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Problems in Positional Games and Extremal Combinatorics

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Preface

This thesis consists of five Chapters.

The main purpose of the first chapter is to serve as an introduction. We will define all necessary concepts and discuss the problems that are studied in this thesis as well as stating the main results of it.

This thesis deals with two different directions of extremal graph theory. In the first part we consider two kinds of positional games, the so-called strong games and the Maker-Breaker games.

In the second chapter we consider the following strong Ramsey game: Two players take turns in claiming a previously unclaimed hyperedge of the complete k-uniform hypergraph on n vertices until all edges have been claimed. The first player to build a copy of a predetermined k-uniform hypegraph is declared the winner of the game. If none of the players win, then the game ends in a draw. The well-known strategy stealing argument shows that the second player cannot expect to ever win this game. Moreover, for sufficiently large n, it follows from Ramsey's Theorem for hypergraphs that the game cannot end in a draw and is thus a first player win. Now suppose the game is played on the infinite k-uniform complete hypergraph. Strategy stealing and Ramsey's Theorem still hold and so we might ask the following question: is this game still a first player win or a draw. In this chapter we construct a 5-uniform hypergraph for which the corresponding game is a draw. This chapter is based on [45].

In the third chapter we consider biased $(1 : q)$ Maker-Breaker games: Two players called Maker and Breaker alternate in occupying previously unoccupied vertices of a given hypergraph $\mathcal{H}$. Maker occupies 1 vertex per round and Breaker occupies q vertices. Maker wins if he fully occupies a hyperedge of $\mathcal{H}$ and Breaker wins otherwise. One of the central questions in this area is to find (or at least approximate) the maximal value of q that allows Maker to win the game. In this chapter we prove two new general winning criteria -one for Maker and one for Breaker- and apply them to two types of games. In the first type, the target is a fixed uniform hypergraph and in the second it is a solution to an arbitrary but fixed linear system of inhomogeneous equations. This chapter is based on [53].

The second part of this thesis deals with two types of questions from extremal combinatorics. The first type is of the following form: suppose we are given a certain property of hypergraphs, what is the minimum possible number of edges a k-uniform hypergraph can have such that it does have the property of interest. The second type goes in the different direction. If a certain family of hypergraphs has the minimum number of edges satisfying a certain property, i.e. if the family is extremal in that sense, then how can we characterise it and what operations can we perform while maintaining extremality?

In the fourth chapter, we investigate the minimum number $f(k)$ of edges a k-uniform hypergaph having *Property O* can have. In [24] it is shown that $k! \leq f(k) \leq (k^2 \ln k)k!$, where the upper bound holds for sufficiently large k. We improve the upper bound by a factor of $k \ln k$ showing $f(k) \leq \left(\lfloor \frac{k}{2} \rfloor + 1\right) k! - \lfloor \frac{k}{2} \rfloor (k-1)!$ for every $k \geq 3$. We also answer a question regarding the minimum number $n(k)$ of vertices a k-uniform hypergaph having Property O can have. This chapter is based on [51].

In the fifth chapter, we consider shattering extremal set systems. A set system $\mathcal{F} \subseteq 2^{[n]}$ is said to *shatter* a given set $S \subseteq [n]$ if $2^S = \{F \cap S : F \in \mathcal{F}\}$. The Sauer-Shelah Lemma states that in general, a set system $\mathcal{F}$ shatters at least $|\mathcal{F}|$ sets. Here we concentrate on the case of equality. A set system is called *shattering-extremal* if it shatters exactly $|\mathcal{F}|$ sets. The so-called *elimination conjecture*, independently formulated by Mészáros and Rónyai as well as by Kuzmin and Warmuth, states that if a family is shattering-extremal then one can delete a set from it and the resulting family is still shattering-extremal. We prove this conjecture for a class of set systems defined from Sperner systems and for Sperner systems of size at most 4. Furthermore we continue the investigation of the connection between shattering extremal set systems and Gröbner bases. This chapter is based on the extended abstract [52].

Notation Throughout this thesis we will use fairly standard notation, generally following [23]. For the convenience of the reader, we have included the required definitions needed to understand a particular chapter in the corresponding sections of the introduction.

In chapter 4 we make use of asymptotic notation. Given two functions $f, g : \mathbb{N} \to \mathbb{R}$ we write $f = O(g)$ if there exists a constant $C > 0$ such that for n sufficiently large $|f(n)| \leq C|g(n)|$ holds. We write $f = \Omega(g)$ if $g = O(f)$, and $f = \theta(g)$ if $f = O(g)$ and $f = \Omega(g)$. We write $f = o(g)$ if $\frac{f(n)}{g(n)} \to 0$ as n tends to infinity. If $g = o(f)$ then we write $f = \omega(g)$. Finally, if $\frac{f(n)}{g(n)} \to 1$ as n tends to infinity, we write $f \sim g$ and say that f and g are *asymptotically equal.*

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Chapter 1

Introduction

Combinatorics is a branch of mathematics that focuses on the study of discrete mathematical objects. Originally, combinatorial problems arose in other areas of pure mathematics, most notably in Probability Theory, Algebra and Geometry to name a few. In the last 60 years, it has experienced an enormous growth and matured into an independent and important branch of mathematics. Its methods and techniques have found numerous applications in Computer Science, Mathematical Optimisation and Statistical Physics.

The work presented in this thesis takes place in two very active areas of Combinatorics: Positional Games and Extremal Combinatorics.

The Theory of Positional Games aims at providing the mathematical foundation for various perfect information games between two players. A positional game is described by a set of elements (or positions) and a family of subsets (the winning sets). Two players alternately occupy elements of the set until all elements have been occupied. There are several different types of positional games with different rules and using the rules one defines which player is considered the winner of the game. Positional games have deep connections to other mathematical disciplines, such as Probability Theory and Ramsey Theory.

The Extremal Theory of Set Systems aims to determine or at least approximate the maximimum or minimum size of a set system satisfying some prescribed conditions. One of the oldest and most famous results in this area is Sperner's Theorem. It determines the largest possible size a set system can have if none of its sets is allowed to be contained in any other set of the set system. This branch too has many connections to other mathematical fields and proof-techniques often use methods from Probability Theory, Algebra and Topology.

The purpose of this chapter is twofold. Firstly, it introduces the problems that are studied in the following chapters and provides their respective background and motivation. Secondly, it provides an overview of almost all main results of this thesis. Most technical details and proofs are omitted (unless needed) and will be dealt with in the individual chapters.

1.1 Strong Ramsey games

1.1.1 Introduction to strong games The theory of positional games on graphs and hypergraphs goes back to the seminal papers of Hales and Jewett [40] and of Erdős and Selfridge [27]. The theory has enjoyed explosive growth in recent years and has matured into an important area of combinatorics (see the monograph of Beck [7], the recent monograph of Hefetz et al. [44] and the survey of Krivelevich [49]). There are several interesting types of positional games, the most natural of which are the so-called *strong games*.
Let X be a (possibly infinite) set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets of X. As is usual, X is referred to as the *board* of the game and the elements of $\mathcal{F}$ are called the *winning sets*. The *strong game* on $(X, \mathcal{F})$ is a perfect information game, played by two players, called FP (the first player) and SP (the second player), who take turns in claiming previously unclaimed elements of the board, one element per move. The winner of the game is the *first* player to claim all elements of a winning set $A \in \mathcal{F}$. If no player wins the game after some finite number of moves, then the game is declared a *draw*. In case the board is finite, it is a draw if, once all elements of X have been claimed, neither player fully claimed a winning set. A very simple but classical example of this setting is the game of Tic-Tac-Toe, which we shall briefly discuss now.
In the game of *Tic-Tac-Toe* or *Noughts and Crosses* the board X is simply given by the (3×3)-grid $[3]^2$ and the family of winning sets $\mathcal{F}$ is given by:

$$\begin{aligned}\mathcal{F} = &\Big\{\{(i,j) : j \in [3]\}\ i \in [3]\Big\} \cup \Big\{\{(i,j) : i \in [3]\}\ j \in [3]\Big\}\\ &\cup \Big\{\{(1,1),(2,2),(3,3)\},\{(1,3),(2,2),(3,1)\}\Big\}\end{aligned}$$

In words, the winning sets are given by all horizontal, vertical and diagonal lines of length three. A straightforward case analysis, done by many of us when we were little, reveals that if both players play optimally, then the game will inescapably end in a draw. In fact, if we assume that both players play optimally or that both players have access to an computationally all-powerful computer, each strong game is determined and has exactly three possible outcomes:

(i) FP has a winning strategy;

(ii) SP has a winning strategy;

(iii) both FP and SP have a drawing strategy.

Intuitively one might ask that, if both players actually play optimally, then how could SP ever have a winning strategy? Having the first move should somehow give FP a big advantage. This is indeed the case and is the content of the so-called *strategy stealing* argument. It says

that in a strong game played on $(X, \mathcal{F})$, FP can guarantee at least a draw. See [7] or [44] for its easy proof. We would like to emphasise that this statement is valid for *every* strong game. In particular it holds when X is infinite as well. Unfortunately this argument has a little flaw: it is a purely existential result and does not tell us anything as to how to find such a strategy that guarantees at least a draw for FP.
A nice consequence of strategy stealing is the following. If in a strong game played on $(X, \mathcal{F})$ there is no final drawing position, then FP has a winning strategy. It turns out that the combination of these results with results from Ramsey Theory is, on the one hand, quite powerful (as we will discuss shortly in 1.1.2), but on the other hand these two tools are the only tools we have to analyse strong games.

1.1.2 The strong Ramsey game For integers $n \geq q \geq 3$, consider the strong Ramsey game $\mathcal{R}(K_q, n)$. The board of this game is the edge set of K_n and the winning sets are the copies of K_q in K_n. As noted above, by strategy stealing, FP has a drawing strategy in $\mathcal{R}(K_q, n)$ for every n and q. Moreover, it follows from Ramsey's famous Theorem [63] (see also [39] and [21] for numerous related results) that, for every q, there exists an $n_0 = n_0(q)$ such that for every $n \geq n_0$, no matter how one two-colours the edges of the complete graph K_n, there exists a monochromatic copy of K_q. If we imagine that the players colour the edges of K_n in order to claim them, then this exactly means that $\mathcal{R}(K_q, n)$ has no final drawing position and is thus FP's win for every $n \geq n_0$. An explicit winning strategy for FP in $\mathcal{R}(K_q, n)$ is currently only known for two values of q. Firstly, for $q = 3$ and every $n \geq 5$ it is an easy exercise to find such an explicit strategy. For $q = 4$ and n sufficiently large this is already non-trivial, but an explicit winning strategy was shown by László Hegedűs as mentioned by Beck in [9]. Moreover, for every $q \geq 4$, we do not know what the *Game Ramsey number of* q is, i.e., the smallest $n_0 = n_0(q)$ such that $\mathcal{R}(K_q, n)$ is FP's win for every $n \geq n_0$. Determining this value seems to be extremely hard even for relatively small values of q. Furthermore, we do not know the smallest number of moves FP needs in order to win. In fact, we do not even know if this number grows with n or is bounded from above by some function of q. This question was posed by Beck [7] as one of his "7 most humiliating open problems", where he already considers the case $q = 5$ to be "hopeless" (see also [55] and [18] for related problems).

1.1.3 The transition to the infinite board Consider now the strong game $\mathcal{R}(K_q, \aleph_0)$. Its board is the edge set of the countably infinite complete graph $K_{\mathbb{N}}$ and its winning sets are the copies of K_q in $K_{\mathbb{N}}$. Even though the board of this game is infinite, strategy stealing still applies, i.e., FP has a strategy which ensures that SP will never win $\mathcal{R}(K_q, \aleph_0)$. Clearly, Ramsey's Theorem holds as well, i.e., any red/blue colouring of the edges of $K_{\mathbb{N}}$ yields a monochromatic copy of K_q. Hence, as in the finite version of the game, one could expect to combine these two arguments to deduce that FP has a winning strategy in $\mathcal{R}(K_q, \aleph_0)$. Indeed, if $n = 18$ it is known that FP has a winning strategy in $\mathcal{R}(K_4, 18)$, since the Ramsey number for the K_4 is known to be 18. The same holds when n is $19, 20$ or 10^{10}. What possible reason could there be that would prevent FP from having a winning strategy in $\mathcal{R}(K_4, \aleph_0)$? Well, SP might be able to delay FP indefinitely by making infinitely many threats, so that FP could never complete a K_q, and the game would be drawn. This does not happen when $q = 3$, as FP can use the same winning strategy as in the finite game. It does not happen when $q = 4$ either (see [9]), so our intuition seems to be correct. However, the question whether $\mathcal{R}(K_q, \aleph_0)$ is a draw or FP's win is wide open for every $q \geq 5$. In fact, it is not hard to see that this question is equivalent to Beck's question of whether the number of moves needed for FP in order to win $\mathcal{R}(K_q, n)$ grows with n (the corresponding infinite game is a draw) or not (the corresponding infinite game is FP's win).

Of course, when playing Ramsey games, there is no reason why we should restrict our attention to cliques, or even to graphs for that matter. We can easily extend the game to a more general setting: For every integer $k \geq 2$ and every k-uniform hypergraph $\mathcal{H}$, we can study the finite strong Ramsey game $\mathcal{R}^{(k)}(\mathcal{H}, n)$ and the infinite strong Ramsey game $\mathcal{R}^{(k)}(\mathcal{H}, \aleph_0)$. The board of the finite game $\mathcal{R}^{(k)}(\mathcal{H}, n)$ is the edge set of the complete k-uniform hypergraph K_n^k and the winning sets are the copies of $\mathcal{H}$ in K_n^k. As in the graph case, strategy stealing and Hypergraph Ramsey Theory (see, e.g., [21]) shows that FP has winning strategies in $\mathcal{R}^{(k)}(\mathcal{H}, n)$ for every $\mathcal{H}$ and every sufficiently large n. The board of the infinite game $\mathcal{R}^{(k)}(\mathcal{H}, \aleph_0)$ is the edge set of the countably infinite complete k-uniform hypergraph $K_{\mathbb{N}}^k$ and the winning sets are the copies of $\mathcal{H}$ in $K_{\mathbb{N}}^k$. As in the graph case, strategy stealing shows that FP has drawing strategies in $\mathcal{R}^{(k)}(\mathcal{H}, \aleph_0)$ for every $\mathcal{H}$. Hence, here too one could expect to combine strategy stealing and Hypergraph Ramsey Theory to deduce that FP has a winning strategy in $\mathcal{R}^{(k)}(\mathcal{H}, \aleph_0)$ for every $\mathcal{H}$.

However, in Chapter 2, we show that, while it might be true that $\mathcal{R}(K_q, \aleph_0)$ is FP's win for any $q \geq 5$, basing this solely on strategy stealing and Ramsey Theory is ill-founded. The following result is joint work with Dan Hefetz, Lothar Narins, Alexey Pokrovskiy, Clément Requilé and Amir Sarid.

Theorem 1.1.1. *There exists a 5-uniform hypergraph $\mathcal{H}$ such that the strong game $\mathcal{R}^{(5)}(\mathcal{H}, \aleph_0)$ is a draw.*

Apart from being very surprising, Theorem 1.1.1 might indicate that strong Ramsey games are even more complicated than we originally suspected. We discuss this further in Section 2.4.

1.2 Maker-Breaker $\mathcal{G}$-games and van der Waerden games

1.2.1 Introduction to Maker-Breaker games As indicated in the previous section, strong games are very hard to analyse, mainly because we only have two tools to tackle them: the strategy stealing argument and some Ramsey-type results. Moreover, if the players play optimally, then SP has no chance of ever winning the game. To resolve this issue, one could redefine the aim of SP: instead of trying to fully claim a winning set, he could simply try to prevent FP from doing so, and consider himself the winner if he succeeds. The resulting game is called a *Maker-Breaker* game and has received great attention in the last decades (see [7], [13], [20], [27], [36], [44] and [50] for many nice results) and matured into an important area in Combinatorics. We start by giving a precise definition of the game.

Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets of X. In a *Maker-Breaker game* over $(X, \mathcal{F})$, the players are called *Maker* and *Breaker* and they take turns in occupying previously unoccupied elements of the board X. Maker starts. The *winner* is

- Maker, if he fully occupies a winning set by the end of the game;
- Breaker, if he occupies one element of every winning set.

Again, the game is a perfect information game. But note that here a draw is of course impossible, because the players have complementary goals.
We would like to point out that in this thesis we speak of 'occupying' an element in the Maker-Breaker setting, whereas in the strong games we use the term 'picking'. Of course this is only done for convenience and these words are often used interchangeably for both type of games in the literature.

Interestingly, Maker can actually benefit from the fact that Breaker does not try to occupy all elements of a winning set any more. Indeed, Maker does not need to worry about Breaker occupying a winning set and so he can concentrate fully on occupying one himself. An easy example is the game of Tic-Tac-Toe with the Maker-Breaker rules, where one can see that Maker has a winning strategy. Unfortunately, one flaw remains: many Maker-Breaker games are fairly easy wins for Maker. Some examples are

- the K_3 game, where $X = E(K_n)$ and Maker's aim is to build a K_3;
- the *connectivity game* $(E(K_n), \mathcal{C}_n)$, where $\mathcal{C}_n$ denotes the edge sets of spanning connected subgraphs of K_n;

- the *perfect matching game* $(E(K_n), \mathcal{PM}_n)$, where $\mathcal{PM}_n$ denotes the edge sets of perfect matchings of K_n (n even);
- the *Hamiltonicity game* $(E(K_n), \mathcal{HC}_n)$, where $\mathcal{HC}_n$ denotes the edge sets of Hamiltonian cycles of K_n.

It is worth noticing that in all of these games Maker not only wins fairly easily, but he also wins 'fast' (see [42] and [43]). For example, in [42] the authors show that Maker can construct a perfect matching in $n/2+1$ moves (n even) and that this is tight. Since Maker obviously needs at least $n/2$ rounds to win, he only needs one additional round and hence wins 'fast' in that sense.

To help Breaker one would like to give him more power in order to improve his winning chances. To achieve this Chvátal and Erdős [20] introduced a *bias* and so-called *biased Maker-Breaker games*, which we shall define below.

Let p and q be positive integers, X be finite set and let $\mathcal{F} \subseteq 2^X$. The *biased* $(p:q)$ *Maker-Breaker game on* $(X, \mathcal{F})$, which we denote by $\mathbf{G}(\mathcal{F}; q)$, is the same as the Maker-Breaker game with one exception: Maker occupies p elements per round and Breaker occupies q elements. We refer to p and q as the bias. If in the last round there are fewer unoccupied elements of the board left then the bias of the player whose turn it is, then he simply occupies the remaining elements.
So by allowing Breaker to occupy more than one element per round, we make it harder for Maker to win.

We remark that throughout this thesis Maker's bias is always 1, i.e. $p = 1$.

Of course, the strong Ramsey games can be played with the rules of the Maker-Breaker setting. It is not hard to see that again, if the board is large enough, then Maker has a winning strategy in the *unbiased* $(1:1)$ Maker-Breaker game. On the other hand, Breaker wins these games when his bias is $q = \binom{n}{2}$ (and the target graph is not just an edge). Another useful observation is that if Breaker wins with a bias of q, then he also wins with a bias of $q^* \geq q$. Conversely, if Maker wins the biased $(1:q)$ game, then he also wins the biased $(1:q-1)$ game. This leads to the following central definition of biased Maker-Breaker games:

Let X be a finite set and let $\emptyset \neq \mathcal{F} \subseteq 2^X$ such that $\min\{|F| : F \in \mathcal{F}\} \geq 2$. The unique positive integer $q_{\mathcal{F}}$ such that Breaker wins the $(1:q)$ game $(X, \mathcal{F})$ if and only if $q \geq q_{\mathcal{F}}$ is called the *threshold bias* of $(X, \mathcal{F})$.

One of the main questions, or perhaps the central question, of biased Maker-Breaker games is to determine the threshold bias of natural games. Indeed, this has successfully been done for the connectivity game, the perfect matching game and the Hamiltonicity game. We have

$$q_{\mathcal{C}_n}, q_{\mathcal{PM}_n}, q_{\mathcal{HC}_n} = (1+o(1))\frac{n}{\ln n}$$

where $q_{\mathcal{C}_n}, q_{\mathcal{PM}_n}, q_{\mathcal{HC}_n}$ denotes the bias for the connectivity game, the perfect matching game and the Hamiltonicity game respectively (see [37] and [50]). However, for the seemingly 'easier' K_3 game, we only know that $q_{\mathcal{K}_3} = \theta\left(\sqrt{n}\right)$ (see [6] and [44] for a detailed discussion). Again it was Chvátal and Erdős [20] who observed a stunning connection to random graphs. Indeed, when studying the connectivity game, they wrote that the threshold bias 'ought to come around $n/\ln n$' and in fact prove just that. They provide the following intuition, which is now called the *probabilistic intuition*. Assume that the two players do not play cleverly, but instead play completely randomly, that is: whenever a player has to occupy an edge, he occupies it uniformly at random from among all edges that have not yet been occupied. Playing the biased $(1:q)$ connectivity game on $X = E(K_n)$, we get that Maker would create a random graph $G \sim G(n,M)$ with $M = \lceil \frac{1}{q+1}\binom{n}{2}\rceil$. Standard facts from random graph theory (see [46] or [44]) now say that such a graph asymptotically almost surely (a.a.s.) is connected if $M \geq (1+o(1))\frac{n\ln n}{2}$ and is a.a.s. disconnected if $M \leq (1-o(1))\frac{n\ln n}{2}$. Hence if $q \leq (1-o(1))\frac{n}{\ln n}$ Breaker wins a.a.s and if $q \geq (1+o(1))\frac{n}{\ln n}$, then a.a.s. Maker wins. The same intuition works for the perfect matching game and for Hamiltonicity game. In other words, for almost every value of the bias q, the outcome of the game played by two clever players is the same as the typical outcome of the game when both players play randomly. Note that this intuition is not very well understood yet and does not always apply. A first easy example is the K_3 game: here we already mentioned that the bias threshold lies around $\sqrt{n}$. However, the probabilistic intuition would suggest that the correct value lies around n, a much larger value than the correct one. More examples when this probabilistic intuition fails are discussed in Sections 1.2.2 and 1.2.3 where a different probabilistic intuition seems to hold. This will be briefly discussed at the end of Chapter 3.

Before moving on, let us introduce some notation. We denote the number of vertices of a hypergraph $\mathcal{H}$ by $v(\mathcal{H})$, the number of edges by $e(\mathcal{H})$ and its *density* by $d(\mathcal{H}) = e(\mathcal{H})/v(\mathcal{H})$. Given a subset $S \subseteq V(\mathcal{H})$ of vertices, let $d(S) = |\{e \in \mathcal{H} : S \subset e\}|$. For any integer $\ell \in \mathbb{N}$ the *maximum ℓ-degree* is given by $\Delta_\ell(\mathcal{H}) = \max\{d(S) : S \subseteq V(\mathcal{H}), |S| = \ell\}$. Note that if $\mathcal{H}$ is k-uniform for some integer $k \in \mathbb{N}$, then $\Delta_k(\mathcal{H}) = 1$ and $\Delta_\ell(\mathcal{H}) = 0$ for all integers $\ell > k$. Furthermore, in order to simply notation, we shall often identify a hypergraph $\mathcal{H}$ with its edge set $E(\mathcal{H})$.

1.2.2 The Maker-Breaker $\mathcal{G}$-game Given a fixed graph G, one can consider the Maker-Breaker G-game. The board of the game is the edge set of a K_n and the winning sets are simply the copies of G in K_n. Another way of phrasing it goes as follows: Consider the hypergraph $\mathcal{H}_G(n)$ encoding the copies of G in K_n, i.e. $V(\mathcal{H}_G(n)) = E(K_n)$ and

$$E(\mathcal{H}_G(n)) = \left\{\{f_1, ..., f_{e(G)}\} \subseteq E(K_n) \ : \ \{f_1, ..., f_{e(G)}\} = E(G)\right\}.$$

Here, Maker and Breaker take turns in occupying vertices of $\mathcal{H}_G$, where Maker's goal is to fully occupy an edge of $\mathcal{H}_G$ (i.e. a copy of G in K_n) and Breaker's goal is to prevent Maker from achieving his. Note that $\mathcal{H}_G$ is a k-uniform hypergaph with $k = e(G)$. Moreover

$$v(\mathcal{H}_G(n)) = \theta(n^2), \qquad e(\mathcal{H}_G(n)) = \theta(n^{v(G)}) \qquad \text{and} \qquad \Delta_1(\mathcal{H}_G(n)) = e(G)d(\mathcal{H}_G(n)).$$

Bednarska and Łuczak determined the threshold bias, $q(G)$, up to a constant factor for every graph G with at least three non-isolated vertices. It turns out that the threshold bias depends mainly on the density of G. Hence, let us introduce the measure of density needed in this context:

Given some graph G on at least 3 non-isolated vertices, one defines its 2-*density* as

$$m_2(G) = \max_{\substack{F \subseteq G \\ v(F) \geq 3}} \frac{e(F) - 1}{v(F) - 2}. \tag{1.1}$$

We can now state the following classical result of Bednarska and Łuczak.

Theorem 1.2.1 (Bednarska-Łuczak [13]). *For every graph G with at least 3 non-isolated vertices, the threshold bias of the Maker-Breaker G-game on K_n satisifies $q(G) = \theta\left(n^{\frac{1}{m_2(G)}}\right)$.*

We consider the following generalisation. Given some r-uniform hypergraph $\mathcal{G}$ on at least $r+1$ non-isolated vertices, the Maker-Breaker $\mathcal{G}$-game is played on the edge set of the complete r-uniform hypergraph on n vertices, denoted $\mathcal{K}_n^{(r)}$. We define the r-*density* of $\mathcal{G}$ to be

$$m_r(\mathcal{G}) = \max_{\substack{\mathcal{F} \subseteq \mathcal{G} \\ v(\mathcal{F}) \geq r+1}} \frac{e(\mathcal{F}) - 1}{v(\mathcal{F}) - r}. \tag{1.2}$$

Note that this is an obvious generalisation of the 2-density of a graph. Furthermore, we call $\mathcal{G}$ *strictly r-balanced* if $m_r(\mathcal{G}) > (e(\mathcal{F}) - 1)/(v(\mathcal{F}) - r)$ for every proper subhypergraph $\mathcal{F}$ of $\mathcal{G}$ on at least $r + 1$ vertices. Let $\mathcal{H}_n(\mathcal{G})$ denote the hypergraph of all copies of $\mathcal{G}$ in $\mathcal{K}_n^{(r)}$ and refer to $\mathbf{G}(\mathcal{H}_n(\mathcal{G}); q)$ as the $(1 : q)$ *Maker-Breaker $\mathcal{G}$-game on $\mathcal{K}_n^{(r)}$*. Using our new general winning criteria for Maker and Breaker, which will be introduced in Subsection 1.2.4, we will generalise the result of Bednarska and Łuczak and prove the following statement.

Theorem 1.2.2. *For any integer $r \geq 2$ the following holds. If $\mathcal{G}$ is an r-uniform hypergraph on at least $r + 1$ non-isolated vertices, then the threshold bias of the Maker-Breaker $\mathcal{G}$-game on $\mathcal{K}_n^{(r)}$ satisfies*

$$q(\mathcal{H}_n(\mathcal{G})) = \Theta\left(n^{1/m_r(\mathcal{G})}\right)$$

.

Lastly, we would like to point out that these games do not obey the probabilistic intuition. First we need the following notion of *density*. Given a graph G, define the *maximum density* as follows:

$$m(G) = \max_{\substack{F \subseteq G \\ v(F) > 0}} \frac{e(F)}{v(F)}.$$

Now, if G is a fixed graph with at least on edge then for $p \ll n^{-\frac{1}{m(G)}}$ we know that the binomial random graph $G(n,p)$ a.a.s. does not contain G as a subgraph, whereas for $p \gg n^{-\frac{1}{m(G)}}$ we know that $G(n,p)$ a.a.s. does contain G as a subgraph. This beautiful result was in full generality first proved by Bollobás in 1981 (see [15] or [46]). It was previously shown for *balanced* graphs (i.e. graphs G with $m(G) = e(G)/v(G)$) by Erdős and Rényi in their groundbreaking 1960 paper 'On the Evolution of Random Graphs' [28]. The probabilistic intuition would now suggest that the threshold bias of the Maker-Breaker G-game lies around $n^{\frac{1}{m(G)}}$ which it does not. This was already hinted at above in context of the triangle game.

1.2.3 The van der Waerden game and its generalisation Let $k \geq 3$. A *k-term arithmetic progression* (k-AP for short) is a set of integers that can be written in the form $\{a, a+d, \ldots, a+(k-1)d\}$ for some $a, d \in \mathbb{Z}$, and $d \neq 0$. Beck introduced the (unbiased) *van der Waerden games* [12] as the Maker-Breaker positional games played on the board $[n] = \{1, ..., n\}$, where Maker's aim is to occupy a k-AP and Breaker tries to prevent Maker from achieving this. The well-known theorem of van der Waerden [75] states that for every $k \geq 3$ there exists an integer N such that any two-colouring of $[N]$ contains a monochromatic k-term arithmetic progression. The smallest such integer is called the *van der Waerden number*, and is usually denoted by $W(k)$. It is not hard to see that Maker wins if $n \geq W(k)$. Beck defined $W^\star(k)$ to be the least integer, such that Maker has a winning strategy in the unbiased game when the board has size $n \geq W^\star(k)$, and established that $W^\star(k) = 2^{(1+o(1))k}$. This moderate growth is in strong contrast to the known bounds of the van der Waerden number(see [73] and [38]).

Again, we will consider this game in terms of a hypergraph. Let $\mathcal{H}_{k\text{-AP}}(n)$ be the hypergraph encoding k-AP's, i.e. $V(\mathcal{H}_{k\text{-AP}}(n)) = [n]$ and

$$E(\mathcal{H}_{k\text{-AP}}(n)) = \left\{ e \in \binom{[n]}{k} \;:\; e = \{a, a+d, ..., a+(k-1)d\} \text{ for some } a, d \in [n] \right\}$$

Note that

$$e(\mathcal{H}_{k\text{-AP}}(n)) = \theta(n^2), \qquad \Delta_1(\mathcal{H}_{k\text{-AP}}(n)) = O(n) \qquad \text{and} \qquad \Delta_2(\mathcal{H}_{k\text{-AP}}(n)) = O(1) \quad (1.3)$$

where the implicit constants can depend on k.

Let $q(\mathcal{H}_{k\text{-AP}}(n))$ denote the threshold bias of the game $\mathbf{G}(\mathcal{H}_{k\text{-AP}}(n);q)$. In Section 3.1 we will prove the following theorem.

