9 Asymptotically flat solutions

In this section we consider the case that $(\Sigma, \tilde{g}, \tilde{u})$ is a complete and asymptotically flat manifold. This is interesting for applications in physics since one class of solutions to the Einstein equations consists of metrics which describe the gravitational field of a single body like a star or a black hole. These are known as isolated systems. The model for such a solution is an asymptotically flat manifold where the metric satisfies certain decay conditions near infinity. We give two precise definitions of solutions which are asymptotically flat and prove that a decay behavior at time $t = 0$ is maintained as long as the solution satisfies a supremum bound. This indicates the usefulness of the flow (0.4) in general relativity and provides a necessary first step in that direction. Note that the following estimate though is applicable for a general solution satisfying the stated bounds. This applies for example to all solutions given by Theorem 3.22.

Lemma 9.1 *Let $(g,u)(t)$ be a solution to (2.5) on $[0,T] \times \Sigma$ with initial data $(\tilde{g}, \tilde{u})$ such that $|Rm| + |\nabla^2 u| + |du|^2 \leq \sqrt{K}$ on $[0,T] \times \Sigma$. Then we get for $t \in [0,T]$, any $x_0 \in \Sigma$, and any $R > 0$ the estimate:*

$$\sup_{x \in B^t_{R/4}(x_0)} |\Phi|^2(t,x) \leq 2 \sup_{x \in B^0_R(x_0)} |\Phi|^2_0(0,x) \cdot e^{C\tilde{C}t/R^2}$$

where $\tilde{C} := \max\{R^2 \cdot \sqrt{K}, 1\}$ is a scaling invariant constant and C depends only on n.

Proof:
From (2.21) we have the evolution inequality

$$\big(\partial_t - \Delta\big)|\Phi|^2 \leq -2|\nabla\Phi|^2 + C|\Phi|^3 + C|du|^2|\Phi|^2 \ .$$

The assumptions imply $R^2\big(|\Phi| + |du|^2\big) \leq R^2 \cdot \sqrt{K}$ on $B(T, x_0, R)$. Set $\tilde{C} := \max\{R^2 \cdot \sqrt{K}, 1\}$.

$$\big(\partial_t - \Delta\big)|\Phi|^2 \leq -2|\nabla\Phi|^2 + C(n)\tilde{C}R^{-2} \cdot |\Phi|^2 \ .$$

Define a cut-off function η for a fixed radius $R > 0$ by

$$\tilde{\eta}(t,x) := R^{-4} \cdot \big(R^2 - r(t,x)\big)^2 \ .$$

This function differs from the one in (6.8) only by the factor R^{-4}, and we therefore can use the results obtained in Lemma 6.8 for η modified by this factor. Setting $f := |\Phi|^2 \cdot \tilde{\eta}$, we compute

$$\begin{aligned}
\big(\partial_t - \Delta\big)f &\leq \big(\partial_t - \Delta\big)|\Phi|^2 \cdot \tilde{\eta} + |\Phi|^2 \cdot \big(\partial_t - \Delta\big)\tilde{\eta} - 2\nabla|\Phi|^2\nabla\tilde{\eta} \\
&\leq -2|\nabla\Phi|^2\tilde{\eta} + C(n)\tilde{C}R^{-2}f + C(n)\tilde{C}R^{-2}|\Phi|^2 + |\nabla\Phi|^2\tilde{\eta} + C(n)R^{-2}f \\
&\leq C(n)\tilde{C}R^{-2}f,
\end{aligned}$$

using the bound $\tilde{\eta}^{-1} \leq \frac{16}{9}$ on $B(T, x_0, R/4)$ and $\tilde{C} \geq 1$. From (6.5) we estimated

$$-2\nabla|\Phi|^2\nabla\eta \leq 8|\nabla\Phi||\Phi| \cdot \frac{(R^2 - r)|\nabla r|}{R^4} = \left(|\nabla\Phi|\frac{R^2 - r}{R^2} \cdot \frac{8|\nabla r||\Phi|}{R^2}\right) \leq |\nabla\Phi|^2\tilde{\eta} + CR^{-2}|\Phi|^2 \ .$$

