6 Interior estimates for the flow

Before we can start with the proof of the interior estimates, we have to introduce auxiliary functions and prove some of their properties. In addition, we get an estimate for changing distances under the flow from these considerations.

6.1 Preparations

The first auxiliary function we need is a time dependent scaling function.

Lemma 6.1 *Let $R > 0$ be a fixed radius and $T > 0$ be a fixed time. We define a scaling function*

$$\varphi : [0,T] \to \mathbb{R}^+, \qquad \varphi(t) := \frac{R^2 t}{R^2 + t} \ . \tag{6.1}$$

It satisfies for all $k \geq 0$ and all $t \in [0,T]$:

$$\partial_t \varphi^{k+1} = (k+1)\varphi^k \tag{6.2}$$

$$\varphi \leq R^2 \ . \tag{6.3}$$

Furthermore φ is invertible for $t > 0$, and the inverse is given by

$$\varphi^{-1} = \left(\frac{1}{R^2} + \frac{1}{t}\right) \qquad \forall t > 0 \ . \tag{6.4}$$

Proof:
These are short calculations.

□

We want to prove local estimates on the following union of metric balls at different times:

Definition 6.2 *Let $g(t)$ be a time dependent Riemannian metric on a complete manifold Σ with distance function $d_t(x,y)$. Let $x_0 \in \Sigma$ and a radius $R > 0$ be given. Then we define*

$$B(\tau, x_0, R) := \bigcup_{t \in [0,\tau]} \bar{B}^t_R(x_0) \subset [0,\tau] \times \Sigma$$

as the union of the geodesic balls $\bar{B}^t_R(x_0) := \left\{(t,x) \in \{t\} \times \Sigma \middle| d_t(x_0,x) \leq R\right\}$ of radius R around a point $x_0 \in \Sigma$ at times $t \in [0,\tau]$.

We need to estimate the time dependent distance function to show that these sets are compact. To this end we use the squared distance:

Lemma 6.3 *Let $B(T, x_0, R)$ be given as above. Then the squared distance function*

$$r : B(T, x_0, R) \to \mathbb{R}^+ \subset \Sigma, \qquad r(t, x) := \tfrac{1}{2} d_t(x_0, x)^2$$

satisfies

$$|\nabla r(t, x)| \leq R \qquad \forall (t, x) \in B(T, x_0, R) \ . \tag{6.5}$$

If $|Rc| \leq (n-1)\kappa^2$ holds on $B(T, x_0, R)$, then there are the estimates

$$\Delta r(t, x) \leq n + (n-1)\kappa \cdot d_t(x_0, x) \tag{6.6}$$

$$-\partial_t r(t, x) \leq (n-1)\kappa^2 \cdot d_t(x_0, x_1)^2 \ . \tag{6.7}$$

Before we prove this lemma, we first need an estimate for the derivative of the maximum of a smooth function.

Lemma 6.4 *Let $F(t) := \sup_{y \in Y}\{f(t, y)\}$ for a smooth function f where Y is a compact set. Then F is Lipschitz continuous in t, and we can estimate the derivative in the sense of difference quotients as follows:*

$$\inf\big\{\partial_t f(t, y) \big| y \in Y(t)\big\} \leq \frac{d}{dt} F(t) \leq \sup\big\{\partial_t f(t, y) \big| y \in Y(t)\big\}$$

where $Y(t) := \big\{y \in Y \big| F(t) = f(t, y)\big\}$.

Proof:
The upper bound for the derivative of F is proven in [Ham86, §3] where also the notion of the derivative of a Lipschitz function is defined. We want to prove the lower bound and proceed along the lines of the proof of [Ham86, Lemma 3.5]:

Choose a sequence $t_j \searrow t$ such that

$$\lim_{j \to \infty} \frac{F(t_j) - F(t)}{t_j - t} = \liminf_{h \searrow 0} \frac{F(t+h) - F(t)}{h} \ .$$

Since Y is compact, the supremum of f at each t_j is attained, and there is a sequence $(y_j) \subset Y$ with $F(t_j) = f(t_j, y_j)$. After passing to a subsequence, we may assume that $y_j \to y^*$. Since F and f are continuous, we have

$$F(t_j) \to F(t) \quad \text{ and } \quad f(t_j, y_j) \to f(t, y^*) \qquad \text{for } j \to \infty$$

which gives $F(t) = f(t, y^*)$ and therefore $y^* \in Y(t)$. Furthermore

$$f(t_j, y^*) \leq f(t_j, y_j) = F(t_j) = \sup\big\{f(t_j, y) \big| y \in Y\big\}$$

holds, meaning that

$$F(t_j) - F(t) = f(t_j, y_j) - f(t, y^*) \geq f(t_j, y^*) - f(t, y^*) \ .$$

We can apply the Mean Value Theorem to f at y^* and find

$$f(t_j, y^*) - f(t, y^*) = \partial_t f(T_j, y^*) \cdot \big(t_j - t\big)$$

for some $T_j \in (t, t_j)$. Since $t_j \searrow t$ implies $T_j \searrow t$, there is the estimate

$$\lim_{j\to\infty} \frac{F(t_j) - F(t)}{t_j - t} \geq \lim_{j\to\infty} \frac{f(t_j, y^*) - f(t, y^*)}{t_j - t} = \lim_{j\to\infty} \big[\partial_t f(T_j, y^*)\big] = (\partial_t f)(t, y^*)$$

for some $y^* \in Y(t)$ from the continuity of $\partial_t f$. This implies

$$\frac{d}{dt} F \geq \inf_{y \in Y(t)} \big\{\partial_t f(t, y)\big\}$$

as required.

□

By choosing F appropriately, the lemma allows us to control the time derivative of the metric distance:

Corollary 6.5 *Let $(g, u)(t)$ be a solution to (2.5) on $[0, T] \times \Sigma$ where Σ is complete. Let x_0, x_1 be two fixed points in Σ and $d_t(x_0, x_1)$ the time dependent distance between x_0 and x_1. Then there is the following bound on its time derivative:*

$$-\Lambda d_t(x_0, x_1) \leq \frac{d}{dt} d_t(x_0, x_1) \leq \Omega d_t(x_0, x_1)$$

whenever $\sup_{[0,T]\times\Sigma} |Rc| \leq \Lambda$ and $\sup_{[0,T]\times\Sigma} |Sy| \leq \Omega$. The result is still true, if the bounds only hold along all minimizing geodesics between x_0 and x_1 at all times $0 \leq t \leq T$.

Proof:
We first compute the evolution of the length of a fixed curve under the flow. Let $\gamma : [0, L] \to \Sigma$ be a smooth curve parametrized by arc length. Then its time dependent length is defined to be

$$L^t(\gamma) := \int_\gamma \sqrt{g_{ij}(t, \gamma(s))\dot\gamma^i(s)\dot\gamma^j(s)}ds \ .$$

We compute the time derivative:

$$\partial_t L^t(\gamma) = \int_\gamma \frac{1}{2|\dot\gamma|_t} \cdot (\partial_t g_{ij})\dot\gamma^i\dot\gamma^j ds = -\int_\gamma Sy(\dot\gamma, \dot\gamma) ds \ .$$

Consider the compact set Γ of smooth curves γ parametrized by arc length and having at most a finite but large length L:

$$\Gamma := \big\{\gamma \big| \gamma : [0, L^0(\gamma)] \to \Sigma, \gamma(0) = x_0, \gamma(L^0(\gamma)) = x_1, L^0(\gamma) \leq L\big\} \ .$$

Recalling that by definition $d_t(x_0, x_1) = \inf_{\gamma\in\Gamma} L^t(\gamma)$, we see that $-d_t(x_0, x_1) = \sup_{\gamma\in\Gamma}\big(-L^t(\gamma)\big)$. Applying Lemma 6.4 with $F(t) := -d_t(x_0, x_1)$, $f(t, \gamma) := -L^t(\gamma)$, and Γ as above, we thus conclude that

$$\inf\big\{-\partial_t L^t(\gamma)\big|\gamma \in \Gamma(t)\big\} \leq -\frac{d}{dt}\big(d_t(x_0, x_1)\big) \leq \sup\big\{-\partial_t L^t(\gamma)\big|\gamma \in \Gamma(t)\big\}$$

which is equivalent to

$$-\sup\left\{\int_\gamma Sy(\dot\gamma,\dot\gamma)ds\big|\gamma\in\Gamma(t)\right\}\le\frac{d}{dt}(d_t(x_0,x_1))\le-\inf\left\{\int_\gamma Sy(\dot\gamma,\dot\gamma)ds\big|\gamma\in\Gamma(t)\right\}\ .$$

Here $\Gamma(t)$ is the set of minimizing geodesics between x_0 and x_1 at time t. Now the estimates

$$\begin{aligned}-\int_\gamma Sy(\dot\gamma,\dot\gamma)ds&=-\int_\gamma Rc(\dot\gamma,\dot\gamma)ds+2\int_\gamma|du(\dot\gamma)|^2ds\ge-\sup_{[0,T]\times\Sigma}|Rc|\cdot L^t(\gamma)\\&=-\sup_{[0,T]\times\Sigma}|Rc|\cdot d_t(x_0,x_1)\ge-\Lambda\cdot d_t(x_0,x_1)\end{aligned}$$

and

$$-\int_\gamma Sy(\dot\gamma,\dot\gamma)ds=\le\sup_{[0,T]\times\Sigma}|Sy|\cdot d_t(x_0,x_1)\le\Lambda\cdot d_t(x_0,x)$$

for all $\gamma\in\Gamma(t)$ imply the desired result.

