5 Nonexistence of periodic solutions

We want to exclude the possibility of periodic geometries on closed manifolds M and follow the ideas in [Per02, §2,§3]. The technical description of such a solution is given by:

Definition 5.1 *A solution of (2.5) is called a* breather*, if there exists* $t_1, t_2 \in [0,T)$*,* $t_1 < t_2$ *such that* $g(t_2) = \alpha \cdot (\varphi^* g)(t_1)$ *and* $u(t_2) = (\varphi^* u)(t_1)$ *hold for a constant* $\alpha \in \mathbb{R}$ *and a diffeomorphism* φ*. The cases* $\alpha = 1, \alpha > 1, \alpha < 1$ *correspond to* steady, expanding*, and* shrinking breathers*.*

We use the monotonicity of E given by Lemma 2.13 and the monotonicity of W from Theorem 4.4 to prove that the only existing breathers are soliton solutions. Defining

$$\lambda(g,u) := \inf_{f \in C^\infty(M)} \left\{ E(g,u,f) \middle| \int_M e^{-f} dV = 1 \right\}$$

for $(g,u) \in \mathcal{M}(M) \times C^\infty(M)$, we get that λ is attained by a smooth function $\bar{f}$. To see this, we replace f by $\phi := e^{-f/2}$ and get a new functional:

$$\tilde{E}(g,u,\phi) := \int_M \big(4|d\phi|^2 + S\phi^2\big) dV \ .$$

This provides us with an equivalent definition for λ:

$$\lambda(g,u) = \inf_{\phi \in C^\infty(M)} \left\{ \tilde{E}(g,u,\phi) \middle| \int_M \phi^2 dV = 1 \right\} \ .$$

Thus λ is the first eigenvalue of the operator $O(\phi) := -4\Delta\phi + S\phi$ which has a smooth positive minimizer $\bar{\phi}$. Since we will prove a similar statement for W in Proposition 5.8, we do not go into details here. Going back to $\bar{f} := -2\ln\bar{\phi}$, we calculate that a minimizer $\bar{f}$ for λ satisfies:

$$2\Delta\bar{f} - |d\bar{f}|^2 + S = \lambda \ . \tag{5.1}$$

Moreover λ is invariant under diffeomorphisms since E is. We also prove that $\lambda(t)$ is monotone when evaluated on a solution to (2.4):

Lemma 5.2 *Let* $(g,u)(t)$ *be a solution to (2.5) on* $[0,T) \times M$*. Then* $\lambda(t) := \lambda(g(t),u(t))$ *is nondecreasing in* t*. If* $\frac{d}{dt}\lambda(t_0) \equiv 0$*, the solution at time* t_0 *is a gradient soliton satisfying*

$$Sy + \nabla^2 f = 0 \quad \text{and} \quad \Delta u - du(\nabla f) = 0$$

where f *is a minimizer for* λ *at time* t_0*.*

Proof:
Fix t_0 and let $\bar{f}$ be a minimizer for λ at time t_0. Solving $\partial_t f = -\Delta f - S$ backwards in time with initial data $\bar{f}$ at t_0, we conclude from Lemma 2.13 for all $t < t_0$ that

$$\lambda(t) \leq E(g,u,f)(t) \leq E(g(t_0),u(t_0),\bar{f}) = \lambda(t_0) \ .$$

Therefore $\lambda(t)$ is nondecreasing in time. The equality case follows directly from the equality case for E.

□

In the following, we identify a soliton solution $g(t)$ on $[0,T) \times M$ with its representative g at a fixed time, for example at time $t = 0$, and work with the corresponding elliptic equation as described after Definition 2.2. We get from the above considerations:

Proposition 5.3 *Let $(g,u)(t)$ be a steady breather on a closed manifold M. Then it necessarily is a steady soliton and, moreover, (M,g) is Ricci-flat and u is constant.*