We consider the associated ordinary differential equation

$$\frac{d}{dt}v(t) = C\tilde{C}R^{-2}v(t)$$

with initial value $v(0) := \frac{9}{8}\sup_{x\in B^0_R(x_0)} f(0,x)$. The solution is given by $v(t) = v(0)\cdot e^{C\tilde{C}t/R^2}$ and satisfies $f(0) < v(0)$ at time $t = 0$. Choose $\varepsilon := \frac{1}{16}\sup_{x\in B^0_R(x_0)} f(0,x)$ and a point (t^*,x^*) such that $f(t^*,x^*) - v(t^*) = -\varepsilon$ is true for the first time on $B(T,x_0,R/4)$. Since x^* then is also a spatial maximum at time t^*, this implies:

$$\big(\partial_t - \Delta\big)\big(f-v\big)(t^*,x^*) \geq 0\ .$$

Altogether we have at (t^*,x^*) that

$$0 \leq \big(\partial_t - \Delta\big)(f-v) \leq C\tilde{C}R^{-2}(f-v) = C\tilde{C}R^{-2}(-\varepsilon) < 0$$

which is a contradiction. Therefore $v(t)$ is a barrier for $f(t)$ for all $t \geq 0$ as follows:

$$\sup_{x\in B^t_{R/4}(x_0)}\big[|\Phi|^2(t,x)\cdot\tilde{\eta}(t,x)\big] \leq \tfrac{9}{8}\sup_{x\in B^0_R(x_0)}\big[|\Phi|^2(0,x)\cdot\tilde{\eta}(0)\big]\, e^{C\tilde{C}t/R^2}\ .$$

Using $\tilde{\eta}(0) \leq 1$ and $\tilde{\eta}^{-1} \leq \frac{16}{9}$ on $B(T,x_0,R/4)$, the desired result follows.

□

In the same way we can proof:

Lemma 9.2 *Let $(g,u)(t)$ be a solution to (2.5) on $[0,T]\times\Sigma$ with initial data $(\tilde{g},\tilde{u})$ such that $|Rc|^2 \leq K$ on $[0,T]\times\Sigma$. Then we get for $t\in[0,T]$, any $x_0\in\Sigma$, and any $R>0$ the estimate:*

$$\sup_{x\in B^t_{R/4}(x_0)}|du|^2(t,x) \leq 2\sup_{x\in B^0_R(x_0)}|d\tilde{u}|^2_0(0,x)\cdot e^{C\tilde{C}t/R^2}$$

where $\tilde{C} := \max\{R^2\cdot\sqrt{K},1\}$ is a scaling invariant constant and C depends only on n.

Proof:
Since $|du|^2$ satisfies the evolution inequality

$$\big(\partial_t - \Delta\big)|du|^2 \leq -2|\nabla^2 u|^2,$$

we only need a bound on the Ricci curvature to apply (6.12). The remaining proof is analogous to the proof of Lemma 9.1.

□

We can now deal with complete asymptotically flat manifolds. A strong definition is given as follows:

Definition 9.3 *Let Σ be a complete n-dimensional connected Riemannian manifold. (Σ, g, u) is called* strongly asymptotically flat of mass m, *if there is a compact subset $K \subset \Sigma$ such that $\Sigma_K := \Sigma \setminus K$ is diffeomorphic to $\mathbb{R}^n \setminus B_1(0)$ where $B_1(0)$ is the Euclidean unit ball, and (g, u) satisfy in the* exterior region Σ_K*:*

$$\left| g - \left(1 - \frac{2m}{\tilde{r}}\right)\delta \right| \le C_0 \cdot \tilde{r}^{-2} \tag{9.1}$$

$$|\partial^k g| \le C_k \cdot \tilde{r}^{-k-1}, \quad k = 1, 2, 3 \tag{9.2}$$

$$\left| u + \frac{m}{\tilde{r}} \right| \le D_0 \cdot \tilde{r}^{-2} \tag{9.3}$$

$$|\partial^k u| \le D_k \cdot \tilde{r}^{-k-1}, \quad k = 1, 2, 3 \tag{9.4}$$

for some constants C_k, D_k, $k = 0, \dots, 3$ where $\tilde{r}(x) := \sqrt{x_1^2 + \cdots + x_n^2}$ is the radial coordinate of $\mathbb{R}^n$ and $|\cdot|$ and ∂ are with respect to δ.

