0 Introduction

0.1 Geometric evolution equations of parabolic type

In the field of differential geometry the search for canonical metrics on a given Riemannian manifold is a central issue. A beautiful and powerful result is the uniformization theorem for surfaces. The canonical metrics are given by the metrics of constant Gauss curvature, and the theorem guarantees that there exists at least one on every surface. This provides a classification of surfaces by the sign of the Gauss curvature of the metrics they can carry. A more general question is whether there exist metrics with other prescribed properties on a given manifold. For example, Lohkamp proved in [Loh92] that the existence of a metric of negative Ricci curvature places no restriction on the topology of the manifold for $n \geq 3$. DeTurck showed in [DeT82] that a nondegenerate symmetric twice covariant tensor can be realized locally as the Ricci curvature of a metric g.

A very successful way to study these problems is to use methods from the theory of partial differential equations. The special properties of geometric quantities like a Riemannian metric which satisfy a partial differential equation yield a lot of useful geometrical and topological information. In the above mentioned examples the equations in question are of elliptic type and their solutions yield the desired canonical object. The approach we are interested in is to use evolution equations, in particular parabolic evolution equations, to tackle geometric problems. Given some arbitrary initial data on the manifold under consideration, we want to construct a more special or canonical object using the smoothing properties of the parabolic operator. This can be thought of as deforming the geometry of the manifold in time in a careful way. Probably the first modern example of such a treatment is the harmonic map heat flow, introduced by Eells and Sampson in [ES64]. They proved that a homotopy class of maps between closed Riemannian manifolds admits a harmonic representative if the target manifold has nonpositive sectional curvature. Here the representative is obtained in a constructive way since the harmonic map is the limit of the solution to the heat flow as time goes to infinity. Other examples are provided by curvature flows such as the Gauss curvature flow which was used in [Fir74] to study the shapes of worn stones, or the mean curvature flow for hypersurfaces, investigated in [Hui84], which describes the motion of a hypersurface in an ambient manifold along its mean curvature vector. This thesis is concerned with an extension of the Ricci flow introduced in [Ham82].

We point out two recent successes in the field of geometric evolution equations. One is the approach of Perelman to prove the Poincaré conjecture and the geometrization conjecture in the context of Hamilton's program for the Ricci flow [Per02],[Per03]. The other is the topological decomposition of mean convex hypersurfaces by Huisken and Sinestrari, using the singularity analysis for the mean curvature flow [HS]. Besides many more beautiful applications of these methods in geometry, they also play an important role in other fields of mathematics and physics: A discrete version of the mean curvature flow is used in mathematical finance for the pricing of options [ST03]. Other curvature flows like the Gauss curvature flow are used in image processing [AGLM93]. Noting just one example of an application in gauge theory, Struwe showed that one can produce nontrivial selfdual Yang-Mills fields, so-called instantons, by using a heat flow method [Str94].

0.2 Hamilton's Ricci flow

One of the most inspiring and successful geometric evolution equations is the Ricci flow introduced in [Ham82]. The motivation to study this particular flow equation was to provide a tool for the geometric classification of manifolds in the same way as the uniformization theorem provides a classification in the case of surfaces. The flow evolves a given initial metric g_0 on a Riemannian manifold M to a simpler and more canonical metric when the right conditions are imposed on g_0. The meaning of "canonical" in this context depends on the properties of the initial data, but in many cases the canonical metrics have constant sectional curvature.

The Ricci flow equations for the time dependent metric $g(t)$ are given as follows:

$$\begin{aligned} \partial_t g(t) &= -2Rc(g(t)) \\ g(0) &= g_0, \end{aligned} \tag{0.1}$$

where Rc is the Ricci tensor of $g(t)$. In some sense, this is the most natural evolution equation for a Riemannian metric, since it can be interpreted as a heat equation for g on M. The heat-type character of the flow (0.1) becomes apparent in the evolution equation for the norm of the curvature tensor Rm which is induced by the evolution of g. It is an example of a reaction-diffusion equation:

$$\partial_t |Rm|_g^2 \leq \Delta^g |Rm|_g^2 + C|Rm|_g^3 \ . \tag{0.2}$$

We can call the Laplacian the diffusion part and the power of Rm the reaction part. The crucial insight is that the behavior of solutions to (0.1) depends on the balance of these two parts. If the diffusion dominates, the metric becomes more uniform as in the case of surfaces of nonpositive curvature [Ham88]. If the reaction wins, then one expects the formation of singularities, where the curvature of the solutions becomes unbounded. This happens on all closed 3-manifolds of positive curvature.