Proof:
The monotonicity of E (see Lemma 2.13) shows that $\lambda(g(t),u(t))$ is nondecreasing in time. On a steady breather we have $\lambda(t_1) = \lambda(t_2)$ for two times t_1, t_2 since λ is invariant under diffeomorphisms. Therefore we can conclude from Lemma 5.2 that on $[t_1, t_2]$

$$Sy + \nabla^2 \bar{f} = 0 \tag{5.2}$$

$$\Delta u - du(\nabla \bar{f}) = 0 \tag{5.3}$$

holds where $\bar{f}(t)$ is a minimizer for $\lambda(t)$. Thus the breather is a steady soliton solution. Taking the trace in equation (5.2), we have

$$0 = S + \Delta \bar{f} \ ,$$

and by (5.1) $\bar{f}$ satisfies

$$\lambda = 2\Delta \bar{f} - |d\bar{f}|^2 + S = \Delta \bar{f} - |d\bar{f}|^2 \ . \tag{5.4}$$

Integrating, we get

$$\lambda \cdot 1 = \int_M \lambda e^{-\bar{f}} dV = \int_M (\Delta \bar{f} - |d\bar{f}|^2) e^{-\bar{f}} dV = 0$$

by (4.3) such that we conclude from (5.4)

$$\Delta \bar{f} = |d\bar{f}|^2 \ .$$

Another integration shows that $\bar{f}$ must be a constant. But then $\Delta u = 0$ from (5.3), showing that u is constant, too. Together this implies that $Rc = 0$.

□

To deal with expanding breathers, we define a scaling invariant quantity

$$\bar{\lambda}(t) := \bar{\lambda}(g,u)(t) := \lambda(g,u)(t) \cdot V(g(t))^{\frac{2}{n}}$$

where V denotes the volume of M with respect to $g(t)$.

Lemma 5.4 *$\bar{\lambda}(t)$ is scaling invariant with respect to the scaling $\tilde{g} := \alpha \cdot g$ and $\tilde{f} := f + \frac{n}{2} \ln \alpha$ for all constants $\alpha > 0$.*

Proof:
Observe that we also have to scale f since it still has to satisfy the normalization constraint

$$\int_M e^{-\tilde{f}} d\tilde{V} = \int_M e^{-f} e^{\ln(\alpha^{-\frac{n}{2}})} \alpha^{\frac{n}{2}} dV = \int_M e^f dV = 1$$

with respect to the new volume element $dV_{\tilde{g}}$. Then we can calculate

$$\begin{aligned} R(\tilde{g}) &= \alpha^{-1} R(g) \\ |df|^2_{\tilde{g}} &= \tilde{g}^{ij} \partial_i f \partial_j f = \alpha^{-1} g^{ij} \partial_i f \partial_j f = \alpha^{-1} |df|^2_g \\ dV_{\tilde{g}} &= \sqrt{\det(\tilde{g})} dx = \sqrt{\det(\alpha \cdot g)} dx = \sqrt{\alpha^n \det(g)} dx = \alpha^{\frac{n}{2}} \sqrt{\det(g)} dx = \alpha^{\frac{n}{2}} dV, \end{aligned}$$

giving us

$$\begin{aligned} \bar{\lambda}(\tilde{g}, u) &= V(\tilde{g})^{\frac{2}{n}} \cdot \lambda(\tilde{g}, u) = \left[\int_M dV_{\tilde{g}} \right]^{\frac{2}{n}} \cdot \inf_{\tilde{f}} \left\{ \int_M (|d\tilde{f}|^2_{\tilde{g}} + R(\tilde{g}) - 2|du|^2_{\tilde{g}}) e^{-\tilde{f}} dV_{\tilde{g}} \middle| \int_M e^{-\tilde{f}} dV_{\tilde{g}} = 1 \right\} \\ &= \left[\int_M \alpha^{\frac{n}{2}} dV \right]^{\frac{2}{n}} \cdot \inf_{\tilde{f}} \left\{ \int_M \alpha^{-1} \big(|df|^2_g + R - 2|du|^2_g \big) \alpha^{-\frac{n}{2}} e^{-f} \alpha^{\frac{n}{2}} dV \middle| \int_M \alpha^{-\frac{n}{2}} e^{-f} \alpha^{\frac{n}{2}} dV = 1 \right\} \\ &= \alpha^{\frac{n}{2} \cdot \frac{2}{n}} V(g)^{\frac{2}{n}} \cdot \alpha^{-1} \lambda(g, u) = V(g)^{\frac{2}{n}} \cdot \lambda(g, u) = \bar{\lambda}(g, u) \end{aligned}$$

as required. Note that there is no difference taking the infimum over f or $\tilde{f}$.