A very important physical property of an asymptotically flat manifold is its ADM mass. For manifolds as defined above it is given by the coefficient m in the expansion of (g, u). In general it is defined as follows:

Definition 9.4 *Let (Σ, g) be asymptotically flat. Then the* ADM mass *of (Σ, g) is defined by:*

$$m_{ADM}(g) := c(n) \lim_{\rho \to \infty} \int_{S_\rho} \sum_{i,j=1}^{n} (\partial_i g_{ij} - \partial_j g_{ii}) N^j dA$$

where $S_\rho \subset \Sigma$ is the coordinate sphere of radius ρ, N its outer unit normal, dA the associated Riemannian metric on S_ρ, and $c(n)$ a dimension-dependent normalization constant.

It is shown in [Bar86] that the ADM mass is invariantly defined and independent of the asymptotic coordinate system. We prove that the flow (2.5) preserves the class of asymptotically flat manifolds as defined above.

Theorem 9.5 *Let $(g, u)(t)$ be the solution to (2.5) on $[0, T] \times \Sigma$ for $T < \infty$ with initial data $(\tilde{g}, \tilde{u})$ given by Theorem 3.22. Assume that $(\Sigma, \tilde{g}, \tilde{u})$ is strongly asymptotically flat of mass $m := m(\tilde{g})$ with $K = \bar{B}^0_\rho(O)$ for constants C_k, D_k. Then $(\Sigma, g(t), u(t))$ is strongly asymptotically flat for all $t \in [0, T]$. In particular there is a constant $C(t) = C(t, k_0, c_0, s_0, C_k, D_k)$ depending only on time, the bounds from Theorem 3.22, and the asymptotic decay of $(\tilde{g}, \tilde{u})$ such that*

$$\left| g(t) - \left(1 - \frac{2m}{\tilde{r}}\right)\delta \right|_0 + \left| u(t) + \frac{m}{\tilde{r}} \right|_0 \le C(t) \cdot \tilde{r}^{-2}$$

$$|\partial^k g|_0 + |\partial^k u|_0 \le C(t) \cdot \tilde{r}^{-k-1}$$

holds on $[0, T] \times \Sigma \setminus B_{2\rho}(O)$ for $k = 1, 2, 3$ with $C(t) \to \infty$ for $t \to \infty$. In particular the ADM mass is preserved by the flow and we have $m(g(t)) = m$ for all $t \in [0, T]$.

Proof:
Let $(\Sigma, \tilde{g}, \tilde{u})$ be asymptotically flat on $\Sigma_K := \Sigma \setminus \bar{B}_\rho(O)$. Assume without loss of generality that $\rho \geq 1$. We know from the definition that there is a constant $\bar{C}$ such that

$$\sup_{x\in\Sigma_K} \left[|\Phi|_0(0,x)\tilde{r}^3(x)\right] \leq C \sup_{x\in\Sigma_K} \left[(|\partial^2\tilde{g}|_0 + |\partial\tilde{g}|_0^2 + |\partial^2\tilde{u}|_0 + |\partial\tilde{g}|_0|\partial\tilde{u}|_0)\tilde{r}^3\right] \leq \bar{C}\ .$$