Theorem 1.2.3. *For every $k \geq 3$, the threshold bias of the van der Waerden game $\boldsymbol{G}(\mathcal{H}_{k\text{-}AP}(n);q)$ satisfies*

$$q(\mathcal{H}_{k\text{-}AP}(n)) = \theta(n^{1/(k-1)}).$$

Remark 1.2.4. *Theorem 1.2.3 will be a special case of a much more general theorem, namely Theorem 3.5.5.*

To motivate its statement and the fact that we refer to it as a 'generalised van der Waerden game', we will define the k-AP game (i.e. the van der Waerden game) in the following different way.
Consider the following $(k-2)\times k$ matrix with integer coefficients:

$$A_{k\text{-AP}} = \begin{pmatrix} 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & & \cdots & & & \\ & & & & 1 & -2 & 1 \end{pmatrix} \in \mathbb{Z}^{(k-2)\times k}$$

It is not hard to see that the solutions with pairwise distinct entries to the corresponding homogeneous system of linear equations $A_{k\text{-AP}}\,\mathbf{x}^T = \mathbf{0}^T$ are k-AP's. Since in the k-AP game, we are only interested in non-trivial k-AP's, we will disregard all solutions with repeated coordinates. We can now define the Maker-Breaker game $\mathbf{G}(A_{k\text{-AP}}, n; q)$ played on the board $[n]$, where Maker's goal is to occupy a solution (without repeated coordinates) to $A_{k\text{-AP}}\,\mathbf{x}^T = \mathbf{0}^T$ and Breaker's goal is to prevent Maker from doing so. Note that this is the exact same game as $\mathcal{G}(\mathcal{H}_{k\text{-AP}}(n);q)$. The advantage of this formulation is that it naturally leads to the following question: Instead of considering the game $\mathbf{G}(A_{k\text{-AP}}, n; q)$ for the k-AP matrix, what happens if we play the game with an arbitrary matrix $A \in \mathbb{Z}^{r\times m}$? In order to do so, one needs to identify certain properties of matrices that give rise to a non-trivial game. For example, if for a matrix A the corresponding system of linear equations $A\mathbf{x}^T = \mathbf{0}^T$ has no solution at all, then clearly this is not a particularly interesting game. These properties will be given in Chapter 3.

Of course one also has to specify what kinds of solutions should count. Starting with the classical result of Rado (see [62]) most combinatorial research has focused on *proper* solutions, that is solutions to the linear system of homogenous equations $A\mathbf{x}^T = \mathbf{0}^T$ without repeated coordinates. Our results will also cover the case where the system is *inhomogenous.* Furthermore we will discuss the effect that repeated coordinates in solutions have on the game. More precisely, we will identify exactly which coordinate-equalities make the game easier for Maker and wich do not. In order to state our results more precisely, we need some fairly technical definitions which we shall present in Chapter 3.

It turns out that for certain matrices one can define a density parameter, first introduced by Rödl and Ruciński for a more restricted class of matrices, just as in the hypergraph case. Our main result, Theorem 3.5.5, shows that as in the Maker-Breaker $\mathcal{G}$-game, the threshold bias of the (non-trivial) games we will define only depends on the density of a given matrix. For completeness, we will also state a result that covers all remaining cases, i.e. the "trivial" games, see Proposition 3.5.7.

1.2.4 General winning criteria When trying to determine the threshold biases for certain natural games, in particular those mentioned above, we have essentially two ways to trying to do so: for each game we could try to come up with some clever ad-hoc argument (cf. Section 3.1) or we could try to find an argument or a strategy that works for many games. The first such argument was proven in [27] and is usually referred to as the *Erdős-Selfridge Criterion*:

Theorem 1.2.5 (Erdős-Selfridge [27])**.** *Let $\mathcal{F}$ be a hypergraph. If $\sum_{F\in\mathcal{F}} 2^{-|F|} < 1/2$, then Breaker wins the unbiased game on $\mathcal{F}$.*

Its proof method is the first example of the *method of conditional expectation.* The power of this method is that it leads to an efficient deterministic strategy for Breaker (see [44] for a detailed discussion).
In 1982 Beck generalised the Erdős-Selfridge criterion to biased games.

Theorem 1.2.6 (Beck [11])**.** *Let X be a finite set set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets of X. Let p and q be positive integers. If $\sum_{F\in\mathcal{F}} (1+q)^{-|F|/p} < \frac{1}{1+q}$, then Breaker has a winning strategy in the $(p:q)$ game on $\mathcal{F}$.*

Beck also proved a sufficient condition for a Maker win, but this condition has found far fewer applications than his biased Erdős-Selfridge criterion. However, since we will see one application of his Maker's win criterion, we state it here.

Theorem 1.2.7 (Beck [11])**.** *Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets of X. Let p and q be positive integers. If*

$$\sum_{F\in\mathcal{F}} \left(\frac{p+q}{p}\right)^{-|F|} > \frac{p^2q^2}{(p+q)^3}\Delta_2(\mathcal{F}) \cdot |X|, \tag{1.4}$$

then Maker has a winning strategy in the $(p:q)$ game on X.

For the proofs of the aforementioned criteria, we refer the reader to [44].

We will now introduce two new general winning criteria for Maker and Breaker, whose proofs will be presented in Chapter 3. The results stated and described in the previous two sections will then be applications of these criteria.

In order to state the winning criterion for Maker, we will need to introduce the following function. For a k-uniform hypergraph $\mathcal{H}$, we define

$$f(\mathcal{H}) = \min_{2\leq \ell \leq k} \left(\frac{d(\mathcal{H})}{\Delta_\ell(\mathcal{H})}\right)^{\frac{1}{\ell-1}} \tag{1.5}$$

and note that $1/f(\mathcal{H}) = \max_{2\leq\ell\leq k} (\Delta_\ell(\mathcal{H})/d(\mathcal{H}))^{1/(\ell-1)}$. As an example, for the hypergraph encoding non-trivial k-AP's we have, using the properties listed in (1.3):

$$f(\mathcal{H}_{k\text{-AP}}(n)) = \theta(n^{1/(k-1)}).$$

Note that this value coincides with the threshold bias given in Theorem 1.2.3 We can now state the first winning criterion.

Theorem 1.2.8 (Maker Win Criterion). *For every $k \geq 2$ and every positive $c_1 \geq k$ there exist $c = c(k, c_1) > 0$ and $\tilde{c} = \tilde{c}(k, c_1) > 0$ such that the following holds. If $\mathcal{H}$ is a k-uniform hypergraph satisfying*

(i) $\Delta_1(\mathcal{H}) \leq c_1\, d(\mathcal{H})$, *(ii)* $1/f(\mathcal{H}) < 1$, *and* *(iii)* $\dfrac{v(\mathcal{H})}{f(\mathcal{H})}\left(1 - \dfrac{1}{f(\mathcal{H})}\right) \geq \tilde{c}$

then Maker has a winning strategy in $\boldsymbol{G}(\mathcal{H}; q)$ if

$$q \leq c\, f(\mathcal{H}) - 1. \tag{1.6}$$

Let us now spend a few sentences in order to give some intuition for these conditions. Note that if $\mathcal{H}$ is a k-uniform hypergraph, then clearly $\Delta_1(\mathcal{H}) \geq k\; d(\mathcal{H})$. If in addition $\mathcal{H}$ is regular, then we in fact have equality. So Condition (i) ensures that the hypergaph is not too far from being regular. Maker's strategy will be a random one and the proof that it succeeds with positive probability is based on a probabilistic statement, namely Theorem 3.2.3. It turns out that when working in the so-called *binomial random subset* model, $1/f(\mathcal{H})$ will be the correct probability to work with. Condition (ii) makes thus sure that this is in fact a non-trivial probability. Finally, Condition (iii) serves two purposes. Firstly, by choosing $\tilde{c}$ to be large, we can enforce that the number of vertices of $\mathcal{H}$ is large too. Secondly, if a random variable is binomially distributed with parameters $v(\mathcal{H})$ and $1/f(\mathcal{H})$, then Condition (iii) precisely says that its variance has to be big, namely at least as big as $\tilde{c}$ which in applications will be exponential in k.

The second statement now gives a winning criterion for Breaker. The proof will be given in Section 3.3 and is based on multiple applications of the biased Erdős-Selfridge Theorem joint with a bias-doubling strategy that mimics a common alteration approach of the probabilistic method.

Theorem 1.2.9 (Breaker Win Criterion)**.** *For every integer $k \geq 2$ and $0 < \epsilon < 1$ there exists $v_0 = v_0(k, \epsilon)$ and a constant $C_1 = C_1(k) > 0$ such that the following holds. If $\mathcal{H}$ is a k-uniform hypergraph on $v(\mathcal{H}) \geq v_0$ vertices, then Breaker has a winning strategy in $\boldsymbol{G}(\mathcal{H}; q)$ provided that*

$$q \geq C_1 \max\left(\Delta_1(\mathcal{H})^{\frac{1}{k-1}}, \max_{2\leq \ell \leq k-1}\left(\Delta_\ell(\mathcal{H})^{\frac{1}{k-\ell}}\right) v(\mathcal{H})^{\epsilon}\right). \tag{1.7}$$

Remark 1.2.10. *We remark that the bounds stated in Theorem 1.2.8 and Theorem 1.2.9 coincide -up to constant factor- in all games we have considered above, namely the Maker-Breaker $\mathcal{G}$-games and the generalised van der Waerden games.*

We remark that the proofs of both these criteria are build on the ideas laid out by Bednarska and Łuczak in [13]. The results presented in the above sections are joint work with Juanjo Rué, Christoph Spiegel and Tibor Szabó.

1.3 Hypergraphs with Property O

1.3.1 Introduction of the problem Recall that a *hypergraph* is a pair $H = (V, E)$, where V is a finite set whose elements are called *vertices* and E is a family of subsets of V, called *edges*. It is *k-uniform* if every edge contains precisely k vertices. When studying certain hypergraph properties, one is often interested in a question of the following form: what is the minimum possible number of edges a k-uniform hypegraph can have, such that it does or doesn't have the property of interest. Arguably the most famous example is *property B*, first introduced by Bernstein [14] in 1908. A hypergraph $\mathcal{H}$ has Property B if there exists a proper two-colouring of its vertex set. One can then ask for the minimum possible number of edges a k-uniform hypergraph can have such that its vertex set is not properly two-colourable. Here, the function of interest is the following. Let $k \geq 2$, one defines

$$m(k) := \min\{|E| \ : \text{ there exists a } k\text{-graph } \mathcal{H} = (V, E) \text{ that does not have property } B\}.$$

In 1963 Erdős [25] proved that every k-graph with fewer than 2^{k-1} edges has Property B, i.e. $m(k) \geq 2^{k-1}$. The proof is very simple, just colour the vertices randomly by two colours and apply the union bound. This was improved by Beck [8] in 1978 to $m(k) = \Omega\left(k^{1/3}2^k\right)$. Building on Beck's idea, Radhakrishnan and Srinivasan [61] in 2000 proved the current best lower bound: $m(k) = \Omega\left(\sqrt{k/\log k}\, 2^k\right)$. Both proofs apply a random recolouring and are

nice examples of the so called *alteration method* (see [2] for more information). In 2015, Cherkashin and Kozik [19] found a very nice proof of this lower bound which is based on a random greedy colouring developed by Pluhár [60].
The best known upper bound, due to Erdős [26] from 1964, is a nice twist of the probabilistic argument, where the k-sets are chosen randomly and each colouring defines an event. He showed that $m(k) = O\left(k^2 2^k\right)$.
Since the expected number of monochromatic edges in a random two-colouring of a hypergraph with m edges is $m/2^{k-1}$, it is natural to consider the following normalised quantity: $m(k)/2^{k-1}$. This quantity measures the ratio between $m(k)$ and its trivial lower bound. From the above we have $m(k)/2^{k-1} = O\left(k^2\right)$ and $m(k)/2^{k-1} = \Omega\left(\sqrt{k/\log k}\right)$. Although these bounds remain quite far apart, they do show that $m(k)$ is bounded away from its trivial lower bound.

In a recent paper, Duffus, Kay and Rödl [24] introduce the following property, called *Property O.* Fix an integer $k \geq 2$ and some finite set V. An *ordered k-set* is a k-tuple $\overline{e} = (x_1, ..., x_k)$ of distinct elements of V. We write e to denote the underlying k-set of $\overline{e}$. An *oriented k-uniform hypergraph*, or *oriented k-graph*, is a pair $\mathcal{H} = (V, \mathcal{E})$, where $\mathcal{E} \subset V^k$ is a family of ordered k-sets with no two k-tuples forming the same k-element set. In the case that $\mathcal{E}$ contains an ordered k-set for every k-subset of V, we call $\mathcal{H}$ a *k-tournament.*
Given a linear order $<$ on V, we say that an ordered k-set $\overline{e} = (x_1, x_2, ..., x_k)$ is *consistent* with $<$, if $x_1 < x_2 < ... < x_k$. For convenience, we shall then simply say that $\overline{e}$ is $<$-consistent.

Definition 1.3.1. *Let $k \geq 2$ and let $\mathcal{H} = (V, \mathcal{E})$ be an oriented k-graph. We say that $\mathcal{H}$ has the* ordering property *or* Property O*, if for every linear order $<$ on V, there exists $\overline{e} \in \mathcal{E}$ that is consistent with $<$. Furthermore, let*

$$f(k) := \min\{|\mathcal{E}| : \textit{there exists an oriented } k\textit{-graph } \mathcal{H} = (V, \mathcal{E}) \textit{ having } \text{Property O}\}.$$

In words, $f(k)$ is the minimum number of edges in an oriented k-graph having Property O. It is easy to check that $f(2) = 3$ and an example for the upper bound is a cyclically ordered triangle. Apart from this trivial case, the following is known about $f(k)$:

Theorem 1.3.2 (Duffus-Kay-Rödl [24])**.** *The function $f(k)$ satisfies $k! \leq f(k) \leq (k^2 \ln(k))k!$ where the lower bound holds for all k and the upper bound for k sufficiently large.*

Their proof of the upper bound is probabilistic: they showed that a randomly chosen k-tournament on n vertices with $(k^2 \ln(k)k!)$ edges has Property O with positive probability (for suitably chosen n and k sufficiently large). Furthermore they showed that almost all k-tournaments with $(1 - o(1))\sqrt{k} \cdot k!$ edges don't have Property O.

1.3.2 Results: an improved upper bound The aim of Chapter 4 is to prove the following improvement, by giving an explicit construction of an oriented k-graph.

Theorem 1.3.3. *Let $k \geq 3$. Then there exists an oriented k-graph with $\left(\lfloor\frac{k}{2}\rfloor + 1\right) k! - \lfloor\frac{k}{2}\rfloor(k-1)!$ edges with* Property O. *Hence*

$$f(k) \leq \left(\lfloor\frac{k}{2}\rfloor + 1\right) k! - \lfloor\frac{k}{2}\rfloor(k-1)!$$

Note that, in contrast to Theorem 1.3.2, our upper bound holds for all $k \geq 3$. However, the question whether $f(k)$ is bounded away from $k!$ remains open.
Unfortunately we are not able to improve the (trivial) lower bound for general k, namely $f(k) \geq k!$. We will include its proof here for the convenience of the reader. Suppose that $\mathcal{H} = ([n], \mathcal{E})$ is an oriented k-graph that has Property O. Then every $\overline{e} \in \mathcal{E}$ is consistent with

$$\binom{n}{k}(n-k)! = \frac{n!}{k!}$$

orders on $[n]$. Since $\mathcal{H}$ has Property O, we must have

$$|\mathcal{E}| \cdot \frac{n!}{k!} \geq n!$$

and hence $f(k) \geq k!$.

Another question posed in [24] is to determine the minimum number of *vertices* a 3-uniform hypergraph having Property O can have.

Definition 1.3.4. *For $k \geq 2$ we define*

$$n(k) := \min\{|V| : \textit{there exists an oriented } k\textit{-graph } \mathcal{H} = (V, \mathcal{E}) \textit{ having } \text{Property O}\}$$

Duffus et al. proved that $6 \leq n(3) \leq 9$. For the upper bound they gave a construction and the lower bound was proved using an exhaustive computer search. In Section 4.2 we prove that $n(3) = 6$ by providing two different constructions. The results of this section are joint work with Gal Kronenberg, Piotr Micek and Tuan Tran.

1.4 Shattering extremal families

1.4.1 Shattering The notion of *shattering* occurs in many mathematical disciplines like combinatorics, statistics and logic as well as other closely related fields such as computer science and machine learning. We will start with some notations and then define the concept of shattering.

Given some sets $X \subseteq [n]$ and $I \subseteq [n] \setminus X$, recall that we write 2^X to denote the power set of X. We write $I + 2^X$ for the family $\{I \cup A \ : \ A \subseteq X\}$ and $\binom{X}{k}$ for the collection of subsets of X of size k. Throughout this thesis the terms 'set system' and 'family' are used interchangeably.

Definition 1.4.1. *A set system $\mathcal{F} \subseteq 2^{[n]}$* shatters *a given set $S \subseteq [n]$ if*

$$2^S = \{F \cap S \ : \ F \in \mathcal{F}\}.$$

Let us remark that we omit multiplicities in the above definition. We denote the family of subsets of $[n]$ shattered by $\mathcal{F}$ by $\mathrm{Sh}(\mathcal{F})$. A set system $\mathcal{F} \subseteq 2^{[n]}$ is a *down-set* (*up-set*) if $G \subseteq F$ and $F \in \mathcal{F}$ ($G \in \mathcal{F}$) implies $G \in \mathcal{F}$ ($F \in \mathcal{F}$). Note that $\mathrm{Sh}(\mathcal{F})$ is always a *down-set*. From here on, if not specified otherwise, $\mathcal{F}$ will always be from $2^{[n]}$.

We would like to point out that the notion of shattering can also be stated in terms of the *trace* of a set system. Given a set $S \subseteq [n]$, the *trace* $\mathcal{F}|_S$ of a set system $\mathcal{F}$ on S is defined as $\mathcal{F}|_S = \{F \cap S \ : \ F \in \mathcal{F}\}$. Then S is shattered by $\mathcal{F}$ precisely if $\mathcal{F}|_S = 2^S$. This naturally leads to certain *forbidden trace problems*. Here, we are given two set systems $\mathcal{F}$ and $\mathcal{G}$ and we say that $\mathcal{F}$ *traces* $\mathcal{G}$ if there is $S \subseteq [n]$ such that $\mathcal{F}|_S$ contains a family isomorphic to $\mathcal{G}$. Then the question is, what is the largest size of a set system $\mathcal{F}$ that does not trace $\mathcal{G}$. For more details we refer the interested reader to the survey of Füredi and Pach [35].

Let us have a look at a small example. If $n = 3$ and $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{1,3\}\}$, then the shattered sets are $\mathrm{Sh}(\mathcal{F}) = \{\emptyset, \{1\}, \{2\}, \{3\}\}$.

A natural first question is to ask how the size of a family $\mathcal{F}$ relates to the size of the family it shatters. This question is answered from one side in the following fundamental result, which is usually referred to as the *Sauer-Shelah lemma*.

Proposition 1.4.2 (Sauer [69], Shelah [72])**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be a set system. Then*

$$|Sh(\mathcal{F})| \geq |\mathcal{F}|.$$

Note that this bound is tight as shown by the example above. This statement was proved independently by several authors, most prominently of course by Sauer [69] and Shelah [72]. Note that the above example shows that this inequality can be achieved and hence is tight. As is usual in Extremal Combinatorics, families that achieve equality for certain inequalities are of special interest and in what follows we shall focus on those.

Definition 1.4.3. *Let $\mathcal{F} \subseteq 2^{[n]}$ be a set system. We call $\mathcal{F}$* shattering extremal*, or* s-extremal *for short, if it shatters exactly $|\mathcal{F}|$ sets, i.e. if $|Sh(\mathcal{F})| = |\mathcal{F}|$.*

An important class of s-extremal families are down-sets. Indeed, if $\mathcal{F}$ is a down-set, then it is not hard to see that $\mathrm{Sh}(\mathcal{F}) = \mathcal{F}$: Every set $S \in \mathcal{F}$ is clearly shattered by $\mathcal{F}$ since S and all its subsets already belong to $\mathcal{F}$. On the other hand, all sets $S \notin \mathcal{F}$ cannot be shattered by $\mathcal{F}$, because $F \cap S \neq S$ for every $F \in \mathcal{F}$ since $\mathcal{F}$ is a down-set. Hence $\mathrm{Sh}(\mathcal{F}) = \mathcal{F}$, and so down-sets are s-extremal.

Many interesting results have been obtained in connection with these combinatorial objects, among others by Bollobás, Leader and Radcliffe in [16], by Bollobás and Radcliffe in [17], by Frankl in [31] and recently Kozma and Moran in [48] provided further interesting examples of s-extremal set systems. Anstee, Rónyai and Sali in [4] related shattering to standard monomials of vanishing ideals, and based on this, Mészáros and Rónyai in [65] developed algebraic methods for the investigation of s-extremal families, which we will briefly recall in Chapter 5.

To broaden the picture, we now mention some well known related results. The *Vapnik-Chervonenkis dimension* of $\mathcal{F}$, denoted by $\dim_{VC}(\mathcal{F})$, is the size of the largest set shattered by $\mathcal{F}$. This notion plays a fundamental role in machine learning. An easy corollary of the Sauer-Shelah lemma is the following result, known as the Sauer-inequality, which has found applications in a variety of contexts.

Proposition 1.4.4 ([69],[72])**.** *Let $0 \leq k \leq n$ and $\mathcal{F} \subseteq 2^{[n]}$. If $\mathcal{F}$ shatters no set of size k, i.e. $dim_{VC}(\mathcal{F}) \leq k-1$, then*

$$|\mathcal{F}| \leq \sum_{i=0}^{k-1} \binom{n}{i}. \tag{1.8}$$

This bound is tight: Let $\mathcal{F}$ be the family of all subsets of $[n]$ of size less than k. Then $\mathcal{F}$ shatters no set of size k and has size exactly $\sum_{i=0}^{k-1} \binom{n}{i}$. Families satisfying (1.8) with equality are called *maximum classes*, and serve as important examples in the theory of machine learning. They have several nice properties, among others they are s-extremal. In the case of uniform families the above bound can be strengthened.

Proposition 1.4.5 (Frankl-Pach [33])**.** *Let $0 \leq k \leq l \leq n$ and $\mathcal{F} \subseteq \binom{[n]}{l}$. If $\mathcal{F}$ shatters no set of size k, i.e. $dim_{VC}(\mathcal{F}) \leq k-1$, then*

$$|\mathcal{F}| \leq \binom{n}{k-1}.$$

A set family $\mathcal{S} \subseteq 2^{[n]}$ is called a *Sperner family*, or an *antichain*, if none of its sets is contained in another. Note that uniform families are Sperner. We define the up-set generated by $\mathcal{S}$ as

$$\mathrm{Up}(\mathcal{S}) = \{F \subseteq [n] \ : \ \exists\, S \in \mathcal{S} \text{ such that } S \subseteq F\}.$$

In connection with Proposition 1.4.5 it is an interesting open problem whether the above bound holds for Sperner families in general and not merely for uniform ones (see [32]). Sperner families will play an important role in our study of s-extremal set systems, since one can use them to 'build' s-extremal set systems.

1.4.2 The elimination conjecture Now let us return to the study of s-extremal families. The main goal here is to find good characterisations of them. A positive answer to the following conjecture, formulated in [59], would be a possible way for this.

Conjecture 1.4.6. *For every s-extremal set system $\mathcal{F} \subsetneq 2^{[n]}$ there exists $F \notin \mathcal{F}$ such that $\mathcal{F} \cup \{F\}$ is again s-extremal.*

As by Theorem 2 in [17] $\mathcal{F}$ is s-extremal if and only if $2^{[n]} \setminus \mathcal{F}$ is so, the above conjecture has an equivalent form, namely that for every non-empty s-extremal set system $\mathcal{F} \subseteq 2^{[n]}$ there exists $F \in \mathcal{F}$ such that $\mathcal{F} \setminus \{F\}$ is again s-extremal. The conjecture was originally formulated like this and that's why we refer to it as the *elimination conjecture*. This latter form was formulated by Litman and Moran independently, and called the *corner peeling conjecture*. For maximum classes essentially the same was conjectured by Kuzmin and Warmuth in [54] and proven by Rubinstein and Rubinstein in [66]. In this thesis we will always consider the version as formulated in Conjecture 1.4.6. There are several other cases when the conjecture is known to be true. First of all it is trivially true for down-sets, as there one can always add any minimal element not belonging to it. Mészáros and Rónyai in [58] and [59], using a graph theoretic approach, proved the conjecture for s-extremal families of VC-dimension at most 2. According to personal communication, the same result was independently proven by Litman and Moran. Some examples of Anstee in [3] and of Füredi and Quinn in [34] also turned out to be s-extremal and they also satisfy the conjecture.

1.4.3 The results In order to state our results, we first introduce some further notation. Suppose we are given a Sperner family $\mathcal{S} \subseteq 2^{[n]}$ and a function $h : \mathcal{S} \to 2^{[n]}$ such that $h(S) \subseteq S$ for every $S \in \mathcal{S}$. For $H \subseteq S \subseteq [n]$ define

$$\mathcal{P}_S = S + 2^{[n]\setminus S} \text{ and } \mathcal{Q}_{S,H} = H + 2^{[n]\setminus S}.$$

Note that $\mathcal{P}_S = \mathrm{Up}(S)$ and that $\mathcal{P}_S$ and $\mathcal{Q}_{S,h(S)}$ are hypercubes of the same dimension, namely $n - |S|$, and in particular $|\mathcal{P}_S| = |\mathcal{Q}_{S,h(S)}|$. Furthermore $\mathcal{Q}_{S,H}$ can be thought of as all sets whose intersection with S equals H. This will be very convenient in future proofs. Lastly, set

$$\mathcal{F}(\mathcal{S}, h) = 2^{[n]} \setminus \bigcup_{S\in\mathcal{S}} \mathcal{Q}_{S,h(S)}.$$

Remark 1.4.7. *We would like to point out that one could of course introduce the above notation for general families* $\mathcal{S}$*. However, in this case, letting* $\mathcal{S}'$ *denote the collection of minimal elements in* $\mathcal{S}$*, we get that* $\mathcal{S}'$ *is a Sperner family,* $\mathcal{H}(S) = \mathcal{H}(\mathcal{S}')$*, and* $\mathcal{F}(\mathcal{S}, h) = \mathcal{F}(\mathcal{S}', h)$*. Using this, most of our results can be formulated and proven for general families* $\mathcal{S}$*. For simplicity and since our interest mostly lies in the case of Sperner families, we will consider only them.*

The following proposition is the starting point for our discussion which might be a good first step towards a nice characterisation of s-extremal families.

Proposition 1.4.8. *Let* $\mathcal{S} \subseteq 2^{[n]}$ *be a Sperner family and let* $h : \mathcal{S} \to 2^{[n]}$ *be a function such that* $h(S) \subseteq S$ *for every* $S \in \mathcal{S}$*. Then* $\mathcal{F} = \mathcal{F}(\mathcal{S}, h)$ *is s-extremal with* $Sh(\mathcal{F}) = 2^{[n]} \setminus \bigcup_{S\in\mathcal{S}} \mathcal{P}_S$ *if and only if*

$$\left|\bigcup_{S\in\mathcal{S}} \mathcal{P}_S\right| = \left|\bigcup_{S\in\mathcal{S}} \mathcal{Q}_{S,h(S)}\right|. \tag{1.9}$$

We will prove this Proposition in Chapter 5.
The reason this might be a good starting point to tackle the elimination conjecture is that every s-extremal family $\mathcal{F} \subseteq 2^{[n]}$ is of the form $\mathcal{F}(\mathcal{S}, h)$ for a unique Sperner system $\mathcal{S}$ and function h. Indeed, the Sperner family is simply the collection of all minimal sets not shattered by $\mathcal{F}$ and we will show the existence and uniqueness of an appropriate h in Chapter 5, see Lemma 5.3.5. This further justifies the fact that we restrict our attention to Sperner families $\mathcal{S}$, as mentioned in the remark above.

We will study the applications of Proposition 1.4.8 in three different ways. Firstly we will prove Conjecture 1.4.6 for a special class motivated from Equation (1.9). More precisely, we will show the following theorem.

Theorem 1.4.9. *Let* $\mathcal{S} = \{S_1, ..., S_N\} \subseteq 2^{[n]}$ *be a Sperner family and* $A \subseteq [n]$ *be a fixed set. Furthermore let* $h_A : \mathcal{S} \to 2^{[n]}$ *be defined as* $h_A(S) = S \cap A$*. Then Conjecture 1.4.6 holds for* $\mathcal{F}(\mathcal{S}, h_A)$*, i.e.* $\mathcal{F}(\mathcal{S}, h_A)$ *is s-extremal and there is* $F \notin \mathcal{F}(\mathcal{S}, h_A)$ *such that* $\mathcal{F}' = \mathcal{F}(\mathcal{S}, h_A) \cup \{F\}$ *is again extremal. Moreover* $\mathcal{F}' = \mathcal{F}(\mathcal{S}', h_A)$ *for some suitable Sperner family* $\mathcal{S}'$*.*

So in fact we will start with a Sperner family $\mathcal{S}$ and define the function h by fixing some set $A \subseteq [n]$ as described in the Theorem. A first step will be to show that the resulting set system $\mathcal{F}(\mathcal{S}, h_A)$ is indeed s-extremal, see Proposition 5.1.3.

Secondly, we will prove the conjecture when the corresponding Sperner family is small. For this, we shall present an equivalent form of Conjecture 1.4.6 which is formulated in terms of the cubes $\mathcal{Q}_{S,h}$.

Theorem 1.4.10. *Let $\mathcal{S} \subseteq 2^{[n]}$ be a Sperner family of size at most four, $h : \mathcal{S} \to 2^{[n]}$ be a function such that $h(S) \subseteq S$ for every $S \in \mathcal{S}$ and suppose that the resulting family $\mathcal{F}(\mathcal{S}, h)$ is s-extremal. Then Conjecture 1.4.6 holds for $\mathcal{F}(\mathcal{S}, h)$, i.e. there is $F \notin \mathcal{F}(\mathcal{S}, h)$ such that $\mathcal{F}' = \mathcal{F}(\mathcal{S}, h) \cup \{F\}$ is again extremal.*

Lastly we continue the study of the connection between so-called Gröbner bases and s-extremal families, initiated by Mészáros and Rónyai [65]. Since the result requires some more definitions we will only state it after introducing Gröbner bases in Chapter 5. The results of this section are joint work with Tamás Mészáros.