□

The quantity $\bar{\lambda}(t)$ is not monotone in general, but we only need to establish the following monotonicity property:

Lemma 5.5 *Let $(g, u)(t)$ be a solution to (2.5). Then $\bar{\lambda}(t)$ is nondecreasing at times t where it is nonpositive. If $\frac{d}{dt}\bar{\lambda}(t_0) = 0$ at a time t_0, then the solution satisfies*

$$\begin{aligned} |\nabla^2 \bar{f} + Sy + \tfrac{1}{n}(S + \Delta \bar{f}) g|^2 &= 0 \\ |\Delta \bar{f} - du(\nabla \bar{f})|^2 &= 0 \\ \Delta \bar{f} + S &= const \end{aligned}$$

where $\bar{f}$ is a minimizer for λ at time $t = t_0$.

Proof:
Since $\bar{\lambda}$ is Lipschitz continuous, the time derivative exists in the sense of forward difference quotients. At a fixed time t, we assume $\bar{\lambda}(t) \leq 0$ and compute

$$\tfrac{d}{dt}\bar{\lambda}(t) = \tfrac{d}{dt} V^{\frac{2}{n}} \cdot \lambda + V^{\frac{2}{n}} \cdot \tfrac{d}{dt}\lambda = \tfrac{2}{n} V^{\frac{2}{n}} \cdot V^{-1} \cdot \tfrac{d}{dt} V + V^{\frac{2}{n}} \tfrac{d}{dt}\lambda \ . \tag{5.5}$$

Using Lemma 1.4, further calculations show

$$\tfrac{d}{dt} V(t) = \partial_t \int_M dV = \int_M \frac{\operatorname{tr} \partial_t g}{2} dV = \int_M \big(-\Delta f - R + 2|du|^2 \big) dV = -\int_M S dV \ .$$

Setting $f(t) := \ln V(t)$, we obtain

$$E(g,u,\ln V) = \int_M \big(S + |d(\ln V)|^2\big)e^{-\ln V} dV = \int_M \big(S+0\big)V^{-1}dV = V^{-1}\int_M SdV$$

since $f(t)$ is independent of $x \in M$. Furthermore, f is properly normalized

$$\int_M e^{-\ln V} dV = V^{-1}\int_M dV = V^{-1}\cdot V = 1$$

and therefore an admissible function. From the definition of λ we conclude that

$$V^{-1}\int_M SdV = E(g,u,\ln V) \geq \inf_{f\in C^\infty(M)} E(g,u,f) = \lambda\ . \tag{5.6}$$

Whenever $\lambda \leq 0$, using (4.3) and (5.6), we compute

$$\begin{aligned} -\lambda V^{-1}\int_M SdV &= |\lambda|V^{-1}\int_M SdV \geq |\lambda|\lambda = -|\lambda|^2 \\ &= -\Big(\int_M (S+|d\bar f|^2)\, e^{-\bar f} dV\Big)^2 = -\Big(\int_M \big(S+\Delta \bar f\big)\, e^{-\bar f}dV\Big)^2 \end{aligned}$$

where $\bar f$ is a minimizer for E at time t. This gives an estimate for the first term in (5.5):

$$\tfrac{d}{dt}V^{\frac{2}{n}}(t)\cdot\lambda \geq -\tfrac{2}{n}V^{\frac{2}{n}}\Big(\int_M \big(S+\Delta\bar f\big)\, e^{-\bar f}dV\Big)^2\ . \tag{5.7}$$