Fix $x_0 \in \Sigma \setminus B^0_{2\rho}(O)$ and choose $R := \frac{1}{4}\tilde{r}(x_0)$. Using the equivalence of the norms $|\cdot|_0$ and $|\cdot|$ and the bounds on $|\Phi| + |du|^2$ from Theorem 3.22, we can apply Lemma 9.1 to find

$$\begin{aligned}|\Phi|_0^2(t,x_0) &\leq e^{ct}|\Phi|^2(t,x_0)\cdot\tilde{\eta}(t,x_0) \leq e^{ct}\cdot \sup_{x\in B^t_{R/4}(x_0)} \left[|\Phi|_0^2(t,x)\cdot\tilde{\eta}(t,x)\right] \\ &\leq e^{ct}\cdot 2\sup_{x\in B^0_R(x_0)}\left[|\Phi|_0^2(0,x)\cdot\tilde{\eta}(0,x)\right]\cdot e^{C\tilde{C}t/R^2} \leq 2e^{ct}\cdot \sup_{x\in B^0_R(x_0)}\left[|\Phi|_0^2(0,x)\right]\cdot e^{C\tilde{C}t/R^2}\end{aligned}$$

for a constant c depending only on n, k_0, c_0 where $c_0 := \sup_\Sigma |d\tilde{u}|_0^2$. Since $\bar{B}^0_R(x_0)$ is compact, the supremum is attained at some point $x^* \in B^0_R(x_0)$. Multiplying by $\tilde{r}^6(x_0)$ leads to

$$|\Phi|_0^2(t,x_0)\cdot\tilde{r}^6(x_0) \leq 2e^{ct}e^{C\tilde{C}t/R^2}\cdot|\Phi|_0^2(0,x^*)\cdot\tilde{r}^6(x_0)\ .$$

Examining the constants, we see that either $\tilde{C} = 1$ and $e^{C\tilde{C}t/R^2} \leq e^{Ct/\rho^2}$ or $\tilde{C} = R^2\cdot\sqrt{K}$ such that $e^{C\tilde{C}t/R^2} \leq e^{C\sqrt{K}t}$. In both cases we get

$$e^{ct}e^{C\tilde{C}t/R^2} \leq e^{\bar{c}t}$$

for a constant $\bar{c} = \bar{c}(n, k_0, c_0, s_0, \rho)$ independent of x_0. Here $s_0 := \sup_\Sigma |\tilde{\nabla}^2\tilde{u}|_0^2$ as in Theorem 3.22. On all of $B^0_R(x_0)$ there is the estimate

$$\tilde{r}(x) \geq \tilde{r}(x_0) - R = \tfrac{3}{4}\tilde{r}(x_0)$$

such that we obtain

$$|\Phi|_0^2(t,x_0)\cdot\tilde{r}^6(x_0) \leq 12e^{\bar{c}t}\cdot\left(|\Phi|_0^2(0,x^*)\cdot\tilde{r}^6(x^*)\right) \leq 12\bar{C}^2e^{\bar{c}t}\ .$$

Note that the choice of R guarantees that the ball $B^0_R(x_0)$ is fully contained in Σ_K. Therefore we can use the asymptotic expressions. Furthermore the constants $\bar{c}$ and $\bar{C}$ do not depend on x_0. This implies that the estimate is uniform for all $x \in \Sigma \setminus B^0_{2\rho}(O)$ and we have

$$|\Phi|_0(t,x) \leq A\cdot\tilde{r}^{-3}(x) \tag{9.5}$$

on $[0,T]\times\Sigma\setminus B^0_{2\rho}(O)$ for a constant $A = A(t)$ depending only on t with $A(t)\to\infty$ for $t\to\infty$.