Chapter 2

Strong Ramsey games: Drawing on an infinite board

Recall that the main aim of this chapter is to prove Thereom 1.1.1, i.e. to show that there exists a 5-uniform hypegraph $\mathcal{H}$ such that the strong Ramsey game $\mathcal{R}^{(5)}(\mathcal{H}, \aleph_0)$ is a draw. We begin by outlining the idea of the construction of $\mathcal{H}$ as well as the strategy for SP. We then prove some sufficient conditions a hypergraph $\mathcal{H}$ has to satisfy such that the game $\mathcal{R}^{(k)}(\mathcal{H}, \aleph_0)$ is a draw. This will be followed by an explicit construction of a 5-graph that satisfies these conditions, thus proving Theorem 1.1.1. The last section will contain some concluding remarks and open problems.

2.1 Overview of the proof

Since the proof of our main result is fairly technical, even though it is based on a very simple idea, we briefly sketch this idea now.

The main properties of the construction are that the hypergraph $\mathcal{H}$ has a distinguished degree 2 vertex z while all other vertices have degree at least 4, and that $\mathcal{H} \setminus z$ is highly *asymmetrical* and still has minimum degree at least 3.

The main idea of the proof is fairly simple and goes as follows: SP can create an *almost* copy of $\mathcal{H}$, namely a copy of $\mathcal{H} \setminus z$, before FP can make a single threat. SP can then make an infinite series of threats, each one forcing FP to respond immediately. Making such a series of threats is possible due to the properties of $\mathcal{H}$ which make z unique. Moreover, since $\mathcal{H} \setminus z$ is asymmetric, when FP blocks a threat, he does not create a threat of his own. It is of course possible that FP will be the first to initiate a (possibly infinite) series of threats. However, for similar reasons, SP will be able to block all of them, thus preventing FP from completing a copy of $\mathcal{H}$. Hence, the rough idea of the strategy of SP may be summarised as follows:

(i) Build a copy of $\mathcal{H} \setminus z$ before FP can create a threat. Otherwise SP essentially ignores the moves of FP.

(ii) Once SP has build $\mathcal{H} \setminus z$, he looks at the hypergraph FP built.

 (a) If FP created a threat, SP blocks it.

 (b) If FP did not create a threat, SP starts an infinite series of threats.

The precise strategy containing all details will be given in the proof of Theorem 2.2.1 in the following section.

2.2 Sufficient conditions for a draw

In this section we list several conditions of a k-uniform hypergraph $\mathcal{H}$ which suffice to ensure that $\mathcal{R}^{(k)}(\mathcal{H}, \aleph_0)$ is a draw. Recall that the *degree* of a vertex $x \in V(\mathcal{H})$ in a hypergraph $\mathcal{H}$, denoted by $d_{\mathcal{H}}(x)$, is the number of edges of $\mathcal{H}$ which are incident with x. The *minimum degree* of $\mathcal{H}$, denoted by $\delta(\mathcal{H})$, is $\min\{d_{\mathcal{H}}(u) : u \in V(\mathcal{H})\}$. In the remainder of this chapter we will often use the terminology *k-graph* or simply *graph* rather than k-uniform hypergraph.

Lastly, we say that a k-graph $\mathcal{F}$ has a *fast winning strategy* if a player can build a copy of $\mathcal{F}$ in $|E(\mathcal{F})|$ moves (note that this player is not concerned about his opponent building a copy of $\mathcal{F}$ first).

Theorem 2.2.1. *Let $\mathcal{H}$ be a k-graph which satisfies all of the following properties:*

(i) *$\mathcal{H}$ has a degree 2 vertex z;*

(ii) *$\delta(\mathcal{H} \setminus \{z\}) \geq 3$ and $d_{\mathcal{H}}(u) \geq 4$ for every $u \in V(\mathcal{H}) \setminus \{z\}$;*

(iii) *$\mathcal{H} \setminus \{z\}$ has a fast winning strategy;*

(iv) *For every two edges $e, e' \in \mathcal{H}$, if $\phi : V(\mathcal{H} \setminus \{e, e'\}) \longrightarrow V(\mathcal{H})$ is a monomorphism, then ϕ is the identity;*

(v) *$e \cap r \neq \emptyset$ and $e \cap g \neq \emptyset$ holds for every edge $e \in \mathcal{H}$, where r and g are the two edges incident with z in $\mathcal{H}$.*

(vi) *$|V(\mathcal{H}) \setminus (r \cup g)| < k - 1$.*

Then $\mathcal{R}^{(k)}(\mathcal{H}, \aleph_0)$ is a draw.

Before proving this theorem, we will introduce some more notation and terminology which will be used throughout this section.

Definition 2.2.2. *Let $e \in \mathcal{H}$ be an arbitrary edge, let $\mathcal{F}$ be a copy of $\mathcal{H} \setminus \{e\}$ in $K_{\mathbb{N}}^k$ and let $e' \in K_{\mathbb{N}}^k$ be an edge such that $\mathcal{F} \cup \{e'\} \cong \mathcal{H}$. If e' is free, i.e. it is not claimed by either player, then it is said to be a* threat *and $\mathcal{F}$ is said to be* open. *If $\mathcal{F}$ is not open, then it is said to be* closed.
Moreover, e' is called a standard threat *if it is a threat and $e \in \{r, g\}$. Similarly, e' is called a* special threat *if it is a threat and $e \notin \{r, g\}$.*

Next, we state and prove two simple technical lemmata. These lemmata nicely highlight how to apply the monomorphism property (iv) of Thereom 2.2.1.

Lemma 2.2.3. *Let $\mathcal{H}$ be a k-graph which satisfies Properties (i), (ii) and (iv) from Theorem 2.2.1. Then, for every edge $e \in \mathcal{H}$, if $\phi : V(\mathcal{H} \setminus \{e\}) \longrightarrow V(\mathcal{H})$ is a monomorphism, then ϕ is the identity.*

Proof. Fix an arbitrary edge $e \in \mathcal{H}$ and an arbitrary monomorphism $\phi : V(\mathcal{H} \setminus \{e\}) \longrightarrow V(\mathcal{H})$. It follows by Properties (i) and (ii) that there exists an edge $f \in \mathcal{H} \setminus \{e\}$ such that $V(\mathcal{H} \setminus \{e, f\}) = V(\mathcal{H})$. Hence, ϕ equals its restriction to $V(\mathcal{H} \setminus \{e, f\})$ which is the identity by Property (iv). □

The next lemma plays an important role in the proof of Theorem 2.2.1. It asserts that given a copy of $\mathcal{H} \setminus \{z\}$ and a vertex x disjoint from $V(\mathcal{H} \setminus \{z\})$, there exists a unique pair of edges r, g completing a copy of $\mathcal{H}$. It is clear from the outlined strategy for SP that this property is crucial for the strategy to work.

Lemma 2.2.4. *Let $\mathcal{H}$ be a k-graph which satisfies Properties (i) and (iv) from Theorem 2.2.1. For any given copy $\mathcal{H}'$ of $\mathcal{H} \setminus \{z\}$ in $K_{\mathbb{N}}^k$ and any vertex $x \in V(K_{\mathbb{N}}^k) \setminus V(\mathcal{H}')$, there exists a unique pair of edges $r', g' \in K_{\mathbb{N}}^k$ such that $x \in r' \cap g'$ and $\mathcal{H}' \cup \{r', g'\} \cong \mathcal{H}$.*

Proof. Let $\mathcal{H}'$ be an arbitrary copy of $\mathcal{H} \setminus \{z\}$ in $K_{\mathbb{N}}^k$ and let $x \in V(K_{\mathbb{N}}^k) \setminus V(\mathcal{H}')$ be an arbitrary vertex. It is immediate from the definition of $\mathcal{H}'$ and Property (i) that there are edges $r', g' \in E(K_{\mathbb{N}}^k)$ such that $x \in r' \cap g'$ and $\mathcal{H}' \cup \{r', g'\} \cong \mathcal{H}$. Suppose for a contradiction that there are edges $r'', g'' \in E(K_{\mathbb{N}}^k)$ such that $\{r'', g''\} \neq \{r', g'\}$, $x \in r'' \cap g''$ and $\mathcal{H}' \cup \{r'', g''\} \cong \mathcal{H}$. Let $\phi : V(\mathcal{H}' \cup \{r', g'\}) \to V(\mathcal{H}' \cup \{r'', g''\})$ be an arbitrary isomorphism. The restriction of ϕ to $V(\mathcal{H}')$ is clearly a monomorphism and is thus the identity by Property (iv). Since x is the only vertex in $(r' \cap g') \setminus V(\mathcal{H}')$ and in $(r'' \cap g'') \setminus V(\mathcal{H}')$, it follows that ϕ itself is the identity and thus $\{r', g'\} = \{r'', g''\}$ contrary to our assumption. □

We are now in a position to prove the main result of this section.
Proof of Theorem 2.2.1. Let $\mathcal{H}$ be a k-graph which satisfies the conditions of the theorem and let $m = |E(\mathcal{H})|$. At any point during the game, let $\mathcal{G}_1$ denote FP's current graph and

let $\mathcal{G}_2$ denote SP's current graph. We will describe a drawing strategy for SP. We begin by a brief description of its main ideas and then detail SP's moves in each case. The strategy is divided into three stages. In the first stage SP quickly builds a copy of $\mathcal{H} \setminus \{z\}$, in the second stage SP defends against FP's threats, and in the third stage (which we might never reach) SP makes his own threats.

Stage I: Let e_1 denote the edge claimed by FP in his first move. In his first $m-2$ moves, SP builds a copy of $\mathcal{H} \setminus \{z\}$ which is vertex-disjoint from e_1. SP then proceeds to Stage II.

Stage II: Immediately before each of SP's moves in this stage, he checks whether there are a subgraph $\mathcal{F}_1$ of $\mathcal{G}_1$ and a free edge $e' \in K_{\mathbb{N}}^k$ such that $\mathcal{F}_1 \cup \{e'\} \cong \mathcal{H}$. If such $\mathcal{F}_1$ and e' exist, then SP claims e' (we will show later that, if such $\mathcal{F}_1$ and e' exist, then they are unique). Otherwise, SP proceeds to Stage III.

Stage III: Let $\mathcal{F}_2$ be a copy of $\mathcal{H} \setminus \{z\}$ in $\mathcal{G}_2$ and let z' be an arbitrary vertex of $K_{\mathbb{N}}^k \setminus (\mathcal{G}_1 \cup \mathcal{G}_2)$. Let $r', g' \in K_{\mathbb{N}}^k$ be free edges such that $z' \in r' \cap g'$ and $\mathcal{F}_2 \cup \{r', g'\} \cong \mathcal{H}$. If, once SP claims r', FP cannot make a threat by claiming g', then SP claims r'. Otherwise he claims g'.

It readily follows by Property (iii) that SP can play according to Stage I of the strategy (since $K_{\mathbb{N}}^k$ is infinite, it is evident that SP's graph can be made disjoint from e_1). It is obvious from its description that SP can play according to Stage II of the strategy. Finally, since SP builds a copy of $\mathcal{H} \setminus \{z\}$ in Stage I and since $K_{\mathbb{N}}^k$ is infinite, it follows that SP can play according to Stage III of the strategy as well.

It thus remains to prove that the proposed strategy ensures at least a draw for SP. Since, trivially, FP cannot win the game in less than m moves, this will readily follow from the next three lemmata which correspond to three different options for FP's $(m-1)$th move. We start with the easiest case and proceed in increasing order of difficulty.

2.2.1 No threat The first lemma deals with the case in which FP does not make a threat with his $(m-1)$th move. According to SP's strategy, he will start making standard threats of his own and so the main point to prove here is that he can do so in such a way, that the answer of FP can't be a "counter-threat", i.e. a move that simultaneously blocks SP's threat and is a threat itself.

Lemma 2.2.5. *If FP's $(m-1)$th move is not a threat, then he cannot win the game.*

Proof. Assume that SP does not win the game. We will prove that, under this assumption, not only does FP not win the game, but in fact he does not even make a single threat throughout the game. We will prove by induction on i that the following two properties hold immediately after FP's ith move for every $i \geq m-1$.

(a) FP has no threat.

(b) Let $\mathcal{G}_1'$ denote FP's graph immediately after his $(m-1)$th move. Then $\mathcal{G}_1 \setminus \mathcal{G}_1'$ consists of $i-m+1$ edges $e_m, \ldots, e_i$, where, for every $m \leq j \leq i$, e_j contains a vertex z_j such that $d_{\mathcal{G}_1}(z_j) = 1$.

Properties (a) and (b) hold for $i = m-1$ by assumption. Assume they hold for some $i \geq m-1$; we will prove they hold for $i+1$ as well. Since FP's $(m-1)$th move is not a threat, SP's ith move is played in Stage III. By the description of Stage III, in his ith move SP claims an edge $e' \in \{r', g'\}$, where both r' and g' contain a vertex z' which is isolated in $\mathcal{G}_1$. If FP does not respond by claiming the unique edge of $\{r', g'\} \setminus \{e'\}$ in his $(i+1)$th move, then SP will claim it in his $(i+1)$th move and win the game contrary to our assumption (by Property (a), FP had no threat before SP's ith move and thus cannot complete a copy of $\mathcal{H}$ in one move). It follows that Property (b) holds immediately after FP's $(i+1)$th move. Suppose for a contradiction that Property (a) does not hold, i.e., that FP makes a threat in his $(i+1)$th move. As noted above, in his ith move, SP claims either r' or g' and, by our assumption that Property (a) does not hold immediately after FP's $(i+1)$th move, in either case FP's response is a threat. Hence, immediately after FP's $(i+1)$th move, there exist free edges r'' and g'' and copies $\mathcal{F}^r$ and $\mathcal{F}^g$ of $\mathcal{H} \setminus \{z\}$ in $\mathcal{G}_1$ such that $\mathcal{F}^r \cup \{r', g''\} \cong \mathcal{H}$ and $\mathcal{F}^g \cup \{r'', g'\} \cong \mathcal{H}$. By Property (ii) and since, by the induction hypothesis, Property (b) holds for i, we have $\mathcal{F}^r \subseteq \mathcal{G}_1'$ and $\mathcal{F}^g \subseteq \mathcal{G}_1'$. Suppose for a contradiction that $e_1 \in \mathcal{F}^r$. Since $\mathcal{F}^r \cup \{r'\}$ is a threat, with $z' \in r'$ in the role of z, it follows by Property (v) that $r' \cap e_1 \neq \emptyset$. However, SP could have created a threat by claiming r' in his ith move which, by Stages I and III of SP's strategy, implies that $r' \cap e_1 = \emptyset$. Hence $e_1 \notin \mathcal{F}^r$ and an analogous argument shows that $e_1 \notin \mathcal{F}^g$. Since $|E(\mathcal{G}_1') \setminus \{e_1\}| = m-2$, it follows that $\mathcal{F}^r = \mathcal{G}_1' \setminus \{e_1\} = \mathcal{F}^g$. Therefore, by Lemma 2.2.4 we have $\{r', g''\} = \{r'', g'\}$. Since, clearly $r' \neq g'$, it follows that $\{r'', g''\} = \{r', g'\}$ contrary to our assumption that both r'' and g'' were free immediately before FP's $(i+1)$th move. We conclude that Property (a) holds immediately after FP's $(i+1)$th move as well. □

2.2.2 Special threat The next lemma deals with the case in which FP makes a special threat in his $(m-1)$th move. According to SP's strategy, he will first block this special threat. Here, a main point is to prove, using the monomorphism property (iv), that FP cannot win in his mth move. In fact, we will show that in this case FP makes exactly one (special) threat.

Lemma 2.2.6. *If FP's $(m-1)$th move is a special threat, then he cannot win the game.*

Proof. Assume that SP does not win the game. We will prove that, under this assumption, FP does not win the game. We begin by showing that he does not win the game in his mth move. Let e' be a free edge such that $\mathcal{G}_1 \cup \{e'\} \cong \mathcal{H}$. Playing according to the proposed strategy, SP responds to this threat by claiming e'. Let f' denote the edge FP claims in his mth move. Suppose for a contradiction that, by claiming f', FP completes a copy of $\mathcal{H}$. Note that $(\mathcal{G}_1 \setminus \{f'\}) \cup \{e'\} \cong \mathcal{H}$ and so there exists an isomorphism $\phi : V((\mathcal{G}_1 \setminus \{f'\}) \cup \{e'\}) \to V(\mathcal{G}_1)$. The restriction of ϕ to $V(\mathcal{G}_1 \setminus \{f'\})$ is clearly a monomorphism and is thus the identity by Lemma 2.2.3. However, $V((\mathcal{G}_1 \setminus \{f'\}) \cup \{e'\}) = V(\mathcal{G}_1 \setminus \{f'\})$ and so ϕ itself is the identity. It follows that $e' \in \mathcal{G}_1$ and thus $e' \in \mathcal{G}_1 \cap \mathcal{G}_2$ which is clearly a contradiction. We conclude that indeed FP does not win the game in his mth move. Next, we prove that, in his mth move, FP does not even make a threat. Suppose for a contradiction that by claiming f' in his mth move, FP does create a threat. Immediately after FP's mth move, let $f'' \in \mathcal{G}_1$ and $f''' \in K_{\mathbb{N}}^k \setminus (\mathcal{G}_1 \cup \mathcal{G}_2)$ be edges such that $\mathcal{H}' := (\mathcal{G}_1 \setminus \{f''\}) \cup \{f'''\} \cong \mathcal{H}$. Recall that $\mathcal{H}'' := (\mathcal{G}_1 \setminus \{f'\}) \cup \{e'\} \cong \mathcal{H}$ as well. Let $\phi : V(\mathcal{H}'') \to V(\mathcal{H}')$ be an isomorphism. The restriction of ϕ to $V(\mathcal{H}'' \setminus \{e', f''\})$ is clearly a monomorphism and is thus the identity by Property (iv). Since FP's $(m-1)$th move was a special threat, it follows that $V(\mathcal{H}'' \setminus \{e', f''\}) = V(\mathcal{H}'')$ and thus ϕ itself is the identity. Therefore $e' \in \mathcal{H}'$. Since $e' \neq f'''$ we then have $e' \in \mathcal{G}_1$ and thus $e' \in \mathcal{G}_1 \cap \mathcal{G}_2$ which is clearly a contradiction. We conclude that indeed FP does not make a threat in his mth move.

It remains to prove that FP cannot win the game in his ith move for any $i \geq m+1$. We will prove by induction on i that the following two properties hold immediately after FP's ith move for every $i \geq m$.

(a) FP has no threat.

(b) $\mathcal{G}_1$ contains at most one copy of $\mathcal{H} \setminus \{z\}$.

Starting with the induction basis $i = m$, note that Property (a) holds by the paragraph above. Moreover, since FP's $(m-1)$th move is a special threat, immediately after this move, there exists a vertex u of degree two in $\mathcal{G}_1$. By Property (ii), this vertex and the two edges incident with it cannot be a part of any copy of $\mathcal{H} \setminus \{z\}$ in $\mathcal{G}_1$ immediately after FP's mth move. Property (b) now follows since FP's graph contains only $m-2$ additional edges. Assume Properties (a) and (b) hold immediately after FP's ith move for some $i \geq m$; we will prove they hold after his $(i+1)$th move as well. As in the proof of Lemma 2.2.5, we can assume that in his $(i+1)$th move FP claims either r' or g'. Since both edges contain a vertex which was isolated in $\mathcal{G}_1$ immediately before FP's $(i+1)$th move, neither edge can be a part of a copy of $\mathcal{H} \setminus \{z\}$ in $\mathcal{G}_1$. Hence, Property (b) still holds. As in the proof of Lemma 2.2.5, if FP does make a threat in his $(i+1)$th move, then $\mathcal{G}_1$ must contain two copies $\mathcal{F}^r \neq \mathcal{F}^g$ of

$\mathcal{H} \setminus \{z\}$ contrary to Property (b). We conclude that Property (a) holds as well. □

2.2.3 Standard threats The following and final lemma deals with the case in which FP makes a standard threat in his $(m-1)$th move. Intuitively, this is the hardest case since here FP has more control over the game.

Lemma 2.2.7. *If FP's $(m-1)$th move is a standard threat, then he cannot win the game.*

Proof. The basic idea behind this proof is that either FP continues making standard threats forever or, at some point, he makes a move which is not a standard threat. We will prove that, assuming SP does not win the game, in the former case there is always a unique threat which SP can block, and in the latter case, by making his own standard threats, SP can force FP to respond to these threats forever, without ever creating another threat of his own.

We first claim that, if FP does win the game in some move s, then there must exist some $m \leq i < s$ such that FP's ith move is not a threat. Suppose for a contradiction that this is not the case. Assume first that, for every $m-1 \leq i < s$, FP's ith move is a standard threat. We will prove by induction on i that, for every $m-1 \leq i < s$, immediately after FP's ith move, $\mathcal{G}_1$ satisfies the following three properties:

(a) $\mathcal{G}_1$ contains a unique copy $\mathcal{F}_1$ of $\mathcal{H} \setminus \{z\}$;

(b) Let $e_{m-1}, \ldots, e_i$ denote the edges of $\mathcal{G}_1 \setminus \mathcal{F}_1$. Then, for every $m-1 \leq j \leq i$, there exists a vertex $z_j \in V(\mathcal{G}_1)$ such that $\{z_j\} = e_j \setminus V(\mathcal{F}_1)$ and $d_{\mathcal{G}_1}(z_j) = 1$;

(c) $\mathcal{F}_1 \cup \{e_i\}$ is open and $\mathcal{F}_1 \cup \{e_j\}$ is closed for every $m-1 \leq j < i$.

Properties (a), (b) and (c) hold by assumption for $i = m-1$. Assume they hold for some $i \geq m-1$; we will prove they hold for $i+1$ as well. Immediately after FP's ith move, let e'_i be a free edge such that $\mathcal{F}_1 \cup \{e_i, e'_i\} \cong \mathcal{H}$. Note that e'_i exists by Property (c) and is unique by Lemma 2.2.4. According to his strategy, SP claims e'_i thus closing $\mathcal{F}_1 \cup \{e_i\}$. By assumption, in his $(i+1)$th move FP makes a standard threat by claiming an edge e_{i+1}. It follows that $e_{i+1} \setminus V(\mathcal{F}_1) = \{z_{i+1}\}$, where, immediately after FP's $(i+1)$th move, $d_{\mathcal{G}_1}(z_{i+1}) = 1$. Hence, Property (b) is satisfied immediately after FP's $(i+1)$th move. Since $\delta(\mathcal{H} \setminus \{z\}) \geq 3$ holds by Property (ii), it follows that Property (a) is satisfied as well. Finally, $\mathcal{G}_1$ satisfies Property (c) by Lemma 2.2.4. Now, by Properties (a), (b) and (c), for every $m-1 \leq i < s$, immediately after FP's ith move there is a unique threat e'_i. According to his strategy, SP claims e'_i in his ith and thus FP cannot win the game in his $(i+1)$th move. In particular, FP cannot win the game in his sth move, contrary to our assumption.

Assume then that there exists some $m \leq i < s$ such that FP makes a special threat in his ith move. We will prove that this is not possible. Consider the smallest such i.

As discussed in the previous paragraph, immediately before FP's ith move, $\mathcal{G}_1$ contained a unique copy $\mathcal{F}_1$ of $\mathcal{H} \setminus \{z\}$, and every vertex of $\mathcal{G}_1 \setminus \mathcal{F}_1$ had degree one in $\mathcal{G}_1$. If FP makes a special threat in his ith move by claiming some edge f'_1, then there exists a free edge f'_2 such that, by claiming f'_2 in his $(i+1)$th move, FP would complete a copy $\mathcal{H}_1$ of $\mathcal{H}$. Since $|V(\mathcal{F}_1)| < |V(\mathcal{H})|$, there is some vertex $u \in V(\mathcal{H}_1) \setminus V(\mathcal{F}_1)$. Immediately after FP's $(i+1)$th move, the degree of u in $\mathcal{G}_1$ is at most three. Hence, by Property (ii), u must play the role of z in $\mathcal{H}_1$. Therefore, $\mathcal{H}_1 = (\mathcal{F}_1 \cup \{f'_1, f'_2, e'_u\}) \setminus \{f'_3\}$, where e'_u is the first edge incident with u which FP has claimed and f'_3 is some edge of $\mathcal{F}_1$. Since, at some point in the game, e'_u was a standard threat, and, at that point, $\mathcal{F}_1$ was the unique copy of $\mathcal{H} \setminus \{z\}$ in $\mathcal{G}_1$, there exists an edge e''_u such that $\mathcal{H}' := \mathcal{F}_1 \cup \{e'_u, e''_u\} \cong \mathcal{H}$. Let $\phi : V(\mathcal{H}') \to V(\mathcal{H}_1)$ be an isomorphism. It is evident that $\mathcal{H}' \setminus \{e''_u, f'_3\} = \mathcal{H}_1 \setminus \{f'_1, f'_2\}$ and that the restriction of ϕ to $V(\mathcal{H}' \setminus \{e''_u, f'_3\})$ is a monomorphism and is thus the identity by Property (iv). However, $V(\mathcal{H}' \setminus \{e''_u, f'_3\}) = V(\mathcal{H}') = V(\mathcal{F}_1) \cup \{u\} = V(\mathcal{H}_1)$ and thus ϕ itself is the identity entailing $e''_u \in \mathcal{G}_1$. However, $e''_u \in \mathcal{G}_2$ holds by the description of the proposed strategy. Hence $e''_u \in \mathcal{G}_1 \cap \mathcal{G}_2$ which is clearly a contradiction.

We conclude that there must exist some $m \leq i < s$ such that FP's ith move is not a threat. Let ℓ denote the first such move. In order to complete the proof of the lemma, we will prove by induction on i that the following two properties hold immediately after FP's ith move for every $i \geq \ell$.

(1) FP has no threat.

(2) Let $\mathcal{G}'_1 = \mathcal{F}_1 \cup \{f\}$, where $\mathcal{F}_1$ is the unique copy of $\mathcal{H} \setminus \{z\}$ FP has built during his first $m-1$ moves and f is the edge FP has claimed in his ℓth move. Then $\mathcal{G}_1 \setminus \mathcal{G}'_1$ consists of $i - m + 1$ edges $e_m, \ldots, e_i$, where, for every $m \leq j \leq i$, e_j contains a vertex z_j such that $d_{\mathcal{G}_1}(z_j) = 1$.

Properties (1) and (2) hold for $i = \ell$ by assumption, by the choice of ℓ and by Properties (a) – (c) above. Assume they hold for some $i \geq \ell$; we will prove they hold for $i+1$ as well. Proving Property (2) can be done by essentially the same argument as the one used to prove Property (b) in Lemma 2.2.5; the details are therefore omitted. Suppose for a contradiction that Property (1) does not hold immediately after FP's $(i+1)$th move. As in the proof of Property (a) in Lemma 2.2.5, it follows that there are free edges r'' and g'' and graphs $\mathcal{F}^r \subseteq \mathcal{G}'_1$ and $\mathcal{F}^g \subseteq \mathcal{G}'_1$ such that $\mathcal{F}^r \cup \{r', g''\} \cong \mathcal{H} \cong \mathcal{F}^g \cup \{r'', g'\}$. Since $\mathcal{F}^r \subseteq \mathcal{G}'_1$ and $\mathcal{F}^g \subseteq \mathcal{G}'_1$, it follows by Property (ii) that $V(\mathcal{F}^r) = V(\mathcal{F}^g)$. Let $\mathcal{F}_2 \subseteq \mathcal{G}_2$ be such that $\mathcal{F}_2 \cup \{r', g'\} \cong \mathcal{H}$ and let z' be the unique vertex in $r' \setminus V(\mathcal{F}_2)$. Note that $r' \setminus \{z'\} \subseteq V(\mathcal{F}^r)$ and $g' \setminus \{z'\} \subseteq V(\mathcal{F}^g)$. Hence $(r' \cup g') \setminus \{z'\} \subseteq V(\mathcal{F}^r)$. By Property (vi), we then have $|V(\mathcal{F}_2) \setminus V(\mathcal{F}^r)| \leq |V(\mathcal{F}_2) \setminus (r' \cup g')| < k - 1$. However, $e_1 \cap V(\mathcal{F}_2) = \emptyset$ holds by the description of the proposed strategy and $|e_1 \cap V(\mathcal{F}^r)| \geq k-1$ holds by our assumption that FP's $(m-1)$th move was a threat. This implies that $k-1 \leq |e_1 \cap V(\mathcal{F}^r)| \leq |V(\mathcal{F}^r) \setminus V(\mathcal{F}_2)| = |V(\mathcal{F}_2) \setminus V(\mathcal{F}^r)| < k-1$ which is clearly a contradiction. We conclude that Property (1) does hold immediately after FP's $(i+1)$th move. □

Since FP's $(m-1)$th move is either a standard threat or a special threat or no threat at all, Theorem 2.2.1 follows immediately from Lemmata 2.2.5, 2.2.6 and 2.2.7. □

2.3 An explicit construction

In this section we will describe a 5-graph $\mathcal{H}$ (Fig. 2.1) which satisfies Properties (i) – (vi) from Theorem 2.2.1 and thus $\mathcal{R}^{(5)}(\mathcal{H}, \aleph_0)$ is a draw, proving Theorem 1.1.1. Before doing so, we need one last definition.

Definition 2.3.1. *A* tight path *is a k-graph with vertex set $\{u_1, \ldots, u_t\}$ and edge set $e_1, \ldots, e_{t-k+1}$ such that $e_i = \{u_i, \ldots, u_{i+k-1}\}$ for every $1 \leq i \leq t-k+1$. The* length *of a tight path is the number of its edges.*

We proceed with the construction of the 5-graph $\mathcal{H}$, see also Figure 2.1. The vertex set of $\mathcal{H}$ is $\{z, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$, and its edges are

$$\begin{aligned}
r &= \{z, v_1, v_3, v_5, v_8\},\\
g &= \{z, v_2, v_4, v_7, v_9\},\\
a &= \{v_1, v_4, v_6, v_8, v_9\},\\
b &= \{v_9, v_1, v_2, v_3, v_4\},\\
e_1 &= \{v_1, v_2, v_3, v_4, v_5\},\\
e_2 &= \{v_2, v_3, v_4, v_5, v_6\},\\
e_3 &= \{v_3, v_4, v_5, v_6, v_7\},\\
e_4 &= \{v_4, v_5, v_6, v_7, v_8\},\\
e_5 &= \{v_5, v_6, v_7, v_8, v_9\}.
\end{aligned}$$

It readily follows from the definition of $\mathcal{H}$ that it satisfies Properties (i), (ii), (v) and (vi) from Theorem 2.2.1. We claim that it satisfies Properties (iii) and (iv) as well. We start with Property (iii).

