3 Short time existence

The goal of this chapter is to prove short time existence for the system (2.5)

$$\partial_t g(t) = -2Rc(g(t)) + 4du(t) \otimes du(t) \tag{3.1}$$

$$\partial_t u(t) = \Delta^{g(t)} u(t) \tag{3.2}$$

with initial data $g(0) = \tilde{g}$ and $u(0) = \tilde{u}$ for $\tilde{g} \in \mathcal{M}(\Sigma)$ and $\tilde{u} \in C^\infty(\Sigma)$. On closed manifolds we obtain a solution on a time interval $[0, T)$ for some $T > 0$. To get existence on noncompact complete manifolds we have to impose additional conditions on the initial data later on.

3.1 The boundary value problem

Since we consider noncompact manifolds Σ, we cannot directly invoke the theory of parabolic systems to get a solution on the whole manifold Σ. Instead, we solve the boundary value problem on a family of discs of increasing radius. This provides a sequence of local solutions. We prove that a limit exists and is the desired global solution on the whole of Σ.

The system (3.1) and (3.2) is only weakly parabolic due to the diffeomorphism invariance of the equations. The principal symbol of (3.1) is the same as the principal symbol of the Ricci operator since the second term is independent of g, hence a lower order term. Therefore the considerations in [Ham82, §4] concerning the Ricci Flow are also true for this system.

Fortunately we can overcome this difficulty using the methods that were developed for the Ricci flow. In particular, we can find a strongly parabolic system which is equivalent to (3.1) and (3.2) by the application of a diffeomorphism. This is referred to as DeTurck's trick [DeT83]. In the noncompact setting this is carefully worked out in [Shi89] which we will strongly refer to in the sequel. We first calculate the evolution equations for solutions pulled back by such a diffeomorphism.

To this end let $V \in \mathcal{X}([0, T] \times \Sigma)$ be a smooth time dependent vector field and denote the induced 1-parameter family of diffeomorphisms by φ_t. Then the diffeomorphisms satisfy at every $x \in \Sigma$ the following ordinary differential equation:

$$\begin{aligned} \frac{d}{dt}\varphi_t(x) &= V(\varphi_t(x)) \\ \varphi_0(x) &= x \ . \end{aligned} \tag{3.3}$$

Lemma 3.1 *Suppose $(\bar{g}, \bar{u})(t)$ is a solution of (3.1) and (3.2) on $[0, T] \times \Sigma$ and $\varphi_t : \Sigma \to \Sigma$ is the 1-parameter family of diffeomorphisms generated by V. Then the pullbacks $g(t) := \varphi_t^* \bar{g}(t)$ and $u(t) := \varphi_t^* \bar{u}(t)$ satisfy the following system of equations:*

$$\partial_t g_{ij} = -2R_{ij} + 4\partial_i u \partial_j u + \nabla_i V_j + \nabla_j V_i \tag{3.4}$$

$$\partial_t u = \Delta^g u + du(V) \tag{3.5}$$

on $[0, T] \times \Sigma$ where $\{V_i\}$ is the associated 1-form to V. Furthermore, $(g, u)(t)$ have the same initial values as $(\bar{g}, \bar{u})(t)$:

$$(g, u)(0) = (\tilde{g}, \tilde{u}) \ . \tag{3.6}$$

Proof:
Denote by $\{y^\alpha\}_{\alpha=1\dots n}$ the coordinates where $\bar g$ and $\bar u$ are represented by $\bar g_{\alpha\beta}$ and $\bar u$. Define new coordinates by $x^i := (y\circ\varphi)^i$ for $i = 1\dots n$. We use the argument in [Shi89, §2] which goes through since the extra term in the equation for the metric is simply

$$\varphi_t^*(d\bar u\otimes d\bar u)_{ij} = \frac{\partial y^\alpha}{\partial x^i}\partial_\alpha \bar u\frac{\partial y^\beta}{\partial x^j}\partial_\beta\bar u = \partial_i u\partial_j u = (du\otimes du)_{ij}$$

in the new coordinates $\{x^i\}$ and the rest of the calculation is unchanged. Moreover, the evolution of u with respect to the new coordinates is given as follows:

$$\begin{aligned}\partial_t u(t,x) &= \partial_t(\bar u(t,\varphi(x))) = \frac{\partial}{\partial t}\bar u + \frac{\partial \bar u}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial t} = \bar\Delta\bar u + \frac{\partial \bar u}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial t}\\ &= \bar g^{\alpha\beta}\bar\nabla_\alpha\bar\nabla_\beta\bar u + \frac{\partial \bar u}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^p}V^p = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}g^{ij}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}\nabla_i\nabla_j u + \frac{\partial u}{\partial x^p}V^p\\ &= \Delta u + du(V)\end{aligned}$$

since we know from (3.3) that

$$\frac{\partial y^\alpha}{\partial t} = \frac{\partial(\varphi_t(x)^\alpha)}{\partial t} = \big(\frac{d}{dt}\varphi_t(x)\big)^\alpha = \big(V\varphi_t(x)\big)^\alpha = (D\varphi_t(V))^\alpha = \frac{\partial y^\alpha}{\partial x^p}V^p\ . \tag{3.7}$$

The initial data remain the same under this coordinate change since $\varphi_0 = id$ on Σ from (3.3) which proves the lemma.

□

There is a suitable vector field V to make the system strictly parabolic. This is an idea from [DeT83]. From now on, we will denote all derivatives with respect to the initial metric $\tilde g$ by $\tilde\nabla$ which is time-independent.

Lemma 3.2 *The choice of* $V^i := g^{pq}(\Gamma^i_{pq} - \tilde\Gamma^i_{pq})$ *makes the system (3.4) and (3.5) strictly parabolic on* $[0,T]\times\Sigma$.

Proof:
To see that the system is strictly parabolic, we rewrite the equations such that all derivatives are with respect to the (fixed) initial metric $\tilde g$ and examine the leading order terms in coordinates. We use the identity

$$\Gamma^k_{ij} - \tilde\Gamma^k_{ij} = \tfrac12 g^{kl}(\tilde\nabla_i g_{jl} + \tilde\nabla_j g_{il} - \tilde\nabla_l g_{ij}) \tag{3.8}$$

to replace the Christoffel symbols of g by derivatives $\tilde\nabla g$ and work in normal coordinates for $\tilde g$ such that $\tilde\Gamma^k_{ij} = 0$ at the base point. We compute the evolution equation for g_{ij} as in [Shi89, Lemma 2.1]. The additional term $4du\otimes du$ is independent of the metric g, giving us altogether:

$$\begin{aligned}\partial_t g_{ij} =& g^{ab}\tilde\nabla_a\tilde\nabla_b\, g_{ij} - g^{ab}g_{ik}\tilde g^{kl}\tilde R_{jalb} - g^{ab}g_{jk}\tilde g^{kl}\tilde R_{ialb} + 4\partial_i u\partial_j u\\ &+ \tfrac12 g^{ab}g^{kl}\left\{\tilde\nabla_i g_{ka}\tilde\nabla_j g_{lb} + 2\tilde\nabla_a g_{jk}\tilde\nabla_l g_{ib} - 2\tilde\nabla_a g_{jk}\tilde\nabla_b g_{il} - 2\tilde\nabla_j g_{ka}\tilde\nabla_b g_{il} - 2\tilde\nabla_i g_{ka}\tilde\nabla_b g_{jl}\right\}\ ,\end{aligned} \tag{3.9}$$

containing only one term of second order and terms quadratic in the gradient of g. The equation for u is computed similarly as follows:

$$\begin{aligned}\partial_t u &= \Delta^g u + du(V) = g^{ij}\nabla_i\nabla_j u + \partial_k u V^k = g^{ij}\big(\partial_i\partial_j u - \Gamma_{ij}^k\partial_k u\big) + \partial_k u V^k \\ &= g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j u - g^{ij}\Gamma_{ij}^k\partial_k u + \partial_k u\cdot g^{ij}(\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k) \\ &= g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j u\end{aligned} \tag{3.10}$$

where we used that $\tilde{\Gamma}_{ij}^k = 0$. The principal symbol of the system in these coordinates is given by the coefficient matrices of the second derivatives of g and u. These are the coordinate representations of the inverse metric g^{-1}, henceforth positive definite, making the symbol invertible over all nonzero cotangent vectors. Therefore the system is strictly parabolic.

□

In the following, let $D \subset \Sigma$ be an bounded open connected subset of Σ with compact closure. The parabolic boundary Γ of $[0,T]\times D$ is defined by:

$$\Gamma := \big(\{0\}\times D\big) \cup \big([0,T]\times\partial D\big)\ . \tag{3.11}$$

We want to solve the initial/boundary value problem (3.1) and (3.2) on $[0,T]\times D$ with initial and boundary data $\tilde{g}$ and $\tilde{u}$, that is

$$(g,u)_{|\Gamma} = (\tilde{g},\tilde{u})\ .$$

3.2 Equivalence of the solution metrics

We prove that the evolving metric is equivalent to the initial metric at least for a short time. This allows us to compare the unknown metric $g(t)$ with the initial $\tilde{g}$. This shows in particular that the system (3.1) and (3.2) is uniformly parabolic. Let in the following $(\Sigma,\tilde{g})$ be a complete Riemannian manifold, $\tilde{u}$ a smooth function and assume $\tilde{g},\tilde{u}$ satisfy the global bounds

$$|\tilde{R}m|_0^2 \le k_0, \qquad |\tilde{\nabla}\tilde{u}|_0^2 \le c_0 \tag{3.12}$$

for some constants $k_0, c_0 \ge 0$ on Σ where $|\cdot|_0$ is the norm associated with $\tilde{g}$. Let $(g,u)(t)$ be a solution to (3.4) and (3.5) on $[0,T]\times\bar{D}$ with initial and boundary values $(\tilde{g},\tilde{u})$. To prove a lower bound for $g(t)$ we use the same test function as in [Shi89, Lemma 2.2]:

Lemma 3.3 *For an integer $m>0$, define a test function φ on $[0,T]\times\bar{D}$ as follows:*

$$\varphi := g^{a_1b_1}\tilde{g}_{b_1a_2}g^{a_2b_2}\cdots g^{a_mb_m}\tilde{g}_{b_ma_1}\ . \tag{3.13}$$

Then φ satisfies:

$$\begin{aligned}&\partial_t\varphi \le g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\varphi + 2mn\sqrt{k_0}\cdot\varphi^{1+1/m}\\ &\varphi_{|\Gamma} \equiv n\end{aligned}$$

where Γ is the parabolic boundary defined in (3.11) and n is the dimension of Σ.

Proof:
Considering the boundary values (3.6), we compute on Γ:

$$\varphi = g^{a_1b_1}\tilde{g}_{b_1a_2}\cdots g^{a_mb_m}\tilde{g}_{b_ma_1} = \tilde{g}^{a_1b_1}\tilde{g}_{b_1a_2}\cdots \tilde{g}^{a_mb_m}\tilde{g}_{b_ma_1} = \tilde{g}^{a_1}{}_{a_1} = n$$

after successive contractions over all indices. The evolution (3.9) of g_{ij} implies an evolution equation for g^{ij}:

$$\begin{aligned}\partial_t g^{ij} =& g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b g^{ij} + g^{ab}g^{ik}g^{jl}g_{kp}\tilde{g}^{pq}\tilde{R}_{laqb} + g^{ab}g^{ik}g^{jl}g_{pl}\tilde{g}^{pq}\tilde{R}_{kaqb} + g^{ab}\tilde{\nabla}_a g^{ip}g^{jq}\tilde{\nabla}_b g_{pq}\\ &+ g^{ab}g^{ip}\tilde{\nabla}_a g^{jq}\tilde{\nabla}_b g_{pq} - 4g^{ik}g^{jl}\tilde{\nabla}_k u\tilde{\nabla}_l u\\ &+ g^{ab}g^{pq}g^{ik}g^{jl}\big(\tilde{\nabla}_a g_{pl}\tilde{\nabla}_b g_{qk} + \tilde{\nabla}_l g_{pa}\tilde{\nabla}_b g_{qk} + \tilde{\nabla}_k g_{pa}\tilde{\nabla}_b g_{ql} - \tilde{\nabla}_a g_{pl}\tilde{\nabla}_q g_{bk} - \tfrac{1}{2}\tilde{\nabla}_k g_{pa}\tilde{\nabla}_l g_{qb}\big).\end{aligned} \tag{3.14}$$

From now on we work in a normal coordinate system for $\tilde{g}$ where in addition g and g^{-1} are diagonal in the pole: (This is possible because g is symmetric and positive definite.)

$$(\tilde{g}_{ij}) = \begin{pmatrix} 1 & & & 0\\ & 1 & & \\ & & \ddots & \\ 0 & & & 1\end{pmatrix}, \quad (g_{ij}) = \begin{pmatrix} \lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} \frac{1}{\lambda_1} & & & 0\\ & \frac{1}{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\lambda_n}\end{pmatrix}. \tag{3.15}$$

Here and in the following, we always contract with the metric $g(t)$. Therefore a repeated lower index q is always paired with a factor λ_q^{-1} in contrast to Einstein's convention. We have for example:

$$g^{pq}\tilde{R}_{ipjq} = \sum_{q=1}^{n}\frac{1}{\lambda_q}\tilde{R}_{iqjq}\ .$$

To avoid misunderstandings, all summations will be explicit in the forthcoming calculations. In the coordinate system introduced above we compute

$$\tilde{\nabla}_a g^{ij} = \sum_{k,l=1}^{n} -\delta^{ik}\lambda_i^{-1}\delta^{jl}\lambda_j^{-1}\tilde{\nabla}_a g_{kl} = -\frac{1}{\lambda_i\lambda_j}\tilde{\nabla}_a g_{ij}\ .$$

Together with other similar calculations, we get from (3.14) that

$$\begin{aligned}\partial_t g^{ij} =& \sum_{a,b=1}^{n} g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b g^{ij} + \sum_{k,l=1}^{n}\left(\frac{1}{\lambda_i\lambda_k}\tilde{R}_{ikjk} + \frac{1}{\lambda_j\lambda_k}\tilde{R}_{jkik} - \frac{2}{\lambda_k\lambda_l\lambda_i\lambda_j}\tilde{\nabla}_k g_{jl}\tilde{\nabla}_k g_{il}\right) - \frac{4}{\lambda_i\lambda_j}\tilde{\nabla}_i u\tilde{\nabla}_j u\\ &+ \sum_{k,l=1}^{n}\frac{1}{\lambda_i\lambda_j\lambda_k\lambda_l}\big(\tilde{\nabla}_k g_{li}\tilde{\nabla}_k g_{il} + \tilde{\nabla}_j g_{lk}\tilde{\nabla}_k g_{il} + \tilde{\nabla}_i g_{lk}\tilde{\nabla}_k g_{jl} - \tilde{\nabla}_k g_{lj}\tilde{\nabla}_l g_{ik} - \tfrac{1}{2}\tilde{\nabla}_i g_{lk}\tilde{\nabla}_j g_{lk}\big)\ .\end{aligned} \tag{3.16}$$

The definition of φ in (3.13) implies at the base point

$$\varphi = \sum_{i=1}^{n}\left(\frac{1}{\lambda_i}\right)^m \tag{3.17}$$

which yields an evolution equation for φ:

$$\partial_t\varphi = \sum_i m\left(\frac{1}{\lambda_i}\right)^{m-1}\cdot\partial_t(\lambda_i^{-1}) = m\sum_i\left(\frac{1}{\lambda_i}\right)^{m-1}\cdot\partial_t g^{ii}\ .$$

Using (3.16) and always summing all indices from 1 to n, we obtain

$$\begin{aligned}\partial_t\varphi &= \sum_{i,a,b}\frac{m}{\lambda_i^{m-1}}g^{ab}\tilde\nabla_a\tilde\nabla_b g^{ii} + \sum_{i,k}\frac{2m}{\lambda_i^m\lambda_k}\tilde R_{ikik} - \sum_i\frac{4m}{\lambda_i^{m+1}}\tilde\nabla_i u\tilde\nabla_i u\\
&\quad - \sum_{i,k,l}\frac{m}{\lambda_i^{m+1}\lambda_k\lambda_l}\left(\tilde\nabla_k g_{il}\tilde\nabla_k g_{il} - 2\tilde\nabla_i g_{lk}\tilde\nabla_k g_{il} + \tilde\nabla_k g_{li}\tilde\nabla_l g_{ik} + \tfrac{1}{2}\tilde\nabla_i g_{lk}\tilde\nabla_i g_{lk}\right)\\
&= \sum_{i,a,b}\frac{m}{\lambda_i^{m-1}}g^{ab}\tilde\nabla_a\tilde\nabla_b g^{ii} + \sum_{i,k}\frac{2m}{\lambda_i^m\lambda_k}\tilde R_{ikik} - \sum_i\frac{4m}{\lambda_i^{m+1}}|\tilde\nabla u|_0^2\\
&\quad - \sum_{i,k,l}\frac{m}{\lambda_i^{m+1}\lambda_k\lambda_l}|\tilde\nabla_k g_{il} + \tilde\nabla_l g_{ik} - \tilde\nabla_i g_{lk}|_0^2\\
&\le \sum_{i,a,b}\frac{m}{\lambda_i^{m-1}}g^{ab}\tilde\nabla_a\tilde\nabla_b g^{ii} + \sum_{i,k}\frac{2m}{\lambda_i^m\lambda_k}\tilde R_{ikik}\ .\end{aligned}$$