□

Proof: (of Lemma (6.3))
From the definition $r(t,x):=\frac{1}{2}d_t(x_0,x)^2$ we get $\nabla r(t,x)=d_t(x_0,x)\cdot\nabla d_t(x_0,x)$ where ∇ denotes the spatial gradient at time t. This implies (6.5):

$$|\nabla r(t,x)|=|d_t(x_0,x)|\cdot|\nabla d_t(x_0,x)|\le R\cdot 1\ .$$

To estimate Δr, we use the Laplacian Comparison Theorem: At each time $t\in[0,T]$ the Laplacian of the distance function can be estimated away from the cut locus as follows:

$$\Delta\big(d_t(x_0,x)\big)\le\frac{(n-1)\big(1+\kappa\cdot d_t(x_0,x)\big)}{d_t(x_0,x)}$$

whenever $\sup_{\bar B^t_R(x_0)}|Rc|\le(n-1)\kappa^2$ holds. This is proven in [SY94, Corollary 1.2]. We compute

$$\begin{aligned}\Delta\big(\tfrac{1}{2}d_t(x_0,x)^2\big)&=\nabla_k\big(d_t(x_0,x)\cdot\nabla_k d_t(x_0,x)\big)=|\nabla d_t(x_0,x)|^2+d_t(x_0,x)\cdot\Delta\big(d_t(x_0,x)\big)\\&\le1+(n-1)\big(1+\kappa\cdot d_t(x_0,x)\big)\le n+(n-1)\kappa\cdot d_t(x_0,x_1)\ ,\end{aligned}$$

proving (6.6). Finally, to prove (6.7), we use Corollary 6.5 and show

$$-\partial_t r(t,x)=-\partial_t\big(\tfrac{1}{2}d_t(x_0,x)^2\big)=d_t(x_0,x)\cdot(-\partial_t d_t(x_0,x))\le(n-1)\kappa^2\cdot d_t(x_0,x)^2\ .$$

This finishes the proof of Lemma 6.3.

□

Corollary 6.5 allows us to compare the distance of points at two different times:

Corollary 6.6 *Assume $(g,u)(t)$ is a solution to (2.5) on $[0,T]\times\Sigma$ where Σ is complete. Let $x_0,x_1\in\Sigma$ be two fixed points and $0\le t_1,t_2\le T$ be two arbitrary times. If $|Rc|\le\Lambda$ and $|Sy|\le\Omega$ hold along all minimizing geodesics γ connecting x_0 and x_1 at times $0\le t\le T$, then the distances at time t_1 and t_2 satisfy:*

$$e^{-\Lambda(t_2-t_1)}d_{t_1}(x_0,x_1)\le d_{t_2}(x_0,x_1)\le e^{\Omega(t_2-t_1)}d_{t_1}(x_0,x_1)\ .$$

Proof:
Given these conditions, we can apply Corollary 6.5 to get

$$-\Lambda \leq \frac{d}{dt}\big(\ln d_t(x_0, x_1)\big) \leq \Omega \ .$$

Integrating on $[t_1, t_2]$, we can estimate

$$-\Lambda(t_2 - t_1) \leq \ln d_{t_2}(x_0, x_1) - \ln d_{t_1}(x_0, x_1) \leq \Omega(t_2 - t_1) \ .$$

After exponentiation this is the desired result:

$$\exp\big(-\Lambda(t_2 - t_1) + \ln d_{t_1}(x_0, x_1)\big) \leq d_{t_2}(x_0, x_1) \leq \exp\big(\Omega(t_2 - t_1) + \ln d_{t_1}(x_0, x_1)\big) \ .$$

□

The corollary implies that the sets $B(\tau, x_0, R)$ are compact.

Corollary 6.7 *Let $x_0 \in \Sigma$ and $R > 0$ be fixed, and suppose $(g, u)(t)$ is a solution to (2.5) on $[0, T] \times \Sigma$ satisfying the bound $\sup_{B(T,x_0,R)} |Sy|^2 \leq \Lambda < \infty$. Then the set $B(T, x_0, R)$ is compact with respect to the manifold topology of $[0, T] \times \Sigma$.*

□

Since we want to prove local estimates, we need a cut-off function.

Lemma 6.8 *Let η be the cut-off function on $B(T, x_0, R)$ defined by*

$$\eta : B(T, x_0, R) \to \mathbb{R}^+, \qquad \eta(t, x) := (R^2 - r(t, x))^2 \ . \tag{6.8}$$

For all $\theta \in [0, 1)$, η has the properties:

$$\eta \leq R^4 \tag{6.9}$$
$$\eta^{-1} \leq (1 - \theta)^{-2} R^{-4} \qquad \text{on } B(T, \theta R, x_0) \tag{6.10}$$
$$|\nabla \eta|^2 \leq 4R^2 \cdot \eta \ . \tag{6.11}$$

Whenever $\sup_{B(T,x_0,R)} |Rc| \cdot R^2 \leq \tilde{C}$ holds, there is the additional estimate

$$\big(\partial_t - \Delta\big)\eta \leq C(n)\tilde{C} \cdot R^2 \tag{6.12}$$

for a scaling invariant constant $\tilde{C}$ and $C(n)$ depending only on n.

Proof:
(6.9) and (6.10) are immediate. To prove (6.11), we use (6.5):

$$\eta^{-1}|\nabla\eta|^2 = (R^2-r)^{-2}\cdot 4(R^2-r)^2|\nabla r|^2 = 4|\nabla r|^2 \le 4R^2 \ .$$

The Laplacian of η is given by

$$\Delta\eta = \nabla_k(-2(R^2-r)\nabla_k r) = 2|\nabla r|^2 - 2(R^2-r)\Delta r \ ,$$

and the time derivative by

$$\partial_t\eta = -2(R^2-r)\partial_t r \ .$$

Using (6.6) and (6.7), we therefore calculate

$$\begin{aligned}\big(\partial_t-\Delta\big)\eta &= -2(R^2-r)\partial_t r - 2|\nabla r|^2 + 2(R^2-r)\Delta r \le 2(R^2-r)\cdot\big(\Delta-\partial_t\big)r\\ &\le 2R^2\cdot\big(n+(n-1)\Big(\frac{\sqrt{\tilde C}}{\sqrt{n-1}R}R + \frac{\tilde C}{(n-1)R^2}\cdot R^2\Big)\big)\\ &\le 2R^2\cdot\big(C(n)+\tilde C\big) \le C(n)\tilde C R^2\end{aligned}$$

where the suprema are on $B(T,x_0,R)$, and we assumed without loss of generality that $\tilde C \ge 1$. This proves (6.12) as required.

□

6.2 Estimates for the Lapse function

The evolution equations for u and $|du|^2$ give us good control on the behavior of the logarithm of the Lapse function. We prove several a priori estimates.

Lemma 6.9 *Let $(g,u)(t)$ be a solution on $[0,T)\times M$ for closed M with initial data $(\tilde g,\tilde u)$. Then, for all $t\in(0,T]$, there are the following a priori estimates:*

$$\sup_{x\in M}|du|^2(t,x) \le \max_{x\in M}|d\tilde u|^2(x) \tag{6.13}$$

$$\sup_{x\in M}|du|^2(t,x) \le \tfrac14 t^{-1} \ . \tag{6.14}$$

Proof:
The first estimate follows straight from the maximum principle for subsolutions applied to the evolution equation (2.11)

$$\big(\partial_t-\Delta\big)|du|^2 \le 0 \ .$$

A closer look at (2.11) reveals

$$\partial_t\big(t\cdot|du|^2\big) \le 1\cdot|du|^2 + t\big(\Delta u - 2|\nabla^2 u|^2 - 4|du|^4\big) \le \Delta\big(t\cdot|du|^2\big) + t^{-1}\big(t|du|^2 - 4t^2|du|^4\big)$$

such that we get at the first point $(t_1,x_1)\in[0,\tau]\times M$, with $\tau<T$ arbitrary, where $f := t\cdot|du|^2$ attains its maximum:

$$0 \le t^{-1}\big(f-4f^2\big) \ .$$

This implies for $t > 0$

$$f\big(1 - 4f^2\big) \geq 0$$

which forces $f \leq \frac{1}{4}$ on $(0, T) \times M$, implying $|du|^2 \leq \frac{1}{4}t^{-1}$ independent of the initial data. In addition this yields a uniform bound on $[\frac{T}{2}, T)$ and therefore the claim for $t = T$.