Lemma 2.3.2. *$\mathcal{H} \setminus \{z\}$ has a fast winning strategy.*

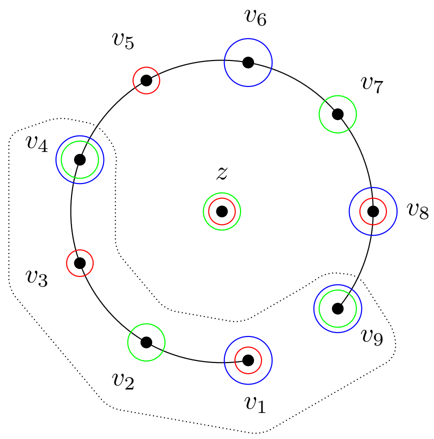

Figure 2.1: The 5-uniform hypergraph $\mathcal{H}$. The black line from v_1 to v_9 represents the tight path consisting of the edges $e_1, \dots, e_5$.

Proof. We describe a strategy for SP to build a copy of $\mathcal{H} \setminus \{z\}$ in seven moves. The basic idea is to build a tight path of length 5 in five moves, and then to use certain symmetries of $\mathcal{H} \setminus \{z\}$ in order to complete a copy of $\mathcal{H} \setminus \{z\}$ in two additional moves. Our strategy is divided into the following three stages.

Stage I: In his first move, SP claims an arbitrary free edge $e_1 = \{v_1, v_2, v_3, v_4, v_5\}$. For every $2 \leq i \leq 5$, in his ith move SP picks a vertex v_{i+4} which is isolated in both his and FP's current graphs and claims the edge $e_i = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}$. If in his 6th move FP claims either $\{v_1, v_2, v_3, v_4, v_9\}$ or $\{v_1, v_4, v_6, v_8, v_9\}$ or $\{v_1, v_3, v_5, v_8, v_9\}$, then SP claims $\{v_1, v_6, v_7, v_8, v_9\}$ and proceeds to Stage II. Otherwise, SP claims $\{v_1, v_2, v_3, v_4, v_9\}$ and skips to Stage III.

Stage II: If in his seventh move FP claims $\{v_1, v_2, v_4, v_6, v_9\}$, then SP claims $\{v_1, v_2, v_5, v_7, v_9\}$. Otherwise, SP claims $\{v_1, v_2, v_4, v_6, v_9\}$.

Stage III: If in his seventh move FP claims $\{v_1, v_4, v_6, v_8, v_9\}$, then SP claims $\{v_1, v_3, v_5, v_8, v_9\}$. Otherwise, SP claims $\{v_1, v_4, v_6, v_8, v_9\}$.

It is easy to see that SP can indeed play according to the proposed strategy and that, in each of the possible cases, the graph he builds is isomorphic to $\mathcal{H} \setminus \{z\}$. □

It remains to prove that $\mathcal{H}$ satisfies Property (iv). We begin by introducing some additional notation. If $\phi : V(\mathcal{H}) \to V(\mathcal{H})$ is a monomorphism, and $e = \{a_1, a_2, \dots, a_5\} \in \mathcal{H}$, then we set $\phi(e) := \{\phi(a_1), \phi(a_2), \dots, \phi(a_5)\}$. For two edges $e, f \in \mathcal{H}$, let $\mathcal{H}_{ef} = \mathcal{H} \setminus \{e, f\}$.

Next, we observe several simple properties of $\mathcal{H}$ and of monomorphisms. These will be crucial for the remainder of the section. Table 2.1 shows the degrees of the vertices in $\mathcal{H}$ and

Table 2.2 shows the sizes of intersections of pairs of edges in $\mathcal{H}$.

Vertex	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	z
Degree	4	4	5	7	6	5	4	4	4	2

Table 2.1: Degrees of vertices in $\mathcal{H}$.

	r	g	a	b	e_1	e_2	e_3	e_4	e_5
r		1	2	2	3	2	2	2	2
g	1		2	3	2	2	2	2	2
a	2	2		3	2	2	2	3	3
b	2	3	3		4	3	2	1	1
e_1	3	2	2	4		4	3	2	1
e_2	2	2	2	3	4		4	3	2
e_3	2	2	2	2	3	4		4	3
e_4	2	2	3	1	2	3	4		4
e_5	2	2	3	1	1	2	3	4	

Table 2.2: Intersection sizes of pairs of edges in $\mathcal{H}$.

Observation 2.3.3. *The hypergraph $\mathcal{H}$ satisfies all of the following properties:*

(1) $V(\mathcal{H}) \setminus \{r, g\} = \{v_6\}$ *and* $r \cap g = \{z\}$.

(2) e_1 *is the unique edge satisfying* $|e_1 \cap r| = 3$ *and* $|e_1 \cap g| = 2$.

(3) b *is the unique edge satisfying* $|b \cap g| = 3$ *and* $|b \cap r| = 2$.

(4) *There are precisely two tight paths of length five in $\mathcal{H}$, namely,* $TP_1 := (e_1, e_2, e_3, e_4, e_5)$ *and* $TP_2 := (b, e_1, e_2, e_3, e_4)$.

(5) *For every two vertices $u, v \in V(\mathcal{H})$, there are three edges $f_1, f_2, f_3 \in \mathcal{H}$ such that* $|f_i \cap \{u, v\}| = 1$ *for every* $1 \leq i \leq 3$.

Observation 2.3.4. *Let $\mathcal{F}$ and $\mathcal{F}'$ be k-graphs, where $\mathcal{F}' \subseteq \mathcal{F}$, and let $\phi : V(\mathcal{F}') \to V(\mathcal{F})$ be a monomorphism. Then*

(a) $d_{\mathcal{F}}(\phi(x)) \geq d_{\mathcal{F}'}(x) \geq d_{\mathcal{F}}(\phi(x)) - |E(\mathcal{F} \setminus \mathcal{F}')|$ *holds for every* $x \in V(\mathcal{F}')$.

(b) *If P is a tight path of length ℓ in $\mathcal{F}'$, then $\phi(P)$ is a tight path of length ℓ in $\mathcal{F}$.*

(c) *Let $P = (f_1, f_2, \dots, f_m)$ be a tight path in $\mathcal{F}'$, where $m \geq k$ and $f_i = \{p_i, \dots, p_{i+k-1}\}$ for every $1 \leq i \leq m$. If $\phi(P) = (e_1, e_2, \dots, e_m)$, where $e_i = \{q_i, \dots, q_{i+k-1}\}$ for every $1 \leq i \leq m$, then either $\phi(p_i) = q_i$ for every $1 \leq i \leq m + k - 1$ or $\phi(p_i) = q_{m+k-i}$ for every $1 \leq i \leq m + k - 1$.*

(d) *For any pair of edges $x, y \in \mathcal{F}'$ we have $|\phi(x) \cap \phi(y)| = |x \cap y|$.*

We prove that $\mathcal{H}$ satisfies Property (iv) in a sequence of lemmata.

Lemma 2.3.5. *Let e and f be two arbitrary edges of $\mathcal{H}$ and let $\phi : V(\mathcal{H}_{ef}) \to V(\mathcal{H})$ be a monomorphism. If $\phi(e') = e'$ holds for every edge $e' \in \mathcal{H}_{ef}$, then ϕ is the identity.*

Proof. Suppose for a contradiction that ϕ is not the identity. Then, there exist distinct vertices $u, v \in V(\mathcal{H}_{ef})$ such that $\phi(u) = v$. By Observation 2.3.3(5), there are three edges $f_1, f_2, f_3 \in \mathcal{H}$ such that $|f_i \cap \{u, v\}| = 1$ for every $1 \leq i \leq 3$. Clearly, we may assume that $f_1 \notin \{e, f\}$ and thus $\phi(f_1) = f_1$ by the assumption of the lemma. Since $\phi(u) = v$, it follows that $\{u, v\} \subseteq f_1$ which is a contradiction. □

Lemma 2.3.6. *Let $\phi : V(\mathcal{H}_{ef}) \to V(\mathcal{H})$ be a monomorphism. Then $\phi(z) = z$.*

Proof. Assume first that $\{e, f\} \cap \{r, g\} \neq \emptyset$. Then $d_{\mathcal{H}_{ef}}(z) \leq 1$. Combined with Observation 2.3.4(a), this implies that $d_{\mathcal{H}}(\phi(z)) \leq 1 + |\{e, f\}| = 3$. Since z is the only vertex of degree at most 3 in $\mathcal{H}$, it follows that $\phi(z) = z$.

Assume then that $\{e, f\} \cap \{r, g\} = \emptyset$. Since ϕ is a monomorphism, there exists a vertex $v \in V(\mathcal{H}_{ef})$ such that $\phi(v) = z$. Suppose for a contradiction that $v \neq z$. By Observation 2.3.4(a), we have $d_{\mathcal{H}_{ef}}(v) \leq 2$ and thus $d_{\mathcal{H}}(v) \leq 4$. Since z is the only vertex of degree less than 4 in $\mathcal{H}$, it follows that $d_{\mathcal{H}}(v) = 4$ and that both e and f contain v. Let $r' = \phi^{-1}(r)$ and $g' = \phi^{-1}(g)$ be the other two edges of $\mathcal{H}$ that contain v. By Observation 2.3.4(d), we have $|r' \cap g'| = |r \cap g| = 1$. Looking at Tables 2.1 and 2.2, we see that the only choice of r', g' and v such that $d_{\mathcal{H}}(v) = 4$ and $r' \cap g' = \{v\}$ is $v = v_9$ and $\{r', g'\} = \{b, e_5\}$. Since both e and f contain v as well, this implies that $\{e, f\} = \{g, a\}$, contrary to our assumption that $\{e, f\} \cap \{r, g\} = \emptyset$. □

Lemma 2.3.7. *Let $\phi : V(\mathcal{H}_{ef}) \to V(\mathcal{H})$ be a monomorphism. If $r, g \in \mathcal{H}_{ef}$, $\phi(r) = r$ and $\phi(g) = g$, then ϕ is the identity.*

Proof. Since ϕ is injective, $\phi(r) = r$, and $\phi(g) = g$, it follows by Observation 2.3.3 (1) that $\phi(v_6) = v_6$.

By Observation 2.3.4(a), we have that $d_{\mathcal{H}}(\phi(v_4)) \geq d_{\mathcal{H}_{ef}}(v_4) \geq 5$ which in turn implies that $\phi(v_4) \in \{v_3, v_4, v_5, v_6\}$. Since, moreover, $\phi(v_4) \in \phi(g) = g = \{z, v_2, v_4, v_7, v_9\}$, it follows that $\phi(v_4) = v_4$. Since e_5 is the unique edge in $\mathcal{H}$ containing v_6 but not v_4, we have that if $e_5 \in \mathcal{H}_{ef}$, then $\phi(e_5) = e_5$.

Since $\phi(r) = r$ and $\phi(g) = g$, it follows by Observation 2.3.4(d) and by Observation 2.3.3(2), that if $e_1 \in \mathcal{H}_{ef}$, then $\phi(e_1) = e_1$. Similarly, using Observation 2.3.3(3), it follows that if $b \in \mathcal{H}_{ef}$, then $\phi(b) = b$. We distinguish between the following three cases.

Case 1: $b, e_3 \in \mathcal{H}_{ef}$. As noted above $\phi(b) = b$. Since $|e_3 \cap b| = 2$, Observation 2.3.4(d) and Table 2.2 imply that $\phi(e_3) \in \{e_3, r\}$. Since, moreover, $\phi(r) = r$ by assumption, we conclude that $\phi(e_3) = e_3$. Observation 2.3.4(d) then implies that $(|\phi(x) \cap b|, |\phi(x) \cap e_3|) = (|x \cap b|, |x \cap e_3|)$ for every edge $x \in \mathcal{H}_{ef}$. Looking at the rows corresponding to b and e_3 in Table 2.2, we see that the pair $(|x \cap b|, |x \cap e_3|)$ is distinct for every $x \in \mathcal{H} \setminus \{r, g\}$. It follows that $\phi(x) = x$ for every $x \in \mathcal{H}_{ef}$. Hence, ϕ is the identity by Lemma 2.3.5.

Case 2: $e_2, e_5 \in \mathcal{H}_{ef}$. As noted above $\phi(e_5) = e_5$. Since $|e_2 \cap e_5| = 2$, Observation 2.3.4(d) and Table 2.2 imply that $\phi(e_2) \in \{e_2, r, g\}$. Since, moreover, $\phi(r) = r$ and $\phi(g) = g$ by assumption, we conclude that $\phi(e_2) = e_2$. Observation 2.3.4(d) then implies that $(|\phi(x) \cap e_5|, |\phi(x) \cap e_2|) = (|x \cap e_5|, |x \cap e_2|)$ for every edge $x \in \mathcal{H}_{ef}$. Looking at the rows corresponding to e_5 and e_2 in Table 2.2, we see that the pair $(|x \cap e_5|, |x \cap e_2|)$ is distinct for every $x \in \mathcal{H} \setminus \{r, g\}$. It follows that $\phi(x) = x$ for every $x \in \mathcal{H}_{ef}$. Hence, ϕ is the identity by Lemma 2.3.5.

Case 3: $\{e, f\} \in \{b, e_3\} \times \{e_2, e_5\}$. Observe that $e_1 \in \mathcal{H}_{ef}$ and thus, as noted above, $\phi(e_1) = e_1$. Looking at the row corresponding to e_1 in Table 2.2 and using Observation 2.3.4(d), we infer that $\phi(e_3) = e_3$, $\phi(e_5) = e_5$, $\{\phi(b), \phi(e_2)\} = \{b, e_2\}$, and $\{\phi(a), \phi(e_4)\} = \{a, e_4\}$. Since $\phi(v_6) = v_6$, it then follows that $\phi(e_2) = e_2$ and thus $\phi(b) = b$. Let x denote the unique edge of $\{e_2, e_5\} \cap \mathcal{H}_{ef}$. Looking at the row corresponding to x in Table 2.2, we see that $|x \cap a| \neq |x \cap e_4|$. Using Observation 2.3.4(d), we conclude that $\phi(a) = a$ and $\phi(e_4) = e_4$. Hence, ϕ is the identity by Lemma 2.3.5.

Since, clearly, at least one of the above three cases must occur, this concludes the proof of the lemma. □

Lemma 2.3.8. *Let $\phi : V(\mathcal{H}_{ef}) \to V(\mathcal{H})$ be a monomorphism. If $b, e_5 \in \mathcal{H}_{ef}$, then $\phi(v_9) = v_9$.*

Proof. Suppose for a contradiction that $\phi(v_9) \neq v_9$. By Lemma 2.3.6 we have $\phi(z) = z$ which implies that $\phi(v_9) \neq z$. By Observation 2.3.4(a) we have $d_{\mathcal{H}}(\phi(v_9)) \leq d_{\mathcal{H}_{ef}}(v_9) + 2 \leq 6$ which implies that $\phi(v_9) \neq v_4$.

Since ϕ is a monomorphism, we have $\{\phi(v_9)\} = \phi(b \cap e_5) = \phi(b) \cap \phi(e_5)$. Since $\{\phi(v_9)\}$ is the intersection of two edges, we must have $\phi(v_9) \in \{v_4, v_5, v_9, z\}$. Combining this with the previous paragraph, we infer that $\phi(v_9) = v_5$.

Note that $6 = d_{\mathcal{H}}(v_5) = d_{\mathcal{H}}(\phi(v_9)) \leq d_{\mathcal{H}_{ef}}(v_9) + 2$. Hence, $d_{\mathcal{H}_{ef}}(v_9) = 4$ which implies that $\{g, a\} \cap \{e, f\} = \emptyset$. Since $\phi(z) = z$ and $\phi(v_9) = v_5$, we must have $\phi(g) = r$.

Since e_1, e_5 is the unique pair of edges satisfying $e_1 \cap e_5 = \{v_5\}$, it follows that $\{\phi(b), \phi(e_5)\} = \{e_1, e_5\}$. Suppose for a contradiction that $\phi(b) = e_5$. Then, by Observation 2.3.4(d) we have $3 = |b \cap g| = |\phi(b) \cap \phi(g)| = |e_5 \cap r| = 2$. We conclude that $\phi(b) = e_1$ and $\phi(e_5) = e_5$. We can now determine the missing edges in $\mathcal{H}_{ef}$ and in $\phi(\mathcal{H}_{ef})$.

Claim 2.3.9. *$e_2, e_4 \notin \mathcal{H}_{ef}$ and $e_2, e_4 \notin \phi(\mathcal{H}_{ef})$.*

Proof. Suppose for a contradiction that $e_2 \in \mathcal{H}_{ef}$. Since $|e_2 \cap b| = 3$ and $v_9 \notin e_2$, it follows by Observation 2.3.4(d) that $|\phi(e_2) \cap e_1| = |\phi(e_2) \cap \phi(b)| = 3$ and $v_5 \notin \phi(e_2)$. This is a contradiction since there is no edge $x \in \mathcal{H}$ such that $|x \cap e_1| = 3$ and $v_5 \notin x$.

Suppose for a contradiction that $e_4 \in \mathcal{H}_{ef}$. It follows by Observation 2.3.4(d) that $4 = |e_4 \cap e_5| = |\phi(e_4) \cap \phi(e_5)| = |\phi(e_4) \cap e_5|$ and thus $\phi(e_4) = e_4$. Since, moreover, $v_9 \notin e_4$ and $\phi(v_9) = v_5$, it follows that $v_5 \notin e_4$, contrary to the definition of e_4.

Suppose for a contradiction that $e_2 \in \phi(\mathcal{H}_{ef})$. Let $x \in \mathcal{H}_{ef}$ be such that $\phi(x) = e_2$. Since $\phi(b) = e_1$, it follows by Observation 2.3.4(d) that $4 = |e_1 \cap e_2| = |b \cap x|$. Looking at the row corresponding to b in Table 2.2, we infer that $x = e_1$. However, since $v_9 \notin e_1$, we then deduce that $v_5 = \phi(v_9) \notin e_2$ which is clearly a contradiction.

Suppose for a contradiction that $e_4 \in \phi(\mathcal{H}_{ef})$. Let $x \in \mathcal{H}_{ef}$ be such that $\phi(x) = e_4$. Since $\phi(e_5) = e_5$, it follows by Observation 2.3.4(d) that $4 = |e_4 \cap e_5| = |x \cap e_5|$. Looking at the row corresponding to e_5 in Table 2.2, we infer that $x = e_4$. However, we already saw before that assuming $e_4 \in \mathcal{H}_{ef}$ results in a contradiction. $\square$

We are now in a position to complete the proof of Lemma 2.3.8. Let $\mathcal{F} = \mathcal{H} \setminus \{e_2, e_4\}$. It follows from Claim 2.3.9 that $\mathcal{H}_{ef} = \phi(\mathcal{H}_{ef}) = \mathcal{F}$ and that ϕ is an automorphism of $\mathcal{F}$. Hence, in particular, $d_{\mathcal{F}}(\phi(v_4)) = d_{\mathcal{F}}(v_4) = 5$. On the other hand, since $\phi(g) = r$, it follows that $\phi(v_4) \in \{v_1, v_3, v_5, v_8\}$. Therefore $d_{\mathcal{F}}(\phi(v_4)) \leq 4$ which is clearly a contradiction. $\square$

Lemma 2.3.10. *Let $\phi : V(\mathcal{H}_{ef}) \to V(\mathcal{H})$ be a monomorphism. Suppose that $\mathcal{H}_{ef}$ contains a tight path of length 5. Then ϕ is either the identity or one of $(v_9v_1v_2v_3v_4v_5v_6v_7v_8)(z)$, $(v_1v_9v_8v_7v_6v_5v_4v_3v_2)(z)$, $(v_9v_8)(v_1v_7)(v_2v_6)(v_3v_5)(v_4)(z)$, $(v_1v_8)(v_2v_7)(v_3v_6)(v_4v_5)(v_9)(z)$, and $(v_1v_9)(v_2v_8)(v_3v_7)(v_4v_6)(v_5)(z)$.*

Proof. By Lemma 2.3.6 we know that $\phi(z) = z$. Moreover, by Observation 2.3.3(4), we know that $\mathcal{H}_{ef}$ contains $TP1$ or $TP2$. Moreover, by Observation 2.3.4(b), if $TP1 \in \mathcal{H}_{ef}$, then $\phi(TP1) \in \{TP1, TP2\}$ and if $TP2 \in \mathcal{H}_{ef}$, then $\phi(TP2) \in \{TP1, TP2\}$. Accordingly, we distinguish between the following four cases.

Case 1: $TP1 \in \mathcal{H}_{ef}$ **and** $\phi(TP1) = TP1$. It follows by Observation 2.3.4(c) that either ϕ is the identity or $\phi = (v_1v_9)(v_2v_8)(v_3v_7)(v_4v_6)(v_5)(z)$.

Case 2: $TP1 \in \mathcal{H}_{ef}$ **and** $\phi(TP1) = TP2$. It follows by Observation 2.3.4(c) that either $\phi = (v_1v_9v_8v_7v_6v_5v_4v_3v_2)(z)$ or $\phi = (v_1v_8)(v_2v_7)(v_3v_6)(v_4v_5)(v_9)(z)$.

Case 3: $TP2 \in \mathcal{H}_{ef}$ **and** $\phi(TP2) = TP1$. It follows by Observation 2.3.4(c) that either $\phi = (v_9v_1v_2v_3v_4v_5v_6v_7v_8)(z)$ or $\phi = (v_1v_8)(v_2v_7)(v_3v_6)(v_4v_5)(v_9)(z)$.

Case 4: $TP2 \in \mathcal{H}_{ef}$ **and** $\phi(TP2) = TP2$. It follows by Observation 2.3.4(c) that either ϕ is the identity or $\phi = (v_8v_9)(v_1v_7)(v_2v_6)(v_3v_5)(v_4)(z)$. □

Lemma 2.3.11. *Let $\phi : V(\mathcal{H}_{ef}) \to V(\mathcal{H})$ be a monomorphism. If $b, e_5 \in \mathcal{H}_{ef}$, then ϕ is the identity.*

Proof. Suppose for a contradiction that ϕ is not the identity. By Lemma 2.3.6 we know that $\phi(z) = z$ and by Lemma 2.3.8 we know that $\phi(v_9) = v_9$. Assume first that $\phi(b) = b$. Since $\phi(v_9) = v_9$, $\phi(b) = b$, and e_5 is the unique edge whose intersection with b is $\{v_9\}$, we infer that $\phi(e_5) = e_5$. Since g is the unique edge containing both v_9 and z, we infer that, if $g \in \mathcal{H}_{ef}$, then $\phi(g) = g$. Since e_2 is the unique edge satisfying $|e_2 \cap b| = 3$, $|e_2 \cap e_5| = 2$, and $|e_2 \cap b \cap e_5| = 0$, it follows by Observation 2.3.4(d) that, if $e_2 \in \mathcal{H}_{ef}$, then $\phi(e_2) = e_2$. Looking at the rows corresponding to e_5 and b in Table 2.2, we see that $(|x \cap e_5|, |x \cap b|)$ is distinct for every $x \in \mathcal{H} \setminus \{g, e_2\}$. This implies that $\phi(x) = x$ for every $x \in \mathcal{H}_{ef}$ and thus ϕ is the identity by Lemma 2.3.5 contrary to our assumption. Therefore, from now on we will assume that $\phi(b) \neq b$. Since $\phi(v_9) = v_9$, it follows that $\phi(b) = e_5$ and $\phi(e_5) = b$. We distinguish between the following three cases.

Case 1: $\{e, f\} \subseteq \{r, g, a\}$. Observe that $\mathcal{H}_{ef}$ contains $TP1$. Since, moreover, $\phi(v_9) = v_9$ and ϕ is not the identity by assumption, it follows from Lemma 2.3.10 that

$$\phi = (v_1v_8)(v_2v_7)(v_3v_6)(v_4v_5)(v_9)(z).$$

Let $x \in \{r, g, a\} \setminus \{e, f\}$. Then $\phi(x)$ is not an edge of $\mathcal{H}$ contrary to ϕ being a monomorphism.

Case 2: $g \in \mathcal{H}_{ef}$. As noted above, $\phi(g) = g$. Since b is the unique edge intersecting g in 3 vertices, we have $\phi(b) = b$, contrary to our assumption that $\phi(b) \neq b$.

Case 3: $g \notin \mathcal{H}_{ef}$ **and** $r, a \in \mathcal{H}_{ef}$. Since r is the unique edge such that $z \in r$ and $v_9 \notin r$, it follows that $\phi(r) = r$. Similarly, since $z \notin a$, $v_9 \in a$, $\phi(b) = e_5$, and $\phi(e_5) = b$, it follows that $\phi(a) = a$. Then

$$\begin{aligned}
\{\phi(v_1)\} &= \phi(b) \cap \phi(r) \cap \phi(a) = e_5 \cap r \cap a = \{v_8\}, \\
\{\phi(v_2)\} &= \phi(b) \setminus (\phi(r) \cup \phi(a)) = e_5 \setminus (r \cup a) = \{v_7\}, \\
\{\phi(v_3)\} &= \phi(b) \cap \phi(r) \setminus \phi(a) = e_5 \cap r \setminus a = \{v_5\}.
\end{aligned}$$

Since, moreover, $\phi(b) = e_5$, it follows that $\phi(v_4) = v_6$. Now, using $\phi(r) = r$ and $\phi(a) = a$, it is easy to see that $\phi(v_8) = v_1$ and thus $\phi(v_6) = v_4$, $\phi(v_5) = v_3$ and $\phi(v_7) = v_2$. However, then neither $\phi(e_1)$ nor $\phi(e_4)$ is an edge of $\mathcal{H}$. Since $\{e_1, e_4\} \setminus \{e, f\} \neq \emptyset$, this contradicts ϕ being a monomorphism. □

Lemma 2.3.12. *Let $\phi : V(\mathcal{H}_{ef}) \to V(\mathcal{H})$ be a monomorphism. If $\{e, f\} \in \{b, e_5\} \times \{r, g, a\}$, then ϕ is the identity.*

Proof. Since $|\{e, f\} \cap \{b, e_5\}| = 1$ by assumption, $\mathcal{H}_{ef}$ contains either $TP1$ or $TP2$. Hence, ϕ must be one of the six permutations listed in Lemma 2.3.10.

Assume first that $r, g \in \mathcal{H}_{ef}$. By Lemma 2.3.6 we know that $\phi(z) = z$ and thus $\{\phi(r), \phi(g)\} \subseteq \{r, g\}$. Therefore $\phi(v_6) = v_6$ holds by Observation 2.3.3(1). This implies that ϕ is the identity since this is the only permutation listed in Lemma 2.3.10 which maps v_6 to itself.

Assume then that $a \in \mathcal{H}_{ef}$. This implies that ϕ is the identity since this is the only permutation listed in Lemma 2.3.10 which maps a to an edge of $\mathcal{H}$. □

Lemma 2.3.13. *Let $\phi : V(\mathcal{H}_{ef}) \to V(\mathcal{H})$ be a monomorphism. Then ϕ is the identity.*

Proof. Let e' and f' denote the two edges of $\mathcal{H} \setminus \phi(\mathcal{H}_{ef})$. Suppose for a contradiction that ϕ is not the identity. Observe that this implies that ϕ^{-1} is a monomorphism from $\phi(\mathcal{H}_{ef})$ to $\mathcal{H}$ which is not the identity.

Since ϕ is not the identity, it follows from Lemma 2.3.11 that $\{b, e_5\} \cap \{e, f\} \neq \emptyset$. By Lemma 2.3.12 we then infer that $\{r, g, a\} \cap \{e, f\} = \emptyset$. Similarly, since ϕ^{-1} is a monomorphism which is not the identity, it follows from Lemma 2.3.11 that $\{b, e_5\} \cap \{e', f'\} \neq \emptyset$ and from Lemma 2.3.12 that $\{r, g, a\} \cap \{e', f'\} = \emptyset$.

By Lemma 2.3.6 we know that $\phi(z) = z$ and thus $\{\phi(r), \phi(g)\} \subseteq \{r, g\}$. Therefore $\phi(v_6) = v_6$ holds by Observation 2.3.3(1). By Lemma 2.3.7 we know that $\phi(r) = g$ and $\phi(g) = r$, which implies that $\phi(x) \neq x$ for every $x \in V(\mathcal{H}) \setminus \{z, v_6\}$.

Since g, a and b are the only edges which do not contain v_5, and $\{e, f\} \setminus \{g, a, b\} \neq \emptyset$, it follows that $d_{\mathcal{H}_{ef}}(v_5) \leq 5$. Since $\{e', f'\} \setminus \{g, a, b\} \neq \emptyset$, an analogous argument shows that $d_{\phi(\mathcal{H}_{ef})}(v_5) \leq 5$.

Suppose for a contradiction that $e_5 \in \{e, f\}$. Then $d_{\mathcal{H}_{ef}}(v_4) \geq 6$ and thus $d_{\phi(\mathcal{H}_{ef})}(\phi(v_4)) \geq 6$ as well. Since, as noted above, $d_{\phi(\mathcal{H}_{ef})}(v_5) \leq 5$, it follows from Table 2.1 that $\phi(v_4) = v_4$. However, this contradicts the fact that ϕ does not fix any vertex of $V(\mathcal{H}) \setminus \{z, v_6\}$. It follows that $b \in \{e, f\}$. An analogous argument shows that $b \in \{e', f'\}$ as well.

Suppose for a contradiction that $e_1 \notin \{e, f\}$. Then $|\phi(e_1) \cap g| = |\phi(e_1) \cap \phi(r)| = |e_1 \cap r| = 3$ holds by Observation 2.3.4(d). Since b is the only edge of $\mathcal{H}$ which intersects g in 3 vertices, it then follows that $\phi(e_1) = b$. However, this contradicts the fact that $b \in \{e', f'\}$. An analogous argument shows that $e_1 \in \{e', f'\}$ as well.

We have thus shown that $\{e, f\} = \{e', f'\} = \{e_1, b\}$. Hence, $P = (e_2, e_3, e_4, e_5)$ is the unique tight path of length 4 in $\mathcal{H}_{ef}$ and in $\phi(\mathcal{H}_{ef})$. Since $\phi(z) = z$ and since $\phi(P) = P$ holds by Observation 2.3.4(b) it follows that $\phi(v_1) = v_1$ contrary to ϕ not fixing any vertex of $V(\mathcal{H}) \setminus \{z, v_6\}$. □

This ends the case analysis and the proof of Theorem 1.1.1.