On the other hand, we can calculate

$$\begin{aligned}\sum_{a,b}g^{ab}\tilde\nabla_a\tilde\nabla_b\varphi &= \sum_{a,b}g^{ab}\tilde\nabla_a\tilde\nabla_b\left(\sum_{i=1}^n\left(\frac{1}{\lambda_i}\right)^m\right) = \sum_{a,b}g^{ab}\tilde\nabla_a\left(m\sum_i\left(\frac{1}{\lambda_i}\right)^{m-1}\tilde\nabla_b g^{ii}\right)\\
&= \sum_{i,a,b}g^{ab}\frac{m}{\lambda_i^{m-1}}\tilde\nabla_a\tilde\nabla_b g^{ii} + \sum_{i,j,a,b}mg^{ab}\tilde\nabla_a g^{ij}\tilde\nabla_b g^{ij}\left(\lambda_i^{(2-m)} + \lambda_i^{(3-m)}\lambda_j^{-1} + \cdots + \lambda_j^{(2-m)}\right)\\
&= \sum_{i,a,b}g^{ab}\frac{m}{\lambda_i^{m-1}}\tilde\nabla_a\tilde\nabla_b g^{ii} + \sum_{i,j,a}m\lambda_a^{-1}\left(\lambda_i^m\lambda_j^{-2} + \lambda_i^{(1-m)}\lambda_j^{-3} + \cdots + \lambda_i^{-2}\lambda_j^m\right)|\tilde\nabla g|_0^2\\
&\ge \sum_{i,a,b}g^{ab}\frac{m}{\lambda_i^{m-1}}\tilde\nabla_a\tilde\nabla_b g^{ii}\ .\end{aligned}$$

We put both inequalities together and get

$$\partial_t\varphi \le \sum_{a,b}g^{ab}\tilde\nabla_a\tilde\nabla_b\varphi + \sum_{i,k}\frac{2m}{\lambda_i^m\lambda_k}\tilde R_{ikik} \le \sum_{a,b}g^{ab}\tilde\nabla_a\tilde\nabla_b\varphi + \frac{2m\sqrt{k_0}}{\lambda_1+\cdots+\lambda_n}\cdot\varphi$$

where we used (3.17) and the global curvature bound (3.12) for $\tilde g$. For all $q = 1\ldots n$ we have

$$\lambda_q^{-1} = \left(\lambda_q^{-m}\right)^{\frac{1}{m}} \le \left(\lambda_1^{-m}+\cdots+\lambda_n^{-m}\right)^{\frac{1}{m}} = \varphi^{\frac{1}{m}}\ ,$$

and it follows that

$$\partial_t\varphi \le \sum_{a,b}g^{ab}\tilde\nabla_a\tilde\nabla_b\varphi + 2mn\sqrt{k_0}\cdot\varphi^{1+1/m}$$

as required.

□

This lemma allows us to estimate g from below by $\tilde g$ in the following way:

Lemma 3.4 *Suppose $(g,u)(t)$ is a solution of (3.4) and (3.5) on $[0,T_0]\times D$. Then for all $\delta>0$ there exists $T=c(n,\delta)\cdot 1/\sqrt{k_0}>0$, independent of D, such that*

$$g_{ij}(t,x)\ge(1-\delta)\tilde g_{ij}(x)\qquad\forall(t,x)\in[0,T]\times D\ .$$

Proof:
Fix a $\delta>0$. Let φ be defined as in (3.13) and choose an integer $m>0$ such that

$$\frac{\ln 2n}{\ln(1/(1-\delta))}\le m\le\frac{\ln 2n}{\ln(1/(1-\delta))}+1$$

is satisfied. Since $\bar D\subset\Sigma$ is compact, we can define the Lipschitz continuous function

$$\bar\varphi(t):=\max_{x\in\bar D}\varphi(t,x)\ .$$

The maximum principle on $\bar D$ together with Lemma 3.3 implies

$$\begin{aligned}\partial_t\bar\varphi(t)&\le 2mn\sqrt{k_0}\cdot\bar\varphi(t)^{1+1/m}\\ \bar\varphi(0)&=n\ .\end{aligned}$$

This allows us to compare φ with the solution of the corresponding ordinary differential equation. Considering the definition of $\bar\varphi$, we get for all $x\in\bar D$:

$$\varphi(t,x)\le\frac{n}{(1-2n^{1+\frac{1}{m}}\sqrt{k_0}t)^m}\ .$$

If we choose T according to

$$T:=\frac{1}{2\sqrt{k_0}}\left(\frac{1}{n}\right)^{1+\frac{1}{m}}\left[1-\left(\frac{1}{2}\right)^{\frac{1}{m}}\right]=c(n,m)\frac{1}{\sqrt{k_0}},$$

we get for $0\le t\le T$ and for all $x\in\bar D$ in the coordinates (3.15) that

$$\sum_{i=1}^n\lambda_i^{-m}=\varphi(t,x)\le 2n\ .$$

By the choice of m this implies

$$\lambda_i\ge\left(\frac{1}{2n}\right)^{\frac{1}{m}}\ge(1-\delta)\qquad\forall i=1\ldots n\ .$$

Considering again the coordinates (3.15), we finally conclude

$$g_{ij}(t,x)\ge(1-\delta)\tilde g_{ij}(x)\qquad(t,x)\in[0,T]\times\bar D\ .$$

By definition T only depends on $n,m,\sqrt{k_0}$ and m depends only on n and δ. Therefore $T=T(n,\delta,\sqrt{k_0})$ is independent of D. More precisely we have $T=c(n,\delta)\cdot 1/\sqrt{k_0}$.

□

Our aim is to show the equivalence of the initial metric and the evolving metric for a short time interval. The last lemma showed that g is bounded from below by $\tilde g$. To obtain an upper bound we need a bound for $|\tilde\nabla u|_0^2$.

Lemma 3.5 *Suppose $(g,u)(t)$ is a solution of (3.4) and (3.5) on $[0,T]\times\bar{D}$ with initial data $(\tilde{g},\tilde{u})$ satisfying $|\tilde{\nabla}\tilde{u}|_0^2 \le c_0$ on Σ and $g(t)\le(1+\varepsilon)\tilde{g}$ on $[0,T]$ for an $\varepsilon>0$. Then*

$$|\tilde{\nabla}u|^2(t,x)\le(1+\varepsilon)c_0 \qquad \forall(t,x)\in[0,T]\times\bar{D}\ .$$

Proof:
Since $|\tilde{\nabla}u|^2$ satisfies (2.11), we can apply the maximum principle on $[0,T]\times\bar{D}$ to get

$$|\tilde{\nabla}u|^2(t,x)\le\max_{\Gamma}|\tilde{\nabla}u|^2(t,x)=\max_{\bar{D}}|\tilde{\nabla}\tilde{u}|_0^2\le c_0\ .$$

Here we used (3.6) and the fact that $|\cdot|^2=|\cdot|_0^2$ on Γ. The upper bound for $g(t)$ implies that we can estimate on $[0,T]\times\bar{D}$:

$$|\tilde{\nabla}u|_0^2(t,x)=\tilde{g}^{pq}\partial_pu\partial_qu\le(1+\varepsilon)g^{pq}\partial_pu\partial_pu=(1+\varepsilon)|\tilde{\nabla}u|^2(t,x)\le(1+\varepsilon)\cdot c_0\ .$$

□

For the lemma we had to assume an upper bound on the metric which is what we wanted to prove in the first place. Fortunately, we can show that this upper bound always exists.

Lemma 3.6 *Suppose $(g,u)(t)$ is a solution for (3.4) and (3.5) on $[0,T_0]\times\bar{D}$ which satisfies $|\tilde{R}m|_0^2\le k_0$ and $|\tilde{\nabla}\tilde{u}|_0^2\le c_0$. Then for all $\varepsilon>0$ there exists $T=c(n,\varepsilon)\cdot 1/(\sqrt{k_0}+c_0)$ such that*

$$g_{ij}(t,x)\le(1+\varepsilon)\tilde{g}_{ij}(x) \qquad \forall(t,x)\in[0,T]\times\bar{D}\ .$$

Proof:
Let $T_1\le T_0$ be the maximal time such that

$$\max_{x\in\bar{D}}|\tilde{\nabla}u|_0^2(t,x)\le 2c_0 \tag{3.18}$$

holds for all $t\in[0,T_1]$. Since $|\tilde{\nabla}u|_0^2$ is continuous, we know that $T_1>0$. To be able to use this estimate in the following reasoning, we will restrict ourselves to the time interval $[0,T_1]$. Later on we show that this places no restriction on the choice of T.

Using the coordinates (3.15), we can rewrite the evolution equation (3.9) for g_{ij} as follows:

$$\partial_tg_{ij}=\sum_{a,b}g^{ab}\tilde{\nabla}_a\tilde{\nabla}_bg_{ij}-\sum_{k,l}\left(\frac{1}{\lambda_j\lambda_k}\tilde{R}_{jkik}-\frac{1}{\lambda_i\lambda_k}\tilde{R}_{ikjk}+\frac{1}{2\lambda_k\lambda_l}(\tilde{\nabla}g*\tilde{\nabla}g)_{kl}\right)+4\partial_iu\partial_ju$$

where $(\tilde{\nabla}g*\tilde{\nabla}g)_{kl}$ is just an abbreviation for the more complicated quadratic terms in (3.9) summed over k and l. Following the ideas in [Shi89, Lemma 2.3], we define

$$F:=\left(1-\left[\frac{\lambda_1^m+\cdots+\lambda_n^m}{n+\sigma}\right]\right)^{-1} \tag{3.19}$$

on $[0, T_1] \times \bar{D}$ for given $\sigma > 0$ and $m \in \mathbb{Z}^+$. By the choice of the boundary values in (3.6) we have $F_{|\Gamma} \equiv (n+\sigma)/\sigma$. The evolution of F is given by

$$\begin{aligned}\partial_t F &= \left(1 - \frac{1}{n+\sigma}\sum_k \lambda_k^m\right)^{-2} \cdot \frac{1}{n+\sigma}\sum_i m\lambda_i^{m-1}\partial_t g_{ii} \\ &= \sum_{i,a,b} F^2 \frac{m\lambda_i^{m-1}}{n+\sigma} g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b g_{ii} + \sum_{i,k,l} F^2 \frac{m\lambda_i^{m-1}}{n+\sigma}\left(\frac{-2}{\lambda_i\lambda_k}\tilde{R}_{ikik} + \frac{1}{2\lambda_k\lambda_l}(\tilde{\nabla} g * \tilde{\nabla} g)_{ikl} + 4\partial_i u \partial_i u\right).\end{aligned}$$

We know from Lemma 3.4 that for all $0 < \delta < 1$ there is a $T_2 = T_2(\delta, n, k_0) > 0$ such that

$$g_{kk}(t,x) \ge (1-\delta)\tilde{g}_{kk}(x) \qquad \forall (t,x) \in [0, T_2] \times \bar{D} .$$

This implies that in the same set $\lambda_k \ge 1 - \delta$ holds for all $k = 1 \dots n$. Additionally, we can assume $F < \infty$ (otherwise $\lambda_i^m \le n + \sigma$ from (3.19)). Since F is continuous and $F(0) > 0$ we therefore get for all $t \in [0, T_2]$:

$$1 - \frac{1}{n+\sigma}\sum_k \lambda_k^m > 0 \quad \Rightarrow \quad \frac{\lambda_i^{m-1}}{n+\sigma} < \lambda_i^{-1} \le \frac{1}{1-\delta} \tag{3.20}$$

for all $i = 1 \dots n$. This allows us to estimate

$$\begin{aligned}&\sum_{i,k,l} \frac{m\lambda_i^{m-1}}{n+\sigma}\left(-\frac{2}{\lambda_i\lambda_k}\tilde{R}_{ikik} + \frac{1}{\lambda_k\lambda_l}\left(\tilde{\nabla} g * \tilde{\nabla} g\right)_{ikl} + 4\partial_i u\partial_i u\right) \\ &\qquad \le \frac{m}{(1-\delta)}\left(\frac{2n^2}{(1-\delta)^2}|\tilde{R}m|_0 + \frac{4}{(1-\delta)^2}|\tilde{\nabla} g|_0^2 + 4|\tilde{\nabla} u|_0^2\right) \le \frac{m}{(1-\delta)^3}\left(c + 4|\tilde{\nabla} g|_0^2\right)\end{aligned}$$

on $[0, \min\{T_1, T_2\}] \times \bar{D}$ where we used the bounds (3.12), (3.18), and the fact that $1 - \delta < 1$. Here c is a constant only depending on n, k_0, c_0. Returning to the evolution of F, we see

$$\partial_t F \le F^2 \sum_{i,a,b} \frac{m\lambda_i^{m-1}}{n+\sigma} g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b g_{ii} + \frac{mF^2}{(1-\delta)^3}\left(c + 4|\tilde{\nabla} g|_0^2\right) . \tag{3.21}$$

On the other hand we compute

$$\begin{aligned}\sum_{a,b} g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b F &= \sum_{a,b} g^{ab}\tilde{\nabla}_a\left(\frac{m}{n+\sigma}F^2\sum_k \lambda_k^{m-1}\tilde{\nabla}_b g_{kk}\right) \\ &= \sum_{i,a,b} g^{ab}\frac{m\lambda_i^{m-1}}{n+\sigma}F^2\tilde{\nabla}_a\tilde{\nabla}_b g_{ii} + \underbrace{\sum_{i,a,b}\frac{2m^2\lambda_i^{m-1}\lambda_j^{m-1}}{(n+\sigma)^2}F^3 g^{ab}\tilde{\nabla}_a g_{ii}\cdot\tilde{\nabla}_b g_{jj}}_{\ge 0} \\ &\quad + \sum_{i,j,a,b}\frac{m}{n+\sigma}F^2\left(g^{ab}\tilde{\nabla}_a g_{ij}\tilde{\nabla}_b g_{ij}\right)\underbrace{\left(\lambda_i^{m-2} + \lambda_i^{m-3}\lambda_j + \cdots + \lambda_j^{m-2}\right)}_{m-1 \text{ terms}} \\ &\ge \sum_{i,a,b} g^{ab}\frac{m\lambda_i^{m-1}}{n+\sigma}F^2\tilde{\nabla}_a\tilde{\nabla}_b g_{ii} + \sum_{i,j,k}\frac{m}{n+\sigma}F^2(m-1)(1-\delta)^{m-2}\frac{1}{\lambda_k}\tilde{\nabla}_k g_{ij}\tilde{\nabla}_k g_{ij} \\ &\ge \sum_{i,a,b} g^{ab}\frac{m\lambda_i^{m-1}}{n+\sigma}F^2\tilde{\nabla}_a\tilde{\nabla}_b g_{ii} + m(m-1)\left(\frac{1}{n+\sigma}\right)^{1+\frac{1}{m}} F^2(1-\delta)^{m-2}|\tilde{\nabla} g|_0^2 .\end{aligned} \tag{3.22}$$

In this step we used (3.20) to estimate $\lambda_a^{-1} \geq (n+\sigma)^{-\frac{1}{m}}$ and $\lambda_i^k \geq (1-\delta)^k$.