□

Since we can estimate $|Rm|^2 + |\nabla^2 u|^2$ for solutions of (2.5) on complete Σ, it is possible to prove a priori bounds for the logarithm of the Lapse function $u(t)$ also on complete, noncompact manifolds. To this end we need a maximum principle for complete noncompact manifolds:

Theorem 6.10 *Let Σ be a complete Riemannian manifold and $(g, u)(t)$ be the solution of (2.5) on $[0, T] \times \Sigma$ constructed in Theorem 3.22 with $T < \infty$. Let f be a smooth function on $[0, T] \times \Sigma$. Assume furthermore that there is a vector field $a \in \mathcal{X}([0, T] \times \Sigma)$, and a function $b \in C^\infty([0, T] \times \Sigma)$ satisfying $\sup_{[0,T]\times\Sigma}\big(|a| + |b|\big) \leq \alpha$, and*

$$\begin{aligned} \big(\partial_t - \Delta\big) f &\leq a \cdot \nabla f + bf \\ f(0) &\leq 0 \qquad \text{on } \Sigma \\ |\nabla f|^2 &\leq \beta \qquad \text{on } [0, T] \times \Sigma \end{aligned}$$

hold for some numbers $\alpha, \beta < \infty$. Then $f \leq 0$ holds on $[0, T] \times \Sigma$.

Proof:
This is a specialization of the quite general maximum principle [EH91, Theorem 4.3]. Using the knowledge on solutions of (2.5), we can settle some of the assumptions there. At first we show two properties of the metric $g(t)$: Since we know that $|Rm|^2 + |du|^2 \leq c(n, k_0, c_0, s_0)$ on $[0, T] \times \Sigma$ from Theorem 3.22, we get

$$\sup_{[0,T]\times\Sigma} |\partial_t g_{ij}| \leq \sup_{[0,T]\times\Sigma} \big(2n|Rm| + 4|du|^2\big) \leq c < \infty \ .$$

Furthermore the curvature bound implies that $Rc \geq -(n-1)\kappa^2$ for some $\kappa > 0$. We get the desired volume growth estimate

$$|B^t_\rho(x_0)| \leq e^{c(1+\rho^2)} \tag{6.15}$$

for an arbitrary $x_0 \in \Sigma$ and all $\rho > 0$ from the Bishop volume comparison theorem for some constant $c = c(n, \kappa)$. A bound $|\nabla f|^2 \leq \beta$ implies the integral bound *(iii)* in [EH91, Theorem 4.3]. In particular we have for every fixed t

$$\begin{aligned} \int_\Sigma e^{-(c+2)d_t^2(x_0,x)}|\nabla f|^2 dV dr &\leq \beta \lim_{\rho\to\infty} \int_{B^t_\rho(x_0)} e^{-(c+2)d_t^2(x_0,x)} dV = \beta \lim_{\rho\to\infty} \int_0^\rho \int_{S^t_r(x_0)} e^{-(c+2)r^2} dA dr \\ &\leq \beta \lim_{\rho\to\infty} \int_0^\rho e^{-(c+2)r^2} |S^t_r(x_0)| dr \leq \beta \lim_{\rho\to\infty} \int_0^\rho e^{c+r^2(c-c-2)} dr \\ &\leq C(n,\kappa)\beta \lim_{\rho\to\infty} \big[-2e^{-2\rho^2} + 2\big] = 2C(n,\kappa)\beta \end{aligned}$$

where we used the volume estimate (6.15). Integrating in time, we therefore get

$$\int_0^T \int_\Sigma e^{-(c+2)d_t^2(x_0,x)} |\nabla f|^2 dV dt \leq C(n,\kappa)\beta T < \infty$$

as required. Now the maximum principle of Ecker and Huisken can be applied to f.

□

As an application we show that u is controlled completely by its initial value.

Lemma 6.11 *Let $(g,u)(t)$ be a solution on $[0,T)\times\Sigma$ where Σ is closed or complete and non-compact. Assume in the second case that $(g,u)(t)$ is the solution from Theorem 3.22. Then we have the following bounds*

$$\inf_{x\in\Sigma}\tilde{u}(x)\leq u(t,x)\leq\sup_{x\in\Sigma}\tilde{u}(x)$$

for all $t>0$ as long as the solution exists.

Proof:
The statement for closed Σ follows directly from the parabolic maximum principle applied to

$$\partial_t u=\Delta^g u\ .$$

In the complete case we want to apply the maximum principle Theorem 6.10 to the two functions $u_1(t,x):=u(t,x)-\sup_{x\in\Sigma}\tilde{u}(x)$ and $u_2(t,x):=\inf_{x\in\Sigma}\tilde{u}(x)-u(t,x)$. For $i=1,2$

$$\partial_t u_i=\partial_t u=\Delta^g u=\Delta^g u_i$$

holds. At time $t=0$ on Σ, we have $u_i\leq 0$ by definition. In addition there is the bound $|du|^2\leq C=C(n,k_0,c_0,s_0)$ from Theorem 3.22. Thus all the assumptions in Theorem 6.10 are satisfied, and the claim follows.

□

A second application of the maximum principle leads to the following time decay estimate for du which is similar to the estimate (6.14) in the closed case.

Lemma 6.12 *Let Σ be a complete Riemannian manifold, and $(g,u)(t)$ the solution to (2.5) on $[0,T]\times\Sigma$ constructed in Theorem 3.22 with initial data $(\tilde{g},\tilde{u})$. Then the derivative of u satisfies:*

$$\sup_{x\in\Sigma}|du|^2(t,x)\leq B\cdot t^{-1}$$

for $B:=\sup_{x\in\Sigma}|\tilde{u}|^2(x)$.

Proof:
We define a test function $f:=t|du|^2+|u|^2-B$. From (2.11) we find

$$\big(\partial_t-\Delta\big)f\leq|du|^2-2t|\nabla^2u|^2-4t|du|^4-2|du|^2\leq 0\ ,$$

and the initial data satisfies

$$f(0)=0\cdot|du|^2+|u|^2(0)-B=|\tilde{u}|^2-B\leq 0\ .$$

Considering the bound $|u|^2+|du|^2+|\nabla^2u|^2\leq C(n,k_0,c_0,s_0)$ from Theorem 3.22, we can apply Theorem 6.10 to conclude for all $t>0$

$$0\geq f(t)=t|du|^2+|u|^2-B\quad\Rightarrow\quad t|du|^2\leq B-|u|^2\leq B\ ,$$

proving the lemma.

□

In the next lemma, we proof a local bound on $|du|^2$. The technique is adapted from [EH91, §3] and goes back to [Shi89]. We will use the same ideas for the more complicated estimates to follow.

Lemma 6.13 *Let $(g,u)(t)$ be a solution to (2.5) on $[0,T)\times\Sigma$ where Σ is complete. Fix $x_0\in\Sigma$ and a radius $R>0$. If there is an estimate*

$$\sup_{B(T,x_0,R)} R^2|Rc| \le \tilde{C}\ ,$$

then for all $\theta\in[0,1)$ and all $t\in(0,T]$ there exists a constant $C(n)$, depending only on n, such that:

$$\sup_{x\in B^t_{\theta R}(x_0)} |du|^2(t,x) \le C(n)(1-\theta)^{-2}\tilde{C}\left(\frac{1}{R^2}+\frac{1}{t}\right)\ .$$

Proof:
Using the scaling function φ from (6.1), we define $f:=\varphi\cdot|du|^2$. A calculation from (2.11) and (6.2) shows that

$$\big(\partial_t-\Delta\big)f \le \varphi^{-1}(f-4f^2)\ .$$

We multiply by the cut-off function η defined in (6.8) and calculate on $B(T,x_0,R)$:

$$\big(\partial_t-\Delta\big)(f\eta) \le \varphi^{-1}(f-4f^2)\cdot\eta - 2\nabla\eta\nabla f + f\cdot C(n)\tilde{C}R^2$$

from (6.12), using the curvature bound. Rewriting the second term and applying (6.11)

$$-2\nabla\eta\nabla f = -2\eta^{-1}\nabla\eta\nabla(\eta f) + 2\eta^{-1}|\nabla\eta|^2 f \le -2\eta^{-1}\nabla\eta\nabla(\eta f) + 8R^2 f\ ,$$

we find

$$\big(\partial_t-\Delta\big)(f\eta) \le -4\varphi^{-1}f^2\eta + \varphi^{-1}f\eta - 2\eta^{-1}\nabla\eta\nabla(f\eta) + C(n)\tilde{C}R^2\cdot f + 8R^2\cdot f\ .$$

Fix $\tau\in[0,T)$. Since $\varphi(0)=0$, $\eta_{|\partial B^t_R(x_0)}=0$ for all $t\in[0,\tau]$ and $f\eta\ge 0$, the first maximum point (t^*,x^*) of $f\eta$ in the compact set $B(\tau,x_0,R)$ (compare Corollary 6.7) must be an interior point. Consequently, we have at this point

$$\partial_t(f\eta)\ge 0,\qquad \Delta(f\eta)\le 0,\qquad \nabla(f\eta)=0,$$

and obtain at (t^*,x^*), assuming $\tilde{C}\ge 1$:

$$4\varphi^{-1}f^2\eta \le \varphi^{-1}f\eta + C(n)\tilde{C}R^2\cdot f\ .$$

Using (6.9) on the right hand side and multiplying the equation by $\varphi\eta$, we find

$$4f^2\eta^2 \le f\eta\cdot R^4 + C(n)\tilde{C}R^2\varphi\cdot f\eta \le 3f^2\eta^2 + C(n)\tilde{C}^2R^8 \tag{6.16}$$

where the second estimate is due to Young's inequality and (6.3). Since (t^*, x^*) was a maximum point, we get

$$\sup_{B(\tau,x_0,R)} f\eta \leq C(n)\tilde{C}R^4 \ .$$

The estimate (6.10) for η^{-1} together with (6.16) implies that for any $\theta \in [0,1)$

$$\sup_{x\in B^t_{\theta R}(x_0)} |du|^2(t,x) \leq C(n)(1-\theta)^{-2}\tilde{C}\varphi^{-1}$$

holds, proving the lemma for all $0 < t < T$. Since the estimate is uniform on $[T/2, T)$, it also holds for $t = T$ completing the proof of Lemma 6.13.