In the same way as above, we get using Lemma 9.2:

$$|du|_0^2(t,x) \leq B\cdot\tilde{r}(x)^{-4} \tag{9.6}$$

on $[0,T]\times\Sigma\setminus\bar{B}_{2\rho}(x_0)$ with $B(t)\to\infty$ for $t\to\infty$. The estimates (9.5) and (9.6) allow us to integrate pointwise for all $x\in\Sigma\setminus B^0_{2\rho}(O)$:

$$\begin{aligned}\left|g(t) - \left(1-\frac{2m}{\tilde{r}}\right)\delta\right|_0 &\leq C_0\tilde{r}^{-2} + \int_0^t |\partial_t g(\tau)|_0 d\tau \leq \tilde{C}_0\tilde{r}^{-2} + 4\int_0^t \left(|Rc|_0 + |du|_0^2\right)(\tau)d\tau \\ &\leq \tilde{C}_0\tilde{r}^{-2} + \left(A\tilde{r}^{-3} + B\tilde{r}^{-4}\right)\cdot T = \tilde{C}_0\cdot\tilde{r}^{-2}\end{aligned} \tag{9.7}$$

where $\tilde{C}_0 = \tilde{C}_0(n, C_0, A, B, T)$ is independent of x. For $u(t)$ we integrate:

$$\begin{aligned}\Big|u(t) + \frac{m}{r}\Big|_0 &\leq D_0\tilde{r}^{-2} + \int_0^t |\partial_t u(\tau)|_0 d\tau \leq D_0\tilde{r}^{-2} + \int_0^t |\Delta u|_0(\tau) d\tau \\ &\leq D_0\tilde{r}^{-2} + nAT \cdot C\tilde{r}^{-3} = \tilde{D}_0 \cdot \tilde{r}^{-2}\end{aligned} \tag{9.8}$$

where $\tilde{D}_0 = \tilde{D}_0(n, D_0, A, B, T)$ also does not depend on x. This shows that $(g, u)(t)$ remains asymptotically flat at zeroth order on $\Sigma \setminus B^0_{2\rho}(O)$.

To estimate the first derivative of g, we need the first derivative of Φ. From (2.21) we have

$$\big(\partial_t - \Delta\big)|\nabla\Phi|^2 \leq -2|\nabla^2\Phi|^2 + C\left\{|\Phi||\nabla\Phi|^2 + |du||\Phi|^2|\nabla\Phi| + |du|^2|\nabla\Phi|^2\right\} \ .$$

Estimating $(|du|^2 + |\Phi|) \leq \tilde{C}R^{-2}$ as before and using Young's inequality, this implies

$$\big(\partial_t - \Delta\big)|\nabla\Phi|^2 \leq -2|\nabla^2\Phi|^2 + C\tilde{C}R^{-2}|\nabla\Phi|^2 + A^4\tilde{r}^{-12} \ .$$

The remaining calculations go through as above such that we find for $f := |\nabla\Phi|^2 \cdot \tilde{\eta}$:

$$\big(\partial_t - \Delta\big)f \leq C\tilde{C}R^{-2}f + CA^4\tilde{r}^{-12}(x_0) \ .$$

Here we used that $\tilde{\eta} \leq 1$ on $B(T, x_0, R)$ and $\tilde{r}^{-1}(x) \leq \frac{4}{3}\tilde{r}^{-1}(x_0)$. Let $v(t)$ be the solution to

$$\frac{d}{dt}v(t) = C\tilde{C}R^{-2}v(t) + CA^4\tilde{r}^{-12}(x_0)$$

with initial value $v(0) := \frac{9}{8}\sup_{x \in B^0_R(x_0)} f(0, x)$. Then v is given explicitly by

$$\begin{aligned}v(t) &= -CA^4\tilde{r}^{-12}(x_0) \cdot \frac{R^2}{C\tilde{C}} + e^{C\tilde{C}t/R^2} \cdot \big(v(0) + CA^4\tilde{r}^{-12}(x_0) \cdot \frac{R^2}{C\tilde{C}}\big) \\ &\leq e^{C\tilde{C}t/R^2} \cdot \big(v(0) + CA^4\tilde{r}^{-10}(x_0)\big)\end{aligned}$$

from the choice of R and since $\tilde{C} \geq 1$. Since v is a barrier for f for all $t > 0$, we find:

$$\sup_{x \in B^t_{R/4}(x_0)} \big[|\nabla\Phi|^2(t, x) \cdot \tilde{\eta}(t, x)\big] \leq 2\big(\sup_{x \in B^0_R(x_0)} \big[|\nabla\Phi|^2_0(0, x)\tilde{\eta}(0, x)\big] + CA^4\tilde{r}^{-10}(x_0)\big) \cdot e^{C\tilde{C}t/R^2} \ .$$