We fix $\varepsilon > 0$, set $\sigma := n$, and choose $m \in \mathbb{Z}^+$ according to

$$20n^2 + \frac{\ln 2n}{\ln(1+\varepsilon)} \leq m \leq \frac{\ln 2n}{\ln(1+\varepsilon)} + 20n^2 + 1 \ .$$

From $m \geq 18n^2 + 1$ we conclude that

$$(m-1)\left(\frac{1}{2n}\right)^2 \geq \frac{9}{2} \geq \frac{4}{(1-\delta)^{m+1}}$$

for some $\delta = \delta(m) > 0$. Therefore we can estimate

$$m(m-1)\left(\frac{1}{2n}\right)^{1+\frac{1}{m}}(1-\delta)^{m-2} \geq \frac{4m}{(1-\delta)^3} \ .$$

Putting together equations (3.21) and (3.22), this implies that

$$\begin{aligned}
\partial_t F &\leq \sum_{a,b} g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b F + \frac{cF^2}{(1-\delta)^3} + F^2|\tilde{\nabla} g|_0^2 \left\{\frac{4m}{(1-\delta)^3} - m(m-1)\left(\frac{1}{2n}\right)^{1+\frac{1}{m}}(1-\delta)^{m-2}\right\} \\
&\leq \sum_{a,b} g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b F + \frac{cF^2}{(1-\delta)^3}
\end{aligned}$$

where $c = c(n, c_0, k_0)$. Considering the initial and boundary data (3.6), we know $F_{|\Gamma} \equiv 2$. From comparison to the appropriate ordinary differential equation we get that

$$F(x,t) \leq 2\left(1 - \frac{ct}{(1-\delta)^3}\right)^{-1}$$

for $0 \leq t \leq \min\{T_1, T_2\}$. Defining $T := \min\{T_1, T_2, \frac{(1-\delta)^3}{2c}\}$, we conclude that $F \leq 4$ on $[0,T]\times\bar{D}$. This implies

$$\lambda_k \leq (2n)^{\frac{1}{m}} \leq 1+\varepsilon \qquad \forall k = 1\ldots n$$

by the choice of m and shows the claim on $[0,T]$, assuming the bound $|\tilde{\nabla} u|_0^2 \leq 2c_0$. If $T \neq T_1$, we are already finished, because T does not depend on T_1 in this case. If $T = T_1$, we apply Lemma 3.5. Because of the equivalence of g and $\tilde{g}$ just proven, we have

$$\sup_{x\in\bar{D}} |\tilde{\nabla} u|_0^2(t,x) \leq (1+\varepsilon)c_0$$

for all $t \in [0, T_1]$. Therefore, T_1 was not maximally chosen at the beginning. We conclude that T does not depend on T_1 and that $T = \min\{T_2, \frac{1}{2}(1-\delta)^3 \cdot c(n)/(\sqrt{k_0}+c_0)\}$, where T_2 is from Lemma 3.4 with $T_2 = c(n,\delta)/\sqrt{k_0}$. This gives $T > c(n,\varepsilon)\cdot 1/(\sqrt{k_0}+c_0)$ since $\delta = \delta(m) = \delta(n,\varepsilon)$.
□

The combination of Lemma 3.4 and Lemma 3.6 shows that there is a short time interval such that the evolving metric is equivalent to the initial one.

Theorem 3.7 *Suppose that $(g, u)(t)$ is a solution of (3.4) and (3.5) on $[0, T_0] \times \bar{D}$ with initial data $(\tilde{g}, \tilde{u})$ satisfying $|\tilde{R}m|_0^2 \leq k_0$ and $|\tilde{\nabla}\tilde{u}|_0^2 \leq c_0$. Then for any $\varepsilon > 0$ there is a time $T = T(n, \varepsilon, k_0, c_0) > c(n, \varepsilon) \cdot 1/(\sqrt{k_0} + c_0)$ independent of D such that*

$$(1 - \varepsilon)\tilde{g}_{ij}(x) \leq g_{ij}(t, x) \leq (1 + \varepsilon)\tilde{g}_{ij}(x) \quad \forall (t, x) \in [0, T] \times \bar{D} \ .$$

□

3.3 Higher order estimates

In order to prove short time existence for the modified flow (3.4) and (3.5), we need a priori bounds for all derivatives of the solution $(g, u)(t)$. Since we want to prove existence on complete manifolds, we need estimates that are independent of the domain D.

For technical convenience we collect the coefficient functions g_{ij} of g together with u in a vector that we call Ψ. In short

$$\Psi := (g_{ij}, u) \qquad i, j = 1, \ldots, n \ . \tag{3.23}$$

We use the abbreviation $\tilde{\nabla}\Psi$ for the collection of first covariant derivatives with respect to $\tilde{g}$ and similar for higher derivatives. The norm of Ψ and its derivatives is computed pointwise with the Euclidean vector norm. This gives the local description

$$|\tilde{\nabla}^k \Psi|_0^2 = |\tilde{\nabla}^k g|_0^2 + |\tilde{\nabla}^k u|_0^2$$

for all $k \geq 0$ in the chosen coordinate system. We prove the following gradient estimate:

Proposition 3.8 *Let $(g, u)(t)$ be a solution of (3.4) and (3.5) on $[0, T] \times \bar{D}$ for $D := B_{\gamma+\delta}(x_0)$ the geodesic ball around x_0 of radius $\gamma + \delta$ with respect to the initial metric $\tilde{g}$. Assume that on $[0, T] \times \bar{D}$ the metrics $g(t)$ are equivalent to $\tilde{g}$ in the sense that*

$$(1 - \varepsilon)\tilde{g}_{ij}(x) \leq g_{ij}(t, x) \leq (1 + \varepsilon)\tilde{g}_{ij}(x) \tag{3.24}$$

for $0 < \varepsilon \leq \varepsilon_0 := 1/416000n^{10}$, and we have bounds $|\tilde{R}m|_0^2 \leq k_0$ and $|\tilde{\nabla}\tilde{u}|_0^2 \leq c_0$. Then there is a constant $C = C(n, \tilde{g}, c_0, \gamma, \delta) > 0$ such that

$$|\tilde{\nabla}\Psi|_0^2(t, x) \leq C$$

holds on $[0, T] \times B(x_0, \gamma + \delta/2)$. In particular the bound is independent of the base point x_0.

Proof:
Fix an ε in the designated range and comparable to $1/n^{10}$. We start with the computation of the evolution of $\tilde{\nabla} g$ and compute from (3.9)

$$\begin{aligned}\partial_t \tilde{\nabla}_k g_{ij} &= \tilde{\nabla}_k \partial_t g_{ij} \\ &= g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\tilde{\nabla}_k g_{ij} + g * \tilde{\nabla}\tilde{R}m + \tilde{\nabla} g * \tilde{R}m + \tilde{\nabla} u * \tilde{\nabla}^2 u + \tilde{\nabla} g * \tilde{\nabla}^2 g + \tilde{\nabla} g * \tilde{\nabla} g * \tilde{\nabla} g\end{aligned} \tag{3.25}$$

where summation over a and b is understood. Consequently we get

$$\begin{aligned}\partial_t|\tilde\nabla g|_0^2 &= 2\tilde\nabla_k g_{ij}\cdot g^{ab}\tilde\nabla_a\tilde\nabla_b\tilde\nabla_k g_{ij} + \tilde\nabla g * \tilde\nabla\tilde Rm + \tilde\nabla g * \tilde\nabla g * \tilde Rm + \tilde\nabla g * \tilde\nabla u * \tilde\nabla^2 u\\ &\quad + \tilde\nabla g * \tilde\nabla g * \tilde\nabla^2 g + \tilde\nabla g * \tilde\nabla g * \tilde\nabla g * \tilde\nabla g,\end{aligned}$$

from which we can deduce the following estimate for the time derivative of $|\tilde\nabla g|_0^2$:

$$\begin{aligned}\partial_t|\tilde\nabla g|_0^2 &\le g^{ab}\tilde\nabla_a\tilde\nabla_b|\tilde\nabla g|_0^2 - |\tilde\nabla^2 g|_0^2 + c(\tilde g)|\tilde\nabla g|_0 + c(\tilde g)|\tilde\nabla g|_0^2 + 16n^3|\tilde\nabla^2 u|_0|\tilde\nabla u|_0|\tilde\nabla g|_0\\ &\quad + 80n^5|\tilde\nabla^2 g|_0|\tilde\nabla g|_0^2 + 144n^6|\tilde\nabla g|_0^4\ ,\end{aligned} \tag{3.26}$$

using (3.24) and counting carefully the number of terms. Here $c(\tilde g)$ depends on derivatives of $\tilde g$ since we have to estimate $\tilde\nabla\tilde Rm$ and $\tilde Rm$ on the compact set $\bar B_{\gamma+\delta}(x_0)$. In addition, we computed

$$g^{ab}\tilde\nabla_a\tilde\nabla_b\tilde\nabla_k g_{ij} = g^{ab}\tilde\nabla_k(\tilde\nabla_a\tilde\nabla_b g_{ij}) + g * \tilde\nabla\tilde Rm + \tilde\nabla g * \tilde Rm$$

and applied the Bochner formula

$$\begin{aligned}\tilde\nabla_k g_{ij}\cdot g^{ab}\tilde\nabla_a\tilde\nabla_b(\tilde\nabla_k g_{ij}) &= g^{ab}\tilde\nabla_a\tilde\nabla_b(\tilde\nabla_k g_{ij}\cdot\tilde\nabla_k g_{ij}) - 2g^{ab}\tilde\nabla_a\tilde\nabla_k g_{ij}\cdot\tilde\nabla_b\tilde\nabla_k g_{ij}\\ &\le g^{ab}\tilde\nabla_a\tilde\nabla_b|\tilde\nabla g|_0^2 - \tilde g^{ab}\tilde\nabla_a\tilde\nabla_k g_{ij}\cdot\tilde\nabla_b\tilde\nabla_k g_{ij}\ .\end{aligned}$$

Using (3.10), the corresponding estimate for $|\tilde\nabla u|_0^2$ is given by

$$\partial_t|\tilde\nabla u|^2 \le g^{ab}\tilde\nabla_a\tilde\nabla_b|\tilde\nabla u|_0^2 - |\tilde\nabla^2 u|_0^2 + c(\tilde g)|\tilde\nabla u|_0^2 + n^3|\tilde\nabla^2 u|_0|\tilde\nabla g|_0|\tilde\nabla u|_0\ . \tag{3.27}$$

Combining equations (3.26) and (3.27), we get an inequality for $|\tilde\nabla\Psi|_0^2$:

$$\partial_t|\tilde\nabla\Psi|_0^2 \le g^{ab}\tilde\nabla_a\tilde\nabla_b|\tilde\nabla\Psi|_0^2 - |\tilde\nabla^2\Psi|_0^2 + 100n^5|\tilde\nabla^2\Psi|_0|\tilde\nabla\Psi|_0^2 + 144n^6|\tilde\nabla\Psi|_0^4 + c|\tilde\nabla\Psi|_0^2 + c|\tilde\nabla\Psi|_0\ .$$

This can be simplified using Young's inequality:

$$|\tilde\nabla^2\Psi|_0\cdot 100n^5|\tilde\nabla\Psi|_0^2 \le \tfrac12|\tilde\nabla^2\Psi|_0^2 + 5000n^{10}|\tilde\nabla\Psi|_0^4\ ,$$

and we get with $c|\tilde\nabla\Psi|_0 \le c|\tilde\nabla\Psi|_0^2 + c$:

$$\partial_t|\tilde\nabla\Psi|_0^2 \le g^{ab}\tilde\nabla_a\tilde\nabla_b|\tilde\nabla\Psi|_0^2 - \tfrac12|\tilde\nabla^2\Psi|_0^2 + 5200n^{10}|\tilde\nabla\Psi|_0^4 + c|\tilde\nabla\Psi|_0^2 + c\ . \tag{3.28}$$

Again calculating in the coordinate system (3.15), we have from (3.24)

$$1-\varepsilon \le \lambda_k \le 1+\varepsilon, \qquad \tfrac12 \le \lambda_k \le 2 \qquad k = 1,\dots,n\ .$$

Following the ideas in [Shi89, §4], we define $m := 41600n^{10}$ and $a := 10400n^{10}$, and a function φ on $[0,T]\times B_{\gamma+\delta}(x_0)$ as follows

$$\varphi(t,x) := a + \sum_{k=1}^n \lambda_k^m\ . \tag{3.29}$$

It satisfies the evolution equation

$$\partial_t\varphi \le g^{ab}\tilde\nabla_a\tilde\nabla_b\varphi + C - \frac{m^2}{8}|\tilde\nabla\Psi|_0^2\ .$$

The argument is similar to [Shi89, §4] where it is proven that

$$\partial_t \varphi \leq g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\varphi + \hat{C} - \frac{m^2}{8}|\tilde{\nabla} g|_0^2 \; .$$

We can add $\frac{m^2}{8}\big(2c_0 - |\tilde{\nabla} u|_0^2\big) > 0$ on the right hand side to get

$$\partial_t \varphi \leq g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\varphi + \left(\hat{C} + \frac{m^2}{8}\cdot 2\,c_0\right) - \frac{m^2}{8}\left(|\tilde{\nabla} g|_0^2 + |\tilde{\nabla} u|_0^2\right)$$

in view of the bound on $|\tilde{\nabla} u|_0^2$ in Lemma 3.5. Note that C depends only on $n, \tilde{g}$ and c_0. Combining the evolution equations, we get

$$\begin{aligned}\partial_t\big(\varphi \cdot |\tilde{\nabla}\Psi|_0^2\big) \leq\ & g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\big(\varphi\cdot|\tilde{\nabla}\Psi|_0^2\big) - 2\sum_{a,b} g^{ab}\tilde{\nabla}_a\varphi\tilde{\nabla}_b|\tilde{\nabla}\Psi|_0^2 - \frac{\varphi}{2}|\tilde{\nabla}^2\Psi|_0^2 + 5200n^{10}\varphi|\tilde{\nabla}\Psi|_0^4 \\ & + c\varphi|\tilde{\nabla}\Psi|_0^2 + c\varphi + C|\tilde{\nabla}\Psi|_0^2 - \frac{m^2}{8}|\tilde{\nabla}\Psi|_0^4 \; .\end{aligned} \tag{3.30}$$

Our task now is to simplify this equation significantly. Calculating $5200n^{10}\varphi \leq \frac{m^2}{16}$ which holds for all $\varepsilon \leq \varepsilon_0$ and using (3.29) we conclude

$$\big(5200n^{10}\varphi - \frac{m^2}{8}\big)|\tilde{\nabla}\Psi|_0^4 \leq -\frac{m^2}{16}|\tilde{\nabla}\Psi|_0^4 \; . \tag{3.31}$$

The cross term can be handled as follows:

$$\begin{aligned}-2g^{ab}\tilde{\nabla}_a\varphi\tilde{\nabla}_b|\tilde{\nabla}\Psi|_0^2 &= -2g^{ab}\tilde{\nabla}_a\left(\sum_{k=1}^n \lambda_k^m\right)\cdot\tilde{\nabla}_b|\tilde{\nabla}\Psi|_0^2 \leq -m\sum_k \lambda_k^{m-1}\cdot\tilde{g}^{ab}\tilde{\nabla}_a g_{kk}\cdot\tilde{\nabla}_b|\tilde{\nabla}\Psi|_0^2 \\ &\leq n\cdot mn(1+\varepsilon)^{m-1}|\tilde{\nabla} g|_0\cdot 2n^3|\tilde{\nabla}\Psi|_0|\tilde{\nabla}^2\Psi|_0 \leq 4mn^5|\tilde{\nabla}\Psi|_0^2\cdot|\tilde{\nabla}^2\Psi|_0 \\ &\leq \frac{\varphi}{2}|\tilde{\nabla}^2\Psi|_0^2 + \frac{1}{2\varphi}16m^2n^{10}|\tilde{\nabla}\Psi|_0^4\end{aligned} \tag{3.32}$$

where we have used Young's inequality in the last step and the fact that $(1+\varepsilon)^{m-1} < 2$ for all $\varepsilon \leq \varepsilon_0$. Because of $\varphi \geq a > 1$, we can also estimate

$$C|\tilde{\nabla}\Psi|_0^2 \leq C\varphi|\tilde{\nabla}\Psi|_0^2 \; . \tag{3.33}$$

Collecting (3.31),(3.32), and (3.33) we get from (3.30) that

$$\partial_t\big(\varphi\cdot|\tilde{\nabla}\Psi|_0^2\big) \leq g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\big(\varphi\cdot|\tilde{\nabla}\Psi|_0^2\big) - \frac{m^2}{16}|\tilde{\nabla}\Psi|_0^4 + \frac{10m^2n^{10}}{\varphi}|\tilde{\nabla}\Psi|_0^4 + c\varphi|\tilde{\nabla}\Psi|_0^2 + c\varphi \; .$$

There is the lower bound $\varphi \geq a > 320n^{10}$ from the choice of a, which allows us to estimate

$$\frac{10m^2n^{10}}{\varphi} \leq \frac{m^2}{32} \; ,$$

giving us

$$\partial_t\big(\varphi\cdot|\tilde\nabla\Psi|_0^2\big)\le g^{ab}\tilde\nabla_a\tilde\nabla_b\big(\varphi\cdot|\tilde\nabla\Psi|_0^2\big)-\frac{m^2}{32}|\tilde\nabla\Psi|_0^4+c\varphi|\tilde\nabla\Psi|_0^2+c\varphi\ . \tag{3.34}$$

Now $m\ge 2[a+n(1+\varepsilon)^m]$ (for all $\varepsilon\le\varepsilon_0$) implies

$$\frac{m^2}{32}|\tilde\nabla\Psi|_0^4=\frac{m^2}{32\varphi^2}|\tilde\nabla\Psi|_0^4\varphi^2\ge\frac{m^2}{32[a+(1+\varepsilon)^m]^2}|\tilde\nabla\Psi|_0^4\varphi^2\ge\frac{1}{8}|\tilde\nabla\Psi|_0^4\varphi^2\ .$$

An application of Young's inequality

$$c\cdot\varphi|\tilde\nabla\Psi|_0^2\le 4c^2+\tfrac{1}{16}\varphi^2|\tilde\nabla\Psi|_0^4$$

leads from (3.34) to

$$\partial_t\big(\varphi\cdot|\tilde\nabla\Psi|_0^2\big)\le g^{ab}\tilde\nabla_a\tilde\nabla_b\big(\varphi\cdot|\tilde\nabla\Psi|_0^2\big)-\tfrac{1}{16}\varphi^2\cdot|\tilde\nabla\Psi|_0^4+c+c\varphi\ .$$

The last term can be estimated for all $\varepsilon\le\varepsilon_0$ as follows:

$$c\varphi\le c[a+(1+\varepsilon)^m]\le c[10400n^{10}+3]\le c\ .$$

Defining $\phi:=\varphi\cdot|\tilde\nabla\Psi|_0^2$, we get in the end

$$\partial_t\phi\le g^{ab}\tilde\nabla_a\tilde\nabla_b\phi-\frac{1}{16}\phi^2+c \tag{3.35}$$

where c still only depends on n, c_0 and the initial metric $\tilde g$.