□

We can localize Proposition 2.17 to find that local control on the Riemann tensor implies local control on the Hessian of u.

Proposition 6.14 *Let Σ be complete and $(g,u)(t)$ be a solution on $[0,T)\times\Sigma$. Suppose that for fixed $x_0 \in \Sigma$ and a fixed radius $R > 0$ there is a bound:*

$$\sup_{B(T,x_0,R)} R^4|Rm|^2 \leq \tilde{C}^2 \ . \tag{6.17}$$

Then for all $t \in (0,T]$ and $\theta \in [0,1)$ there is a constant $C(n)$, depending only on n, such that we have the estimate

$$\sup_{x\in B^t_{\theta R}(x_0)} |\nabla^2 u|^2(t,x) \leq C(n)(1-\theta)^{-2}\tilde{C}^2\left(\frac{1}{R^2}+\frac{1}{t}\right)^2 \ .$$

Proof:
We prove the theorem by using a trick of [Shi89, §7]. We combine the evolution inequalities (2.12) and (2.11) for $|\nabla^2 u|^2$ and $|du|^2$ in a clever way. To this end, define the test function $f := \varphi^2|\nabla^2 u|^2\big(\lambda + \varphi|du|^2\big)$ where the constant λ will be chosen later. In the following, C will denote a constant only depending on n that can change from line to line. Using (6.17), we get from Lemma 6.13 for all $\theta \in [0,1)$ the estimate

$$\sup_{B(T,x_0,\theta R)} \varphi|du|^2 \leq C(n)(1-\theta)^{-2}\tilde{C} \ . \tag{6.18}$$

The evolution of f is given by

$$\begin{aligned}\big(\partial_t-\Delta\big)f = \cdot\big(\partial_t-\Delta\big)(\varphi^2|\nabla^2 u|^2)\cdot(\lambda+\varphi|du|^2) - 2\varphi^2\nabla|\nabla^2 u|^2\cdot\varphi\nabla|du|^2 \\ +\,\varphi^2|\nabla^2 u|^2\cdot\big(\partial_t-\Delta\big)(\lambda+\varphi|du|^2)\ ,\end{aligned} \tag{6.19}$$

and we compute for the first term on $B(T,x_0,\theta R)$

$$\begin{aligned}\big(\partial_t-\Delta\big)(\varphi^2|\nabla^2 u|^2) &\leq -2\varphi^2|\nabla^3 u|^2 + C\varphi^{-1}\cdot\varphi^2|\nabla^2 u|^2(1+\varphi|Rm|+\varphi|du|^2)\\ &\leq -2\varphi^2|\nabla^3 u|^2 + C\varphi^{-1}\tilde{C}\cdot\varphi^2|\nabla^2 u|^2\end{aligned}$$

from (6.17) and (6.18). Note that we can estimate

$$\varphi|Rm| = \frac{R^2 t}{R^2+t}|Rm| = R^2|Rm| \cdot \frac{t}{R^2+t} \le \tilde{C} \cdot 1 \ . \tag{6.20}$$

Multiplying by $(\lambda + \varphi|du|^2)$, we get

$$\big(\partial_t - \Delta\big)(\varphi^2|\nabla^2 u|^2) \cdot (\lambda + \varphi|du|^2) \le -2\varphi^2|\nabla^3 u|^2 \cdot (\lambda + \varphi|du|^2) + C\varphi^{-1}\tilde{C} \cdot f \ .$$

For the last term in (6.19) we compute

$$\big(\partial_t - \Delta\big)(\lambda + \varphi|du|^2) \le -2\varphi|\nabla^2 u|^2 + C|du|^2 \le -2\varphi|\nabla^2 u|^2 + C\varphi^{-1} \cdot (\lambda + \varphi|du|^2) \ ,$$

such that multiplication by $\varphi^2|\nabla^2 u|^2$ gives

$$\big(\partial_t - \Delta\big)(\lambda + \varphi|du|^2) \cdot \varphi^2|\nabla^2 u|^2 \le -2\varphi^3|\nabla^2 u|^4 + C\varphi^{-1} \cdot f \ .$$

The cross term in (6.19) can be estimated using Kato's inequality $\big|\nabla|(\cdot)|\big| \le |\nabla(\cdot)|$ as follows:

$$\begin{aligned} -2\varphi^2\nabla|\nabla^2 u|^2 \cdot \varphi\nabla|du|^2 &\le 8\varphi^2|\nabla^2 u||\nabla^3 u| \cdot \varphi|du||\nabla^2 u| \\ &\le 2\varphi|\nabla^3 u|(\lambda + \varphi|du|^2)^{\frac{1}{2}} \cdot 4(\lambda + \varphi|du|^2)^{-\frac{1}{2}}\varphi^{\frac{1}{2}}|du|\varphi^{\frac{3}{2}}|\nabla^2 u|^2 \\ &\le 2\varphi^2|\nabla^3 u|^2 \cdot (\lambda + \varphi|du|^2) + \frac{8\varphi|du|^2}{\lambda + \varphi|du|^2}\varphi^3|\nabla^2 u|^4 \ . \end{aligned}$$

We choose $\lambda := 7\tilde{C} \ge 7\varphi|du|^2$ and compute

$$\frac{8\varphi|du|^2}{\lambda + \varphi|du|^2} - 2 \le \frac{8\varphi|du|^2}{8\varphi|du|^2} - 2 \le 1 - 2 \le -1 \ .$$

This simplifies (6.19) to

$$\big(\partial_t - \Delta\big) f \le -\varphi^{-1}(\lambda + \varphi|du|^2)^{-2} \cdot f^2 + C\tilde{C}\varphi^{-1} \cdot f \ ,$$

assuming $\tilde{C} \ge 1$ without loss of generality. We estimate further

$$-(\lambda + \varphi|du|^2)^{-2} \le -\tilde{C}^{-2}$$

and find using Young's inequality:

$$\big(\partial_t - \Delta\big) f \le -\tfrac{1}{2}\varphi^{-1}\tilde{C}^{-2} \cdot f^2 + C\tilde{C}^4\varphi^{-1} \ .$$

The product of f and the cut-off function η from (6.8) satisfies

$$\big(\partial_t - \Delta\big)(f\eta) \le -\tfrac{1}{2}\varphi^{-1}\tilde{C}^{-2} \cdot f^2\eta + C\tilde{C}^4\varphi^{-1}\eta - 2\eta^{-1}\nabla\eta\nabla(f\eta) + C\tilde{C}R^2 \cdot f \ .$$

At the first maximum point of $f\eta$ on $B(\tau, x_0, R)$, one has similarly to Lemma 6.13

$$2f^2\eta^2 \le C\tilde{C}^6\eta^2 + C\tilde{C}^3R^2\varphi \cdot f\eta \le f^2\eta^2 + C\tilde{C}^6R^8$$

again using (6.3), (6.9) and Young's inequality. As before, this leads to

$$\sup_{B(\tau,x_0,\theta R)} \varphi^2|\nabla^2 u|^2 \le C(1-\theta)^{-2}\tilde{C}^3(\lambda + \varphi|du|^2)^{-1} \ ,$$

and we can estimate $\tilde{C}(\lambda + \varphi|du|^2)^{-1} \le \tilde{C}/\lambda = \frac{1}{7}$. The remainder of the proof is analogous to the proof of Lemma 6.13.

□

6.3 Interior a priori estimates

Having done all necessary preparations, we finally prove local a priori estimates for solutions with bounded curvature. To this end, we estimate the derivatives of Φ where Φ is defined as in (2.20), giving an explicit dependence of the result on the initial curvature bound. This constitutes a regularity theory for the solutions of (2.5) in the sense that solutions with bounded curvature are always smooth.

Theorem 6.15 *Let $(\Sigma, \tilde{g})$ be a complete Riemannian manifold. Suppose that $(g, u)(t)$ is a solution on $[0, T) \times \Sigma$ satisfying*

$$\sup_{B(T,x_0,R)} R^4 |Rm|^2 \le \tilde{C}^2 \tag{6.21}$$

for some radius $R > 0$ and some point $x_0 \in \Sigma$. Then the derivatives of Φ satisfy for all $m \ge 0$ and for all $t \in (0, T]$ the estimates

$$\sup_{x \in B^t_{R/2}(x_0)} |\nabla^m \Phi|^2(t, x) \le C(n, m)\tilde{C}^{m+2} \left(\frac{1}{R^2} + \frac{1}{t} \right)^{m+2}$$

where $C = C(n, m)$ is a constant depending only on n, and m.