The second term in (5.5) comes down to

$$\begin{aligned} V^{\frac{2}{n}}\tfrac{d}{dt}\lambda(t) &= V^{\frac{2}{n}}\cdot 2\int_M |\nabla^2\bar f + Sy|^2 + 2|\Delta u - du(\nabla\bar f)|^2\, e^{-\bar f} dV \\ &= 2V^{\frac{2}{n}}\int_M |\nabla^2\bar f + Sy + \tfrac{1}{n}(S+\Delta\bar f)g|^2 + 2|\Delta\bar f - du(\nabla\bar f)|^2 + \tfrac{1}{n}(S+\Delta\bar f)^2\, e^{-\bar f}dV \end{aligned} \tag{5.8}$$

where $\bar f$ is the same minimizer as above. We also used the equation for $\partial_t E$ in Lemma 2.13 and

$$|\nabla^2\bar f + Sy + \tfrac{1}{n}(S+\Delta\bar f)g|^2 = |\nabla^2\bar f + Sy|^2 - \tfrac{1}{n}(S+\Delta\bar f)^2\ .$$

The combination of (5.7) and (5.8) proves

$$\begin{aligned} \tfrac{d}{dt}\bar\lambda(t) &\geq 2V^{\frac{2}{n}}\int_M |\nabla^2\bar f + Sy + \tfrac{1}{n}(S+\Delta\bar f)g|^2 + 2|\Delta u - du(\nabla\bar f)|^2\, e^{-\bar f}dV \\ &\quad + \tfrac{2}{n}V^{\frac{2}{n}}\Big\{\int_M (\Delta\bar f + S)^2\, e^{-\bar f}dV - \Big(\int_M (\Delta\bar f + S)\, e^{-\bar f}dV\Big)^2\Big\} \geq 0 \end{aligned} \tag{5.9}$$

where the non-negativity of the second line is due to Hölder's inequality. Therefore $\bar\lambda$ is nondecreasing at time t. If $\frac{d}{dt}\bar\lambda(t) = 0$, all individual terms have to vanish. Note that the second line can only be zero if $\Delta\bar f + S \equiv const$. This proves the lemma.

□

Proposition 5.6 *Let $(g,u)(t)$ be an expanding breather on a closed manifold M. Then it necessarily is an expanding gradient soliton and, moreover, (M,g) is an Einstein manifold and u is constant.*

Proof:
Assume there are times t_1, t_2 and $\alpha > 1$ such that $g(t_2) = \alpha \cdot (\phi^* g)(t_1)$. We have $\bar{\lambda}(t_2) = \bar{\lambda}(t_1)$ since $\bar{\lambda}$ is invariant under scaling and diffeomorphisms. Since $\alpha > 1$, we know in addition that $V(t_2) > V(t_1)$, implying that there is a time $t_0 \in [t_1, t_2]$ such that

$$\tfrac{d}{dt} V(t_0) = -\int_M S dV(t_0) > 0 \ .$$

Therefore we can conclude at time t_0:

$$\bar{\lambda} \leq V^{\frac{2-n}{n}} \int_M S dV < 0$$

where we used (5.6) in the first step. Consider two cases: If $\bar{\lambda}(t_1) \geq 0$, $\bar{\lambda}$ can never decrease below 0 again, in particular not at time t_0. On the other hand, if $\bar{\lambda}(t_1) < 0$, then it has to increase up to time t_0 where it is still negative. Therefore it cannot decrease back to its old value at time t_2 as required. Lemma 5.5 then shows that $\bar{\lambda}(t) \equiv const$ for $t \in [t_1, t_2]$. This implies $\frac{d\bar{\lambda}}{dt} = 0$ on $[t_1, t_2]$, and we get for a constant $c \in \mathbb{R}$

$$\begin{aligned} Sy + \nabla^2 \bar{f} - \tfrac{1}{n}(S + \Delta \bar{f}) g &= 0 \\ \Delta u - du(\nabla \bar{f}) &= 0 \\ S + \Delta \bar{f} &= c \end{aligned} \tag{5.10}$$

because we have equality in (5.9). Since the minimizer $\bar{f}$ satisfies (5.1), we get

$$2\Delta \bar{f} - |d\bar{f}|^2 + S = \lambda = \int_M \left(S + |d\bar{f}|^2\right) e^{-\bar{f}} dV = \int_M \left(S + \Delta \bar{f}\right) e^{-\bar{f}} dV = c \cdot \int_M e^{-\bar{f}} dV = c$$

where we used (4.3). This implies

$$\Delta \bar{f} - |d\bar{f}|^2 + c = c \quad \Rightarrow \quad \Delta \bar{f} = |d\bar{f}|^2 \ .$$