The next step is to localize this equation. To this end we use the same cut-off function ξ on $B_{\gamma+\delta}(x_0)$ as in [Shi89, §4(38)] and define

$$F(t,x):=\xi(x)\cdot\phi(t,x)\qquad\forall(t,x)\in[0,T]\times B_{\gamma+\delta}(x_0)\ .$$

We can compute in the same way that

$$\begin{aligned}&F=\xi\cdot\varphi\cdot|\tilde\nabla\Psi|_0^2\ge 0 &&\text{on } [0,T]\times B_{\gamma+\delta}(x_0)\\ &F\equiv 0 &&\text{on } [0,T]\times\big(\Sigma\setminus B_{\gamma+3\delta/4}(x_0)\big)\ .\end{aligned}$$

In addition, there is a point $(t_0,x_0)\in[0,T]\times B_{\gamma+3\delta/4}(x_0)$ such that

$$F(t_0,x_0)=\max_{[0,T]\times B_{\gamma+\delta}(x_0)}F(t,x)\ . \tag{3.36}$$

We would like to assume that $t_0>0$, but there is a difference to Shi's work here when considering the initial value of F at time $t=0$

$$F(0)=\xi\varphi(0)|\tilde\nabla\Psi|_0^2(0)=\xi\varphi(0)|\tilde\nabla\tilde u|_0^2$$

so we cannot be sure that the maximum does not occur at $t=0$. However, this is not a problem because of our initial bound on $\tilde\nabla\tilde u$. If the maximum of F on $[0,T]\times B_{\gamma+\delta}(x_0)$ should occur at $t=0$, we easily get

$$\begin{aligned}&\xi\cdot\varphi(t)\cdot|\tilde\nabla\Psi|_0^2(t)\le\xi\cdot\varphi(0)\cdot|\tilde\nabla\tilde u|_0^2\\ \Rightarrow\quad &|\tilde\nabla\Psi|_0^2(t)\le\frac{\varphi(0)}{\varphi(t)}|\tilde\nabla\tilde u|_0^2\le\frac{a+n}{a+n(1-\varepsilon)^m}\cdot c_0=c(n,c_0)\end{aligned}$$

and are already done. Otherwise, we deduce from (3.36) that

$$\xi(x_0) \cdot \partial_t \phi(t_0, x_0) = \partial_t F(t_0, x_0) \geq 0 \ .$$

Additionally, we get from (3.36) that

$$0 \geq g^{ab} \tilde{\nabla}_a \tilde{\nabla}_b F(t_0, x_0) = \xi \cdot g^{ab} \tilde{\nabla}_a \tilde{\nabla}_b \phi + \phi \cdot g^{ab} \tilde{\nabla}_a \tilde{\nabla}_b \xi + 2 g^{ab} \tilde{\nabla}_a \phi \cdot \tilde{\nabla}_b \xi \ .$$

Since $\xi \geq 0$, we conclude from (3.35) that

$$\frac{1}{16} \xi \phi^2 \leq c\xi - \phi \cdot g^{ab} \tilde{\nabla}_a \tilde{\nabla}_b \xi - 2 g^{ab} \tilde{\nabla}_a \phi \cdot \tilde{\nabla}_b \xi \ .$$

This can be simplified since we know from (3.36) that

$$0 = \tilde{\nabla} F(t_0, x_0) = \xi \cdot \tilde{\nabla} \phi + \phi \cdot \tilde{\nabla} \xi \ ,$$

giving us

$$\frac{1}{16} \xi \phi^2 \leq c\xi - \phi \cdot g^{ab} \tilde{\nabla}_a \tilde{\nabla}_b \xi + 2 \xi^{-1} \phi g^{ab} \tilde{\nabla}_a \xi \cdot \tilde{\nabla}_b \xi \ .$$

We have to use estimates on ξ to continue which are proven in [Shi89, §4(40),(45)]:

$$|\tilde{\nabla} \xi|_0^2 \leq \frac{16^2}{\delta^2} \xi, \qquad \tilde{\nabla}_a \tilde{\nabla}_b \xi \geq -c(\gamma, \delta, k_0) \tilde{g}_{ab} \ .$$

Using $0 \leq \xi \leq 1$ and (3.24), we therefore get at (t_0, x_0) that

$$\frac{1}{16} \xi \phi^2 \leq c + c\phi$$

which implies

$$\frac{1}{16} F^2 \leq c + cF$$

and therefore

$$F(t, x) \leq F(t_0, x_0) \leq c(n, \tilde{g}, c_0, \gamma, \delta)$$

holds for all $(t, x) \in [0, T] \times B_{\gamma+\delta}(x_0)$. Since $\xi \equiv 1$ on $B_{\gamma+\delta/2}(x_0)$, we have there

$$|\tilde{\nabla} \Psi|_0^2 \leq c \varphi^{-1} \leq c \cdot a = c(n, \tilde{g}, c_0, \gamma, \delta)$$

from the definition of φ and ϕ as required.

□

Having established the bound for the first derivative of Ψ, we prove the boundedness of higher derivatives by induction.

Proposition 3.9 *Suppose $(g, u)(t)$ is a solution as in Proposition 3.8 and $\tilde{\nabla}^m \Psi$ is constructed as above. Then for any $m \in \mathbb{Z}_0^+$ there is a $C_m = C_m(n, \gamma, \delta, \tilde{g}, c_0)$ independent of D such that*

$$|\tilde{\nabla}^m \Psi|_0^2 \leq C_m$$

on $[0, T] \times B_{\gamma+\delta/(m+1)}(x_0)$ where $|\cdot|_0$ is the norm associated with $\tilde{g}$.

Proof:
To start the induction argument, we need a global bound on the norm of $\tilde{u}$ itself, not only on $\tilde{\nabla}\tilde{u}$. We therefore make the additional assumption

$$|\tilde{u}|_0^2 \le c_0$$

on Σ. From (3.10) we get the evolution equation

$$\partial_t |u|_0^2 = g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b|u|_0^2 - |\tilde{\nabla}u|_0^2 \ ,$$

and an application of the maximum principle on $[0,T]\times B_{\gamma+\delta}(x_0)$ proves the boundedness of u also for later times. Thus we get for $m=0$ on $[0,T]\times B_{\gamma+\delta}(x_0)$ that

$$|\Psi|_0^2 = |g|_0^2 + |u|_0^2 \le 4|\tilde{g}|_0^2 + |\tilde{u}|_0^2 \le 4n + c_0 = C_0(n,c_0) \ . \tag{3.37}$$

The case $m=1$ is proven in Proposition 3.8.

Suppose that the statement is true for $k=0,\dots,m-1$. Then there exist constants $C_0,\dots,C_{m-1}$ such that

$$|\tilde{\nabla}^k\Psi|_0^2(t,x) \le C_k \qquad \forall (t,x)\in[0,T]\times B_{\gamma+\delta/(k+1)}(x_0) \ . \tag{3.38}$$

We want to prove the statement for $k=m$. Assume $m\ge 2$ and calculate

$$\partial_t\big(\tilde{\nabla}^m g_{ij}\big) = g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\big(\tilde{\nabla}^m g_{ij}\big) + \sum_I \tilde{\nabla}^{k_1}g * \dots * \tilde{\nabla}^{k_{m+2}}g * P_{k_1k_2\dots k_{m+2}} + \sum_J \tilde{\nabla}^{l_1}u * \tilde{\nabla}^{l_2}u \tag{3.39}$$

from (3.9). The sets of indices are defined as follows:

$$I := \big\{(k_1,\dots,k_{m+2})\in\mathbb{N}^{m+2} : 0\le k_i\le m+1 \quad \forall i=1,\dots,m+2;\ k_1+\dots+k_{m+2}\le m+2\big\}$$
$$J := \big\{(l_1,l_2)\in\mathbb{N}^2 : 1\le l_i\le m+1 \quad \forall i=1,\dots,m+2;\ l_1+l_2=m+2\big\} \ .$$

We denote by $P_{k_1k_2\dots k_{m+2}}$ a polynomial in $g, g^{-1}, \tilde{R}m, \tilde{\nabla}\tilde{R}m, \dots, \tilde{\nabla}^m\tilde{R}m$. Then the formula can easily be proven by induction. Analogously, we get from (3.10) that

$$\partial_t\big(\tilde{\nabla}^m u\big) = g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\big(\tilde{\nabla}^m u\big) + \sum_K \tilde{\nabla}^{k_1}g * \dots * \tilde{\nabla}^{k_m}g * \tilde{\nabla}^l u * P_{k_1\dots k_m l} \tag{3.40}$$

holds where in this case the set of indices is

$$K := \big\{(k_1,\dots,k_m,l)\in\mathbb{N}^{m+1} : 0\le k_i\le m-1;\ 1\le l\le m+1;\ k_1+\dots+k_m+l\le m+2\big\} \ .$$

We get an equation for $\tilde{\nabla}^m\Psi$ out of (3.39) and (3.40) as follows:

$$\partial_t\big(\tilde{\nabla}^m\Psi\big) = g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\big(\tilde{\nabla}^m\Psi\big) + \sum_I \tilde{\nabla}^{k_1}\Psi * \dots * \tilde{\nabla}^{k_{m+2}}\Psi * P_{k_1k_2\dots k_{m+2}} \tag{3.41}$$

where $P_{k_1k_2\dots k_{m+2}}$ again is a polynomial as above. From this we can deduce an evolution equation for the norm squared of $\tilde{\nabla}^m\Psi$:

$$\begin{aligned}\partial_t|\tilde{\nabla}^m\Psi|_0^2 =& g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b|\tilde{\nabla}^m\Psi|_0^2 - 2g^{ab}\tilde{\nabla}_a\big(\tilde{\nabla}^m\Psi\big)\cdot\tilde{\nabla}_b\big(\tilde{\nabla}^m\Psi\big)\\ &+ \sum_I \tilde{\nabla}^{k_1}\Psi * \dots * \tilde{\nabla}^{k_{m+2}}\Psi * \tilde{\nabla}^m\Psi * P_{k_1k_2\dots k_{m+2}}\end{aligned}$$

which we can refine as before:

$$-2g^{ab}\tilde{\nabla}_a\big(\tilde{\nabla}^m\Psi\big)\cdot\tilde{\nabla}_b\big(\tilde{\nabla}^m\Psi\big)\le-|\tilde{\nabla}^{m+1}\Psi|_0^2$$
$$|P_{k_1k_2\dots k_{m+2}}|_0\le c(n,m,|\tilde{g}|_{C^\infty})\ ,$$

using the boundedness of the derivatives $\tilde{\nabla}^k\tilde{R}m$ of the initial curvature on the compact set $\bar{B}_{\gamma+\delta}(x_0)$. Therefore c depends on the whole C^∞-norm of $\tilde{g}$. Making use of the induction hypotheses (3.38), we find on $[0,T]\times B_{\gamma+\delta/m}(x_0)$ that:

$$\begin{aligned}\partial_t|\tilde{\nabla}^m\Psi|_0^2\le{}& g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b|\tilde{\nabla}^m\Psi|_0^2-|\tilde{\nabla}^{m+1}\Psi|_0^2\\&+C\cdot\Big\{|\tilde{\nabla}^m\Psi|_0|\tilde{\nabla}^{m+1}\Psi|_0\big(1+|\tilde{\nabla}\Psi|_0\big)+|\tilde{\nabla}^m\Psi|_0^2\big(1+|\tilde{\nabla}\Psi|_0^2+|\tilde{\nabla}^2\Psi|_0\big)+|\tilde{\nabla}^m\Psi|_0\Big\}\end{aligned}\tag{3.42}$$

where $C=C(C_0,\dots,C_{m-1},n,m,|\tilde{g}|_{C^\infty},c_0,\delta,\gamma)$. The first summand in parentheses corresponds to the terms in the big sum that contain one factor of order $m+1$, the second to the terms that contain one factor of order m and the third of all other terms that only contain factors of orders less than m and can be dealt with by (3.38). We first consider the case $m=2$:

$$\begin{aligned}\partial_t|\tilde{\nabla}^2\Psi|_0^2\le{}& g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b|\tilde{\nabla}^2\Psi|_0^2-|\tilde{\nabla}^3\Psi|_0^2\\&+C\cdot\Big\{|\tilde{\nabla}^2\Psi|_0|\tilde{\nabla}^3\Psi|_0\big(1+|\tilde{\nabla}\Psi|_0\big)+|\tilde{\nabla}^2\Psi|_0^2\big(1+|\tilde{\nabla}\Psi|_0^2+|\tilde{\nabla}^2\Psi|_0\big)+|\tilde{\nabla}^2\Psi|_0\Big\}\\\le{}& g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b|\tilde{\nabla}^2\Psi|_0^2-|\tilde{\nabla}^3\Psi|_0^2+C\cdot\Big\{|\tilde{\nabla}^3\Psi|_0|\tilde{\nabla}^2\Psi|_0+|\tilde{\nabla}^2\Psi|_0^3+|\tilde{\nabla}^2\Psi|_0^2+|\tilde{\nabla}^2\Psi|_0\Big\}\ .\end{aligned}$$

We can estimate by Young's inequality

$$C\cdot|\tilde{\nabla}^3\Psi|_0|\tilde{\nabla}^2\Psi|_0\le\tfrac{1}{2}|\tilde{\nabla}^3\Psi|_0^2+\tfrac{1}{2}C^2|\tilde{\nabla}^2\Psi|_0^2$$
$$C\cdot(|\tilde{\nabla}^2\Psi|_0+|\tilde{\nabla}^2\Psi|_0^2)\le C\cdot\big(|\tilde{\nabla}^2\Psi|_0^3+1\big)$$

and get

$$\partial_t|\tilde{\nabla}^2\Psi|_0^2\le g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b|\tilde{\nabla}^2\Psi|_0^2-\tfrac{1}{2}|\tilde{\nabla}^3\Psi|_0^2+C\cdot|\tilde{\nabla}^2\Psi|_0^2+C\ .$$

The same reasoning applies in the case $m\ge3$, and we conclude that

$$\partial_t|\tilde{\nabla}^m\Psi|_0^2\le g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b|\tilde{\nabla}^m\Psi|_0^2-\tfrac{1}{2}|\tilde{\nabla}^{m+1}\Psi|_0^2+C\cdot|\tilde{\nabla}^m\Psi|_0^2+C\tag{3.43}$$

is valid for all $m\ge2$ on $[0,T]\times B_{\gamma+\delta/m}(x_0)$. Using (3.38) we compute:

$$\partial_t|\tilde{\nabla}^{m-1}\Psi|_0^2\le g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b|\tilde{\nabla}^{m-1}\Psi|_0^2-\tfrac{1}{2}|\tilde{\nabla}^m\Psi|_0^2+C\tag{3.44}$$

for all $m\ge2$ on $[0,T]\times B_{\gamma+\delta/(m-1)}(x_0)$. We want to use (3.44) to cancel the bad term in (3.43). To do that, we define a new test function

$$\phi:=\big(\lambda+|\tilde{\nabla}^{m-1}\Psi|_0^2\big)\cdot|\tilde{\nabla}^m\Psi|_0^2$$

where the constant $\lambda > 0$ will be chosen later, again following the ideas of [Shi89, Lemma 4.2]. The evolution equation for ϕ is given as follows:

$$\begin{aligned}\partial_t\phi &= (\partial_t|\tilde\nabla^{m-1}\Psi|_0^2)\cdot|\tilde\nabla^m\Psi|_0^2 + \big(\lambda+|\tilde\nabla^{m-1}\Psi|_0^2\big)\cdot(\partial_t|\tilde\nabla^m\Psi|_0^2)\\ &\le g^{ab}\tilde\nabla_a\tilde\nabla_b|\tilde\nabla^{m-1}\Psi|_0^2\cdot|\tilde\nabla^m\Psi|_0^2 - \tfrac12|\tilde\nabla^m\Psi|_0^4 + C|\tilde\nabla^m\Psi|_0^2\\ &\quad + g^{ab}\tilde\nabla_a\tilde\nabla_b|\tilde\nabla^m\Psi|_0^2\cdot\big(\lambda+|\tilde\nabla^{m-1}\Psi|_0^2\big) - \tfrac12|\tilde\nabla^{m+1}\Psi|_0^2\cdot\big(\lambda+|\tilde\nabla^{m-1}\Psi|_0^2\big)\\ &\quad + C|\tilde\nabla^m\Psi|_0^2\cdot\big(\lambda+|\tilde\nabla^{m-1}\Psi|_0^2\big) + C\cdot(\lambda+|\tilde\nabla^{m-1}\Psi|_0^2)\ .\end{aligned}$$