Proof:
The proof is an induction argument, using similar techniques as in the proof of Proposition 6.14. The curvature bound (6.21) together with Lemma 6.13 provides the estimate

$$\sup_{B(T,x_0,\theta_{-1}R)} \varphi |du|^2 \le C\tilde{C}$$

for $\theta_{-1} := \frac{5}{6}$. From Proposition 6.14 we know, using (6.21) in combination with (6.20), and setting $\theta_0 := \frac{3}{4}$, that

$$\sup_{B(T,x_0,\theta_0 R)} \varphi^2 |\Phi|^2 = \sup_{B(T,x_0,\theta_0 R)} \varphi^2 \big(|Rm|^2 + |\nabla^2 u|^2 \big) \le \tilde{C}^2 \big(1 + C(1 - \theta_0)^{-2} \big) \le C\tilde{C}^2$$

holds where C depends only on n. This proves the theorem in the case where $m = 0$.

In the following C denotes a constant depending only on n and m which can change its value from line to line. In the induction step we assume that

$$\sup_{x \in B^t_{\theta_s R}(x_0)} \varphi^{s+2} |\nabla^s \Phi|^2(t, x) \le C(1 - \theta_s)^{-2} \tilde{C}^{s+2} \le C\tilde{C}^{s+2} \tag{6.22}$$

holds for all $t \in (0, T]$ and all $0 \le s \le m$. The choice of $\theta_s := \frac{1}{2} + \frac{1}{s+4}$ guarantees that $\frac{1}{2} < \theta_s < 1$ is true for all s and $\theta_i > \theta_j$ for all $i < j$. We assume without loss of generality that $\tilde{C} \ge 1$ in the following.

To prove the estimate for $s = m + 1$, we define a test function

$$f(t, x) := \varphi^{m+3}(t) |\nabla^{m+1} \Phi|^2(t, x) \big(\lambda + \varphi^{m+2}(t) |\nabla^m \Phi|^2(t, x) \big)$$

where λ is a constant that will be chosen later. The evolution of f is given by

$$\begin{aligned}\big(\partial_t - \Delta\big) f = \big(\partial_t - \Delta\big)(\varphi^{m+3}|\nabla^{m+1}\Phi|^2) \cdot (\lambda + \varphi^{m+2}|\nabla^m\Phi|^2) \\ + \varphi^{m+3}|\nabla^{m+1}\Phi|^2 \cdot \big(\partial_t - \Delta\big)(\varphi^{m+2}|\nabla^m\Phi|^2) - 2\varphi^{m+3}\nabla|\nabla^{m+1}\Phi|^2\varphi^{m+2}\nabla|\nabla^m\Phi|^2 \ .\end{aligned} \tag{6.23}$$

We want to estimate the individual terms on $B^t_{\theta_{m+1}R}(x_0)$, using the estimate for the derivatives of Φ from Lemma 2.23 and the evolution equation (6.2) for powers of φ. We start with

$$\begin{aligned}&\big(\partial_t - \Delta\big)(\varphi^{m+3}|\nabla^{m+1}\Phi|^2) \\ &\quad \le (m+3)\varphi^{m+2} \cdot |\nabla^{m+1}\Phi|^2 - 2\varphi^{m+3}|\nabla^{m+2}\Phi|^2 \\ &\quad\quad + C\varphi^{m+3}\Big\{ \sum_{\alpha+\beta=m+1} |\nabla^\alpha\Phi||\nabla^\beta\Phi||\nabla^{m+1}\Phi| + \sum_{\alpha+\beta=m} |du||\nabla^\alpha\Phi||\nabla^\beta\Phi||\nabla^{m+1}\Phi| \\ &\quad\quad\quad + \sum_{\alpha+\beta+\gamma=m-1} |\nabla^\alpha\Phi|\nabla^\beta\Phi||\nabla^\gamma\Phi||\nabla^{m+1}\Phi| + |du|^2|\nabla^{m+1}\Phi|^2\Big\} \\ &\quad \le -2\varphi^{m+3}|\nabla^{m+2}\Phi|^2 + C\varphi^{-1}\cdot\varphi^{m+3}|\nabla^{m+1}\Phi|^2\big(1 + \varphi|\Phi| + \varphi|du|^2\big) \\ &\quad\quad + C\varphi^{-1} \sum_{\substack{\alpha+\beta=m+1\\ \alpha>0,\beta>0}} \varphi^{\frac{\alpha+1}{2}}|\nabla^\alpha\Phi|^{\frac{\alpha+1}{\alpha+2}}\varphi^{\frac{\beta+2}{2}}|\nabla^\beta\Phi|\cdot\varphi^{\frac12}|\nabla^\alpha\Phi|^{\frac{1}{\alpha+2}}\varphi^{\frac{m+3}{2}}|\nabla^{m+1}\Phi| \\ &\quad\quad + C\varphi^{-1}\sum_{\alpha+\beta=m}\varphi^{\frac{\alpha+2}{2}}|\nabla^\alpha\Phi|\varphi^{\frac{\beta+2}{2}}|\nabla^\beta\Phi|\cdot\varphi^{\frac12}|du|\varphi^{\frac{m+3}{2}}|\nabla^{m+1}\Phi| \\ &\quad\quad + C\varphi^{-1}\sum_{\alpha+\beta+\gamma=m-1}\varphi^{\frac{\alpha+1}{2}}|\nabla^\alpha\Phi|^{\frac{\alpha+1}{\alpha+2}}\varphi^{\frac{\beta+2}{2}}|\nabla^\beta\Phi|\varphi^{\frac{\gamma+2}{2}}|\nabla^\gamma\Phi|\cdot\varphi^{\frac12}|\nabla^\alpha\Psi|^{\frac{1}{\alpha+2}}|\varphi^{\frac{m+3}{2}}|\nabla^{m+1}\Phi| \ .\end{aligned}$$

Having paired the correct powers of the scaling function with the derivatives of Φ, we can apply the induction hypotheses (6.22) and find

$$\begin{aligned}&\big(\partial_t - \Delta\big)(\varphi^{m+3}|\nabla^{m+1}\Phi|^2) \\ &\quad \le -2\varphi^{m+3}|\nabla^{m+2}\Phi|^2 + C\varphi^{-1}\cdot\varphi^{m+3}|\nabla^{m+1}\Phi|^2(1+\tilde C) \\ &\quad\quad + C\varphi^{-1}\Big\{\sum_{\substack{\alpha+\beta=m+1\\ \alpha>0,\beta>0}}\tilde C^{\alpha+1}\tilde C^{\beta+2} + \sum_{\alpha+\beta=m}\tilde C^{\alpha+2}\tilde C^{\beta+2} + \sum_{\alpha+\beta+\gamma=m-1}\tilde C^{\alpha+1}\tilde C^{\beta+2}\tilde C^{\gamma+2}\Big\} \\ &\quad \le -2\varphi^{m+3}|\nabla^{m+2}\Phi|^2 + C\tilde C\varphi^{-1}\cdot\varphi^{m+3}|\nabla^{m+1}\Phi|^2 + C\tilde C^{m+4}\varphi^{-1}\end{aligned}$$

where we used Young's inequality and $\tilde C \ge 1$. Multiplication by $(\lambda + \varphi^{m+2}|\nabla^m\Phi|^2)$ gives

$$\begin{aligned}\big(\partial_t - \Delta\big)|\nabla^{m+3}\Phi|^2\cdot(\lambda + \varphi^{m+2}|\nabla^m\Phi|^2) \le -2\varphi^{m+3}|\nabla^{m+2}\Phi|^2(\lambda + \varphi^{m+2}|\nabla^m\Phi|^2) + C\tilde C\varphi^{-1}f \\ + C\tilde C^{m+4}\varphi^{-1}(\lambda + \varphi^{m+2}|\nabla^m\Phi|^2) \ .\end{aligned}$$

The second term in (6.23) can be estimated as follows:

$$
\begin{aligned}
&\big(\partial_t - \Delta\big)(\lambda + \varphi^{m+2}|\nabla^m\Phi|^2) \\
&\quad \le (m+2)\varphi^{m+1}|\nabla^m\Phi|^2 - 2\varphi^{m+2}|\nabla^{m+1}\Phi|^2 \\
&\qquad + C\varphi^{-1}\Big\{ \sum_{\alpha+\beta=m} \varphi^{\frac{\alpha+1}{2}}|\nabla^\alpha\Phi|^{\frac{\alpha+1}{\alpha+2}}\varphi^{\frac{\beta+2}{2}}|\nabla^\beta\Phi| \cdot \varphi^{\frac{1}{2}}|\nabla^\alpha\Phi|^{\frac{1}{\alpha+2}}\varphi^{\frac{m+2}{2}}|\nabla^m\Phi| \\
&\qquad\qquad + \sum_{\alpha+\beta=m-1} \varphi^{\frac{\alpha+2}{2}}|\nabla^\alpha\Phi|\varphi^{\frac{\beta+2}{2}}|\nabla^\beta\Phi| \cdot \varphi^{\frac{1}{2}}|du|\varphi^{\frac{m+2}{2}}|\nabla^m\Phi| \\
&\qquad\qquad + \sum_{\alpha+\beta+\gamma=m-2} \varphi^{\frac{\alpha+1}{2}}|\nabla^\alpha\Phi|^{\frac{\alpha+1}{\alpha+2}}\varphi^{\frac{\beta+2}{2}}|\nabla^\beta\Phi|\varphi^{\frac{\gamma+2}{2}}|\nabla^\gamma\Phi| \cdot \varphi^{\frac{1}{2}}|\nabla^\alpha\Phi|^{\frac{1}{\alpha+2}}\varphi^{\frac{m+2}{2}}|\nabla^m\Phi| \\
&\qquad\qquad + \varphi|du|^2\varphi^{m+2}|\nabla^m\Phi|^2\Big\} \\
&\quad \le -2\varphi^{m+2}|\nabla^{m+1}\Phi|^2 + C\varphi^{-1}\cdot\varphi^{m+2}|\nabla^m\Phi|^2\big(1+\tilde{C}\big) \\
&\qquad + C\varphi^{-1}\Big\{ \sum_{\alpha+\beta=m} \tilde{C}^{\alpha+1}\tilde{C}^{\beta+2} + \sum_{\alpha+\beta=m-1} \tilde{C}^{\alpha+2}\tilde{C}^{\beta+2} + \sum_{\alpha+\beta+\gamma=m-2} \tilde{C}^{\alpha+1}\tilde{C}^{\beta+2}\tilde{C}^{\gamma+2}\Big\} \\
&\quad \le -2\varphi^{m+2}|\nabla^{m+1}\Phi|^2 + C\varphi^{-1}\tilde{C}\varphi^{m+2}|\nabla^m\Phi|^2 + C\varphi^{-1}\tilde{C}^{m+3},
\end{aligned}
$$

and we get for the product

$$
\begin{aligned}
\big(\partial_t - \Delta\big)&(\lambda + \varphi^{m+2}|\nabla^m\Phi|^2)\cdot\varphi^{m+3}|\nabla^{m+1}\Phi|^2 \\
&\le -2\varphi^{2m+5}|\nabla^{m+1}\Phi|^4 + C\varphi^{-1}\tilde{C}f + C\varphi^{-1}\tilde{C}^{m+3}\varphi^{m+3}|\nabla^{m+1}\Phi|^2 \\
&\le -\tfrac{3}{2}\varphi^{2m+5}|\nabla^{m+1}\Phi|^4 + C\varphi^{-1}\tilde{C}f + C\varphi^{-1}\tilde{C}^{2(m+3)} .
\end{aligned}
$$

Here we used Young's inequality in the following way:

$$
\varphi^{-1}(C\tilde{C}^{m+3}\cdot\varphi^{m+3}|\nabla^{m+1}\Phi|^2) \le \varphi^{-1}\cdot\tfrac{1}{2}\varphi^{2m+6}|\nabla^{m+1}\Phi|^4 + \varphi^{-1}C\tilde{C}^{2(m+3)} .
$$

The cross term in (6.23) is controlled in the same way as in Proposition 6.14:

$$
\begin{aligned}
-2\varphi^{m+3}&\nabla|\nabla^{m+1}\Phi|^2\varphi^{m+2}\nabla|\nabla^m\Phi|^2 \\
&\le 2\varphi^{m+3}|\nabla^{m+2}\Phi|^2(\lambda + \varphi^{m+2}|\nabla^m\Phi|^2) + \frac{8\varphi^{m+2}|\nabla^m\Phi|^2}{\lambda + \varphi^{m+2}|\nabla^m\Phi|^2}\cdot\varphi^{2m+5}|\nabla^{m+1}\Phi|^4 .
\end{aligned}
$$

Altogether (6.23) comes down to

$$
\begin{aligned}
\big(\partial_t - \Delta\big)f \le \varphi^{2m+5}|\nabla^{m+1}\Phi|^4\left(\frac{8\varphi^{m+2}|\nabla^m\Phi|^2}{\lambda + \varphi^{m+2}|\nabla^m\Phi|^2} - \frac{3}{2}\right) + C\tilde{C}\varphi^{-1}f \\
+ C\varphi^{-1}\tilde{C}^{m+4}(\lambda + \varphi^{m+2}|\nabla^m\Phi|^2) + C\varphi^{-1}\tilde{C}^{2(m+3)} .
\end{aligned}
$$

We choose $\lambda := 7\tilde{C}^{m+2} \ge 7\varphi^{m+2}|\nabla^m\Psi|^2 \ge 1$ and compute

$$
\begin{aligned}
\frac{8\varphi^{m+2}|\nabla^{m+2}\Phi|^2}{\lambda + \varphi^{m+2}|\nabla^m\Phi|^2} - \frac{3}{2} &\le 8\tilde{C}^{m+2}\left(7\tilde{C}^{m+2} + \tilde{C}^{m+2}\right)^{-1} - \frac{3}{2} = 1 - \frac{3}{2} = -\frac{1}{2} \\
(\lambda + \varphi^{m+2}|\nabla^m\Phi|^2) &\le 8\tilde{C}^{m+2} .
\end{aligned}
$$

This simplifies the above equation to

$$\big(\partial_t - \Delta\big) f \leq -\tfrac{1}{2}\varphi^{2m+5}|\nabla^{m+1}\Phi|^4 + C\tilde{C}\varphi^{-1} f + C\varphi^{-1}\tilde{C}^{2(m+3)} \ . \tag{6.24}$$

Estimating

$$-(\lambda + \varphi^{m+2}|\nabla^m\Phi|^2)^{-2} \leq -(8\tilde{C}^{m+2})^{-2} = -\tfrac{1}{64}\tilde{C}^{-2(m+2)},$$

we can complete the first term in (6.24) to f^2:

$$\big(\partial_t - \Delta\big) f \leq -\tfrac{1}{128}\varphi^{-1}\tilde{C}^{-2(m+2)} f^2 + C\tilde{C}\varphi^{-1} f + C\varphi^{-1}\tilde{C}^{2(m+3)} \ .$$

Applying Young's inequality

$$\varphi^{-1} C\tilde{C} f \leq \tfrac{1}{256}\varphi^{-1}\tilde{C}^{-2(m+2)} f^2 + C\tilde{C}^{2(m+3)}\varphi^{-1}$$

to the second term in (6.24), we conclude that

$$\big(\partial_t - \Delta\big) f \leq -\tfrac{1}{256}\tilde{C}^{-(m+2)}\varphi^{-1} f^2 + C\tilde{C}^{m+3}\varphi^{-1} \ .$$

To localize this estimate, we multiply f by η defined in (6.8) and get on the ball $B^t_{\theta_{m+1}R}(x_0)$:

$$\big(\partial_t - \Delta\big)(f\eta) \leq -\tfrac{1}{256}\tilde{C}^{-2(m+2)}\varphi^{-1} f^2\eta + C\tilde{C}^{2(m+3)}\varphi^{-1}\eta - 2\eta^{-1}\nabla\eta\nabla(\eta f) + C(1+\tilde{C})R^2\cdot f \tag{6.25}$$

where we used the evolution equation (6.12) for η. From the definition of φ and η we conclude that the function $f\eta$ attains its maximum on the compact set $B(\tau, x_0, \theta_{m+1}R)$ for an arbitrary $\tau < T$ in an interior point (t^*, x^*). At this point, we therefore have

$$\nabla(f\eta) = 0, \qquad \big(\partial_t - \Delta\big)(f\eta) \geq 0 \ .$$

Putting this into (6.25), we get with $\tilde{C} \geq 1$ that

$$0 \leq -\tfrac{1}{256}\tilde{C}^{-2(m+2)}\varphi^{-1} f^2\eta + C\tilde{C}^{2(m+3)}\varphi^{-1}\eta + C\tilde{C}R^2 f \ .$$

Multiplying by $256\tilde{C}^{2(m+2)}\varphi\eta$ and using (6.3) and (6.9), we get

$$f^2\eta^2 \leq C\tilde{C}^{4m+10}R^8 + C\tilde{C}^{2m+5}R^4 f\eta \ .$$

Estimating $C\tilde{C}^{2m+5}R^4 \cdot f\eta \leq \frac{1}{2}f^2\eta^2 + \frac{1}{2}C\tilde{C}^{4m+10}R^8$, we conclude

$$f\eta \leq C\tilde{C}^{2m+5}R^4 \ .$$

By choice of θ_{m+1}, (6.10) implies $\eta^{-1} \leq (1-\theta_{m+1})^{-2}R^{-4} \leq CR^{-4}$ on $B^t_{\theta_{m+1}R}(x_0)$, and we therefore get

$$|\nabla^{m+1}\Phi|^2 \leq C\tilde{C}^{2m+5}\big(\lambda + \varphi^{m+2}|\nabla^m\Phi|^2\big)^{-1}\varphi^{-(m+3)} \leq C\tilde{C}^{2m+5}(7\tilde{C}^{m+2})^{-1}\varphi^{-(m+3)} \ .$$

Plugging in the expression for φ^{-1} from (6.4), we finally arrive at

$$|\nabla^{m+1}\Phi|^2(t,x) \leq C\tilde{C}^{m+3}\left(\frac{1}{R^2} + \frac{1}{t}\right)^{m+3}$$

for all $(t,x) \in B(\tau, x_0, \theta_{m+1}R)$ since (t^*, x^*) was maximal in $B(\tau, x_0, \theta_{m+1}R)$. Since $\tau \in [0,T)$ is arbitrary, this provides a uniform estimate for all $t \in [\frac{T}{2}, T)$:

$$\sup_{x \in B^t_{\theta_{m+1}R}(x_0)} |\nabla^{m+1}\Phi|^2(t,x) \leq C\tilde{C}^{m+3}\left(\frac{1}{R^2} + \frac{1}{T}\right)^{m+3},$$

and we can conclude that the same estimate is valid for $t = T$. This proves the induction step, and since $\theta_m > \frac{1}{2}$ for all $m \geq 0$, it also proves the theorem.