Integrating as before, we know that $\bar{f}$ is constant. Inserting this into (5.10) yields

$$\begin{aligned} Rc - 2du \otimes du - \tfrac{c}{n} g &= 0 \\ \Delta u &= 0, \end{aligned}$$

and we conclude that u has to be constant, too. This leaves

$$Rc - \tfrac{c}{n} g = 0,$$

and g has to be an Einstein metric on M.

□

The remaining case are shrinking breathers which we want to handle with help of the functional (4.1) and its monotonicity proven in Theorem 4.4. We first give a definition:

Definition 5.7 *Let $(g,u,\tau) \in \mathcal{M}(M) \times C^\infty(M) \times \mathbb{R}^+$ be a configuration. Then we define:*

$$\mu := \mu(g,u,\tau) := \inf_{f \in C^\infty(M)} \left\{ W(g,u,f,\tau) \,:\, \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV = 1 \right\} .$$

Proposition 5.8 *Let M be closed and connected. Then μ is attained by a smooth function $\bar{f} \in C^\infty(M)$ satisfying the normalization constraint.*

Proof:
We adapt the method from [Rot81]. Replacing $\phi := e^{-f/2}$ as before, we equivalently can minimize the integral

$$\tilde{W}(g,u,\phi,\tau) := \int_M \left[4\tau |d\phi|^2 + \tau S \phi^2 - \phi^2 \ln \phi^2 - n\phi^2 \right] (4\pi\tau)^{-\frac{n}{2}} dV$$

for functions $\phi \in W^{1,2}(M)$. In the following, all L^p and $W^{k,p}$ spaces are be with respect to the measure $dm = (4\pi\tau)^{-\frac{n}{2}} dV$. Analogously to Definition 5.7 we set for fixed (g,u,τ):

$$\tilde{\mu} := \tilde{\mu}(g,u,\tau) := \inf_{\phi \in W^{1,2}(M)} \left\{ \tilde{W}(g,u,\phi,\tau) \,:\, \int_M \phi^2 (4\pi\tau)^{-\frac{n}{2}} dV = 1 \right\}$$

and show that $\tilde{W}$ is bounded below for $\phi \in W^{1,2}(M)$. Choose $p := \frac{2}{n-2}$. Using Jensen's inequality for the logarithm with respect to the measure $\phi^2 dm$, we get

$$\begin{aligned} \int_M \phi^2 \ln \phi^2 dm &= \int_M \phi^2 \ln \left[(\phi^{2p})^{1/p} \right] dm = \tfrac{n-2}{2} \int_M \left[\ln |\phi|^{2p} \right] \phi^2 dm \le \tfrac{n-2}{2} \ln \left[\int_M |\phi|^{2p+2} dm \right] \\ &= \tfrac{n-2}{2} \ln \left[\|\phi\|_{2p+2}^{2p+2} \right] = n \ln \|\phi\|_{\frac{2n}{n-2}} . \end{aligned}$$

The Sobolev embedding $W_0^{1,2}(M) \subset L^{\frac{2n}{n-2}}(M)$ for $n > 2$ is proven for example in [Aub82, Theorem 2.20]. By the choice of ε we can estimate:

$$\|\phi\|_{\frac{2n}{n-2}} \le c(n) \|\phi\|_{W^{1,2}},$$

such that together with the monotonicity of the logarithm we have

$$\int_M \phi^2 \ln \phi^2 dm \le \tfrac{n-2}{2} \ln \left[c(n) \|\phi\|_{W^{1,2}} \right] .$$