Collecting terms and using the induction hypotheses (3.38), we find

$$\begin{aligned}\partial_t\phi \le\ & g^{ab}\tilde\nabla_a\tilde\nabla_b\phi - 2g^{ab}\tilde\nabla_a|\tilde\nabla^{m-1}\Psi|_0^2\cdot\tilde\nabla_b|\tilde\nabla^m\Psi|_0^2 - \tfrac12|\tilde\nabla^m\Psi|_0^4 - \tfrac12|\tilde\nabla^{m+1}\Psi|_0^2\cdot\big(\lambda+|\tilde\nabla^{m-1}\Psi|_0^2\big)\\ & + C|\tilde\nabla^m\Psi|_0^2 + C\end{aligned}\tag{3.45}$$

where $C = C(\lambda)$ now also depends on λ. We estimate $C|\tilde\nabla^m\Psi|_0^2 \le \frac14|\tilde\nabla^m\Psi|_0^4 + C$ and use Kato's inequality $\big|\nabla|(\cdot)|\big| \le |\nabla(\cdot)|$ to control the cross term:

$$\begin{aligned}-2g^{ab}\tilde\nabla_a|\tilde\nabla^{m-1}\Psi|_0^2\cdot\tilde\nabla_b|\tilde\nabla^m\Psi|_0^2 &\le 8g^{ab}|\tilde\nabla^{m-1}\Psi|_0\tilde\nabla_a|\tilde\nabla^{m-1}\Psi|_0\cdot|\tilde\nabla^m\Psi|_0\tilde\nabla_b|\tilde\nabla^m\Psi|_0\\ &\le 16c|\tilde\nabla^m\Psi|_0^2|\tilde\nabla^{m+1}\Psi|_0 \le \tfrac{\lambda}{2}|\tilde\nabla^{m+1}\Psi|_0^2 + \tfrac{1}{2\lambda}c^2|\tilde\nabla^m\Psi|_0^4\ .\end{aligned}$$

Here c depends only on n and C_{m-1}. Together with (3.45) this implies

$$\partial_t\phi \le g^{ab}\tilde\nabla_a\tilde\nabla_b\phi + (\tfrac{1}{2\lambda}c^2 - \tfrac14)|\tilde\nabla^m\Psi|_0^4 + C\ .$$

Choosing $\lambda := 4(c^2+1)$, we get

$$\partial_t\phi \le g^{ab}\tilde\nabla_a\tilde\nabla_b\phi - \frac18\phi^2\cdot\big(\lambda+|\tilde\nabla^{m-1}\Psi|_0^2\big)^{-1} + C = g^{ab}\tilde\nabla_a\tilde\nabla_b\phi - \tilde C\phi^2 + C$$

on $[0,T]\times B_{\gamma+\delta/(m+1)}(x_0)$ where C and $\tilde C$ are constants only depending on $C_0,\dots,C_{m-1}$, $n, m, \delta, \gamma, |\tilde g|_{C^\infty}, c_0$. By an application of the maximum principle as in Proposition 3.8, we finally get

$$|\tilde\nabla^m\Psi|_0^2(t,x) \le C_m(C_0,\dots,C_{m-1},n,\delta,\gamma,|\tilde g|_{C^\infty},c_0) = C_m(n,\delta,\gamma,|\tilde g|_{C^\infty},c_0)$$

on $[0,T]\times B_{\gamma+\delta/(m+1)}(x_0)$ as required. This finishes the induction argument.

□

Given these a priori estimates, the theory of parabolic systems on compact domains provides the existence of a solution to the modified system (3.4) and (3.5) on a finite time interval $0 \le t \le T$ where T is determined by Theorem 3.7. We use the existence theorem for quasilinear parabolic systems in [LSU68, Theorem VII 7.1] together with the remarks in [Shi89, chap. 3].

Theorem 3.10 *Let Σ be a complete Riemannian manifold and $D \subset \Sigma$ be a connected open subset with compact closure. Then the system (3.4) and (3.5) together with smooth initial/boundary data $(\tilde g,\tilde u)$, satisfying $|\tilde{Rm}|_0^2 \le k_0$ and $|\tilde u|_0^2 + |\tilde\nabla\tilde u|_0^2 \le c_0$, has a unique smooth solution $(g,u)(t)$ on a time interval $[0,T]$ for some $T = T(n,k_0,c_0) > c(n)\cdot 1/(\sqrt{k_0}+c_0)$.*

We note a direct corollary of these results in the case of closed manifolds:

Theorem 3.11 *Let M be a closed Riemannian manifold and $(\tilde{g}, \tilde{u}) \in \mathcal{M}(M) \times C^\infty(M)$ be given. Then the initial value problem*

$$\begin{aligned} \partial_t g &= -2Rc(g) + 4du \otimes du \\ \partial_t u &= \Delta^g u \end{aligned}$$

with initial data $g(0) = \tilde{g}$ and $u(0) = \tilde{u}$ has a unique smooth solution on a time interval $[0, T)$ for some maximal $T > c(n) \cdot 1/(\sqrt{k_0} + c_0)$ where $k_0 := \max_M |\tilde{Rm}|_0^2$ and $c_0 := \max_M |d\tilde{u}|_0^2$.

Proof:
This easily follows from the fact that (3.4) and (3.5) form a uniformly parabolic system on a compact domain without boundary as was proven in Theorem 3.7. Furthermore, the solution of (3.4) has bounded curvature and the vector field V together with its first derivative ∇V is smooth and bounded on M. This implies that the 1-parameter group of diffeomorphisms generated by V exists and is smooth on $[0, T) \times M$ and that the pullback of the solution satisfies the original system (2.5). A more detailed exposition is given for the complete case in the proof of Theorem 3.22.

□

3.4 Short time existence on complete manifolds

The a priori estimates for Ψ in the last chapter enable us to prove the existence of a solution to the initial value problem

$$\begin{aligned} \partial_t g_{ij} &= -2R_{ij} + 4\partial_i u \partial_j u + \nabla_i V_j + \nabla_j V_i \\ \partial_t u &= \Delta u + du(V) \end{aligned} \tag{3.46}$$

on the whole space $[0, T] \times \Sigma$ with initial values $g(0) = \tilde{g}$ and $u(0) = \tilde{u}$ for a given smooth Riemannian metric $\tilde{g}$ on Σ and function $\tilde{u} \in C^\infty(\Sigma)$. To this end we fix a point $x_0 \in \Sigma$ and choose a family of domains $\{D_k \subset \Sigma : k = 1, 2, 3, \ldots\}$ such that for each k

1. the boundary ∂D_k is a smooth $(n-1)$-dimensional submanifold of Σ
2. the closure $\bar{D}_k$ is compact in Σ
3. $B_k(x_0) \subset D_k$

where $B_k(x_0)$ is a geodesic ball with respect to $\tilde{g}$ of radius k. This family exists since Σ is complete. Theorem 3.10 shows the existence of a solution on each D_k for $0 \leq t \leq T$ where T depends only on n and the initial data $(\tilde{g}, \tilde{u})$. Furthermore, Theorem 3.7 implies that these solutions satisfy

$$(1 - \varepsilon)\tilde{g}_{ij}(x) \leq g_{ij}(t, x) \leq (1 + \varepsilon)\tilde{g}_{ij}(x) \qquad \forall (t, x) \in [0, T] \times \bar{D}_k$$

uniformly in k for an arbitrary fixed ε satisfying $0 < \varepsilon < \varepsilon_0 := 1/(416000n^{10})$.

In addition, we have for any integer $l \geq 1$

$$B_l(x_0) \subset D_k \qquad \forall k \geq l \ .$$

Proposition 3.9 gives domain independent a priori estimates for these solutions

$$|\tilde{\nabla}^m \Psi(k,t,x)|_0^2 \leq C_m(n,l,\tilde{g},c_0) \qquad \forall (t,x) \in [0,T] \times B_l(x_0)$$

for all $k > l$ where C_m and T do not depend on k. Therefore, all derivatives of $\Psi(k)$ are uniformly bounded on any compact subset of $[0,T] \times \Sigma$. Since we have

$$\bigcup_{k=1}^{\infty} D_k = \Sigma \ ,$$

we can take the limit $k \to \infty$ and get convergence of a subsequence of the solutions $\Psi(k)(t)$ in the C^∞ topology on compact subsets of $[0,T] \times \Sigma$ to a smooth solution $(g_\infty, u_\infty)(t)$ on $[0,T] \times \Sigma$ by the theorem of Arzela-Ascoli. This proves

Theorem 3.12 *Let $(\Sigma, \tilde{g})$ be a smooth, complete Riemannian manifold with bounded curvature $|\tilde{R}m|_0^2 \leq k_0$. Let $\tilde{u} \in C^\infty(\Sigma)$ satisfy $|\tilde{u}|_0^2 + |d\tilde{u}|_0^2 \leq c_0$ and $V \in \mathcal{X}([0,T] \times \Sigma)$ be as in Lemma 3.2. Then there is a time $T = T(n, k_0, c_0) > c(n) \cdot 1/(\sqrt{k_0} + c_0)$ such that the initial value problem*

$$\begin{aligned} \partial_t g &= -2Rc + 4du \otimes du + \mathcal{L}_V g \\ \partial_t u &= \Delta^g u + \mathcal{L}_V u \end{aligned}$$

on $[0,T] \times \Sigma$ with initial data $g(0) = \tilde{g}$ and $u(0) = \tilde{u}$ has a smooth solution $(g,u)(t)$ satisfying

$$(1-\varepsilon)\tilde{g}_{ij}(x) \leq g_{ij}(t,x) \leq (1+\varepsilon)\tilde{g}_{ij}(x)$$

for all $(t,x) \in [0,T] \times \Sigma$ and for all $\varepsilon \leq 1/(416000n^{10})$.

3.5 Global estimates for complete solutions

In order to construct a solution of the original system (3.1) and (3.2) from a solution of the modified system, we have to prove the existence of the diffeomorphisms we want to use to pull back the solution. To this end we need to assume the additional bound

$$\sup_{x \in \Sigma} |\tilde{\nabla}^2 \tilde{u}|_0^2(x) \leq s_0 \tag{3.47}$$

on the initial data from now on. The aim of this section is to prove global estimates for $|Rm|$, $|\nabla^2 u|$ and $|\nabla V|$.

Remark 3.13 *We do not want to use the interior estimates in Proposition 3.9. Although these indeed imply a global bound for the derivatives of $(g,u)(t)$ on Σ, the constant depends on the whole C^∞-norm of the initial metric $\tilde{g}$. We want to prove an estimate that depends only on the curvature of $\tilde{g}$.*

As a first step we prove a global bound on the first derivative of the solution $(g, u)(t)$. This is done in the following proposition.

Proposition 3.14 *Let $(g, u)(t)$ be a solution satisfying the assumptions in Theorem 3.12. Assume furthermore a bound $|\tilde{\nabla}^2 u|_0^2 \leq s_0$ on Σ. Then there exists a constant $c = c(n, k_0, c_0, s_0)$ such that*

$$\sup_{[0,T]\times\Sigma} |\tilde{\nabla}\Psi|^2 \leq c(n, k_0, c_0, s_0) \ .$$

Proof:
From Theorem 3.7 we know that for $T = T(n, k_0, c_0)$ small enough all the approximating solutions $g(k, t)$ are equivalent to the initial metric $\tilde{g}$ independent of k in the sense that

$$(1 - \varepsilon)\tilde{g} \leq g(k, t) \leq (1 + \varepsilon)\tilde{g} \tag{3.48}$$

for the same arbitrary fixed ε as in the theorem. Because of the uniform convergence $g(k) \to g$ this also holds for the limit. Let $H_{ij}(t, x) := \frac{1}{\varepsilon}\left(g_{ij}(t, x) - \tilde{g}_{ij}(x)\right)$ and compute

$$\begin{aligned} \partial_t H_{ij} &= g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b H_{ij} + A_{ij} \qquad \text{on } [0, T] \times \Sigma \\ H_{ij}(0) &\equiv 0 \qquad \text{on } \Sigma \end{aligned}$$

where we define

$$\begin{aligned} A_{ij} :=& \tfrac{1}{\varepsilon} \cdot (-g^{ab} g_{ik}\tilde{g}^{kl}\tilde{R}_{jalb} - g^{ab} g_{jk}\tilde{g}^{kl}\tilde{R}_{jalb} + 4\partial_i u \partial_j u) \\ &+ \varepsilon g^{ab} g^{kl}\big(\tfrac{1}{2}\tilde{\nabla}_i H_{ka}\tilde{\nabla}_j H_{lb} + \tilde{\nabla}_a H_{jk}\tilde{\nabla}_l H_{ib} - \tilde{\nabla}_a H_{jk}\tilde{\nabla}_b H_{il} - \tilde{\nabla}_j H_{ka}\tilde{\nabla}_b H_{il} - \tilde{\nabla}_i H_{ka}\tilde{\nabla}_b H_{jl}\big). \end{aligned}$$

In addition, we get for $w(t, x) := \varepsilon\big(u(t, x) - \tilde{u}(x)\big)$

$$\begin{aligned} \partial_t w &= g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b w + B \qquad \text{on } [0, T] \times \Sigma \\ w(0) &\equiv 0 \qquad \text{on } \Sigma \end{aligned}$$

where $B := \varepsilon g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\tilde{u}$. To estimate the coefficients of the differential operator, we calculate

$$-\left(\frac{8n\sqrt{k_0} + 2c_0}{\varepsilon} + 20\varepsilon|\tilde{\nabla}H|_0^2\right)\tilde{g}_{ij} \leq A_{ij}(t) \leq \left(\frac{8n\sqrt{k_0} + 2c_0}{\varepsilon} + 20\varepsilon|\tilde{\nabla}H|_0^2\right)\tilde{g}_{ij} \ .$$

Using (3.47), we can estimate

$$|B|_0 = |\varepsilon g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\tilde{u}|_0 \leq 2\varepsilon|\tilde{g}^{ab}|_0|\tilde{\nabla}^2\tilde{u}|_0 \leq \sqrt{4ns_0} \cdot \varepsilon$$

on $[0, T] \times \Sigma$. Furthermore, we can derive bounds on $g(t)$ from (3.48):

$$-\tilde{g} = \tfrac{1}{\varepsilon}\big((1 - \varepsilon)\tilde{g} - \tilde{g}\big) \leq \tfrac{1}{\varepsilon}(g(t) - \tilde{g}) = H(t) = \tfrac{1}{\varepsilon}(g(t) - \tilde{g}) \leq \tfrac{1}{\varepsilon}\big((1 + \varepsilon)\tilde{g} - \tilde{g}\big) = \tilde{g}$$

which provides an estimate for the coefficients of the second derivatives

$$\frac{1}{1 + \varepsilon}\tilde{g}^{ab} \leq g^{ab}(t) \leq \frac{1}{1 - \varepsilon}\tilde{g}^{ab},$$

implying

$$|\tilde{\nabla}_i g^{ab}|_0^2 \leq \frac{\varepsilon^2}{(1-\varepsilon)^4} |\tilde{\nabla} H|_0^2$$

on $[0,T] \times \Sigma$. In addition, we know from the smooth convergence $u(k) \to u$ that the bound $|u(k,t)|^2 \leq c_0$ for the approximating solutions on $[0,T] \times B_k$ is preserved in the limit. This implies that

$$|u|_0^2(t) \leq c_0 \qquad \Rightarrow \qquad |w|_0^2(t) \leq \varepsilon^2 |u|_0^2(t) + \varepsilon^2 |\tilde{u}|_0^2 \leq 2\varepsilon^2 c_0$$

for all $t \in [0,T]$. Then the arguments in [Shi89, §5] applied to the system (H_{ij}, w) show that for $\varepsilon < \varepsilon_0(n)$ small enough

$$\sup_{[0,T]\times\Sigma} \left(|\tilde{\nabla} H|_0^2 + |\tilde{\nabla} w|_0^2 \right) \leq c(n, k_0, c_0, s_0) \ .$$

This yields

$$|\tilde{\nabla} u|_0^2 \leq |\tilde{\nabla} u - \tilde{\nabla} \tilde{u}|_0^2 + |\tilde{\nabla} \tilde{u}|_0^2 \leq \varepsilon^{-2} \cdot c(n, k_0, c_0, s_0) + c_0 = c(n, k_0, c_0, s_0)$$

since ε depends only on n. A similar estimate on $|\tilde{\nabla} g|_0^2$ proves the proposition.