One hopes that the diffusion character of the equation is sufficient to "improve" the metric. This works in several cases. The first and very important case was proven by Hamilton in [Ham82]. He showed that the Ricci flow for a metric g_0 of positive Ricci curvature on a closed 3-manifold converges to one of constant positive curvature after a rescaling process. In dimension 4, a similar result holds for metrics g_0 with positive curvature operator [Ham86]. It should be mentioned that there is also a proof of the uniformization theorem using the Ricci flow in [Ham88] together with the remarks in [CLT05]. There are further results confirming the usefulness of the Ricci Flow to find canonical geometries. Several of these are collected in [CCCY03] and a good survey is given in [Ham95b]. Finally, Perelman's work attacks Thurston's conjecture on the geometrization (and classification) of closed 3-manifolds [Thu97].

It is possible to normalize the Ricci flow such that the manifold has constant volume under the evolution. One interesting property of this flow is that stationary points are precisely the Einstein metrics on the manifold M. This is another indication for the usefulness of the Ricci flow to find canonical geometries on manifolds in the sense described above (considering Einstein metrics as particularly simple or canonical).

0.3 Main results for the extended system

We introduce an evolution for a system closely related to (0.1) which has a different set of stationary points. In Einstein's theory of General Relativity there exist special solutions to the Einstein field equations which are independent of time. An important subclass of these so-called stationary solutions are the static solutions. Although the Einstein evolution problem in general is a hyperbolic system of partial differential equations such that the solution describes a Lorentzian 4-manifold, in this case the system reduces to a weakly elliptic system on a 3-dimensional spacelike slice Σ of the spacetime which is a Riemannian manifold. The remaining freedom for the spacetime metric consists of the Riemannian metric g on the slice and a function u. This function is the logarithm of the lapse function e^u of the spacetime which measures the speed of the evolution of the slice in time direction. If in addition the spacetime does not contain any matter, we obtain the static Einstein vacuum equations for the pair (g, u):

$$\begin{aligned} Rc(g) &= 2du \otimes du \\ \Delta^g u &= 0 \ . \end{aligned} \tag{0.3}$$

This reduction is made in [EK62]. Note that (0.3) is conformally equivalent to the standard form of the static Einstein vacuum equations. We will give more information on static vacuum solutions in section 1.2.

In this thesis, we consider a system of evolution equations such that the stationary points satisfy (0.3) and thus can be interpreted as static solutions of the Einstein vacuum equations. To this end, we extend the Ricci flow (0.1) to the system

$$\begin{aligned} \partial_t g(t) &= -2Rc(g(t)) + 4du(t) \otimes du(t) \\ \partial_t u(t) &= \Delta^{g(t)} u(t) \end{aligned} \tag{0.4}$$

for a Riemannian metric $g(t)$, a function $u(t)$, and given initial data $g(0), u(0)$. This is a quasilinear, weakly parabolic, coupled system of second order.

We prove a short time existence result for very general initial data in Theorem 3.22. On compact manifolds solutions exist for all given smooth initial data up to a maximal time T. Since the flow equations are weakly parabolic, we show that we can apply DeTurck's trick to (0.4) to find an equivalent parabolic system. In the case of complete noncompact manifolds we only need to assume mild conditions on the initial data to prove existence for short times. We first solve the boundary value problem for the parabolic system corresponding to (0.4) on discs. This is done in the proof of Theorem 3.10. Using interior a priori estimates, we prove convergence of a sequence of solutions on an exhaustion of the noncompact manifold to a limit solution defined globally in space (Theorem 3.12 and 3.22). This technique is similar to the one Shi uses for the Ricci flow [Shi89].

