}$. We will use the notation $U_{a,b}$ for $(U^{\otimes a})^{\oplus b}$. Denote by $\mathcal{Q}^0$ the quasi-projective scheme parametrizing equivalence classes of quotients $\mathfrak{q}\colon U \otimes \pi_Y^*\mathcal{O}_Y(-n) \to \mathcal{E}$ where $\mathcal{E}$ is a locally free sheaf of uniform multirank r and multidegree $(d_1, \ldots, d_l)$ on Y, and such that the induced map $U \to H^0(Y, \mathcal{E}(n))$ is an isomorphism. On $\mathcal{Q}^0 \times Y$, we have the universal quotient

$$\mathfrak{q}_{\mathcal{Q}^0}\colon U \otimes \pi_Y^*\mathcal{O}_Y(-n) \to \mathcal{E}_{\mathcal{Q}^0}.$$

Since $n > n_0$, the following sheaf is locally free,

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_{\mathcal{Q}^0}}(U_{a,b} \otimes \mathcal{O}_{\mathcal{Q}^0}, \pi_{\mathcal{Q}^0 *}(\det(\mathcal{E})^{\otimes c} \otimes \pi_Y^*\mathcal{L} \otimes \pi_Y^*\mathcal{O}_Y(na))).$$

Consider the corresponding projective bundle $\pi'\colon \mathfrak{h} = \mathbb{P}(\mathcal{H}^\vee) \to \mathcal{Q}^0$ and let

$$\mathfrak{q}_{\mathfrak{h}}\colon U \otimes \pi_Y^*\mathcal{O}_Y(-n) \to \mathcal{E}_{\mathfrak{h}}$$

be the pullback of the universal bundle to $\mathfrak{h} \times Y$. Now, the tautological invertible quotient on $\mathfrak{h}$,

$$\pi'^*\mathcal{H}^\vee \to \mathcal{O}_{\mathfrak{h}}(1) \to 0,$$

induces a tautological morphism on $\mathfrak{h} \times Y$,

$$s_{\mathfrak{h}} : U_{a,b} \otimes \mathcal{O}_{\mathfrak{h}} \to \det(\mathcal{E}_{\mathfrak{h}})^{\otimes c} \otimes \pi_Y^* \mathcal{L} \otimes \pi_Y^* \mathcal{O}_Y(na) \otimes \pi_{\mathfrak{h}}^* \mathcal{O}_{\mathfrak{h}}(1).$$

Now, from the universal quotient we get a surjective morphism

$$(\mathfrak{q}^{\otimes a})^{\oplus b} : U_{a,b} \otimes \pi_Y^* \mathcal{O}_Y(-na) \to (\mathcal{E}_{\mathfrak{h}}^{\otimes a})^{\oplus b}.$$

Denoting by K its kernel, we get a diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[dr, dashed] & U_{a,b} \otimes \pi_Y^* \mathcal{O}_Y(-na) \arrow[r] \arrow[d, "s_{\mathfrak{h}} \otimes \pi_Y^* id_{\mathcal{O}_Y(-na)}"] & (\mathcal{E}_{\mathfrak{h}}^{\otimes a})^{\oplus b} \arrow[r] & 0 \\ & & \det(\mathcal{E}_{\mathfrak{h}})^{\otimes c} \otimes \pi_Y^* \mathcal{L} \otimes \pi_{\mathfrak{h}}^* \mathcal{O}_{\mathfrak{h}}(1) & & \end{tikzcd}$$

Applying Lemma 2.1.35 to the morphism $K \to \det(\mathcal{E}_{\mathfrak{h}})^{\otimes c} \otimes \pi_Y^* \mathcal{L} \otimes \pi_{\mathfrak{h}}^* \mathcal{O}_{\mathfrak{h}}(1)$, where X is, in this case, the curve Y and X' is $\mathfrak{h}$, we get a closed subscheme $\mathfrak{G} \subset \mathfrak{h}$. Note that $\mathfrak{G}$ is the closed subscheme over which $s_{\mathfrak{h}} \otimes \pi_Y^* \mathrm{id}_{\mathcal{O}_Y(-na)}$ factorizes through a morphism

$$\phi_{\mathfrak{G}} : (\mathcal{E}_{\mathfrak{G}}^{\otimes a})^{\oplus b} \to \det(\mathcal{E}_{\mathfrak{G}})^{\otimes c} \otimes \pi_Y^* \mathcal{L} \otimes \pi_{\mathfrak{G}}^* \mathfrak{N}_{\mathfrak{G}}, \tag{3.8}$$

$\pi_{\mathfrak{G}}^* \mathfrak{N}_{\mathfrak{G}}$ being the restriction of $\mathcal{O}_{\mathfrak{h}}(1)$ to $\mathfrak{G}$. Then, on the scheme $\mathfrak{G} \times Y$ we have a family of tensor fields $(\mathcal{E}_{\mathfrak{G}}, \mathfrak{N}_{\mathfrak{G}}, \phi_{\mathfrak{G}})$ parametrized by $\mathfrak{G}$. In order to include the parabolic structure, we need to consider the grassmannian $\mathcal{G}r := \mathrm{Grass}_r(U^{\oplus 2})$ of r dimensional quotients of $U^{\oplus 2}$. Recall that ν is the number of nodes of the curve, so that we have ν divisors, $D_i = y_1^i + y_2^i$, in the normalization Y. Define,

$$Z := \mathfrak{G} \times \overbrace{\mathcal{G}r \times \cdots \times \mathcal{G}r}^{\nu}$$

and denote $c_i : Z \to \mathcal{G}r$ the ith projection. Consider the pullback of the universal quotient of the grassmannian $\mathcal{G}r$ by the projection c_i,

$$q_Z^i : U^{\oplus 2} \otimes \mathcal{O}_Z \to R_Z,$$

and take the direct sum

$$q_Z : U^{\oplus 2\nu} \otimes \mathcal{O}_Z \to \bigoplus_1^{\nu} R_Z.$$

Consider now the two natural projections

$$\begin{tikzcd} \mathfrak{G} \times Y \arrow[d] & Z \times Y \arrow[d] \\ \mathfrak{G} & Z. \end{tikzcd}$$

Denote by $\mathfrak{N}_{\mathfrak{G} \times \mathcal{G}r}$ the pullbak of $\mathfrak{N}_{\mathfrak{G}}$ to $\mathfrak{G} \times Y$ and by $\mathfrak{q}_Z$, $\mathcal{E}_Z$ and ϕ_Z the pullbacks to $Z \times Y$. Look at the following conmutative diagram

$$\begin{tikzcd} Z \times Y \arrow[d, "\pi"] & Z \times \{y_1^i, y_2^i\} \arrow[l, hook', "\overline{j}^i"'] \arrow[d, "\pi^i"'] \\ Z \times X & Z \times \{x_i\} \simeq Z. \arrow[l, hook', "j^i"'] \end{tikzcd}$$

For each i, there are quotients $f_i : U^{\oplus 2} \times \mathcal{O}_Z \to \pi^i_*(\mathcal{E}_Z|_{y^i_1,y^i_2})$. Tanking the direct sum over all i we get $f\colon = \oplus(f_i) : U^{\oplus 2\nu} \times \mathcal{O}_Z \to \bigoplus \pi^i_*(\mathcal{E}_Z|_{y^i_1,y^i_2})$. Consider the following diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(f) \arrow[r] \arrow[dr, dotted, "q'"] & U^{\oplus 2\nu} \times \mathcal{O}_Z \arrow[r, "f"] \arrow[d, "q_Z"] & \bigoplus \pi^i_*(\mathcal{E}_Z|_{y^i_1,y^i_2}) \arrow[r] & 0 \\
& & \bigoplus_1^\nu R_Z & &
\end{tikzcd}$$

Denote by $\mathfrak{I}_{\underline{d}} \subset Z$ the closed subscheme given by the zero locus of the morphism q' (see Lemma 2.1.35). Then the restriction of q to $\mathfrak{I}_{\underline{d}}$ factorizes through

$$q_{\mathfrak{I}_{\underline{d}}}\colon \bigoplus_1^\nu \pi^i_*(\mathcal{E}_Z|_{y^i_1,y^i_2})|_{\mathfrak{I}_{\underline{d}}} = \bigoplus_1^\nu \pi^i_{\mathfrak{I}_{\underline{d}}*}(\mathcal{E}_{\mathfrak{I}_{\underline{d}}}|_{y^i_1,y^i_2}) \to \bigoplus_1^\nu R_Z|_{\mathfrak{I}_{\underline{d}}} = \bigoplus_1^\nu R_{\mathfrak{I}_{\underline{d}}}.$$

Since f and q_Z are diagonal morphisms we deduce that $q_{\mathfrak{I}_{\underline{d}}}$ is also diagonal. Therefore $q_{\mathfrak{I}_{\underline{d}}}$ is determined by ν morphisms

$$q^i_{\mathfrak{I}_{\underline{d}}} : \pi^i_{\mathfrak{I}_{\underline{d}}*}(\mathcal{E}_{\mathfrak{I}_{\underline{d}}}|_{y^i_1,y^i_2}) \to R_{\mathfrak{I}_{\underline{d}}}.$$

Denote by $(\mathcal{E}_{\mathfrak{I}_{\underline{d}}}, \mathfrak{N}_{\mathfrak{I}_{\underline{d}}}, \phi_{\mathfrak{I}_{\underline{d}}})$ the restriction of $(\mathcal{E}_Z, \mathfrak{N}_Z, \phi_Z)$ to $\mathfrak{I}_{\underline{d}}$. Then we have a universal family of generalized parabolic tensor fields,

$$(\mathcal{E}_{\mathfrak{I}_{\underline{d}}}, \underline{q}_{\mathfrak{I}_{\underline{d}}}, \mathfrak{N}_{\mathfrak{I}_{\underline{d}}}, \phi_{\mathfrak{I}_{\underline{d}}}), \tag{3.9}$$

with rank r and multidegree $(d_1, \dots, d_l)$. Let us denote

$$I(r,d,\delta) = \{(d_1,\dots,d_l) | \sum_{i=1}^{l} d_i = d \text{ and } A_- \le d_i \le A_+\}, \tag{3.10}$$

where A_- and A_+ are as in Remark 3.1.14. Then we define

$$\mathfrak{I}_{r,d,\delta} := \coprod_{\underline{d} \in I} \mathfrak{I}_{\underline{d}}. \tag{3.11}$$

3.1.5 The Gieseker Space

We will show that there is a natural closed embedding of the parameter space $\mathfrak{I}_{\underline{d}}$ which is $\mathrm{SL}(U)$-equivariant.

Fix a Poincaré line bundle $\mathcal{P}_i$ on $Y_i \times \mathrm{Pic}^{d_i}(Y_i)$ and let $n \in \mathbb{Z}$. Define the sheaf

$$\mathcal{G}^i_1 = \mathcal{H}om_{\mathcal{O}_{\mathrm{Pic}^{d_i}(Y_i)}}(\bigwedge^r U \otimes \mathcal{O}_{\mathrm{Pic}^{d_i}(Y_i)}, \pi_{\mathrm{Pic}^{d_i}(Y_i)*}(\mathcal{P}_i \otimes \pi^*_{Y_i}\mathcal{O}_{Y_i}(rn))). \tag{3.12}$$

The natural number we have fixed satisfies $n > \max\{n_0, n_1\}$, therefore the above sheaf is locally free, and we can consider the corresponding projective bundle on $\mathrm{Pic}^{d_i}(Y_i)$

$$\mathbb{G}^i_1 = \mathbf{P}(\mathcal{G}^{i\vee}_1).$$

Recall the functor of points of $\mathbb{G}_1^i$. For each $\mathrm{Pic}^{d_i}(Y_i)$-scheme $f : S \to \mathrm{Pic}^{d_i}(Y_i)$ we have

$$\mathbb{G}_1^{i\bullet}(S) = \left\{ \begin{array}{c} \text{equivalence classes of invertible} \\ \text{quotients } f^*\mathcal{G}_1^{i\vee} \to \mathcal{N} \to 0 \\ \text{on the fibers of } f : S \to \mathrm{Pic}^{d_i}(Y_i) \end{array} \right\}.$$

Since $\mathcal{H}om_{\mathcal{O}_S}(f^*\mathcal{G}_1^{i\vee}, \mathcal{N}) \simeq f^*\mathcal{G}_1^i \otimes \mathcal{N}$ we find that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_S}(f^*\mathcal{G}_1^{i\vee}, \mathcal{N}) &\simeq H^0(S, f^*\mathcal{G}_1^i \otimes \mathcal{N}) \simeq \\ &\simeq \mathrm{Hom}(\bigwedge^r U \otimes \mathcal{O}_S, f^*\pi_{\mathrm{Pic}^{d_i}(Y_i)*}(\mathcal{P}_i \otimes \pi_{Y_i}^*\mathcal{O}_{Y_i}(rn)) \otimes \mathcal{N}) \simeq \\ &\simeq \mathrm{Hom}(\bigwedge^r U \otimes \mathcal{O}_S, \pi_{S*}(\mathrm{id} \times f)^*(\mathcal{P}_i \otimes \pi_{Y_i}^*\mathcal{O}_{Y_i}(rn)) \otimes \mathcal{N}). \end{aligned}$$

Therefore

$$\mathbb{G}_1^{i\bullet}(S) = \left\{ \begin{array}{c} \text{equivalence classes of non zero morphisms} \\ \bigwedge^r U \otimes \mathcal{O}_S \to \pi_{S*}(\mathrm{id} \times f)^*(\mathcal{P}_i \otimes \pi_{Y_i}^*\mathcal{O}_{Y_i}(rn)) \otimes \mathcal{N} \\ \text{on the fibers of } f : S \to \mathrm{Pic}^{d_i}(Y_i) \end{array} \right\}.$$

Note that the determinant map $\mathcal{E}_{\mathfrak{I}_{\underline{d}}} \mapsto \bigwedge \mathcal{E}_{\mathfrak{I}_{\underline{d}}}|_{Y_i} = \bigwedge(\mathcal{E}_{\mathfrak{I}_{\underline{d}}}|_{Y_i})$ defines a morphism

$$\mathfrak{d}_i : \mathfrak{I}_{\underline{d}} \to \mathrm{Pic}^{d_i}(Y_i).$$

Consider now on $\mathfrak{I}_{\underline{d}} \times Y$ the universal quotient $q_{\mathfrak{I}_{\underline{d}}} : U \otimes \pi_Y^*\mathcal{O}_Y(-n) \to \mathcal{E}_{\mathfrak{I}_{\underline{d}}}$. Restricting to the ith component, twisting by n and taking the determinant we find

$$\bigwedge q^i_{\mathfrak{I}_{\underline{d}}}(n) : \bigwedge^r U \otimes \mathcal{O}_{\mathfrak{I}_{\underline{d}} \times Y_i} \to \bigwedge^r \mathcal{E}_{\mathfrak{I}_{\underline{d}}}|_{Y_i} \otimes \pi_{Y_i}^*\mathcal{O}_{Y_i}(nr).$$

Let $\mathcal{N}_i$ be a line bundle on $\mathfrak{I}_{\underline{d}}$ such that $\bigwedge^r \mathcal{E}_{\mathfrak{I}_{\underline{d}}}|_{Y_i} = (\mathfrak{d}_i \times id_{Y_i})^*\mathcal{P}_i \otimes \pi_{\mathfrak{I}_{\underline{d}}}^*\mathcal{N}_i$. Then we have defined a morphism

$$\bigwedge q^i_{\mathfrak{I}_{\underline{d}}}(n) : \bigwedge^r U \otimes \mathcal{O}_{\mathfrak{I}_{\underline{d}} \times Y_i} \to (\mathfrak{d}_i \times id_{Y_i})^*(\mathcal{P}_i \otimes \pi_{Y_i}^*\mathcal{O}_{Y_i}(nr)) \otimes \pi_{\mathfrak{I}_{\underline{d}}}^*\mathcal{N}_i,$$

from what we get the morphism

$$\pi_{\mathfrak{I}_{\underline{d}}*}(\bigwedge q^i_{\mathfrak{I}_{\underline{d}}}(n)) : \bigwedge^r U \otimes \mathcal{O}_{\mathfrak{I}_{\underline{d}}} \to \pi_{\mathfrak{I}_{\underline{d}}*}(\mathfrak{d}_i \times id_{Y_i})^*(\mathcal{P}_i \otimes \pi_{Y_i}^*\mathcal{O}_{Y_i}(nr)) \otimes \mathcal{N}_i.$$

Each of these morphisms of locally free sheaves on $\mathfrak{I}_{\underline{d}}$ gives a point in $\mathbb{G}_1^{i\bullet}(\mathfrak{I}_{\underline{d}})$ and, therefore, a morphism

$$\begin{tikzcd} \mathfrak{I}_{\underline{d}} \arrow[rr, "i_1"] \arrow[dr, "\mathfrak{d}"'] & & \mathbb{G}_1^1 \times \ldots \times \mathbb{G}_1^l \arrow[dl] \\ & \mathrm{Pic}^{\underline{d}}(Y) & \end{tikzcd}$$

Define now the sheaf,

$$\mathcal{G}_2 = \mathcal{H}om_{\mathcal{O}_{\mathrm{Pic}^d(Y)}}(U_{a,b} \otimes \mathcal{O}_{\mathrm{Pic}^d(Y)}, \pi_{\mathrm{Pic}^d(Y)*}(\mathcal{P}^{\otimes c} \otimes \pi_Y^*\mathcal{L} \otimes \pi_Y^*\mathcal{O}_Y(na))).$$

For $n > \max\{n_0, n_1\}$, $\mathcal{G}_2$ is also locally free and we can consider the corresponding projective bundle on $\mathrm{Pic}^d(Y)$,

$$\mathbb{G}_2 = \mathbf{P}(\mathcal{G}_2^\vee).$$

Recall the functor of points of $\mathbb{G}_2$. For each $\mathrm{Pic}^{\underline{d}}(Y)$-scheme $f : S \to \mathrm{Pic}^{\underline{d}}(Y)$ we have

$$\mathbb{G}_2^\bullet(S) = \left\{ \begin{array}{c} \text{equivalence classes of non zero morphisms} \\ U_{a,b} \otimes \mathcal{O}_S \to \pi_{S*}(\mathrm{id} \times f)^*(\mathcal{P}^{\otimes c} \otimes \pi_Y^* \mathcal{L} \otimes \pi_Y^* \mathcal{O}_Y(rn)) \otimes \mathcal{N} \\ \text{on the fibers of } f : S \to \mathrm{Pic}^{\underline{d}}(Y) \end{array} \right\}.$$

Consider now the universal quotient $\mathfrak{q}_{\mathfrak{I}_{\underline{d}}} : U \otimes \mathcal{O}_{\mathfrak{I}_{\underline{d}} \times Y}(-n) \to \mathcal{E}_{\mathfrak{I}_{\underline{d}}}$ and the universal tensor field

$$\phi_{\mathfrak{I}_{\underline{d}}} : (\mathcal{E}^{\otimes a})^{\oplus b} \to \det(\mathcal{E})^{\otimes c} \otimes \pi_Y^* \mathcal{L} \otimes \pi_{\mathfrak{I}_{\underline{d}}}^* \mathcal{N}_{\mathfrak{I}_{\underline{d}}}.$$

Composing $(\mathfrak{q}_{\mathfrak{I}_{\underline{d}}}(n)^{\otimes a})^{\oplus b}$ with the tensor field $\phi_{\mathfrak{I}_{\underline{d}}}$ we find

$$\phi_{\mathfrak{I}_{\underline{d}}} \circ (\mathfrak{q}_{\mathfrak{I}_{\underline{d}}}(n)^{\otimes a})^{\oplus b} : U_{a,b} \otimes \mathcal{O}_{\mathfrak{I}_{\underline{d}} \times Y} \to \det(\mathcal{E})^{\otimes c} \otimes \pi_Y^* \mathcal{L} \otimes \pi_{\mathfrak{I}_{\underline{d}}}^* \mathfrak{N}_{\mathfrak{I}_{\underline{d}}} \otimes \pi_Y^* \mathcal{O}_Y(na).$$

Let $\mathcal{N}$ be an invertible sheaf on $\mathfrak{I}_{\underline{d}}$ such that $\det(\mathcal{E}_{\mathfrak{I}_{\underline{d}}}) = (\mathfrak{d} \times \mathrm{id})^* \mathcal{P} \otimes \pi_{\mathfrak{I}_{\underline{d}}}^* \mathcal{N}$. Then we have

$$U_{a,b} \otimes \mathcal{O}_{\mathfrak{I}_{\underline{d}} \times Y} \to (\mathfrak{d} \times \mathrm{id})^* \mathcal{P}^{\otimes c} \otimes \pi_Y^* \mathcal{L} \otimes \pi_{\mathfrak{I}_{\underline{d}}}^* (\mathfrak{N}_{\mathfrak{I}_{\underline{d}}} \otimes \mathcal{N}^{\otimes c}) \otimes \pi_Y^* \mathcal{O}_Y(na).$$

Note that $U_{a,b} \otimes \mathcal{O}_{\mathfrak{I}_{\underline{d}} \times Y} \simeq \pi_{\mathfrak{I}_{\underline{d}}}^*(U_{a,b} \otimes \mathcal{O}_{\mathfrak{I}_{\underline{d}}})$. Therefore taking $\pi_{\mathfrak{I}_{\underline{d}}*}$ and composing with the adjuntion morphism $\psi : U_{a,b} \otimes \mathcal{O}_{\mathfrak{I}_{\underline{d}}} \to \pi_{\mathfrak{I}*_{\underline{\mathfrak{d}}}} \pi_{\mathfrak{I}_{\underline{d}}}^*(U_{a,b} \otimes \mathcal{O}_{\mathfrak{I}_{\underline{d}}})$ we get

$$\pi_{\mathfrak{I}_{\underline{d}}*}(\phi_{\mathfrak{I}_{\underline{d}}} \circ (q^i_{\mathfrak{I}_{\underline{d}}}(n)^{\otimes a})^{\oplus b}) \circ \psi : U_{a,b} \otimes \mathcal{O}_{\mathfrak{I}_{\underline{d}}} \to \pi_{\mathfrak{I}_{\underline{d}}*}(\mathfrak{d} \times \mathrm{id})^*(\mathcal{P}^{\otimes c} \otimes \pi_Y^* \mathcal{L} \otimes \pi_Y^* \mathcal{O}_Y(na)) \otimes \mathfrak{N}_{\mathfrak{I}_{\underline{d}}} \otimes \mathcal{N}^{\otimes c}$$

hence, a morphism

$$\pi_{\mathfrak{I}_{\underline{d}}*}(\phi_{\mathfrak{I}_{\underline{d}}} \circ (q^i_{\mathfrak{I}_{\underline{d}}}(n)^{\otimes a})^{\oplus b}) \circ \psi : \mathfrak{I}_{\underline{d}} \to \mathbb{G}_2.$$

Altogether, with the obvious morphism $\mathfrak{I}_{\underline{d}} \to (\mathcal{G}r)^\nu$, give us the so called Gieseker morphism

$$\mathrm{Gies} : \mathfrak{I}_{\underline{d}} \longrightarrow (\mathbb{G}_1^1 \times \ldots \times \mathbb{G}_1^l) \times_{\mathrm{Pic}^{\underline{d}}(Y)} \mathbb{G}_2 \times (\mathcal{G}r)^\nu =: \mathbb{G}.$$

Proposition 3.1.17. *The Gieseker morphism* Gies : $\mathfrak{I}_{\underline{d}} \to \mathbb{G}$ *is injective and* $\mathrm{SL}(U)$-*equivariant.*

Proof. Follows as in the connected case (see for instance [16, Lemma 4.3]). □

3.1.6 Semistability in the Gieseker Space

In this section we will compute the semistability function for those points in the Gieseker space which lie in the image of the Gieseker map. In fact, we will compute the semistability function just with respect to some special filtrations. This calculation will become important in later results.

Let $b_1, \ldots, b_l, c, k_1, \ldots, k_\nu$ be positive integers and consider the ample invertible sheaf on $\mathbb{G}$,

$$\mathcal{O}_{\mathbb{G}}(b_1, \ldots, b_l, c, k_1, \ldots, k_\nu).$$

Consider the obvious linearization of $\mathcal{O}_{\mathbb{G}}(b_1, \ldots, b_l, c, k_1, \ldots, k_\nu)$, and let $\mathbb{G}^{(s)s}$ be the set of points which are (semi)stable with respect to the given linearization. Consider a weighted flag $(U^\bullet, \underline{m})$, being

$$U^\bullet : (0) \subset U_1 \subset \ldots \subset U_s \subset U \tag{3.13}$$

and $\underline{m} = (m_1, \ldots m_s)$. Let $\lambda : \mathbb{G}_m \to \mathrm{SL}(U)$ be a one parameter subgroup whose weighted flag is $(U^\bullet, \underline{m})$. Let t be a rational point of $\mathfrak{I}_{\underline{d}}$ and $\mathrm{Gies}(t) = (t_{1,1}, \ldots, t_{1,l}, t_2, t_{3,1}, \ldots, t_{3,\nu})$ its image in $\mathbb{G}$. Let

$$q_t : U \otimes \mathcal{O}_Y(-n) \to \mathcal{E}$$

be the locally free quotient sheaf corresponding to t. The weighted filtration (3.13) induces a filtration of $\mathcal{E}$ defined by $\mathcal{E}_u := q(U_u \otimes \mathcal{O}_Y(-n)) \subset \mathcal{E}$. Assume that

$$\begin{aligned} l_u := \dim(U_u) = h^0(\mathcal{E}_u(n)) \\ 0 = h^1(\mathcal{E}_u(n)). \end{aligned}$$

Then, the semistability function is given by

$$\begin{aligned} \mu(\lambda, \mathrm{Gies}(t)) &= \sum_{i=1}^{l} b_i \mu_{\mathbb{G}_1}(\lambda, t_{1,i}) + c\mu_{\mathbb{G}_2}(\lambda, t_2) + \sum_{i=1}^{\nu} k_i \mu_{\mathcal{G}r}(\lambda, t_{3,i}) = \\ &= \sum_{i=1}^{l} b_i \sum_{u=1}^{s} m_u (\mathrm{rk}(\mathcal{E}_u^i)p - rh^0(\mathcal{E}_u(n))) + \\ &+ c\sum_{u=1}^{s} m_u(\nu(I_0, l_u)p - ah^0(\mathcal{E}_u(n))) + \\ &+ \sum_{i=1}^{\nu} k_i \sum_{u=1}^{s} m_u (p \dim(q_i(\mathcal{E}_u(y_1^i) \oplus \mathcal{E}_u(y_2^i))) - rh^0(\mathcal{E}_u(n))). \end{aligned} \tag{3.14}$$

We fix now a concrete polarization, defined as follows,

$$\begin{cases} b_i = bd_i,\ b = p - b',\ b' = b_1' + b_2',\ b_1' = a\delta,\ b_2' = r\sum_{i=1}^{\nu} \kappa_i \\ c = \delta r d = \sum_{i=1}^{l} \delta r d_i \\ k_i = \kappa_i \alpha \end{cases}$$

Then, Eq 3.14 becomes into,

$$\begin{aligned} \mu(\lambda, \mathrm{Gies}(t)) = \sum_{u=1}^{s} m_u \Bigg\{ &\sum_{i=1}^{l} b_i (\mathrm{rk}(\mathcal{E}_u^i)p - rh^0(\mathcal{E}_u(n))) + \\ &c(\nu(I_0, l_u)p - ah^0(\mathcal{E}_u(n))) + \\ &\sum_{i=1}^{\nu} k_i (p\mathrm{dim}(q_i(\mathcal{E}_u(y_1^i) \oplus \mathcal{E}_u(y_2^i))) - rh^0(\mathcal{E}_u(n))) \Bigg\}. \end{aligned}$$

Note that $r(\sum_{i=1}^{l} b_i) + ac + r(\sum_{i=1}^{\nu} k_i) = \alpha p$. Then,

$$\begin{aligned} \mu(\lambda, \mathrm{Gies}(t)) = \sum_{u=1}^{s} m_u \Bigg\{ &(\sum_{i=1}^{l} b_i \mathrm{rk}(\mathcal{E}_u^i)p) - h^0(\mathcal{E}_u(n))\alpha p + \\ &c\nu(I_0, l_u)p + \sum_{i=1}^{\nu} \alpha\kappa_i p\mathrm{dim}(q_i(\mathcal{E}_u(y_1^i) \oplus \mathcal{E}_u(y_2^i)) \Bigg\}. \end{aligned}$$

Now, since $\sum_{i=1}^{l}(b_i \mathrm{rk}(\mathcal{E}_u^i)) = b\alpha_u$ we find

$$\mu(\lambda, \mathrm{Gies}(t)) = \sum_{u=1}^{s} m_u \Bigg\{ b\alpha_u p - h^0(\mathcal{E}_u(n))\alpha p + \\ c\nu(I_0, l_u)p + \sum_{i=1}^{\nu} \alpha\kappa_i p \dim(q_i(\mathcal{E}_u(y_1^i) \oplus \mathcal{E}_u(y_2^i)) \Bigg\}.$$

Again, since $b = p - b'_1 - b'_2$, $b'_1 = a\delta$ and $\alpha_u = \sum_{i=1}^{l} d_i \mathrm{rk}(\mathcal{E}_u^i)$, we get

$$\frac{\mu(\lambda, \mathrm{Gies}(t))}{p} = \sum_{u=1}^{s} m_u \Bigg\{ p\alpha_u - \alpha h^0(\mathcal{E}_u(n)) + \\ + \delta \sum_{i=1}^{l} d_i (r\nu(I_0, l_u) - a\mathrm{rk}(\mathcal{E}_u^i)) + \\ + \sum_{i=1}^{\nu} \alpha\kappa_i \dim(q_i(\mathcal{E}_u(y_1^i) \oplus \mathcal{E}_u(y_2^i)) - b'_2 \alpha_u \Bigg\}.$$

Since the first cohomology groups are assumed to be 0, we find

$$p\alpha_u - \alpha h^0(\mathcal{E}_u(n)) = \alpha_u P_{\mathcal{E}}(n) - \alpha P_{\mathcal{E}_u}(n) = \alpha_u \deg(\mathcal{E}) - \alpha \deg(\mathcal{E}_u).$$

We also know that

$$\mathrm{pardeg}(\mathcal{E}) = \deg(\mathcal{E}) - r(\sum_{i=1}^{\nu} \kappa_i)$$
$$\mathrm{pardeg}(\mathcal{E}_u) = \deg(\mathcal{E}_u) - \sum_{i=1}^{\nu} \kappa_i \dim(q_i(\mathcal{E}_u(y_1^i) \oplus \mathcal{E}_u(y_2^i))$$

Then, we finally get

$$\frac{\mu(\lambda, \mathrm{Gies}(t))}{p} = \sum_{u=1}^{s} m_u \Bigg\{ (\alpha_u \mathrm{pardeg}(\mathcal{E}) - \alpha \mathrm{pardeg}(\mathcal{E}_u)) + \\ \delta(\alpha\nu(I_0, l_u) - a\alpha_u) \Bigg\}.$$

3.1.7 (κ, δ)-Semistability and Hilbert-Mumford Semistability

The goal of this subsection is to prove Theorem 3.1.24, which shows that $(\underline{\kappa}, \delta)$-(semi)stability is equivalent to GIT (semi)stability in the Gieseker space under some conditions.

Let B be the constant given in Theorem 2.1.18, α, d, h, a, ν, δ as always, and let K' be a constant with the property

$$\alpha K' > \max \Bigg\{ d(w - \alpha) + \alpha r\nu + a\delta(\alpha - 1) + \tag{3.15}$$

$$+ B\alpha(\alpha - 1) | w = 1 \ldots \alpha - 1 \Bigg\}, \tag{3.16}$$

and such that $d + K' > 0$.