□

Having obtained the bound on the first derivatives of Ψ, we aim at an estimate for $|\nabla \tilde{\nabla} \Psi|^2$. In the following $|\cdot|$, $|\cdot|_0$ denote the norms with respect to $g(t)$, $\tilde{g}$ and dV, dV_0 the corresponding volume elements.

Lemma 3.15 *Let $(g,u)(t)$ be a solution as in Proposition 3.14 and Ψ be defined as in (3.23). Then we have for any $x_0 \in \Sigma$ and any radius $0 < R < \infty$:*

$$\int_0^T \int_{B_R(x_0)} |\tilde{\nabla}^2 \Psi|_0^2 \, dV_0 \, dt \leq c = c(n, k_0, c_0, R) \ .$$

Proof:
The metrics $g(t)$ on $[0,T] \times \Sigma$ are equivalent in the following sense:

$$\tfrac{1}{2}\tilde{g} \leq g(t) \leq 2\tilde{g} \ . \tag{3.49}$$

Let $\xi \in C_c^\infty(\Sigma)$ be a cutoff function satisfying $|\tilde{\nabla}\xi|_0 \leq 8$ and $0 \leq \xi \leq 1$ on Σ, $\xi \equiv 1$ on $B_R(x_0)$ and $\xi \equiv 0$ on $\Sigma \setminus B_{R+1/2}(x_0)$. From now on, c will always denote (different) constants only depending on n, k_0, c_0, s_0, R. Abbreviate $\Omega := B_{R+1}(x_0)$ and note that, in view of the curvature bound, we can compare the volume of Ω with that of a ball in the model space of constant sectional curvature by the volume comparison theorem of Bishop and Gromov [SY94, Theorem 1.3]. This implies that $vol_0(\Omega) \leq c$ holds for a constant $c = c(n, k_0, R)$ independent of the base point x_0. We will use that in the next sections without further comment.

Using the evolution equation (3.25) for $\tilde{\nabla} g$, a calculation shows

$$\begin{aligned}
\partial_t \int_\Omega |\tilde{\nabla} g|_0^2 \, \xi^2 dV_0 &= 2 \int_\Omega \tilde{\nabla}_k g_{ij} \cdot (\partial_t \tilde{\nabla}_k g_{ij}) \, \xi^2 dV_0 \\
&= 2 \int_\Omega \tilde{\nabla}_k g_{ij} \cdot g^{ab} \tilde{\nabla}_a \tilde{\nabla}_b (\tilde{\nabla}_k g_{ij}) \, \xi^2 dV_0 + \int_\Omega \tilde{\nabla} g * \tilde{\nabla} \tilde{R}m \, \xi^2 dV_0 \\
&\quad + \int_\Omega \tilde{\nabla} g * \left(\tilde{\nabla} u * \tilde{\nabla}^2 u + \tilde{\nabla} g * \tilde{\nabla}^2 g + \tilde{\nabla} g * \tilde{R}m + \tilde{\nabla} g * \tilde{\nabla} g * \tilde{\nabla} g \right) \xi^2 dV_0 \ .
\end{aligned}$$

We deal with all terms individually. Integrating by parts, we get for the first

$$\begin{aligned}
&2\int_\Omega \tilde\nabla_k g_{ij}\cdot g^{ab}\tilde\nabla_a\tilde\nabla_b(\tilde\nabla_k g_{ij})\,\xi^2 dV_0\\
&\quad= -2\int_\Omega \tilde\nabla_b(\tilde\nabla_k g_{ij})\cdot\tilde\nabla_a\left(g^{ab}\tilde\nabla_k g_{ij}\,\xi^2\right)dV_0\\
&\quad= -2\int_\Omega g^{ab}\tilde\nabla_b\tilde\nabla_k g_{ij}\cdot\tilde\nabla_a\tilde\nabla_k g_{ij}\,\xi^2 dV_0+\int_\Omega \tilde\nabla^2 g*\left(\tilde\nabla g*\tilde\nabla g\cdot\xi+\tilde\nabla g*\tilde\nabla\xi\right)\cdot\xi dV_0\\
&\quad\le -\int_\Omega |\tilde\nabla^2 g|_0^2\,\xi^2 dV_0+c\int_\Omega |\tilde\nabla^2 g|_0\,\xi dV_0\ .
\end{aligned}$$

This follows from the global bound on $|\tilde\nabla g|_0$ in Proposition 3.14, the properties of ξ, and (3.49). The second term is taken care of by an integration by parts

$$\begin{aligned}
\int_\Omega \tilde\nabla g*\tilde\nabla\tilde{R}m\,\xi^2 dV_0 &= -\int_\Omega \tilde{R}m*\tilde\nabla\left(\tilde\nabla g\cdot\xi^2\right)dV_0=\int_\Omega \tilde{R}m*\left(\tilde\nabla^2 g\cdot\xi+\tilde\nabla g*\tilde\nabla\xi\right)\xi dV_0\\
&\le c\int_\Omega(1+|\tilde\nabla^2 g|_0)\xi dV_0\le c\cdot vol_0(\Omega)+c\int_\Omega |\tilde\nabla^2 g|_0\,\xi dV_0
\end{aligned}$$

where we also used $|\tilde{R}m|_0^2\le k_0$. A similar reasoning implies for the last term:

$$\begin{aligned}
&\int_\Omega \tilde\nabla g*\left(\tilde\nabla u*\tilde\nabla^2 u+\tilde\nabla g*\tilde\nabla^2 g+\tilde\nabla g*\tilde{R}m+\tilde\nabla g*\tilde\nabla g*\tilde\nabla g\right)\xi^2 dV_0\\
&\quad\le c\int_\Omega\left(|\tilde\nabla^2 g|_0+|\tilde\nabla^2 u|_0+1\right)\xi^2 dV_0\le c\int_\Omega\left(|\tilde\nabla^2 g|_0+|\tilde\nabla^2 u|_0\right)\xi dV_0+c\ ,
\end{aligned}$$

and we get altogether

$$\partial_t\int_\Omega |\tilde\nabla g|_0^2\,\xi^2 dV_0\le -\int_\Omega |\tilde\nabla^2 g|_0^2\,\xi^2 dV_0+c\int_\Omega |\tilde\nabla^2 g|_0\,\xi dV_0+c\int_\Omega |\tilde\nabla^2 u|_0\,\xi dV_0+c\ . \tag{3.50}$$

Doing the same calculation for $\tilde\nabla u$, we obtain

$$\begin{aligned}
\partial_t\int_\Omega |\tilde\nabla u|_0^2\,\xi^2 dV_0 &= 2\int_\Omega \partial_i u\cdot(\partial_i\partial_t u)\xi^2 dV_0=2\int_\Omega \partial_i u\cdot\tilde\nabla_i(g^{ab}\tilde\nabla_a\tilde\nabla_b u)\xi^2 dV_0\\
&= 2\int_\Omega g^{ab}\partial_i u\cdot\tilde\nabla_a\tilde\nabla_b\tilde\nabla_i u\,\xi^2 dV_0+\int_\Omega\left(\tilde\nabla u*\tilde\nabla g*\tilde\nabla^2 u+\tilde\nabla u*\tilde{R}m\right)\xi^2 dV_0\ .
\end{aligned}$$

An integration by parts gives for the first term

$$\begin{aligned}
&2\int_\Omega g^{ab}\partial_i u\cdot\tilde\nabla_a\tilde\nabla_b\tilde\nabla_i u\,\xi^2 dV_0\\
&\quad= -2\int_\Omega g^{ab}\tilde\nabla_a\tilde\nabla_i u\cdot\tilde\nabla_b\tilde\nabla_i u\,\xi^2 dV_0+\int_\Omega \tilde\nabla u*\left(\tilde\nabla g*\tilde\nabla^2 u\cdot\xi+\tilde\nabla^2 u*\tilde\nabla\xi\right)\xi dV_0\\
&\quad\le -\int_\Omega |\tilde\nabla^2 u|_0^2\,\xi^2 dV_0+c\int_\Omega |\tilde\nabla^2 u|_0\xi dV_0
\end{aligned}$$

as before. The second term can be estimated in exactly the same way as above. Thus there is the estimate:

$$\partial_t\int_\Omega |\tilde\nabla u|_0^2\,\xi^2 dV_0\le -\int_\Omega |\tilde\nabla^2 u|_0^2\,\xi^2 dV_0+c\int_\Omega |\tilde\nabla^2 u|_0\,\xi dV_0+c\ . \tag{3.51}$$

This allows us to combine (3.50) and (3.51). Using Young's inequality, we estimate Ψ as follows:

$$\begin{aligned}\partial_t \int_\Omega |\tilde\nabla\Psi|_0^2\,\xi^2 dV_0 &\le -\int_\Omega |\tilde\nabla^2\Psi|_0^2\,\xi^2 dV_0 + c\int_\Omega |\tilde\nabla^2 g|_0\,\xi dV_0 + c\int_\Omega |\tilde\nabla^2 u|_0\,\xi dV_0 + c\\ &\le -\int_\Omega |\tilde\nabla^2\Psi|_0^2\,\xi^2 dV_0 + \tfrac12\int_\Omega |\tilde\nabla^2 g|_0^2\,\xi^2 dV_0 + \tfrac12\int_\Omega |\tilde\nabla^2 u|_0^2\,\xi^2 dV_0 + c\\ &\le -\tfrac12\int_\Omega |\tilde\nabla^2\Psi|_0^2\,\xi^2 dV_0 + c\;.\end{aligned}$$

Since we have at time $t=0$ that

$$\int_\Omega |\tilde\nabla\Psi|_0^2(0)\,\xi^2 dV_0 = \int_\Omega |\tilde\nabla\tilde g|_0^2\,\xi^2 dV_0 + \int_\Omega |\tilde\nabla\tilde u|_0^2\,\xi^2 dV_0 \le 0 + c\;,$$

we can integrate from 0 to T to find

$$\begin{aligned}\int_\Omega |\tilde\nabla\Psi|_0^2(T)\,\xi^2 dV_0 &= \int_\Omega |\tilde\nabla\Psi|_0^2(0)\,\xi^2 dV_0 + \int_0^T \partial_t\int_\Omega |\tilde\nabla\Psi|_0^2(t)\,\xi^2 dV_0\,dt\\ &\le c - \tfrac12\int_0^T\int_\Omega |\tilde\nabla^2\Psi|_0^2(t)\,\xi^2 dV_0\,dt + c\int_0^T dt\;.\end{aligned}$$

Since $T = T(n,k_0,c_0)$, we get the desired result

$$\int_0^T\int_\Omega |\tilde\nabla^2\Psi|_0^2(t)\,\xi^2 dV_0\,dt + \int_\Omega |\tilde\nabla\Psi|_0^2(T)\,\xi^2 dV_0 \le c,$$

proving the lemma.

□

This estimate is still valid if we change to the time dependent norm and volume form.

Lemma 3.16 *Let $(g,u)(t)$ be a solution as in Proposition 3.14 on $[0,T]\times\Sigma$. Then, for any $x_0\in\Sigma$ and any radius $0<R<\infty$, we can estimate*

$$\int_0^T\int_{B_R(x_0)} |\tilde\nabla^2\Psi|^2 dV\,dt + \int_0^T\int_{B_R(x_0)} |\nabla\tilde\nabla\Psi|^2 dV\,dt \le c$$

where $c = c(n,k_0,c_0,s_0,R)$, and $|\cdot|$ is the norm associated to $g(t)$.

Proof:
Because of the equivalence (3.49) for the metrics $g(t)$, we can estimate $|\tilde\nabla^2\Psi|^2 \le 16|\tilde\nabla^2\Psi|_0^2$ and $dV \le 2^{n/2}dV_0$ on the whole space $[0,T]\times\Sigma$. This proves the bound for the first term using Lemma 3.15. For the second term we estimate

$$\begin{aligned}|\nabla\tilde\nabla\Psi|^2 &= |\tilde\nabla^2\Psi + (\nabla-\tilde\nabla)\tilde\nabla\Psi|^2 = |\tilde\nabla^2\Psi + \tilde\nabla g * \tilde\nabla\Psi|^2\\ &\le |\tilde\nabla^2\Psi|^2 + |\tilde\nabla^2\Psi|\cdot c|\tilde\nabla\Psi| + c \le 2|\tilde\nabla^2\Psi|^2 + c\end{aligned} \tag{3.52}$$

in view of the bound for $|\tilde\nabla\Psi|_0^2$, the equivalence (3.49) and Young's inequality. Thus the boundedness of the second term follows from that of the first.

□

The next step is to bound the norm of the vector field V and its gradient since we want to use V for the construction of the pullback diffeomorphisms. It is convenient to simultaneously prove a bound on $|Rm|^2$, $|\nabla^2 u|^2$, and $|\nabla V|^2$. First we need to deduce evolution equations for Rm and $\nabla^2 u$ from the modified flow (3.46):

Lemma 3.17 *Let $(g,u)(t)$ be a solution to the modified flow (3.46) and assume V is given by $V_i := g_{ij} g^{pq}(\Gamma^j_{pq} - \tilde{\Gamma}^j_{pq})$. Then Rm, $\nabla^2 u$, and ∇V satisfy the following evolution equations:*

$$\partial_t R_{ijkl} = \Delta R_{ijkl} + Rm * Rm + \nabla^2 u * \nabla^2 u + Rm * \nabla V + \nabla Rm * V \tag{3.53}$$

$$\partial_t(\nabla_i \nabla_j u) = \Delta(\nabla_i \nabla_j u) + Rm * \nabla^2 u + du * du * \nabla^2 u + \nabla^2 u * \nabla V + V * \nabla^3 u + du * V * Rm \tag{3.54}$$

$$\begin{aligned}\partial_t(\nabla_i V_j) = {} & \Delta(\nabla_i V_j) + Rm * \nabla V + \nabla V * \nabla V + du * du * \nabla V + \nabla^2 u * \nabla^2 u + du * \nabla^3 u \\ & + \tilde{\nabla} g * (\nabla Rm + du * \nabla^2 u + \nabla^2 V) + \nabla \tilde{\nabla} g * (Rm + du * du + \nabla V)\end{aligned} \tag{3.55}$$

Proof:
We already did most of the calculations in the proofs of Lemma 2.14, Lemma 2.7, and Lemma 2.6. The remaining terms are handled similarly. We also used the identity (3.8) for $\Gamma - \tilde{\Gamma}$.

□

Since we need integral estimates, we have to compute the evolution of the volume element.

Lemma 3.18 *The volume element $dV(t) := \sqrt{\det(g_{ij}(t))} dx^1 \wedge \dots dx^n$ associated to the evolving metric $g(t)$ satisfies:*

$$\partial_t dV = \big(-R + 2|du|^2 + \operatorname{div}(V)\big) dV \ .$$

Proof:
This is a short calculation:

$$\begin{aligned}\partial_t dV = \partial_t \left(\sqrt{\det(g)} dx\right) &= \frac{1}{2\sqrt{\det(g)}} \cdot \det(g) \cdot g^{pq}(-2R_{pq} + 4\partial_p u \partial_q u + \nabla_p V_q + \nabla_q V_p) dx \\ &= \big(-R + 2|du|^2 + \operatorname{div}(V)\big)\sqrt{\det(g)} dx \ .\end{aligned}$$

□

Using the integral estimate for $|\nabla \tilde{\nabla} \Psi|^2$, we can prove:

Lemma 3.19 *Let $(g,u)(t)$ be a solution as in Proposition 3.14 on $[0,T] \times \Sigma$. Then we have for any $x_0 \in \Sigma$ and any radius $0 < R < \infty$:*

$$\int_{B_R(x_0)} \left(|Rm|^2 + |\nabla^2 u|^2 + |\nabla V|^2\right) dV \le c$$

where $c = c(n, k_0, c_0, s_0, R)$ and V is defined as in Lemma 3.17.