2.4 Concluding remarks and open problems

As noted in the introduction, the work in this chapter originated from Beck's open problem of deciding whether $\mathcal{R}(K_q, \aleph_0)$ is a draw or FP's win. While it would be very interesting to solve this challenging problem, there are several natural intermediate steps one could make in order to improve one's understanding of the problem. In this chapter we constructed a 5-uniform hypergraph $\mathcal{H}_5$ such that $\mathcal{R}^{(5)}(\mathcal{H}_5, \aleph_0)$ is a draw, thus refuting the intuition that, due to strategy stealing and Ramsey-type arguments, $\mathcal{R}^{(k)}(\mathcal{H}, \aleph_0)$ is FP's win for every k and every k-graph $\mathcal{H}$. It would be interesting to replace $\mathcal{H}_5$ with a graph.

Question 2.4.1. *Is there a graph G such that $\mathcal{R}^{(2)}(G, \aleph_0)$ is a draw?*

Our proof that $\mathcal{R}^{(5)}(\mathcal{H}_5, \aleph_0)$ is a draw, relies heavily on the fact that $\mathcal{H}_5$ has a vertex of degree 2. Since this is clearly not the case with K_q, for $q \geq 5$, it would be interesting to determine whether this condition is necessary.

Question 2.4.2. *Given an integer $d \geq 3$, is there a k-graph $\mathcal{H}$ such that $\delta(\mathcal{H}) \geq d$ and $\mathcal{R}^{(k)}(\mathcal{H}, \aleph_0)$ is a draw?*

At this point it is worth emphasising that strong games are not monotone and so one cannot simply take a 5-graph $\mathcal{G}$ containing the one constructed in Section 2.3 to boost the minimum degree and argue that FP cannot build $\mathcal{H}_5$ and hence cannot build $\mathcal{G}$ either. For example, a key point in SP strategy was to threaten to build a copy of $\mathcal{H}$, which FP had to block earlier. When the aim is to build $\mathcal{G}$, then threatening to build $\mathcal{H}$ is of course meaningless since FP does not care whether SP builds $\mathcal{H}$.
Another important ingredient in our proof that $\mathcal{R}^{(5)}(\mathcal{H}_5, \aleph_0)$ is a draw, is the fact that SP can build $\mathcal{H}_5 \setminus \{z\}$ very quickly. A similar idea was used in [30] and in [29] to devise explicit winning strategies for FP in various natural strong games. On the other hand, it was proved by Beck in [9] that building a copy of K_q takes time which is at least exponential in q. Intuitively, not being able to build a winning set quickly, should not be beneficial to FP. This leads us to raise the following question.

Question 2.4.3. *Is there a k-graph $\mathcal{H}$ with minimum degree at least 3 such that $\mathcal{R}^{(k)}(\mathcal{H}, \aleph_0)$ is a draw and, for every positive integer n, FP cannot win $\mathcal{R}^{(k)}(\mathcal{H}, n)$ in less than, say, $1000|V(\mathcal{H})|$ moves?*

Chapter 3

General Winning Criteria for Maker-Breaker games

This Chapter is organised as follows. First, as a warm up, we will consider a special case of Theorem 3.5.5, namely the k-AP game. We then continue and provide a proof of Theorem 1.2.8, the general criterion for Maker, based on a probabilistic result of Janson, Łuczak and Ruciński in Section 3.2. This will be followed in Section 3.3 by a proof of Theorem 1.2.9, the general criterion for Breaker, through a combination of several smaller strategies aimed at avoiding clustering of solutions. Subsequently we will consider the Maker-Breaker $\mathcal{G}$ games and prove Theorem 1.2.2. We then define our generalisation of Beck's van der Waerden games and prove Theorem 3.5.5. These proofs are done mostly through applications of the two general winning criteria for Maker and Breaker. Section 3.6 will include several common minor results regarding the counting of solutions to a linear system of inhomogeneous equations as well as the notions of induced subsystems. Lastly, Section 3.7 will contain some open questions as well as a reflection on the strong connection between these results and some well-known recent sparse statements in extremal combinatorics.

3.1 The threshold bias of the k-AP game

The reason for this section is twofold. On the one hand it introduces some of the main ideas of the proofs of both Maker's and Breaker's winning criterion, and thus serves as a leisurely paced introduction to the next two sections. On the other hand, in this special case we actually have a deterministic strategy for Maker which is in general not the case. Since the results proven in this section are a special case of Theorem 3.5.5, this chapter may be skipped. In order to enable the reader to do so, all necessary definitions and theorems will be stated (and proved) both here and in Section 3.3.

We will now present two strategies for Maker and one for Breaker which together imply that the threshold bias of the k-AP game $\mathbf{G}(\mathcal{H}_{k\text{-AP}}(n); q)$ satisfies $q(\mathcal{H}_{k\text{-AP}}(n)) = \Theta\left(n^{1/(k-1)}\right)$, proving Theorem 1.2.3.

3.1.1 Maker's strategy - a deterministic approach As mentioned at the beginning of this section, a particularly nice thing about the k-AP game is that we actually have a deterministic strategy for Maker. Indeed, Maker can simply play according to Beck's Maker criterion from Theorem 1.2.7.
Note that in this case we have $p = 1$ and recall that $v(\mathcal{H}_{k\text{-AP}}) = n$ and $\Delta_2(\mathcal{H}_{k\text{-AP}}) \leq \binom{k}{2}$. In order to apply Theorem 1.2.7, we need a lower bound on $e(\mathcal{H}_{k\text{-AP}})$, i.e. on the number of k-AP's in $[n]$. We claim that $e(\mathcal{H}_{k\text{-AP}}) \geq n^2/(4(k-1))$ is such a lower bound: Indeed, first choose the starting point of the k-AP in $[n/2]$ and consider the largest possible common difference that guarantees that the k-AP fits into $[n]$ which is obviously $n/(2(k-1))$. Now we just plug these values into Beck's Maker criterion to get a constant $c_M = c_M(k)$ such that Maker has a winning strategy provided the bias of Breaker satisfies $q < c_M n^{\frac{1}{k-1}}$. □

Unfortunately, this approach does not carry over to the general framework. The main reason for this is that the maximum 2-degree is in general non-constant.

3.1.2 Maker's strategy - a probabilistic approach The following notion plays a crucial role in the probabilistic approach.

Definition 3.1.1 ((δ, k)-Szemerédi)**.** *Let $0 < \delta < 1$ be given. We say that a set $T \subseteq \mathbb{N}$ is* (δ, k)-Szemerédi *if every subset $S \subseteq T$ of size $|S| \geq \delta|T|$ contains an arithmetic progression of length k.*

Schacht [70] and independently Conlon and Gowers [22] proved a sparse random analogue of Szemerédi's theorem. The result was formulated for the *binomial random subset* model $[n]_p$.

Given some finite set A of size $|A| = n$ and $0 < p < 1$ we will use the notation A_p to refer to the *binomial random set* that is obtained by picking each element of A independently with probability p. For any given subset $T \subset A$ we therefore have $\mathbb{P}(A_p = T) = p^{|T|}(1-p)^{|A|-|T|}$. On the other hand, given $0 \leq M \leq n$ the *uniform random set* A_M is obtained by assigning each subset $T \subset A$ of size $|T| = M$ the same probability $\mathbb{P}(A_M = T) = 1/\binom{n}{M}$.

Theorem 3.1.2 (Schacht [70], Conlon-Gowers [22])**.** *For every $0 < \delta < 1$ and every $k \in \mathbb{N}$ there exists a constant $C = C(\delta, k)$ such that*

$$\mathbb{P}\left([n]_M \text{ is } (\delta, k)\text{-Szemerédi }\right) \to 1$$

for every sequence $M = M(n)$ satisfying $M(n) \geq Cn^{1-1/(k-1)}$.

We can now prove Maker's part of Theorem 1.2.3.
Fix an arbitrary strategy SB for Breaker. Maker will play according to the following random strategy: in each round he *picks* an element uniformly at random from among all elements of $[n]$ that he has not previously picked. If this element was not already occupied by Breaker, then Maker occupies it. Otherwise he occupies an arbitrary free vertex and forgets about it for the rest of the game (and so he might pick it in a later round). Note the subtle difference between picking and occupying an element. We label an element picked by Maker as a *failure*, if that element was already occupied by Breaker. We will show that this random strategy succeeds with positive probability against S_B, so that S_B is not a winning strategy. Since S_B was arbitrary, this shows that Maker must have a winning strategy.
Pick any $0 < \delta < 1$ and let $C = C(\delta, k)$ be the corresponding constant from Theorem 3.1.2. Fix a positive constant $c_M = c_M(C, \delta) < (1-\delta)/(2C)$ and consider for $q \leq c_M n^{1/(k-1)} - 1$ the first

$$M = C\lfloor n^{1-1/(k-1)} \rfloor \leq (1-\delta)/2 \cdot \frac{n}{q+1} \tag{3.1}$$

rounds of the game. We may consider the set of elements that Maker picked in these M rounds as the uniform random set $[n]_M$. Note that some of his elements may be failures. We will now upper bound the probability that Maker's i-th move, which we refer to as m_i, was a failure. Clearly this probability is upper bounded by the probability that his M-th move is a failure since in every round the number of 'potential failures' does not decrease and the number of vertices Maker picks from strictly decreases. Note that in the first $M-1$ rounds, Maker picked exactly $M-1$ vertices. So, in round M, there are $n-M+1$ available vertices to pick from. The potential failures are among the vertices occupied by Breaker and hence their number is at most $q\,(M-1)$. Using Equation 3.1, it follows for every $i \in [M]$ that

$$\mathbb{P}\,(m_i \text{ a failure}) \leq \mathbb{P}\,(m_M \text{ a failure}) \leq \frac{q\,(M-1)}{n-(M-1)} \leq \frac{q\,M}{n-M} \leq \frac{1-\delta}{2}.$$

The probability that Maker has more than $(1-\delta)\,M$ failures is now at most the probability that among M independent Bernoulli trials with failure probability $(1-\delta)/2$ there exist more than $(1-\delta)\,M$ failures, which is less than $1/2$ by Markov's inequality. In other words, with probability at least $1/2$, at least $\delta\,M$ elements picked by Maker are not failures.

By Theorem 3.1.2 we have $\mathbb{P}\,([n]_M \text{ is not}(\delta, k)\text{-Szemerédi}) \leq 1/4$ for n sufficiently large. Consequently, with probability at least $1/4$, the at least δM elements occupied by Maker contain a k-AP. □

Note that the proof in fact shows that for q as above, Maker's random strategy succeeds with probability tending to 1 as n tends to infinity.
In contrast to the deterministic approach, this approach does generalise to general systems. For this the main tasks are to generalise the notion of (δ, k)-Szemerédi sets (see Defini-

tion 3.2.1) and to prove an appropriate version of Theorem 3.1.2 in this general context (see Theorem 3.2.3). Here, a small change compared to Theorem 3.1.2 is that in the general context the statement does not hold for every $\delta \in (0, 1)$ as above, but only for a specific value. However, this clearly will be enough for our purposes.

3.1.3 Breaker's strategy The main ingredient of Breaker's strategy is the following consequence of Beck's biased Erdős-Selfridge criterion, see Theorem 3.3.2. It says that

for every hypergraph $\mathcal{H}$ and integer $q \geq 1$ the following holds. If Breaker plays as the second player, he can keep Maker from covering more than

$$(q+1) \sum_{H \in \mathcal{H}} \left(\frac{1}{q+1} \right)^{|H|} \tag{3.2}$$

winning sets in $\mathbf{G}(\mathcal{H}; q)$.

We also need the following definitions.

Definition 3.1.3 (Set-Theoretic Definitions)**.** *Given some hypergraph $\mathcal{H}$, we define the following:*

- *an* almost complete solution (H°, h) *is a tuple consisting of a set $H^\circ \subseteq V(\mathcal{H})$ as well as an element $h \notin H^\circ$ so that $H = H^\circ \cup \{h\}$ is a edge in $\mathcal{H}$,*
- *a t-*fan *is a family of distinct almost complete solutions $\{(H_1^\circ, h_1), \ldots, (H_t^\circ, h_t)\}$ in $\mathcal{H}$ satisfying $|\bigcap_{i=1}^t H_i^\circ| \geq 1$ and it is called* simple *if $|H_i^\circ \cap H_j^\circ| = 1$ for all $1 \leq i < j \leq t$.*

Given a t-fan in $\mathcal{H}$ we call the h_i the open elements*, the H_i° the* major parts *and the elements of the intersection $\bigcap_{i=1}^t H_i^\circ$ the* common elements*.*

Definition 3.1.4 (Game-Theoretic Definitions)**.** *At any given point in a positional game on a given hypergraph $\mathcal{H}$, we call an almost complete solution (H°, h)* dangerous *if all elements of H° have been picked by Maker and h has not yet been picked by either player. A fan is dangerous if all of its almost complete solutions are.*

Observe that for a dangerous fan we must have $h_i \notin H_j^\circ$ for all $1 \leq i, j \leq t$. In the following we will always assume that Breaker plays as second player. We say that a player *occupied* a given t-fan $(H_1^\circ, h_1), \ldots, (H_t^\circ, h_t)$ if his selection of vertices contains $\bigcup_{i=1}^t H_i^\circ$.

Already at this point, we would like to stress a key difference in Breaker's strategy for k-AP's compared to the general situation. As already seen in (1.3), the maximum 2 degree of the hypergraph $\mathcal{H}_{k\text{-AP}}$ encoding non-trivial k-APs is constant. So in this case we do not need to worry about many almost complete solutions having common intersection larger than 1.

In general, however, we do have to deal with such structures and it is because of this that the analysis of Breaker's strategy will be much more complicated in general.

In order to prove Breaker's part of Theorem 1.2.3, note that if Maker succeeds in occupying a k-AP, then in some previous round he must have built a dangerous t-fan for some integer $t \geq 1$. The key point is to show that t cannot be too large.
Recall that $\mathcal{H}_{k\text{-AP}}$ is the hypergraph encoding non-trivial k-AP's and that $\Delta_2(\mathcal{H}_{k\text{-AP}}) \leq \binom{k}{2}$. Firstly, we will show, using (3.2), that Breaker has a strategy that prevents Maker from building a large dangerous fan.

Note that a direct application of (3.2) is difficult, due to overlaps of the not necessarily simple fans. To resolve this issue, we will consider a dangerous and not necessarily simple t-fan and extract from it a large number of 'large' dangerous simple fans.
First, recall that any two elements in $[n]$ are contained in at most $\binom{k}{2}$ k-AP's. Therefore, any t-fan must contain a simple s-fan, where $s = s(t) \geq \frac{t}{(k-2)\binom{k}{2}}$. To artificially increase the number of simple fans, we will consider this simple s-fan as a collection of $\binom{s}{w}$ simple w-fans, where $w = w(s) = s^{\alpha}$, for some arbitrary but fixed $0 < \alpha < 1$. In other words, if there is a 'large' t-fan, then there is a 'large' collection of simple w-fan.

Lemma 3.1.5. *For every integer $k \geq 3$ the following holds. There exists a constant $C'_B = C'_B(k) > 0$ such that Breaker with a bias of $q \geq C'_B n^{\frac{1}{k-1}}$ has a strategy that prevents Maker from occupying a dangerous t-fan with $t = t(q) \geq q/2$.*

Proof. Let w and s be as defined above and let

$$\mathcal{F} = \Big\{ \bigcup_{i=1}^{w} H_i^\circ \mid \{(H_1^\circ, h_1), \ldots, (H_w^\circ, h_t)\} \text{ simple } w\text{-fan in } \mathcal{H}_{k\text{-AP}} \Big\}$$

to be the hypergraph of all simple w-fans in $\mathcal{H}_{k\text{-AP}}$.
Now assume that Maker succeeds in building a dangerous t-fan, where $t = t(q) \geq q/2$. From the above discussion, we know that this t-fan contains a collection of $\binom{s}{w}$ simple w-fans. However, we will show that Breaker can in fact keep Maker from occupying $\binom{s}{w}$ simple w-fans and hence showing that the dangerous t-fan Maker built satisfies $t = t(q) < q/2$.
Set $C'_B = C'_B(k) > (k^4 \cdot e)^{1/(k-1)}$ and let $q \geq C'_B n^{1/(k-1)}$. For convenience, let $\bar{q} = q/2$.

By applying (3.2) to $\mathcal{F}$ we get that Breaker can keep Maker from building

$$(\bar{q}+1) \sum_{A \in \mathcal{F}} (\bar{q}+1)^{-|A|} \leq (\bar{q}+1) \left(n \, \frac{\left(k \, \frac{n}{k-1}\right)^w (k-1)^w}{w!} \right) (\bar{q}+1)^{-w(k-2)-1} \tag{3.3}$$

simple w-fans. This inequality holds because there are n ways to fix a common element of a simple w-fan, $(k(n/(k-1)))^w$ is an upper bound on the number of w-tuples of k-AP's containing the fixed common element, since $\Delta_1(\mathcal{H}_{k\text{-AP}}(n) \leq k \cdot n/(k-1)$, and there are $(k-1)^w$ ways of fixing the corresponding open elements. Lastly, the $w!$ factor takes care of symmetries and each simple w-fan has size $w(k-2)+1$. We therefore get, using $s \geq t/((k-2)\binom{k}{2}) \geq \bar{q}/((k-2)\binom{k}{2})$, that

$$(\bar{q}+1) \sum_{A \in \mathcal{F}} (\bar{q}+1)^{-|A|} \leq n \left(\frac{k\ n\ e}{w\ \bar{q}^{k-2}} \right)^w = n \left(\frac{k\ n\ e}{\bar{q}^{k-1}} \right)^w \left(\frac{\bar{q}}{w} \right)^w$$
$$\leq n \left(\frac{k(k-2)\binom{k}{2}\ n\ e}{C_B'^{k-1} n} \right)^w \left(\frac{s}{w} \right)^w \leq \frac{1}{2} \binom{s}{w}.$$

for $n = n(k)$ sufficiently large. Note that we have used that w grows with n and that

$$n \left(\frac{k(k-2)\binom{k}{2}\ n\ e}{C_B'^{k-1} n} \right)^w \leq n \left(\frac{k^4\ n\ e}{C_B'^{k-1} n} \right)^w \to 0,$$

as n tends to infinity since $C_B' > (k^4 \cdot e)^{1/(k-1)}$. Therefore $t = t(q) < q/2$ as claimed. □

Remark 3.1.6. *Note that Inequality 3.3 used almost no specific information about $\mathcal{H}_{k\text{-}AP}(n)$. Indeed the only time we did use such information was to bound the maximum degree of $\mathcal{H}_{k\text{-}AP}(n)$ by $k \cdot n/(k-1)$. This observation is crucial in the general context and in particular for Lemma 3.3.6.*

We are now in a position to describe the strategy of Breaker, which consists of two parts. Let $q \geq C_B n^{1/(k-1)}$, where $C_B = 2\ C_B'$ and C_B' is as defined in Lemma 3.1.5.

SB1: Using $q/2$ moves, Breaker will play according to Lemma 3.1.5, thus preventing Maker from occupying a dangerous t-fan, $t = t(q) \geq q/2$.

SB2: With the remaing $q/2$ moves, Breaker will occupy all open elements of any dangerous almost complete solution.

Note that Breaker can indeed play according to $SB1$ and $SB2$ and achieve the described goals by choice of q. We will now prove, by induction, that after each of Breaker's moves there is no dangerous almost complete solution. Clearly this implies that Breaker's strategy is indeed a winning strategy. Initially there is obviously no dangerous almost complete solution. So suppose the result is true in round $r-1$. In round r Maker claims an element b say. Then every new dangerous almost complete solution must contain b. Therefore they all belong to the same dangerous fan (with common element b). By Lemma 3.1.5 the size of this dangerous fan is not more than $q/2$ (SB1) and hence Breaker can occupy all open elements in this dangerous fan (SB2), which completes the inductive step and hence Breaker's part of Theorem 1.2.3.

3.2 Proof of Theorem 1.2.8 – Maker's Strategy

The following notion plays a crucial role in the proof of Theorem 1.2.8 and is a natural generalization of the notion that a set is (δ, k)-*Szeméredi* as defined by Conlon and Gowers [22] (cf. Definition 3.1.1).

Definition 3.2.1 (δ-stable)**.** *Let $\mathcal{F}$ be a hypergraph, $T \subseteq V(\mathcal{F})$ be a subset of its vertices and $0 < \delta < 1$. We say T is* δ-stable *if every subset of $S \subseteq T$ of size $|S| \geq \delta|T|$ contains an edge of $\mathcal{F}$.*

Remark 3.2.2. *Equivalently, T is called δ-stable, if the hypergraph $\mathcal{F}$ induced by T has independence number less than $\delta|T|$.*

Maker's strategy will consist of picking elements uniformly at random from among all elements he has not previously picked. Some of these elements could have been occupied previously by Breaker, in which case Maker occupies an arbitrary free element and forgets about it for the remainder of the game. We will prove that with positive probability Maker wins using this strategy. To do so, we will show that at least a δ-fraction of the elements Maker picked were 'legal' moves (for some $0 < \delta < 1$). Furthermore we will ensure that the set of vertices occupied by Maker is δ-stable. It then follows that Maker's set of vertices contain an edge with positive probability.

Given some finite set A and $0 < p < 1$ we will use the notation A_p to refer to the *binomial random set* that is obtained by picking each element of A independently with probability p. For any given subset $T \subset A$ we therefore have $\mathbb{P}(A_p = T) = p^{|T|}(1-p)^{|A|-|T|}$. On the other hand, given $0 \leq M \leq n$ the *uniform random set* A_M is obtained by assigning each subset $T \subset A$ of size $|T| = M$ the same probability $\mathbb{P}(A_M = T) = 1/\binom{n}{M}$.

The key ingredient to prove the existence of a winning strategy for Maker is the following statement that says that for $\mathcal{H}$ as in Theorem 1.2.8, $V(\mathcal{H})_M$ is δ-stable for suitable M and δ.

Theorem 3.2.3. *For every $k \geq 2$ and for every constant $c_1 \geq k$ there exists constants $\delta = \delta(k, c_1) \in (0,1)$ and $\tilde{c} = \tilde{c}(k, c_1) > 0$ such that the following holds. Let $\mathcal{H}$ be a k-uniform hypergraph satisfying conditions (i),(ii) and (iii) from Theorem 1.2.8. Then*

$$\mathbb{P}\left(V(\mathcal{H})_M \text{ is not } \delta\text{-stable}\right) < 3\exp\left(-M\,\frac{1}{c_1\, 2^{k+2}}\right),$$

for every $M \geq 2\left\lfloor v(\mathcal{H})/f(\mathcal{H})\right\rfloor$.

3.2.1 Proof of Theorem 1.2.8 from Theorem 3.2.3 We will now prove Theorem 1.2.8 assuming the statement of Theorem 3.2.3, whose proof will be presented in the next section.

Proof of Theorem 1.2.8. Fix an arbitrary strategy S_B for Breaker. Maker will play according to the following random strategy: in each round he *picks* an element uniformly at random from among all elements of $V(\mathcal{H})$ that he has not previously picked. If this element was not already occupied by Breaker, then Maker occupies it. Otherwise he occupies an arbitrary free vertex and forgets about it for the rest of the game. Note the subtle difference between picking and occupying a vertex. We label an element picked by Maker as a *failure*, if that element was already occupied by Breaker. We will show that this random strategy succeeds with positive probability against S_B, so that S_B is not a winning strategy. Since S_B was arbitrary, this shows that Maker must have a winning strategy.
Let $\delta = \delta(k, c_1)$ be chosen according to Theorem 3.2.3 and define $c = (1-\delta)/4 > 0$ to be a constant. Let $q \leq cf(\mathcal{H}) - 1$ and consider the first

$$M = 2 \left\lfloor \frac{v(\mathcal{H})}{f(\mathcal{H})} \right\rfloor \leq \frac{1-\delta}{2} \frac{v(\mathcal{H})}{q+1} \tag{3.4}$$

rounds of the game. We may consider the set of elements that Maker picked in these M rounds as the uniform random set $V(\mathcal{H})_M$. Note that some of his elements may be failures. We will now upper bound the probability that Maker's i-th move, which we refer to as m_i, was a failure. Clearly this probability is upper bounded by the probability that his M-th move is a failure since in every round the number of potential failures does not decrease and the number of vertices Maker picks from strictly decreases. Note that in the first $M-1$ rounds, Maker picked exactly $M-1$ vertices. So, in round M, there are $v(\mathcal{H}) - M + 1$ available vertices to pick from. The potential failures are among the vertices occupied by Breaker and hence their number is at most $q\,(M-1)$. Using Equation (3.4) it follows for every $i \in [M]$ that

$$\mathbb{P}\,(m_i \text{ a failure}) \leq \mathbb{P}\,(m_M \text{ a failure}) \leq \frac{q\,(M-1)}{v(\mathcal{H}) - (M-1)} \leq \frac{q\,M}{v(\mathcal{H}) - M} \leq \frac{1-\delta}{2}.$$

The probability that Maker has more than $(1-\delta)\,M$ failures is now at most the probability that among M independent Bernoulli trials with failure probability $(1-\delta)/2$ there exist more than $(1-\delta)\,M$ failures, which is less than $1/2$ by Markov's inequality. In other words, with probability at least $1/2$, at least $\delta\,M$ elements picked by Maker are not failures.

By Theorem 3.2.3 we have $\mathbb{P}\,(V(\mathcal{H})_M \text{ is not } \delta\text{-stable}) < 3\exp\left(-M(c_1\, 2^{k+2})^{-1}\right)$. Now setting $\tilde{c}$ to be the maximum of $c_1 2^{k+1}(\log(3) + \log(4)) + 1$ and the corresponding value of $\tilde{c}$ in Theorem 3.2.3, and using that $f(\mathcal{H}) \leq \tilde{c}^{-1} v(\mathcal{H})$ by Condition (ii), one can verify that the probability that the uniform random set $V(\mathcal{H})_M$ is not δ-stable is at most $1/4$. Consequently, with probability at least $1/4$, the at least δM vertices occupied by Maker contain an edge of $\mathcal{H}$. □

Remark 3.2.4. *Note that with a little more effort, the proof in fact shows that if* $q \leq cf(\mathcal{H})-1$*, then Maker's random strategy succeeds with probability tending to 1 as* $v(\mathcal{H})$ *tends to infinity.*

3.2.2 Proof of Theorem 3.2.3 The heart of the proof of Theorem 3.2.3 is the following statement due to Janson, Łuczak and Ruciński, see Theorem 2.18, (*ii*) in [46].

Theorem 3.2.5 (Janson-Łuczak-Ruciński [46])**.** *Let* A *be a set and let* $0 < p < 1$*. For a set* $S \subseteq A$ *denote by* $\mathbb{1}_S$ *the indicator random variable for the event that* $S \subseteq A_p$*. Let* $\mathcal{S} \subset \mathcal{P}(A)$ *be a family of subsets of* A *and let* $X = \sum_{S \in \mathcal{S}} \mathbb{1}_S$*. Then*

$$\mathbb{P}(X = 0) \leq \exp\left(- \frac{\mathbb{E}(X)^2}{\sum\sum_{\substack{S,S' \in \mathcal{S} \\ S \cap S' \neq \emptyset}} \mathbb{E}(\mathbb{1}_S \mathbb{1}_{S'})} \right)$$

where the sum is taken over all ordered pairs S, S' *with nonempty intersection.*

Proof of Theoerem 3.2.3. We want to apply Theorem 3.2.5 with $A = V(\mathcal{H})$, $\mathcal{S} = E(\mathcal{H})$ and $p = 1/f(\mathcal{H})$. Note that $p < 1$ due to Condition (ii). For $X = \sum_{e \in \mathcal{H}} \mathbb{1}_e$ we have by linearity of expectation $\mathbb{E}(X) = e(\mathcal{H})\, p^k$. Note that $\mathbb{E}(\mathbb{1}_e \mathbb{1}_{e'}) = p^{2k-|e \cap e'|}$ for $e, e' \in \mathcal{H}$ and therefore

$$\begin{aligned}
\sum\sum_{\substack{e,e' \in \mathcal{H} \\ e \cap e' \neq \emptyset}} \mathbb{E}(\mathbb{1}_e \mathbb{1}_{e'}) &= \sum_{e \in \mathcal{H}} \sum_{\emptyset \neq S \subseteq e} \sum_{\substack{e' \in \mathcal{H} \\ e \cap e' = S}} p^{2k-|S|} \leq \sum_{e \in \mathcal{H}} \sum_{\emptyset \neq S \subseteq e} d(S)\, p^{2k-|S|} \\
&\leq e(\mathcal{H}) \left(2^k - 1\right) \max_{\emptyset \neq S \subseteq V(\mathcal{H})} \left(d(S)\, p^{2k-|S|}\right) \\
&\leq 2^k\, e(\mathcal{H}) \max_{1 \leq \ell \leq k} \left(\Delta_\ell(\mathcal{H})\, p^{2k-\ell}\right) \\
&= 2^k\, e(\mathcal{H})\, p^{2k-1} \max\left(\max_{2 \leq \ell \leq k} \left(\frac{\Delta_\ell(\mathcal{H})}{p^{\ell-1}}\right), \Delta_1(\mathcal{H}) \right) \\
&= 2^k \frac{\mathbb{E}(X)^2}{p\, v(\mathcal{H})\, d(\mathcal{H})} \max\left(\max_{2 \leq \ell \leq k} \left(\frac{\Delta_\ell(\mathcal{H})}{p^{\ell-1}}\right), \Delta_1(\mathcal{H}) \right)
\end{aligned}$$

Using Condition (i) we now get

$$\begin{aligned}
\sum\sum_{\substack{e,e' \in \mathcal{H} \\ e \cap e' \neq \emptyset}} \mathbb{E}(\mathbb{1}_e \mathbb{1}_{e'}) &\leq 2^k \frac{\mathbb{E}(X)^2}{p\, v(\mathcal{H})} \max\left(\max_{2 \leq \ell \leq k} \left(\frac{\Delta_\ell(\mathcal{H})}{d(\mathcal{H})\; p^{\ell-1}}\right), c_1 \right) \\
&\leq 2^k \frac{\mathbb{E}(X)^2}{p\, v(\mathcal{H})} \max\left(\max_{2 \leq \ell \leq k} \left(\frac{1}{(f(\mathcal{H})\; p)^{\ell-1}}\right), c_1 \right) = c_1\, 2^k \frac{\mathbb{E}(X)^2}{p\, v(\mathcal{H})},
\end{aligned}$$

where the last inequality follows from the definition of $f(\mathcal{H})$ and the last equality follows from the fact that $f(\mathcal{H})\,p = 1$ and $c_1 \geq k$. Now, using the estimate from Theorem 3.2.5, we get

$$\mathbb{P}\,(V(\mathcal{H})_p \text{ contains no edge of } \mathcal{H}) = \mathbb{P}\,(X = 0) \leq \exp(-c'\,v(\mathcal{H})\,p)$$

where $c' = 1/(c_1\,2^k)$. Since the property of 'not containing an edge of $\mathcal{H}$' is monotone decreasing, we use Lemma 3.8.1 from the Appendix to restate this for the uniform random set model as follows:

$$\mathbb{P}\,(V(\mathcal{H})_{\bar{M}} \text{ contains no edge of } \mathcal{H}) \leq 3\,\mathbb{P}\left(V(\mathcal{H})_p \text{ contains no edge of } \mathcal{H}\right) \leq 3\,\exp(-c'\bar{M}) \tag{3.5}$$

for any $\bar{M} \geq \lfloor v(\mathcal{H})/f(\mathcal{H}) \rfloor$.