A main motivation to study this system stems from its connection to general relativity. An important issue in the numerical evolution of the Einstein equations is the construction of good initial data sets which have to satisfy the so-called constraint equations. In general, this is a hard problem [Ren05, §2.1], [Coo00]. A parabolic system could be used to improve these data sets. Since this should work in particular for static solutions, our system is an interesting candidate for such a smoothing operator. It should be possible to approximate static solutions by solutions to (0.4).

We take advantage of the the fact that our system has a similar structure as the Ricci flow (0.1). This not only provides a source of inspiration but also allows us to adapt techniques developed over the last twenty years in that field [CCCY03], [CK04]. A further crucial property of (0.4) is the following: We prove in section 2.1 that it can be obtained from a variational principle. In particular, we show that it is equivalent to the L^2-gradient flow of the entropy integral

$$E(g,u,f) := \int_\Sigma \left(R - 2|du|^2 + |df|^2\right) e^{-f} dV$$

with respect to the fixed measure $dm := e^{-f} dV$. The additional function f is introduced as a potential for a diffeomorphism. Taking into account the diffeomorphisms of the manifold Σ is necessary for our proof of the existence of a variational structure. This part of the work has been inspired by the ideas in [Per02]. We want to mention that our approach immediately provides a variational characterization of solutions to the static Einstein vacuum equations (0.3).

In the field of geometric evolution equations the value of monotone quantities cannot be overestimated. In our case the variational formulation readily provides us with the monotone entropy E. If we evaluate $E(g(t), u(t), f(t))$ on a solution, where $f(t)$ satisfies an auxiliary evolution equation, its value is nondecreasing in time. This implies a lot of additional information about the behavior of solutions to (0.4). We deduce the nonexistence of periodic solutions from monotonicity considerations for E and some extended entropy functional in Proposition 5.3, 5.6 and 5.11. The proofs rely on our precise characterization of the equality case when the entropy is constant. This leads to the notion of soliton solutions which evolve only by scaling and the action of diffeomorphisms. In fact our result says that a periodic solution already must be a soliton.

Even more importantly, we also prove the noncollapse of solutions to (0.4) in the sense of Cheeger-Gromov for finite times on closed manifolds in Theorem 7.2. This result is very important in the study of singularity formation in the flow since it implies a lower bound on the injectivity radius of such a solution at a singular time. Other important examples for monotonicity formulas are the formula for the mean curvature flow in [Hui90] and its local version [Eck01]. More formulas for parabolic flows are collected in [Ham93]. In his recent work [Per02], Perelman crucially invokes monotone quantities. In particular, he uses the "reduced volume" to prove noncollapse of solutions to the Ricci flow.

Having established that solutions to (0.4) exist up to some maximal time T, we want to know under which conditions T is infinite and we thus obtain a solution which is global in time. To this end, we take a closer look at the reaction diffusion equation for the curvature of a solution to (0.4). We expect from comparison with the ordinary differential equation

$$\partial_t v^2(t) = v^3(t)$$

that the curvature of the solution can blow up after finite time. Therefore, solutions will exist in general only locally in time. A necessary and sufficient condition is given by Theorem 6.22. In particular, the solution is global if and only if the curvature of the metric $g(t)$ is uniformly bounded on every time interval.

The next step to understand solutions to (0.4) better, is to analyze the structure of the finite-time singularities. This is a broad field of research for all kinds of geometric evolution equations. One uses rescaling techniques to understand the behavior of a solution near a singular point. Roughly speaking, one magnifies a neighborhood of a singular point in the manifold. This procedure provides a sequence of solutions whose time of existence gets larger and larger due to the parabolic nature of the flow. The limit of such a sequence of rescalings has several nice properties making the analysis easier. This knowledge provides useful information about the microstructure around a singularity.