Using the fact that S is smooth and the normalization $\int_M \phi^2 dm = 1$, we get altogether:

$$\begin{aligned} &\int_M \left(4\tau |d\phi|^2 - \phi^2 \ln \phi^2 + \phi^2 (\tau S - n) \right) dm \\ &\quad = 4\tau \int_M |d\phi|^2 dm + 4\tau \int_M |\phi|^2 dm - 4\tau \int_M |\phi|^2 dm - \int_M \phi^2 \ln \phi^2 dm + \int_M \phi^2 (\tau S - n) dm \\ &\quad \ge 4\tau \|\phi\|_{W^{1,2}}^2 - 4\tau - \tfrac{n-2}{2} \ln \left[c(n) \|\phi\|_{W^{1,2}} \right] + \tau \min_{x \in M} S(x) \int_M |\phi|^2 dm - n \cdot 1 \\ &\quad \ge 4\tau \|\phi\|_{W^{1,2}}^2 - \tfrac{n-2}{2} \ln \|\phi\|_{W^{1,2}} - \tfrac{n-2}{2} \ln c(n) - 4\tau + \tau \min_{x \in M} S(x) - n \\ &\quad \ge 4\tau \|\phi\|_{W^{1,2}}^2 - \tfrac{n-2}{2} \|\phi\|_{W^{1,2}} + C(n,\tau,S_{min}) \\ &\quad \ge C(n,\tau,S_{min}) \end{aligned}$$

independent of ϕ. Here we set $S_{min} := \min_{x\in M} S(x)$ and used that $f(x) = Ax^2 - Bx \geq -\frac{B^2}{4A}$ on $\mathbb{R}^+$. Therefore $\tilde{W}$ is bounded below for $\phi \in W^{1,2}(M)$.

In view of the Sobolev embedding, Hölder's inequality, and the mean value theorem the functional $F : W^{1,2}(M) \to \mathbb{R}$ given by

$$F(\phi) := \int_M \big((\tau S - n)\phi^2 - \phi^2 \ln \phi^2\big) dm \tag{5.11}$$

is continuous in L^p for all $p > 2$. Let us assume that $\phi_i \subset W^{1,2}(M)$ is a minimizing sequence for $\tilde{W}$ such that $\tilde{W}(g, u, \phi_i, \tau) \leq \tilde{\mu} + \frac{1}{i}$ holds for all $i \geq 0$. We calculate

$$\begin{aligned}
&\int_M 4\tau|d\phi_i|^2 - \phi_i^2 \ln \phi_i^2 + (\tau S - n)\phi_i^2 dm \\
&\quad = \int_M 2\tau|d\phi_i|^2 - \phi_i^2 \ln \phi_i^2 + (\tfrac{\tau}{2} S - n)\phi_i^2 dm + \int_M 2\tau|d\phi_i|^2 dm + \tfrac{\tau}{2} \int_M S\phi_i^2 dm \\
&\quad \geq \tilde{\mu}(g, u, \phi, \tau/2) + \frac{\tau}{2} S_{min} + 2\tau \int_M \big(|d\phi_i|^2 + |\phi_i|^2\big) dm - 2\tau
\end{aligned}$$

where S_{min} is defined as above. This implies that

$$\begin{aligned}
\|\phi\|_{W^{1,2}} &\leq \frac{1}{2\tau} \Big\{ -\tilde{\mu}(g, u, \phi, \tau/2) - \frac{\tau}{2} S_{min} + 2\tau + \tilde{W}(g, u, \phi_i, \tau) \Big\} \\
&\leq \frac{1}{2\tau} \Big\{ -\tilde{\mu}(g, u, \phi, \tau/2) - \frac{\tau}{2} S_{min} + 2\tau + \tilde{\mu}(g, u, \phi_i, \tau) + \frac{1}{i} \Big\} \\
&\leq C
\end{aligned}$$

holds for a constant C independent of i. Therefore the sequence (ϕ_i) is uniformly bounded in $W^{1,2}(M)$ and converges weakly to a function $\bar{\phi} \in W^{1,2}(M)$. By the compactness of the embedding $W^{1,2}(M) \hookrightarrow L^p(M)$ for all $1 < p < \frac{2n}{n-2}, n \geq 3$, we know that $\phi_i \to \bar{\phi}$ strongly in L^p for p in this range. Since $\frac{2n}{n-2} > 2$, the functional F defined in (5.11) is continuous in $L^p(M)$, and we get

$$\begin{aligned}
\tilde{\mu} &= \inf_{\phi} \tilde{W}(g, u, \phi, \tau) = \lim_{i\to\infty} \tilde{W}(g, u, \phi_i, \tau) \\
&= \Big[\lim_{i\to\infty} \int_M 4\tau|d\phi_i|^2 dm\Big] - \int_M \bar{\phi}^2 \ln \bar{\phi}^2 dm + \int_M \bar{\phi}^2(\tau S - n) dm \ .
\end{aligned}$$