Lemma 3.1.18. *There exists $n \gg 0$ such that for any triple $t = (q : U \otimes \mathcal{O}_Y(-n) \to \mathcal{E}, \underline{q}, \phi)$ of degree $d = \deg(\mathcal{E})$ and multiplicity α whose induced map $U \to H^0(\mathcal{E}(n))$ is injective, and giving a semistable point in the Gieseker space, $\mathbb{G}^{(s)s}$, the following holds*

$$\forall \mathcal{E}' \subset \mathcal{E},\ \mu(\mathcal{E}') < d + K'.$$

Proof. We follow [50, Section 2.3.8]. It is enough to show that $\deg(\mathcal{E}') < \deg(\mathcal{E}) + K'$ for the maximal destabilizing subsheaf, since

$$\mu(\mathcal{E}'') \leq \mu(\mathcal{E}') < \frac{\deg(\mathcal{E}) + K'}{\alpha'} \leq \deg(\mathcal{E}) + K'.$$

Let $\mathcal{Q} := \mathcal{E}/\mathcal{E}'$ be the quotient vector bundle. We know that it is semistable. Assume that $\deg(\mathcal{E}') \geq d + K'$. For all $n \in \mathbb{N}$ we have

$$h^0(Y, \mathcal{Q}(n)) \leq \alpha(\mathcal{Q})[\mu(\mathcal{Q}) + n + B]_+ = \begin{cases} 0 & \text{if } \mu(\mathcal{Q}) + n + B \leq 0 \\ \alpha(\mathcal{Q})(\mu(\mathcal{Q}) + n + B) & \text{otherwise} \end{cases}$$

We have to study both cases separately.

a) $\underline{h^0(Y, \mathcal{Q}(n)) \leq \alpha(\mathcal{Q})(\mu(\mathcal{Q}) + n + B)}$. Let us denote $U' := H^0(\mathcal{E}'(n)) \cap U$. Then we have,

$$\begin{aligned} \dim(U') &\geq p - h^0(Y, \mathcal{Q}(n)) \geq \\ &\geq \alpha(\frac{1-g}{h}) + d + \alpha n - \alpha(\mathcal{Q})(\mu(\mathcal{Q}) + n + B) = \\ &\geq \alpha(\frac{1-g}{h} + n) + d - \alpha(\mathcal{Q})(\frac{1-g}{h} + n) - d(\mathcal{Q}) - \alpha(\mathcal{Q})B = \\ &= \alpha'(\frac{1-g}{h} + n) + d(\mathcal{E}') - \alpha(\mathcal{Q})B \geq \\ &\geq \alpha'(\frac{1-g}{h} + n) + d + K' - B(\alpha - 1). \end{aligned}$$

Consider the locally free sheaf $\widehat{\mathcal{E}} := \mathrm{Im}(U' \otimes \mathcal{O}_Y(-n) \to \mathcal{E}_t)$. Thus $U' \subset H^0(Y, \widehat{\mathcal{E}}(n)) \cap U$ (see Lemma 2.1.38), $\mathrm{rk}(\widehat{\mathcal{E}}|_{Y_i}) \leq \mathrm{rk}(\mathcal{E}'|_{Y_i})$ and $\widehat{\mathcal{E}}$ is generically generated by global sections. Let $\{u_1, \ldots, u_i\}$ be a basis for U' and complete it to a basis $\underline{u} = \{u_1, \ldots, u_p\}$ of U. Consider the associated one parameter subgroup $\lambda = \lambda(\underline{u}, \gamma_p^{(i)})$. Then

$$\begin{aligned} \mu_{\mathbb{G}_1^i}(\lambda, i_{1,i}(t)) &= p\mathrm{rk}(\widehat{\mathcal{E}}|_{Y_i}) - r\dim(U') \leq \\ &\leq p\mathrm{rk}(\mathcal{E}'|_{Y_i}) - r\dim(U'). \end{aligned}$$

Also, since $\nu(I, i) \leq a$, we have

$$\mu_{\mathbb{G}_2}(\lambda, i_2(t)) \leq a(p - \dim(U')).$$

Therefore,

$$\begin{aligned}
\mu_{\mathbb{G}}(\lambda,\mathrm{Gies}(t)) = & \sum_{i=1}^{l} b_i \mu_{\mathbb{G}_1^i}(\lambda, i_{1,i}(t)) + c\mu_{\mathbb{G}_2}(\lambda, i_2(t)) + \\
& + \sum_{i=1}^{\nu} k_i(p\dim(q_i(\widehat{\mathcal{E}}(y_1^i)\oplus\widehat{\mathcal{E}}(y_2^i))) - r\dim(U')) \leq \\
& \leq \sum_{i=1}^{l} b_i(p\mathrm{rk}(\mathcal{E}'|_{Y_i}) - r\dim(U')) + \\
& + ca(p - \dim(U')) + \\
& + \sum_{i=1}^{\nu} k_i(p\dim(q_i(\mathcal{E}'(y_1^i)\oplus\mathcal{E}'(y_2^i))) - r\dim(U')) = \\
& = \sum_{i=1}^{l} d_i(p - a\delta - r(\sum_{i=1}^{l}\kappa_i))(p\mathrm{rk}(\mathcal{E}'|_{Y_i}) - r\dim(U')) + \\
& + \sum_{i=1}^{l} d_i\delta ra(p - \dim(U')) + \\
& + \sum_{i=1}^{\nu} \kappa_i\alpha(p\dim(q_i(\mathcal{E}'(y_1^i)\oplus\mathcal{E}'(y_2^i)))) - rh\dim(U')).
\end{aligned}$$

An easy calculation gives us

$$\begin{aligned}
\frac{\mu_{\mathbb{G}}(\lambda,\mathrm{Gies}(t))}{p} \leq & \alpha'\{p - r(\sum_{i=1}^{\nu}\kappa_i)\} - \alpha\{\dim(U') - \\
& - \sum_{i=1}^{\nu}\kappa_i\alpha(\dim(q_i(\mathcal{E}'(y_1^i)\oplus\mathcal{E}'(y_2^i))))\} + a\delta(\alpha-\alpha').
\end{aligned} \tag{3.17}$$

Since $p = \alpha(n + \frac{1-g}{h}) + \deg(\mathcal{E})$ and $\dim(U') \geq d + K' + \alpha'(n + \frac{1+g}{h}) - B(\alpha - 1)$, we deduce that,

$$\begin{aligned}
\frac{\mu_{\mathbb{G}}(\lambda,\mathrm{Gies}(t))}{p} \leq & a\delta(\alpha-\alpha') - \alpha K' + B\alpha(\alpha-1) - \\
& - r\alpha'(\sum_{i=1}^{\nu}\kappa_i) + \alpha(\sum_{i=1}^{\nu}\kappa_i\dim(q_i(\mathcal{E}'(y_1^i)\oplus\mathcal{E}'(y_2^i)))) \\
& + d(\alpha'-\alpha).
\end{aligned}$$

Since $\alpha' r(\sum_{i=1}^{\nu}\kappa_i) > 0$, $\alpha\sum_{i=1}^{\nu}\kappa_i\dim(q_i(\mathcal{E}'(y_1^i)\oplus\mathcal{E}'(y_2^i))) < \alpha\nu r$ (because $\kappa_i < 1$), $\alpha - \alpha' < \alpha - 1$, and the definition of K' (see Equation 3.15), we get

$$\frac{\mu_{\mathbb{G}}(\lambda,\mathrm{Gies}(t))}{p} < 0.$$

But we know that $\mathrm{Gies}(t)$ is semistable so we get a contradiction.

b) $\underline{h^0(Y, \mathcal{Q}(n)) = 0}$. In this case, assuming $n > \dfrac{g-1}{h}$, we have $\dim(U') = p$. The same calculation as before (3.17) shows that

$$\begin{aligned}\frac{\mu_{\mathbb{G}}(\lambda, \mathrm{Gies}(t))}{p} &\leq \alpha'(p - r(\sum_{i=1}^{\nu} \kappa_i)) - \alpha\{\dim(U') - \\ &\quad - \sum_{i=1}^{\nu} \kappa_i \dim(q_i(\mathcal{E}'(y_1^i) \oplus \mathcal{E}'(y_2^i)))\} + a\delta(\alpha - \alpha') \leq \\ &\leq (\alpha' - \alpha)(p - a\delta).\end{aligned}$$

Assume n is large enough so that $p - a\delta > 0$ (recall that $p = r\chi(\mathcal{O}_Y) + d + \alpha n$). Then, $\mu_{\mathbb{G}}(\lambda, \mathrm{Gies}(t))/p < 0$ and we get a contradiction. □

Proposition 3.1.19. *There exists n large enough and a constant C_3 so that for any triple $t = (q : U \otimes \mathcal{O}_Y(-n) \to \mathcal{E}, \underline{q}, \phi)$ of degree d and multiplicity α whose induced map $U \to H^0(\mathcal{E}(n))$ is injective, giving a semistable point in the Gieseker space, $\mathbb{G}^{(s)s}$, the following holds*

$$\mu_{\max}(\mathcal{E}) \leq \frac{\deg(\mathcal{E})}{\alpha} + C_3.$$

Proof. Because of Lemma 3.1.18 there exist a constant K' such that $\mu(\mathcal{E}') < \deg(\mathcal{E}) + K'$ for all $\mathcal{E}' \subset \mathcal{E}$. We can assume that K' is large enough so that $\deg(\mathcal{E}) + K' > 0$. Then

$$\begin{aligned}\mu(\mathcal{E}') &< \deg(\mathcal{E}) + K' = \\ &= \deg(\mathcal{E}) + K' - \frac{\deg(\mathcal{E})}{\alpha} + \frac{\deg(\mathcal{E})}{\alpha} = \\ &= \frac{\deg(\mathcal{E})}{\alpha}(\alpha - 1) + \frac{\deg(\mathcal{E})}{\alpha} + K'.\end{aligned}$$

Defining $C_3 := \frac{\deg(\mathcal{E})}{\alpha}(\alpha - 1) + K'$ we get the result. □

Given a tensor field with generalized parabolic structure, $(\mathcal{E}, \underline{q}, \tau)$, we define the *parabolic slope* as

$$\mathrm{par}\mu(\mathcal{E}) := \frac{\mathrm{pardeg}(\mathcal{E})}{\alpha}.$$

Lemma 3.1.20. *Let $(\mathcal{E}_\bullet, \underline{m})$ be a weighted filtration such that*

$$\mathrm{par}\mu(\mathcal{E}_i) < \mathrm{par}\mu(\mathcal{E}) - C_1, \quad C_1 = a\delta + r\nu,$$

for all $i = 1, \dots, s$. Then $P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m}) + \delta\mu(\mathcal{E}_\bullet, \underline{m}, \phi) \geq 0$.

Proof. By Lemma 3.1.12, we have

$$\begin{aligned}P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m}) + \delta\mu(\mathcal{E}_\bullet \underline{m}, \phi) &\geq \sum_{i=1}^{s} m_i(\mathrm{pardeg}(\mathcal{E})\alpha_i - \mathrm{pardeg}(\mathcal{E}_i)\alpha) - \\ &\quad - (\sum_{i=1}^{s-1} m_i)\delta a(\alpha - 1),\end{aligned}$$

and, because of the condition of the statement,

$$\mathrm{pardeg}(\mathcal{E}_i)\alpha - \mathrm{pardeg}(\mathcal{E})\alpha_i < -C_1\alpha\alpha_i.$$

Then, since $C_1 = a\delta + r\nu$ (see Proposition3.1.13), we get

$$\begin{aligned} P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m}) + \delta\mu(\mathcal{E}_\bullet \underline{m}, \phi) \geq & \sum_{i=1}^{s-1} m_i(C_1\alpha\alpha_i - \delta a(\alpha - 1)) = \\ = & \sum_{i=1}^{s-1} m_i(a\delta(\alpha\alpha_i - \alpha + 1) + r\nu\alpha\alpha_i]) \geq 0. \end{aligned}$$

□

As a trivial consequence we have

Corollary 3.1.21. *$(\mathcal{E}, \underline{q}, \phi)$ is $(\underline{\kappa}, \delta)$-(semi)stable if and only if for any weighted filtration $(\mathcal{E}_\bullet, \underline{m})$, such that $\mathrm{par}\mu(\mathcal{E}_i) \geq \mathrm{par}\mu(\mathcal{E}) - C_1$, where $C_1 = a\delta + r\nu$, the inequality*

$$P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m}) + \delta\mu(\mathcal{E}_\bullet, \underline{m}, \phi)(\geq)0$$

holds.

Proof. Follows from Lemma 3.1.20. □

Proposition 3.1.22. *There exists n large enough so that, if $t \in Gies^{-1}(\mathbb{G}^{(s)s})$ then $(\mathcal{E}_t, \underline{q}_t, \phi_t)$ is $(\underline{\kappa}, \delta)$-(semi)stable.*

Proof. By Lemma 3.1.18, we know that for all $\mathcal{E}' \subset \mathcal{E}_t$, $\deg(\mathcal{E}') < \deg(\mathcal{E}_t) + K'$. Then, by Proposition 3.1.19, $\mathrm{Gies}(t) \in \mathbb{G}^{\delta-(s)s}$ implies $\mu_{\max}(\mathcal{E}_t) \leq \frac{\deg(\mathcal{E})}{\alpha} + C_3$. We also know, by Corollary 3.1.21, that $(\mathcal{E}_t, \underline{q}_t, \phi_t)$ is $(\underline{\kappa},\delta)$-(semi)stable if and only if

$$P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m}) + \delta\mu(\mathcal{E}_\bullet, \underline{m}, \phi)(\geq)0$$

for every $(\mathcal{E}_\bullet, \underline{m})$ with $\mathrm{par}\mu(\mathcal{E}_j) \geq \mathrm{par}\mu(\mathcal{E}) - C_1$. Observe that, in this case,

$$\mu(\mathcal{E}_j) > \mathrm{par}\mu(\mathcal{E}_j) \geq \mathrm{par}\mu(\mathcal{E}) - C_1 \geq \mu(\mathcal{E}) - \frac{\nu}{h} - C_1,$$

and denote $\overline{C}_1 = \frac{\nu}{h} + C_1$. Consider the family of locally free sheaves satisfying

$$\begin{aligned} &\mu_{\max}(\mathcal{E}') \leq \frac{\deg(\mathcal{E})}{\alpha} + C_3, \\ &\mathrm{par}\mu(\mathcal{E}') \geq \mathrm{par}\mu(\mathcal{E}) - C_1, \\ &1 \leq \alpha' \leq \alpha - 1. \end{aligned} \tag{3.18}$$

This family is clearly bounded (see proof of Theorem 3.1.16). Therefore, there is a natural number, $n \in \mathbb{N}$, large enough such that $\mathcal{E}'(n)$ is globally generated and $h^1(\mathcal{E}'(n)) = 0$ for any $\mathcal{E}'$ of this family. From the construction of the parameter space, we know that q_t induces an isomorphism $U \simeq H^0(\mathcal{E}_t(n))$.

Now, fix a weighted filtration $(\mathcal{E}_\bullet, \underline{m})$ satisfying the second condition in (3.18) (thus the three conditions). Let $\underline{u} = \{u_1, \ldots, u_p\}$ be a basis of U, such that there are indices $l_1, \dots, l_s$ with $U^{(l_j)} := \langle u_1, \dots, u_{l_j} \rangle \simeq H^0(\mathcal{E}_j(n))$ for each j. Define $\underline{\gamma} = \sum_{j=1}^{s} \alpha_j \gamma_p^{(l_j)}$ and consider the one parameter subgroup, $\lambda(\underline{u}, \gamma_p^{(l_j)})$. Let I_0 be a multiindex giving the minimum in $\mu_{\mathbb{G}_2}(\lambda(\underline{u}, \underline{\gamma}))$. Then $\mu_{\mathbb{G}}(\lambda(\underline{u}, \gamma), \mathrm{Gies}(t))(\geq 0)$ if and only if $\mu_{\mathbb{G}}(\lambda(\underline{u}, \gamma), \mathrm{Gies}(t))/p$ (≥ 0). But looking at the calculations at the beginning of Section 3.1.6, we have

$$\begin{aligned}
0(\leq)&\frac{\mu_{\mathbb{G}}(\lambda(\underline{u}, \gamma), \mathrm{Gies}(t))}{p} = \\
=&\sum_{u=1}^{s} m_u \{(\widehat{\alpha}_u \mathrm{pardeg}(\mathcal{E}) - \alpha \mathrm{pardeg}(\widehat{\mathcal{E}}_u)) + \delta(\alpha\nu(I_0, l_u) - a\widehat{\alpha}_u)\},
\end{aligned}$$

being $\widehat{\mathcal{E}_i}$ the saturated subsheaf generated by $\mathcal{E}_i$. Since $\widehat{\alpha_i} := \alpha(\widehat{\mathcal{E}_i}) = \alpha_i$ and $\mathrm{pardeg}(\widehat{\mathcal{E}_i}) \geq \mathrm{pardeg}(\mathcal{E}_i)$, then

$$\begin{aligned}
0(\leq)&\frac{\mu_{\mathbb{G}}(\lambda(\underline{u}, \gamma), \mathrm{Gies}(t))}{p} = \\
=&\sum_{u=1}^{s} m_u \{(\alpha \mathcal{E}_u \mathrm{pardeg}(\mathcal{E}) - \alpha \mathrm{pardeg}(\widehat{\mathcal{E}}_u)) + \delta(\alpha\nu(I_0, l_u) - a\widehat{\alpha}_u)\} \leq \\
\leq&\sum_{u=1}^{s} m_u \{(\alpha_u \mathrm{pardeg}(\mathcal{E}) - \alpha \mathrm{pardeg}(\mathcal{E}_u)) + \delta(\alpha\nu(I_0, l_u) - a\alpha_u)\} = \\
=&P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m}) + \delta\mu(\mathcal{E}_\bullet, \underline{m}, \tau).
\end{aligned}$$

Thus, the tensor field is $(\underline{\kappa}, \delta)$-semistable. □

Proposition 3.1.23. *There exists $n \gg 0$ such that, if $(\mathcal{E}_t, \underline{q}_t, \tau_t)$ is $(\underline{\kappa}, \delta)$-(semi)stable, then $t \in \mathrm{Gies}^{-1}(\mathbb{G}^{(s)s})$.*

Proof. By Theorem 3.1.16 we deduce that

$$\sum_{i=1}^{s} m_i(\mathrm{par}\chi(\mathcal{E}(n))\alpha_i - \mathrm{par}h^0(\mathcal{E}_i(n))\alpha) + \delta\mu(\mathcal{E}_\bullet, \underline{m}, \phi)(\geq) 0 \tag{3.19}$$

for any weighted filtration $(\mathcal{E}_\bullet, \underline{m})$. Let λ be a one parameter subgroup and $(U_\bullet, \underline{m}')$ a weighted filtration such that $\lambda = \lambda(U_\bullet, \underline{m}')$, being

$$U_\bullet \equiv (0) \subset U_1 \subset \ldots \subset U_{s'} \subset U.$$

The quotient $q_t : U \otimes \mathcal{O}_{Y_t}(-n) \to \mathcal{E}_t$ induces a chain

$$(0) \subseteq \mathcal{E}'_1 \subseteq \ldots \subseteq \mathcal{E}'_{s'} \subseteq \mathcal{E} \tag{3.20}$$

and, therefore, a filtration

$$\mathcal{E}_\bullet \equiv (0) \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_s \subset \mathcal{E},$$

formed by the different subsheaves collected in the above chain. Let $J = (i_1, \dots, i_s)$ be the multiindex defined by the following condition: $i_j \in \{1, \dots, s'\}$ is the maximum index among those $k \in \{1, \dots, s'\}$ such that $\mathcal{E}_j = \mathcal{E}'_k$. Denote by m_j the sum of the numbers m'_k corresponding to those sheaves in the chain (3.20) which are equal to $\mathcal{E}_i$, i.e.,

$$m_j = m_k + m_{k+1} + \ldots + m_{i_j}$$

$(k, k+1, \ldots, i_j)$ being the indices such that $\mathcal{E}'_k = \mathcal{E}'_{k+1} = \ldots = \mathcal{E}'_{i_j} = \mathcal{E}_j$. We get in this way a weighted filtration $(\mathcal{E}_\bullet, \underline{m})$. Multiplying by p in Equation (3.19) we get

$$0 \le \sum_{i=1}^{s} m_i \Bigg\{ p^2\alpha_i - ph^0(\mathcal{E}_i(n))\alpha + \delta p(\alpha\nu_i(I_0) - a\alpha_i) + \\ + p\sum_{j=1}^{\nu} \kappa_i \mathrm{dim}(q_j(\mathcal{E}_i(y_1^j) \oplus \mathcal{E}_i(y_2^j)))\alpha - rp(\sum_{j=1}^{\nu} \kappa_j)\alpha_i \Bigg\}.$$

The reverse calculation done in Section 3.1.6 gives

$$\begin{aligned} 0 \le &\sum_{u=1}^{l} b_u \sum_{i=1}^{s} m_i(\mathrm{rk}(\mathcal{E}_i^u)p - rh^0(\mathcal{E}_i(n))) + \\ &+ c\sum_{i=1}^{s} m_i(\nu_i(I_0)p - ah^0(\mathcal{E}_i(n))) + \\ &+ \sum_{j=1}^{\nu} k_j \sum_{i=1}^{s} m_i(p\mathrm{dim}(q_j(\mathcal{E}_i(y_1^j) \oplus \mathcal{E}_i(y_2^j))) - rh^0(\mathcal{E}_i(n))). \end{aligned} \tag{3.21}$$

By the fact that $l_i := \mathrm{dim} U_i \le h^0(\mathcal{E}_i(n))$ and by the definition of the numbers m_i, inequality (3.21) turns into

$$\begin{aligned} 0 \le &\sum_{u=1}^{l} b_u \sum_{i=1}^{s'} m'_i(\mathrm{rk}(\mathcal{E}_i^u)p - rl_i) + \\ &+ c\sum_{i=1}^{s'} m'_i(\nu_i(I_0)p - al_i) + \\ &+ \sum_{j=1}^{\nu} k_j \sum_{i=1}^{s'} m'_i(p\mathrm{dim}(q_j(\mathcal{E}_i(y_1^j) \oplus \mathcal{E}_i(y_2^j))) - rl_i) = \\ &= \mu_{\mathbb{G}}(\lambda(U_\bullet, \underline{m}'), \mathrm{Gies}(t)), \end{aligned} \tag{3.22}$$

and the proposition is proved. $\square$

Finally we have,

Theorem 3.1.24. *There exists n large enough so that, $(\mathcal{E}_t, \underline{q}_t, \tau_t)$ is $(\underline{\kappa}, \delta)$-(semi)stable if and only if $t \in \mathrm{Gies}^{-1}(\mathbb{G}^{(s)s})$.*

Proof. Follows from Proposition 3.1.22 and Proposition 3.1.23. $\square$

3.1.8 Properness of the Gieseker Map.

We will show that the Gieseker morphism Gies: $\mathfrak{I}_{\underline{d}} \to \mathbb{G}$ is proper (Theorem 3.1.25). Let us fix an effective divisor H of degree h in Y such that $\mathcal{O}_Y(H) \simeq \mathcal{O}_Y(1)$.

Theorem 3.1.25. *There exists n large enough such that, the Gieseker map,*

$$\mathrm{Gies} : \mathfrak{I}_{\underline{d}} \to \mathbb{G}$$

is proper for any $\underline{d} \in I_{r,d,\delta}$.

Proof. For the sake of notation we drop the subindex $\underline{d}$. We use the the valuative criterion for properness. Let $(\mathcal{O}, \mathfrak{m}, k)$ be a DVR being K its function field and assume we have a conmutative diagram

$$\begin{tikzcd}
 & \mathrm{Spec}(K) \arrow[r, "h_K"] \arrow[d] & \mathfrak{I}_{\underline{d}}^{\delta-(s)s} \arrow[d] \\
\{0,\eta\} = S : \arrow[r, equal] & \mathrm{Spec}(\mathcal{O}) \arrow[r, "h"] & \mathbb{G}^{(s)s}.
\end{tikzcd}$$

The morphism h_K is given by a family $(q_K, \underline{q}_K, \phi_K)$ over $Y_K := Y \times \mathrm{Spec}(K)$, where

$$\begin{aligned}
q_K &: U \otimes \mathcal{O}_{Y_K}(-n) \twoheadrightarrow \mathcal{E}_K \\
\phi_K &: (\mathcal{E}_K^{\otimes a})^{\oplus b} \to \det(\mathcal{E}_K)^{\otimes c} \otimes \mathcal{L}_K \\
q_{iK} &: \Gamma(\mathcal{E}_K|_{y_1^i, y_2^i}) \to R_K
\end{aligned} \tag{3.23}$$

Let us see that h_K can be extended to a family, $\widehat{h} = (q_S, \phi_S, \underline{q}_S)$, over $Y \times S$. The quotient q_K defines a point in the Quot scheme of quotients of $U \otimes \mathcal{O}_Y(-n)$ with the fixed Hilbert polynomial $P(n)$. Therefore, there exists a (unique) flat extension

$$q_S : U \otimes \pi^* \mathcal{O}_Y(-n) \twoheadrightarrow \mathcal{E}_S \tag{3.24}$$

over $Y \times S$. Define now the sheaves

$$\begin{aligned}
\mathcal{G} &= \pi_{S*}((U^{\otimes a})^{\oplus b} \otimes \pi_Y^* \mathcal{O}_Y) \\
\mathcal{H} &= \pi_{S*}(\det(\mathcal{E}_S)^{\otimes c} \otimes \pi_Y^* \mathcal{L} \otimes \pi_Y^* \mathcal{O}_Y(an))
\end{aligned}$$

Both sheaves are locally free, so we can form the projective space over S

$$\mathrm{pr}_S : \mathbb{P} := \mathbf{P}(\mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, \mathcal{H})^{\vee}) \to S$$

which carries a tautological morphism over $\mathbb{P} \times Y$,

$$\mathrm{pr}_{\mathbb{P}}^* \mathrm{pr}_{\mathbb{P}*}((U^{\otimes a})^{\oplus b} \otimes \mathrm{pr}_Y^* \mathcal{O}_Y) \to (\mathrm{id}_Y \times \mathrm{pr}_S)^* \det(\mathcal{E}_S)^{\otimes c} \otimes \mathrm{pr}_Y^* \mathcal{O}_Y(an) \otimes \mathrm{pr}_Y^* \mathcal{L} \otimes \mathrm{pr}_{\mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(1)$$

Now, the canonical morphism $\Delta : \mathrm{pr}_{\mathbb{P}}^* \mathrm{pr}_{\mathbb{P}*}(U^{\otimes a})^{\oplus b} \otimes \pi_Y^* \mathcal{O}_Y \to (U^{\otimes a})^{\oplus b} \otimes \pi_Y^* \mathcal{O}_Y$ induces a diagram

$$\begin{tikzcd}
K \arrow[r] \arrow[dr, dotted, "g"'] & \mathrm{pr}_{\mathbb{P}}^* \mathrm{pr}_{\mathbb{P}*}((U^{\otimes a})^{\oplus b} \otimes \mathrm{pr}_Y^* \mathcal{O}_Y) \arrow[r] \arrow[d] & (\mathrm{id}_Y \times \mathrm{pr}_S)^* (\mathcal{E}_S(n)^{\otimes a})^{\oplus b} \\
 & \mathcal{H}' &
\end{tikzcd},$$

where $\mathcal{H}' = (\mathrm{id}_Y \times \mathrm{pr}_S)^*\det(\mathcal{E}_S)^{\otimes c} \otimes \mathrm{pr}_Y^*\mathcal{O}_Y(an) \otimes \mathrm{pr}_Y\mathcal{L} \otimes \mathrm{pr}_{\mathbb{P}}^*\mathcal{O}_{\mathbb{P}}(1)$. Let $\mathbb{S} \subset \mathbb{P}$ be the closed subscheme over which g is the zero morphism, i.e., over which the tautological morphism factorizes through $(\mathrm{id}_Y \times \mathrm{pr}_S)^*(\mathcal{E}_S(n)^{\otimes a})^{\oplus b}$. Thus, we have over $\mathbb{S} \times Y$ a tautological morphism

$$(\mathrm{id}_Y \times \mathrm{pr}_S)^*(\mathcal{E}_S^{\otimes a})^{\oplus b} \to (\mathrm{id}_Y \times \mathrm{pr}_S)^*\det(\mathcal{E}_S)^{\otimes c} \otimes \mathrm{pr}_Y\mathcal{L} \otimes \mathrm{pr}_{\mathbb{P}}^*\mathcal{O}_{\mathbb{P}}(1)$$

Note now that the morphism $\phi_K : (\mathcal{E}_K^{\otimes a})^{\oplus b} \to \det(\mathcal{E}_K)^{\otimes c} \otimes \mathcal{L}_K$ defines a point $\mathrm{Spec}(K) \to \mathbb{S}$. Since $\mathbb{S}$ is projective this point extends (uniquely) to a point $\mathrm{Spec}(\mathcal{O}) \to \mathbb{S}$, i.e., to a morphism

$$\phi_S : (\mathcal{E}_S^{\otimes a})^{\otimes b} \to \det(\mathcal{E}_S)^{\otimes c} \otimes \pi_Y^*\mathcal{L} \otimes \mathcal{N} \tag{3.25}$$

Let us extend now the parabolic structure. Since $\mathcal{E}_{S,\eta} \simeq \mathcal{E}_K$ we have $\pi_{K*}(\mathcal{E}_{S,\eta}|_{D_i}) \simeq \pi_{K*}(\mathcal{E}_K|_{D_i})$. Thus composing with $\pi_{K*}(\mathcal{E}_K|_{D_i}) \twoheadrightarrow R_K$, we get a surjection

$$\pi_{K*}(\mathcal{E}_{S,\eta}|_{D_i}) \twoheadrightarrow R_K$$

Notice that the morphism $\pi_S : D_i \times S \to S$ is finite, thus affine and proper. By flat base change, we know that

$$\pi_{K*}(\mathcal{E}_{S,\eta}|_{D_i}) = j^*\pi_{S*}(\mathcal{E}_S|_{D_i}).$$

j being the open embedding $j : \eta \hookrightarrow S$. Now we can form the morphism

$$\begin{tikzcd} \pi_{S*}(\mathcal{E}_S|_{D_i}) \arrow[r] \arrow[rr, bend right, "q_{iS}"'] & j_*j^*\pi_{S*}(\mathcal{E}_S|_{D_i}) \arrow[r] & j_*R_K \end{tikzcd}$$

Let $R_S \subset j_*R_K$ be its image. Then by [26, Proposition 2.8.1], R_S is S-flat (thus a free $\mathcal{O}$-module) and the quotient

$$q_{iS} : \pi_{S*}(\mathcal{E}_S|_{D_i}) \twoheadrightarrow R_S \tag{3.26}$$

extends $q_{iS} : \pi_{K*}(\mathcal{E}_K|_{D_i}) \twoheadrightarrow R_K$ (thus $\mathrm{rk}(R_S) = r$). Then the family $\widehat{h} = (q_S, \phi_S, \underline{q}_S)$ given in Equations (3.24), (3.25), (3.26) extend the family given in Eq. (3.23). Clearly, the family $(q_S, \phi_S, \underline{q}_S)$ defines an S-valued $t : S \to \mathbb{G}$ in the Gieseker space. Since $t(\eta) = h(\eta)$ we deduce that $t(0) = h(0)$, thus it defines a semistable point in the Gieseker space $\mathbb{G}$. Let us show that $q_{(0)}$ induces an isomorphism $U \simeq H^0(Y, \mathcal{E}_{(0)}(n))$. To show that it is injective, we consider the kernel, $H \subset U$, of $H^0(q_{(0)}(n)) : Y \to H^0(Y, \mathcal{E}_{(0)}(n))$. Since $t(0)$ is semistable we have,

$$\begin{aligned}
\mu_{\mathbb{G}}(\lambda, t(0)) &= \sum_{i=1}^{l} b_i \mu_{\mathbb{G}_1^i}(\lambda, t_{1,i}(0)) + c\mu_{\mathbb{G}_2}(\lambda, t_2(0)) + \sum_{i=1}^{\nu} k_i \mu_{\mathcal{G}r}(\lambda, t_{3,i}(0)) = \\
&= \sum_{i=1}^{l} b_i(-r\dim(H)) + ca(-\dim(H)) + \\
&+ \sum_{i=1}^{\nu} k_i(p\dim(t_{i0}(H \oplus H) - r\dim(H)) = \\
&= \sum_{i=1}^{l} d_i(p - a\delta - r(\sum_{i=1}^{\nu} \kappa_i))(-r\dim(H)) + \sum_{i=1}^{l} d_i\delta ra(-\dim(H)) + \\
&+ \sum_{i=1}^{\nu} \kappa_i\alpha(-r\dim(H)) = -\alpha p\dim(H) \geq 0
\end{aligned}$$

so we must have $\dim(H)=0$, i.e, $U\to H^0(Y,\mathcal{E}_{(0)}(n))$ is injective.
Let us show that it is in fact an isomorphism. For that we just need to show that $h^1(Y,\mathcal{E}_{(0)})=0$. Suppose it does not. Then, by Serre duality, there is a non trivial morphism $\mathcal{E}_{(0)}\to\omega_Y$. Let $\mathcal{G}$ be its image, and consider the linear map

$$\Omega: U\hookrightarrow H^0(Y,\mathcal{E}_{(0)}(n))\to H^0(Y,\mathcal{G}(n))$$

Let $H\subset U$ be the kernel of Ω, let λ be the corresponding one parameter subgroup and let $\mathcal{F}\subset\mathcal{E}_{(0)}$ be the subsheaf generated by H. Since $t(0)$ is semistable, we get:

$$\begin{aligned}0\leq\frac{\mu(\lambda,\mathrm{Gies}(t))}{p}=&\Bigg\{p\alpha_{\mathcal{F}}-\alpha\dim(H)+\\&+\delta\sum_{i=1}^{l}d_i(r\nu(I_0,\dim(H))-a\mathrm{rk}(\mathcal{F}^i))+\\&+\sum_{i=1}^{\nu}\alpha\kappa_i\dim(q_i(\mathcal{F}(y_1^i)\oplus\mathcal{F}(y_2^i))-b_2'\alpha_{\mathcal{F}}\Bigg\}.\end{aligned}$$

Since $h^0(Y,\mathcal{G}(n))\geq p-\dim(H)$, we get

$$\begin{aligned}0\leq&\Bigg\{-p\alpha_{\mathcal{G}}+\alpha h^0(Y,\mathcal{G}(n))+\\&+\delta\sum_{i=1}^{l}d_i(r\nu(I_0,\dim(H))-a\mathrm{rk}(\mathcal{F}^i))+\\&+\sum_{i=1}^{\nu}\alpha\kappa_i\dim(q_i(\mathcal{F}(y_1^i)\oplus\mathcal{F}(y_2^i))-b_2'\alpha_{\mathcal{F}}\Bigg\}.\end{aligned}$$

and therefore

$$h^0(Y,\mathcal{G}(n))\geq p\alpha_{\mathcal{G}}+B$$

B being a constant not depending on $\mathcal{G}$. Note that $p=\alpha n+d+r(1-g)$ and that we can assume $h^0(Y,\omega_Y)\geq h^0(Y,\mathcal{G}(n))$. Then, if n is l arge enough we get a contradiction, so $h^1(Y,\mathcal{E}_{(0)})=0$.