Proof:
Suppose ξ is given as in the proof of Lemma 3.15 and c is a changing constant depending only on n, k_0, c_0, s_0, R and $\Omega := B_{R+1}(x_0)$. Recalling the evolution equation (3.53), we compute

$$\begin{aligned}
\partial_t \int_\Omega &|Rm|^2 \xi^2 dV \\
&= 2\int_\Omega R_{ijkl}\cdot(\partial_t R_{ijkl})\xi^2 dV + \int_\Omega Rm * Rm * (\partial_t g^{-1})\xi^2 dV + \int_\Omega Rm * Rm \cdot \xi^2(\partial_t dV) \\
&= 2\int_\Omega R_{ijkl}\cdot \Delta R_{ijkl}\,\xi^2 dV + \int_\Omega Rm * \big(Rm * Rm + \nabla^2 u * \nabla^2 u + Rm * \nabla V + \nabla Rm * V\big)\xi^2 dV \\
&\quad + \int_\Omega Rm * Rm * \big(Rm + du * du + \nabla V - R + 2|du|^2 + \mathrm{div}(V)\big)\,\xi^2\,dV \\
&= 2\int_\Omega R_{ijkl}\cdot \Delta R_{ijkl}\cdot \xi^2 dV + \int_\Omega Rm * Rm * \nabla V \cdot \xi^2 dV \\
&\quad + \int_\Omega \big(\nabla Rm * Rm * V + du * du * Rm * Rm\big)\xi^2 dV \\
&\quad + \int_\Omega \big(Rm * Rm * Rm + \nabla^2 u * \nabla^2 u * Rm\big)\xi^2 dV\ .
\end{aligned}$$

We take care of the individual terms. Integration by parts yields for the first one:

$$\begin{aligned}
2\int_\Omega R_{ijkl}\cdot \Delta R_{ijkl}\,\xi^2 dV &= -2\int_\Omega g^{ab}\nabla_b R_{ijkl}\cdot \nabla_a(R_{ijkl}\cdot\xi^2) dV \\
&= -2\int_\Omega |\nabla Rm|^2\xi^2 dV + \int_\Omega Rm * \nabla Rm * \nabla\xi\cdot\xi dV \\
&\le -2\int_\Omega |\nabla Rm|^2\xi^2 dV + \int_\Omega |\nabla Rm|\xi\cdot c|Rm| dV \\
&\le -\frac{3}{2}\int_\Omega |\nabla Rm|^2\xi^2 dV + c\int_\Omega |Rm|^2 dV,
\end{aligned}$$

since we can estimate $|\nabla\xi| = |\tilde\nabla\xi| \le \sqrt{2}|\tilde\nabla\xi|_0 \le 12$. Performing an integration by parts on the second term gives in the same way as for the first

$$\begin{aligned}
\int_\Omega Rm * Rm * \nabla V\cdot\xi^2 dV &= \int_\Omega V\cdot\nabla\left(Rm * Rm\cdot\xi^2\right) dV \\
&= \int_\Omega V * [Rm * \nabla Rm\cdot\xi + Rm * Rm * \nabla\xi]\xi dV \\
&\le \int_\Omega |\nabla Rm|\xi\cdot c|Rm| dV + c\int_\Omega |Rm|^2 dV \\
&\le \frac{1}{8}\int_\Omega |\nabla Rm|^2\xi^2 dV + c\int_\Omega |Rm|^2 dV
\end{aligned}$$

where we used that

$$|V|^2 = g_{ij}V^iV^j \le 2\tilde g_{ij}V^iV^j = 2|V|_0^2 \le c|\tilde\nabla g|_0^2 \le c|\tilde\nabla\Psi|_0^2 \le c \tag{3.56}$$

holds on $[0,T]\times\Sigma$. The third term is straightforward:

$$\int_\Omega \big(\nabla Rm * Rm * V + du * du * Rm * Rm\big)\xi^2 dV \le \frac{1}{16}\int_\Omega |\nabla Rm|^2\xi^2 dV + c\int_\Omega |Rm|^2 dV\ .$$

For the last terms we have to replace one occurrence of Rm by second derivatives of g as follows:

$$Rm = \tilde{\nabla}^2 g + \tilde{\nabla} g * \tilde{\nabla} g = \nabla \tilde{\nabla} g + \tilde{\nabla} g * \tilde{\nabla} g \ . \tag{3.57}$$

Integrating by parts, this allows us to estimate:

$$\begin{aligned}
&\int_\Omega Rm * (Rm * Rm + \nabla^2 u * \nabla^2 u)\xi^2 dV \\
&\quad \leq - \int_\Omega \tilde{\nabla} g * \nabla(Rm * Rm + \nabla^2 u * \nabla^2 u \cdot \xi^2) dV + c\int_\Omega |Rm|^2 dV + c\int_\Omega |\nabla^2 u|^2 dV \\
&\quad \leq \frac{1}{16}\int_\Omega |\nabla Rm|^2 \xi^2 dV + \frac{1}{8}\int_\Omega |\nabla^3 u|^2\xi^2 dV + c\int_\Omega |Rm|^2 dV + c\int_\Omega |\nabla^2 u|^2 dV \ .
\end{aligned}$$

Altogether we find

$$\partial_t \int_\Omega |Rm|^2\xi^2 dV \leq -\frac{10}{8}\int_\Omega |\nabla Rm|^2\xi^2 dV + \frac{1}{8}\int_\Omega |\nabla^3 u|^2 \xi^2 dV + c\int_\Omega |Rm|^2 dV + c\int_\Omega |\nabla^2 u|^2 dV \ . \tag{3.58}$$

Similar computations for the Hessian of u using (3.54) yield:

$$\partial_t \int_\Omega |\nabla^2 u|^2 \xi^2 dV \leq -\frac{10}{8}\int_\Omega |\nabla^3 u|^2\xi^2 dV + c\int_\Omega |\nabla^2 u|^2 dV + c\int_\Omega |Rm|^2 dV \ . \tag{3.59}$$

We use (3.55) to estimate ∇V in the same way:

$$\begin{aligned}
\partial_t \int_\Omega |\nabla V|^2\xi^2 dV \leq & -\int_\Omega |\nabla^2 V|^2 \xi^2 dV + \frac{2}{8}\int_\Omega |\nabla Rm|^2\xi^2 dV + \frac{1}{8}\int_\Omega |\nabla^3 u|^2\xi^2 dV \\
& + c\int_\Omega (|Rm|^2 + |\nabla^2 u|^2 + |\nabla V|^2) dV \ .
\end{aligned} \tag{3.60}$$

Combining (3.58), (3.59), and (3.60), we estimate altogether

$$\begin{aligned}
\partial_t \int_\Omega \big(|Rm|^2 + |\nabla^2 u|^2 + |\nabla V|^2\big)\xi^2 dV \leq & -\int_\Omega \big(|\nabla Rm|^2 + |\nabla^3 u|^2 + |\nabla^2 V|^2\big)\xi^2 dV \\
& + c\int_\Omega \big(|Rm|^2 + |\nabla^2 u|^2 + |\nabla V|^2\big) dV \ .
\end{aligned}$$

Since $V(0) \equiv 0$, we have at time $t = 0$:

$$\int_\Omega \big(|\tilde{Rm}|_0^2 + |\tilde{\nabla}^2 \tilde{u}|_0^2 + |\tilde{\nabla} V(0)|_0^2\big)\xi^2 dV_0 \leq (k_0 + s_0 + 0)\int_\Omega \xi^2 dV_0 \leq c \ .$$

Therefore we can integrate at every time $\tau \in [0, T]$ and estimate:

$$\begin{aligned}
&\int_\Omega \big(|Rm|^2 + |\nabla^2 u|^2 + |\nabla V|^2\big)\xi^2 dV(\tau) \\
&\quad \leq c + \int_0^\tau \partial_t \int_\Omega \big(|Rm|^2 + |\nabla^2 u|^2 + |\nabla V|^2\big)\xi^2 dV dt \\
&\quad \leq c - \int_0^\tau \int_\Omega \big(|\nabla Rm|^2 + |\nabla^3 u|^2 + |\nabla^2 V|^2\big)\xi^2 dV dt + c\int_0^\tau \int_\Omega \big(|Rm|^2 + |\nabla^2 u|^2 + |\nabla V|^2\big) dV dt \ .
\end{aligned} \tag{3.61}$$

Using the identities

$$Rm = \tilde{\nabla}^2 g + \tilde{\nabla} g * \tilde{\nabla} g, \qquad \nabla^2 u = \tilde{\nabla}^2 u + \tilde{\nabla} g * \tilde{\nabla} u, \tag{3.62}$$

we obtain the estimate (using Young's inequality)

$$\begin{aligned}\int_0^\tau \int_\Omega \big(|Rm|^2 + |\nabla^2 u|^2\big) dV dt &\leq \int_0^\tau \int_\Omega \big(|\tilde{\nabla}^2 g|^2 + |\tilde{\nabla}^2 u|^2\big) dV dt + c \int_0^\tau \int_\Omega dV dt \\ &\leq \int_0^\tau \int_\Omega |\tilde{\nabla}^2 \Psi|^2 dV dt + c \leq c\end{aligned}$$

in view of Proposition 3.14, Lemma 3.16, and $\tau \leq T = T(n, c_0, k_0, s_0)$. Since $\nabla V \simeq \nabla \tilde{\nabla} g$, the last part can be dealt with by applying Lemma 3.16:

$$\int_0^\tau \int_\Omega |\nabla V|^2 dV dt = \int_0^\tau \int_\Omega \nabla \tilde{\nabla} g * \nabla \tilde{\nabla} g \, dV dt \leq c \int_0^\tau \int_\Omega |\nabla \tilde{\nabla} g|^2 dV dt \leq c \ .$$

Rearranging (3.61), we have

$$\int_\Omega \big(|Rm|^2 + |\nabla^2 u|^2 + |\nabla V|^2\big) \xi^2 dV(\tau) + \int_0^\tau \int_\Omega \big(|\nabla Rm|^2 + |\nabla^3 u|^2 + |\nabla^2 V|^2\big) \xi^2 dV dt \leq c \ . \tag{3.63}$$

Taking into account that the second integral is positive and that the right hand side does not depend on time, we have

$$\max_{0 \leq \tau \leq T} \int_\Omega \big(|Rm|^2 + |\nabla^2 u|^2 + |\nabla V|^2\big) \xi^2 dV(\tau) \leq c = c(n, k_0, c_0, s_0, R)$$

which is what we wanted to show.

□

To be able to prove supremum bounds, we need higher integrability. We simplify the notation and define $\psi = (R_{ijkl}, \nabla_p \nabla_q u, \nabla_r V_s)$ from now on where all indices are running from 1 to n. This is a useful collection since the evolutions of all parts have the same structure:

$$\begin{aligned}\partial_t Rm &= \Delta Rm + \psi * \psi + \tilde{\nabla} g * \nabla \psi \\ \partial_t \nabla^2 u &= \Delta \nabla^2 u + \psi * \psi + \tilde{\nabla} g * \nabla \psi + du * du * \psi + du * \tilde{\nabla} g * \psi \\ \partial_t \nabla V &= \Delta \nabla V + \psi * \psi + \tilde{\nabla} g * \nabla \psi + du * \nabla \psi + du * du * \psi + du * \tilde{\nabla} g * \psi + \nabla \tilde{\nabla} g * \psi \\ &\quad + du * du * \nabla \tilde{\nabla} g \ .\end{aligned}$$

Therefore the norm $|\psi|^2$ satisfies the evolution equation:

$$\begin{aligned}\partial_t |\psi|^2 &= 2\psi * \Delta \psi + \psi * \psi * \psi + (du * du + du * \tilde{\nabla} g + \nabla \tilde{\nabla} g) * \psi * \psi + (du + \tilde{\nabla} g) * \psi * \nabla \psi \\ &\quad + du * du * \psi * \nabla \tilde{\nabla} g \ .\end{aligned}$$

By Young's inequality and the Bochner formula, we can estimate

$$\partial_t |\psi|^2 \leq \Delta |\psi|^2 - 2|\nabla \psi|^2 + \psi * \psi * \psi + c|\psi|^2 + \nabla \tilde{\nabla} g * \psi * \psi + \tfrac{1}{8} |\nabla \psi|^2 + du * du * \psi * \nabla \tilde{\nabla} g$$

in view of the global bounds on $\tilde{\nabla} u = du$ and $\tilde{\nabla} g$ from Proposition 3.14. In addition, we have from (3.62) and $\nabla V \simeq \nabla\tilde{\nabla} g$ the identity

$$\psi = \nabla\tilde{\nabla}\Psi + \tilde{\nabla} g * \tilde{\nabla}\Psi \qquad (3.64)$$

since we can compare

$$\tilde{\nabla}^2\Psi = \nabla\tilde{\nabla}\Psi + (\tilde{\nabla} - \nabla)\tilde{\nabla}\Psi = \nabla\tilde{\nabla}\Psi + \tilde{\nabla} g * \tilde{\nabla}\Psi \ .$$

Lemma 3.20 *Let $(g,u)(t)$ be a solution to (2.5) as in Proposition 3.14 on $[0,T]\times\Sigma$. Then for any $x_0 \in \Sigma$, any radius $0 < R < \infty$, and all $m \geq 1$ we can estimate*

$$\int_0^T \int_{B_R(x_0)} |\psi|^{2(m-1)} \cdot \Big(|\nabla\psi|^2 + |\nabla\tilde{\nabla}\Psi|^2 \Big) dV dt \leq c$$

and

$$\max_{0\leq t\leq T} \int_{B_R(x_0)} |\psi|^{2m} dV \leq c$$

where $c = c(n, m, k_0, c_0, s_0, R)$ and Ψ is defined in (3.23).

Proof:
We just give a short proof of this lemma since the techniques are exactly the same as in the last one. We use the global bound $|\tilde{\nabla}\Psi|^2 \leq c$ and the evolution inequality (3.28) to obtain:

$$\partial_t |\tilde{\nabla}\Psi|^2 = \partial_t |\tilde{\nabla}\Psi|_0^2 + (\partial_t g^{-1}) * \tilde{\nabla}\Psi * \tilde{\nabla}\Psi \leq g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b |\tilde{\nabla}\Psi|^2 - |\tilde{\nabla}^2\Psi|^2 + c \ .$$

Note that we have $\partial_t g^{-1} \leq c\cdot\tilde{\nabla}^2 g + \tilde{\nabla}\Psi * \tilde{\nabla}\Psi \leq \frac{1}{2}|\tilde{\nabla}\Psi|^2 + c$ from (3.14). The proof is by induction and the case $m = 1$ is proven in Lemma 3.15 and (3.63). To do the induction step, we assume that the lemma is true for all $s \leq m-1$. We start the computation with

$$\begin{aligned}\partial_t \int_\Omega |\tilde{\nabla}\Psi|^2 |\psi|^{2(m-1)} \xi^2 dV = & \int_\Omega \partial_t |\tilde{\nabla}\Psi|^2 \cdot |\psi|^{2(m-1)}\xi^2 dV + \int_\Omega |\tilde{\nabla}\Psi|^2 \cdot \partial_t |\psi|^{2(m-1)}\xi^2 dV \\ & + \int_\Omega |\tilde{\nabla}\Psi|^2 |\psi|^{2(m-1)} \xi^2 (\partial_t dV) \ .\end{aligned} \qquad (3.65)$$

We will use Young's inequality and the bound $|\tilde{\nabla}\Psi|^2 \leq c$ frequently in the following calculations. Integrating by parts, we have

$$\begin{aligned}&\int_\Omega \big(|\psi|^{2(m-1)} \cdot g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b |\tilde{\nabla}\Psi|^2\big)\xi^2 dV \\ &\quad = -\int_\Omega g^{ab}\tilde{\nabla}_a |\tilde{\nabla}\Psi|^2 \cdot \Big\{ \tilde{\nabla}_b |\psi|^{2(m-1)}\xi^2 + |\psi|^{2(m-1)}\tilde{\nabla}_b\xi\cdot\xi \Big\} \\ &\quad \leq c\int_\Omega \big(|\tilde{\nabla}^2\Psi||\psi|\cdot|\tilde{\nabla}\psi|\big)|\psi|^{2(m-2)}\xi^2 + |\tilde{\nabla}^2\Psi||\psi|^{2(m-1)}\xi \, dV \\ &\quad \leq \frac{2}{16}\int_\Omega |\psi|^{2(m-1)}\cdot|\tilde{\nabla}^2\Psi|^2\xi^2 dV + c\int_\Omega |\psi|^{2(m-2)}|\nabla\psi|^2\xi^2 dV + c\int_\Omega |\psi|^{2(m-1)} dV \ .\end{aligned}$$

Here we used $(\tilde\nabla - \nabla)\psi = \tilde\nabla g * \psi$ to estimate $|\tilde\nabla\psi|^2 \le |\nabla\psi|^2 + c|\psi|^2$. Then the first term in (3.65) comes down to:

$$\begin{aligned}
&\int_\Omega \partial_t|\tilde\nabla\Psi|^2 \cdot |\psi|^{2(m-1)}\xi^2 dV \\
&\qquad = \int_\Omega |\psi|^{2(m-1)} \left\{ g^{ab}\tilde\nabla_a\tilde\nabla_b|\tilde\nabla\Psi|^2 - |\tilde\nabla^2\Psi|^2 + c \right\} \xi^2 dV \\
&\qquad \le -\frac{7}{8}\int_\Omega |\psi|^{2(m-1)}|\tilde\nabla^2\Psi|^2\xi^2 dV + c\int_\Omega |\psi|^{2(m-1)}dV + c\int_\Omega |\psi|^{2(m-2)}|\nabla\psi|^2\xi^2 dV \ .
\end{aligned}$$