□

A version with scaling dependent bounds is given as follows:

Corollary 6.16 *Let $(\Sigma, \tilde{g})$ be a complete Riemannian manifold. Suppose that $(g,u)(t)$ is a solution on $[0,T) \times \Sigma$ satisfying*

$$\sup_{B(T,x_0,R)} |Rm|^2 \leq k_0^2$$

for a radius $R > 0$ and some point $x_0 \in \Sigma$. Assume furthermore that there is a constant $c > 1$ such that $R \leq c\sqrt{T}$. Then the derivatives of Φ satisfy for all $m \geq 0$ and for all $t \in (0,T]$ the estimates

$$\sup_{x \in B^t_{R/2}(x_0)} |\nabla^m\Phi|^2(t,x) \leq Ck_0^{m+2}$$

where $C = C(n,m,c)$ is a constant depending only on n, m, and c.

Proof:
From a bound $|Rm|^2 \leq k_0^2$ we get $R^4 \cdot |Rm|^2 \leq R^4k_0^2 =: \tilde{C}^2$. Applying Theorem 6.15, we conclude for all $t \in (0,T]$:

$$\begin{aligned}\sup_{B^t_{R/2}(x_0)} |\nabla^m\Phi|^2 &\leq C(n,m)\tilde{C}^{m+2}\left(\frac{1}{R^2} + \frac{1}{T}\right)^{m+2} \\ &\leq C(n,m)k_0^{m+2}R^{2(m+2)} \cdot (2c)^{m+2}R^{-2(m+2)} = C(n,m,c)k_0^{m+2}\end{aligned}$$

since we can estimate $\varphi^{-1} \leq 2cR^{-2}$.

□

A slightly modified version of Theorem 6.15 deduces a bound on $|\nabla^{m+1}\Phi|^2$ from bounds on all derivatives of smaller order. Here the dependencies are not as explicit as above.

Theorem 6.17 *Let $(\Sigma, \tilde{g})$ be a complete Riemannian manifold and $m \geq 0$ a fixed number. Suppose $(g,u)(t)$ is a solution to (2.5) on $[0,T) \times \Sigma$ which satisfies*

$$\sup_{B(T,x_0,R)} R^4|Rm|^2 \leq \tilde{C_0}^2,$$

and for $k = 1 \ldots m$:

$$\sup_{B(T,x_0,R)} \varphi^{k+2}|\nabla^k\Phi|^2 \leq \tilde{C}_k^2$$

for a radius $R > 0$ around some point $x_0 \in \Sigma$. Then $\nabla^{m+1}\Phi$ can be estimated as follows

$$\sup_{x \in B^t_{\theta R}(x_0)} |\nabla^{m+1}\Phi|^2(t,x) \leq C(1-\theta)^{-2}\left(\frac{1}{R^2}+\frac{1}{t}\right)^{m+3}$$

for all $t \in (0,T]$ and all $\theta \in [0,1)$ where C is a constant depending only on n, m and $\tilde{C}_0, \ldots, \tilde{C}_m$.

Proof:
The proof is the same as for the induction step in the proof of Theorem 6.15.

□

6.4 Long time existence

As an application of the interior estimates we prove a characterization of long time existence for solutions of (2.5). To this end, we first deduce some general properties for solutions on compact manifolds.

Proposition 6.18 *Let $(g,u)(t)$ be a solution to (2.5) on $[0,T) \times M$ where M is compact. Define $K(t) := \sup_{x \in M} |Rm|(t,x)$. Then there exists a constant $c = c(n)$ such that we get for all $0 \leq t \leq \min\{T, c/K(0)\}$ that:*

$$K(t) \leq 2K(0)\ .$$

Proof:
From the evolution equation (2.17) we have:

$$\partial_t |Rm|^2 \leq \Delta |Rm|^2 + C|Rm|^3 + C|Rm||\nabla^2 u|^2 + C|du|^2|Rm|^2\ .$$

Since $K(t) = \sup_{x \in M} |Rm|^2(t,x)$ is Lipschitz continuous, we can define its derivative in the sense of difference quotients as described in Lemma 6.4. A computation shows

$$\frac{d}{dt}K(t)^2 \leq CK(t)^3$$

since we can estimate $|du|^2(t) \leq C\sup_M |Rm|(t) = CK(t)$ from Lemma 6.13 and $|\nabla^2 u|^2(t) \leq C\sup_M |Rm|^2(t) = CK^2(t)$ from Proposition 6.14. Simplifying, we get

$$\frac{d}{dt}K(t) \leq CK(t)^3 \cdot \tfrac{1}{2}K(t)^{-1} = \tfrac{1}{2}CK(t)^2$$

where C depends only on n. Solving the associated ordinary differential equation, we find

$$K(t) \leq \left(\frac{1}{K(0)} - \frac{C}{2}t\right)^{-1}$$

for $t < \frac{2}{CK(0)}$. Taking $c = 1/C$ yields the claim.

□

Corollary 6.19 *Let $(g,u)(t)$ be a solution to (2.5) on $[0,T)\times M$ where M is compact and T is the maximal time of existence. Define $k_0 := \max_{x\in M}|Rm|^2(0,x)$. Then there exists a constant $c = c(n)$, depending only on n, such that $T > c/\sqrt{k_0}$.*

Proof:
From Proposition 6.18 we know that the solution has bounded curvature on $[0, c/\sqrt{k_0}]$. This implies the smoothness of $(g,u)(t)$ up to that time using Corollary 6.16.

□

Corollary 6.20 *Let $(g,u)(t)$ be a solution to (2.5) on $[0,T]\times\Sigma$ with $k_0 := \sup_\Sigma |Rm|^2(0) < \infty$ and $c_0 := \sup_\Sigma |du|^2(0) < \infty$ where T is the maximal time of existence. Then we have $T > c/(\sqrt{k_0}+c_0)$ for a constant $c = c(n)$ depending only on the dimension.*

Proof:
This was proven in Theorem 3.22 and Corollary 6.19.

□

For the proof of the long time existence result, we need a technical lemma.

Lemma 6.21 *Let $(g,u)(t)$ be a solution to (2.5) on $[0,T)\times M$ with initial data $(\tilde g,\tilde u)$ where M is closed. Suppose there is a uniform curvature bound $|Rm|\le k_0$ on $[0,T)\times M$. Then for any fixed metric $\bar g$ with connection $\bar\nabla$ on M, we have for all $t\in[0,T)$ and all $m\ge 1$:*

$$\sup_{x\in M}\left(|\bar\nabla^m\Psi|^2_{\bar g} + |\bar\nabla^m Rm|^2_{\bar g}\right)(t,x) \le C(n,m,k_0,T,\tilde g,\tilde u)$$

where $\Psi = (\bar\nabla g, \bar\nabla u)$ is defined in (3.23).

Proof:
We work in normal coordinates for $\bar g$ such that $\bar\Gamma = 0$ at the base point. In particular we assume that $\Gamma = \Gamma - \bar\Gamma$ is a tensor in this chart. From Lemma 2.8 we know that the metrics $g(t)$ are uniformly equivalent for $(t,x)\in[0,T)\times M$, giving us

$$c^{-1}\bar g(x) \le g(t,x) \le c\bar g(x) \tag{6.26}$$

for some $c = c(k_0,T,\tilde u)$ only depending on k_0 and T and an initial bound on $d\tilde u$. In the following $C = C(n,m,k_0,\tilde g,\tilde u,T)$ denotes a constant only depending on n,m, the curvature bound k_0, the initial data $\tilde g,\tilde u$, and the final time T. Applying the interior estimates from Corollary 6.16, we get uniform bounds for $t\in[0,T)$ and all $k\ge 0$:

$$\sup_{x\in M}\left(|\nabla^k Rm|^2 + |\nabla^{2+k}u|^2\right)(t,x) \le C(n,k,k_0)\ .$$

Using (6.13), we get an estimate for $|du|^2$:

$$\sup_{x\in M}|\bar\nabla u|^2_{\bar g}(t,x) \le C\sup_{x\in M}|du|^2(t,x) \le C \tag{6.27}$$

for all $t \in [0, T)$. In addition, the curvature bound and Proposition 2.17 imply the following estimate for the Hessian:

$$|\bar{\nabla}^2 u|_{\bar{g}} \le C|\bar{\nabla}^2 u| \le C|\nabla^2 u| + C|\bar{\nabla} g||du| \le C + C|\bar{\nabla} g|$$

since in these coordinates

$$\bar{\nabla}_i \bar{\nabla}_j u = \partial_i \partial_j u = \nabla_i \nabla_j u + \Gamma_{ij}^k \partial_k u = \nabla_i \nabla_j u + g^{kl}\big(\bar{\nabla}_i g_{jl} + \bar{\nabla}_j g_{il} - \bar{\nabla}_l g_{ij}\big)\partial_k u \; . \tag{6.28}$$