The weak convergence in $W^{1,2}(M)$ gives

$$\lim_{i\to\infty} \int_M 4\tau|d\phi_i|^2 dm \geq \int_M 4\tau|d\bar{\phi}|^2 dm \ ,$$

implying

$$\tilde{\mu} \geq \int_M 4\tau|d\bar{\phi}|^2 - \bar{\phi}^2 \ln \bar{\phi}^2 + (\tau S - n)\bar{\phi}^2 dm \ ,$$

and the infimum is indeed attained at $\bar{\phi}$. The strong convergence also implies that the limit $\bar{\phi}$ satisfies the normalization condition. We can assume in addition that $\bar{\phi} \geq 0$. If (ϕ_i) is a minimizing sequence, then also $\big(|\phi_i|\big)$ because we have $\|d\phi_i\|_{L^2} = \|d|\phi_i|\|_{L^2}$ and $\|\phi_i\|_{L^2} = \||\phi_i|\|_{L^2}$.

As a critical point of $\tilde{W}$, $\bar{\phi}$ satisfies the Euler-Lagrange equation

$$-4\tau\Delta\bar{\phi} - 2\bar{\phi}\ln\bar{\phi} + (\tau S - n - \tilde{\mu})\bar{\phi} = 0 \qquad (5.12)$$

weakly in $W^{1,2}$. To be able to go back to the original functional W, it remains to show that $\bar{\phi}$ is smooth and positive. To this end we rewrite the equation:

$$\Delta\bar{\phi} = -\tfrac{1}{2\tau}\cdot\bar{\phi}\ln\bar{\phi} + \tfrac{1}{4\tau}(\tau S - n - \tilde{\mu})\cdot\bar{\phi} = c\cdot\bar{\phi}\ln\bar{\phi} + \omega(x)\cdot\bar{\phi} =: P$$

where $\omega(x) := \frac{1}{4\tau}(\tau\cdot S(x) - n - \mu)$ is smooth. Since $\bar{\phi} \in W_0^{1,2}(M) \hookrightarrow L^{\frac{2n}{n-2}}(M)$ by the Sobolev embedding, and since $\bar{\phi}\cdot\ln\bar{\phi} \in L^{p-\delta}(M)$ for all $\delta > 0$ whenever $\bar{\phi} \in L^p(M)$, we know that $P \in L^{\frac{2n}{n-2+\varepsilon}}$ for all $\varepsilon > 0$. Now the regularity result [GT98, Theorem 9.15] implies that $\bar{\phi} \in W^{2,\frac{2n}{n-2+\varepsilon}}(M)$. Again by the Sobolev embedding we know that $\bar{\phi} \in L^{\frac{2n}{n-6+\varepsilon}}(M)$, proving that $P \in L^{\frac{2n}{n-6+2\varepsilon}}$ by the previous argument. After a finite number of iterations we get $P \in L^p(M)$ for some $p > 2n$. This implies that $\bar{\phi} \in C^\alpha(M)$ for some $\alpha > 0$ by [GT98, Theorem 8.22].