We prove in Theorem 8.6 that we can take a limit at a finite time singularity which is complete, ancient and noncollapsed. Moreover, it not only satisfies (0.4) but in fact is a solution to (0.1) which shows the deep connections between these two systems of equations. This is very useful, because a lot is already known about singularity formation in the Ricci flow.

Why it is important to understand these singularities can be seen in the applications of the Ricci flow. The idea of Hamilton to attack the geometrization conjecture using the Ricci flow heavily relies on the study of singularity formation. Here the singularities in some sense reflect the topological decomposition of the manifold. Another example is the mean curvature flow on mean convex surfaces. In this case the rescaling limit at a singularity (a so-called singularity model) is a round infinite evolving cylinder. Therefore a small piece in the original solution looks like a piece of this cylinder shortly before the singular time. Using this knowledge, one can perform a very careful surgery procedure to cut away singular parts of the manifold. Then one continues the flow on the remaining pieces of the original manifold beyond the singular time. This intricate procedure is proven in [HS]. We expect that the system (0.4) provides useful information about the topology and geometry of the underlying manifold in a similar way as the Ricci flow does. Moreover, since there is an additional data, there is a richer structure of solutions.

To be able to take a rescaling limit as mentioned above, we prove in Theorem 8.2 the compactness of a subset of solutions to (0.4). A crucial ingredient in the proof are interior a priori estimates for the curvature, the logarithm u of the lapse function and all their derivatives. We prove these estimates in chapter 6. The noncollapsing result of Theorem 7.2 and Hamilton's convergence theorem for Riemannian manifolds in [Ham95a, Theorem 2.3] complete the argument.

Besides studying the singularities, it is interesting to examine the properties of global solutions which exist for all time. Here the variational structure of (0.4) comes back in. Since (0.4) is in some sense equivalent to a gradient flow, we expect that for suitably chosen initial data the solution converges as $t \to \infty$ to a critical point of the entropy E. In our case this means that the limit is a solution to the static Einstein vacuum equations (0.3). We provide tools that can be used for further study in that area. Since the a priori estimates in chapter 6 are local, they are applicable on arbitrary complete manifolds and therefore also usable for the study of asymptotically flat manifolds. In particular, we show in Lemma 6.13, Proposition 6.14, and Theorem 6.15 that the behavior of a solution $(g, u)(t)$ is determined by the curvature of $g(t)$. Therefore a solution with bounded curvature is smooth.

In general relativity an interesting class of solutions to the Einstein equations is given by isolated systems consisting of a central body like a star or a black hole in an otherwise empty universe.

Geometrically, solutions of this type are modeled by asymptotically flat manifolds where certain boundary conditions at infinity are imposed.

We show that the flow (0.4) preserves two classes of asymptotically flat solutions in Theorem 9.5 and Theorem 9.7 assuming a uniform curvature bound. This is a reasonable assumption since the formation of singularities has to be expected otherwise as mentioned above. Moreover, since we prove that the solution constructed on complete manifolds in Theorem 3.22 has bounded curvature, we immediately get existence of solutions for asymptotically flat initial data where the asymptotical flatness is preserved in time. These results also imply that the ADM mass of the initial data stays constant under the flow at least for finite time. More generally, we prove in Lemma 9.1 and Lemma 9.2 that any decay behavior of the initial data (g_0, u_0) is preserved under the curvature condition.

At the end of this introduction we like to propose directions for future studies. Note that an asymptotically flat limit for $t \to \infty$ of a global asymptotically flat solution to (0.4) satisfies (0.3) and therefore must be the Minkowski space due to the vacuum theorem of Lichnerowicz [BS99, §2.3]. Other physically relevant solutions to the static Einstein vacuum equations have boundaries in the interior (for example the horizon of a black hole). Therefore it is interesting to consider the system (0.4) as a boundary value problem for some given boundary in the interior of Σ together with suitable boundary conditions. Another useful extension of this work would be to drop the vacuum assumption. To this end, one would need to modify the entropy and the flow equations in a useful way. Finally, the complete structure of singularities at finite times and the question of convergence of a general solution for $t \to \infty$ deserves attention.