Using $\tilde\nabla g * \tilde\nabla g * \psi * \psi \le c|\psi|^2$ and $(\frac{1}{8} - 2)|\nabla\psi|^2 \le 0$, we calculate for the second term of (3.65):

$$\begin{aligned}
\int_\Omega |\tilde\nabla\Psi|^2 \cdot \partial_t|\psi|^{2(m-1)}\xi^2 dV &= (m-1)\int_\Omega |\tilde\nabla\Psi|^2|\psi|^{2(m-2)} \cdot \Delta|\psi|^2\xi^2 dV \\
&\quad + c\int_\Omega |\psi|^{2(m-2)}\big(\nabla\tilde\nabla\Psi * \psi * \psi + c|\psi|^2 + \psi * \nabla\tilde\nabla\Psi\big)\xi^2 dV \ .
\end{aligned} \tag{3.66}$$

A partial integration takes care of the Laplacian term:

$$\begin{aligned}
&(m-1)\int_\Omega |\tilde\nabla\Psi|^2|\psi|^{2(m-2)} \cdot \Delta|\psi|^2\xi^2 dV \\
&\qquad \le -(m-1)\int_\Omega \Big(\nabla|\tilde\nabla\Psi|^2|\psi|^{2(m-2)} \cdot \nabla|\psi|^2\xi^2 + |\tilde\nabla\Psi|^2\nabla|\psi|^{2(m-2)}\nabla|\psi|^2\xi^2 \\
&\qquad\qquad + |\tilde\nabla\Psi|^2|\psi|^{2(m-2)}\nabla|\psi|^2\nabla\xi \cdot \xi \Big) dV \\
&\qquad \le \int_\Omega \Big(\big(|\nabla\tilde\nabla\Psi||\psi| \cdot c|\nabla\psi|\big)|\psi|^{2(m-2)}\xi^2 \underbrace{-(m-1)(m-2)|\tilde\nabla\Psi|^2|\psi|^{2(m-3)}\big|\nabla|\psi|^2\big|^2\xi^2}_{\le 0} \\
&\qquad\qquad + c|\psi|^{2(m-2)}\big(|\psi| \cdot |\nabla\psi|\xi\big) \Big) dV \\
&\qquad \le \frac{1}{16}\int_\Omega |\psi|^{2(m-1)}|\nabla\tilde\nabla\Psi|^2\xi^2 dV + c\int_\Omega |\psi|^{2(m-2)}|\nabla\psi|^2\xi^2 dV + c\int_\Omega |\psi|^{2(m-1)}dV,
\end{aligned}$$

and the remaining terms in (3.66) give

$$\begin{aligned}
&c\int_\Omega |\psi|^{2(m-2)}\big(\nabla\tilde\nabla\Psi * \psi * \psi + c|\psi|^2 + \psi * \nabla\tilde\nabla\Psi\big)\xi^2 dV \\
&\qquad \le c\int_\Omega \big(|\psi|^{2(m-2)}|\nabla\tilde\nabla\Psi||\psi|^2 + |\psi|^{2(m-1)} + |\psi|^{2(m-2)} \cdot |\psi||\nabla\tilde\nabla\Psi|\big)\xi^2 dV \\
&\qquad \le \frac{2}{16}\int_\Omega |\psi|^{2(m-1)}|\nabla\tilde\nabla\Psi|^2\xi^2 dV + c\int_\Omega (1 + |\psi|^2)|\psi|^{2(m-2)}dV \ .
\end{aligned}$$

Using (3.52), we therefore get for the second term in (3.66):

$$\begin{aligned}
\int_\Omega |\tilde\nabla\Psi|^2 \cdot \partial_t|\psi|^{2(m-1)}\xi^2 dV &\le \frac{3}{16}\int_\Omega |\psi|^{2(m-1)}|\tilde\nabla^2\Psi|^2\xi^2 dV + c\int_\Omega |\psi|^{2(m-2)}|\nabla\psi|^2\xi^2 dV \\
&\quad + c\int_\Omega (1 + |\psi|^2)|\psi|^{2(m-2)}dV \ .
\end{aligned}$$

It remains to estimate the third term in (3.65):

$$\begin{aligned}\int_\Omega |\tilde\nabla\Psi|^2|\psi|^{2(m-1)}\xi^2(\partial_t dV) &\le c\int_\Omega |\psi|^{2(m-1)}\xi^2(g^{-1}*Rm+du*du+\nabla V)dV\\ &\le c\int_\Omega |\psi|^{2(m-1)}|\nabla\tilde\nabla\Psi|\xi^2 dV + c\int_\Omega|\psi|^{2(m-1)}dV\\ &\le \frac{1}{16}\int_\Omega |\psi|^{2(m-1)}|\tilde\nabla^2\Psi|^2\xi^2 dV + c\int_\Omega |\psi|^{2(m-1)}dV,\end{aligned}$$

using (3.64). Collecting terms, (3.65) therefore comes down to:

$$\begin{aligned}\partial_t\int_\Omega |\tilde\nabla\Psi|^2|\psi|^{2(m-1)}\xi^2 dV \le -\frac{1}{2}\int_\Omega |\psi|^{2(m-1)}|\tilde\nabla^2\Psi|^2\xi^2 dV + c\int_\Omega (1+|\psi|^2)|\psi|^{2(m-2)}dV\\ + c\int_\Omega |\psi|^{2(m-2)}|\nabla\psi|^2\xi^2 dV,\end{aligned}$$

such that we can estimate for all $\tau\in[0,T]$:

$$\begin{aligned}&\int_\Omega |\tilde\nabla\Psi|^2|\psi|^{2(m-1)}\xi^2 dV(\tau)\\ &\le \int_\Omega |\tilde\nabla\Psi(0)|_0^2|\psi(0)|_0^{2(m-1)}\xi^2 dV_0 - \frac{1}{2}\int_0^\tau\int_\Omega |\psi|^{2(m-1)}|\tilde\nabla^2\Psi|^2\xi^2 dV dt\\ &\quad + c\int_0^\tau\int_\Omega (1+|\psi|^2)|\psi|^{2(m-2)}dV dt + c\int_0^\tau\int_\Omega |\psi|^{2(m-2)}|\nabla\psi|^2\xi^2 dV dt\ .\end{aligned}$$

This is the same as

$$\int_0^\tau\int_\Omega |\psi|^{2(m-1)}|\tilde\nabla^2\Psi|^2\xi^2 dV dt + \int_\Omega |\tilde\nabla\Psi|^2|\psi|^{2(m-1)}\xi^2 dV(\tau) \le c$$

in view of the induction hypotheses. Using (3.52), we obtain the first claim of the lemma

$$\int_0^\tau\int_\Omega |\psi|^{2(m-1)}|\nabla\tilde\nabla\Psi|^2\xi^2 dV dt \le c\ . \tag{3.67}$$

This estimate allows us to prove all remaining inequalities at the same time. Integrating in time and performing several integrations by part in space, we finally get in a similar way as before

$$\int_\Omega |\psi|^{2m}\xi^2 dV(\tau) \le -m\int_0^\tau\int_\Omega |\psi|^{2(m-1)}|\nabla\psi|^2\xi^2 dV + c\int_0^\tau\int_\Omega |\psi|^{2m}dV + c\ .$$

Because of (3.67), we can estimate further

$$\int_0^\tau\int_\Omega |\psi|^{2m}dV \le \int_0^\tau\int_\Omega |\psi|^{2(m-1)}|\nabla\tilde\nabla\Psi|^2 dV dt + c\int_0^\tau\int_\Omega |\psi|^{2(m-1)}dV dt \le c\ ,$$

and the desired result

$$\int_\Omega |\psi|^{2m}\xi^2 dV(\tau) + m\int_0^\tau\int_\Omega |\psi|^{2(m-1)}|\nabla\psi|^2\xi^2 dV \le c$$

follows. This finishes the proof of Lemma 3.20.

□

We can now prove the supremum bounds:

Proposition 3.21 *Let $(g,u)(t)$ be a solution as in Proposition 3.14 on $[0,T]\times\Sigma$. Then there is a constant $c = c(n,k_0,c_0,s_0)$ such that*

$$\sup_{[0,T]\times\Sigma} |\psi|^2 \leq c \ .$$

Proof:
We rewrite the evolution equations for Rm, $\nabla^2 u$ and ∇V in the same way as in [Shi89, Theorem 6.6] to collect the terms of equal order:

$$\begin{aligned}
\partial_t R_{ijkl} &= \Delta R_{ijkl} + \nabla(g^{-2} * Rm * \tilde{\nabla} g) + (g^{-2} * Rm * Rm + \nabla^2 u * \nabla^2 u + g^{-1} * Rm * \nabla V) \\
\partial_t(\nabla_i\nabla_j u) &= \Delta(\nabla_i\nabla_j u) + \nabla(g^{-2} * \nabla^2 u * \tilde{\nabla} g) \\
&\quad + (g^{-2} * Rm * \nabla^2 u + g^{-1} * du * du * \nabla^2 u + g^{-1} * \nabla^2 u * \nabla V + g^{-3} * \tilde{\nabla} g * Rm * du) \\
\partial_t(\nabla_i V_j) &= \Delta(\nabla_i V_j) + \nabla(g^{-2} * \tilde{\nabla} g * \big[Rm + du * du + \nabla V\big] + g^{-1} * du * \nabla^2 u) \\
&\quad + g^{-2} * Rm * \nabla V + g^{-1} * \nabla V * \nabla V + g^{-1} * du * du * \nabla V \ .
\end{aligned}$$

We consider these equations as linear equations as follows:

$$\begin{aligned}
\partial_t Rm &= \Delta Rm + \nabla A_1 + B_1 \\
\partial_t \nabla^2 u &= \Delta \nabla^2 u + \nabla A_2 + B_2 \\
\partial_t \nabla V &= \Delta \nabla V + \nabla A_3 + B_3
\end{aligned}$$

where $\nabla A_i, B_i$ are free terms defined accordingly. Choosing the radius $R = \frac{1}{8}(1/k_0)^{1/4}$, we get from Lemma 3.20 that

$$\max_{t\in[0,T]} \int_{B_R(x_0)} \big(|A_i|^m + |B_i|^m\big) dV \leq c(n,m,k_0,c_0,s_0)$$

for all $m \geq 1$ and arbitrary $x_0 \in \Sigma$. This is sufficient to apply [LSU68, Theorem III.8.1]. Using the same arguments as in [Shi89, Theorem 6.6], we finally get the estimate:

$$\sup_{[0,T]\times B_{R/2}(x_0)} \big(|Rm|^2 + |\nabla^2 u|^2 + |\nabla V|^2\big) \leq c = c(n,k_0,c_0,s_0) \ .$$

Since $x_0 \in \Sigma$ is arbitrary, the desired result follows.

□

These bounds allow us to translate the results for the solutions of the modified flow (3.46) to the pulled back solutions of the original flow (3.1) and (3.2).

Theorem 3.22 *Let $(\Sigma,\tilde{g})$ be a smooth complete noncompact n-dimensional Riemannian manifold with bounded curvature $|\tilde{Rm}|_0^2 \leq k_0$. Additionally, let $\tilde{u} \in C^\infty(\Sigma)$ be a smooth function satisfying $|\tilde{u}|_0^2 + |\tilde{\nabla}\tilde{u}|_0^2 \leq c_0$ and $|\tilde{\nabla}^2 u|_0^2 \leq s_0$, where $|\cdot|_0$ is the norm given by $\tilde{g}$. Then there exists a constant $T = T(n,k_0,c_0) > c(n)\cdot 1/(\sqrt{k_0}+c_0)$ such that the initial value problem*

$$\begin{aligned}
\partial_t \bar{g} &= -2\bar{R}c + 4 d\bar{u} \otimes d\bar{u} \\
\partial_t \bar{u} &= \bar{\Delta}\bar{u}
\end{aligned}$$

with initial values $\bar{g}(0) = \tilde{g}$ *and* $\bar{u}(0) = \tilde{u}$ *on* Σ *has a smooth solution* $(\bar{g}, \bar{u})(t)$ *on* $[0, T] \times \Sigma$*. Moreover the solution satisfies*

$$C^{-1}\tilde{g} \leq \bar{g}(t) \leq C\tilde{g} \qquad \forall\, t \in [0, T]$$

for some constant $C = C(n, k_0, c_0, s_0)$*, and on* $[0, T] \times \Sigma$ *there is a bound*

$$|\bar{R}m|^2 + |\bar{u}|^2 + |d\bar{u}|^2 + |\bar{\nabla}^2 \bar{u}|^2 \leq c = c(n, k_0, c_0, s_0) \ .$$

Proof:
We take the solution $(g, u)(t)$ constructed in the proof of Theorem 3.12 and apply the inverse of the diffeomorphisms $\varphi(t)$ constructed in Lemma 3.1. The pullback metric $\bar{g} := (\varphi^{-1})^* g$ and the pullback function $\bar{u} := (\varphi^{-1})^* u$ are solutions to the unmodified system (3.1) and (3.2) with the desired properties. In more detail:

The infinitesimal generator of $\varphi(t)$ was chosen to be the vector field

$$V^k = g^{ij}\big(\bar{\Gamma}^k_{ij} - \tilde{\Gamma}^k_{ij}\big)$$

where $\bar{\Gamma}$ is the Christoffel symbol of $\bar{g}$. Using coordinates $x = \{x^1, \ldots, x^n\}$ for g and defining $y := \varphi(x)$, the pointwise equations for the diffeomorphisms are given by (3.7) as follows:

$$\frac{\partial}{\partial t} y^\alpha = \frac{\partial y^\alpha}{\partial x^k} g^{ij} (\bar{\Gamma}^k_{ij} - \tilde{\Gamma}^k_{ij}), \qquad y^\alpha(0) = \delta^\alpha_i x^i \ . \tag{3.68}$$

This establishes a first order system of ordinary differential equations in t for $y^\alpha(x)$, $x \in \Sigma$. From (3.56) and Proposition 3.21 the smooth vector field $V^k = g^{kl} V_l$ satisfies $|V|^2 + |\nabla V|^2 \leq c(n, k_0, c_0, s_0)$ uniformly in $x \in \Sigma$. Therefore the theory of ordinary differential equations provides a unique smooth solution to (3.68) on $[0, T] \times \Sigma$. This implies that the diffeomorphisms $\varphi(t)$ are smooth as long as the solution $(g, u)(t)$ exists. Then the pullbacks $\bar{g} := (\varphi^{-1})^* g$ and $\bar{u} := (\varphi^{-1})^* u$ are well defined, smooth and satisfy the flow equations (2.5) together with the right initial conditions since $\varphi(0) = id$ from Lemma 3.1.

The bound on $|\bar{u}|_0$ is obtained via the maximum principle as in (3.37). Furthermore we have from the usual transformation formulas

$$\begin{aligned} |\bar{\nabla}^2 \bar{u}|^2_{\bar{g}} &= \bar{g}^{ac} \bar{g}^{bd} \bar{\nabla}_a \bar{\nabla}_b \bar{u} \bar{\nabla}_c \bar{\nabla}_d \bar{u} = g^{\alpha\gamma} \frac{\partial x^a}{\partial y^\alpha} \frac{\partial x^c}{\partial y^\gamma} g^{\beta\delta} \frac{\partial x^b}{\partial y^\beta} \frac{\partial x^d}{\partial y^\delta} \cdot \frac{\partial y^\alpha}{\partial x^a} \frac{\partial y^\beta}{\partial x^b} \nabla_\alpha \nabla_\beta u \frac{\partial y^\gamma}{\partial x^c} \frac{\partial y^\delta}{\partial x^d} \nabla_\gamma \nabla_\delta u \\ &= g^{\alpha\gamma} g^{\beta\delta} \nabla_\alpha \nabla_\beta u \nabla_\gamma \nabla_\delta u = |\nabla^2 u|^2_g \leq c \end{aligned}$$

on $[0, T] \times \Sigma$ applying Proposition 3.21. Similarly we get for the curvature

$$|\bar{R}m|^2_{\bar{g}} = |Rm|^2 \leq c \ .$$

Since we also have $|d\bar{u}|^2_{\bar{g}} = |du|^2_g \leq c$ from Lemma 3.5, we can estimate further

$$|\partial_t \bar{g}|^2_{\bar{g}} \leq 4|\bar{R}c|^2 + 16|d\bar{u}|^2_{\bar{g}} \leq 4n^2 |\bar{R}m|^2 + c \leq c \ .$$

Thus we get on $[0, T] \times \Sigma$ analogously to Lemma 2.8 that

$$e^{-cT} \tilde{g} \leq \bar{g}(t) \leq e^{cT} \tilde{g},$$

finishing the proof of the theorem.

□