Let us show now that $\mathcal{E}_{(0)}$ has no torsion. Assume it has torsion, $\mathcal{T}\subset\mathcal{E}_{(0)}(n)$, supported on the divisors D_i, and let $T=H^0(\mathcal{T})$. Let now $H:=H0(q_{(0)}(n))^{-1}(T)\subset$

U. Again, since $t_{(0)}$ is semistable, we have

$$
\begin{aligned}
0 \leq \mu_{\mathbb{G}}(\lambda, t(0)) &= \sum_{i=1}^{l} b_i \mu_{\mathbb{G}_1^i}(\lambda, t_{1,i}(0)) + c\mu_{\mathbb{G}_2}(\lambda, t_2(0)) + \sum_{i=1}^{\nu} k_i \mu_{\mathcal{G}r}(\lambda, t_{3,i}(0)) = \\
&= \sum_{i=1}^{l} b_i(-r\mathrm{dim}(H)) + ca(-\mathrm{dim}(H)) + \\
&+ \sum_{i=1}^{\nu} k_i(p\mathrm{dim}(t_{i0}(H \oplus H) - r\mathrm{dim}(H)) = \\
&= \sum_{i=1}^{l} d_i(p - a\delta - r(\sum_{i=1}^{\nu} \kappa_i))(-r\mathrm{dim}(H)) + \sum_{i=1}^{l} d_i \delta r a(-\mathrm{dim}(H)) + \\
&+ \sum_{i=1}^{\nu} \kappa_i \alpha(-r\mathrm{dim}(H)) + \sum_{i=1}^{\nu} k_i(p\mathrm{dim}(t_{i0}(H \oplus H)) = \\
&= \sum_{i=1}^{\nu} \kappa_i \alpha(p\mathrm{dim}(t_{i0}(H \oplus H)) - \alpha p\mathrm{dim}(H) \leq \\
&= \sum_{i=1}^{\nu} \kappa_i \alpha p\mathrm{dim}(T_{D_i}) - \sum_{i=1}^{\nu} \alpha p\mathrm{dim}(T_{D_i}) = \\
&= \alpha p \sum_{i=1}^{\nu} (\kappa_i - 1)\mathrm{dim}(T_{D_i})
\end{aligned}
$$

Since $\kappa_i - 1 < 0$ we must have $\mathrm{dim}(T_{D_i}) = 0$, that is $\mathcal{T} = 0$, so $\mathcal{E}_{(0)}$ has no torsion supported on the divisors D_i. Furthermore, from the last calculation it is clear that there can no be any torsion subsheaf supported outside the divisors D_i, Therefore $\mathcal{E}_{(0)}$ is locally free.
Thus, the extended family defines a point in $\mathfrak{I}_{\underline{d}}$. Since the corresponding point in $\mathbb{G}$ lies in the semistable part we deduce that the extended family also lies in $\mathbb{G}^{s(s)}$, and by Theorem 3.1.24 we are done. □

3.1.9 Construction of the Moduli Space

Let $\delta \in \mathbb{Q}_{>0}$, $r, d, a, b, c \in \mathbb{N}$ and $\underline{d} \in I_{r,d,\delta}$ be as in Section 3.1.4, Equation 3.10. Let $\mathfrak{I}_{\underline{d}}$ be the parameter space constructed in Section 3.1.4. Over $Y \times \mathfrak{I}_{\underline{d}}$ there is a universal family satisfying the local universal property,

Proposition 3.1.26. (Local Universal Property) *Let S be a scheme of finite type over $\mathbb{C}$ and $(\mathcal{E}_S, q_S, \mathfrak{N}_S, \phi_S)$ a family of $(\underline{\kappa}, \delta)$-(semi)stable generalized parabolic tensor fields of rank r and multidegree $\underline{d}$ with a decoration of type $a, b, c, \mathcal{L}$ parametrized by S. Then there exists an open covering S_i, $i \in I$ of S and morphisms $\beta_i : S_i \to \mathfrak{I}_{\underline{d}}$, $i \in I$ such that the restriction of the family $(\mathcal{E}_S, q_S, \mathfrak{N}_S, \phi_S)$ to $S_i \times Y$ is equivalent to the pullback of $(\mathcal{E}_{\mathfrak{I}_{\underline{d}}}, \underline{q}_{\mathfrak{I}_{\underline{d}}}, \mathfrak{N}_{\mathfrak{I}_{\underline{d}}}, \phi_{\mathfrak{I}_{\underline{d}}})$ via $\beta_i \times \mathrm{id}_Y$ for all $i \in I$*

Proof. Follows using the standard arguments given in Proposition 2.2.16. □

Note that the natural $\mathrm{SL}(V)$ action on $\mathcal{Q}^0$, $\mathfrak{h}$ and Gr determines an action on the space $\mathfrak{I}_{\underline{d}}$,

$$\Gamma : \mathrm{SL}(V) \times \mathfrak{I}_{\underline{d}} \to \mathfrak{I}_{\underline{d}}.$$

Then we have

Proposition 3.1.27. (Glueing Property) *Let S be a scheme of finite type over $\mathbb{C}$ and $s_1, s_2 : S \to \mathfrak{I}_{\underline{d}}$ two morphisms such that the pullbacks of $(\mathcal{E}_{\mathfrak{I}_{\underline{d}}}, \underline{q}_{\mathfrak{I}_{\underline{d}}}, \mathfrak{N}_{\mathfrak{I}_{\underline{d}}}, \phi_{\mathfrak{I}_{\underline{d}}})$ via $s_1 \times \mathrm{id}_X$ and $s_2 \times \mathrm{id}_X$ are isomorphic. Then there exists an étalé covering $c : T \to S$ and a morphism $g : T \to \mathrm{SL}(V)$ such the triangle*

$$\begin{tikzcd} \mathrm{SL}(V) \times \mathfrak{I}_{\underline{d}} \arrow[rr, "\Gamma"] & & \mathfrak{I}_{\underline{d}} \\ & T \arrow[ul, "g\times(s_1\circ c)"] \arrow[ur, "s_2\circ c"'] & \end{tikzcd}$$

is commutative.

Proof. Follows using the standard arguments given in Proposition 2.2.15. □

Finally, we have

Theorem 3.1.28. *There is a projective scheme $\mathcal{PTF}_P^{(\underline{\kappa},\delta)\text{-}ss}$ and an open subscheme $\mathcal{PTF}_P^{(\underline{\kappa},\delta)\text{-}s}$ together with natural tranformations*

$$\alpha^{(s)s} : \mathbf{ParTensors}^{(\underline{k},\delta)-(s)s}_{P,\mathcal{L},\{D_i\}} \to h_{\mathcal{PTF}_P^{(\underline{\kappa},\delta)\text{-}(s)s}}$$

with the following propoerties:

1) For every scheme $\mathcal{N}$ and every natural transformation $\mathbf{ParTensors}^{(\underline{k},\delta)-(s)s}_{P,\mathcal{L},\{D_i\}} \to h_{\mathcal{N}}$, there exists a unique morphism $\varphi : \mathcal{PTF}_P^{(\underline{\kappa},\delta)\text{-}(s)s} \to \mathcal{N}$ with $\alpha' = h(\varphi) \circ \alpha^{(s)s}$.

2) The scheme $\mathcal{PTF}^{(\underline{\kappa},\delta)\text{-}s}_{P,\mathcal{L},\{D_i\}}$ is a coarse moduli space for $\mathbf{ParTensors}_P^{(\underline{k},\delta)\text{-}s}$.

Proof. Consider the Gieseker map Gies : $\mathfrak{I}_{\underline{d}} \hookrightarrow \mathbb{G}$, which is injective and $\mathrm{SL}(U)$-equivariant (see Proposition 3.1.17). Consider on $\mathbb{G}$ the polarization,

$$\mathcal{O}(b_1, \dots, b_l, c, k_1, \dots, k_\nu)$$

given in Subsection 3.1.6 and let $\mathcal{L} := \mathrm{Gies}^*\mathcal{O}(b_1, \dots, b_l, c, k_1, \dots, k_\nu)$. By Proposition 1.1.21, we know that $\mathrm{Gies}^{-1}(\mathbb{G}^{(s)s}) = \mathfrak{I}_{\underline{d}}^{(s)s}$, and therefore Theorem 3.1.24 implies that $\mathfrak{I}_{\underline{d}}^{(s)s} = \mathfrak{I}_{\underline{d}}^{(\underline{\kappa},\delta)\text{-(s)s}}$. By Theorem 3.1.25, we deduce that the restriction of the Gieseker map to the semistable locus is a SL(U)-equivariant injective and proper morphism. Thus

1) the good quotient $\mathcal{PTF}_{\underline{d}}^{(\underline{\kappa},\delta)\text{-ss}} := \mathfrak{I}_{\underline{d}}^{(\underline{\kappa},\delta)\text{-ss}} /\!\!/ \mathrm{SL}(U)$ exists and is projective,

2) the geometric quotient $\mathcal{PTF}_{\underline{d}}^{(\underline{\kappa},\delta)\text{-s}} := \mathfrak{I}_{\underline{d}}^{(\underline{\kappa},\delta)\text{-s}} / \mathrm{SL}(U)$ exists and is an open subscheme of $\mathcal{PTF}_{\underline{d}}^{(\underline{\kappa},\delta)\text{-ss}}$.

Define

$$\mathcal{PTF}_P^{(\underline{\kappa},\delta)\text{-(s)s}} := \coprod_{\underline{d} \in I(r,d,\delta)} \mathcal{PTF}_{\underline{d}}^{(\underline{\kappa},\delta)\text{-(s)s}}.$$

Now, 1) and 2) follow from this construction, Proposition 3.1.26 and Proposition 3.1.27. □

3.2 Moduli Space of Singular Principal G-Bundles with Generalised Parabolic Structures

The aim of this section is to prove Theorem 3.2.8, which shows the existence of a coarse moduli space of $(\underline{\kappa},\delta)$-(semi)stable singular principal G-bundles with generalized parabolic structure. The strategy we will follow here is the same one as we have followed in Chapter 2, for the construction of the moduli space of δ-(semi)stable singular principal G-bundles.

3.2.1 Singular Principal G-Bundles with Generalized Parabolic Structures on Non Connected Smooth Curves

Definition 3.2.1. A singular principal G-bundle with a generalized parabolic structures over Y is a triple $(\mathcal{E},\tau,\underline{q})$ where $(\mathcal{E},\underline{q})$ is a generalized parabolic bundle of rank r (see Definition 3.1.1) and $(\mathcal{E},\tau)$ is a singular principal G-bundle.

Definition 3.2.2. Let $(\mathcal{E},\tau,\underline{q})$ and $(\mathcal{G},\lambda,\underline{p})$ be singular principal G-bundles with generalized parabolic structure on Y. A morphism between them is a morphism of $\mathcal{O}_Y$-modules $f:\mathcal{F}\to\mathcal{G}$ compatible with both structures. The isomorphisms are the obvious ones.

Definition 3.2.3. We say that a singular principal G-bundle with a generalized parabolic structure on Y, $(\mathcal{E},\tau,\underline{q})$, is honest if the singular principal G-bundle $(\mathcal{E},\tau)$ is. We say that it is quasi-honest if it is honest over some subcurve $Y'\subset Y$.

Following Section 2.2.3, we can assign to any singular principal G-bundle with generalized parabolic structure a tensor field with a generalized parabolic structure,

$$(\mathcal{E},\tau,\underline{q})\mapsto(\mathcal{E},\varphi_\tau,\underline{q}),$$

this map being injective. Also, we can define, for any weighted filtration $(\mathcal{E}_\bullet,\underline{m})$, the semistability function $\mu(\mathcal{E}_\bullet,\underline{m},\tau)$ as in Section 2.2.3 (see Definition (2.2.13)).

Definition 3.2.4. We fix numbers $\delta\in\mathbb{Q}_{>0}$ and $\kappa_i\in(0,1)\cap\mathbb{Q}$, $i=1,\dots,\nu$. Denote $\underline{\kappa}=(\kappa_1,\dots,\kappa_\nu)$. A singular principal G-bundle with generalized parabolic structure, $(\mathcal{E},\underline{q},\tau)$, is $(\underline{\kappa},\delta)$-(semi)stable if for every weigted filtration $(\mathcal{E}_\bullet,\underline{m})$ of $\mathcal{E}$, the inequality

$$P_{\underline{\kappa}}(\mathcal{E}_\bullet,\underline{m})+\delta\mu(\mathcal{E}_\bullet,\underline{m},\tau)(\geq)0 \tag{3.27}$$

holds true.

3.2.2 The Parameter Space

The aim of this section is to prove the existence of a coarse projective moduli space for the moduli functor given by

$$\mathbf{SPBGPS}(\rho)^{(\underline{\kappa},\delta)\text{-(s)s}}_P(S)=\left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{families of }(\underline{\kappa},\delta)\text{-(semi)stable singular}\\ \text{principal }G\text{-bundles with}\\ \text{generalized parabolic structure on }Y\\ \text{parametrized by }S\text{ with Hilbert polynomial }P\end{array}\right\}$$

We will use the same strategy as in Section 2.1.4 for the construction of this moduli space. Therefore, we need to rigidify the moduli problem. Let $m \in \mathbb{N}$ and W a vector space of dimension $P(m)$. Consider the functor

$$ {}^{rig}\mathbf{SPBGPS}(\rho)^n_P(S) = \left\{ \begin{array}{c} \text{isomorphism classes of tuples } (\mathcal{E}_S, \underline{q}_S, \tau_S, g_S) \text{ where} \\ (\mathcal{E}_S, \tau_S) \text{ is a family of singular principal } G\text{-bundles} \\ \text{parametrized by } S \text{ with Hilbert polynomial } P \\ (\mathcal{E}_S, \underline{q}_S) \text{ is a family of generalized parabolic bundles} \\ \text{and } g_S : W \otimes \mathcal{O}_S \to \pi_{S*}\mathcal{E}_S(n) \text{ is a morphism} \\ \text{such that the induced morphism} \\ W \otimes \mathcal{O}_{Y\times S}(-n) \to \mathcal{E}_S \text{ is surjective} \end{array} \right\}. \tag{3.28} $$

and let us show that there is a representative for it.

We will reproduce the construction given in Chapter 2, Section 2.2. Recall from Proposition 3.1.15 that the family of locally free sheaves $\mathcal{E}$ of rank r and degree d which appear in $(\underline{\kappa}, \delta)$-(semi)stable tensor fields with generalized parabolic structure is bounded. In consequence, there is a natural number $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $\mathcal{E}(n)$ is globally generated and $H^1(\mathcal{E}(n)) = 0$.

Fix $n > \max\{n_0, n_1\}$ and $\underline{d} = (d_1, \dots, d_l) \in \mathbb{N}^l$ with $d = \sum_{i=1}^l d_i$, and let $p = r\chi(\mathcal{O}_Y) + d + \alpha n$ (recall $\alpha = hr$). Let W be the vector space $k^{\oplus p}$. Denote by $\mathcal{Q}^0$ the quasi-projective scheme parametrizing equivalence classes of quotients $\mathfrak{q} : U \otimes \pi_Y^* \mathcal{O}_Y(-n) \to \mathcal{E}$ where $\mathcal{E}$ is a locally free sheaf of uniform multirank r and multidegree $(d_1, \dots, d_l)$ on Y, and such that the induced map $U \to H^0(Y, \mathcal{E}(n))$ is an isomorphism. On $\mathcal{Q}^0 \times Y$, we have the following morphism on $Q^0 \times Y$,

$$ h : S^\bullet(V \otimes W \otimes \pi_Y^* \mathcal{O}_Y(-n)) \to S^\bullet(V \otimes \mathcal{E}_{\mathcal{Q}^0}) \to S^\bullet(V \otimes \mathcal{E}_{\mathcal{Q}^0})^G. $$

Let $s \in \mathbb{N}$ be as in Theorem 2.2.6. Then

$$ h(\bigoplus_{i=1}^s S^i(V \otimes W \otimes \pi_Y \mathcal{O}_Y(-n))), $$

contains a set of generators of $S^\bullet(V \otimes \mathcal{E}_{\mathcal{Q}^0})^G$. A morphism $k\colon \oplus_{i=1}^s S^i(V \otimes W \otimes \mathcal{O}_Y(-n)) \to \mathcal{O}_Y$ breaks into a family of morphisms

$$ k^i : S^i(V \otimes W) \otimes \mathcal{O}_Y(-in) \simeq S^i(V \otimes W \otimes \mathcal{O}_Y(-n)) \to \mathcal{O}_Y $$

and therefore into morphisms

$$ k^i : S^i(V \otimes W) \overset{\Delta}{\hookrightarrow} S^i(V \otimes W) \otimes k^{\oplus l} \to H^0(\mathcal{O}_Y(in)), $$

Δ being the diagonal morphism. Consider the space

$$ \mathcal{Q}^* := \mathcal{Q}^0 \times \bigoplus_{i=1}^s \underline{\mathrm{Hom}}(S^i(V \otimes W), H^0(\mathcal{O}_Y(in))). $$

We want to put a scheme structure on the locus given by the points $([q], [k]) \in \mathcal{Q}^*$ that comes from a morphism of algebras

$$ S^\bullet(V \otimes \mathcal{E}_{\mathcal{Q}^0|_{[q]} \times Y})^G \to \mathcal{O}_Y. $$

On $\mathcal{Q}^* \times Y$ there are universal morphisms

$$\varphi'^i : S^i(V \otimes W) \otimes \mathcal{O}_{\mathcal{Q}^* \times Y} \to H^0(\mathcal{O}_Y(in)) \otimes \mathcal{O}_{\mathcal{Q}^* \times Y}.$$

Consider the pullbacks of the evaluation maps to $\mathcal{Q}^* \times Y$,

$$H^0(\mathcal{O}_Y(in)) \otimes \mathcal{O}_{\mathcal{Q}^* \times Y} \to \pi_Y^* \mathcal{O}_Y(in).$$

Composing we get

$$\varphi^i : S^i(V \otimes W) \otimes \mathcal{O}_{\mathcal{Q}^* \times Y} \to \pi_Y^* \mathcal{O}_Y(in).$$

and summing up,

$$\varphi : V_{\mathcal{Q}^*} := \bigoplus_{i=1}^{s} S^i(V \otimes W \otimes \pi_Y^* \mathcal{O}_Y(-n)) \to \mathcal{O}_{\mathcal{Q}^* \times Y}.$$

Now, φ gives a morphism

$$\tau'_{\mathcal{Q}^*} : S^\bullet(V_{\mathcal{Q}^*}) \to \mathcal{O}_{\mathcal{Q}^* \times Y}.$$

Consider again the universal quotient $\pi_{\mathcal{Q}}$ and the following chain of surjections

$$\begin{tikzcd}
S^\bullet(V \otimes W \otimes \otimes \pi_Y^* \mathcal{O}_Y(-n)) \arrow[r, "S^\bullet(\pi_{\mathcal{Q}})"] & S^\bullet(V \otimes \pi_{\mathcal{Q}^* \times Y}^* \mathcal{E}_{\mathcal{Q}^*}) \arrow[d] \\
S^\bullet V_{\mathcal{Q}^*} \arrow[u] & S^\bullet(V \otimes \pi_{\mathcal{Q}^* \times Y}^* \mathcal{E}_{\mathcal{Q}^*})^G
\end{tikzcd}$$

Let us denote by β the composition of these morphisms and consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\beta) \arrow[r, hook] \arrow[dr, dashed, "\tau'_{\mathcal{Q}^*}"] & S^\bullet V_{\mathcal{Q}^*} \arrow[r] \arrow[d, "\tau'_{\mathcal{Q}^*}"] & S^\bullet(V \otimes \pi_{\mathcal{Q}^* \times Y}^* \mathcal{E}_{\mathcal{Q}^*})^G \arrow[r] & 0 \\
& & \mathcal{O}_{\mathcal{Q}^* \times Y} & &
\end{tikzcd}$$

Define $\mathbb{D} = \{c = ([q],[h]) | \tau'_{\mathcal{Q}^*}|_c = 0\}$. This is a closed subscheme of $\mathcal{Q}^*$ over which $\tau_{\mathcal{Q}^*}$ lifts to

$$\tau_{\mathbb{D}} : S^\bullet(V \otimes \pi_{\mathcal{Q}^* \times Y}^* \mathcal{E}_{\mathcal{Q}^*}) \to \mathcal{O}_{\mathcal{Q}^* \times Y}.$$

Thus the pullback of $(\mathcal{E}_{\mathcal{Q}^*}, \tau'_{\mathcal{Q}^*})$ to $\mathbb{D} \times Y$ gives a universal family $(\mathcal{E}_{\mathbb{D}}, \tau_{\mathbb{D}})$. In order to include the parabolic structure as well we need to consider the Grassmannian $\mathcal{G}r := \mathrm{Grass}_r(U^{\oplus 2})$ of r dimensional quotients of $U^{\oplus 2}$. Define

$$Z := \mathbb{D} \times \overbrace{\mathcal{G}r \times \ldots \times \mathcal{G}r}^{\nu},$$

and denote by $c_i : Z \to \mathcal{G}r$ the projection onto the ith Grassmannian. Consider the pullback of the universal quotient of the ith grassmannian to Z:

$$q_Z^i : U^{\oplus 2} \otimes \mathcal{O}_Z \to R_Z,$$

and take the direct sum

$$q_Z : U^{\oplus 2\nu} \otimes \mathcal{O}_Z \to \bigoplus_1^{\nu} R_Z.$$

Consider now the two natural projections

$$\begin{tikzcd} \mathbb{D} \times Y \arrow[d] & Z \times Y \arrow[d] \\ \mathbb{D} & Z. \end{tikzcd}$$

Denote by $\mathfrak{q}_Z$, $\mathcal{E}_Z$ and τ_Z the pullbacks to $Z \times Y$ and look at the following commutative diagram

$$\begin{tikzcd} Z \times Y \arrow[d, "\pi"] & Z \times \{y_1^i, y_2^i\} \arrow[l, hook', "\overline{\alpha}^i"'] \arrow[d, "\pi^i"'] \\ Z \times X & Z \times \{x_i\} = Z. \arrow[l, hook', "\alpha^i"'] \end{tikzcd}$$

For each i, there are quotients

$$f_i : U^{\oplus 2} \times \mathcal{O}_Z \to \pi^i_*(\mathcal{E}_Z|_{y_1^i, y_2^i}).$$

Tanking the direct sum over all i we get

$$f := \oplus(f_i) : U^{\oplus 2\nu} \times \mathcal{O}_Z \to \bigoplus \pi^i_*(\mathcal{E}_Z|_{y_1^i, y_2^i}).$$

Consider the following diagram,

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(f) \arrow[r] \arrow[dr, dotted, "q'"] & U^{\oplus 2\nu} \times \mathcal{O}_Z \arrow[r, "f"] \arrow[d, "q_Z"] & \bigoplus \pi^i_*(\mathcal{E}_Z|_{y_1^i, y_2^i}) \arrow[r] & 0 \\ & & \bigoplus R_Z & & . \end{tikzcd}$$

Denote by $\mathfrak{M}_{\underline{d}}(G) \subset Z$ the closed subscheme given by the zero locus of the morphism q' (see Lemma 2.1.35). Then, the restriction of q_Z to $\mathfrak{M}_{\underline{d}}(G)$ factorizes

$$\bigoplus \pi^i_*(\mathcal{E}_Z|_{y_1^i, y_2^i})|_{\mathfrak{M}_{\underline{d}}(G)} = \bigoplus \pi^i_{\mathfrak{M}_{\underline{d}}(G)*}(\mathcal{E}_{\mathfrak{M}_{\underline{d}}(G)}|_{y_1^i, y_2^i}) \xrightarrow{q_{\mathfrak{M}_{\underline{d}}(G)}} \bigoplus R_Z|_{\mathfrak{M}_{\underline{d}}(G)} = \bigoplus R_{\mathfrak{M}_{\underline{d}}(G)}.$$

Since f and q_Z are diagonal morphisms we deduce that $q_{\mathfrak{M}_{\underline{d}}(G)}$ is also diagonal. Therefore $q_{\mathfrak{M}_{\underline{d}}(G)}$ is determined by ν morphisms

$$q^i_{\mathfrak{M}_{\underline{d}}(G)} : \pi^i_{\mathfrak{M}_{\underline{d}}(G)*}(\mathcal{E}_{\mathfrak{M}_{\underline{d}}(G)}|_{y_1^i, y_2^i}) \to R_{\mathfrak{M}_{\underline{d}}(G)}.$$

Denote by $(\mathcal{E}_{\mathfrak{M}_{\underline{d}}(G)}, \tau_{\mathfrak{M}_{\underline{d}}(G)})$ the restriction of $(\mathcal{E}_Z, \tau_Z)$ to $\mathfrak{M}_{\underline{d}}(G)$. Then we have a universal family of singular principal G-bundles with generalized parabolic structure

$$(\mathcal{E}_{\mathfrak{M}_{\underline{d}}(G)}, \underline{q}_{\mathfrak{M}_{\underline{d}}(G)}, \tau_{\mathfrak{M}_{\underline{d}}(G)}). \tag{3.29}$$

Theorem 3.2.5. *The functor* ${}^{rig}\mathbf{SPBGPS}^n_P$ *is representable.*

Proof. Follows from the construction of $\mathfrak{M}_{\underline{d}}(G)$ and taking the disjoint union over all the possible multidegrees as in Theorem 3.1.28, which we denote by $\mathfrak{M}(G)$. $\square$

3.2.3 Construction of the Moduli Space

Recall from Proposition 3.1.15 that the family of locally free sheaves $\mathcal{E}$ which appears in a $(\underline{\kappa}, \delta)$-(semi)stable tensor field with generalized parabolic structure is bounded. As a consequence, there is a natural number $n \in \mathbb{N}$ such that for $n \geq n_0$, $\mathcal{E}(n)$ is globally generated and $h^1(Y, \mathcal{E}(n)) = 0$. Fix such natural number n and consider the functors

$$
{}^{\mathrm{rig}}\mathbf{ParTensors}^n_{P,\mathcal{O}_Y}(S) = \left\{ \begin{array}{c} \text{isomorphism classes of tuples } (\mathcal{E}_S, \underline{q}_S, \phi_S, N, g_S) \\ \text{where } (\mathcal{E}_S, \underline{q}_S) \text{ is a family of generalized parabolic} \\ \text{bundles with polynomial } P \text{ and } g_S \text{ is a morphism} \\ g_S : W \otimes \mathcal{O}_S \to \pi_{S*}\mathcal{E}_S(m) \text{ such that} \\ \text{the induced morphism } W \otimes \mathcal{O}_{Y \times S}(-n) \to \mathcal{E}_S \\ \text{is surjective and } \phi_S \text{ is a morphism} \\ \phi_S : ((V \otimes \mathcal{E}_S)^{\otimes d!})^{\oplus N} \to \pi_S^* N \end{array} \right\}
$$

$$
{}^{\mathrm{rig}}\mathbf{SPBGPS}(\rho)^n_P(S) = \left\{ \begin{array}{c} \text{isomorphism classes of tuples } (\mathcal{E}_S, \underline{q}_S, \tau_S, g_S) \text{ where} \\ (\mathcal{E}_S, \tau_S) \text{ is a family of singular principal } G\text{-bundles} \\ (\mathcal{E}_S, \underline{q}_S) \text{is a family of generalized parabolic bundles} \\ \text{with Hilbert polynomial P and } g_S : W \otimes \mathcal{O}_S \to \pi_{S*}\mathcal{E}_S(n) \\ \text{is a morphism such that the induced morphism} \\ W \otimes \mathcal{O}_{Y \times S}(-n) \to \mathcal{E}_S \text{ is surjective} \end{array} \right\}
$$

Note that there is a natural $\mathrm{GL}(W)$ action the space $\mathfrak{M}(G)$,

$$\Gamma : \mathrm{GL}(W) \times \mathfrak{M}(G) \to \mathfrak{M}(G).$$

We can view the $\mathrm{GL}(W)$-action as a $(\mathbb{C}^* \times \mathrm{SL}(W))$-action. Thus, we will construct the quotient of $\mathfrak{M}(G)$ by $\mathrm{GL}(W)$ in two steps, considering the actions of $\mathbb{C}^*$ and $\mathrm{SL}(W)$ separately. Consider the action of $\mathbb{C}^*$ on ${}^{\mathrm{rig}}\mathbf{SPBGPS}(\rho)^n_P$. By Theorem 2.2.12 and Definition 2.1.2, there is an injective $\mathbb{C}^*$-invariant natural transformation

$${}^{\mathrm{rig}}\mathbf{SPBGPS}(\rho)^n_P \hookrightarrow {}^{\mathrm{rig}}\mathbf{ParTensors}^n_{P,\mathcal{O}_Y}.$$

Moreover, the morphism induced between the representatives is a $\mathrm{SL}(W)$-equivariant injective and proper morphism,

$$\beta : \mathfrak{M}(G) /\!\!/ \mathbb{C}^* \hookrightarrow \coprod \mathfrak{I}_{r,d,\delta}. \tag{3.30}$$

Then we have

Proposition 3.2.6. (Local Universal Property) *Let S be a scheme of finite type over $\mathbb{C}$ and $(\mathcal{E}_S, q_S, \tau_S)$ a family of $(\underline{\kappa}, \delta)$-(semi)stable singular principal G-bundles with generalized parabolic structure parametrized by S. Then there exists an open covering S_i, $i \in I$ of S and morphisms $\beta_i : S_i \to \mathfrak{M}(G)$, $i \in I$ such that the restriction of the family $(\mathcal{E}_S, q_S, \tau_S)$ to $S_i \times Y$ is equivalent to the pullback of $(\mathcal{E}_{\mathfrak{M}(G)}, \underline{q}_{\mathfrak{M}(G)}, \tau_{\mathfrak{M}(G)})$ via $\beta_i \times \mathrm{id}_Y$ for all $i \in I$*

Proof. Follows using the standard arguments given in Proposition 2.2.16. ☐

Note that there is a natural $\mathrm{SL}(W)$-action on the space $\mathfrak{M}(G)$,

$$\Gamma : \mathrm{SL}(W) \times \mathfrak{M}(G) \to \mathfrak{M}(G).$$

Then we have,

Proposition 3.2.7. (Glueing Property) *Let S be a scheme of finite type over $\mathbb{C}$ and $s_1, s_2 : S \to \mathfrak{M}(G)$ two morphisms such that the pullbacks of $(\mathcal{E}_{\mathfrak{M}(G)}, \underline{q}_{\mathfrak{M}(G)}, \tau_{\mathfrak{M}(G)})$ via $s_1 \times \mathrm{id}_X$ and $s_2 \times \mathrm{id}_X$ are isomorphic. Then there exists an étalé covering $c : T \to S$ and a morphism $g : T \to \mathrm{SL}(W)$ such the triangle*

$$\begin{tikzcd} \mathrm{SL}(W) \times \mathfrak{M}(G) \arrow[rr, "\Gamma"] & & \mathfrak{M}(G) \\ & T \arrow[ul, "g\times(s_1\circ c)"] \arrow[ur, "s_2\circ c"'] & \end{tikzcd}$$

is commutative.

Proof. Follows using the standard arguments given in Proposition 2.2.15. □

Consider the linearized invertible sheaf $\mathcal{L}$ given in the proof of Theorem 3.1.28 and let $\mathcal{L}' := \beta^*\mathcal{L}$. We finally have

Theorem 3.2.8. *There is a projective scheme* $\mathrm{SPBGPS}(\rho)^{(\underline{\kappa},\delta)-\mathrm{ss}}$ *and an open subscheme* $\mathrm{SPBGPS}(\rho)^{(\underline{\kappa},\delta)-\mathrm{s}} \subset \mathrm{SPBGPS}(\rho)^{(\underline{\kappa},\delta)-\mathrm{ss}}$ *together with a natural tranformation*

$$\alpha^{(\mathrm{s})\mathrm{s}} : \mathbf{SPBGPS}(\rho)_P^{(\underline{\kappa},\delta)\text{-}(\mathrm{s})\mathrm{s}} \to h_{\mathrm{SPBGPS}(\rho)^{(\underline{\kappa},\delta)\text{-}(\mathrm{s})\mathrm{s}}}$$

with the following properties:

1) For every scheme $\mathcal{N}$ and every natural transformation $\alpha' : \mathbf{SPBGPS}(\rho)_P^{(\underline{\kappa},\delta)\text{-}(\mathrm{s})\mathrm{s}} \to h_{\mathcal{N}}$, there exists a unique morphism $\varphi : \mathrm{SPBGPS}^{(\underline{\kappa},\delta)\text{-}(\mathrm{s})\mathrm{s}}(\rho) \to \mathcal{N}$ with $\alpha' = h(\varphi)\circ\alpha^{(\mathrm{s})\mathrm{s}}$.

2) The scheme $\mathrm{SPBGPS}^{(\underline{\kappa},\delta)\text{-}\mathrm{s}}(\rho)$ is a coarse moduli space for $\mathbf{SPBGPS}^{(\underline{\kappa},\delta)\text{-}\mathrm{s}}(\rho)$.

Proof. By Proposition 3.2.6, Proposition 3.2.7 and Proposition 2.2.17, the quotients $\mathrm{SPBGPS}^{\delta\text{-}(\mathrm{s})\mathrm{s}}(\rho) := \mathfrak{M}(G)^{(s)s}/\!\!/\mathrm{GL}(V)$ exist, $\mathfrak{M}(G)^{\mathrm{ss}}/\!\!/\mathrm{GL}(V)$ is a projective scheme, $\mathfrak{M}(G)^s/\mathrm{GL}(V)$ is an open subscheme, and 1), 2) hold true. □

3.3 Moduli Space for Large Values of δ

We will define the notions of generic semistability and asymptotic semistability. In [52], it is shown that, for the connected case, generic semistability corresponds to GIT semistability of the corresponding tensor field restricted to the generic point, η, of the base curve Y. In our case, we have many connected components, $Y = \amalg_{i=1}^{l} Y_i$, making it impossible to reproduce the known arguments. We solve the problem restricting to each connected component and changing the base field to the field of fractions of the l-dimensional projective smooth variety $Y_1 \times \ldots \times Y_l$.