This allows us to estimate

$$|\partial_t \bar{\nabla} g| = |\bar{\nabla} \partial_t g| \le C(|\bar{\nabla} Rm| + |du||\bar{\nabla}^2 u|) \le C|\nabla Rm| + C|\Gamma||Rm| + C|\bar{\nabla}^2 u| \le C + C|\bar{\nabla} g| \; .$$

By an application of Gronwall's lemma we get at most exponential growth for $\partial_t \bar{\nabla} g$. This shows that on finite time intervals:

$$|\partial_t \bar{\nabla} g| \le C(n, m, k_0, \tilde{g}, \tilde{u}, T) \; .$$

Using (6.26), an integration gives for arbitrary $\tau \in [0, T)$:

$$|\bar{\nabla} g|_{\bar{g}}(\tau) \le C|\bar{\nabla} g|(\tau) \le C|\bar{\nabla} \tilde{g}| + C\int_0^\tau |\partial_t \bar{\nabla} g|(t)dt \le C(n, m, k_0, \tilde{g}, \tilde{u}, T) \; . \tag{6.29}$$

Therefore the claim for $m = 1$ follows from (6.27) and (6.29). We assume that the claim is true for $s = 1 \ldots m - 1$. To do the induction step, we need the evolution equations for higher derivatives. A calculation shows that we have for $m \ge 2$:

$$\begin{aligned} \partial_t \bar{\nabla}^m g &= Rc * \bar{\nabla}^m g + du * \bar{\nabla}^{m+1} u + Rc * P(\bar{\nabla}^0 g, \ldots, \bar{\nabla}^{m-1} g) \\ &\quad + \sum_{i=1}^{m} \nabla^i Rc * P(\bar{\nabla}^0 g, \ldots, \bar{\nabla}^{m-i} g) + \sum_{i=1}^{m-1} \bar{\nabla}^{1+i} u * \bar{\nabla}^{1+m-i} u \\ \partial_t \bar{\nabla}^{m+1} u &= \nabla^3 u * \bar{\nabla}^m g + \sum_{i=2}^{m+1} \nabla^{i+2} u * P(\bar{\nabla}^0 g, \ldots, \bar{\nabla}^{m+1-i} g) + \nabla^3 u * P(\bar{\nabla}^0 g, \ldots, \bar{\nabla}^{m-1} g) \; . \end{aligned} \tag{6.30}$$

To derive these equations, we used the equivalence $\bar{\nabla} g \simeq \nabla - \bar{\nabla}$ to replace derivatives with respect to g by derivatives with respect to $\bar{g}$ in a similar way to (6.28). Furthermore P is a polynomial in the components of the derivatives of g of the designated order as in (8.10). This allows us to estimate

$$\begin{aligned} \big|\partial_t\big(|\bar{\nabla}^m g|_{\bar{g}} + |\bar{\nabla}^{m+1} u|_{\bar{g}}\big)\big| &\le \big||\partial_t \bar{\nabla}^m g|_{\bar{g}} + |\partial_t \bar{\nabla}^{m+1} u|_{\bar{g}}\big| \le C\big(|\partial_t \bar{\nabla}^m g| + |\partial_t \bar{\nabla}^{m+1} u|\big) \\ &\le C\big(|\bar{\nabla}^m g|_{\bar{g}} + |\bar{\nabla}^{m+1} u|_{\bar{g}}\big) + C, \end{aligned}$$

using the induction hypotheses and the equivalence (6.26) of $\bar{g}$ and the metrics $g(t)$. Here C depends only on $n, m, k_0, \tilde{g}, \tilde{u}$, and T. This proves the desired result using Gronwall's lemma. In addition the curvature satisfies

$$\bar{\nabla}^m Rm = \bar{\nabla}^m g + \sum_{i=0}^{m-1} \nabla^i Rm * P(\bar{\nabla}^0 g, \ldots, \bar{\nabla}^{m-i-1} g) + \nabla^m Rm$$

which can be proven by induction and the identity $(\bar\nabla - \nabla) \simeq \bar\nabla g$. Using the estimates for $|\nabla^s Rm|^2$ and $|\bar\nabla^s g|^2_{\bar g}$ for $s = 0, \dots, m$, the bound for $|\bar\nabla^m Rm|^2_{\bar g}$ follows in the same way as above. This finishes the proof of the proposition.

□

The interior estimates provide a necessary and sufficient condition for the long time existence of solutions on a closed manifold M.

Theorem 6.22 *Let $(g,u)(t)$ be a solution to (2.5) on $[0,T)\times M$ for closed M with initial data $(\tilde g, \tilde u)$. Assume that $T<\infty$ is maximally chosen such that the solution cannot be extended beyond T. Then the curvature of $g(t)$ has to become unbounded for $t \to T$ in the sense that*

$$\lim_{t\nearrow T}\big[\sup_{x\in M}|Rm|^2(t,x)\big] = \infty \ .$$

Proof:
We partly follow the proof of [CK04, Theorem 6.45] and show first that

$$\limsup_{t\nearrow T}\big[\sup_{x\in M}|Rm|^2(t,x)\big] = \infty \ . \tag{6.31}$$

Suppose to the contrary that the curvature stays bounded on $[0,T]$, say $|Rm|^2 \le k_0$. We prove that the solution can be extended smoothly beyond T, contradicting the choice of T.

The inequality (6.13) implies a bound on du such that $|Rm| + |du|^2 \le \tilde C$ holds on $[0,T]\times M$. For arbitrary $X \in \mathcal{X}(M)$ define a function $g(T)(X,X) := \lim_{t\to T} g(t)(X,X)$ on $M\times\mathcal{X}(M)$. By integration we get for $t\nearrow T$

$$\begin{aligned}|g(T)(X,X) - g(t)(X,X)| &\le \int_t^T |\partial_t g(\tau)(X,X)|d\tau \le c\int_t^T \big(|Rc|(\tau) + |du|^2(\tau)\big)d\tau\\ &\le c\tilde C(T-t) \longrightarrow 0\end{aligned}$$

uniformly in $x\in M$. Therefore the limit $g(T)(X,X)$ is well defined and continuous in x. By polarization we can construct a continuous limit $g(T)\in \mathrm{Sym}_2(M)$. This tensor is a Riemannian metric because of the uniform equivalence of the metrics $g(t)$ from Lemma 2.8. Similarly we find a continuous limit $u(T) := \lim_{t\to T} u(t)$ on M. From Lemma 6.21 we get for $t\in[0,T]$ the bounds

$$\sup_{x\in M}\big(|\bar\nabla^m g|^2_{\bar g} + |\bar\nabla^m u|^2_{\bar g} + |\bar\nabla^m Rm|^2_{\bar g}\big)(t,x) \le C(n,m,k_0,T,\tilde g,\tilde u)$$

for an arbitrary background metric $\bar g$ in a coordinate chart. Therefore $(g,u)(T)$ is smooth. In addition $(g,u)(t)\to(g,u)(T)$ is a smooth limit since we can estimate for all $m\ge 1$:

$$\begin{aligned}|\bar\nabla^m g(T) - \bar\nabla^m g(\tau)|_{\bar g} &\le \int_\tau^T |\partial_t \bar\nabla^m g(t)|_{\bar g}dt = \int_\tau^T |\bar\nabla^m(\partial_t g(t))|_{\bar g}dt\\ &\le C\int_\tau^T \left(|\bar\nabla^m Rm(t)|_{\bar g} + \sum_{i=0}^m |\bar\nabla^{1+i}u(t)|_{\bar g}|\bar\nabla^{1+m-i}u(t)|_{\bar g}\right)dt\\ &\le C(T-\tau) \longrightarrow 0\end{aligned}$$

for $\tau \nearrow T$, using Lemma 6.21 again. Taking $(g, u)(T)$ as initial data, the short time existence result in Theorem 3.11 provides a solution on a small time interval $[T, T + \delta)$. This solution extends the original one smoothly beyond T since the bounds in Lemma 6.21 together with (6.30) imply bounds for all time derivatives of $g(T)$ at time $t = T$. This gives the contradiction.

We can replace the limes superior (6.31) with a proper limit. Define $K(t) := \sup_{x \in M} |Rm|(t, x)$. Suppose

$$\lim_{t \nearrow T} \big[\sup_{x \in M} |Rm|^2(t, x) \big] = \infty$$

does not hold. Then there exists $B < \infty$ and a sequence $t_k \nearrow T$ such that $K(t_k) \le B$. From Proposition 6.18 we get a constant $c = c(n)$ such that

$$K(t) \le 2K(t_k) \le 2B$$

for all $t \in [t_k, T_k)$ where $T_k := \min\{T, t_k + c/B\}$. Since $t_k \nearrow T$, there is an index k_0 such that $t_{k_0} + c/B \ge T$. Therefore we get

$$\sup_{t_{k_0} \le t < T} K(t) \le 2B \ ,$$

contradicting (6.31). This completes the proof of the theorem.

□