Since $\bar{\phi}$ therefore is continuous, we can prove that it is pointwise positive. We have the following lemma from Rothaus which analogously holds in our situation:

Lemma 5.9 *[Rot81, page 114] Assume $\bar{\phi} \in W^{1,2}(M) \cap C^0(M)$ is a nonnegative minimizer for $\tilde{\mu}$ and $\bar{\phi}(p) = 0$. Then there exists a neighborhood of p where $\bar{\phi}$ vanishes identically.*

This shows that $\bar{\phi}$ is positive everywhere on M since defining

$$\Omega := \{p \in M \big| \bar{\phi}(p) = 0\},$$

we see from the lemma that Ω is open. But it is also closed since $\bar{\phi}$ is continuous. Because M is connected, there are only two possibilities, either $\Omega = \emptyset$ and $\bar{\phi} > 0$ on M, or $\Omega = M$ and $\bar{\phi} \equiv 0$ which is impossible since $\|\bar{\phi}\|_{L^2} = 1$. Furthermore we know that $\bar{\phi}$ is uniformly bounded below away from 0 since M is compact.

Using this information, we see that $P = c\cdot\bar{\phi}\ln\bar{\phi} + \omega\cdot\bar{\phi}$ is Hölder continuous for $\bar{\phi} \in C^\alpha(M)$ positive. This implies that $\bar{\phi}$ is a classical solution of (5.12) in $C^{2,\alpha}(M)$ by [GT98, Theorem 9.19] and satisfies

$$\Delta\bar{\phi} = c\cdot\bar{\phi}\ln\bar{\phi} + \omega\cdot\bar{\phi}$$

in the classical sense. Repeating this argument, we learn that $\bar{\phi} \in C^\infty(M)$. Since $\bar{\phi}$ is positive, we can define $\bar{f} := -2\ln\bar{\phi}$ and $\bar{f}$ is a smooth minimizer for μ as required.

□

Now that we have understood the variational problem, we can investigate the remaining case of shrinking breathers. For the proof we are going to need the following lemma:

Lemma 5.10 *Suppose $(g, u)(t)$ is a solution to (2.5) on $[0, T) \times M$ where M is closed. Fix a $\bar{\tau} \in [0, T)$ and define $\tau(t) := \bar{\tau} - t$. Then $\mu(g, u, \tau)(t)$ is nondecreasing in t. If $\frac{d}{dt}\mu(t) = 0$ the solution is a gradient shrinking soliton.*

Proof:
Let $\tilde{f}$ be a minimizer of $\mu(t_0)$ for $t_0 < \bar{\tau}$ arbitrary. We can solve the equation for f backwards in time with initial value $\tilde{f}$ at $t = t_0$. The monotonicity (4.6) implies that $W(g, u, f, \tau)(t)$ is nondecreasing in time. We get for $t < t_0$:

$$\mu(g, u, \tau)(t) \leq W(g, u, f, \tau)(t) \leq W(g, u, f, \tau)(t_0) = W(g(t_0), u(t_0), \tilde{f}, \tau(t_0)) = \mu(g, u, \tau)(t_0) \ ,$$

proving the lemma.

□

Proposition 5.11 *Let $(g, u)(t)$ be a shrinking breather on a closed manifold M. Then it necessarily is a gradient shrinking soliton.*

Proof:
The breather $(g, u)(t)$ satisfies $g(t_2) = \alpha \cdot \varphi^* g(t_1)$ and $u(t_2) = \varphi^* u(t_1)$ for two times t_1, t_2 and a constant $\alpha < 1$. Define the reference time

$$\bar{\tau} := \frac{t_2 - \alpha t_1}{1 - \alpha} \ ,$$

and set $\tau(t) := \bar{\tau} - t$. It follows that

$$\alpha = \frac{\bar{\tau} - t_2}{\bar{\tau} - t_1} = \frac{\tau(t_2)}{\tau(t_1)} \ .$$

Using Lemma 4.2 and Lemma 5.10, we conclude that

$$\mu(g, u, \tau)(t_2) = \mu(\alpha \cdot \varphi^* g, \varphi^* u, \alpha \tau(t_1)) = \mu(\varphi^* g, \varphi^* u, \tau(t_1)) = \mu(g, u, \tau)(t_1) \ .$$

By the equality case of the monotonicity formula, $(g, u)(t)$ must be a gradient shrinking soliton.

□

We cannot draw further conclusions as we did for steady and expanding breathers since we cannot use the Euler-Lagrange equation satisfied by the minimizer $\bar{f}$ in the way we did for minimizers of E.