3.3.1 Generic Semistability

Fix natural numbers $a, b, c, r, d, h_1, \ldots, h_l \in \mathbb{N}$ and a line bundle $\mathcal{L}$ on Y. Denote $h = h_1 + \cdots + h_l$, and $\alpha = hr$.

Definition 3.3.1. Let $\mathcal{E}$ be a locally free sheaf of rank r and $\phi : (\mathcal{E}^{\otimes a})^{\oplus b} \to \det(\mathcal{E})^{\otimes c} \otimes \mathcal{L}$ a tensor field. We say that $(\mathcal{E}, \varphi)$ is *generically semistable* if $\mu(\mathcal{E}_\bullet, \underline{m}, \phi) \geq 0$ for every weighted filtration $(\mathcal{E}_\bullet, \underline{m})$.

The aim of this section is to characterize geometrically the generic semistability condition. Let $Y = Y_1 \amalg \cdots \amalg Y_l$ be a disjoint union of smooth projective curves and let $\mathcal{O}_Y(1)$ be a very ample invertible sheaf of multidegree $(h_1, \ldots, h_l)$. Set $V := \mathbb{C}^r$, and let $\overline{H} \subset \mathrm{GL}(V) \times \cdots \times \mathrm{GL}(V)$ be the subgroup defined by $\iota^{-1}(\mathrm{SL}(V^h))$, ι being the obvious morphism of groups $\iota : \mathrm{GL}(V) \times \cdots \times \mathrm{GL}(V) \to \mathrm{GL}(V^h)$.

Consider now the diagram,

$$\begin{tikzcd}
& S \arrow[d, equal] & \\
& Y_1 \times \cdots \times Y_l \arrow[dl, "q_1"'] \arrow[dr, "q_l"] & \\
Y_1 \arrow[dr, "\iota_1"'] & \cdots & Y_l \arrow[dl, "\iota_l"] \\
& Y_1 \amalg \cdots \amalg Y_l \arrow[d, equal] & \\
& Y &
\end{tikzcd} \tag{3.31}$$

Note that S is a smooth projective variety, and that its field of functions is given by $K := (K_1 \otimes_k \cdots \otimes_k K_l)_{(0)}$, K_i being the field of functions of the ith component Y_i. We will show that a tensor field $(\mathcal{E}, \phi)$ defines a K-valued point, $[\phi]$, in the projective space $\mathbf{P}((V^{\otimes a})^{\oplus b} \oplus \cdots \oplus (V^{\otimes a})^{\oplus b})$, with $V = \mathbb{C}^r$, and that generic semistability of $(\mathcal{E}, \phi)$ is equivalent to semistability of this point with respect to the action of $\overline{H}$ defined through the homogeneous representation

$$\rho : \overline{H} \subset \mathrm{GL}(V) \times \cdots \times \mathrm{GL}(V) \hookrightarrow \mathrm{GL}((V^{\otimes a})^{\oplus b} \times \cdots \times (V^{\otimes a})^{\oplus b})$$

of degree a. Furthermore, we will show that this representation is a direct summand of the natural representation $\rho' : \overline{H} \hookrightarrow \mathrm{GL}(((V^{\oplus h})^{\otimes a})^{\oplus b})$, where $V^{\oplus h} = V^{\oplus h_1} \oplus \cdots \oplus V^{\oplus h_l}$. This permits us to define an $\overline{H}$-equivariant closed immersion

$$\sigma \colon \mathbf{P}((V^{\otimes a})^{\oplus b} \oplus \cdots \oplus (V^{\otimes a})^{\oplus b}) \hookrightarrow \mathbf{P}(((V^{\oplus h})^{\otimes a})^{\oplus b})$$

and, with this in hand, to show that $\mu(\mathcal{E}_\bullet, \underline{m}, \phi) = \mu(\sigma([\phi]), \lambda)$, λ being the one parameter subgroup induced by the filtration $(\mathcal{E}_\bullet, \underline{m})$, as we will explain later.

Preliminaries

Here we will show how $(\mathcal{E}, \phi)$ defines a K-valued point in $\mathbf{P}((V^{\otimes a})^{\oplus b} \oplus \cdots \oplus (V^{\otimes a})^{\oplus b})$. For that, we will need to trivialize the vector bundles $\mathcal{E}$ and $\mathcal{L}$, so we will need to make some choices. However, we will prove that the semistability function of this point does not depend on them.

Consider a tensor field $\phi : (\mathcal{E}^{\otimes a})^{\oplus b} \to \det(\mathcal{E})^{\otimes c} \otimes \mathcal{L}$ on Y, being $\mathcal{E}$ a locally free sheaf of rank r, and a weighted filtration $(\mathcal{E}_\bullet, \underline{m})$. The tensor field ϕ determines a tensor field $\phi'_i : (\mathcal{E}|_{Y_i}^{\otimes a})^{\oplus b} \to \det(\mathcal{E}|_{Y_i})^{\otimes c} \otimes \mathcal{L}|_{Y_i}$ on Y_i. Likewise, the filtration $(\mathcal{E}_\bullet, \underline{m})$ induces a weighted filtration, $(\mathcal{E}^i_\bullet, \underline{m}^i)$, of $\mathcal{E}|_{Y_i}$ for each $i \in \{1, \ldots, l\}$ (see Equation 3.38). Let V be the vector space $\mathbb{C}^r$ and let $\underline{w} = \{w_1, \ldots, w_r\}$ be a basis of V. For each

$i \in \{1,\dots,r\}$, we define $V_i = \langle w_1,\dots,w_i\rangle$, and for each $i \in \{1,\dots,l\}$, the weighted flag $(\mathcal{E}^i_\bullet, \underline{m}^i)$ determines a weighted flag $(V^i_\bullet, \underline{m}^i)$ of V, being $V^i_j = V_{\mathrm{rk}(\mathcal{E}^i_j)}$. If the weights are integral, these weighted flags define one parameter subgroups, $\lambda(\underline{w}, \underline{m}^i) : \mathbb{G}_m \to \mathrm{GL}(V)$. Otherwise, we can find a natural number $k \in \mathbb{N}$ such that $k\underline{m}^i$ are all of them integral, and they define again one parameter subgroups, $\lambda(\underline{w}, k\underline{m}^i) : \mathbb{G}_m \to \mathrm{GL}(V)$. In any case, we can form a one parameter subgroup, $\mathbb{G}_m \to \mathrm{GL}(V) \times \cdots \times \mathrm{GL}(V)$ (actually, it factorizes through $\overline{H}$), which we denote by $\lambda(\underline{w}, \underline{m})$ (resp. $\lambda(\underline{w}, k\underline{m})$).
For each $i \in \{1,\dots,l\}$, we can find an open subset $U_i \subset Y_i$ over which,

(*i*) there is an isomorphism $\Psi^i_1 : \mathcal{E}|_{U_i} \simeq V \otimes \mathcal{O}_{U_i}$ such that $\Psi^i_1(\mathcal{E}^i_\bullet|_{U_i}) = V^i_\bullet \otimes \mathcal{O}_{U_i}$,

(*ii*) there is an isomorphism $\Psi^i_2 : \mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_i}$,

(*iii*) ϕ'_i is surjective over U_i.

Let $U \subset S$ be the dense open subset defined by $q_1^{-1}(U_1) \cap \cdots \cap q_l^{-1}(U_l)$. Over U we have l quotients,

$$\phi_i := (q_i^*(\det((\Psi^i_1)^{\otimes c}) \otimes \Psi^i_2 \circ \phi'_i|_{U_i} \circ ((\Psi^{i^{-1}}_1)^{\otimes a})^{\oplus b}))|_U : (V^{\otimes a})^{\oplus b} \otimes \mathcal{O}_U \to \mathcal{O}_U,$$

and the sum of all of them defines a surjection,

$$\underline{\phi} \colon = (\phi_1,\dots,\phi_l) : (V^{\otimes a})^{\oplus b} \oplus \cdots \oplus (V^{\otimes a})^{\oplus b} \otimes \mathcal{O}_U \to \mathcal{O}_U.$$

This, in turn, induces a morphism, which we denote by the same symbol,

$$\begin{tikzcd} U \arrow[r] \arrow[rr, bend right, "\underline{\phi}"'] & U \times \mathbf{P}((V^{\otimes a})^{\oplus b} \oplus \cdots \oplus (V^{\otimes a})^{\oplus b}) \arrow[r, "\mathrm{pr}_2"] & \mathbf{P}((V^{\otimes a})^{\oplus b} \oplus \cdots \oplus (V^{\otimes a})^{\oplus b}). \end{tikzcd}$$

Then, following [50], we define

$$\overline{\mu}(\mathcal{E}_\bullet, \underline{m}, \phi) := \max\{\mu(\underline{\phi}(s), \lambda(\underline{w}, \underline{m}))|\ s \in S\}$$

if all the weights are integral, or

$$\overline{\mu}(\mathcal{E}_\bullet, \underline{m}, \phi) := \max\{\frac{1}{k}\mu(\underline{\phi}, \lambda(\underline{w}, k\underline{m}))|\ s \in S\}$$

otherwise, being k as we said before. Let us show that the last definition does not depend on the basis $\underline{w}$ and the trivializations Ψ^i_1, Ψ^i_2 we have chosen. Let $\underline{\widehat{w}} = \{\widehat{w}_1,\dots,\widehat{w}_r\}$ be a different basis, and let $g \in \mathrm{GL}(V)$ be a matrix mapping w_i to $\widehat{w}_i$. Define $\widehat{\Psi}^i_1 := (g \otimes \mathrm{id}_{\mathcal{O}_{U_i}}) \circ \Psi^i_1$. Then we get l quotients

$$\widehat{\phi}_i : (V^{\otimes a})^{\oplus b} \otimes \mathcal{O}_U \to \mathcal{O}_U,$$

where $\widehat{\phi}_i = \phi_i \cdot g = \det(g)^c \phi_i \circ ((g^{-1} \otimes \mathrm{id}_{\mathcal{O}_U})^{\otimes a})^{\oplus b}$, and therefore a quotient

$$\underline{\widehat{\phi}} \colon = (\phi_1,\dots,\phi_l) \cdot g : (V^{\otimes a})^{\oplus b} \oplus \cdots \oplus (V^{\otimes a})^{\oplus b} \otimes \mathcal{O}_U \to \mathcal{O}_U,$$

g acting componentwise, which, in turn, induces a morphism (denoted, again, by the same symbol)

$$\begin{tikzcd} U \arrow[r] \arrow[rr, bend right, "\underline{\widehat{\phi}} = g \cdot \underline{\phi}"'] & U \times \mathbf{P}((V^{\otimes a})^{\oplus b} \oplus \cdots \oplus (V^{\otimes a})^{\oplus b}) \arrow[r, "\mathrm{pr}_2"] & \mathbf{P}((V^{\otimes a})^{\oplus b} \oplus \cdots \oplus (V^{\otimes a})^{\oplus b}). \end{tikzcd}$$

On the other hand, we have $\lambda(\underline{w}',\underline{m}^i) = g\cdot\lambda(\underline{w},\underline{m}^i)\cdot g^{-1}$ for each $i=1,\dots,l$. Thus, $\lambda(\underline{w}',\underline{m}) = g\cdot\lambda(\underline{w},\underline{m})\cdot g^{-1}$, where g is acting again componentwise. By [50, Section 1.5.1],

$$\mu(g\cdot\underline{\phi}, g\cdot\lambda(\underline{w},\underline{m})\cdot g^{-1}) = \mu(\underline{\phi},\lambda(\underline{w},\underline{m}))$$

so, we deduce that the definition does not depend on the basis $\underline{w}$. Consider now different trivializations $\overline{\Psi}^i_1$. For each $i\in\{1,\dots,l\}$, $\overline{\Psi}^i_1$ differs from Ψ^i_1 by a family of flag automorphisms parametrized by U_i, i.e., by a morphism $U_i\to P_i$, being $P_i\subset \mathrm{GL}(V)$ the parabolic subgroup associated to the weighted flag $(V^i_\bullet,\underline{m}^i)$. Therefore, we get l morphisms (see 3.31), $U\to P_i$, and finally a morphism

$$U\to P_1\times\cdots\times P_l\subset \mathrm{GL}(V)\times\cdots\times\mathrm{GL}(V),$$

being $P_1\times\cdots\times P_l$ a parabolic subgroup of $\mathrm{GL}(V)\times\cdots\times\mathrm{GL}(V)$. Following the same argument as in [50], we show that the definition of $\overline{\mu}(\mathcal{E}_\bullet,\underline{m},\phi)$ does no depend on the trivializations Ψ^i_1. Let now $\overline{\Psi}^i_2$ be a different trivialization of $\mathcal{L}|_{Y_i}$. Then, Ψ^i_2 differs from Ψ^i_2 by a non zero regular function over U_i, $\alpha_i\in\mathcal{O}^*_U$. Let $\widehat{\phi}_i : (V^{\otimes a})^{\oplus b}\otimes\mathcal{O}_U\to\mathcal{O}_U$ be the corresponding quotients. Then, clearly $\widehat{\phi}_i=\alpha_i\phi_i$, and

$$\widehat{\phi}=(\alpha_1\phi_1,\dots,\alpha_i\phi_i)$$

We have to show that $\mu(\phi(s),\lambda(\underline{w},\underline{m}))=\mu(\widehat{\phi}(s),\lambda(\underline{w},\underline{m}))$ for all $s\in S$. Note that the basis $\underline{w}=\{w_1,\dots,w_r\}$ induces a basis of $(V^{\otimes a})^{\oplus b}\oplus\cdots\oplus(V^{\otimes a})^{\oplus b}$

$$\{w^{j,t}_I=(\overbrace{0,\dots,w^j_I,\dots,0}^{t)})|t=1,\dots,l,\text{and } j=1,\dots,b\}$$

with $w^j_I=(\underbrace{0,\dots,w_I,\dots,0}_{j)})$, and $w_I=w_{i_1}\otimes\cdots\otimes w_{i_a}$, which diagonalizes $\lambda(\underline{w},\underline{m})$. By Remark 1.1.23, it is enough to show that $\phi(s)(w^{j,t}_I)\neq 0$ if and only if $\widehat{\phi}(s)(w^{j,t}_I)\neq 0$. Observe that

$$\begin{aligned}\phi(s)(w^{j,t}_I)&=\phi_t(s)(w^j_I)\\ \widehat{\phi}(s)(w^{j,t}_I)&=\alpha_t(s)\phi_t(s)(w^t_I).\end{aligned}$$

Since $\alpha_t\in\mathcal{O}^*_{U_t}$ for all t, we clearly have $\phi(s)(w^{j,t}_I)\neq 0$ if and only if $\widehat{\phi}(s)(w^{j,t}_I)\neq 0$. Finally, it remains to show that the definition of $\overline{\mu}(\mathcal{E}_\bullet,\underline{m},\phi)$ does not depend on the open subsets U_i we have chosen for the trivializations. Note that any other open subsets lead to an open subset $U'\subset S$ and a morphism

$$\phi : U'\longrightarrow\mathbf{P}((V^{\otimes a})^{\oplus b}\oplus\cdots\oplus(V^{\otimes a})^{\oplus b})$$

Since S is irreducible, we conclude, by [50, Remark 1.5], that $\overline{\mu}(\mathcal{E}_\bullet,\underline{m},\phi)$ is precisely the semistability function μ corresponding to the generic point. Thus, it does not depend on the open subsets U_i.

Analysis of the semistability function $\overline{\mu}$ (I).

The aim of this section is to analyze the semistability function $\overline{\mu}(\mathcal{E}_\bullet, \underline{m}, \phi)$. In the last section, we have shown that the problem is reduced to the analysis of the semistability condition in the projective space $\mathbf{P}((V^{\otimes a})^{\oplus b} \oplus \cdots \oplus (V^{\otimes a})^{\oplus b})$ with respect to the action of $\overline{H} \subset H$.

The following result can help us to relate the semistability condition of points $[\phi]$ with respect to the $\overline{H}$-action and the semistability condition of its coordinates $[\phi_i]$ with respect to to the SL(V)-action.

Corollary 3.3.2. *Consider a point* $[\phi = (\phi_1, \ldots, \phi_l)] \in \mathbf{P}((V^{\otimes a})^{\oplus b} \oplus \cdots \oplus (V^{\otimes a})^{\oplus b})$. *Then, we have,*

(*i*) $[\phi]$ *is semistable with respect to* $\overline{H}$ *if and only if* $[\phi_i] \in \mathbf{P}((V^{\otimes a})^{\oplus b})$ *is semistable with respect to* $\mathrm{SL}(V)$ *for all* $i = 1 \ldots, l$. *Notice that this in particular means that every* ϕ_i *is non-zero.*

(*ii*) *Suppose that* $[\phi_i]$ *is semistable for all* $i = 1, \ldots, l$, *and let* $\lambda = (\lambda_1, \ldots, \lambda_l)$ *be a one parameter subgroup of* $\overline{H}$. *Then,* $\mu([\phi], \lambda) = 0$ *if and only if* $\mu([\phi_i], \lambda_i) = 0$ *for all* $i = 1, \ldots, l$.

(*iii*) *If* $[\phi_i]$ *is stable for all* $i = 1, \ldots, l$, *then* $[\phi]$ *is stable.*

Proof. (i) This part follows from Proposition 1.1.28, V_i being now equal to $((V^{\otimes a})^{\oplus b})^\vee$.
(ii) By Lemma 1.1.27, there is an index $t \in \{1, \ldots, l\}$, the one giving the maximum, such that $\mu([\phi_t], \lambda_t) = 0$. This implies that, for every $i = 1, \ldots, l$, we have

$$\mu([\phi_i], \lambda_i) \leq \mu([\phi_t], \lambda_t) = 0.$$

Since every $[\phi_i]$ is semistable, we also have $\mu([\phi_i], \lambda_i) \geq 0$. Therefore, $\mu([\phi_i], \lambda_i) = 0$ for all $i = 1, \ldots, l$. The other direction is trivial using again Lemma 1.1.27.
(iii) This follows from Lemma 1.1.27 and (ii). □

Analysis of the semistability function $\overline{\mu}$ (II).

In the last section we have analyze the function $\overline{\mu}(\mathcal{E}_\bullet, \underline{m}, \phi)$, and we have studied the relation between generic semistability of the tensor field $(\mathcal{E}, \phi)$ and generic semistability of its components $(\mathcal{E}|_{Y_i}, \phi_i)$ (see [52] for generic semistability in a single curve). However, in order to compare the functions $\overline{\mu}(\mathcal{E}_\bullet, \underline{m}, \phi)$ and $\mu(\mathcal{E}_\bullet, \underline{m}, \phi)$, we need a numerical description of $\overline{\mu}(\mathcal{E}_\bullet, \underline{m}, \phi)$. I order to do so, we will show that there is an H-equivariant closed immersion,

$$\mathbf{P}((V^{\otimes a})^{\oplus b} \oplus \cdots \oplus (V^{\otimes a})^{\oplus b}) \hookrightarrow \mathbf{P}(((V^{\oplus h})^{\otimes a})^{\oplus b})$$

that will allow us to compute $\overline{\mu}(\mathcal{E}_\bullet, \underline{m}, \phi)$ as a multiple of the semistability function of the corresponding point in $\mathbf{P}(((V^{\oplus h})^{\otimes a})^{\oplus b})$ (see [42, Chapter 2, §3]), which, as we will show, coincide with $\mu(\mathcal{E}_\bullet, \underline{m}, \phi)$.
The aim of this section is to describe the above immersion. Consider

$$V^{\oplus h} = V^{\oplus h_1} \oplus \cdots \oplus V^{\oplus h_l}.$$

The group $\overline{H}$ acts on $V^{\oplus h}$ in the obvious way

$$\begin{aligned} \overline{H} \times V^{\oplus h} &\to V^{\oplus h} \\ (g_1, \dots, g_l), (\underline{v}_1, \dots, \underline{v}_l) &\mapsto (g_1(\underline{v}_1), \dots, g_l(\underline{v}_l)) \end{aligned},$$

being $g_i(\underline{v}_i) = g_i(v_i^1, \dots, v_i^{h_i}) = (g_i(v_i^1), \dots, g_i(v_i^{h_i}))$ and this action induces an action on $((V^{\oplus h})^{\otimes a})^{\oplus b}$. We want to define an $\overline{H}$-equivariant surjection:

$$\sigma : ((V^{\oplus h})^{\otimes a})^{\oplus b} \twoheadrightarrow (V^{\otimes a})^{\oplus b} \oplus \cdots \oplus (V^{\otimes a})^{\oplus b}.$$

For the definition of σ we will fix first some notation. Note that an element in $((V^{\oplus h})^{\otimes a})^{\oplus b} = ((V^{\oplus h_1} \oplus \cdots \oplus V^{\oplus h_l})^{\otimes a})^{\oplus b}$ is a linear combination of elements of the form

$$\begin{aligned}
\Big(&((v^1_{1,1,1}, \dots, v^1_{h_1,1,1}), \dots, (v^1_{1,l,1}, \dots, v^1_{h_l,l,1}))\otimes \\
\otimes &((v^1_{1,1,2}, \dots, v^1_{h_1,1,2}), \dots, (v^1_{1,l,2}, \dots, v^1_{h_l,l,2}))\otimes \\
&\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
\otimes &((v^1_{1,1,a}, \dots, v^1_{h_1,1,a}), \dots, (v^1_{1,l,a}, \dots, v^1_{h_l,l,a})), \\
&\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
&\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
&((v^b_{1,1,1}, \dots, v^b_{h_1,1,1}), \dots, (v^b_{1,l,1}, \dots, v^b_{h_l,l,1}))\otimes \\
\otimes &((v^b_{1,1,2}, \dots, v^b_{h_1,1,2}), \dots, (v^b_{1,l,2}, \dots, v^b_{h_l,l,2}))\otimes \\
&\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
\otimes &((v^b_{1,1,a}, \dots, v^b_{h_1,1,a}), \dots, (v^b_{1,l,a}, \dots, v^b_{h_l,l,a}))\Big) = \underline{v}
\end{aligned}$$

where every $v^i_{j,k,q}$ belongs to V. We denote $\underline{v}(i,j,k) = v^i_{j,k,1} \otimes \cdots \otimes v^i_{j,k,a}$, and define its image in $(V^{\otimes a})^{\oplus b} \oplus \cdots \oplus (V^{\otimes a})^{\oplus b}$ by,

$$\sigma(\underline{v}) := \Big(\sum_{j=1}^{h_1} \underline{v}(1,j,1), \dots, \sum_{j=1}^{h_1} \underline{v}(b,j,1), \dots, \sum_{j=1}^{h_l} \underline{v}(1,j,l), \dots, \sum_{j=1}^{d_l} \underline{v}(b,j,l) \Big) \tag{3.32}$$

and extend it by linearity.

Lemma 3.3.3. *The linear map σ is surjective and $\overline{H}$-equivariant.*

Proof. The $\overline{H}$-equivariance is clear from the definition of σ. Consider the elements $\underline{v}(i,j) := v^i_{j,1} \otimes \cdots \otimes v^i_{j,a} \in V^{\otimes a}$, and let

$$\underline{w} = (\underline{v}(1,1), \dots, \underline{v}(b,1), \dots, \underline{v}(1,l), \dots, \underline{v}(b,l)) \in (V^{\otimes a})^{\oplus b} \oplus \cdots \oplus (V^{\otimes a})^{\oplus b}$$

Then the image of the element,

$$\begin{aligned}
&\Big(((v^1_{1,1},\dots,0),\dots,(v^1_{l,1},\dots,0))\otimes\\
&\otimes((v^1_{1,2},\dots,0),\dots,(v^1_{l,2},\dots,0))\otimes\\
&\qquad\qquad\vdots\qquad\qquad\vdots\\
&\otimes((v^1_{1,a},\dots,0),\dots,(v^1_{l,a},\dots,0)),\\
&\qquad\qquad\vdots\qquad\qquad\vdots\\
&\qquad\qquad\vdots\qquad\qquad\vdots\\
&((v^b_{1,1},\dots,0),\dots,(v^b_{l,1},\dots,0))\otimes\\
&\otimes((v^b_{1,2},\dots,0),\dots,(v^b_{l,2},\dots,0))\otimes\\
&\qquad\qquad\vdots\qquad\qquad\vdots\\
&\otimes((v^b_{1,a},\dots,0),\dots,(v^b_{l,a},\dots,0))\Big)=\underline{v}
\end{aligned}$$

is precisely $\underline{w}$, i.e., $\sigma(\underline{v})=\underline{w}$. □

By Lemma 3.3.3, σ defines an $\overline{H}$-equivariant closed embedding,

$$\sigma:\mathfrak{X}:=\mathbf{P}(\bigoplus_1^l((V^{\otimes a})^{\oplus b}))\hookrightarrow\mathbf{P}((((V^{\oplus h})^{\otimes a})^{\oplus b}))=:\mathfrak{Y}. \tag{3.33}$$

Note that the map, $\underline{w}\mapsto\underline{v}$, we have defined in the last proof can by extended by linearity, and it is clearly $(\mathrm{GL}(V)\times\cdots\times\mathrm{GL}(V))$-equivariant. Moreover, it is a retract of σ, so $\bigoplus_1^l((V^{\otimes a})^{\oplus b})$ is a *direct summand* of $(((V^{\oplus h})^{\otimes a})^{\oplus b})$.

Weighted Flags of Tuples of Vector Spaces

In order to compute properly the semistability function in $\mathbf{P}((((V^{\oplus h})^{\otimes a})^{\oplus b}))=:\mathfrak{Y}$, we need to understand the relation between weighted flags of $V^{\oplus h}$ and one parameter subgroup of $\lambda:\mathbb{G}_m\to\overline{H}\xrightarrow{\iota}\mathrm{SL}(V^{\oplus h})$, where $\iota:\overline{H}\hookrightarrow\mathrm{SL}(V^{\oplus h})$ sends $(g_1,\dots,g_l)$ to the block diagonal matrix in which g_1 is first repeated h_1 times, then g_2 is repeated h_2 times, and so on.

On $\mathfrak{Y}$ we consider the canonical ample invertible sheaf $\mathcal{O}_{\mathfrak{Y}}(1)$ and we denote $\mathcal{N}=\sigma^*\mathcal{O}_{\mathfrak{Y}}(1)$. Then we know that $\mu^{\mathcal{N}}(\phi,\lambda)=\mu^{\mathcal{O}_{\mathfrak{Y}}(1)}(\sigma(\phi),\lambda)$ for all one parameter subgroup $\lambda:\mathbb{G}_m\to\overline{H}$.

Let $\lambda=(\lambda_1,\dots,\lambda_l)$ be a one parameter subgroup of $\overline{H}$. Then λ_i provides us with a *weighted flag for all* $i=1,\dots,l$

$$\begin{aligned}&V^i_\bullet\equiv(0)\subset V^i_1\subset\ldots\subset V^i_{s(i)}\subset V,\\ &\underline{\gamma}^i=(\gamma^i_1,\dots,\gamma^i_{s(i)+1})\quad.\end{aligned}\tag{3.34}$$

Let us see how to construct the weighted flag of $V^{\oplus h}$ corresponding to λ (see [53] p. 212). Let $\gamma_1<\ldots<\gamma_{s+1}$ be the different weights ocurring among the γ^i_j. Define

$$W^i_j:=V^i_{\theta_i(j)},\text{ with }\theta_i(j)=\max\{\theta=1,\dots,s(i)+1|\gamma^i_\theta\le\gamma_j\}.$$

This gives to us chains of subspaces

$$(0) \subseteq W_1^i \subseteq \ldots \subseteq W_s^i \subseteq W_{s+1}^i = V, \ \forall\, i = 1, \ldots, l, \tag{3.35}$$

and therefore a *weighted flag of the tuple of vector spaces* $(V, \overset{l)}{\ldots}, V)$

$$(0) \subset (W_1^i,\ i \in \{1, \ldots, l\}) \subset \ldots \subset (W_s^i,\ i \in \{1, \ldots, l\}) \subset (V, \overset{l)}{\ldots}, V) \tag{3.36}$$

and weights given by $\gamma_1 < \ldots < \gamma_{s+1}$. Now define $W_j^{\mathrm{tot}} = \bigoplus_{i=1}^{l} (W_j^i)^{\oplus h_i}$ and we get

$$V_\bullet^{\oplus h} \equiv (0) \subset W_1^{\mathrm{tot}} \subset \ldots \subset W_s^{\mathrm{tot}} \subset V^{\oplus h},\ \underline{\gamma} = (\gamma_1, \ldots, \gamma_{s+1}). \tag{3.37}$$

This is the weighted flag associated to

$$\lambda : \mathbb{G}_m \to \overline{H} \overset{\rho}{\hookrightarrow} \mathrm{SL}(V^{\oplus d}).$$

Conversely, given a weighted flag of the tuple of vector spaces $(V \overset{l)}{\ldots}, V)$ as in (3.36) we get chains of subspaces as in Equation (3.35) by projecting onto the ith component. Eliminating the improper inclusions and defining

$$\begin{aligned} &\gamma_j^i := \min\{\gamma_t |\ W_t^i = V_j^i\ t = 1, \ldots, s+1\} \\ &\text{with}\ \ j = 1, \ldots, s(i)+1 \text{ and } i = 1, \ldots, l \end{aligned} \tag{3.38}$$

we get weighted flags as in (3.34).

Analysis of the semistability function $\overline{\mu}$ (III).

Since we already know the semistability function for points in $\mathbf{P}(((V^{\oplus h})^{\otimes a})^{\oplus b})$ with respect to the natural action of $\mathrm{SL}(V^{\oplus h})$ (see [53, Section 2.3.2]), it is easy to find out the semistability function with respect to the action of $\overline{H}$ (through ρ). Let $\lambda = (\lambda_1, \ldots, \lambda_l) : \mathbb{G}_m \to \overline{H}$ be a one parameter subgroup. This defines a weighted flag as we have seen before in Equation (3.37). Giving $\underline{\gamma} = (\gamma_1, \ldots, \gamma_{s+1})$ is equivalent to giving $\underline{m} = (m_1, \ldots, m_s)$ defined by

$$(\gamma_1, \overset{\alpha_1)}{\ldots}, \gamma_1, \ldots, \gamma_{s+1}, \overset{\alpha-\alpha_s)}{\ldots}, \gamma_{s+1}) = \sum_{i=1}^{s} m_i \gamma_\alpha^{(\alpha_i)},$$

where $\alpha = \sum d_i r = dr$, $\alpha_j = \sum d_i r_i^j$ (being $r_i^j = \dim(V_j^i)$) and

$$\gamma_\alpha^{(j)} = (\overbrace{j-\alpha, \ldots, j-\alpha}^{j}, \overbrace{j \ldots, j}^{\alpha-j})$$

Then,

$$\begin{aligned} \mu^{\mathcal{O}_{\mathfrak{Y}}(1)}([l], \lambda) = -\min\{\gamma_{i_1} + \ldots + \gamma_{i_a} |\ l|_{(W_{i_1}^{\mathrm{tot}} \otimes \ldots \otimes W_{i_a}^{\mathrm{tot}})^{\oplus b}} \neq 0\} = \\ = \sum_{j=1}^{s} m_j(\nu_j(I)\alpha - a\alpha_j). \end{aligned}$$

with $\nu_j(I) = \#\{i_t \leq j | t = 1, \dots, a\}$, and I the multiindex giving the minimum. This gives to us the semistability function for $[l] \in \mathfrak{Y}$ with respect to a one parameter subgroup.
Let $\phi_i : (V^{\otimes a})^{\oplus b} \to k$, $i = 1, \dots, l$, be l linear forms. These define a linear form

$$\begin{aligned} \phi : (V^{\otimes a})^{\oplus b} \oplus \dots \oplus (V^{\otimes a})^{\oplus b} &\to k \\ (\underline{v}_1, \dots, \underline{v}_l) &\mapsto \phi_1(\underline{v}_1) + \dots + \phi_l(\underline{v}_l) \end{aligned} \tag{3.39}$$

whose equivalence class defines a point in $\mathfrak{X}$. The corresponding point in $\mathfrak{Y}$ is given by composing ϕ with the surjection σ,

$$\sigma(\phi) = \phi \circ \sigma : ((V^{\oplus d})^{\otimes a})^{\oplus b} \to k.$$

Suppose we are given weighted flags as in Equation (3.34). Then we get a weighted flag as in Equation (3.37). We find

Lemma 3.3.4. $\sigma(\phi)|_{(W^{\mathrm{tot}}_{i_1} \otimes \ldots \otimes W^{\mathrm{tot}}_{i_a})^{\oplus b}} \neq 0 \Leftrightarrow \phi_j|_{(V^j_{\theta_j(i_1)} \otimes \ldots \otimes V^j_{\theta_j(i_a)})^{\oplus b}} \neq 0$ *for some* j.