We are now ready to finish the proof. Let $M \geq 2\,\lfloor v(\mathcal{H})/f(\mathcal{H}) \rfloor$ and let $\delta = \delta(k, c_1) > 1/2$ be such that $(1-\delta)(1-\ln(1-\delta)) < c'/4$. To see that this is indeed possible, note that for $x \in (0,1]$ the function $f(x) = (1-x)(1-\ln(1-x))$ satisfies $|f(x)| < 1$ and $f(x) \to 0$ as $x \to 1$. Consider pairs (S, S') where $S \subset V(\mathcal{H})$ with $|S| = M$ and $S' \subseteq S$ is such that $|S'| = \delta\,M$ and S' does not contain an edge of $\mathcal{H}$. Using Inequality (3.5) with $\delta\,M > \lfloor v(\mathcal{H})/f(\mathcal{H}) \rfloor$, we can estimate the number of choices for a set S' of size $\delta\,M$ that contains no edge of $\mathcal{H}$ by

$$3\,\exp(-c'\delta M)\binom{v(\mathcal{H})}{\delta\,M} \leq 3\,\exp\left(-\,c'\frac{M}{2}\right)\binom{v(\mathcal{H})}{\delta\,M}.$$

Hence, we can upper bound the number of pairs (S, S') as described above by

$$3\,\exp\left(-\,c'\frac{M}{2}\right)\binom{v(\mathcal{H})}{\delta\,M}\binom{v(\mathcal{H})-\delta M}{(1-\delta)\,M} = 3\,\exp\left(-\,c'\frac{M}{2}\right)\binom{M}{(1-\delta)\,M}\binom{v(\mathcal{H})}{M}.$$

We can therefore upper bound the number of choices for a set S of size M containing a subset of size δM that does not contain an edge of $\mathcal{H}$ by

$$3\,\exp\left(-\,c'\frac{M}{2}\right)\binom{M}{(1-\delta)\,M}\binom{v(\mathcal{H})}{M} \leq 3\,\exp\left(M(-c'/2 + (1-\delta)(1-\ln(1-\delta)))\right)\binom{v(\mathcal{H})}{M}.$$

Hence we get

$$\mathbb{P}\,(V(\mathcal{H})_M \text{ is not } \delta\text{-stable }) \leq 3\,\exp\left(M(-c'/2 + (1-\delta)(1-\ln(1-\delta)))\right) \leq 3\exp\left(-M\frac{c'}{4}\right)$$

where the last inquality follows by choice of $\delta = \delta(k, c_1)$. □

3.3 Proof of Theorem 1.2.9 – Breaker's Strategy

We will in fact derive Theorem 1.2.9 as a corollary of the following stronger statement.

Theorem 3.3.1 (Breaker Win Criterion)**.** *For every $k \geq 2$ and $t > (2k)^k$ the following holds. If $\mathcal{H}$ is a k-uniform hypergraph, then Breaker has a winning strategy in $\boldsymbol{G}(\mathcal{H}; q)$ provided that*

$$q > 4 \max\left(\left((2\,v(\mathcal{H}))^{1/t}\,\Delta_1(\mathcal{H})\;ke\right)^{\frac{1}{k-1}},\; 2k^2t^3\left(\max_{2\leq\ell\leq k-1}\left(\Delta_\ell(\mathcal{H})\left((tk)^{tk}\,k^t\,v(\mathcal{H})^2\right)^{\frac{k}{t^{1/k}}}\right)^{\frac{1}{k-\ell}} + 2\right)\right).$$

Note that e denotes Euler's constant and should not be confused with the number of edges. We start by giving a proof of Theorem 1.2.9 using Theorem 3.3.1. Then we define the necessary concepts for the remainder of the section. Following this we present the two main strategies for Breaker and prove their correctness. Finally we prove Theorem 3.3.1 using these ingredients.

Proof of Theorem 1.2.9 from Theorem 3.3.1. Let $k \geq 2$ and $\epsilon > 0$ be given and set $t = \log v(\mathcal{H})$. Assume that $v(\mathcal{H})$ is large enough such that $\log v(\mathcal{H}) > (2k)^k$. Using $e = v(\mathcal{H})^{1/\log v(\mathcal{H})}$ it is straightforward to check that

$$\left((2n)^{1/t}\Delta_1(\mathcal{H})\;ke\right)^{\frac{1}{k-1}} \leq C_1'\,\Delta_1(\mathcal{H})^{\frac{1}{k-1}}$$

for some constant $C_1' = C_1'(k) > 0$. Similarly for $v(\mathcal{H})$ sufficiently large we can upper bound the term

$$2k^2t^3\left(\max_{2\leq\ell\leq k-1}\left(2k\;\Delta_\ell(\mathcal{H})\left(v(\mathcal{H})^2\;(tk)^{tk}\right)^{\frac{k}{t^{1/k}}}\right)^{\frac{1}{k-\ell}} + 2\right)$$

by

$$C_2'\,v(\mathcal{H})^{C_3'\frac{\log\log v(\mathcal{H})}{\log^{1/k} v(\mathcal{H})}}\max_{2\leq\ell\leq k-1}\left(\Delta_\ell(\mathcal{H})\right)^{\frac{1}{k-\ell}}$$

for some constants $C_2' = C_2'(k) > 0$ and $C_3' = C_3'(k) > 0$. Note that $\log\log v(\mathcal{H})/\log^{1/k} v(\mathcal{H}) \to 0$ and so for $v(\mathcal{H})$ large enough this will be at most $v(\mathcal{H})^\epsilon \max_{2\leq\ell\leq k-1}\left(\Delta_\ell(\mathcal{H})^{\frac{1}{k-\ell}}\right)$. Choose $C_1 = C_1(k) \geq \max(C_1', C_2', 4)$ and $v_0 = v_0(k)$ large enough, proving Theorem 1.2.9. □

3.3.1 Preliminaries and Definitions One of the most important results in the area of positional games is the Erdős-Selfridge Theorem [27], the biased version of which is due to Beck [10]. It ensures that Breaker can do at least as well as the expected outcome when both players act randomly. We will use the following consequence of it heavily in the proof of Theorem 1.2.9.

Theorem 3.3.2 (A biased Erdős-Selfridge Theorem [10])**.** *For every hypergraph $\mathcal{H}$ and integer $q \geq 1$ the following holds. If Breaker plays as the second player, he can keep Maker from covering more than*

$$(q+1)\sum_{H\in\mathcal{H}}\left(\frac{1}{q+1}\right)^{|H|} \tag{3.6}$$

winning sets in $\boldsymbol{G}(\mathcal{H}; q)$.

Proof. Beck showed that if $\sum_{H\in\mathcal{H}}(q+1)^{-|H|} < (q+1)^{-1}$ then Breaker has a winning strategy in $\mathbf{G}(\mathcal{H};q)$ (see [10] or [44]). The crucial point in his argument is to show that the function

$$g(M,B) = \sum_{\substack{H\in\mathcal{H}\\ H\cap B=\emptyset}} (1+q)^{-|H\setminus M|}$$

never increases if it's evaluated after every move of Maker. Here, M and B denote the sets of Maker's and Breaker's vertices respectively. Note that before the game starts we have $g(\emptyset,\emptyset) = \sum_{H\in\mathcal{H}}(q+1)^{-|H|}$ and after Maker's first move, in which he occupied the vertex m say, $g(\{m\},\emptyset) \le (1+q)\sum_{H\in\mathcal{H}}(1+q)^{-|H|}$.

Now, if Maker managed to occupy $k \ge 0$ edges at some point during the game, then clearly $g(M,B) \ge k$ since for every such edge H we have $|H\setminus M| = 0$.

Hence, if Breaker tries to minimise the value of g in each move, he can prevent Maker from occupying more than $(q+1)\sum_{H\in\mathcal{H}}(q+1)^{-|H|}$ edges as claimed. □

We will also need the following simple yet powerful remark.

Remark 3.3.3. *If Breaker has a winning strategy for some positional game $\boldsymbol{G}(\mathcal{H};q_0)$, then he trivially still has a winning strategy in $\boldsymbol{G}(\mathcal{H};q)$ where $q \ge q_0$. It follows that if he has a winning strategy for some game $\boldsymbol{G}(\mathcal{H}_1;q_1)$ and a winning strategy for another game $\boldsymbol{G}(\mathcal{H}_2;q_2)$, then he can combine these two strategies to form a winning strategy in $\boldsymbol{G}(\mathcal{H}_1\cup\mathcal{H}_2;q_1+q_2)$.*

This remark will be used extensively throughout the proof. Furthermore, we will need the following definitions.

Definition 3.3.4 (Set-Theoretic Definitions)**.** *Given some hypergraph $\mathcal{H}$, we define the following:*

- *a t-*cluster *is any family of distinct edges $\{H_1,\dots,H_t\}\subset\mathcal{H}$ satisfying $|\bigcap_{i=1}^t H_i| \ge 2$,*
- *an* almost complete solution *(H°,h) is a tuple consisting of a set $H^\circ\subseteq V(\mathcal{H})$ as well as an element $h\notin H^\circ$ so that $H = H^\circ\cup\{h\}$ is a edge in $\mathcal{H}$,*
- *a t-*fan *is a family of distinct almost complete solutions $\{(H_1^\circ,h_1),\dots,(H_t^\circ,h_t)\}$ in $\mathcal{H}$ satisfying $|\bigcap_{i=1}^t H_i^\circ| \ge 1$ and it is called* simple *if $|H_i^\circ\cap H_j^\circ| = 1$ for all $1\le i<j\le t$,*
- *a t-*flower *is a t-fan satisfying $|\bigcap_{i=1}^t H_i^\circ| \ge 2$.*

For each t-fan in $\mathcal{H}$ we call the h_i the open elements*, the H_i° the* major parts *and the elements of the intersection $\bigcap_{i=1}^t H_i^\circ$ the* common elements*.*

Definition 3.3.5 (Game-Theoretic Definitions)**.** *At any given point in a positional game on a given hypergraph $\mathcal{H}$, we call an almost complete solution (H°,h)* dangerous *if all elements of H° have been occupied by Maker and h has not yet been occupied by either player. A fan or flower is dangerous if their respective almost complete solutions are.*

Observe that for a dangerous fan or flower we must have $h_i \notin H_j^\circ$ for all $1 \leq i, j \leq t$. In the following we will always assume that Breaker plays as second player. We say that a player *occupied* a given t-fan or t-flower $(H_1^\circ, h_1), \ldots, (H_t^\circ, h_t)$ if his selection of vertices contains $\bigcup_{i=1}^t H_i^\circ$. Similarly a player *occupied* a t-cluster $H_1, \ldots, H_t$ if his selection of vertices contains $\bigcup_{i=1}^t H_i$.

3.3.2 Two important strategies for Breaker The following two lemmata give us strategies that we will use to construct a larger strategy in the proof of Theorem 1.2.9. Note that in the statement of the lemma we do not care about which player covers the open elements of a fan.

Lemma 3.3.6. *For every integer $k \geq 2$ and $t \geq 1$ the following holds. If $\mathcal{H}$ is a k-uniform hypergraph, then Breaker with a bias of $q > \left((2\,v(\mathcal{H}))^{1/t}\,\Delta_1(\mathcal{H})\,ke\right)^{1/(k-1)}$ has a strategy that prevents Maker from occupying $1/2\binom{q}{t}$ simple t-fans in the game $\boldsymbol{G}(\mathcal{H};q)$.*

Proof. Let $\mathcal{F} = \left\{ \bigcup_{i=1}^t H_i^\circ \mid \{(H_1^\circ, h_1), \ldots, (H_t^\circ, h_t)\} \text{ simple } t\text{-fan in } \mathcal{H} \right\}$ be the hypergraph of all simple t-fans in $\mathcal{H}$. We want to apply Theorem 3.3.2, so we estimate

$$(q+1) \sum_{F \in \mathcal{F}} \left(\frac{1}{q+1}\right)^{|F|} \leq (q+1) \left(v(\mathcal{H})\, \frac{\Delta_1(\mathcal{H})^t\,(k-1)^t}{t!}\right) \left(\frac{1}{q+1}\right)^{t(k-2)+1}.$$

This inequality holds because there are $v(\mathcal{H})$ ways to fix the common element of a simple t-fan, $\Delta_1(\mathcal{H})^t$ is an upper bound on the number of t-tuples of edges containing the fixed common element and there are $(k-1)^t$ ways of fixing the corresponding open elements. Note that an open element is never a common element by definition. Furthermore, $t!$ takes care of the symmetry and each simple t-fan is of size $t(k-2)+1$. We therefore get, using $t! \geq (t/e)^t$

$$\begin{aligned}(q+1) \sum_{F \in \mathcal{F}} \left(\frac{1}{q+1}\right)^{|F|} &\leq v(\mathcal{H}) \left(\frac{\Delta_1(\mathcal{H})\,ke}{t\,q^{k-2}}\right)^t = v(\mathcal{H}) \left(\frac{\Delta_1(\mathcal{H})\,ke}{q^{k-1}}\right)^t \left(\frac{q}{t}\right)^t \\ &< v(\mathcal{H})\,\frac{1}{2\,v(\mathcal{H})} \left(\frac{q}{t}\right)^t \leq \frac{1}{2}\binom{q}{t},\end{aligned}$$

where the strict inequality follows from the choice of q. The claim now follows by applying Theorem 3.3.2. □

Lemma 3.3.7. *For every integer $k \geq 2$ and $t > (2k)^k$ the following holds. If $\mathcal{H}$ is a k-uniform hypergraph, then Breaker with a bias of*

$$q > \max_{2 \leq \ell \leq k-1} \left(\Delta_\ell(\mathcal{H})\,\left((tk)^{tk}\,k^t\,v(\mathcal{H})^2\right)^{\frac{k}{t^{1/k}}}\right)^{\frac{1}{k-\ell}} \tag{3.7}$$

has a strategy that prevents dangerous $t(q+1)$-flowers in $\boldsymbol{G}(\mathcal{H};q)$

Proof. Let $\mathcal{F} = \left\{ \bigcup_{i=1}^t H_i \mid \{H_1, \dots, H_t\} \text{ } t\text{-cluster in } \mathcal{H} \right\}$ be the hypergraph of all t-clusters in $\mathcal{H}$. First we will show that Breaker can prevent t-clusters. Given some t-cluster $H_1, \dots, H_t$ let $\ell_i = \left| H_i \cap \bigcup_{j=1}^{i-1} H_j \right|$ for all $2 \le i \le t$. We call $(2, \ell_2, \dots, \ell_t)$ its *intersection characteristic* and observe that $2 \le \ell_i \le k$ for $2 \le i \le t$. We will set $\ell_1 = 2$ for notational convenience. For any $\boldsymbol{\ell} = (\ell_1, \dots, \ell_t) \in \{2\} \times [2, k]^{t-1}$ let $\mathcal{F}(\boldsymbol{\ell})$ denote the set of edges in $\mathcal{F}$ which come from some t-cluster with the intersection characteristic $\boldsymbol{\ell}$ and observe that it is $v(\boldsymbol{\ell})$-uniform where

$$v(\boldsymbol{\ell}) = 2 + \sum_{i=1}^{t} (k - \ell_i) = k + \sum_{i=2}^{t} (k - \ell_i). \tag{3.8}$$

This follows since given any cluster $H_1, \dots, H_t$ with intersection characteristic $\boldsymbol{\ell}$ we have $|\bigcup_{i=1}^t H_i| = v(\boldsymbol{\ell})$. There is the trivial upper bound $v(\boldsymbol{\ell}) \le tk$ for all $\boldsymbol{\ell} \in \{2\} \times [2, k]^{t-1}$. Let $L = \{\boldsymbol{\ell} : \mathcal{F}(\boldsymbol{\ell}) \ne \emptyset\} \subseteq \{2\} \times [2, k]^{t-1}$ be the set of all intersection characteristics that actually occur in $\mathcal{H}$. Now for any $\boldsymbol{\ell} \in L$ we trivially have $t \le \binom{v(\boldsymbol{\ell})-2}{k-2}$ which we restate as the lower bound

$$v(\boldsymbol{\ell}) \ge t^{1/k} \text{ for all } \boldsymbol{\ell} \in L. \tag{3.9}$$

Now for $\boldsymbol{\ell} = (\ell_1, \dots, \ell_t) \in L$ observe that

$$\begin{aligned} |\mathcal{F}(\boldsymbol{\ell})| &\le \binom{v(\mathcal{H})}{2} \Delta_2(\mathcal{H}) \prod_{i=2}^{t} \binom{k + \sum_{j=2}^{i-1} (k - \ell_i) - 2}{\ell_i - 2} \Delta_{\ell_i}(\mathcal{H}) \\ &\le \binom{v(\mathcal{H})}{2} \binom{v(\boldsymbol{\ell}) - 2}{k - 2}^{t-1} \Delta_2(\mathcal{H}) \prod_{i=2}^{t} \Delta_{\ell_i}(\mathcal{H}) \le v(\mathcal{H})^2 \, (tk)^{tk} \prod_{i=1}^{t} \Delta_{\ell_i}(\mathcal{H}). \end{aligned}$$

Here, the first inequality is justified by observing that there are $\binom{v(\mathcal{H})}{2}$ ways to fix two common elements and at most $\Delta_2(\mathcal{H})$ ways to choose the first edge H_1 of a t-cluster. The product counts ways to add the i-th additional edge H_i for $2 \le i \le t$ by first fixing the intersection with the already established parts $\bigcup_{j=1}^{i-1} H_j$ and then adding one of the at most Δ_{ℓ_i} possible ways of picking H_i. The second inequality follows since by assumption $t > (2k)^k$ so that Equation (3.9) gives us $v(\boldsymbol{\ell}) \ge 2k$ from which it follows that for all $2 \le i \le t$ we have

$$\binom{k + \sum_{j=2}^{i-1} (k - \ell_i) - 2}{\ell_i - 2} \le \binom{v(\boldsymbol{\ell}) - 2}{k - 2}.$$

We now want to apply Theorem 3.3.2, so we estimate

$$\begin{aligned} (q+1) \sum_{F \in \mathcal{F}} \left(\frac{1}{q+1} \right)^{|F|} &\le (q+1) \sum_{\ell_2 \in [2,k]} \cdots \sum_{\ell_t \in [2,k]} |\mathcal{F}(\boldsymbol{\ell})| \left(\frac{1}{q+1} \right)^{v(\boldsymbol{\ell})} \\ &\le (tk)^{tk} \, v(\mathcal{H})^2 \, (q+1) \sum_{\boldsymbol{\ell} \in L} \prod_{i=1}^{t} \Delta_{\ell_i}(\mathcal{H}) \left(\frac{1}{q+1} \right)^{v(\boldsymbol{\ell})}. \end{aligned}$$

where we have just inserted the previously stated upper bound on $|\mathcal{F}(\boldsymbol{\ell})|$. We now split up the factor $(1/(q+1))^{v(\boldsymbol{\ell})}$ using Equation (3.8) to obtain

$$(q+1)\sum_{F\in\mathcal{F}}\left(\frac{1}{q+1}\right)^{|F|} \leq (tk)^{tk}\, v(\mathcal{H})^2\, \frac{1}{q+1}\sum_{\boldsymbol{\ell}\in L}\prod_{i=1}^{t}\left(\Delta_{\ell_i}(\mathcal{H})\left(\frac{1}{q+1}\right)^{k-\ell_i}\right).$$

Note that we have $\Delta_\ell(\mathcal{H})\,(1/(q+1))^{k-\ell} = 1$ for $\ell = k$ and $\Delta_\ell(\mathcal{H})\,(1/(q+1))^{k-\ell} < 1$ for $2 \leq \ell < k$ due to the lower bound on q. Furthermore, since $\boldsymbol{\ell} \in L$ is the intersection characteristic of a t-cluster in $\mathcal{H}$, the number of indices $1 \leq i \leq t$ for which $\ell_i < k$ must be at least $\lceil v(\boldsymbol{\ell})/k\rceil \geq \lceil t^{1/k}/k\rceil$. Now, due to Equation (3.7) it follows that

$$(q+1)\sum_{F\in\mathcal{F}}\left(\frac{1}{q+1}\right)^{|F|} \leq (tk)^{tk}\, v(\mathcal{H})^2\, k^t\left(\max_{2\leq\ell\leq k-1}\Delta_\ell(\mathcal{H})\left(\frac{1}{q}\right)^{k-\ell}\right)^{\frac{t^{1/k}}{k}} < 1.$$

It follows, by applying Theorem 3.3.2, that using a bias of q, Breaker has a strategy to keep Maker from fully covering any t-cluster. Following this strategy, it is easy to see that Breaker will also keep Maker from creating a dangerous $t(q+1)$-flower at any point in the game. To see this, suppose that this is not the case and that Maker succeeds in creating such a dangerous flower. By repeatedly claiming the open element of this dangerous flower which has not yet been claimed and is the open element of the most almost complete solutions in the flower, Maker would be able to cover a $t(q+1)/(q+1) = t$-cluster, which is a contradiction. □

3.3.3 Proof of Theorem 3.3.1 In order to join the previous two strategies together, we will need the following simple auxiliary statement. We include its simple proof for the convenience of the reader.

Lemma 3.3.8. *For every $q \geq 2$ and $t \geq 2$ the following holds: If F is a graph on q vertices with $e(F) < q^2/2t^2$ then F has at least $1/2\binom{q}{t}$ independent sets of size t.*

Proof. The number of subsets of $V(F)$ of size t that are not independent is upper bounded by

$$e(F)\binom{q-2}{t-2} \leq e(F)\left(\frac{t^2}{q^2}\right)\binom{q}{t} < \frac{1}{2}\binom{q}{t}$$

since $e(F) < q^2/2t^2$. □

We are now ready to prove Theorem 1.2.9. Let $k \geq 2$ and $t > (2k)^k$ be given and let

$$q > 4\max\left(\left((2\,v(\mathcal{H}))^{1/t}\Delta_1(\mathcal{H})\,ke\right)^{\frac{1}{k-1}},\ 2k^2t^3\left(\max_{2\leq\ell\leq k-1}\left(\Delta_\ell(\mathcal{H})((tk)^{tk}\,k^t\,v(\mathcal{H})^2)^{\frac{k}{t^{1/k}}}\right)^{\frac{1}{k-\ell}}+2\right)\right).$$

Breaker will play according to the following three strategies, splitting his bias as $q = q/2 + q/4 + q/4$. Note that in case Breaker does not need all his moves to play according to one of the strategies, he plays them arbitrarily, which cannot hurt him.

SB1: Using $q/4$ moves, he will play according to Lemma 3.3.6 and thus preventing Maker from occupying $1/2\binom{q/4}{t}$ simple t-fans.

SB2: Using $\overline{q} = \max_{2\le\ell\le k-1}\left(\Delta_\ell(\mathcal{H})\left((tk)^{tk}k^t\,v(\mathcal{H})^2\right)^{k/t^{1/k}}\right)^{1/(k-\ell)} + 1 < q/4$ moves, he will play according to Lemma 3.3.7 and hence preventing dangerous $t(\overline{q}+1)$-flowers to appear.

SB3: Using $q/2$ moves, Breaker will occupy all open elements of any dangerous almost complete solution.

First of all note that Maker can successfully play according to *SB1* and *SB2* since

$$q/4 > \left((2\,v(\mathcal{H}))^{1/t}\Delta_1(\mathcal{H})\;ke\right)^{1/(k-1)} \text{ and } \overline{q} > \max_{2\le\ell\le k-1}\left(2k\;\Delta_\ell(\mathcal{H})\;\left(v(\mathcal{H})^2\;(tk)^{tk}\right)^{\frac{k}{t^{1/k}}}\right)^{1/(k-\ell)}.$$

We can combine these strategies due Remark 3.3.3 and will now prove by induction, that after each of Breaker's moves there is no dangerous almost complete solution. Clearly this implies that Breaker's strategy is indeed a winning strategy. Initially there is obviously no dangerous almost complete solution. So suppose the result is true in round $r-1$. In round r Maker occupies an element w say. Then every new dangerous almost complete solution must contain w. Therefore they all belong to the same dangerous fan (with common element w). In order to complete the inductive step, we have to show that the size of this dangerous fan is not more than $q/2$ as Breaker can then occupy all open elements in this dangerous fan (SB3), which completes the inductive step. Indeed, using a bias of $q/2$ Breaker has a strategy that avoids dangerous $q/2$-fans at any point in the game.

Suppose Maker succeeds in occupying a dangerous $q/2$-fan $(H_1^\circ, h_1), \ldots, (H_{q/2}^\circ, h_{q/2})$. Construct an auxiliary graph F whose vertices are the almost complete solutions of this fan and an edge between (H_i°, h_i) and (H_j°, h_j) indicates that $|H_i^\circ \cap H_j^\circ| \ge 2$ where $1 \le i < j \le q/2$. Recall that using $\overline{q}$ moves according to *SB2*, Breaker prevents dangerous $t(\overline{q}+1)$-flowers from appearing. Therefore the maximum degree in F is bounded by $\Delta(F) \le (t(\overline{q}+1)-2)\binom{k-1}{2} \le t(\overline{q}+1)k^2$ and hence $e(F) \le \frac{1}{2}\frac{q}{2}t(\overline{q}+1)k^2 < \frac{1}{2}\frac{(q/2)^2}{t^2}$ by choice of q. Therefore, by Lemma 3.3.8, F has at least $\frac{1}{2}\binom{q/4}{t}$ independent sets of size t. But that means that Maker occupied $\frac{1}{2}\binom{q/4}{t}$ simple t-fans contradicting SB1. This establishes the claim that Breaker has a strategy that avoids dangerous $q/2$-fans and finishes the proof. □

3.4 Proof of Theorem 1.2.2 - Maker-Breaker $\mathcal{G}$-game

Recall that the aim of this section is to use our general winning criteria to show that if $\mathcal{G}$ is an r-uniform hypergraph on at least $r+1$ non-isolated vertices, then the threshold bias of the Maker-Breaker $\mathcal{G}$-game on $\mathcal{K}_n^{(r)}$ satisfies $q(\mathcal{H}_n(\mathcal{G})) = \Theta\left(n^{1/m_r(\mathcal{G})}\right)$.

Proof of Theorem 1.2.2. We recall that $\mathcal{H}_n(\mathcal{G})$ is the hypergraph of all copies of $\mathcal{G}$ in $\mathcal{K}_n^{(r)}$. Throughout this proof we will shorten the notation and simply write $\mathcal{H}_n$ for $\mathcal{H}_n(\mathcal{G})$. We observe that $\mathcal{H}_n$ is $e(\mathcal{G})$-uniform and clearly

$$v(\mathcal{H}_n) = \binom{n}{r} = \Theta(n^r) \qquad \text{and} \qquad e(\mathcal{H}_n) = \binom{n}{v(\mathcal{G})}\frac{v(\mathcal{G})!}{\mathrm{aut}(\mathcal{G})} = \Theta(n^{v(\mathcal{G})}).$$

In particular, this implies that

$$d(\mathcal{H}_n) = \Theta(n^{v(\mathcal{G})-r}). \tag{3.10}$$

The last estimate we will need is the following. For every $1 \le \ell \le e(\mathcal{G})$, we have

$$\Delta_\ell(\mathcal{H}_n) = \Theta\left(\max_{\substack{\mathcal{F}\subset\mathcal{G}\\ e(\mathcal{F})=\ell}} n^{v(\mathcal{G})-v(\mathcal{F})}\right) \tag{3.11}$$

We will prove that the threshold bias satisfies $q(\mathcal{H}_n) = \Theta(n^{1/m_r(\mathcal{G})})$ by showing the existence of two constants $C_{\mathrm{M}} = C_{\mathrm{M}}(k, e(\mathcal{G})) > 0$ and $C_{\mathrm{B}} = C_{\mathrm{B}}(\mathcal{G}) > 0$ as well as $n_{0,\mathrm{M}} = n_{0,\mathrm{M}}(\mathcal{G}) \in \mathbb{N}$ and $n_{0,\mathrm{B}} = n_{0,\mathrm{B}}(\mathcal{G}) \in \mathbb{N}$ such that for $n \ge n_{0,\mathrm{M}}$ and $q \le C_{\mathrm{M}}\, n^{1/m_r(\mathcal{G})}$ Maker has a winning strategy in $\mathbf{G}(\mathcal{H}_n; q)$ and for $n \ge n_{0,\mathrm{B}}$ and $q \ge C_{\mathrm{B}}\, n^{1/m_r(\mathcal{G})}$ the same holds for Breaker.