The asymptotic conditions for $|\partial^3 g|$ and $|\partial^3 u|$ imply that for $x \in \Sigma \setminus B^0_\rho(O)$ at $t = 0$ we have

$$|\nabla\Phi|_0(0, x) \leq \bar{C} \cdot \tilde{r}^{-4}$$

for a constant E independent of x. We finally find at a point $x^* \in B^0_R(x_0)$ where the supremum is attained that

$$\begin{aligned}|\nabla\Phi|^2(t, x_0) \cdot \tilde{r}^8(x_0) &\leq \big(2|\nabla\Phi|^2_0(0, x^*) \cdot \tilde{r}^8(x_0) + CA^4\tilde{r}^{-10} \cdot \tilde{r}^8(x_0)\big) \cdot e^{C\tilde{C}t/R^2} \\ &\leq (32\bar{C} + CA^4\tilde{r}^{-2}(x_0))e^{C\tilde{C}t/R^2} =: E(t)\end{aligned}$$

holds as required. Here we used $\tilde{r}^{-1}(x_0) \leq 1$.

This implies the decay

$$|\nabla\Phi|_0(t,x_0) \le E(t)\cdot\tilde r^{-4}(x_0)$$

on $[0,T]\times B_{2\rho}(x_0)$. Observe that $E(t)$ only depends on t and $E(t)\to\infty$ for $t\to\infty$. An integration gives

$$\begin{aligned}|\partial g|_0(t) &\le |\partial\tilde g|_0 + \int_0^t |\partial_t\partial g|_0(\tau)d\tau \le C_1\cdot\tilde r^{-3} + 4\sup_{\tau\in[0,T]}(|\nabla Rc|_0 + |du|_0|\nabla^2 u|_0)(\tau)\cdot T\\ &\le C_1\cdot\tilde r^{-3} + 4T\cdot E\tilde r^{-4} + 4T\sqrt{B}A\tilde r^{-5} \le \tilde C_1\cdot\tilde r^{-3}\ .\end{aligned}$$

Together with (9.6) we get the desired asymptotics for ∂g and ∂u. We have from (9.7) and (9.8):

$$\begin{aligned}|\partial^2 g|_0 &\le |Rm|_0 + |\partial g|_0^2 \le A\cdot\tilde r^{-3} + \tilde C_1^2\cdot\tilde r^{-4} \le \tilde C_2\cdot\tilde r^{-3}\\ |\partial^2 u|_0 &\le |\nabla^2 u|_0 + |\partial g|_0|\partial u|_0 \le A\cdot\tilde r^{-3} + \tilde C_1\cdot\tilde r^{-2}\cdot\tilde D_1\cdot\tilde r^{-2} \le D_2\cdot\tilde r^{-3}\end{aligned}$$

establishing the claim for $k=2$. The remaining case $k=3$ is handled similarly and we obtain:

$$\begin{aligned}|\partial^3 g|_0 &\le |\nabla Rm|_0 + |\partial g|_0|Rm|_0 + |\partial g|_0|\partial^2 g|_0\\ &\le E\cdot\tilde r^{-4} + \tilde C_1\tilde r^{-2}\cdot\tilde C_2\tilde r^{-3} + \tilde C_1\tilde r^{-2}\cdot\tilde C_2\tilde r^{-3} \le \tilde C_3\cdot\tilde r^{-4}\\ |\partial^3 u|_0 &\le |\nabla^3 u|_0 + |\partial g|_0|\partial^2 u|_0 + |\partial^2 g|_0|\partial u|_0 + |\partial^2 g|_0|\partial g|_0|\partial u|_0\\ &\le E\cdot\tilde r^{-4} + \tilde C_1\tilde r^{-2}\cdot\tilde D_2\tilde r^{-3} + \tilde C_2\tilde r^{-3}\cdot\tilde D_1\tilde r^{-2} + \tilde C_2\tilde r^{-3}\cdot\tilde C_1\tilde r^{-2}\cdot\tilde D_1\tilde r^{-2} \le \tilde D_3\cdot\tilde r^{-4}\ .\end{aligned}$$

Since $\tilde g$ and δ are equivalent, this proves that the asymptotic flatness is preserved.