Proof. Consider the diagram

$$\begin{tikzcd}
 & k \\
(W^{\mathrm{tot}}_{i_1} \otimes \dots \otimes W^{\mathrm{tot}}_{i_a})^{\oplus b} \arrow[r, hook, "\Phi"] \arrow[d, dashed, "\sigma'"] \arrow[dr, "\sigma\circ\Phi"] & ((V^{\oplus d})^{\otimes a})^{\oplus b} \arrow[u, "\sigma(\phi)"'] \arrow[d, "\sigma"] \\
(V^1_{\theta_1(i_1)} \otimes \dots \otimes V^1_{\theta_1(i_a)})^{\oplus b} \oplus \dots \oplus (V^l_{\theta_l(i_1)} \otimes \dots \otimes V^l_{\theta_l(i_a)})^{\oplus b} \arrow[r, hook, "\Phi'"'] & (V^{\otimes a})^{\oplus b} \oplus \dots \oplus (V^{\otimes a})^{\oplus b} \arrow[uu, bend right, "\phi"']
\end{tikzcd}$$

Let us see that $\sigma \circ \Phi$ factorizes, thus giving rise to a surjection σ'. Recall that

$$\begin{aligned} W^{\mathrm{tot}}_{i_1} &= (V^1_{\theta_1(i_1)})^{\oplus d_1} \oplus \dots \oplus (V^l_{\theta_l(i_1)})^{\oplus d_l} \\ &\vdots \qquad\qquad\qquad \vdots \\ W^{\mathrm{tot}}_{i_a} &= (V^1_{\theta_1(i_a)})^{\oplus d_1} \oplus \dots \oplus (V^l_{\theta_l(i_a)})^{\oplus d_l}. \end{aligned}$$

Now, fix a simple element in $(W^{\mathrm{tot}}_{i_1} \otimes \dots \otimes W^{\mathrm{tot}}_{i_a})^{\oplus b}$,

$$\begin{gathered} \Big(((v^1_{1,1,1}, \dots, v^1_{d_1,1,1}), \dots, (v^1_{1,l,1}, \dots, v^1_{d_l,l,1})) \otimes \\ \otimes ((v^1_{1,1,2}, \dots, v^1_{d_1,1,2}), \dots, (v^1_{1,l,2}, \dots, v^1_{d_l,l,2})) \otimes \\ \vdots \qquad\qquad \vdots \\ \otimes ((v^1_{1,1,a}, \dots, v^1_{d_1,1,a}), \dots, (v^1_{1,l,a}, \dots, v^1_{d_l,l,a})), \\ \vdots \qquad\qquad \vdots \\ \\ \vdots \qquad\qquad \vdots \\ ,((v^b_{1,1,1}, \dots, v^b_{d_1,1,1}), \dots, (v^b_{1,l,1}, \dots, v^b_{d_l,l,1})) \otimes \\ \otimes ((v^b_{1,1,2}, \dots, v^b_{d_1,1,2}), \dots, (v^b_{1,l,2}, \dots, v^b_{d_l,l,2})) \otimes \\ \vdots \qquad\qquad \vdots \\ \otimes ((v^b_{1,1,a}, \dots, v^b_{d_1,1,a}), \dots, (v^b_{1,l,a}, \dots, v^b_{d_l,l,a})) \Big) = \underline{v} \end{gathered}$$

then, clearly $\underline{v}(i,j,k) = v^i_{j,k,1} \otimes \cdots \otimes v^i_{j,k,a} \in V^k_{\theta_k(i_1)} \otimes \cdots \otimes V^k_{\theta_k(i_a)}$, and we can easily show that the image,

$$\sigma(\underline{v}) := \Big(\textstyle\sum_{j=1}^{d_1} \underline{v}(1,j,1), \dots, \sum_{j=1}^{d_1} \underline{v}(b,j,1), \dots, \sum_{j=1}^{d_l} \underline{v}(1,j,l), \dots, \sum_{j=1}^{d_l} \underline{v}(b,j,l) \Big),$$

belongs to $(V^1_{\theta_1(i_1)} \otimes \ldots \otimes V^1_{\theta_1(i_a)})^{\oplus b} \oplus \ldots \oplus (V^l_{\theta_l(i_1)} \otimes \ldots \otimes V^l_{\theta_l(i_a)})^{\oplus b}$. Therefore, $\sigma \circ \Phi$ factorizes through $(V^1_{\theta_1(i_1)} \otimes \ldots \otimes V^1_{\theta_1(i_a)})^{\oplus b} \oplus \ldots \oplus (V^l_{\theta_l(i_1)} \otimes \ldots \otimes V^l_{\theta_l(i_a)})^{\oplus b}$ and since σ is surjective we find that its image is the whole space, i.e., the induced morphism σ' is surjective. That means that

$$\sigma(\phi)|_{(W^{tot}_{i_1} \otimes \ldots \otimes W^{tot}_{i_a})^{\oplus b}} \neq 0 \Leftrightarrow (\phi_1, \dots, \phi_l)|_{(V^1_{\theta_1(i_1)} \otimes \ldots \otimes V^1_{\theta_1(i_a)})^{\oplus b} \oplus \ldots \oplus (V^l_{\theta_l(i_1)} \otimes \ldots \otimes V^l_{\theta_l(i_a)})^{\oplus b}} \neq 0$$
$$\Leftrightarrow \phi_j|_{(V^j_{\theta_j(i_1)} \otimes \ldots \otimes V^j_{\theta_j(i_a)})^{\oplus b}} \neq 0 \text{ for some } j.$$

□

Comparison with the semistability fuction of tensor fields

Suppose now that a weighted filtration $(\mathcal{E}_\bullet, \underline{m})$ of $\mathcal{E}$ is given. Denote r^j_i the rank $\mathrm{rk}(\mathcal{E}_i|_{Y_j})$ and $\alpha_i = \sum_{j=1}^l r^j_i d_j$ the multiplicity of $\mathcal{E}_i$. As we have seen before, this data defines a one parameter subgroup, $\lambda : \mathbb{G}_m \to \overline{H} \hookrightarrow \mathrm{SL}(V^{\oplus d})$. From $(\mathcal{E}_\bullet, \underline{m})$ and λ we get the two quantities (which do not depend on the fixed trivializations, as we have seen at the beginning of this section)

$$\mu(\mathcal{E}_\bullet, \underline{m}, \phi) = \sum_{j=1}^{s} m_j(\nu_j(I')\alpha - a\alpha_j),$$
$$\mu^{\mathcal{O}_{\mathfrak{Y}}(1)}(\sigma(\phi), \lambda) = \sum_{j=1}^{s} m_j(\nu_j(I)\alpha - a\alpha_j).$$

Then, we have,

Proposition 3.3.5. *The indices I and I' are the same. Thus $\mu^{\mathcal{O}_{\mathfrak{Y}}(1)}(\sigma(\phi), \lambda) = \mu(\mathcal{E}_\bullet, \underline{m}, \phi)$.*

Proof. Follows from Lemma 3.3.4 and the definition of $\mu(\mathcal{E}_\bullet, \underline{m}, \phi)$. □

Corollary 3.3.6. *$(\mathcal{E}, \phi)$ is generically semistable, if and only if $\sigma(\phi) \in \mathbf{P}(((\mathbb{V}^{\oplus d})^{\otimes a})^{\oplus b})$ is semistable.*

Proof. Note that any one parameter subgroup can be constructed from some weighted filtration $(\mathcal{E}_\bullet, \underline{m})$ as above. Now the result follows from Proposition 3.3.5. □

The following corollary will be crucial for the proof of the main result of this work

Corollary 3.3.7. *Let $\mathcal{E}$ be a locally free sheaf of uniform multirank r and degree d on Y, and let $(\mathcal{E}, \tau)$ be a singular principal G-bundle. Let $(\mathcal{E}, \phi_\tau)$ be the associated tensor field. Then $(\mathcal{E}, \phi_\tau)$ is generically semistable if and only if $(\mathcal{E}, \tau)|_{Y_i}$ is an honest singular principal G-bundle for all i, i.e., if and only if it is an honest singular principal G-bundle.*

Proof. Follows from [53, Corollary 4.1.2.], Corollary 3.3.2 (i), and [53, Lemma 4.2.1.] applied to each connected component. □

3.3.2 Asymptotic Semistability

We fix again constants $a, b, c, r \in \mathbb{N}$, $d \in \mathbb{Z}$, $\kappa_1, \ldots, \kappa_l \in \mathbb{Q} \cap (0,1)$ and an invertible sheaf $\mathcal{L}$ on Y. As always g denotes the genus of Y and g_i the genus of Y_i. Recall that we have a polarization on Y, $\mathcal{O}_Y(1)$ whose degree is denoted by h. Likewise, the degree of the polarization restricted to Y_i is denoted by h_i.

Definition 3.3.8. A generalized parabolic tensor field of rank, $(\mathcal{E}, \underline{q}, \phi)$, is $\underline{\kappa}$- *asymptotically (semi)stable* if a) $(\mathcal{E}, \phi)$ is generically semistable and b) for every weighted filtration $(\mathcal{E}_\bullet, \underline{m})$ such that $\mu(\mathcal{E}_\bullet, \underline{m}, \phi) = 0$, it holds $P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m})(\geq)0$.

Remark 3.3.9. Since every $\underline{\kappa}$-asymptotically semistable parabolic tensor field is generically semistable, every component $\phi|_{Y_i}$ is non-zero as well (see Corollary 3.3.2).

Definition 3.3.10. Let K_+, K_- be integers such that $d \in [K_-, K_+]$. A type $K_\bullet$ tensor field is a tensor field

$$\phi\colon (\mathcal{E}^{\otimes})^{\oplus b} \to \det(\mathcal{E})^{\otimes c} \otimes \mathcal{L}$$

such that $\deg(\mathcal{E}|_{Y_i}) \in [K_-, K_+]$.

Lemma 3.3.11. *Assume $a - rc \neq 0$. There are constants K_-, K_+ depending only on the input data such that any generically semistable tensor field of rank r and degree d is of type $K_\bullet$.*

Proof. Let $(\mathcal{E}, \phi)$ be a generically semistable tensor field of rank r and degree d. By Lemma 3.3.2, $(\mathcal{E}, \phi)$ is generically semistable if and only if $(\mathcal{E}|_{Y_i}, \phi|_{Y_i})$ is generically semistable for each $i = 1, \ldots, l$. Since $d = \deg(\mathcal{E}|_1) + \cdots + \deg(\mathcal{E}|_{Y_l})$, it is enough to show that for any generically semistable tensor field of a given rank on a smooth connected projective curve, the degree of the locally free sheaf is bounded (from below if $a - rc < 0$, or from above if $a - rc > 0$). For that we have to distinguish two cases.
1) Assume $\mathcal{E}$ is semistable. Since $\phi : (\mathcal{E}^{\otimes a})^{\oplus b} \to \det(\mathcal{E})^{\otimes c} \otimes \mathcal{L}$ is non-zero, we deduce that

$$\mu_{\min}(\mathcal{E}^{\otimes a}) \leq c \cdot \deg(\mathcal{E}) + \deg(\mathcal{L}),$$

but we know that $\mu_{\min}(\mathcal{E}^{\otimes a}) = a\mu_{\min}(\mathcal{E}) = a\mu(\mathcal{E})$. Therefore

$$(a - cr)\mu(\mathcal{E}) \leq \deg(\mathcal{L}).$$

2) Assume now that $\mathcal{E}$ is not semistable. Consider its Harder-Narasimhan filtration

$$\mathcal{E}_\bullet = \ 0 \subset H_1 \subset \cdots \subset H_s \subset H_{s+1} = \mathcal{E}.$$

We use the following notation: $H^i = H_i/H_{i-1}$, $r_i = \mathrm{rk}(H_i)$, $r^i = \mathrm{rk}(H^i)$, and $\mu^i = \mu(H^i)$. Define now

$$\mathcal{C}(\mathcal{E}_\bullet) = \{\gamma = (\gamma_1, \ldots, \gamma_{s+1}) \in \mathbb{R}^{s+1} | \gamma_1 \leq \cdots \leq \gamma_{s+1} \text{ with } \sum_{i=1}^{s+1} \gamma_i r^i = 0\}.$$

Since $(\mathcal{E}, \phi)$ is generically semistable, we have

$$f(\gamma) := \mu(\mathcal{E}_\bullet, m_\bullet(\gamma), \phi) > 0, \ m_\bullet(\gamma) = (\frac{\gamma_2 - \gamma_1}{r}, \ldots, \frac{\gamma_{s+1} - \gamma_s}{r})$$

for all $\gamma \in \mathcal{C}(\mathcal{E}_\bullet)\setminus 0$. Take a multiindex $I=(i_1,\dots,i_a)$ with $\phi|_{(H_{i_1}\otimes\cdots\otimes H_{i_a})^{\oplus b}} \neq 0$. Thus,

$$\mu_{\min}(H_{i_1}\otimes\cdots\otimes H_{i_a}) \leq c\cdot\deg(\mathcal{E})+\deg(\mathcal{L}).$$

But $\mu_{\min}(H_{i_1}\otimes\cdots\otimes H_{i_a}) = \mu_{\min}(H_{i_1})+\cdots+\mu_{\min}(H_{i_a}) = \mu^{i_1}+\cdots+\mu^{i_a}$, and we deuce that $\mu^{i_1}+\cdots+\mu^{i_a} \leq c\cdot\deg(\mathcal{E})+\deg(\mathcal{L})$. Take the point $\gamma=(\mu(\mathcal{E})-\mu^1,\dots,\mu(\mathcal{E})-\mu^{s+1}) \in \mathcal{C}(\mathcal{E}_\bullet)$, and let $m_1,\dots,m_s$ be such that

$$\Gamma=\sum_{j=1}^{s} m_j\Gamma^{(r_i)} = (\overbrace{\gamma_1,\dots,\gamma_1}^{r_1},\overbrace{\gamma_2,\dots,\gamma_2}^{r_2-r_1},\dots,\overbrace{\gamma_{s+1},\dots,\gamma_{s+1}}^{r-r_s}).$$

Then we know that

$$\begin{aligned}\mu(\mathcal{E}_\bullet,\underline{m}_\bullet(\gamma),\phi) &= -\min_I\{\Gamma_{\alpha_{i_1}}+\cdot+\Gamma_{\alpha_{i_a}} \mid \phi|_{(H_{i_1}\otimes\cdots H_{i_a})^{\oplus b}}\neq 0\} = \\ &= -(\mu(\mathcal{E})-\mu^{i_1}+\cdots+\mu(\mathcal{E})-\mu^{i_a}) = \\ &= \mu^{i_1}+\cdots+\mu^{i_a}-a\mu(\mathcal{E}) \leq \\ &\leq c\cdot\deg(\mathcal{E})+\deg(\mathcal{L})-a\mu(\mathcal{E}) = \\ &= (cr-a)\mu(\mathcal{E})+\deg(\mathcal{L}).\end{aligned}$$

Since $\mu(\mathcal{E}_\bullet,\underline{m}_\bullet(\gamma),\phi)\geq 0$ we deduce that

$$0\leq (cr-a)\mu(\mathcal{E})+\deg(\mathcal{L}) \Rightarrow (a-cr)\mu(\mathcal{E})\leq\deg(\mathcal{L}).$$

□

Lemma 3.3.12. *Let $(\mathcal{E},\underline{q},\varphi)$ be a generalized parabolic tensor field and suppose that Y_1 is a component over which $\deg(q_{i*}\mathcal{E}|_{Y_1})\geq\deg(q_{j*}\mathcal{E}|_{Y_j})$ for every $j=1,\dots,l$. There exists $\delta_0\in\mathbb{Q}_{>0}$ depending only on the input data such that if $\delta>\delta_0$ and $(\mathcal{E},\underline{q},\varphi)$ is $(\underline{\kappa},\delta)$-semistable then the restriction of φ to Y_1 is non-zero.*

Proof. Suppose that $\varphi|_{Y_1}=0$ and consider the one-step filtration given by

$$\mathcal{E}_\bullet=(0)\subset q_{1*}\mathcal{E}|_{Y_1}\subset\mathcal{E}$$

Recall that

$$\begin{aligned}P_{\underline{\kappa}}(\mathcal{E}_\bullet,\underline{m}) &= \sum_{i=1}^{s}(\alpha_i\mathrm{pardeg}(\mathcal{E})-\alpha\mathrm{pardeg}(\mathcal{E}_i)) = \\ &= \sum_{i=1}^{s} m_i\alpha_i\alpha(\mu(\mathcal{E})-\mu(\mathcal{E}_i)) - \sum_{i=1}^{s} m_i(\alpha_i\sum_{j=1}^{\nu}k_jr-\alpha\sum_{j=1}^{\nu}\kappa_j\dim(q_j(\mathcal{E}_i(y_1^j)\oplus\mathcal{E}_i(y_2^j))))\end{aligned}$$

for any weighted filtration. Since

$$\begin{aligned}\mu(\mathcal{E})-\mu(q_{1*}\mathcal{E}|_{Y_1}) &= \sum_{i=1}^{l}\frac{h_i}{h}\mu(q_{i*}\mathcal{E}|_{Y_i})-\mu(q_{1*}\mathcal{E}|_{Y_1}) = \\ &= \sum_{i\neq 1}\frac{h_i}{h}(\mu(q_{i*}\mathcal{E}|_{Y_i})-\mu(q_{1*}\mathcal{E}|_{Y_1})) = \sum_{i\neq 1}\frac{h_i}{h}(\mu(\mathcal{E}|_{Y_i})-\mu(\mathcal{E}|_{Y_1}))\leq 0,\end{aligned}$$

and $(\mathcal{E}, \underline{q}, \varphi)$ is $(\underline{\kappa}, \delta)$-semistable we deduce that

$$-(\alpha_1 \sum_{j=1}^{\nu} k_j r - \alpha \sum_{j=1}^{\nu} \kappa_j \dim(q_j(\mathcal{E}_1(y_1^j) \oplus \mathcal{E}_1(y_2^j)))) + \delta\mu(\mathcal{E}_\bullet, 1, \varphi) \geq 0$$

Now, since $\varphi|_{Y_1} = 0$ we now that $\mu(\mathcal{E}_\bullet, 1, \varphi) = -arh_1$. But there are only finitely many possible values for the first summand, which depend only on r, h, ν so, denoting by M the maximum value and $\delta_0 = \dfrac{M}{arh_1}$, we get a contradiction. Therefore $\varphi|_{Y_1} \neq 0$. □

Lemma 3.3.13. *Let $\delta > \delta_0$ and let $(\mathcal{E}, \underline{q}, \phi)$ be a $(\underline{\kappa}, \delta)$-semistable generalized parabolic tensor field of rank r and degree d. Then there are constants K'_-, K'_+ depending only on the input data, such that $(\mathcal{E}, \underline{q}, \varphi)$ is of type $K'_\bullet$.*

Proof. Recall that

$$P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m}) = \sum_{i=1}^{s} (\alpha_i \mathrm{pardeg}(\mathcal{E}) - \alpha \mathrm{pardeg}(\mathcal{E}_i)) =$$
$$= \sum_{i=1}^{s} m_i \alpha_i \alpha(\mu(\mathcal{E}) - \mu(\mathcal{E}_i)) - \sum_{i=1}^{s} m_i (\alpha_i \sum_{j=1}^{\nu} k_j r - \alpha \sum_{j=1}^{\nu} \kappa_j \dim(q_j(\mathcal{E}_i(y_1^j) \oplus \mathcal{E}_i(y_2^j))))$$

We can assume without lost of generality that $\deg(\mathcal{E}|_{Y_1}) \geq \deg(\mathcal{E}|_{Y_i})$ for all $i = 1, \ldots, l$. Let

$$H'_\bullet = 0 \subset H'_1 \subset \cdots \subset H'_s \subset H'_{s+1} = \mathcal{E}|_{Y_1}$$

be the Harder-Narasimhan filtration of $\mathcal{E}|_{Y_1}$. Denote $H_t := q_{t*} H'_t$, with $t = 1, \ldots, s$, and by $H_\bullet$ the corresponding filtration. Notice that for this filtration we have $\mu(q_{1*}\mathcal{E}|_{Y_1}) - \mu(H_t) \leq 0$ for all $t = 1, \ldots, s$. Then, we have

$$\mu(\mathcal{E}) - \mu(H_t) = \sum_{i=1}^{l} \frac{h_i}{h} \mu(q_{i*}\mathcal{E}|_{Y_i}) - \mu(H_t) \leq \sum_{i=1}^{l} \frac{h_i}{h} \mu(q_{i*}\mathcal{E}|_{Y_i}) - \mu(q_{1*}\mathcal{E}|_{Y_1}) =$$
$$= \sum_{i \neq 1} \frac{h_i}{h} (\mu(q_{i*}\mathcal{E}|_{Y_i}) - \mu(q_{1*}\mathcal{E}|_{Y_1})) = \sum_{i \neq 1} \frac{h_i}{h} (\mu(\mathcal{E}|_{Y_i}) - \mu(\mathcal{E}|_{Y_1})) \leq 0,$$

so the summand $\sum_{i=1}^{s} m_i \alpha_i \alpha(\mu(\mathcal{E}) - \mu(\mathcal{E}_i)) < 0$ is negative. Since $(\mathcal{E}, \underline{q}, \phi)$ is $(\underline{\kappa}, \delta)$-semistable we must have

$$\sum_{i=1}^{s} m_i (\alpha_i \sum_{j=1}^{\nu} k_j r - \alpha \sum_{j=1}^{\nu} \kappa_j \dim(q_j(H_i(y_1^j) \oplus H_i(y_2^j)))) + \delta\mu(H_\bullet, \underline{m}, \phi) \geq 0$$

for all vector of weights $\underline{m}$. Let us show that $\mu(H_\bullet, \underline{m}, \phi)$ must be non-negative for all vector of weights. Suppose there is $\underline{m}$ such that $\mu(H_\bullet, \underline{m}, \phi) < 0$. Then there is an index $j = 1, \ldots, s$ such that $\nu_j(H_\bullet)\alpha - a\alpha_i < 0$. Then we can find positive rational numbers $m'_1, \ldots, m'_s$ with $m_j \gg 0$ and m_i very close to 0 for all $i \neq j$, such that

$$\sum_{i=1}^{s} m'_i (\alpha_i \sum_{j=1}^{\nu} k_j r - \alpha \sum_{j=1}^{\nu} \kappa_j \dim(q_j(H_i(y_1^j) \oplus H_i(y_2^j)))) + \delta\mu(H_\bullet, \underline{m}', \phi) < 0$$

which is not possible. Therefore $\mu(H_\bullet, \underline{m}, \phi) \geq 0$ for all $\underline{m}$. Now, by Lemma 3.3.12, we can proceed as in Lemma 3.3.11 to show that $\deg(\mathcal{E}|_{Y_1}) \leq r\dfrac{cd + \deg(\mathcal{L})}{a}$. Since $\deg(\mathcal{E}|_{Y_i}) \leq \deg(\mathcal{E}|_{Y_1})$ and $d = \deg(\mathcal{E}|_{Y_1}) + \cdots + \deg(\mathcal{E}|_{Y_l})$ we conclude. □

Theorem 3.3.14. *There is a rational number $\delta^1_\infty \in \mathbb{Q}_{>0}$, depending only on the input data, such that for any $\delta > \delta^1_\infty$, if a tensor field with generalized parabolic structure, $(\mathcal{E}, \underline{q}, \varphi)$, of uniform multirank r and degree d, is $(\underline{\kappa}, \delta)$-semistable then it is $\underline{\kappa}$-asymptotically semistable.*

Proof. First we assume that $\delta > \delta_0$ so Lemma 3.3.13 holds true. It is enough to show that it is generically semistable. Suppose it is not. Then by Corollary 3.3.2 and Corollary 3.3.6, it implies that the restriction of $(\mathcal{E}, \underline{q}, \varphi)$ some component is not generically semistable. We can assume without lost of generality that it is not generically semistable on the first component. By [52, Proposition 3.3.3] there is a constant C_1, depending only on the input data, and a weighted filtration $(\mathcal{E}^1_\bullet, \underline{m^1})$, such that

(i) $C_1 \geq \sum_{i=1}^s m^1_i(\mathrm{rk}(\mathcal{E}^1_i)\deg(\mathcal{E}|_{Y_1}) - r\deg(\mathcal{E}^1_i))$
(ii) $\mu(\mathcal{E}^1_\bullet, \underline{m^1}, \varphi_1) < 0$

where $(\mathrm{rk}(\mathcal{E}^1_1), \ldots, \mathrm{rk}(\mathcal{E}^1_s), m^1_1, \ldots, m^1_s)$ belongs to a finite set which depends only on the numerical input data a, b, c. In particular $m^1_i \leq A$, where A is a constant which depends only on a, b, c. From the filtration $(\mathcal{E}^1_\bullet, \underline{m^1})$ we can construct a weighted filtration $(\mathcal{E}_\bullet, \underline{m})$ of $\mathcal{E}$ defining

$$\begin{aligned} \mathcal{E}_i\colon &= \iota_{1*}\mathcal{E}^1_i \oplus \iota_{2*}\mathcal{E}|_{Y_2} \oplus \cdots \oplus \iota_{l*}\mathcal{E}|_{Y_l} \\ m_i\colon &= m^1_i \end{aligned}$$

By Proposition 1.1.28, we can assume without lost of generality that $\mu(\mathcal{E}_\bullet, \underline{m}, \varphi) < -1$. Let us give a bound for $P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m})$. Recall that we may assume that $m_i \leq A$ for some constant A, and

$$\begin{aligned} P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m}) &= \sum_{i=1}^s m_i(\alpha_i\underline{\kappa}\text{-pardeg}(\mathcal{E}) - \alpha\underline{\kappa}\text{-pardeg}(\mathcal{E})) = \\ &= \sum_{i=1}^s m_i(\alpha_i\deg(\mathcal{E}) - \alpha\deg(\mathcal{E}_i)) + \sum_{i=1}^s m_i(\alpha\underline{\kappa}\text{-par}(\mathcal{E}_i) - \alpha_i\underline{\kappa}\text{-par}(\mathcal{E})) = \\ &= P(\mathcal{E}_\bullet, \underline{m}) + \sum_{i=1}^s m_i(\alpha\underline{\kappa}\text{-par}(\mathcal{E}_i) - \alpha_i\underline{\kappa}\text{-par}(\mathcal{E})), \end{aligned}$$

where $\underline{\kappa}\text{-par}(\mathcal{F}) = \sum_{i=1}^\nu \kappa_i \dim(q(\mathcal{F}(y^1_i) \oplus \mathcal{F}(y^2_i)))$ for any $\mathcal{F} \subset \mathcal{E}$. Let us give bounds for both terms. For the right hand side, we have

$$\begin{aligned} \sum_{i=1}^s m_i(\alpha\underline{\kappa}\text{-par}(\mathcal{E}_i) - \alpha_i\underline{\kappa}\text{-par}(\mathcal{E})) &= hr\sum_{i=1}^s m_i(\mathrm{par}(\mathcal{E}_i) - \frac{h_i}{h}\mathrm{rk}(\mathcal{E}_i)\nu\sum_{j=1}^\nu \kappa_j) \leq \\ \leq hr\sum_{i=1}^s m_i\underline{\kappa}\text{-par}(\mathcal{E}_i) &\leq \alpha^2 A\nu r. \end{aligned} \tag{3.40}$$

For the left hand side, a short calculation shows that

$$\begin{aligned}
\alpha\ \deg(\mathcal{E}_i) &= \sum_{j=1}^{l} \alpha\ \deg(\mathcal{E}_i|_{Y_j}) + \alpha(\alpha(\mathcal{E}_i|_{Y_j})B^j(\mathcal{O}_Y(1))) \\
&= \alpha\deg(\mathcal{E}_i^1) + \sum_{j=2}^{l} \alpha\ \deg(\mathcal{E}|_{Y_j}) + \alpha(\alpha(\mathcal{E}_i|_{Y_j})B^j(\mathcal{O}_Y(1))), \qquad (3.41) \\
\alpha_i\ \deg(\mathcal{E}) &= \sum_{j=1}^{l} h_j r(\mathcal{E}_i|_{Y_j})d,
\end{aligned}$$

where $B^j(\mathcal{O}_Y(1)$ is the constant defined in Section 1.3. Therefore,

$$\begin{aligned}
&\alpha_i\ \deg(\mathcal{E}) \leq \sum_{j=1}^{l} h_j r(\mathcal{E}_i|_{Y_j})d \leq \alpha\widehat{d},\ \text{ where } \widehat{d} = \max = \{0, d\}, \\
&\alpha\ \deg(\mathcal{E}_i) \geq \alpha\deg(\mathcal{E}_i^1) + \alpha(l-1)K_- + \alpha^2\widehat{B},\ \text{ where } \widehat{B} = \min_j\{0, B^j(\mathcal{O}_Y(1))\}.
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{s} m_i(\alpha_i\ \deg(\mathcal{E}) - \alpha\ \deg(\mathcal{E}_i)) &\leq \sum_{i=1}^{s} m_i(\alpha\widehat{d} - \alpha(l-1)K_- + \alpha^2\widehat{B} - \alpha\deg(\mathcal{E}_i^1)) = \\
&= (\alpha\widehat{d} - \alpha(l-1)K_- + \alpha^2\widehat{B})\sum_{i=1}^{s}(m_i) - h\sum_{i=1}^{s} m_i^1 r\deg(\mathcal{E}_i^1)
\end{aligned}$$

In the other hand, condition (i) implies that

$$C_1 - \sum_{i=1}^{s} m_i^1\mathrm{rk}(\mathcal{E}_i^1)K_- \geq C_1 - \sum_{i=1}^{s} m_i^1(\mathrm{rk}(\mathcal{E}_i^1)\deg(\mathcal{E}|_{Y_1}) \geq \sum_{i=1}^{s} m_i^1 r\deg(\mathcal{E}_i^1)).$$

If we define $\widehat{K} := \min\{iK_-|i = 0, \dots, r\}$, we finally have

$$C_1 - \widehat{K}\sum_{i=1}^{s} m_i^1 \geq \sum_{i=1}^{s} m_i^1 r\deg(\mathcal{E}_i^1)).$$

Thus,

$$\sum_{i=1}^{s} m_i(\alpha_i\ \deg(\mathcal{E}) - \alpha\ \deg(\mathcal{E}_i)) \leq (\alpha\widehat{d} - \alpha(l-1)K_- + \alpha^2\widehat{B} - h\widehat{K})\alpha A - hC_1$$

All of this together shows that there exists a constant C which depends only on the input data such that $P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m}) < C$. Now, since $\mu(\mathcal{E}_\bullet, \underline{m}, \varphi) < -1$, taking $\delta_\infty^1 > \max\{\delta_0, C\}$ we get

$$P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m}) + \delta_\infty\mu(\mathcal{E}_\bullet, \underline{m}, \varphi) < C - \delta_\infty^1 < 0$$

which contradicts the fact that $(\mathcal{E}, \underline{q}, \varphi)$ is $(\underline{\kappa}, \delta)$-semistable.

□

Theorem 3.3.15. *Assume $a-cr\neq 0$. There exists $\delta^2_\infty \in \mathbb{Q}_{>0}$ depending on a,b,c,d,r, $\mathcal{L},l,g,g_i,h,h_i$ such that for each $\delta>\delta^2_\infty$, any tensor field with generalized parabolic structure $(\mathcal{E},\underline{q},\phi)$ of degree d and rank r which is $\underline{\kappa}$-asymptotically semistable is $(\underline{\kappa},\delta)$-semistable as well.*

Proof. First of all, note that, fixed $h_1,\ldots,h_l,h,g$ there are only finitely many values for the constants $B^j(Y,\mathcal{O}_Y(1)) = \dfrac{\chi(Y_i,\mathcal{O}_{Y_i})}{h_i} - \dfrac{\chi(Y,\mathcal{O}_Y)}{h}$. Let us denote by $\{\omega_1,\ldots,\omega_q\}$ these possible values, and define $B\colon = \max\{0,\omega_1,\ldots,\omega_q\}$. Thus, we know that

$$B^j(Y,\mathcal{O}_Y(1)) \le B,\ \forall\, j=1,\ldots,l.$$

Let us show that for any weighted filtration $(\mathcal{E}_\bullet,\underline{m})$ there are constants E_1,E_2 such that,

$$\begin{aligned}\alpha_i\deg(\mathcal{E})-\alpha\deg(\mathcal{E}_i)&\ge E_1,\\ \alpha\underline{\kappa}\text{-par}(\mathcal{E}_i)-\alpha_i\underline{\kappa}\text{-par}(\mathcal{E})&\ge E_2.\end{aligned}$$

(i) For the parabolic part we have,

$$\alpha\underline{\kappa}\text{-par}(\mathcal{E}_i)-\alpha_i\underline{\kappa}\text{-par}(\mathcal{E}) \ge -\alpha_i\underline{\kappa}\text{-par}(\mathcal{E}) \ge -\alpha_i\nu r \ge -\alpha\nu r =: E_2.$$

(ii) For the non parabolic part we have,

$$\begin{aligned}\alpha_i\deg(\mathcal{E})-\alpha\deg(\mathcal{E}_i) &\ge \alpha_i\widehat{d}-\alpha\deg(\mathcal{E}_i) =\\ &= \alpha_i\widehat{d}-\alpha\sum_{j=1}^{l}(\deg(\mathcal{E}_i|_{Y_j})+\alpha(\mathcal{E}_i|_{Y_j})B^j(Y,\mathcal{O}_Y(1))),\end{aligned}$$

where the last equality follows from Equation (1.26). Since $B^j(Y,\mathcal{O}_Y(1))\le B$ and $\alpha(\mathcal{E}_i|_{Y_j})\le\alpha$, we get

$$\alpha_i\deg(\mathcal{E})-\alpha\deg(\mathcal{E}_i)\ge \alpha\widehat{d}-\alpha^2B-\sum_{j=1}^{l}\alpha\deg(\mathcal{E}_i|_{Y_j}).$$

Now by Lemma 3.3.11 and [53, Theorem 2.3.4.3], we know that $\deg(\mathcal{E}_i|_{Y_j})\le K''$ for some constant depending only on the input data. Therefore,

$$\alpha_i\deg(\mathcal{E})-\alpha\deg(\mathcal{E}_i)\ge\alpha\widehat{d}-\alpha^2B-\alpha lK'' =: E_1.$$

Now we have,

$$P_{\underline{\kappa}}(\mathcal{E}_\bullet,\underline{m})+\delta\mu(\mathcal{E}_\bullet,\underline{m},\phi)\ge E_1+E_2+\delta\mu(\mathcal{E}_\bullet,\underline{m},\varphi).$$

Since the tensor field is generically semistable we can assume $\mu(\mathcal{E}_\bullet,\underline{m},\phi)\ge 1$. Thus,

$$P_{\underline{\kappa}}(\mathcal{E}_\bullet,\underline{m})+\delta\mu(\mathcal{E}_\bullet,\underline{m},\phi)\ge E_1+E_2+\delta.$$

If we define $\delta^2_\infty := -E_1-E_2$, we deduce that

$$P_{\underline{\kappa}}(\mathcal{E}_\bullet,\underline{m})+\delta\mu(\mathcal{E}_\bullet,\underline{m},\phi)\ge 0.$$

□

Corollary 3.3.16. *Assume $a - cr \neq 0$. The family of isomorphism classes of locally free sheaves $\mathcal{E}$ of degree d and rank r appearing in $\underline{\kappa}$-asymptotically semistable parabolic tensor fields is bounded.*

Proof. Follows from Theorem 3.3.15 and Proposition 3.1.15. □

Corollary 3.3.17. *Assume $a - cr \neq 0$. Let $(\mathcal{E}, \underline{q}, \phi)$ be a tensor field with generalized parabolic structure of rank r and degree d. Then $(\mathcal{E}, \underline{q}, \phi)$ is $\underline{\kappa}$-asymptotically semistable if and only if it is $(\underline{\kappa}, \delta)$-semistable for all $\delta > \delta_\infty$*

Proof. The direct part is Theorem 3.3.15. Let us see the inverse. Suppose that $(\mathcal{E}, \underline{q}, \phi)$ is $(\underline{\kappa}, \delta)$-semistable for all $\delta > \delta_\infty$. If there exists a weighted filtration $(\mathcal{E}_\bullet, \underline{m})$ such that $\mu(\mathcal{E}_\bullet, \underline{m}, \phi) = 0$ then obviously $P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m}) \geq 0$. Suppose that $(\mathcal{E}_\bullet, \underline{m})$ is such that $\mu(\mathcal{E}_\bullet, \underline{m}, \phi) < 0$ then we can find δ large enough such that

$$P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m}) + \delta\mu(\mathcal{E}_\bullet, \underline{m}, \phi) < 0$$

which is a contradiction, so $\mu(\mathcal{E}_\bullet, \underline{m}, \phi)$ must be non-negative. □

Lemma 3.3.18. *Let S be a scheme and let $\phi : (\mathcal{E}_S^{\otimes a})^{\oplus b} \to \det(\mathcal{E}_S)^{\otimes c} \otimes \pi^*(\mathcal{L})$ be a family of tensor fields parametrized by S. Then the set*

$$\{s \in S | \mu(\mathcal{E}_{s\bullet}, \underline{\beta}, \phi_s) \geq 0 \text{ for all weighted filtrations}\}$$

is open.