We start with Maker's strategy by checking that $\mathcal{H}_n$ satisfies the three conditions of Theorem 1.2.8 for n large enough. First observe that if $\mathcal{G}$ is a collection of $e(\mathcal{G})$ independent edges, then Maker has a winning strategy if $q < \binom{n-r(e(\mathcal{G})-1)}{r}/(e(\mathcal{G})-1)$. So we may assume that this is not the case. It is easy to see that $\mathcal{H}_n$ satisfies Condition (i) for $c_1 = k = e(\mathcal{G})$. Just note that $\Delta_1(\mathcal{H}_n) = e(\mathcal{G})\, d(\mathcal{H}_n)$, since $\mathcal{H}_n$ is a regular. It remains to be shown that $\mathcal{H}_n$ satisfies Conditions (ii) and (iii) as well. Note that, using (3.10) and (3.11),

$$\begin{aligned} f(\mathcal{H}_n) = \min_{2\le\ell\le e(\mathcal{G})}\left(\frac{d(\mathcal{H}_n)}{\Delta_\ell(\mathcal{H}_n)}\right)^{\frac{1}{\ell-1}} &= \min_{2\le\ell\le e(\mathcal{G})}\left(\frac{\Theta(n^{v(\mathcal{G})-r})}{\Theta(\max_{\substack{\mathcal{F}\subset\mathcal{G}\\ e(\mathcal{F})=\ell}} n^{v(\mathcal{G})-v(\mathcal{F})})}\right)^{\frac{1}{\ell-1}} \\ &= \min_{\substack{\mathcal{F}\subset\mathcal{G}\\ e(\mathcal{F})\ge 2}} \Theta\left(n^{\frac{v(\mathcal{F})-r}{e(\mathcal{F})-1}}\right) = \Theta\left(n^{1/m_r(\mathcal{G})}\right). \end{aligned}$$

Clearly we have $1 < f(\mathcal{H}_n)$ and so $\mathcal{H}_n$ satisfies Condition (ii).
Let $\tilde{c} = \tilde{c}(k, e(\mathcal{G}))$ be as given by Theorem 1.2.8. Since the maximum in $m_r(\mathcal{G})$ is attained for a connected subhypergraph of $\mathcal{G}$, we have $m_r(\mathcal{G}) > 1/r$ so that $f(\mathcal{H}_n) \to \infty$ as $n \to \infty$ and $f(\mathcal{H}_n) = o(v(\mathcal{H}_n))$. It follows that there exists $n_{0,\mathrm{M}} \in \mathbb{N}$ so that for $n \ge n_{0,\mathrm{M}}$ we have

$$\frac{v(\mathcal{H}_n)}{f(\mathcal{H}_n)}\left(1 - \frac{1}{f(\mathcal{H}_n)}\right) \ge \tilde{c},$$

so Condition (iii) is satisfied as well. Hence, by Theorem 1.2.8, there exists a constant $c = c(k, e(\mathcal{G})) > 0$ such that Maker has a winning strategy in $\mathbf{G}(\mathcal{H}_n; q)$ if $q \leq c\, n^{1/m_r(\mathcal{F})} - 1$ and $n \geq n_{0,\mathrm{M}}$. The result follows for $C_\mathrm{M} = C_\mathrm{M}(k, e(\mathcal{G})) = c - 1$.

Now we provide Breaker's strategy. Note that we can restrict our attention to the case in which $\mathcal{G}$ is strictly r-balanced, as otherwise we can replace $\mathcal{G}$ with a strictly r-balanced suphypergraph $\mathcal{F} \subset \mathcal{G}$. Indeed, if Breaker can keep Maker from occupying $\mathcal{F}$, then he clearly also succeeds in keeping Maker from occupying a copy of $\mathcal{G}$. So we may assume that $m_r(\mathcal{G}) = (e(\mathcal{G}) - 1)/(v(\mathcal{G}) - r)$ and that $m_r(\mathcal{F}) = (e(\mathcal{F}) - 1)/(v(\mathcal{F}) - r) < m_r(\mathcal{G})$ for all subgraphs $\mathcal{F} \subsetneq \mathcal{G}$ on at least $r + 1$ vertices.

First of all, note that

$$\Delta_1(\mathcal{H}_n)^{\frac{1}{e(\mathcal{G})-1}} \leq c_1\, n^{\frac{v(\mathcal{G})-r}{e(\mathcal{G})-1}} = c_1\, n^{\frac{1}{m_r(\mathcal{G})}}$$

for some $c_1 = c_1(r, \mathcal{G}) > 0$. Note that since $\mathcal{G}$ is strictly r-balanced, we have for every $2 \leq \ell \leq e(\mathcal{G})$ and every subhypergraph $\mathcal{F} \subset \mathcal{G}$ with $e(\mathcal{F}) = \ell$ that

$$\frac{v(\mathcal{G}) - v(\mathcal{F})}{e(\mathcal{G}) - \ell} = \frac{(v(\mathcal{G}) - r) - (v(\mathcal{F}) - r)}{(e(\mathcal{G}) - 1) - (e(\mathcal{F}) - 1))} = \frac{1}{m_r(\mathcal{G})} \frac{1 - \frac{v(\mathcal{F})-r}{v(\mathcal{G})-r}}{1 - \frac{e(\mathcal{F})-1}{e(\mathcal{G})-1}} < \frac{1}{m_r(\mathcal{G})}$$

where the last strict inequality follows since $v(\mathcal{F}) - r > (e(\mathcal{F}) - 1)\frac{v(\mathcal{G})-r}{e(\mathcal{G})-1}$. Therefore there exists a sufficiently small $\epsilon = \epsilon(r, \mathcal{G})$ such that for every $2 \leq \ell \leq e(\mathcal{G})$ and every subhypergraph $\mathcal{F} \subset \mathcal{G}$ with $e(\mathcal{F}) = \ell$ we have, using (3.11)

$$\Delta_\ell(\mathcal{H}_n)^{\frac{1}{e(\mathcal{G})-\ell}}\, n^\epsilon \leq c_\ell \max_{\substack{\mathcal{F}\subset\mathcal{G} \\ e(\mathcal{F})=\ell}} n^{\frac{v(\mathcal{G})-v(\mathcal{F})}{e(\mathcal{G})-\ell}+\epsilon} \leq c_\ell\, n^{\frac{1}{m_r(\mathcal{G})}}$$

for some constant $c_\ell = c_\ell(r, \mathcal{G}) > 0$. Now let $C_1 = C_1(k)$ and $v_0 = v_0(k, \epsilon)$ be as given by Theorem 1.2.9. For $C_\mathrm{B} \geq C_1 \max(c_1, \max_{2\leq\ell\leq e(\mathcal{G})} c_\ell)$ and $n_{0,\mathrm{B}} \geq r\, v_0^{1/r}$ it follows by applying Theorem 1.2.9 that Breaker indeed has a winning strategy in $\mathbf{G}(\mathcal{H}_n; q)$ for $q \geq C_\mathrm{B}\, n^{1/m_r(\mathcal{G})}$ and $n \geq n_{0,\mathrm{B}}$ since

$$q \geq C_\mathrm{B}\, n^{\frac{1}{m_r(\mathcal{G})}} \geq C_1 \max\left(\Delta_1(\mathcal{H}_n)^{\frac{1}{r-1}}, \max_{2\leq\ell\leq r-1}\left(\Delta_\ell(\mathcal{H}_n)^{\frac{1}{r-\ell}}\right) n^\epsilon\right)$$

as well as $v(\mathcal{H}_n) \geq v_0$. This finishes the proof. □

3.5 Generalising the van der Waerden game

As already mentioned in Section 1.2.3, we will now discuss what happens when we play the Maker-Breaker game $\mathbf{G}(A, n; q)$ for an arbitrary matrix $A \in \mathbb{Z}^{r\times m}$. This requires a certain amount of preparation.

3.5.1 Proper solutions We start by stating our results in the most natural case, i.e. when Maker's aim is to occupy a proper solution to $A \cdot \mathbf{x}^T = \mathbf{b}^T$ for some $\mathbf{b} \in \mathbb{Z}^r$. We will then see what effect repeated coordinates of solutions have in the subsequent subsection.

We begin by introducing further notation for the sets of different kinds of solutions and the game hypergraph. For an integer-valued matrix $A \in \mathbb{Z}^{r \times m}$ and integer-valued vector $\mathbf{b} \in \mathbb{Z}^r$, we denote by

$$S(A, \mathbf{b}) = \{\mathbf{x} \in \mathbb{Z}^m : A \cdot \mathbf{x}^T = \mathbf{b}^T\} \tag{3.12}$$

the set of *all* integer solutions and let

$$S_0(A, \mathbf{b}) = \{\mathbf{x} = (x_1, \ldots, x_m) \in S(A, \mathbf{b}) : x_i \neq x_j \text{ for } i \neq j\} \tag{3.13}$$

denote the set of all *proper* integer solutions. The m-uniform hypergraph of the game that accepts only proper solutions from $[n]$ is denoted by

$$\mathcal{S}_0(A, \mathbf{b}, n) = \big\{\{x_1, \ldots, x_m\} : (x_1, \ldots, x_m) \in S_0(A, \mathbf{b}) \cap [n]^m\big\}. \tag{3.14}$$

Next, after introducing some notation, we give the definition of some basic properties of a matrix $A \in \mathbb{Z}^{r \times m}$ that are needed for the game not to be trivial. For a subset $Q \subseteq [m]$, let A^Q denote the matrix obtained from A by keeping only the columns indexed by Q, where $A^\emptyset$ is the empty matrix.

Definition 3.5.1. *Let $A \in \mathbb{Z}^{r \times m}$. We call A*

(i) positive *if $S(A, \mathbf{0}) \cap \mathbb{N}^m \neq \emptyset$, that is, there are solutions whose entries lie in the positive integers,*

(ii) abundant *if $rk(A) > 0$ and $rk(A^Q) = rk(A)$ for all $Q \subseteq [m]$ satisfying $|Q| \geq m - 2$, that is, A has rank strictly greater than 0 and every submatrix obtained from A by deleting at most two columns must be of the same rank as A.*

The importance of the first notion should be clear: if the homogeneous solution space is disjoint from the positive quadrant in which we are playing, then the (inhomogeneous) game hypergraph will contain at most finitely many winning sets for all n, so that the bias threshold will also be bounded by some positive constant.

On the other hand, the second definition might initially be less clear, but we will see that this is the key definition. First of all, the notion of abundancy is required for our density parameter below to be well-defined. Secondly, non-abundant systems turn out to be "degenerate" in some sense and in particular, Breaker wins the game with a bias of at most 2. We would like to highlight the following.

Observation 3.5.2. *Let $A \in \mathbb{Z}^{r\times m}$ be abundant. Then $m \geq rk(A) + 2$, that is, the number of columns is at least the rank of the matrix plus two.*

Proof. Suppose for the sake of contradiction that $m \leq \mathrm{rk}(A) + 1$. Now note that if we delete any two columns i, j we get $\mathrm{rk}(A^{\overline{\{i,j\}}}) \leq m - 2 < \mathrm{rk}(A)$. However, this contradicts the fact that A is abundant. □

For readers familiar with the notion of *partition* and *density regular* (or *invariant*) matrices in the homogeneous setting, note that they are trivially positive. See [74] for an easy proof that they are also abundant.

The observant reader will have noticed that even though the game hypergraph only accepts proper solutions, we did not specifically introduce a notion for a matrix having a proper solution. Indeed, this would be superfluous. In Lemma 3.6.1 we will show that if a matrix A is positive and abundant then there exists a proper solution in the positive integers to the associated homogeneous system of equations. Based on this we will in fact show in Lemma 3.6.3 that for any positive and abundant matrix and vector $\mathbf{b}$ there is not just one, but many proper solutions in the positive integers to the associated inhomogeneous system.

Before we continue, we collect some elementary observations, whose proofs we've all seen in Linear Algebra. If $A, B \in \mathbb{Z}^{r\times m}$ and $\mathbf{b}, \mathbf{b}' \in \mathbb{Z}^r$ are such that $S(A, \mathbf{b}) = S(B, \mathbf{b}')$, then the corresponding game hypergraphs are the same, i.e. $\mathcal{S}_0(A, \mathbf{b}, n) = \mathcal{S}_0(B, \mathbf{b}', n)$, and so the resulting games are identical. In particular, if the extended coefficient matrix $(A|\mathbf{b})$ can be transformed into $(B|\mathbf{b}')$ by a sequence of *elementary row operations* (multiplying a row by a non-zero constant, adding a row to another row and switching rows), then $\mathcal{S}_0(A, \mathbf{b}, n) = \mathcal{S}_0(B, \mathbf{b}', n)$ and the resulting games are identical again. More generally, if $P \in \mathbb{Q}^{r\times r}$ is an invertible matrix, then $S(A, \mathbf{b}) = S(PA, P\mathbf{b})$, so multiplying from the left with an invertible matrix does not change the game. Furthermore, these operations do not change the rank of the matrix. Similarly, switching two columns does not change the solution space (up to relabelling the variables) nor its rank. Hence we will apply these operations whenever advantageous. Finally, a crucial point in our arguments in the following sections is the following.

Observation 3.5.3. *Let $A \in \mathbb{Z}^{r\times m}$ be an arbitrary matrix. If the matrix $B \in \mathbb{Z}^{r\times m}$ can be obtained from A by a sequence of elementary row operations and switching columns, then A is abundant if and only if B is abundant.*

Proof. Let $i, j \in [m]$ be arbitrary column indices. We may assume that A can be transformed into B by elementary row operations only. Note that $B^{\overline{\{i,j\}}}$ can be obtained from A by either first applying the elementary row operations needed to transform it into B and then deleting columns i, j or by first deleting columns i, j from A and then applying the exact same row operations to $A^{\overline{\{i,j\}}}$. Since elementary row operations preserve the rank, we get that A is abundant if and only if B is abundant as claimed. □

Next, in order to state our main theorem for van der Waerden type games, we define a parameter for abundant matrices. Let $r_Q = \mathrm{rk}(A) - \mathrm{rk}(A^{\overline{Q}})$ for any set of column indices $Q \subseteq [m]$ where we set $\mathrm{rk}(A^{\emptyset}) = 0$.

Definition 3.5.4. *The* maximum 1-density *of an abundant matrix $A \in \mathbb{Z}^{r\times m}$ is defined as*

$$m_1(A) = \max_{\substack{Q\subseteq[m]\\ 2\leq|Q|}} \frac{|Q|-1}{|Q|-r_Q-1}. \tag{3.15}$$

We will show in Lemma 3.6.6 that this parameter is indeed well-defined, that is $|Q|-r_Q-1>0$ for all $Q \subseteq [m]$ satisfying $|Q| \geq 2$ if A is abundant.

Note that this parameter has some clear parallels to the 2-density of a graph and was originally introduced by Rödl and Rucinski for partition regular systems. For more details on the connections to their result and others in the area of random sets and graphs, we refer to the remarks given in Subsection 3.7.4.

We refer to the biased Maker-Breaker game played on the hypergraph $\mathcal{S}_0(A, \mathbf{b}, n)$ as the *Maker-Breaker $(A, \boldsymbol{b})$-game on $[n]$*. Our main result regarding van der Waerden games states the asymptotic behaviour of the threshold bias of these games depends only on the density of A when A is positive and abundant.

Theorem 3.5.5. *For every positive and abundant matrix $A \in \mathbb{Z}^{r\times m}$ and vector $\boldsymbol{b} \in \mathbb{Z}^r$ such that $S(A, \boldsymbol{b}) \neq \emptyset$, the threshold bias of the Maker-Breaker $(A, \boldsymbol{b})$-game on $[n]$ satisfies*

$$q(\mathcal{S}_0(A, \boldsymbol{b}, n)) = \Theta\left(n^{1/m_1(A)}\right).$$

Remark 3.5.6. *Note that $m_1(A_{k\text{-}AP}) = k-1$, so Theorem 1.2.3 is really just a special case of Theorem 3.5.5, as already mentioned in the introduction.*

The following result deals with those positive matrices not covered by the previous result. This will be much easier to prove than the result of Theorem 3.5.5 and nicely highlights that the simple structure of non-abundant matrices makes it much easier for Breaker.

Proposition 3.5.7. *For every $m \geq 2$, vector $\boldsymbol{b} \in \mathbb{Z}^r$ and positive but non-abundant matrix $A \in \mathbb{Z}^{r\times m}$, the threshold bias of the Maker-Breaker $(A, \boldsymbol{b})$-game on $[n]$ satisfies*

$$q(\mathcal{S}_0(A, \boldsymbol{b}, n)) \leq 2.$$

3.5.2 Solutions with repeated entries From a game theoretic point of view it may seem somewhat artificial to allow Maker to use certain elements multiple times to form a solution with repeated entries. However, in this section we shall try to motivate the study of repeated entries, the effect they have on the game and what kind of component-equalities do not make it any easier for Maker. To do so, we will consider three different well-known matrices and the effects repeated entries have in the corresponding games. This way the reader hopefully finds the notion of *non-degenerate* solutions somewhat intuitive.

Recall that when we defined the k-AP game in terms of the van der Waerden matrix $A_{k\text{-AP}}$ we required that Maker needs to occupy a proper solution to $A_{k\text{-AP}}\mathbf{x}^T = \mathbf{0}^T$ in order to win. If we defined the game to be a Maker's win even if he occupies a non-proper solution, then the game becomes trivial. Indeed, the vector $(z, ..., z)$ is a solution to the above equation system and hence Maker wins with his first move. On the other hand, constant vectors are the only non-proper solutions of $A_{k\text{-AP}}\mathbf{x}^T = \mathbf{0}^T$. Hence, if we allowed that some but not all entries could be equal, the resulting game is identical to the game in which Maker needs to occupy a proper solution.

For another fairly simple example, let us consider the so-called *Schur triples*. A Schur triple is a solution to $B\mathbf{x}^T = \mathbf{0}^T$, where $B = (1, 1, -1)$. In the resulting game it turns out that allowing non-proper solutions does not make it significantly easier for Maker. Indeed, there are no positive solutions with $x_1 = x_3$ or $x_2 = x_3$ and to block solutions with $x_1 = x_2$ Breaker only needs at most two extra moves in each round, since he might need to occupy the double and the half of Maker's previous move.

To see that things can get more complicated, we will consider another classic equation, the Sidon equation $x_1 + x_2 = x_3 + x_4$. In the resulting game the outcome of it changes greatly depending on which component-equalities we allow. According to Theorem 3.5.5 the game with proper solutions has threshold of the order $n^{2/3}$. If we allowed $x_1 = x_3$ and $x_2 = x_4$, then occupying any two different integers would provide Maker with a win, so the threshold bias would grow to $n - 1$. If we were to allow for Maker solutions with repeated coordinates $x_1 = x_2$, (but required $x_3 \neq x_1, x_1 \neq x_4, x_4 \neq x_3$), then it turns out that the game's threshold bias is the same order of magnitude as the one of the game with proper solutions. Indeed, in this case Maker would win the game also by occupying a 3-AP. However, since $m_1(A_{3\text{-AP}}) = 2 \geq 3/2$ (where 3/2 is the maximums 1-density of the matrix corresponding to the Sidon equation), Breaker can block all 3-AP's with a bias of order only $n^{1/2}$. This is a key point for the definition of *non-degenerate* solutions below.

Motivated by the above examples we will be after identifying exactly which component-equalities make the game easier for Maker and which do not. More precisely which one of them change the order of the threshold bias compared to the bias $q(\mathcal{S}_0(A, \mathbf{b}, n))$ and which do not. Identifying this correct notion of "non-degenerate" solution for our setup takes a few definitions.

Definition 3.5.8. *Given a solution $\boldsymbol{x} = (x_1, \ldots, x_m) \in S(A, \boldsymbol{b})$ for an integer-valued matrix $A \in \mathbb{Z}^{r \times m}$ and vector $\boldsymbol{b} \in \mathbb{Z}^r$, let*

$$p(\boldsymbol{x}) = \big\{\{1 \leq j \leq m : x_i = x_j\} : 1 \leq i \leq m\big\} \tag{3.16}$$

denote the set partition of the column indices $[m]$ indicating the repeated entries in $\boldsymbol{x}$.

Note that for a proper solution $\mathbf{x} \in S_0(A, \mathbf{b})$ we have $p(\mathbf{x}) = \{\{1\}, \ldots, \{m\}\}$. Given some set partition p of $\{1, \ldots, m\}$, let A_p denote the matrix obtained by summing up the columns of A according to p, that is for $p = \{T_1, \ldots, T_s\}$ such that $\min(T_1) < \cdots < \min(T_s)$ for some $1 \leq s \leq m$ and $\mathbf{c}_i$ the i-th column vector of A for every $1 \leq i \leq m$, we have

$$A_p = \left(\sum_{i \in T_1} \mathbf{c}_i \;\middle|\; \sum_{i \in T_2} \mathbf{c}_i \;\middle|\; \cdots \;\middle|\; \sum_{i \in T_s} \mathbf{c}_i \right). \tag{3.17}$$

Note that the assumption $\min(T_1) < \cdots < \min(T_s)$ ensures that this notion is well-defined and that $A_p = A$ for $p = \{\{1\}, \ldots, \{m\}\}$.

Using these definitions we can now define when a solution is considered to be non-degenerate:

1. If A is positive and abundant, then a solution $\mathbf{x} \in S(A, \mathbf{b}) \cap \mathbb{N}^m$ is defined to be *non-degenarate* if $|p(\mathbf{x})| \geq 2$ and A_p is either non-abundant or it is abundant and satisfies $m_1(A_p) \geq m_1(A)$.
2. If A is positive and non-abundant, then a solution $\mathbf{x} \in S(A, \mathbf{b}) \cap \mathbb{N}^m$ is defined to be *non-degenarate* if $|p(\mathbf{x})| \geq 2$.

For example for the (positive and abundant) matrices associated to k-APs and Schur triples the only non-degenerate solutions are the proper ones. For the matrix associated with the Sidon equation $x_1 + x_2 = x_3 + x_4$ the discussion above shows that 3-term arithmetic progressions are non-degenerate solutions with repeated entries.

The main result of this section shows that this is the right definition for those solutions, which do not make the game any easier for Maker. One way of interpreting this is that if $\mathbf{x}$ is a non-degenerate solution, then the system of equations given by $A_{p(\mathbf{x})}$ does not loose any of the "complexity" compared to the original system given by A. Note that our definition includes solutions $\mathbf{x} \in S(A, \mathbf{b})$ for which $\mathrm{rk}(A_{p(\mathbf{x})}) = \mathrm{rk}(A)$. These were called *non-trivial* by Rué et al. [67]. Both definitions extend a previous definition for single-line equations due to Ruzsa [68]. With this in mind we define

$$S_1(A, \mathbf{b}) = \big\{\mathbf{x} \in S(A, \mathbf{b}) : \mathbf{x} \text{ is non-degenerate}\big\} \tag{3.18}$$

and remark that

$$S(A, \mathbf{b}) \supseteq S_1(A, \mathbf{b}) \supseteq S_0(A, \mathbf{b}).$$

Furthermore we denote by $\mathcal{S}_1(A, \mathbf{b}, n)$ the hypergraph containing all non-degenerate solutions in $[n]$, that is

$$\mathcal{S}_1(A, \mathbf{b}, n) = \big\{\{x_1, \dots, x_m\} : (x_1, \dots, x_m) \in \mathcal{S}_1(A, \mathbf{b}) \cap [n]^m\big\}. \tag{3.19}$$

Note that $\mathcal{S}_1(A, \mathbf{b}, n)$ in contrast to $\mathcal{S}_0(A, \mathbf{b}, n)$ is not necessarily uniform.

The following result can be proven as a corollary to Theorem 3.5.5 and shows that allowing non-degenerate solutions for Maker does not change the order of the threshold bias compared to the proper game. In other words, with only a constant factor times the original threshold bias, Breaker is able to block not just all proper solutions but also every non-degenerate solution.

Corollary 3.5.9. *For every positive matrix $A \in \mathbb{Z}^{r \times m}$ and vector $\boldsymbol{b} \in \mathbb{Z}^r$ the threshold bias of the Maker-Breaker $(A, \boldsymbol{b})$-game on $[n]$ allowing non-degenerate solutions satisfies*

$$q(\mathcal{S}_1(A, \boldsymbol{b}, n)) = \Theta\left(q(\mathcal{S}_0(A, \boldsymbol{b}, n))\right).$$

Let us motivate why this is the "right" notion of non-degenerate solutions. Observe that given some partition p of $[m]$ the set $\{\mathbf{x} \in S(A, \mathbf{b}) : p(\mathbf{x}) = p\}$ is either empty or it trivialy consists only of non-degenerate or only of degenerate solutions. We therefore respectively also refer to p as either *vacant*, *non-degenerate* or *degenerate*. We will see later in Subsection 3.7.3 that allowing Maker to also win by occupying any solution belonging to a fixed degenerate partition p does change the order of the threshold bias. In other words, non-degenerate solutions indeed provide an exact characterisation for classes of solutions with repeated components that do not change the complexity of the original linear homogenous system.

3.6 Proof of Theorem 3.5.5 – Generalised van der Waerden games

The goal of this section is to show that the threshold bias of the Maker-Breaker $(A, \mathbf{b})$-game on $[n]$ satisfies $q(\mathcal{S}_0(A, \mathbf{b}, n)) = \Theta\left(n^{1/m_1(A)}\right)$ for a given positive and abundant matrix $A \in \mathbb{Z}^{r \times m}$ and vector $\mathbf{b} \in \mathbb{Z}^r$. We will prove this by showing the existence of constants $C_\mathrm{M} = C_\mathrm{M}(A, \mathbf{b}) > 0$ and $C_\mathrm{B} = C_\mathrm{B}(A, \mathbf{b}) > 0$ as well as $n_{0,\mathrm{M}} = n_{0,\mathrm{M}}(A, \mathbf{b}) \in \mathbb{N}$ and $n_{0,\mathrm{B}} = n_{0,\mathrm{B}}(A, \mathbf{b}) \in \mathbb{N}$ such that for $n \geq n_{0,\mathrm{M}}$ and $q \leq C_\mathrm{M}\, n^{1/m_1(A)}$ Maker has a winning strategy in the $(1:q)$ Maker-Breaker $(A, \mathbf{b})$-game on $[n]$ and for $n \geq n_{0,\mathrm{B}}$ and $q \geq C_\mathrm{B}\, n^{1/m_1(A)}$ the same holds for Breaker.

We start by establishing some preliminary results regarding linear systems of equations. This will include a proof of Proposition 3.5.7 as well as a proof of how to derive Corollary 3.5.9 from Theorem 3.5.5 and Proposition 3.5.7. We then obtain Maker's strategy through an application of Theorem 1.2.8. This will be followed by Breaker's strategy, which is obtained through an application of Theorem 1.2.9.

3.6.1 Preliminaries for Linear Systems In this subsection we mostly leave the world of positional games and establish some elementary, though slightly technical, results using Linear Algebra. In particular, we have to come up with a notion of a "dense substructure" of a matrix which will be needed for the proof of Breaker's part in case the matrix contains a part which is "much denser" then the overall matrix.

We will first state the following lemma that shows that if a matrix is positive and abundant, then the corresponding homogenous system of linear equations has a proper solution. This will also be needed for the subsequent lemma.

Lemma 3.6.1. *For every positive and abundant matrix $A \in \mathbb{Z}^{r\times m}$ there exists a proper solution in the positive integers to the associated homogeneous system of equations, that is $S_0(A, \mathbf{0}) \cap \mathbb{N}^m \neq \emptyset$.*

Proof. Since A is positive, there exists $\mathbf{x} \in S(A, \mathbf{0}) \cap \mathbb{N}^m$ and so $S(A, \mathbf{0}) \cap \mathbb{N}^m \neq \emptyset$. For the sake of contradiction, let us assume that no such solution is proper. Under this assumption, we will prove the following.

Claim 3.6.2. *There exist column indices $1 \leq i \neq j \leq m$ such that $x_i = x_j$ for all $\boldsymbol{x} = (x_1, \dots, x_m) \in S(A, \mathbf{0}) \cap \mathbb{N}^m$.*

Proof of Claim 3.6.2. Suppose the claim is false, i.e. for all $1 \leq i \neq j \leq m$ there exists $\mathbf{x} = (x_1, \dots, x_m) \in S(A, \mathbf{0}) \cap \mathbb{N}^m$ with $x_i \neq x_j$, and let $\mathbf{x}^* \in S(A, \mathbf{0}) \cap \mathbb{N}^m$ be a solution that has the fewest number of pairs of equal entries. Since there are no proper solutions, there exists $1 \leq i < j \leq m$ such that $x_i^* = x_j^*$. By assumption, there exists a solution $\mathbf{y} \in S(A, \mathbf{0}) \cap \mathbb{N}^m$ with $y_i \neq y_j$. Now let

$$\mathbf{w} = \mathbf{x}^* + \alpha \mathbf{y},$$

where α is to be determined. Clearly we have $w_i \neq w_j$.
Suppose there exists $1 \leq r < s \leq m$ such that $x_r^* \neq x_s^*$ but $w_r = w_s$. Note that this forces $y_r \neq y_s$. From the definition of w, we get

$$\alpha = \frac{x_r^* - x_s^*}{y_s - y_r}.$$

Hence we can avoid all of these values in our choice of α, by choosing $\alpha \in \mathbb{N} \setminus \{\frac{x_r^*-x_s^*}{y_s-y_r} : 1 \leq r, s \leq m, y_s \neq y_r\}$ arbitrarily and so $\mathbf{w}$ has fewer equal pairs of entries than $\mathbf{x}^*$, contradicting our choice of $\mathbf{x}^*$. Thus, if $x_r^* \neq x_s^*$ then $w_r \neq w_s$.
Continuing in this way finitely many times, we end up with a proper solution to $A\mathbf{x}^T = \mathbf{0}^T$, a contradiction. □

Due to Claim 3.6.2 we can transform A into a matrix $B \in \mathbb{Z}^{r\times m}$ with $\text{rk}(B) = \text{rk}(A)$ and $S(B, 0) = S(A, 0)$ through a sequence of elementary row operations and by permuting the columns appropriately, such that B is again abundant (see Observation 3.5.3) and contains a row of the form $(0, ..., 0, 1, -1)$. However, deleting the last two columns of B decreases its rank, which yields a contradiction. □

We continue by giving two basic bounds for the number of proper solutions. Given any matrix $A \in \mathbb{Z}^{r\times m}$ and vector $\mathbf{b} \in \mathbb{Z}^r$, we remark that we have the basic upper bound

$$\left| S_0(A, \mathbf{b}) \cap [n]^m \right| \leq \left| S(A, \mathbf{b}) \cap [n]^m \right| \leq n^{m-\text{rk}(A)}. \tag{3.20}$$

Indeed, taking a subset $Q \subseteq [m]$ of the column indices with $rk(A) = |Q| = rk(A^Q)$ and setting the $m - \text{rk}(A)$ entries in $\bar{Q}$ of a solution $\mathbf{x} \in S(A, 0)$ arbitrarily, the entries in Q are determined uniquely.

The next lemma, whose proof is based on a construction by Janson and Ruciński [47], establishes a lower bound that matches the upper bound up to a constant for positive and abundant matrices and vectors $\mathbf{b}$ with $S(A, \mathbf{b}) \neq \emptyset$. This will allow us to to estimate the number of proper solutions.