The estimates (9.7) and (9.8) show that the change of (g,u) in time is of order $\tilde r^{-3}$ and that therefore the asymptotic form (9.1) and (9.3) of the initial data is preserved. It immediately follows that the ADM mass stays constant under the flow (2.5). This finishes the proof of the theorem.

□

The same can be proved for the following weaker definition of asymptotic flatness:

Definition 9.6 *Let Σ be a complete n-dimensional connected Riemannian manifold. (Σ,g,u) is called* asymptotically flat, *if there is a compact subset $K\subset\Sigma$ such that $\Sigma_K := \Sigma\setminus K$ is diffeomorphic to $\mathbb{R}^n\setminus B_1(0)$ where $B_1(0)$ is the Euclidean unit ball, and (g,u) satisfy in the exterior region Σ_K:*

$$|g-\delta| \le C_0\cdot\tilde r^{-1} \tag{9.9}$$

$$|\partial^k g| \le C_k\cdot\tilde r^{-k-1},\quad k=1,2,3 \tag{9.10}$$

$$|u| \le D_0\cdot\tilde r^{-1} \tag{9.11}$$

$$|\partial^k u| \le D_k\cdot\tilde r^{-k-1},\quad k=1,2,3 \tag{9.12}$$

for some constants C_k, D_k where $\tilde r(x) := \sqrt{x_1^2+\cdots+x_n^2}$ is the radial coordinate of $\mathbb{R}^n$ and $|\cdot|$ and ∂ are with respect to the Euclidean metric.

Theorem 9.7 *Let $(g,u)(t)$ be the solution to (2.5) on $[0,T]\times\Sigma$ for $T<\infty$ with initial data $(\tilde{g},\tilde{u})$ given by Theorem 3.22. Assume that $(\Sigma,\tilde{g},\tilde{u})$ is asymptotically flat with $K=\bar{B}^0_\rho(O)$ for constants C_k, D_k. Then $(\Sigma, g(t), u(t))$ is asymptotically flat for all $t\in[0,T]$. In particular there is a constant $C=C(t,k_0,c_0,s_0,C_k,D_k)$ depending only on time, the bounds from Theorem 3.22, and the asymptotic decay of $(\tilde{g},\tilde{u})$ such that*

$$\begin{aligned}|g(t)-\delta|_0+|u(t)|_0 &\le C(t)\cdot\tilde{r}^{-1}\\ |\partial^k g|_0+|\partial^k u|_0 &\le C(t)\cdot\tilde{r}^{-k-1}\end{aligned}$$

holds on $[0,T]\times\Sigma\setminus B_{2\rho}(O)$ for $k=1,2,3$, where $C(t)\to\infty$ for $t\to\infty$. In addition, the ADM mass stays constant in time

$$m_{ADM}(g(t))=m_{ADM}(\tilde{g})$$

for all $t\in[0,T]$.

Proof:
The proof is the same as the proof of Theorem 9.5. A careful investigation shows that the integration (9.7) of the metric does not change the first order term in $\tilde{r}$ which determines the mass of $g(t)$.

□

Remark 9.8 *Although the mass is constant for solutions of (2.5) on finite time intervals, we expect that the mass $m_{ADM}(g_\infty)$ of the limit (g_∞,u_∞) of a global solution $(g,u)(t)$ for $t\to\infty$ jumps. In particular it is not clear that the limit in time commutes with the spatial limit.*