Proof. Define $S_i = \{s \in S |\ \mu(\mathcal{E}_{s,i\bullet}, \underline{m}, \phi_{s,i}) \geq 0,\ \forall\ (\mathcal{E}_{s,i\bullet}, \underline{m}) \text{ weighted filtration of } \mathcal{E}_{s,i}\}$. These subsets are open subschemes (see [53, Lemma 2.3.6.8.]). Define $S_g := \cap S_i$ which is also open. Clearly

$$S_g = \{s \in S | \text{for all } \ i = 1, \dots, l, \mu(\mathcal{E}_{s,i\bullet}, \underline{m}, \phi_{s,i}) \geq 0,\ \forall \text{ weighted filtration of } \mathcal{E}_{s,i}\}$$

Corollary 3.3.6 and Corollary 3.3.2 implies that

$$S_g = \{s \in S | \mu(\mathcal{E}_{s\bullet}, \underline{m}, \phi_s) \geq 0,\ \forall \text{ weighted filtrations}\}$$

and we are done.

□

Lemma 3.3.19. *Let S be a scheme and let $(\mathcal{E}_S, \underline{q}_S, \phi_S)$ be a family of tensor fields with gerneralized parabolic structure parametrized by S. Then the set*

$$\overline{S} := \{s \in S | (\mathcal{E}_{S,s}, \underline{q}_{S,s}, \phi_{S,s}) \textit{ is } \underline{\kappa}\text{-asymptotically semistable}\}$$

is open.

Proof. Consider the open subscheme S_g. For any $\delta \in \mathbb{Q}_{>0}$ consider the open subscheme S_δ consisting on those points $s \in S$ for which $(\mathcal{E}_{S,s}, \underline{q}_{S,s}, \phi_{S,s})$ is $(\underline{\kappa}, \delta)$-semistable tensor fields with generalized parabolic structure. Then we can consider the open subscheme $S' = \bigcup_{\delta \in \mathbb{Q}_{>0}} (S_g \cap S_\delta)$ and, obviously, we have $S' = \overline{S}$. □

The aim of this section is to prove the following theorem.

Theorem 3.3.20. *There exists a rational number $\delta_\theta \in \mathbb{Q}_{\geq 0}$ depending only on the numerical input data such that, for every tensor field with generalized parabolic structure, $(\mathcal{E}, \underline{q}, \phi)$, of degree $\deg(\mathcal{E}) = d$, rank $\mathrm{rk}(\mathcal{E}) = r$, and every rational number $\delta \geq \delta_\theta$, the following two conditions are equivalent:*

(i) $(\mathcal{E}, \underline{q}, \phi)$ is $(\underline{\kappa}, \delta)$-(semi)stable.

(ii) $(\mathcal{E}, \underline{q}, \phi)$ is $\underline{\kappa}$-asymptotically (semi)stable.

Proof. Follows from Theorem 3.3.14 and Theorem 3.3.15 taking $\delta_\theta := \max\{\delta^1_\infty, \delta^2_\infty\}$. □

3.3.3 Semistability for Large Values of the Numerical Parameters

Consider a tensor field on Y, $\phi : (\mathcal{E}^{\otimes a})^{\oplus b} \to \det(\mathcal{E})^{\otimes c} \otimes \mathcal{L}$, and a weighted filtration $(\mathcal{E}_\bullet, \underline{\gamma})$. Restricting the filtration to each component we get

$$\mathcal{E}^j_\bullet \equiv (0) \subseteq \mathcal{E}_1|_{Y_j} \subseteq \ldots \subseteq \mathcal{E}_s|_{Y_j} \subseteq \mathcal{E}|_{Y_j}.$$

Eliminating the proper inclusions and defining γ^i_j as in (3.38) we get weighted flags

$$(\mathcal{E}, \underline{\gamma})|_{Y_j} := \left\{ \begin{array}{c} \mathcal{E}^j_\bullet \equiv (0) \subset \mathcal{E}^j_1 \subset \ldots \subset \mathcal{E}^j_{s(j)} \subset \mathcal{E} \\ \underline{\gamma}^j = (\gamma^j_1, \ldots, \gamma^j_{s(j)+1}). \end{array} \right\}, \ j = 1, \ldots, l \tag{3.42}$$

Proposition 3.3.21. *Let $(\mathcal{E}, \tau)$ be an honest singular principal G-bundle and $(\mathcal{E}_\bullet, \underline{m})$ a weighted filtration. Then the following are equivalent:*

1) $\mu(\mathcal{E}_\bullet, \underline{m}, \phi_\tau) = 0$.

2) There are one parameter subgroups λ_i, $i = 1, \ldots l$ and reductions β_i of the G-bundles $(\mathcal{E}|_{Y_i}, \tau_i)$ to the one parameter subgroup λ_i of G such that $(\mathcal{E}_\bullet, \underline{m})|_{Y_i} = (\mathcal{E}_{\beta_i \bullet}, \underline{m}_{\beta_i})$.

Proof. 1) ⇒ 2) Suppose $\mu(\mathcal{E}_\bullet, \underline{m}, \phi_\tau) = 0$. Since $(\mathcal{E}, \tau)$ is an honest singular principal G-bundle we know that $(\mathcal{E}|_{Y_i}, \tau_i)$ is an honest singular principal G-bundle on Y_i and therefore (see [52, Corollary 4.1.2])

$$[\phi_{\tau_i}|_{\eta_i} \otimes_{K_1} K] \in \mathbf{P}((\mathbb{V}^{\otimes a})^{\oplus b})^{ss}, \ \ \forall i = 1, \ldots, l. \tag{3.43}$$

Then by Corollary 3.3.2 we deduce that $\mu((\mathcal{E}_\bullet, \underline{m})|_{Y_i}, \phi_{\tau_i}) = 0$ and by [52, Proposition 4.2.2.], we know that there exists a reduction β_i to a one parameter subgroup λ_i of G such that $(\mathcal{E}_\bullet, \underline{m})|_{Y_i} = (\mathcal{E}_{i,\beta\bullet}, \underline{m}^i_\beta)$.

2) ⇒ 1) Over each component Y_i we have that $\mu((\mathcal{E}_\bullet, \underline{m})|_{X_i}, \phi_{\tau_i}) = 0$ because of [52, Proposition 4.2.2.]. Since $(\mathcal{E}|_{X_i}, \tau_i)$ is an honest singular principal G-bundle we deduce by Corollary 3.3.2 that $\mu(\mathcal{E}_\bullet, \underline{m}, \phi_\tau) = 0$. □

Theorem 3.3.22. *There is a rational number $\delta_\theta > 0$ depending only on the input data such that for all $\delta > \delta_\theta$ and every singular principal G-bundle with a generalized parabolic structure of rank r and degree d, the property A) and the property B) are equivalent:*

A) $(\mathcal{E}, \underline{q}, \tau)$ is $(\underline{\kappa}, \delta)$-(semi)stable.

B) $(\mathcal{E}, \underline{q}, \tau)$ is $\underline{\kappa}$-(semi)stable: $(\mathcal{E}, \underline{q}, \tau)$ is honest and for every weighted filtration $(\mathcal{E}_\bullet, \underline{m})$ such that $(\mathcal{E}_\bullet, \underline{m})|_{Y_i}$ is induced by a reduction to a one parameter subgroup, one has $P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m})(\geq)0$.

Proof. Consider δ_θ to be the rational number such that $\forall \delta > \delta_\theta$ $(\underline{\kappa}, \delta)$-(semi)stable is equivalent to $\underline{\kappa}$-asymptotically (semi)stable (see Theorem 3.3.20).
$A) \Rightarrow B)$ By Theorem 3.3.20, $(\mathcal{E}, \underline{q}, \phi_\tau)$ is $\underline{\kappa}$-asymptotically semistable. In particular, it is generically semistable. By Corollary 3.3.7 this implies that $(\mathcal{E}, \underline{q}, \tau)$ is honest. Now $(\underline{\kappa}, \delta)$-semistability and Proposition 3.3.21 implie condition B).
$B) \Rightarrow A)$ Since $(\mathcal{E}, \underline{q}, \tau)$ is honest, $(\mathcal{E}, \underline{q}, \phi_\tau)$ is generically semistable (see Corollary 3.3.7). Condition B) and Proposition 3.3.21 show that if $\mu(\mathcal{E}_\bullet, \underline{m}, \phi_\tau) = 0$ then $P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m})$ is grater or equal than 0. Therefore $(\mathcal{E}, \underline{q}, \phi_\tau)$ is $\underline{\kappa}$-asymptotically semistable. Finally, by Theorem 3.3.20 this implies condition A). $\square$

Chapter 4

Compactification of the Moduli Space of Principal G-Bundles on Nodal Curves

The main result of this chapter is Theorem 4.4.18, which states the existence of a projective coarse moduli space for semistable honest singular principal G-bundles over the nodal projective curve X.

4.1 Descending G-Bundles

Let k be an algebraically closed field of characteristic 0, G a semisimple linear algebraic group and $\rho : G \to \mathrm{SL}(V) \subset \mathrm{GL}(V)$ a faithful representation, V being a k-vector space of dimension n.

Let X be a reduced connected nodal curve with ν nodes. We denote by $x_1, \ldots, x_\nu$ the nodes of X. Denote by Y the normalization, $X_1, \ldots, X_l$ the irreducible components of X and $Y_1, \ldots, Y_l$ the irreducible components of Y. Recall that Y_i is the normalization of X_i and that $Y = \coprod Y_i$. Denote g, $\widetilde{g}$, g_i and $\widetilde{g}_i$ the arithmetic genus of X, Y, X_i and Y_i respectively. The relation between the genus of the curve X and its normalization Y is

$$g = \widetilde{g} + \nu$$

We use the following notation: $j_i : X_i \hookrightarrow X$ is the natural inclusion of the ith irreducible component of X, $q_i : Y_i \hookrightarrow Y$ is the natural inclusion of the ith irreducible component of Y, $\pi : Y \to X$ is the normalization map of X and $\pi_i : Y_i \to X_i$ is the normalization map for X_i. For each torsion free sheaf $\mathcal{F}$ we denote

$$\mathcal{F}_i = j_i^* \mathcal{F} / T_i$$

T_i being the torsion subsheaf of $j_i^* \mathcal{F}$. We call $\mathcal{F}_i$ the restriction of $\mathcal{F}$ to the ith component. There is always an exact sequence (see [57])

$$0 \to \mathcal{F} \hookrightarrow \bigoplus p_{i*} \mathcal{F}_i \to T \to 0 \tag{4.1}$$

where T is a torsion sheaf with support contained in $\{x_1, \ldots, x_\nu\}$. Recall that if $r_i = \mathrm{rk}(\mathcal{F}_i)$, we say that $\mathcal{F}$ is of multirank $\underline{r} = (r_1, \ldots, r_l)$. If $r_i = r$ we say that $\mathcal{F}$ has uniform multirank equal to r or that $\mathcal{F}$ has rank r.

We will denote by y_1^i, y_2^i the points in Y lying on the ith nodal point x_i. We also denote by $D_i = y_1^i + y_2^i$ the corresponding divisor on Y and by $D = \sum D_i$ the total divisor.

Let us see how to construct singular principal G-bundles on X from singular principal G-bundles with generalized parabolic structure on Y.

Let $(\mathcal{E}, \underline{q}, \tau)$ be a singular principal G-bundle with generalized parabolic structure on Y with $\mathrm{rk}(\mathcal{E}) = r$. Consider the natural surjection

$$\mathrm{ev}_D = \oplus \mathrm{ev}_i : \mathcal{E} \to \mathcal{E}|_D = \bigoplus \mathcal{E}|_{D_i}$$

and take the push-forward via π (the structure morphism of X)

$$\pi_*(\mathrm{ev}_D) : \pi_*(\mathcal{E}) \to \pi_*(\mathcal{E}|_D).$$

Since $\pi_*(\mathcal{E}|_D)$ is precisely the vector space $\bigoplus(\mathcal{E}(y_1^i) \oplus \mathcal{E}(y_2^i))$ supported on the nodes, we can consider $R = \bigoplus R_i$ as a skycraper sheaf supported on the nodes and compose $\pi_*(\mathrm{ev}_D)$ with q

$$q \circ \pi_*(\mathrm{ev}_D) : \pi_*(\mathcal{E}) \to R \to 0.$$

Defining $\mathcal{F} = \mathrm{Ker}(q \circ \pi_*(\mathrm{ev}_D))$, we get an exact sequence

$$0 \to \mathcal{F} \hookrightarrow \pi_*(\mathcal{E}) \to R \to 0 \tag{4.2}$$

where $\mathcal{F}$ is a torsion free sheaf of rank r and R is a torsion sheaf supported on the nodes and of length $l(R) = r\nu$.

It remains to construct $\tau' : S^\bullet(\mathcal{F} \otimes V)^G \to \mathcal{O}_X$ from the data $(\mathcal{E}, \underline{q}, \tau)$. Consider the $\mathcal{O}_X$-algebra $S^\bullet(\mathcal{F} \otimes V)^G$ and take the pull-back via π, $\pi^*(S^\bullet(\mathcal{F} \otimes V)^G)$. Since the functor $(-)^G$ is exact (see Theorem 1.1.11) and the symmetric algebra functor behaves well under base change, we get an isomorphism,

$$\pi^*(S^\bullet(\mathcal{F} \otimes V)^G) \simeq S^\bullet(\pi^*(\mathcal{F}) \otimes V)^G. \tag{4.3}$$

Now, the injection $\mathcal{F} \hookrightarrow \pi_*(\mathcal{E})$ defines, by adjunction, a morphism of $\mathcal{O}_Y$-modules $\pi^*(\mathcal{F}) \to \mathcal{E}$ and, therefore, a morphism of $\mathcal{O}_Y$-algebras $v : S^\bullet(\pi^*(\mathcal{F}) \otimes V)^G \to S^\bullet(\mathcal{E} \otimes V)^G$. Taking the composition with v in (4.3) we get

$$\pi^*(S^\bullet(\mathcal{F} \otimes V)^G) \to S^\bullet(\mathcal{E} \otimes V)^G.$$

Finally if we take the composition with τ, we get

$$\pi^*(S^\bullet(\mathcal{F} \otimes V)^G) \to \mathcal{O}_Y$$

and, again by adjunction, a morphism of $\mathcal{O}_X$-algebras

$$\tau' : S^\bullet(\mathcal{F} \otimes V)^G \to \pi_*(\mathcal{O}_Y). \tag{4.4}$$

Definition 4.1.1. A descending G-bundle of rank r on Y is a singular principal G-bundle with generalized parabolic structure, $(\mathcal{E}, \underline{q}, \tau)$, such that τ' (see (4.4)) takes values in $\mathcal{O}_X \subset \pi_*(\mathcal{O}_Y)$. A descending G-bundle, $(\mathcal{E}, \underline{q}, \tau)$, is principal if $(\mathcal{E}, \tau)$ is honest.

Definition 4.1.2. A descending G-bundle is $(\underline{\kappa}, \delta)$-(semi)stable if it is as singular principal G-bundle with generalized parabolic structure (see Definition 3.2.4).

A family of descending G-bundles is defined in the obvious way. Thus, we have the moduli functor

$$
{}^{\mathrm{rig}}\mathbf{D}(G)(S) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{descending } G\text{-bundles, } (\mathcal{E}_S, \underline{q}_S, \tau_S) \\ \text{with Hilbert polynomial } P \text{ and} \\ g_S : W \otimes \mathcal{O}_S \to \pi_{S*}\mathcal{E}_S(n) \\ \text{a morphism such that the induced} \\ \text{morphism } W \otimes \mathcal{O}_{Y\times S}(-n) \to \mathcal{E}_S \\ \text{is surjective} \end{array} \right\} \subset^{\mathrm{rig}} \mathbf{SPBGPS}(\rho)^n_P(S)
$$

The functors $\mathbf{D}(G)^{(\underline{\kappa},\delta)-(s)s}$ are defined is the obvious way.

4.2 Construction of Torsion Free Sheaves from Parabolic Structures

Let $\mathcal{F}$ be a torsion free sheaf on the nodal curve X and consider the pull-back via the normalization map, $\pi^*(\mathcal{F})$. Since it may have torsion elements, we have the exact sequence

$$0 \to T \hookrightarrow \pi^*(\mathcal{F}) \to \pi^*(\mathcal{F})/T \to 0,$$

T being the torsion module. Taking now the push-forward of the exact sequence, we get

$$0 \to \pi_*T \hookrightarrow \pi_*\pi^*(\mathcal{F}) \to \pi_*(\pi^*(\mathcal{F})/T) \to 0 \tag{4.5}$$

where the zero at the right hand side comes from the fact that, since π is affine the higher direct images $R^i\pi_*(-)$ are zero for $i > 0$.

Lemma 4.2.1. *The torsion module T is concentrated on the points of the preimages, $\{y^i_j\}$, of the nodal points x_i and π_*T is concentrated at the nodal points.*

Proof. Let $y \in Y - \pi^{-1}(\mathrm{Sing}(X))$ and consider $(\pi^*\mathcal{F})_y$. By definition $(\pi^*\mathcal{F})_y = \mathcal{F}_y \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,y}$, where $x = \pi(y)$. Since x is a smooth point we have an isomorphism of rings $\mathcal{O}_{X,x} \simeq \mathcal{O}_{Y,y}$ induced by the morphism of schemes $\pi : Y \to X$ so that $\mathcal{F}$ is canonically an $\mathcal{O}_{Y,y}$-module and $(\pi^*\mathcal{F})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,y} \simeq \mathcal{F}_y$ which is torsion free. Then we deduce that $(T)_y = 0$ outside $\pi^{-1}(\mathrm{Sing}(X))$. □

Lemma 4.2.2. *Let $\mathcal{F}$ be a torsion free sheaf on X of rank r. The canonical map $\alpha : \mathcal{F} \to \pi_*\pi^*(\mathcal{F})$ is injective.*

Proof. Since the canonical map $\mathcal{F} \to \pi_*\pi^*(\mathcal{F})$ is an isomorphism outside the nodes and both have the same (uniform multi-) rank we deduce that the kernel must be a torsion sheaf but, since $\mathcal{F}$ is torsion free, this module must be zero. □

Lemma 4.2.3. *Let $\mathcal{F}$ be a torsion free sheaf on X of rank r and types $\{a_i\}$. Then the torsion sheaf of the canonical exact sequence*

$$0 \to \mathcal{F} \hookrightarrow \pi_*\pi^*\mathcal{F} \to T \to 0$$

has length equal to $\sum_{i=1}^{\nu}(2r - a_i)$.

Proof. Consider the singular point x_i and localize the exact sequence

$$0 \to \mathcal{F}_{x_i} \hookrightarrow \mathcal{F}_{x_i} \otimes_{\mathcal{O}_{x_i}} \widetilde{\mathcal{O}}_{x_i} \to T_{x_i} \to 0.$$

Note that this is the exact sequence we get from

$$0 \to \mathcal{O}_{x_i} \hookrightarrow \widetilde{\mathcal{O}}_{x_i} \to C_{x_i} \to 0$$

tensoring by $\mathcal{F}_{x_i}$. Since $C_{x_i} = k$ we find that $T_{x_i} = \mathcal{F}_{x_i} \otimes_{\mathcal{O}_{x_i}} k = \mathcal{F}(x)$, that is, the fiber of $\mathcal{F}$ at x_i. But we know $\dim_k \mathcal{F}(x) = 2r - a_i$ (it does not matter if the nodal point lies in one component or in two components because $\mathcal{F}$ is of uniform multirank). From this we get the result. □

Lemma 4.2.4. *For every torsion free sheaf of rank r on X with types $\{a_i\}$ there exists a locally free sheaf of the same rank $\mathcal{G}$ and an injective morphism $\mathcal{F} \hookrightarrow \mathcal{G}$ such that $T := \mathcal{F}/\mathcal{G} = \bigoplus_{\mathrm{Sing}(X)} k(x_i)^{r-a_i}$.*

Proof. From [57], p. 173, there exists a locally free sheaf $\mathcal{G}'$ and an injective morphism $\mathcal{F} \hookrightarrow \mathcal{G}'$ with $Q := \mathcal{G}'/\mathcal{F} = T' \oplus T$ a torsion sheaf, where $\mathrm{supp}(T') \cap \mathrm{Sing}(X) = \varnothing$. Take the projection $Q \to T'$ and let $\mathcal{G}$ be the kernel of the composition $\mathcal{G}' \to Q \to T'$. Clearly $\mathcal{G}$ is torsion free of rank r, hence $\mathcal{G}_x \simeq \mathcal{O}_{X,x}^{\oplus r}$ for all $x \in X \setminus \mathrm{Sing}(X)$. Consider now the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & T \arrow[d] & \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G}' \arrow[r] \arrow[d, equal] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{G}' \arrow[r] \arrow[d] & T' \arrow[r] \arrow[d] & 0 \, . \\
 & T & 0 & 0 &
\end{tikzcd}$$

Then it follows that $\mathcal{G}$ is locally free, since $\mathcal{G}_x \simeq \mathcal{G}'_x$ for all $x \in \mathrm{Sing}(X)$, and is the sheaf we where looking for. □

Lemma 4.2.5. *Let $\mathcal{F}$ be a torsion free sheaf of rank r and types $\{a_i\}$, and denote $T(\mathcal{F})$ the torsion subsheaf of $\pi^*\mathcal{F}$. Then we have*

$$\deg(\pi^*\mathcal{F}) = \deg(\mathcal{F}) + r\nu - \sum a_i,$$
$$\deg(T(F)) = 2(r\nu - \sum a_i).$$

Proof. Follows as in [4, Proposition 2.1.] using Lemma 4.2.4. □

Let us see that for each $\mathcal{F}$ there is a canonical locally free sheaf, $\mathcal{E}_0$, on the normalization such that $\mathcal{F} \hookrightarrow \pi_* \mathcal{E}_0$.

Proposition 4.2.6. *For every torsion free sheaf $\mathcal{F}$ on X there exists an immersion*

$$\beta : \mathcal{F} \hookrightarrow \pi_*(\mathcal{E}_0).$$

being $\mathcal{E}_0 = \pi^*(\mathcal{F})/T$.

Proof. Form (4.5) and Lemma 4.2.2 we get a map by composition

$$\begin{tikzcd} 0 \arrow[r] & \pi_* T \arrow[r, hook] & \pi_*\pi^*(\mathcal{F}) \arrow[r] & \pi_*(\pi^*(\mathcal{F})/T) \arrow[r] & 0 \\ & & \mathcal{F} \arrow[u, hook, "\alpha"] \arrow[ur, dashed, "\beta"] & & \end{tikzcd}.$$

Consider the kernel $\mathrm{Ker}(\beta)$. For each smooth point $x \in X$ we have

$$\begin{tikzcd} (\pi_*\pi^*(\mathcal{F}))_x \arrow[r] & (\pi_*(\pi^*(\mathcal{F})/T))_x \\ (\mathcal{F})_x \arrow[u] \arrow[ur, "\beta_x"] & \end{tikzcd}$$

where the horizontal and vertical arrows are isomorphisms because of Lemma 4.2.1 and the definition of α. Then $\mathrm{Ker}(\beta_x) = \mathrm{Ker}(\beta)_x = 0$ for every smooth point. That means that $\mathrm{Ker}(\beta)$ is a torsion sheaf concentrated on the nodal points but, since $\mathcal{F}$ is torsion free, it must be zero. □

Proposition 4.2.7. *If $\mathcal{F}$ is a rank r torsion free sheaf of types $\{a_i\}$ then the cokernel of the canonical injection β is a torsion sheaf of length*

$$l(\mathrm{Coker}(\beta)) = a := \sum a_i$$

Proof. $\mathrm{Coker}(\beta)$ is a torsion sheaf because $\mathcal{F}$ and $\pi_*(\pi^*\mathcal{F}/T)$ have the same rank.

Consider the exact sequence

$$0 \to \mathcal{F} \hookrightarrow \pi_*(\pi^*(\mathcal{F})/T) \to \mathrm{Coker}(\beta) \to 0. \tag{4.6}$$

Since π is finite $\chi(\pi_*(G)) = \chi(G)$ for each locally free sheaf on the normalization Y. Then, taking the Euler-Poincaré characterisitc in (4.6) we find

$$\chi(\pi^*(\mathcal{F})/T) = \chi(\mathcal{F}) + l(\mathrm{Coker}(\beta))$$

so

$$r(\chi(\mathcal{O}_Y)) + \deg(\pi^*(\mathcal{F})/T) = r(\chi(\mathcal{O}_X)) + \deg(\mathcal{F}) + l(\mathrm{Coker}(\beta)).$$

From the exact sequence

$$0 \to \mathcal{O}_X \to \pi_*(\mathcal{O}_Y) \to \bigoplus_{i=1}^{\nu} k \to 0$$

we get $\chi(\mathcal{O}_Y) - \chi(\mathcal{O}_X) = \nu$. Therefore

$$l(\mathrm{Coker}(\beta)) = r\nu + \deg(\pi^*(\mathcal{F})/T) - \deg(\mathcal{F})$$

and applying Lemma 4.2.5 we get the result. □

Theorem 4.2.8. (Factorization) *Let $\mathcal{F}$ be a torsion free sheaf on X and suppose there exists a locally free sheaf of uniform multirank equal to r, $\mathcal{E}$, on the normalization and an injection $f : \mathcal{F} \hookrightarrow \pi_*(\mathcal{E})$. Then f factorizes through β.*

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, hook, "f"] \arrow[dr, hook, "\beta"'] & & \pi_*(\mathcal{E}) \\
& \pi_*(\mathcal{E}_0) \arrow[ur, hook, dashed] &
\end{tikzcd}$$

in a canonical way.

Proof. Consider the morphism $\mathcal{F} \hookrightarrow \pi_*\mathcal{E}$. Taking the pullback we find $\pi^*\mathcal{F} \to \pi^*\pi_*\mathcal{E}$ and composing with the canonical morphism $\pi^*\pi_*\mathcal{E} \to \mathcal{E}$ we get

$$\pi^*\mathcal{F} \to \pi^*\pi_*\mathcal{E} \to \mathcal{E}$$

Since for every locally free sheaf $\mathcal{G}$ of uniform multirank we have an isomorphism $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{E}_0, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_Y}(\pi^*\mathcal{F}, \mathcal{G})$ given by the composition with the projection $\pi^*\mathcal{F} \xrightarrow{p} \mathcal{E}_0$, the above morphism factorizes

$$\begin{tikzcd}
\pi^*\mathcal{F} \arrow[r] \arrow[dr, "p"'] & \pi^*\pi_*\mathcal{E} \arrow[r] & \mathcal{E} \\
& \mathcal{E}_0 \arrow[ur, "\lambda"'] &
\end{tikzcd}.$$

Taking the pushforward we get

$$\pi_*\pi^*\mathcal{F} \to \pi_*\mathcal{E}_0 \xrightarrow{\pi_*(\lambda)} \pi_*\mathcal{E}$$

and composing with the canonical morphism $\mathcal{F} \to \pi_*\pi^*\mathcal{F}$

$$\mathcal{F} \to \pi_*\pi^*\mathcal{F} \to \pi_*\mathcal{E}_0 \xrightarrow{\pi_*(\lambda)} \pi_*\mathcal{E}$$

which is the original morphism. $\square$

Corollary 4.2.9. *Let $\mathcal{F}$ be a torsion free sheaf of rank r and types $\{a_i\}$ on X. Suppose there exists a locally free sheaf $\mathcal{E}$ on Y with the same rank and an injection $i : \mathcal{F} \hookrightarrow \pi_*\mathcal{E}$. Then* $\mathrm{length}(\mathrm{Coker}(i)) = l$ *if and only if* $\mathrm{length}(\mathrm{Coker}(\pi_*(\lambda))) = l - a$ *being $a = \sum a_i$.*

Proof. Because of Theorem 4.2.8 we can construct the following diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F} \arrow[r, hook] \arrow[d, equal] & \pi_*\mathcal{E}_0 \arrow[r] \arrow[d, hook, "\pi_*(\lambda)"] & Q \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r, hook, "i"] \arrow[d] & \pi_*\mathcal{E} \arrow[r] \arrow[d] & Q' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

so $P \simeq P'$. Hence, we deduce that $\text{length}(P) = \text{length}(P') = \text{length}(Q') - \text{length}(Q)$. Since $\text{length}(Q') = l$ by hypothesis and $\text{length}(Q) = a$, we can conclude using Lemma 4.2.7. □

Theorem 4.2.10. *Let $\mathcal{F}$ be a torsion free sheaf on X of rank r and types $\{a_i\}$. There exists a locally free sheaf $\mathcal{E}$ on Y of rank r and an exact sequence*

$$0 \to \mathcal{F} \hookrightarrow \pi_* \mathcal{E} \to Q' \to 0$$

where Q is a torsion sheaf supported on the nodal points and of length $r\nu$.

Proof. For each nodal point x_i fix a point in the preimage $y^i_j \in \pi^{-1}(x_i)$. Denote by H the resulting divisor $\sum_i y^i_j$. For each integer $m > 0$ denote by $\mathcal{E}_0(m)$ the locally free sheaf $\mathcal{E}_0 \otimes \mathcal{O}_Y(mH)$. Then we have an exact sequence

$$0 \to \mathcal{E}_0 \hookrightarrow \mathcal{E}_0(m) \to P \to 0$$

where P is a torsion sheaf supported on H. For each point $y^i_j \in \text{Supp}(P)$ we have $\text{length}(P_{y^i_j}) = rm$. Therefore,

$$\text{length}(P) = r\nu m.$$

Since $r - a_i \leq rm$ for each $m > 0$ we can fix a vector subspace $V_{y^i_j} \subset P_{y^i_j}$ of dimension $r - a_i$. Denote by V the associated torsion sub-sheaf (supported on H), $V \hookrightarrow P$. Let Q the cokernel of the above injection. Q is a torsion sheaf supported on H and $\text{length}(Q) = r\nu m - (r\nu - a)$. Consider the composition

$$\mathcal{E}_0(m) \to P \to Q \to 0,$$

and denote by $\mathcal{E}$ the corresponding kernel

$$0 \to \mathcal{E} \hookrightarrow \mathcal{E}_0(m) \to Q \to 0.$$

Since the composition

$$\mathcal{E}_0 \hookrightarrow \mathcal{E}_0(m) \to P \to Q \to 0.$$

is the zero morphism there exists an injection $\mathcal{E}_0 \hookrightarrow \mathcal{E}$ making the diagram commutative

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E} \arrow[r, hook] & \mathcal{E}_0(m) \arrow[r] & Q \arrow[r] & 0 \\
 & & \mathcal{E}_0 \arrow[ul, hook] \arrow[u, hook] & &
\end{tikzcd}$$

.

Denote by T the cokernel of the injection $\mathcal{E}_0 \hookrightarrow \mathcal{E}$ and consider the following diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & V \arrow[d] & \\
0 \arrow[r] & \mathcal{E}_0 \arrow[r, hook] \arrow[d, hook] & \mathcal{E}_0(m) \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r, hook] \arrow[d] & \mathcal{E}_0(m) \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
 & T \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd}$$

Then by the snake lemma we deduce that $T \simeq V$. Take the push-forward of the first (column) exact sequence

$$0 \to \pi_*\mathcal{E}_0 \hookrightarrow \pi_*\mathcal{E} \to \pi_*V \to 0.$$

Now π_*V is a torsion sheaf supported on the nodes and $\mathrm{length}(\pi_*V) = \mathrm{length}(V) = r\nu - a$. Consider the composition

$$\mathcal{F} \hookrightarrow \pi_*\mathcal{E}_0 \hookrightarrow \pi_*\mathcal{E}.$$

Observe that the length of the cokernel of $\pi_*\mathcal{E}_0 \hookrightarrow \pi_*\mathcal{E}$ has length $r\nu - a$ by construction. Then, by Corollary 4.2.9, we deduce that the cokernel of the injection $\mathcal{F} \hookrightarrow \pi_*\mathcal{E}$ has length $r\nu$. □

Remark 4.2.11. Regarding the above construction, we can show that it does not depend on the chosen natural number m and it depends only on the divisor H. Fix the divisor H. For the sake of notation let us say $H = \sum_{i=1}^{\nu} z_i$. Fix two natural numbers $m' > m$. The above construction provide us with two exact sequences:

$$0 \to \mathcal{E} \hookrightarrow \mathcal{E}_0(m) \to Q \to 0,$$
$$0 \to \mathcal{E}' \hookrightarrow \mathcal{E}_0(m') \to Q' \to 0.$$

Note that for any point z_i we have $\mathrm{length}(Q_{z_i}) = rm - r + a_i < rm' - r + a_i = \mathrm{length}(Q'_{z_i})$ and that there is an injective morphism $P \hookrightarrow P'$ making the following diagrams commutative

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}_0 \arrow[r] \arrow[d, equal] & \mathcal{E}_0(m) \arrow[r] \arrow[d, hook] & P \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{E}_0 \arrow[r] & \mathcal{E}_0(m') \arrow[r] & P' \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, hook] & Q \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & V \arrow[r] & P' \arrow[r] & Q' \arrow[r] & 0
\end{tikzcd}$$

from what we find a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d, hook] & \mathcal{E}_0(m) \arrow[r] \arrow[d, hook] & Q \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{E}' \arrow[r] \arrow[d] & \mathcal{E}_0(m') \arrow[r] \arrow[d] & Q' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{coker}_1 \arrow[r] & \mathrm{coker}_2 \arrow[r] & \mathrm{coker}_3 \arrow[r] & 0
\end{tikzcd} .$$

From the first two commutative diagrams we deduce that $\mathrm{length}(\mathrm{coker}_2) = \mathrm{length}(\mathrm{coker}_3)$ so $\mathrm{coker}_1 = 0$ and therefore $\mathcal{E} \simeq \mathcal{E}'$.

Corollary 4.2.12. *Let $\mathcal{F}$ be a torsion free sheaf of rank r on X. The surjective morphism p of an exact sequence*

$$0 \to \mathcal{F} \hookrightarrow \pi_*\mathcal{E} \xrightarrow{p} Q' \to 0$$

as in Theorem 4.2.10 always factorizes through the canonical morphism

$$\pi_*\mathcal{E} \to \pi_*(\mathcal{E}|_D) \to 0$$

i.e. there exists a surjection $\pi_*(\mathcal{E}|_D) \to Q' \to 0$ such that the triangle

$$\begin{tikzcd}
\pi_*\mathcal{E} \arrow[rr] \arrow[dr, "p"'] & & \pi_*(\mathcal{E}|_D) \arrow[dl] \\
& Q' &
\end{tikzcd}$$

is commutative.

Proof. Consider the composition morphism

$$\mathcal{F} \hookrightarrow \pi_*\mathcal{E} \to \pi_*(\mathcal{E}|_D)$$

and denote $\mathcal{F}(\mathcal{E})$ the image (which is a torsion sub-sheaf of $\pi_*(\mathcal{E}|_D)$). Let $\mathcal{F}'$ be the kernel of the surjection $\pi_*\mathcal{E} \to \pi_*(\mathcal{E}|_D)/\mathcal{F}(\mathcal{E}) \to 0$. Since the composition

$$\mathcal{F} \to \pi_*\mathcal{E} \to \pi_*(\mathcal{E}|_D)/\mathcal{F}(\mathcal{E}) \to 0$$

is the zero morphism there exists an injective morphism $\mathcal{F} \hookrightarrow \mathcal{F}'$ and therefore a diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & T \arrow[d] & \\
0 \arrow[r] & \mathcal{F} \arrow[r, hook] \arrow[d, hook] & \pi_*\mathcal{E} \arrow[r] \arrow[d, equal] & Q' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r, hook] \arrow[d] & \pi_*\mathcal{E} \arrow[r] \arrow[d] & \pi_*(\mathcal{E}|_D)/\mathcal{F}(\mathcal{E}) \arrow[r] \arrow[d] & 0 \\
& T' & 0 & 0 &
\end{tikzcd}$$

so that $T \simeq T'$. Let us see that $T = T' = 0$. For that it is enough to see that $\mathrm{length}(\pi_*(\mathcal{E}|_D)/\mathcal{F}(\mathcal{E})) = \mathrm{length}(Q') = r\nu$.

Restricting the exact sequence

$$0 \to \mathcal{F} \hookrightarrow \pi_*\mathcal{E} \xrightarrow{p} Q' \to 0 \tag{4.7}$$

to the nodal point x_i we obtain the long exact sequence

$$\begin{tikzcd} {} \arrow[r] & \mathrm{Tor}^1_{\mathcal{O}_X}(Q', k(x_i)) \arrow[r] & \mathcal{F} \otimes_{\mathcal{O}_X} k(x_i) \arrow[r, "u"] \arrow[d, equal] & \pi_*\mathcal{E} \otimes_{\mathcal{O}_X} k(x_i) \arrow[r, "p(x_i)"] \arrow[d, equal] & Q'_{x_i} \arrow[r] & 0 . \\ & & \mathcal{F}(x_i) \arrow[r, "u"] & (\mathcal{E}(y_1) \oplus \mathcal{E}(y_2)) & & \end{tikzcd}$$

Now, it is clear that $\mathrm{Im}(u) = \mathcal{F}(\mathcal{E})_{x_i}$ so

$$\begin{aligned} \dim(\mathcal{F}(\mathcal{E})_{x_i}) &= \dim\mathrm{Ker}(p(x_i)) = \\ &= \dim((\mathcal{E}(y_1^i) \oplus \mathcal{E}(y_2^i))) - \dim(Q'_{x_i}) = \\ &= 2r - r = r. \end{aligned}$$

Then

$$\begin{aligned} &\mathrm{length}(\pi_*(\mathcal{E}(y_1^i) \oplus \mathcal{E}(y_2^i))/\mathcal{F}(\mathcal{E})_{x_i}) = \\ &\dim((\mathcal{E}(y_1^i) \oplus \mathcal{E}(y_2^i))) - \dim(\mathcal{F}(\mathcal{E})_{x_i}) = \\ &2r - r = r = \mathrm{length}(Q'_{x_i}), \end{aligned}$$

so $l(Q') = r\nu$, and thus we have an isomorphism

$$\phi : Q' \simeq \pi_*(\mathcal{E}|_D)/\mathcal{F}(\mathcal{E}).$$

Then we obtain the desired factorization

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r, hook] & \pi_*\mathcal{E} \arrow[r] \arrow[d] & Q' \arrow[r] & 0 \\ & & \pi_*(\mathcal{E}|_D) \arrow[r] & \pi_*(\mathcal{E}|_D)/\mathcal{F}(\mathcal{E}) \arrow[r] \arrow[u, "\phi^{-1}"] & 0. \end{tikzcd}$$

□

We can summarize the above results in the following theorem, which were proven in [6, Theorem 3] for the irreducible case, and more generally in [7, Proposition 3.9].

Theorem 4.2.13. *Let $\mathcal{F}$ be a torsion free sheaf of rank r on the nodal curve X. Then there exists a locally free sheaf $\mathcal{E}$ of rank r on the normalization Y and a surjection $\mathcal{E}|_D \to Q$ of dimension $r\nu$ such that $\mathcal{F}$ is equal to the kernel of the morphism*

$$\pi_*\mathcal{E} \to \pi_*(\mathcal{E}|_D) \to Q \to 0 \tag{4.8}$$

Q being a skycraper sheaf supported on the nodes with fibre Q.

Proof. Follows trivially from Theorem 4.2.10 and Corollary 4.2.12. □

4.3 Construction of Honest Singular Principal G-bundles from Parabolic Structures

Recall that a singular principal G-bundle over the nodal curve X is a pair $(\mathcal{F}, \tau)$, τ being a non-trivial morphism of algebras $S^\bullet(V \otimes \mathcal{F})^G \to \mathcal{O}_X)$. Giving $\tau : S^\bullet(V \otimes \mathcal{F})^G \to \mathcal{O}_X$ is the same as giving a morphism $\tau : X \to \underline{\mathrm{Hom}}_X(V \otimes \mathcal{O}_X, \mathcal{F}^\vee) /\!\!/ G$. We say that the pair $(\mathcal{F}, \tau)$ is quasi-honest if the image of $\tau|_U$ is contained in the open subscheme $\underline{\mathrm{Isom}}_X(\mathcal{O}_X|_U \otimes V, \mathcal{F}|_U^\vee)/G$, $U \subset X$ being some open (not necessary dense) subset. If $U = X \setminus \mathrm{Sing}(X)$ then we say that it is honest. We denote by W the subset $Y - \pi^{-1}(\mathrm{Sing}(X))$. Then we have the diagram

$$\begin{tikzcd} W \arrow[r, hook, "j"] \arrow[d, "\pi", "\wr"'] & Y \arrow[d, "\pi"] \\ U \arrow[r, hook, "i"] & X \end{tikzcd}$$

Recall the following lemma

Lemma 4.3.1. ([49, Corollary 3.4]) *Let $(\mathcal{E}, \tau : Y \to \underline{\mathrm{Hom}}_Y(V \otimes \mathcal{O}_Y, \mathcal{E}^\vee))$ be a singular principal G-bundle. Then for all $i = 1, \dots, l$, $\tau(Y_i)$ is either contained in $\underline{\mathrm{Isom}}_{Y_i}(V \otimes \mathcal{O}_{Y_i}, \mathcal{E}|_{Y_i}^\vee)$ or in the complement.*

We start with the following

Lemma 4.3.2. *Let $\mathcal{F}$ be a torsion free sheaf of rank r on the nodal curve X. Then every morphism of $\mathcal{O}_Y$-algebras $\tau_0 : S^\bullet(V \otimes \mathcal{E}_0)^G \to \mathcal{O}_Y$ descends to a morphism of $\mathcal{O}_X$-algebras $\tau : S^\bullet(V \otimes \mathcal{F})^G \to \pi_* \mathcal{O}_Y$ via the projection $\pi^* \mathcal{F} \to \mathcal{E}_0 \to 0$.*

Proof. Consider a morphism of $\mathcal{O}_Y$-algebras $S^\bullet(V \otimes \mathcal{E}_0)^G \to \mathcal{O}_Y$. From the projection $\pi^* \mathcal{F} \xrightarrow{j} \mathcal{E}_0 \to 0$ we get a surjective morphism of $\mathcal{O}_Y$-algebras

$$S^\bullet(V \otimes \pi^* \mathcal{F})^G \xrightarrow{j} S^\bullet(V \otimes \mathcal{E}_0)^G \to 0$$

and we get an injective map

$$\mathrm{Hom}_{\mathcal{O}_Y\text{-alg}}(S^\bullet(V \otimes \mathcal{E}_0), \mathcal{O}_Y) \overset{\circ j}{\hookrightarrow} \mathrm{Hom}_{\mathcal{O}_Y\text{-alg}}(S^\bullet(V \otimes \pi^* \mathcal{F}), \mathcal{O}_Y)$$

so from τ_0 we get a morphism $\tau' := \tau_0 \circ j : S^\bullet(V \otimes \pi^* \mathcal{F})^G \to \mathcal{O}_Y$.

Since $\pi^*(-)$ commutes with $(-)^G$ (see [49]) and the symmetric algebra commutes with base change ([9, Chapter 3, §6, Proposition 7]), we have $S^\bullet(V \otimes \pi^* \mathcal{F})^G = \pi^* S^\bullet(V \otimes \mathcal{F})^G$ so τ' gives a morphism of $\mathcal{O}_Y$-algebras, $\tau' : \pi^* S^\bullet(V \otimes \mathcal{F})^G \to \mathcal{O}_Y$, and then, by adjunction, a morphism of $\mathcal{O}_X$-algebras

$$\tau : S^\bullet(V \otimes \mathcal{F})^G \to \pi_* \mathcal{O}_Y.$$

□

The following lemma is proved in [51], in case X is irreducible,

Lemma 4.3.3. *Let $(\mathcal{F}, \tau)$ be a singular principal G-bundle on the nodal curve X. Then there exists a morphism of $\mathcal{O}_Y$-algebras $\tau_0 : S^\bullet(V \otimes \mathcal{E}_0) \to \mathcal{O}_Y$ which descends to $\tau : S^\bullet(V \otimes \mathcal{F}) \to \mathcal{O}_X$ via the projection $\pi^* \mathcal{F} \to \mathcal{E}_0 \to 0$.*

Proof. Consider the singular principal G-bundle $(\mathcal{F}, \tau)$ and the exact sequence

$$0 \longrightarrow T \longrightarrow \pi^*(\mathcal{F}) \longrightarrow \mathcal{E}_0 = \pi^*\mathcal{F}/T \longrightarrow 0\ .$$

Since $S^\bullet(V \otimes \pi^*\mathcal{F})^G \to S^\bullet(V \otimes \mathcal{E}_0)^G \to 0$ is still surjective we find a closed immersion

$$\underline{\mathrm{Spec}}(S^\bullet(V \otimes \mathcal{E}_0)^G) \hookrightarrow \underline{\mathrm{Spec}}(S^\bullet(V \otimes \pi^*\mathcal{F})^G)\ .$$

We have the following diagram

$$\begin{tikzcd}
\underline{\mathrm{Spec}}(S^\bullet(V \otimes \mathcal{E}_0)^G) \arrow[r, hook] & \underline{\mathrm{Spec}}(S^\bullet(V \otimes \pi^*\mathcal{F})^G) \arrow[r] \arrow[d] & \underline{\mathrm{Spec}}(S^\bullet(V \otimes \mathcal{F})^G) \arrow[d] \\
& Y \arrow[r, "\pi"] \arrow[u, dashed, bend left, "\pi^*(\tau)"] & X \arrow[u, dashed, bend left, "\tau"]
\end{tikzcd}\ ,$$

$\pi^*(\tau)$ being the induced morphism of algebras

$$\pi^*(\tau) : \pi^*(S^\bullet(V \otimes \mathcal{F})^G) = S^\bullet(V \otimes \pi^*\mathcal{F})^G \to \pi^*\mathcal{O}_X = \mathcal{O}_Y.$$

This morphism is also the one that we obtain by adjuntion when we take the composition of $S^\bullet(V \otimes \mathcal{F})^G \to \mathcal{O}_X$ with the natural inclusion of rings $\mathcal{O}_X \subset \pi_*\mathcal{O}_Y$.

Let us denote W the open subset $Y - \pi^{-1}(\mathrm{Sing}(X))$. Restricting the exact sequence

$$0 \longrightarrow T \longrightarrow \pi^*(\mathcal{F}) \longrightarrow \mathcal{E}_0 = \pi^*\mathcal{F}/T \longrightarrow 0$$

to this open subset we get

$$\pi^*\mathcal{F}|_W = \mathcal{E}_0|_W$$

so $\underline{\mathrm{Spec}}(S^\bullet(V \otimes \mathcal{E}_0|_W)^G) = \underline{\mathrm{Spec}}(S^\bullet(V \otimes \pi^*\mathcal{F}|_W)^G)$ which means that the restriction $\pi^*(\tau|_W)$ takes values in $\underline{\mathrm{Spec}}(S^\bullet(V \otimes \mathcal{E}_0|_W))$. The chain of immersions

$$\underline{\mathrm{Spec}}(S^\bullet(V \otimes \mathcal{E}_0|_V)^G) \hookrightarrow \underline{\mathrm{Spec}}(S^\bullet(V \otimes \mathcal{E}_0)^G) \overset{\text{closed}}{\hookrightarrow} \underline{\mathrm{Spec}}(S^\bullet(V \otimes \pi^*\mathcal{F})^G)$$

implies that $\pi^*(\tau)$ must then take values in $\underline{\mathrm{Spec}} S^\bullet(V \otimes \mathcal{E}_0)^G$, that is, the morphism $S^\bullet(V \otimes \pi^*\mathcal{F})^G \to \mathcal{O}_Y$ factorizes through the surjection

$$S^\bullet(V \otimes \pi^*\mathcal{F})^G \to S^\bullet(V \otimes \mathcal{E}_0)^G \to 0$$

and we denote with the same symbol the induced morphism, $\pi^*(\tau) : S^\bullet(V \otimes \mathcal{E}_0)^G \to \mathcal{O}_Y$. Since the procedure used here is precisely the inverse of the procedure used in Lemma 4.3.2 we find that the morphism of $\mathcal{O}_X$-algebras that corresponds to $\pi^*(\tau)$ is precisely τ (in particular the resulting morphism takes values in $\mathcal{O}_X \subset \pi_*\mathcal{O}_Y$). $\square$

Proposition 4.3.4. *The normalization map $\pi : Y \to X$ induces isomorphisms:*

a) $\underline{\mathrm{Isom}}_U(V \otimes \mathcal{O}_X|_U, \mathcal{F}|_U^\vee)/G \simeq \underline{\mathrm{Isom}}_W(V \otimes \mathcal{O}_Y|_W, \pi^*\mathcal{F}|_W^\vee)/G.$

b) $\underline{\mathrm{Isom}}_W(V \otimes \mathcal{O}_Y|_W, \pi^*\mathcal{F}|_W^\vee)/G \simeq \underline{\mathrm{Isom}}_W(V \otimes \mathcal{O}_Y|_W, \mathcal{E}_0|_W^\vee)/G.$

Proof. a) Consider the diagram

$$\begin{tikzcd}
\underline{\mathrm{Spec}} S^\bullet(V \otimes \mathcal{F})^G \times_X Y \arrow[r, equal] & \underline{\mathrm{Spec}} S^\bullet(V \otimes \pi^*\mathcal{F})^G \arrow[r] \arrow[d, "p_2"] & \underline{\mathrm{Spec}} S^\bullet(V \otimes \mathcal{F})^G \arrow[d, "p_1"] \\
& Y \arrow[r, "\pi"] & X \\
& W := Y - \pi^{-1}(\mathrm{Sing}(X)) \arrow[u, hook] \arrow[r, "\sim", "\pi|_W"'] & X - \mathrm{Sing}(X) := U \arrow[u, hook] .
\end{tikzcd}$$

It is clear that

$$\begin{aligned}
p_1^{-1}(U) &= \underline{\mathrm{Spec}} S^\bullet(V \otimes \mathcal{F}|_U)^G = \underline{\mathrm{Hom}}_U(V \otimes \mathcal{O}_U, \mathcal{F}|_U^\vee) /\!\!/ G \\
p_2^{-1}(W) &= \underline{\mathrm{Spec}} S^\bullet(V \otimes \pi^*\mathcal{F}|_W)^G = \underline{\mathrm{Hom}}_W(V \otimes \mathcal{O}_W, \pi^*\mathcal{F}|_W^\vee) /\!\!/ G
\end{aligned}$$

(the second equatility follows in both cases because $\mathcal{F}|_U^\vee$ and $\pi^*\mathcal{F}|_W^\vee$ are locally free). Since $\pi|_W : W \simeq U$ we have $(\pi|_W)^{-1} p_1^{-1}(U) = p_2^{-1}(W)$. Now the result follows again because $\pi|_W$ is an isomorphism.

b) It is also clear since $\pi^*\mathcal{F}|_W^\vee \simeq \mathcal{E}_0|_W^\vee$. □

Proposition 4.3.5. *For every singular principal G-bundle $(\mathcal{F}, \tau)$ on the nodal curve X there exists a morphism of $\mathcal{O}_Y$-algebras $\tau_0 : S^\bullet(V \otimes \mathcal{E}_0) \to \mathcal{O}_Y$ which descends to $\tau : S^\bullet(V \otimes \mathcal{F}) \to \mathcal{O}_X$ via the projection $\pi^*\mathcal{F} \to \mathcal{E}_0 \to 0$ and such that $(\mathcal{E}_0, \tau_0)$ is a quasi-honest singular principal G-bundle on Y if $(\mathcal{F}, \tau)$ is.*

Proof. From Proposition 4.3.4 we know that if $\tau(X) \subset \underline{\mathrm{Isom}}_X(V \otimes \mathcal{O}_X, \mathcal{F}^\vee)/G$ then $\pi^*(\tau)(W)$ is contained in $\underline{\mathrm{Isom}}_W(V \otimes \mathcal{O}_Y|_W, \mathcal{E}_0|_W^\vee)/G$ and by Lemma 4.3.1 we deduce that $\pi^*(\tau)(Y)$ is contained in $\underline{\mathrm{Isom}}_Y(V \otimes \mathcal{O}_Y, \mathcal{E}_0^\vee)/G$ so $(\mathcal{E}_0, \pi^*(\tau))$ is honest. The proposition now follows trivially from the above results and Lemma 4.3.3. □

Remark 4.3.6. Let $(\mathcal{E}, \underline{q}, \tau')$ be a descending principal G-bundle, and consider its direct image on the nodal curve, $(\mathcal{F}, \tau) := \pi_*(\mathcal{E}, \underline{q}, \tau')$. We can construct the diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(V \otimes \mathcal{O}_Y, \mathcal{E}^\vee) /\!\!/ G \arrow[r, equal] & \underline{\mathrm{Spec}}(S^\bullet(V \otimes \mathcal{E}))^G \arrow[d, shift left] & \underline{\mathrm{Spec}}(S^\bullet(V \otimes \mathcal{F}))^G \arrow[r, equal] \arrow[d, shift right] & \underline{\mathrm{Hom}}_{\mathcal{O}_X}(V \otimes \mathcal{O}_X, \mathcal{F}^\vee) /\!\!/ G \\
& Y \arrow[u, bend left, "\tau'"] \arrow[r, "\pi"] & X \arrow[u, bend right, "\tau"'] & .
\end{tikzcd}$$

Let $U = X - \mathrm{Sing}(X)$ and $V = Y - \pi^{-1}(\mathrm{Sing}(X))$. Recall that π induces an isomorphism $V \simeq U$. Moreover, if we restrict the above diagram to V and U we get an isomorphism $\underline{\mathrm{Spec}}(S^\bullet(V \otimes \mathcal{E}|_V))^G \simeq \underline{\mathrm{Spec}}(S^\bullet(V \otimes \mathcal{F}|_U))^G$ from which it follows that $(\mathcal{F}, \tau)$ is honest.

4.4 Semistable Singular *G*-Bundles on Nodal Curves

We present now the main result of this work, Theorem 4.4.8. Results given in Section 4.1, Section 4.2 and Section 4.3 permit the construction of a morphism

$$\mathcal{M}(G)^{(\underline{\kappa},\delta)\text{-(s)s}} \to \mathrm{SPB}(\rho)_P^{\delta\text{-}(s)s}$$

between the space of descending G-bundles on Y and the space of singular principal G-bundles on X.

4.4.1 Construction of the Moduli Space of Descending G-Bundles

Let $\mathfrak{M}(G)$ be the parameter space for singular principal G-bundles with generalized parabolic structure (see Theorem 3.2.5). Recall now that $\mathfrak{M}(G)$ carries a universal family $(\mathcal{E}_{\mathfrak{M}(G)}, \underline{q}_{\mathfrak{M}(G)}, \tau_{\mathfrak{M}(G)})$, being

$$\begin{aligned} &q^i_{\mathfrak{M}(G)} : \pi^i_{\mathfrak{M}(G)*}(\mathcal{E}_{\mathfrak{M}(G)}|_{y_1^i, y_2^i}) \to R_{\mathfrak{M}(G)}, \ i = 1, \dots, \nu, \\ &\tau_{\mathfrak{M}(G)} : S^{\bullet}(V \otimes \mathcal{E}_{\mathfrak{M}(G)})^G \to \mathcal{O}_{\mathfrak{M}(G)\times Y}. \end{aligned} \tag{4.9}$$

Let $\mathcal{F}_{\mathfrak{M}(G)}$ be the induced family of torsion free sheaves on $\mathfrak{M}(G) \times X$. Recall that we have an induced morphism

$$\tau'_{\mathfrak{M}(G)} : S^{\bullet}(\mathcal{F}_{\mathfrak{M}(G)} \otimes V)^G \to (\mathrm{id}_{\mathfrak{M}(G)} \times \pi)_* \mathcal{O}_{\mathfrak{M}(G)\times Y}.$$

The natural inclusion $\mathcal{O}_{\mathfrak{M}(G)\times X} \hookrightarrow (id_{\mathfrak{M}(G)} \times \pi)_* \mathcal{O}_{\mathfrak{M}(G)\times Y}$ induces a quotient and composing with $\tau'_{\mathfrak{M}(G)}$ we get a morphism of $\mathcal{O}_{\mathfrak{M}(G)\times X}$-algebras

$$S^{\bullet}(\mathcal{F}_{\mathfrak{M}(G)} \otimes V)^G \to ((\mathrm{id}_{\mathfrak{M}(G)} \times \pi)_* \mathcal{O}_{\mathfrak{M}(G)\times Y})/\mathcal{O}_{\mathfrak{M}(G)\times X} \simeq \pi_X^*((\pi_* \mathcal{O}_Y)/\mathcal{O}_X)$$

Let us define $\mathfrak{D}(G) \subset \mathfrak{M}(G)$ as the closed subscheme where the above morphism vanishes (see Lemma 2.1.35). Denote by

$$(\mathcal{E}_{\mathfrak{D}(G)}, \underline{q}_{\mathfrak{D}(G)}, \tau_{\mathfrak{D}(G)})$$

the restriction of the universal family to this subscheme. Clearly, $\mathfrak{D}(G)$ represents the moduli functor ${}^{\mathrm{rig}}\mathbf{D}(G)$. Note that there is a natural $\mathrm{GL}(W)$-action on the space $\mathfrak{D}(G)$,

$$\Gamma : \mathrm{GL}(W) \times \mathfrak{D}(G) \to \mathfrak{D}(G).$$

We can view the $\mathrm{GL}(W)$-action as a $(\mathbb{C}^* \times \mathrm{SL}(W))$-action. Thus, we will construct the quotient of $\mathfrak{D}(G)$ by $\mathrm{GL}(W)$ in two steps, considering the actions of $\mathbb{C}^*$ and $\mathrm{SL}(W)$ separately. Consider the action of $\mathbb{C}^*$ on $\mathfrak{D}(G)$, such that the closed immersion

$$\mathfrak{D}(G) \hookrightarrow \mathfrak{M}(G)$$

is $\mathbb{C}^*$-equivariant. Moreover, the morphism induced between the quotients is a $\mathrm{SL}(W)$-equivariant injective (since both are inside $\coprod \mathfrak{I}_{r,d,\delta}$, see Equation 3.30) and proper morphism

$$\beta : \mathfrak{D}(G) /\!\!/ \mathbb{C}^* \hookrightarrow \mathfrak{M}(G) /\!\!/ \mathbb{C}^*.$$

Then we have,

Proposition 4.4.1. *Let S be a scheme of finite type over $\mathbb{C}$ and $(\mathcal{E}_S, \underline{q}_S, \tau_S)$ a family of $(\underline{\kappa}, \delta)$-(semi)stable singular principal G-bundles with generalized parabolic structure parametrized by S. Then there exists an open covering S_i, $i \in I$, of S and morphisms $\beta_i : S_i \to \mathfrak{D}(G)$, $i \in I$, such that the restriction of the family $(\mathcal{E}_S, \underline{q}_S, \tau_S)$ to $S_i \times Y$ is equivalent to the pullback of $(\mathcal{E}_{\mathfrak{D}(G)}, \underline{q}_{\mathfrak{D}(G)}, \tau_{\mathfrak{D}(G)})$ via $\beta_i \times \mathrm{id}_Y$ for all $i \in I$*

Proof. See [52] Proposition 6.1.2 and [50] Proposition 2.8 . □

Proposition 4.4.2. (Glueing Property) *Let S be a scheme of finite type over $\mathbb{C}$ and $s_1, s_2 : S \to \mathfrak{D}(G)$ two morphisms such that the pullbacks of $(\mathcal{E}_{\mathfrak{D}(G)}, \underline{q}_{\mathfrak{D}(G)}, \tau_{\mathfrak{D}(G)})$ via $s_1 \times \mathrm{id}_X$ and $s_2 \times \mathrm{id}_X$ are isomorphic. Then there exists an étalé covering $c : T \to S$ and a morphism $g : T \to \mathrm{SL}(W)$ such the triangle*

$$\begin{tikzcd}
\mathrm{SL}(W) \times \mathfrak{D}(G) \arrow[rr, "\Gamma"] & & \mathfrak{D}(G) \\
& T \arrow[ul, "g\times(s_1\circ c)"] \arrow[ur, "s_2\circ c"'] &
\end{tikzcd}$$

is commutative.

Proof. Follows using the standard arguments given in Proposition 2.2.15. □

Consider the linearized invertible sheaf $\mathcal{L}'$ given in the proof of Theorem 3.2.8 and let $\mathcal{L}'' := \beta^*\mathcal{L}'$. We finally have

Theorem 4.4.3. *There is a projective scheme $\mathcal{M}(\rho)_P^{(\underline{\kappa},\delta)\text{-}ss}$ and an open subscheme $\mathcal{M}(\rho)_P^{(\underline{\kappa},\delta)\text{-}s} \subset \mathcal{M}(\rho)_P^{(\underline{\kappa},\delta)\text{-}ss}$ together with a natural tranformation*

$$\alpha^{(s)s} : \mathbf{D}(\rho)^{(\underline{\kappa},\delta)\text{-}(s)s} \to h_{\mathcal{M}(\rho)^{(\underline{\kappa},\delta)\text{-}(s)s}}$$

with the following properties:

1) For every scheme $\mathcal{N}$ and every natural transformation $\alpha' : \mathbf{D}(\rho)^{(\underline{\kappa},\delta)\text{-}(s)s} \to h_{\mathcal{N}}$, there exists a unique morphism $\varphi : \mathcal{M}(\rho)_P^{(\underline{\kappa},\delta)\text{-}(s)s} \to \mathcal{N}$ with $\alpha' = h(\varphi) \circ \alpha^{(s)s}$.

2) The scheme $\mathcal{M}(\rho)_P^{(\underline{\kappa},\delta)\text{-}s}$ is a coarse moduli space for the functor $\mathbf{D}(\rho)^{(\underline{\kappa},\delta)\text{-}s}$

Proof. Follows by the same standard argument given in Theorem 2.1.44, with $\mathcal{M}(\rho)^{(\underline{\kappa},\delta)\text{-(s)s}} = \mathfrak{D}(\rho)^{(\underline{\kappa},\delta)-\text{(s)s}} /\!\!/ (\mathbb{C}^* \times \mathrm{SL}(V))$. □

4.4.2 Compactification of the Space of Principal G-Bundles on the Nodal Curve

Let $(\mathcal{E}, \underline{q}, \tau)$ be a descending G-bundle and $(\mathcal{F}, \tau')$ the induced singular principal G-bundle. Recall that both sheaves, $\mathcal{E}$ and $\mathcal{F}$, are related through the following exact sequence (see Equation (4.2))

$$0 \to \mathcal{F} \to \pi_*(\mathcal{E}) \xrightarrow{p} R \to 0$$

where the morphism p factorizes over the surjection $q : \pi_*(\mathcal{E}|_D) \to R$. For any subsheaf $\mathcal{G} \subset \mathcal{E}$, the image of p restricted to $\pi_*(\mathcal{G}) \subset \pi_*(\mathcal{E})$ is precisely $\bigoplus_{i=1}^{\nu} q_i(\mathcal{G}(y_1^i) \oplus \mathcal{G}(y_2^i))$. Therefore we can construct the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, hook] & \pi_*(\mathcal{E}) \arrow[r] & R \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ker}(p') \arrow[r, hook] \arrow[u, hook, dashed] & \pi_*(\mathcal{G}) \arrow[r] \arrow[u, hook] & \bigoplus_{i=1}^{\nu} q_i(\mathcal{G}(y_1^i) \oplus \mathcal{G}(y_2^i)) \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

and we define $S(\mathcal{G}) := \mathrm{Ker}(p')$. If $\mathcal{G}$ is saturated then $S(\mathcal{G})$ is clearly saturated. This construction allows us to attach to any weighted filtration $(\mathcal{E}_\bullet, \underline{m})$ of $\mathcal{E}$ by saturated sheaves a weighted filtration $(S(\mathcal{E}_\bullet), \underline{m})$ of $\mathcal{F}$ by saturated sheaves. Moreover, any saturated subsheaf can by constructed from a saturated subsheaf of $\mathcal{E}$ as follows. Let $\mathcal{F}' \subset \mathcal{F}$ be a saturated subsheaf and consider the composition

$$\mathcal{F}' \hookrightarrow \mathcal{F} \hookrightarrow \pi_*\mathcal{E}$$

Denote $\mathcal{E}' = \pi^*\mathcal{F}'/T(\mathcal{F}')$. Then we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, hook] & \pi_*(\mathcal{E}) \arrow[r, "p"] & R \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r, hook] \arrow[u, hook] & \pi_*(\mathcal{E}') \arrow[r, "p'"] \arrow[u] & R' \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

Since R and R' are torsion sheaves and $\pi_*(\mathcal{E}')$ is torsion free we deduce that the arrow in the middle is injective. Since $\mathcal{F}'$ is a saturated subsheaf of $\mathcal{F}$ and R' is a torsion sheaf we also deduce that the arrow in the right is injective as well. Thus, $\mathcal{F}' = S(\mathcal{E}')$.

Lemma 4.4.4. *Let $(\mathcal{E}, \underline{q}, \tau)$ be a descending G-bundle of rank r and degree d, and $(\mathcal{F}, \tau')$ the induced singular G-bundle on X. For any saturated subsheaf $\mathcal{G} \subset \mathcal{E}$ we have*

$$P_{\underline{1}}(\mathcal{G} \subset \mathcal{E}, 1) = P_{\mathcal{F}}(n)\alpha(S(\mathcal{G})) - P_{S(\mathcal{G})}(n)\alpha(\mathcal{F}).$$

Proof. We follow [52, Proposition 5.2.2]. First of all we compute the Euler characteristic of $S(\mathcal{G})$:

$$\begin{aligned}
\chi(S(\mathcal{G})) = \chi(\pi_*(\mathcal{G})) - \sum_{i=1}^{\nu} \dim\, q(\mathcal{G}(y_1^i) \oplus \mathcal{G}(y_2^i)) = \\
= r(\mathcal{G})\chi(Y, \mathcal{O}_Y) + \deg(\mathcal{G}) - \sum_{i=1}^{\nu} \dim\, q(\mathcal{G}(y_1^i) \oplus \mathcal{G}(y_2^i)) = \\
= r(\mathcal{G})\nu + \deg(\mathcal{G}) - \sum_{i=1}^{\nu} \dim\, q(\mathcal{G}(y_1^i) \oplus \mathcal{G}(y_2^i)) + r(\mathcal{G})\chi(X, \mathcal{O}_X).
\end{aligned}$$

Note that the rank $r(\mathcal{G})$ is equal to the rank $r(S(\mathcal{G}))$ since $\bigoplus_{i=1}^{\nu} q_i(\mathcal{G}(y_1^i) \oplus \mathcal{G}(y_2^i))$ is a torsion sheaf, and π is birational. From the definition of $\mathcal{F}$ we also know that $\deg(\mathcal{F}) = \deg(\mathcal{E})$. Therefore

$$\begin{aligned}
P_{\mathcal{F}}(n)\alpha(S(\mathcal{G})) - P_{S(\mathcal{G})}(n)\alpha(\mathcal{F}) =& \deg(\mathcal{F})\alpha(S(\mathcal{G})) - \deg(S(\mathcal{G}))\alpha(\mathcal{F}) = \\
=& \deg(\mathcal{E})\alpha(\mathcal{G}) - \deg(S(\mathcal{G}))\alpha(\mathcal{F}) = \\
=& \deg(\mathcal{E})\alpha(\mathcal{G}) - (r(G)\nu + \deg(\mathcal{G}) - \\
& - \sum_{i=1}^{\nu} \mathrm{dim} q(\mathcal{G}(y_1^i) \oplus \mathcal{G}(y_2^i))) = \\
=& (\deg(\mathcal{E}) - \nu r)\alpha(\mathcal{G}) - (\deg(\mathcal{G}) - \\
& - \sum_{i=1}^{\nu} \dim\, q(\mathcal{G}(y_1^i) \oplus \mathcal{G}(y_2^i)))\alpha(\mathcal{F}) = P_{\underline{1}}(\mathcal{G} \subset \mathcal{E}, 1).
\end{aligned}$$

□

Lemma 4.4.5. *Let $(\mathcal{E}, \underline{q}, \tau)$ be a descending G-bundle of rank r and degree d on Y and $(\mathcal{F}, \tau')$ the induced singular principal G-bundle on X. For any weighted filtration $(\mathcal{E}_\bullet, \underline{m})$ we have*

$$\mu(\mathcal{E}_\bullet, \underline{m}, \tau) = \mu(S(\mathcal{E}_\bullet), \underline{m}, \tau').$$

Proof. The multiindices I giving the minimum in both semistability functions are the same because of the construction of $(\mathcal{F},\, \tau')$ and the flag $S(\mathcal{E}_\bullet)$. Then the result follows from the fact that the weights $\underline{m}$ are in both cases the same. □

Proposition 4.4.6. *Let $(\mathcal{E}, \underline{q}, \tau)$ be a descending G-bundle and $(\mathcal{F}, \tau')$ the induced singular principal G-bundle on X. Then,*

(i) if $(\mathcal{F}, \tau')$ is δ-(semi)stable, then $(\mathcal{E}, \underline{q}, \tau)$ is a $(1, \delta)$-(semi)stable G-bundle with a generalized parabolic structure.

(ii) if $(\mathcal{E}, \underline{q}, \tau)$ is a $(1, \delta)$-(semi)stable G-bundle with a generalized parabolic structure, then $(\mathcal{F}, \tau')$ is a δ-(semi)stable singular G-bundle.

Proof. Following [52, Proposition 5.2.2], the result follows trivially from Lemma 4.4.4 and Lemma 4.4.5 □

Proposition 4.4.7. *There is a number $1 > \epsilon > 0$, such that for any $\underline{\kappa} = (\kappa_1, \ldots, \kappa_\nu) \in ((1-\epsilon, 1) \cap \mathbb{Q})^\nu$, any integral parameter δ and any singular principal G-bundle $(\mathcal{E}, \underline{q}, \tau)$ with a generalized parabolic structure, we have*

1) If $(\mathcal{E}, \underline{q}, \tau)$ is $(\underline{\kappa}, \delta)$-semistable, then it is $(1, \delta)$-semistable.

2) If $(\mathcal{E}, \underline{q}, \tau)$ is $(1, \delta)$-stable, then it is $(\underline{\kappa}, \delta)$-stable

Proof. We follow [52, Proposition 5.2.3.]. Recall that the $(\underline{\kappa}, \delta)$-(semi)stability condition for a singular principal G-bundle with a generalized parabolic structure has to be checked just for the weighted filtrations $(\mathcal{E}^\bullet, \underline{m})$ of $\mathcal{E}$ for which $m_i < A$ for suitable constant A depending only on the numerical input data. This implies that we can find a natural number n such that

$$P_1(\mathcal{E}^\bullet, \underline{m}) + \delta\mu(\mathcal{E}^\bullet, \underline{m}, \tau) \in \mathbb{Z}[\frac{1}{n}]$$

for all such weighted filtrations. For every generalized parabolic bundle $(\mathcal{E}, \underline{q})$ and every weighted filtration $(\mathcal{E}_\bullet, \underline{m})$ we have

$$
\begin{aligned}
&P_1(\mathcal{E}^\bullet,\underline{m}) - P_{\underline{\kappa}}(\mathcal{E}^\bullet,\underline{m}) = \\
=&-\sum_{i=1}^{s} m_i \Bigg\{ \alpha_i \left(\alpha - \sum_{j=1}^{\nu} \kappa_j \mathrm{dim}(q(\mathcal{E}(y_j^1)\oplus\mathcal{E}(y_j^2))) \right) \\
&\qquad -\alpha\left(\alpha_i - \sum_{j=1}^{\nu} \kappa_j \mathrm{dim}(q(\mathcal{E}_i(y_j^1)\oplus\mathcal{E}_i(y_j^2))) \right) \Bigg\} + \\
&+\sum_{i=1}^{s} m_i \Bigg\{ \alpha_i \left(\alpha - \sum_{j=1}^{\nu} \mathrm{dim}(q(\mathcal{E}(y_j^1)\oplus\mathcal{E}(y_j^2))) \right) \\
&\qquad -\alpha\left(\alpha_i - \sum_{j=1}^{\nu} \mathrm{dim}(q(\mathcal{E}_i(y_j^1)\oplus\mathcal{E}_i(y_j^2))) \right) \Bigg\} = \\
=&\sum_{i=1}^{s} m_i \Bigg\{ \alpha \sum_{j=1}^{\nu} (1-\kappa_j) \mathrm{dim}(q(\mathcal{E}_i(y_j^1)\oplus\mathcal{E}_i(y_j^2))) \\
&\qquad -\alpha_i \sum_{j=1}^{\nu} (1-\kappa_j) \mathrm{dim}(q(\mathcal{E}(y_j^1)\oplus\mathcal{E}(y_j^2))) \Bigg\} = \\
=&\alpha \sum_{i=1}^{s} m_i \left\{ \sum_{j=1}^{\nu} (1-\kappa_j)\mathrm{dim}(q(\mathcal{E}_i(y_j^1)\oplus\mathcal{E}_i(y_j^2))) - \mathrm{rk}(\mathcal{E}_i)\sum_{j=1}^{\nu}(1-\kappa_j) \right\} \overset{\kappa_j\leq 1}{\leq} \\
\leq&\alpha(\sum_{j=1}^{\nu}(1-\kappa_j)) \sum_{i=1}^{s} m_i \left\{ \sum_{j=1}^{\nu} \mathrm{dim}(q(\mathcal{E}_i(y_j^1)\oplus\mathcal{E}_i(y_j^2))) - \mathrm{rk}(\mathcal{E}_i) \right\}.
\end{aligned}
$$

For any choice of ϵ we have $\kappa_j > 1-\epsilon$. Since $\mathrm{dim}(q(\mathcal{E}_i(y_j^1)\oplus\mathcal{E}_i(y_j^2))) - \mathrm{rk}(\mathcal{E}_i) \geq 0$ and $m_i < A$, we get

$$
P_1(\mathcal{E}^\bullet,\underline{m}) - P_{\underline{\kappa}}(\mathcal{E}^\bullet,\underline{m}) \leq \alpha\nu\epsilon \sum_{i=1}^{s} m_i r \leq A\alpha^2 r\nu\epsilon.
$$

In fact we can also show that $P_1(\mathcal{E}^\bullet,\underline{m}) - P_{\underline{\kappa}}(\mathcal{E}^\bullet,\underline{m}) \geq -A\alpha^2 r\nu\epsilon$. Take ϵ so that the inequality $A\alpha^2 r\nu\epsilon < \dfrac{1}{n}$ holds.

1) Let $(\mathcal{E},\underline{q},\tau)$ be a $(\underline{\kappa},\delta)$-semistable tensor field with generalized parabolic structure. Suppose it is not $(1,\delta)$-semistable. Then there is a weighted filtration $(\mathcal{E}_\bullet,\underline{m})$ with $m_i < A$ such that

$$
\begin{aligned}
&P_{\underline{\kappa}}(\mathcal{E}_\bullet,\underline{m}) + \delta\mu(\mathcal{E}_\bullet,\underline{m},\tau) \geq 0 \\
&P_1(\mathcal{E}_\bullet,\underline{m}) + \delta\mu(\mathcal{E}_\bullet,\underline{m},\tau) = \frac{n_1}{n} < 0
\end{aligned}
\tag{4.10}
$$

Since $P_{\underline{\kappa}}(\mathcal{E}_\bullet,\underline{m}) + \delta\mu(\mathcal{E}_\bullet,\underline{m},\tau)$ is a positive rational number, we can find $n_{\underline{\kappa}} \in \mathbb{Q}_{>0}$ such that $P_{\underline{\kappa}}(\mathcal{E}_\bullet,\underline{m}) + \delta\mu(\mathcal{E}_\bullet,\underline{m},\tau) = \dfrac{n_{\underline{\kappa}}}{n}$. Therefore

$$
-\frac{1}{n} < P_1(\mathcal{E}_\bullet,\underline{m}) - P_{\underline{\kappa}}(\mathcal{E}_\bullet,\underline{m}) = \frac{n_1 - n_{\underline{\kappa}}}{n} \leq -\frac{1}{n}
$$

which is a contradiction. Thus $(\mathcal{E}, \underline{q}, \tau)$ must be $(1, \delta)$-semistable.

2) Let $(\mathcal{E}, \underline{q}, \tau)$ be a $(1, \delta)$-stable tensor field with generalized parabolic structure. Suppose it is not $(\underline{\kappa}, \delta)$-stable. Then there is a weighted filtration $(\mathcal{E}_\bullet, \underline{m})$ with $m_i < A$ such that

$$\begin{aligned} P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m}) + \delta\mu(\mathcal{E}_\bullet, \underline{m}, \tau) &= \frac{n_{\underline{\kappa}}}{n} \leq 0, \ n_{\underline{\kappa}} \in \mathbb{Q}_{<0} \\ P_1(\mathcal{E}_\bullet, \underline{m}) + \delta\mu(\mathcal{E}_\bullet, \underline{m}, \tau) &= \frac{n_1}{n} > 0. \end{aligned} \tag{4.11}$$

Therefore

$$\frac{1}{n} > P_1(\mathcal{E}_\bullet, \underline{m}) - P_{\underline{\kappa}}(\mathcal{E}_\bullet, \underline{m}) = \frac{n_1 - n_{\underline{\kappa}}}{n} \geq \frac{1}{n}$$

which is a contradiction.

□

Let ϵ be as in Proposition 4.4.7. Choose $\kappa_i \in (1-\epsilon, 1) \cap \mathbb{Q}$ and $\delta \in \mathbb{Z}_{>0}$ so large that Theorem 3.3.22 holds. Then Proposition 4.4.6 and Proposition 4.4.7 imply that there is a well defined functor

$$\mathbf{D}(G)^{(\underline{\kappa},\delta)\text{-(s)s}} \to \mathbf{SPB}(\rho)_P^{\delta-(s)s} \tag{4.12}$$

and thus a morphism

$$\Theta : \mathcal{M}(G)^{(\underline{\kappa},\delta)\text{-(s)s}} \to \mathrm{SPB}(\rho)_P^{\delta\text{-(s)s}} \tag{4.13}$$

between the moduli spaces. We define $\mathcal{M}_X(\rho)$ as the schematic image of Θ. Then

Theorem 4.4.8. *The projective scheme $\mathcal{M}_X(\rho)$ consists of (semi)stable honest singular principal G-bundles, and every stable honest singular principal G-bundle lies in it.*

Proof. This follows by Proposition 4.4.6, Proposition 4.4.7. □

This was proved in the irreducible case in [52], and this space was considered as a good candidate for a compact moduli space for principal G-bundles on the singular curve X. In a later work A. Schmitt observed that if we fix a representation of G into the symplectic group, one can prove that $\mathrm{SPB}(\rho)_P^{\delta\text{-(s)s}}$ fulfils the desired properties to being such compactification (see [51]).

The goal of the last result of this work is to show that $\mathrm{SPB}(\rho)_P^{\delta\text{-(s)s}}$ parametrizes precisely semistable honest singular principal G-bundles.

To get started, let us make some minor changes on the definition of parabolic structure on the normalization.

Definition 4.4.9. Let $\underline{s} = \{s_1, \cdots, s_\nu\}$ be a set of natural numbers and denote $s = \sum_{i=1}^{\nu} s_i$. A generalized parabolic bundle of type $(r, \underline{s})$ on the smooth curve Y is a tuple $(\mathcal{E}, q_1, \ldots, q_\nu)$ where $\mathcal{E}$ is a locally free sheaf of rank r (that is, uniform multirank equal to r) and degree $-s$, and q_i is a quotient of dimension $r - s_i$

$$q_i \colon \Gamma(D_i, \mathcal{E}|_{D_i}) = \mathcal{E}(y_1^i) \oplus \mathcal{E}(y_2^i) \to R_i \to 0,$$

$\mathcal{E}(y_j^i)$ being the fibre of $\mathcal{E}$ over y_j^i.

As in Chapter 3, we denote by $R := \oplus R_i$ the total vector space. Since the supports of the divisors D_i are disjoint we get the equality $\Gamma(D, \mathcal{E}|_D) = \bigoplus \Gamma(D_i, \mathcal{E}|_{D_i})$. From this, we can form a quotient of dimension s.

$$q\colon = \oplus q_i \colon \Gamma(D, \mathcal{E}|_D) \to R \to 0.$$

Definition 4.4.10. Let $(\mathcal{E}, \underline{q})$ be a generalized parabolic bundle of type $(r, \underline{s})$. We define the parabolic degree for any subsheaf $\mathcal{F} \subseteq \mathcal{E}$ as

$$\operatorname{pardeg}(\mathcal{F}) = \deg(\mathcal{F}) - \sum_{i=1}^{\nu} \dim\, q_i(\mathcal{F}(y_1^i) \oplus \mathcal{F}(y_2^i)). \tag{4.14}$$

Observe that this coincides with the old $\underline{1}$-pardeg$(\mathcal{F})$.

Now, the definition of singular principal G-bundle of type $(r, \underline{s})$ is the obvious one.

Definition 4.4.11. A singular principal G-bundle with a generalized parabolic structure of type (r, s) over Y is a triple $(\mathcal{E}, \underline{q}, \tau)$ where $(\mathcal{E}, \underline{q})$ is a generalized parabolic bundle of type (r, s) (see Definition 4.4.9) and $(\mathcal{E}, \tau)$ is a singular principal G-bundle.

Finally, the semistability condition that we take is the old $(\underline{1}, \delta)$-semistability condition.

Definition 4.4.12. We fix $\delta \in \mathbb{Q}_{>0}$. A singular principal G-bundle with generalized parabolic structure of type (r, s), $(\mathcal{E}, \underline{q}, \tau)$, is δ-(semi)stable if for every weighted filtration $(\mathcal{E}_\bullet, \underline{m})$ of $\mathcal{E}$, the inequality

$$P(\mathcal{E}_\bullet, \underline{m}) + \delta\mu(\mathcal{E}_\bullet, \underline{m}, \tau)(\geq)0 \tag{4.15}$$

holds true.

The analysis of the semistability condition done in Chapter 3 is valid in this situation, and we can prove the analogous theorem to Theorem 3.3.22 without any extra work.

Theorem 4.4.13. *There is a rational number $\delta_\theta > 0$ depending only on the input data such that for all $\delta > \delta_\theta$ and every singular principal G-bundle with a generalized parabolic structure of type $(r, \underline{s})$, the property A) and the property B) are equivalent:*

A) $(\mathcal{E}, \underline{q}, \tau)$ is δ-(semi)stable.

B) $(\mathcal{E}, \underline{q}, \tau)$ is (semi)stable: $(\mathcal{E}, \underline{q}, \tau)$ is honest and for every weighted filtration $(\mathcal{E}_\bullet, \underline{m})$ such that $(\mathcal{E}_\bullet, \underline{m})|_{Y_i}$ is induced by a reduction to a one parameter subgroup, one has $P(\mathcal{E}_\bullet, \underline{m})(\geq)0$.

Let us define now descending G-bundles of type $(r, \underline{s})$.

Definition 4.4.14. A descending G-bundle of type (r, s) on Y is a singular principal G-bundle with generalized parabolic structure of type $(r, \underline{s})$, $(\mathcal{E}, \underline{q}, \tau)$, such that τ (see (4.4)) takes values in $\mathcal{O}_X \subset \pi_*(\mathcal{O}_Y)$. A descending G-bundle, $(\mathcal{E}, \underline{q}, \tau)$, is principal if $(\mathcal{E}, \tau)$ is honest.

Definition 4.4.15. A descending G-bundle is δ-(semi)stable if it is as singular principal G-bundle with generalized parabolic structure (see Definition 4.4.12).

Recall Definition 1.2.27. Now we have.

Theorem 4.4.16. *Let $(\mathcal{F},\tau)$ a singular principal G-bundle of degree 0 and suppose that $\mathcal{F}$ has types $\underline{a}=\{a_i\}$. Then there is always a descending G-bundle of type $(r,\underline{a})$, $(\mathcal{E},\underline{q},\tau')$, such that $\pi_*(\mathcal{E},\underline{q},\tau')=(\mathcal{F},\tau)$.*

Proof. Follows from Proposition 4.3.5. □

Theorem 4.4.17. *Let $\mathcal{F}$ be a torsion free sheaf of degree 0 and types $\underline{a}=\{a_i\}$, and fix $\delta\in\mathbb{Q}_{>0}$. Let $(\mathcal{F},\tau)$ be a δ-semistable singular principal G-bundle of degree 0 and multirank r. Then we have:*

A) There is a δ-semistable descending G-bundle type $(r,\underline{a})$, $(\mathcal{E},\underline{q},\tau')$, such that $\pi_*(\mathcal{E},\underline{q},\tau')=(\mathcal{F},\tau)$.

B) There is a positive rational number δ_θ depending only on the input data such that if $\delta>\delta_\theta$ then $(\mathcal{F},\tau)$ is honest.

Proof. A) This is Proposition 4.4.6. B) Follows from Proposition 4.3.5 and Theorem 4.4.13. □

We finally have.

Theorem 4.4.18. *Let $\delta>\delta_\theta$. An honest singular principal G-bundle, $(\mathcal{F},\tau)$, is δ-semistable if and only if it is semistable (see Definition 2.2.4). Therefore,* $\mathrm{SPB}(\rho)_P^{\delta\text{-(s)s}}$ *is a coarse moduli space for (semi)stable honest singular principal G-bundles.*

Proof. 1) Let $(\mathcal{F},\tau)$ be an honest singular principal G-bundle and suppose that it is δ-(semi)stable. Since $\delta>\delta_\theta$, we already know that there is a δ-semistable descending principal G-bundle of type $(r,\underline{a})$, $(\mathcal{E},\underline{q},\tau')$, on the normalization Y such that $\pi_*(\mathcal{E},\underline{q},\tau')=(\mathcal{F},\tau)$. Let $\lambda=(\lambda_1,\ldots,\lambda_l):\mathbb{G}_m\to G\times\ldots\times G$ be a one parameter subgroup and let $\beta=(\beta_1,\ldots,\beta_l)$ be a reduction of $(\mathcal{F},\tau)$ to λ. This defines a weighted filtration $(\mathcal{F}_{\beta\bullet},\underline{m}_\beta)$ (see Chapter 2, Section 2.2.1), and, thus, a weighted filtration $(\mathcal{E}_\bullet,\underline{m}_\beta)$ of $\mathcal{E}$ following the procedure given at the beginning of this section. Thus, $\mathcal{F}_i=S(\mathcal{E}_i)$. The key point is to show the following claim: the filtration $\mathcal{E}_\bullet$ corresponds to a reduction of $(\mathcal{E},\underline{q},\tau')$ to the one parameter subgroup λ. Thus

$$\mu(\mathcal{F}_{\beta\bullet},\underline{m}_\beta,\tau)=\mu(\mathcal{E}_\bullet,\underline{m}_\beta,\tau')=0$$

because of Lemma 4.4.5 and Proposition 3.3.21, hence $L(\mathcal{F}_{\beta\bullet},\underline{m}_\beta)(\geq)0$ since $(\mathcal{F},\tau)$ is δ-(semi)stable. Let us prove the claim. The reduction β is given by sections

$$\beta_i:U_i\to\underline{\mathrm{Isom}}(V\otimes\mathcal{O}_{U_i},\mathcal{F}|_{U_i}^{\vee})/Q_G(\lambda_i).$$

The injection $i:\mathcal{F}\hookrightarrow\pi_*\mathcal{E}$ determines an isomorphism $i_{U_i}:\mathcal{F}|_{U_i}\simeq\mathcal{E}|_{V_i}$ (being $V_i=Y_i\cap\pi^{-1}(U)$) since its cokernel is supported on the singular points. Thus, we get a commutative diagram

$$\begin{tikzcd}
U_i \arrow[d, equal, "\pi" left, "\wr" right] \arrow[r, bend left, "\beta_i"] & \underline{\mathrm{Isom}}(V\otimes\mathcal{O}_{U_i},\mathcal{F}|_{U_i}^{\vee})/Q_G(\lambda_i) \arrow[l] \arrow[d, equal, "\wr" right, "i^\sharp"] \\
V_i \arrow[ur, phantom, "\circlearrowleft"] & \underline{\mathrm{Isom}}(V\otimes\mathcal{O}_{V_i},\mathcal{E}|_{V_i}^{\vee})/Q_G(\lambda_i) \arrow[l]
\end{tikzcd} \tag{4.16}$$

and we can define a section $\beta_i' = i^\sharp \circ \beta_i \circ \pi^{-1} : V_i \to \underline{\mathrm{Isom}}(V \otimes \mathcal{O}_{V_i}, \mathcal{E}|_{V_i}^\vee)/Q_G(\lambda_i)$. This section defines a weighted filtration $(\mathcal{G}_{\beta\bullet}, \underline{m}_\beta)$ and the isomorphisms $i_{U_i} : \mathcal{F}|_{U_i} \simeq \mathcal{E}|_{V_i}$ induce a flag isomorphism

$$\begin{tikzcd}
(0) \arrow[r, hook] & \mathcal{F}_1|_{U_i} \arrow[r, hook] \arrow[d, phantom, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & \dots \arrow[r, hook] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F}_\theta|_{U_i} \arrow[r, hook] \arrow[d, phantom, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F}|_{U_i} \arrow[d, phantom, "\wr"] \\
(0) \arrow[r, hook] & \mathcal{G}_1|_{V_i} \arrow[r, hook] & \dots \arrow[r, hook] & \mathcal{G}_\theta|_{V_i} \arrow[r, hook] & \mathcal{E}|_{V_i}.
\end{tikzcd}$$

On the other hand, the filtration constructed at the beginning, $\mathcal{E}_\bullet$, satisfies

$$\mathcal{F}_i|_U = S(\mathcal{E}_i)|_{U_i} \simeq \mathcal{E}_i|_V$$

where the isomorphism is induced, again, by the injection $i : \mathcal{F} \hookrightarrow \pi_*\mathcal{E}$. Everything together tell us that we have two weighted filtrations $(\mathcal{E}_\bullet, \underline{m}_\beta)$ and $(\mathcal{G}_{\beta\bullet}, \underline{m}_\beta)$ which are isomorphic over V. Therefore $\mu(\mathcal{E}_\bullet, \underline{m}_\beta, \tau') = \mu(\mathcal{G}_{\beta\bullet}, \underline{m}_\beta, \tau')$ because of Proposition 3.3.5, hence $\mu(\mathcal{E}_\bullet, \underline{m}_\beta, \tau') = 0$.

2) Let us prove the inverse. Suppose that $(\mathcal{F}, \tau)$ is a (semi)stable honest singular principal G-bundle. By Theorem 4.4.16, there exists a descending principal G-bundle of type $(r, \underline{a})$ on Y, $(\mathcal{E}, \underline{q}, \tau')$, such that $\pi_*(\mathcal{E}, \underline{q}, \tau') = (\mathcal{F}, \tau)$. First note that $\mu(\mathcal{E}_\bullet, \underline{m}, \tau') \geq 0$ for any weighted filtration, since $(\mathcal{E}, \underline{q}, \tau')$ is an honest singular principal G-bundle (Corollary 3.3.7). Suppose now that $\mu(\mathcal{E}_\bullet, \underline{m}, \tau') = 0$. By Proposition 3.3.21, it follows that the filtration $\mathcal{E}_\bullet$ comes from a reduction $\beta = (\beta_1 \dots, \beta_l)$ to a 1-PS $\lambda = (\lambda_1, \dots, \lambda_l)$. Using the diagram (4.16) in the other way around, we find a section

$$\beta_i' : U_i \to \underline{\mathrm{Isom}}(V \otimes \mathcal{O}_{U_i}, \mathcal{F}^\vee|_{U_i})/Q_G(\lambda_i)$$

for each i, and, therefore, a weighted filtration $(\mathcal{F}_{\beta\bullet}, \underline{m}_\beta = \underline{m})$ such that $i : \mathcal{F}_j|_U \simeq \pi_*(\mathcal{E}_j|_W)$. On the other hand, we can construct from $(\mathcal{E}_\bullet, \underline{m})$ a weighted filtration of $\mathcal{F}$, $(S(\mathcal{E}_\bullet), \underline{m})$, such that $i : S(\mathcal{E}_j)|_U \simeq \pi_*(\mathcal{E}_j|_W)$. Thus, we get a commutative triangle

$$\begin{tikzcd}
\mathcal{F}_j|_U \arrow[dr, hook] \arrow[dd, phantom, "\wr"] & \\
& \mathcal{F}|_U. \\
S(\mathcal{E}_j)|_U \arrow[ur, hook] &
\end{tikzcd}$$

Applying Proposition 1.2.21 we find a global isomorphism

$$\begin{tikzcd}
\mathcal{F}_j \arrow[dr, hook] \arrow[dd, phantom, "\wr"] & \\
& \mathcal{F}. \\
S(\mathcal{E}_j) \arrow[ur, hook] &
\end{tikzcd}$$

Therefore,

$$P(\mathcal{E}_\bullet, \underline{m}) = L(\mathcal{F}_{\beta\bullet}, \underline{m}_\beta) \geq 0,$$

so $(\mathcal{E}, \underline{q}, \tau')$ is semistable. By Theorem 4.4.13, we know that $(\mathcal{E}, \underline{q}, \tau')$ is also δ-semistable, so by Lemma 4.4.4 and Lemma 4.4.5 we deduce that $(\mathcal{F}, \tau)$ is δ-semistable. □

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Zusammenfassung

In dieser Arbeit beschäftigen wir uns mit der Konstruktion eines kompakten Modulraums für G-Prinzipalbündel über einer Knotenkurve X. Der Prozess der Konstruktion dieser Modulräume basiert auf der Arbeit von A. Schmitt.

In Kapitel 1 geben wir den Hintergrund in GIT, kohärente Garben über reduzierte projektiven Kurven und G-Prinzipalbündeln. Wir präsentieren einige Beispiele für die Berechnung der Hilbert-Mumford Semistabilität, die in Kapitel 3 wichtig sein wird. Wir stellen auch eine GIT-Analyse von direkten Summenrepräsentationen vor, die zu Proposition 1.1.28 führt und die in Kapitel 3 entscheidend sein werden.

Kapitel 2 widmet sich der Konstruktion von $\mathrm{SPB}(\rho)_P^{\delta\text{-(s)s}}$. In Abschnitt 1 konstruieren wir nach [8, 17] den Modulraum von δ-semistabilen Tensorfeldern über X, $\mathcal{T}_P^{\delta\text{-(s)s}}$ (Theorem 2.1.44). Da unsere Kurve X nicht irreduzibel ist, müssen wir die Rang durch Multiplizität in der Definition der δ-Semistabilität ändern. (siehe Definition 2.1.9). In Abschnitt 2 konstruieren wir den Modulraum von δ-semistabilen singulären G-Prinzipalündeln, $\mathrm{SPB}(\rho)_P^{\delta\text{-(s)s}}$ (Theorem 2.2.18). Zuerst zeigen wir, wie man jedem singulären G-Prinzipalündel ein Tensorfeld zuordnen kann, für das, was wir brauchen, um das Problem zu linearisieren (Theorem 2.2.6). Dies geschieht durch Verwendung eines Ergebnisses auf graduierten Algebren (Lemma 2.2.5). Nach, müssen wir zeigen, dass diese Zuordnung injektiv ist (Theorem 2.2.12), unter Verwendung von Lemma 1.2.28. Auf diese Weise konstruieren wir den Modulraum als geschlossenes Teilschema des Modulraumes von Tensorfeldern.

In Kapitel 3 beschäftigen wir uns mit Objekten auf der Normalisierung von X. In Abschnitt 1 konstruieren wir den Modulraum von Tensorfeldern mit verallgemeinerten parabolische Strukturen über eine (möglicherweise) nicht zusammenhängende glatte projektive Kurve Y. Die Semistabilitätsbedingung hängt nun von $\nu + 1$ (rationalen) Parametern ab, $\kappa_1 \ldots, \kappa_\nu, \delta$, aufgrund der Anwesenheit der zusätzlichen Struktur, die durch die parabolische Struktur gegeben ist. Der Modulraum von $(\underline{\kappa}, \delta)$-semistabilen singulären G-Prinzipalbündeln mit verallgemeinerten parabolischen Strukturen auf Y ist wie im Knotenfall als geschlossenes Subschema des Modulraumes von Tensorfeldern mit generalisierter parabolischer Struktur aufgebaut. Schließlich studieren wir der Stabilitätsbegriffe für große Werte der Semistabilitätsparameter. Die Existenz mehrerer Minimalpunkte in der Kurve Y macht es unmöglich, die Ergebnisse von [52] zu übersetzen. Hier, das technische Ergebnis, das es uns ermöglicht, das Problem zu lösen, ist Proposition 1.1.28.

In Kapitel 4 beschreiben wir explizit ein Verfahren zur Darstellung eines gegebenen singulären G-Prinzipalbündel durch ein absteigendes G-Prinzipalbündel und vergleichen den Semistabilitätsbegriff beider Objekte für große Werte der Semistabilitätsparameter. Mit diesem in der Hand, können wir die endgültigen Ergebnisse, Theorem 4.4.8 and Theorem 4.4.18, präsentieren.

Erklärung

Gemäß §7 (4), der Promotionsordnung des Fachbereichs Mathematik und Informatik der Freien Universität Berlin vom 8. Januar 2007 versichere ich hiermit, dass ich alle Hilfsmittel und Hilfen angegeben und auf dieser Grundlage die vorliegende Arbeit selbstständig verfasst habe. Des Weiteren versichere ich, dass ich diese Arbeit nicht bereits zu einem früheren Promotionsverfahren eingereicht habe.

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