Lemma 3.6.3. *For every positive and abundant matrix $A \in \mathbb{Z}^{r\times m}$ and vector $\boldsymbol{b} \in \mathbb{Z}^r$ such that $S(A, \boldsymbol{b}) \neq \emptyset$ there exist constants $c_0 = c_0(A, \boldsymbol{b}) > 0$ and $n_0 = n_0(A, \boldsymbol{b}) \in \mathbb{N}$ such that for every $n \geq n_0$*

$$| S_0(A, \boldsymbol{b}) \cap [n]^m| \geq c_0 \, n^{m-rk(A)}. \tag{3.21}$$

Proof. By Lemma 3.6.1 we can pick some $\hat{\mathbf{x}}_\mathbf{0} = (\hat{x}_{\mathbf{0}1}, \ldots, \hat{x}_{\mathbf{0}m}) \in S_0(A, \mathbf{0}) \cap \mathbb{N}^m$. Choose furthermore some $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_m) \in S(A, \mathbf{b})$ as well as some $m - \text{rk}(A)$ linearly independent solutions $\mathbf{x}_1, \ldots \mathbf{x}_{m-\text{rk}(A)} \in S(A, \mathbf{0})$. Let $\hat{s}_\mathbf{0}$ and $\hat{s}$ the maximum *absolute* entries of $\hat{\mathbf{x}}_\mathbf{0}$ and $\hat{\mathbf{x}}$ respectively and s the maximum *absolute* entry in any of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_{m-\text{rk}(A)}$. Define $a(n) = \lfloor n/(\hat{s}_\mathbf{0} + 1)\rfloor$ and set

$$S(n) = \left\{ \hat{\mathbf{x}} + a(n) \cdot \hat{\mathbf{x}}_\mathbf{0} + \sum_{i=1}^{m-\text{rk}(A)} w_i \cdot \mathbf{x}_i : w_i \in \mathbb{Z}, \, |w_i| < \frac{a(n) - 2\hat{s}}{2s(m - \text{rk}(A))} \right\} \subseteq \mathbb{Z}^m.$$

Trivially we have $S(n) \subseteq S(A, \mathbf{b})$. We will show that all solutions in $S(n)$ are proper and that $S(n) \subseteq S_0(A, \mathbf{b}) \cap [n]^m$ for sufficiently large n.

Let $\mathbf{x} = (x_1, \dots, x_m) \in S(n)$ and observe that for n large enough

$$x_i > \hat{x}_i + a(n)\,\hat{x}_{\mathbf{0}i} - (m - \mathrm{rk}(A))\,s\,\frac{a(n) - 2\hat{s}}{2s(m - \mathrm{rk}(A))} \geq -\hat{s} + a(n) - \frac{a(n)}{2} + \hat{s} \geq 1$$

as well as

$$x_i < \hat{x}_i + a(n)\,\hat{x}_{\mathbf{0}i} + (m - \mathrm{rk}(A))\,s\,\frac{a(n) - 2\hat{s}}{2s(m - \mathrm{rk}(A))} \leq \hat{s} + n\,\frac{\hat{s}_{\mathbf{0}}}{\hat{s}_{\mathbf{0}} + 1} + n\,\frac{1}{2(\hat{s}_{\mathbf{0}} + 1)} - \hat{s} \leq n$$

for every $i \in [m]$, so that $S(n) \subseteq [n]^m$ for n large enough. Now assume without loss of generality that $\hat{x}_{\mathbf{0}1} < \dots < \hat{x}_{\mathbf{0}m}$. It follows that

$$\begin{aligned} x_i &< \hat{x}_i + a(n)\,\hat{x}_{\mathbf{0}i} + (m - \mathrm{rk}(A))\,s\,\frac{a(n) - 2\hat{s}}{2s(m - \mathrm{rk}(A))} \\ &= (\hat{x}_i - \hat{s}) + a(n)\left(\hat{x}_{\mathbf{0}i} + \frac{1}{2}\right) \leq (\hat{x}_{i+1} + \hat{s}) + a(n)\left(\hat{x}_{\mathbf{0}(i+1)} - \frac{1}{2}\right) \\ &= \hat{x}_{i+1} + a(n)\,\hat{x}_{\mathbf{0}(i+1)} - (m - \mathrm{rk}(A))\,s\,\frac{a(n) - 2\hat{s}}{2s(m - \mathrm{rk}(A))} < x_{i+1} \end{aligned}$$

for every $i \in [m-1]$ so that $\mathbf{x}$ is proper and therefore $S(n) \subseteq S_0(A, \mathbf{b}) \cap [n]^m$, which of course implies that $|\,\mathcal{S}_0(A, \mathbf{b}) \cap [n]^m| \geq |S(n)|$. Lastly observe that since $\mathbf{x}_1, \dots \mathbf{x}_{m-\mathrm{rk}(A)}$ are linearly independent, $S(n)$ contains

$$\left(2\left\lfloor \frac{a(n) - 2\hat{s}}{2s(m - \mathrm{rk}(A))} \right\rfloor + 1\right)^{m-\mathrm{rk}(A)} \geq \left(\frac{1/(4\hat{s}_{\mathbf{0}} + 4)}{s\,(m - \mathrm{rk}(A))}\,n\right)^{m-\mathrm{rk}(A)}$$

elements (by choice of the w_i), where the lower bound holds for $n \geq 4\,\hat{s}\,(\hat{s}_{\mathbf{0}} + 1)$. It follows that for

$$c_0 = c_0(A, \mathbf{b}) = \left(\frac{1/(4\hat{s}_{\mathbf{0}} + 4)}{s\,(m - \mathrm{rk}(A))}\right)^{m-\mathrm{rk}(A)} < 1$$

and $n_0 = \lceil 4\,\hat{s}\,(\hat{s}_{\mathbf{0}} + 1)\rceil$ we have $|\,S_0(A, \mathbf{b}) \cap [n]^m| \geq c_0\,n^{m-\mathrm{rk}(A)}$ for all $n \geq n_0$. □

Let us now develop the notion of an *induced submatrix* originally introduced (though not explicitly referred to as such) by Rödl and Ruciński [64] (see also Definition 7.1 in [67]). Recall the definitions from Section 3.5, especially $r_Q = \mathrm{rk}(A) - \mathrm{rk}(A^{\overline{Q}})$. We also introduce the additional notation that for any matrix $A \in \mathbb{Z}^{r \times m}$ and selection of row indices $R \subseteq [r]$, we let A_R denote the matrix obtained by only keeping the rows indexed by R. Furthermore, a matrix A is called *strictly balanced* if for every non-empty proper subset $Q \subsetneq [m]$ we have that

$$\frac{|Q| - 1}{|Q| - r_Q - 1} < m_1(A). \tag{3.22}$$

The following lemma now develops the notion of an *induced submatrix* originally introduced (though not explicitly referred to as such) by Rödl and Ruciński [64] for partition regular matrices. Their proofs, adapted for the full generality of abundant matrices and the inhomogeneous case, are included here for completeness. Before stating the lemma, we quickly describe the idea of the construction.

Idea of the construction. For a fixed set of column indices $Q \subseteq [m]$ we want to maximise the number of rows whose entries in columns indexed by $\overline{Q}$ can be set to zero by a sequence of elementary row operations. In other words, for any selection of rows we want to set as many columns of them to zero as possible, again through elementary row operations.

Lemma 3.6.4. *For every matrix $A \in \mathbb{Z}^{r\times m}$ and set of column indices $Q \subseteq [m]$ satisfying $r_Q > 0$ there exists an invertible matrix $P \in \mathbb{Z}^{r\times r}$ such that the submatrix*

$$(P\cdot A)^Q_{[r_Q]} = B(P, A, Q) = B \in \mathbb{Z}^{r_Q\times|Q|} \tag{3.23}$$

is of rank r_Q while the submatrix $(P\cdot A)^{\overline{Q}}_{[r_Q]}$ is of rank 0. For every such P the following hold:

(i) We have rk$((P\cdot A)^{\overline{Q}}_{[r]\setminus[r_Q]}) =$ rk$(A) - r_Q$.

(ii) If A is abundant then B is abundant.

(iii) For any vector vector $\boldsymbol{b} \in \mathbb{Z}^r$ and any solution $\boldsymbol{x} \in S(A, \boldsymbol{b})$ we have that $\boldsymbol{x}^Q \in S(B, \mathbf{c})$, where $\mathbf{c} = \mathbf{c}(P, A, Q, \boldsymbol{b}) = (P \cdot \boldsymbol{b})_{[r_Q]} \in \mathbb{Z}^{r_Q}$. In particular, if A is positive then so is B.

(iv) For any $Q' \subseteq \{1, ..., |Q|\}$ there exists $Q'' \subseteq Q$ such that $|Q''| = |Q'|$ and $r_{Q''}(A) = r_{Q'}(B)$.

Note that there can of course exist multiple P for each A and Q satisfying these properties, but for our further endeavours we will fix one particular such $P = P(A, Q)$ and denote $B(P, A, Q)$ by $B(A, Q)$ as well as $\mathbf{c}(P, A, Q, \mathbf{b})$ by $\mathbf{c}(A, Q, \mathbf{b})$. The following block decomposition demonstrates the situation for $Q = \{1, \ldots, |Q|\}$.

$$P\cdot A = \begin{pmatrix} B & \mathbf{0} \\ X & Y \end{pmatrix} \begin{matrix}]\ r_Q \\ ]\ r - r_Q \end{matrix} \tag{3.24}$$

Proof. We will construct P by using a sequence of elementary row operations. We denote the rows of A by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r$. Among the rows $\mathbf{a}_1^{\overline{Q}}, \mathbf{a}_2^{\overline{Q}}, \ldots, \mathbf{a}_r^{\overline{Q}}$ of $A^{\overline{Q}}$ we choose rk$(A^{\overline{Q}})$ linearly independent rows and express each of the remaining $r -$ rk$(A^{\overline{Q}})$ rows as a rational linear combination of them. To get an integer linear combination we simply multiply with the least common multiple of the denominators. In fact, we will apply these elementary row operations to every full row of A, not just the parts indexed by $\overline{Q}$. By construction, this turns each entry in the $\overline{Q}$-columns of these $r -$ rk$(A^{\overline{Q}})$ rows into a 0. Hence the dimension of these rows must be $\mathrm{rk}(A) - \mathrm{rk}(A^{\overline{Q}}) = r_Q$. We now choose a set of r_Q linearly independent rows and permute them to the top of the matrix, hence obtaining the block decomposition as depicted in (3.24). Note that the rank of B is r_Q by construction.

To prove *(i)* note that $\mathrm{rk}((P \cdot A)^{\overline{Q}}_{[r]\setminus[r_Q]}) = \mathrm{rk}((P \cdot A)^{\overline{Q}}) = \mathrm{rk}(A^{\overline{Q}}) = \mathrm{rk}(A) - r_Q$ by the definition of r_Q.

To prove *(ii)* note that we now have $\mathrm{rk}(A) = \mathrm{rk}(P \cdot A) = \mathrm{rk}(B) + \mathrm{rk}(A^{\overline{Q}})$, by *(i)*, so if deleting some two columns of B decreases its rank then deleting the same columns of $P \cdot A$ decreases its rank. Hence if B is not abundant then $P \cdot A$ is not abundant and consequently A is not abundant either. Thus *(ii)* follows.

For *(iii)* note that from (3.24) it follows that for any solution $\mathbf{x} \in S(A, \mathbf{b}) = S(P \cdot A, P \cdot \mathbf{b})$ we have that $(P \cdot \mathbf{b})_{[r_Q]} = (P \cdot (A \cdot \mathbf{x}))_{[r_Q]} = (P \cdot A)_{[r_Q]} \cdot \mathbf{x} = B \cdot \mathbf{x}^Q$, since $(P \cdot A)^Q_{[r_Q]} = B$ and $(P \cdot A)^{\overline{Q}}_{[r_Q]}$ is the 0-matrix. Therefore $\mathbf{x}^Q \in S(B, \mathbf{c})$. The second statement follows by noting that $\mathbf{c}(P, A, Q, \mathbf{b}) = \mathbf{0}$ provided that $\mathbf{b} = \mathbf{0}$.

Lastly, to show *(iv)* let us without loss of generality assume that the columns are permuted in such a way that $Q = \{1, \ldots, |Q|\}$, so that we may simply choose $Q'' = Q'$. Note that by *(i)* we can extend the r_Q linearly independent rows of $(P \cdot A)_{[r_Q]}$ with $\mathrm{rk}(A^{\overline{Q}})$ linearly independent rows from $(P \cdot A)_{[r]\setminus[r_Q]}$ to a basis of the row space of A. Let $\mathbf{r}_1, \ldots, \mathbf{r}_{r_Q}, \mathbf{r}_{r_Q+1}, \ldots, \mathbf{r}_{\mathrm{rk}(A)}$ be this basis. By construction the rows $\mathbf{r}^{\overline{Q}}_{r_Q+1}, \ldots, \mathbf{r}^{\overline{Q}}_{\mathrm{rk}(A)}$ are linearly independent from each other, and so, in particular, the rows $\mathbf{r}^{\overline{Q''}}_{r_Q+1}, \ldots, \mathbf{r}^{\overline{Q''}}_{\mathrm{rk}(A)}$ are linearly independent as well, since $Q'' \subseteq Q$. Hence $\mathrm{rk}((P \cdot A)^{\overline{Q''}}_{[r]\setminus[r_Q]}) = \mathrm{rk}((P \cdot A)^{\overline{Q}})$.

Note that any linear combination of the vectors $\mathbf{r}^{\overline{Q''}}_1, \ldots, \mathbf{r}^{\overline{Q''}}_{r_Q}$ has the last $|\overline{Q}|$ entries equal to zero, by construction. Therefore they cannot be expressed as a linear combination of $\mathbf{r}^{\overline{Q''}}_{r_Q+1}, \ldots, \mathbf{r}^{\overline{Q''}}_{\mathrm{rk}(A)}$, because the rows $\mathbf{r}^{\overline{Q}}_{r_Q+1}, \ldots, \mathbf{r}^{\overline{Q}}_{\mathrm{rk}(A)}$ were linearly independent. It follows that we can add $\mathrm{rk}((P \cdot A)^{\overline{Q''}}_{[r_Q]}) = \mathrm{rk}(B^{\overline{Q'}})$ linearly independent vectors from $\mathbf{r}^{\overline{Q''}}_1, \ldots, \mathbf{r}^{\overline{Q''}}_{r_Q}$ to the $\mathrm{rk}((P \cdot A)^{\overline{Q}}) = \mathrm{rk}(A^{\overline{Q}})$ linearly independent vectors $\mathbf{r}^{\overline{Q''}}_{r_Q+1}, \ldots, \mathbf{r}^{\overline{Q''}}_{\mathrm{rk}(A)}$ to form a basis of the row space of $\mathrm{rk}(A^{\overline{Q''}})$. This implies that $\mathrm{rk}(A^{\overline{Q''}}) = \mathrm{rk}(A^{\overline{Q}}) + \mathrm{rk}(B^{\overline{Q'}})$ from which we can conclude that

$$r_{Q''}(A) = \mathrm{rk}(A) - \mathrm{rk}(A^{\overline{Q''}}) = r_Q + \mathrm{rk}(A^{\overline{Q}}) - \mathrm{rk}(A^{\overline{Q''}}) = \mathrm{rk}(B) - \mathrm{rk}(B^{\overline{Q'}}) = r_{Q'}(B)$$

as desired. □

Recall that in the proof of Breaker's part in the Maker-Breaker $\mathcal{G}$ games, we had to assume, without loss of generality, that $\mathcal{G}$ was strictly balanced. For this to be valid, it was crucial that if Breaker can keep Maker from occupying a (proper) subhypergraph, then he could also keep him from occupying $\mathcal{G}$. The following corollary to the above lemma will allow us to do the same in Breaker's strategy in the generalised van der Waerden games.

Corollary 3.6.5. *If $A \in \mathbb{Z}^{r\times m}$ is positive and abundant then there exists some non-empty set of column indices $Q \subseteq [m]$ such that $B = B(A, Q)$ is abundant, positive, strictly balanced and satisfies $m_1(B) = m_1(A)$. Furthermore, for $\boldsymbol{c} = \boldsymbol{c}(A, Q, \boldsymbol{b})$ any subset $T \subseteq \mathbb{N}$ such that $S_0(B, \boldsymbol{c}) \cap T^m = \emptyset$ also satisfies $S_0(A, \boldsymbol{b}) \cap T^m = \emptyset$.*

Proof. Choose $Q \subseteq [m]$ such that $(|Q| - 1)/(|Q| - r_Q - 1) = m_1(A)$ and $|Q|$ is minimal with this property. By *(ii)* and *(iii)* we know that B is abundant and positive. Assume for the sake of contradiction that there exists $Q' \subsetneq \{1, \dots, |Q|\}$ such that

$$\frac{|Q'| - 1}{|Q'| - r_{Q'}(B) - 1} \geq \frac{|Q| - 1}{|Q| - r_Q - 1}.$$

By *(iv)* there must exists $Q'' \subseteq [m]$ with $|Q''| = |Q'| < |Q|$ such that $r_{Q''}(A) = r_{Q'}(B)$. It follows that $(|Q''| - 1)/(|Q''| - r_{Q''}(A) - 1) \geq m_1(A)$, giving us a contradiction to our choice of Q. Finally, the last statement readily follows from *(iii)*. □

Finally, the following lemma now establishes some results regarding the rank of induced submatrices of abundant matrices. It in particular verifies that the maximum 1-density parameter given in the introduction is indeed well-defined for abundant matrices. Rödl and Ruciński [64] verified this for partition regular matrices. Here we provide a proof for abundant matrices.

Lemma 3.6.6. *For any abundant matrix $A \in \mathbb{Z}^{r\times m}$ and subset of column indices $Q \subseteq [m]$ the following holds. If $|Q| \geq 2$ then $|Q| - r_Q - 1 > 0$. If $|Q| \leq 2$ then $r_Q = 0$.*

Proof. By the previous lemma $B(A, Q)$ is abundant, has rank r_Q and the number of its columns is $|Q|$. Hence, by Observation 3.5.2 we have $|Q| \geq r_Q+2$ and therefore $|Q|-r_Q-1 > 0$.

If $|Q| \leq 2$ then, since A is abundant, deleting the columns in Q does not reduce the rank of A. Hence $\mathrm{rk}(A^{\overline{Q}}) = \mathrm{rk}(A)$ and therefore $r_Q = 0$. □

We now return to the world of games. Before proving Theorem 3.5.5 we provide the proof of Proposition 3.5.7, which follows from the previous definitions and observations, as well as a proof of Corollary 3.5.9 assuming Theorem 3.5.5.

Proof of Proposition 3.5.7. Let us start by noting that if $S(A, \mathbf{b}) = \emptyset$, then trivially the game hypergraph $\mathcal{S}_0(A, \mathbf{b}, n)$ is empty and the game trivially is an immediate win for Breaker.

Let us therefore consider the case where $S(A, \mathbf{b}) \neq \emptyset$ and A is positive and non-abundant. It follows that $(A, \mathbf{b})$, after some elementary row operations, must yield the equation $v_1x_i + v_2x_j = v_3$ for some constants $v_1, v_2, v_3 \in \mathbb{Z}$ and column indices $1 \leq i \neq j \leq m$. Note that as A is positive we must have $v_1, v_2 \neq 0$. If $\{v_1, v_2\} = \{-1, 1\}$ and $v_3 = 0$, then $S_0(A, \mathbf{b}) \cap [n]^m = \emptyset$ and the game hypergraph again is empty. If this is not the case, then whenever Maker occupies an element $y \in [n]$, Breaker can simply pick $(v_3 - v_1\, y)/v_2$ and $(v_3 - v_2\, y)/v_1$ (if these are indeed integer values in $[n]$) and thus block Maker's ability to cover any solution. It follows that Breaker has a winning strategy with a bias of at most 2. $\square$

Proof of Corollary 3.5.9. The central observation necessary to prove this corollary is that for all non-degenerate partitions p (and in fact for all non-vacant partitions) we have

$$\big\{\{x_1, \ldots, x_m\} : (x_1, \ldots, x_m) \in S(A, \mathbf{b}) \cap [n]^m \text{ and } p(\mathbf{x}) = p\big\} = \mathcal{S}_0(A_p, \mathbf{b}, n). \tag{3.25}$$

Now if A is positive and abundant then for non-degenerate p the bias threshold of the game played on $\mathcal{S}_0(A_p, \mathbf{b}, n)$ either satisfies $q(\mathcal{S}_0(A_p, \mathbf{b}, n)) \leq 2$ by Proposition 3.5.7 if A_p is non-abundant or it satisfes $q(\mathcal{S}_0(A_p, \mathbf{b}, n)) = \Theta(n^{1/m_1(A_p)}) = O(n^{1/m_1(A)})$ by Theorem 3.5.5 if A_p is abundant since we required that $m_1(A_p) \geq m_1(A)$. Noting that the number of possible non-degenerate partitions of $[m]$ is clearly bounded from above by $m!$ gives the desired result through Breaker's possibility to use strategy splitting, see Remark 3.3.3.

If A is non-abundant, then for any non-degenerate p the matrix A_p is also non-abundant and hence $q(\mathcal{S}_0(A_p, \mathbf{b}, n)) = q(\mathcal{S}_0(A, \mathbf{b}, n)) = \Theta(1)$. $\square$

3.6.2 Proof of Theorem 3.5.5 Using the notions and results from the previous section we will now show how to apply our general winning criteria to prove Theorem 3.5.5.

Proof of Theorem 3.5.5. We start by establishing some estimates of the required parameters of the sequence of m-uniform hypergraphs $\mathcal{S}_0(A, \mathbf{b}, n)$.

Recall that $v(\mathcal{S}_0(A, \mathbf{b}, n)) = n$ and observe that each edge in $\mathcal{S}_0(A, \mathbf{b}, n)$ can stem from at most $m!$ solutions in $S_0(A, \mathbf{b}) \cap [n]^m$, so that we have

$$|\, S_0(A, \mathbf{b}) \cap [n]^m|/m! \leq e(\mathcal{S}_0(A, \mathbf{b}, n)) \leq |\, S_0(A, \mathbf{b}) \cap [n]^m|. \tag{3.26}$$

Using Equation (3.20) and Lemma 3.6.3 (i.e. the lower and upper bound on $|\, S_0(A, \mathbf{b}) \cap [n]^m|$) there therefore exists a constant $c_0 = c_0(A, \mathbf{b}) > 0$ so that

$$c_0/m!\; n^{m-\mathrm{rk}(A)-1} \leq d(\mathcal{S}_0(A, \mathbf{b}, n)) \leq n^{m-\mathrm{rk}(A)-1}. \tag{3.27}$$

We also need upper bounds for the maximum ℓ-degrees in $\mathcal{S}_0(A, \mathbf{b}, n)$. For $\ell \in [m]$ we have

$$\begin{aligned}
\Delta_\ell(\mathcal{S}_0(A, \mathbf{b}, n)) &\leq \max_{(x_1,\ldots,x_\ell)\in[n]^\ell} \big|\{\mathbf{x} \in S_0(A, \mathbf{b}) \cap [n]^m : \exists\, Q \subseteq [m] \text{ s.t. } \mathbf{x}^Q = (x_1, \ldots, x_\ell)\}\big| \\
&\leq \binom{m}{\ell} \max_{\substack{(x_1,\ldots,x_\ell)\in[n]^\ell \\ Q\subseteq[m], |Q|=\ell}} \big|\{\mathbf{x} \in [n]^{m-\ell}\ : A^{\overline{Q}} \cdot \mathbf{x}^T = \mathbf{b} - A^Q \cdot (x_1, \ldots, x_\ell)^T\}\big| \\
&\leq m^\ell \max_{\substack{Q\subseteq[m] \\ |Q|=\ell}} \max_{\mathbf{b}'\in\mathbb{Z}^r} \big|\, S(A^{\overline{Q}}, \mathbf{b}') \cap [n]^m\big|
\end{aligned}$$

Using Equation (3.20) as well as the fact that $|\overline{Q}| = m - |Q|$ and $r_Q = \mathrm{rk}(A) - \mathrm{rk}(A^{\overline{Q}})$, it follows that

$$\Delta_\ell(\mathcal{S}_0(A, \mathbf{b}, n)) \leq m^\ell \max_{Q \subseteq [m],\, |Q| = \ell} n^{|\overline{Q}| - \mathrm{rk}(A^{\overline{Q}})} \leq m^\ell \max_{Q \subseteq [m],\, |Q| = \ell} n^{(m - \mathrm{rk}(A)) - (|Q| - r_Q)}. \tag{3.28}$$

We now begin with Maker's part. We will show that the sequence of m-uniform hypergraphs $\mathcal{S}_0(A, \mathbf{b}, n)$ satisfies Conditions (i), (ii) and (iii) for $k = m$ and $c_1 = c_1(A, \mathbf{b})$ as defined in Equation (3.29) below. The constants $C_{\mathrm{M}} = C_{\mathrm{M}}(A, \mathbf{b})$ and $n_{0,\mathrm{M}} = n_{0,\mathrm{M}}(A, \mathbf{b})$ will also be given later in Equation (3.30).

Let us now combine the previous observations. Since A is abundant we have $r_Q = 0$ for any $Q \subseteq [m]$ satisfying $|Q| = 1$ due to Lemma 3.6.6, so that Condition (i) easily follows for

$$c_1 = c_1(c_0, m) = m \, \frac{m!}{c_0} \tag{3.29}$$

using Equation (3.28) as well as the lower bound in Equation (3.27) since

$$\Delta_1(\mathcal{S}_0(A, \mathbf{b}, n)) \leq m \, n^{m - \mathrm{rk}(A) - 1} = c_1 \, \frac{c_0}{m!} \, n^{m - \mathrm{rk}(A) - 1} \leq c_1 \, d(\mathcal{S}_0(A, \mathbf{b}, n))\,.$$

To see that that Condition (ii) holds, let $\tilde{c} = \tilde{c}(m, c_1)$ be as given by Theorem 1.2.8. We use Equation (3.28) as well as the lower bound in Equation (3.27) to obtain

$$f(\mathcal{S}_0(A, \mathbf{b}, n)) = \min_{2 \leq \ell \leq m} \left(\frac{d(\mathcal{S}_0(A, \mathbf{b}, n))}{\Delta_\ell(\mathcal{S}_0(A, \mathbf{b}, n))} \right)^{\frac{1}{\ell - 1}} \geq \min_{\substack{Q \subseteq [m] \\ |Q| \geq 2}} \left(c_0 / (m! \, m^{|Q|}) \, n^{|Q| - r_Q - 1} \right)^{\frac{1}{|Q| - 1}} \to \infty$$

where the last step follows as $(|Q| - r_Q - 1)/(|Q| - 1) > 0$ for any $Q \subseteq [m]$ satisfying $|Q| \geq 2$ due to Lemma 3.6.6. Choose $n_0' \in \mathbb{N}$ large enough such that $f(\mathcal{S}_0(A, \mathbf{b}, n)) > 1$ holds for all $n \geq n_0'$. As $\Delta_m(\mathcal{S}_0(A, \mathbf{b}, n)) = 1$ and $(m - \mathrm{rk}(A) - 1)/(m - 1) < 1$ we have

$$f(\mathcal{S}_0(A, \mathbf{b}, n)) \leq \left(n^{m - \mathrm{rk}(A) - 1} \right)^{\frac{1}{m - 1}} = o(v(\mathcal{S}_0(A, \mathbf{b}, n))).$$

We can therefore pick some $n_0'' \in \mathbb{N}$ large enough such that

$$\frac{v(\mathcal{S}_0(A, \mathbf{b}, n))}{f(\mathcal{S}_0(A, \mathbf{b}, n))} \left(1 - \frac{1}{f(\mathcal{S}_0(A, \mathbf{b}, n))} \right) \geq \tilde{c}$$

so that Condition (iii) is satisfied for $n \geq \max(n_0', n_0'')$. Now let $c = c(m, c_1)$ be as given by Theorem 1.2.8. For

$$C_{\mathrm{M}} = \frac{c}{m^m} \, \frac{c_0}{m!} - 1 \quad \text{and} \quad n_{0,\mathrm{M}} = \max(n_0', n_0'') \tag{3.30}$$

the bias bound given by Theorem 1.2.8 can now be rewritten as

$$c \, f(\mathcal{S}_0(A, \mathbf{b}, n)) - 1 = c \min_{2 \leq \ell \leq m} \left(\frac{d(\mathcal{S}_0(A, \mathbf{b}, n))}{\Delta_\ell(\mathcal{S}_0(A, \mathbf{b}, n))} \right)^{\frac{1}{\ell - 1}} - 1$$

$$\geq c \min_{\substack{Q\subseteq[m]\\|Q|\geq 2}} \left(\frac{c_0/m!\; n^{m-\mathrm{rk}(A)-1}}{m^{|Q|}\; n^{(m-\mathrm{rk}(A))-(|Q|-r_Q)}} \right)^{\frac{1}{|Q|-1}} - 1$$

$$\geq \left(\frac{c}{m^m}\,\frac{c_0}{m!} - 1 \right) \min_{\substack{Q\subseteq[m]\\|Q|\geq 2}} n^{\frac{|Q|-r_Q-1}{|Q|-1}} = C_{\mathrm{M}}\, n^{1/m_1(A)}.$$

where we have used the lower bound given by Equation (3.27) as well as Equation (3.28). This proves Maker's part of Theorem 3.5.5.

We will now turn to Breaker's part. Let Q and the corresponding B and $\mathbf{c}$ be as given by Corollary 3.6.5. Considering the vertices occupied by Maker as the set T in this corollary, it is clear that if Breaker can keep Maker from occupying any solution in $S_0(B,\mathbf{c})$ then he can also keep Maker from occupying any solution in $S_0(A,\mathbf{b})$. For this part, we can therefore without loss of generality assume that A is not just positive and abundant, but also strictly balanced, that is $(m-1)/(m-\mathrm{rk}(A)-1) = m_1(A)$ as well as $(|Q|-1)/(|Q|-r_Q-1) < m_1(A)$ for any $Q \subsetneq [m]$ satisfying $|Q| \geq 2$.

We will apply Theorem 1.2.9 to show that there exists $n_0 \in \mathbb{N}$ and $C_1' = C_1'(m) > 0$ such that Breaker has a winning strategy in $\mathcal{S}_0(A,n)$ for $q \geq C_1'\, n^{1/m_1(A)}$ and $n \geq n_0$. It follows from Equation (3.28) that

$$\Delta_1(\mathcal{S}_0(A,\mathbf{b},n))^{\frac{1}{m-1}} \leq \left(m \max_{Q\subseteq[m],\, |Q|=1} n^{(m-\mathrm{rk}(A))-(1-r_Q)} \right)^{\frac{1}{m-1}} = m^{\frac{1}{m-1}}\, n^{\frac{m-\mathrm{rk}(A)-1}{m-1}} = m^{\frac{1}{m-1}}\, n^{1/m_1(A)}$$

since $r_Q = 0$ for any $Q \subseteq [m]$ such that $|Q| = 1$ due to Lemma 3.6.6 and since we have assumed at the beginning that $m_1(A) = (m-1)/(m-\mathrm{rk}(A)-1)$. That same assumption also states that for all $Q \subseteq [m]$ satisfying $2 \leq |Q| < m$ we have $(|Q|-1)/(|Q|-r_Q-1) < m_1(A)$ so that

$$\begin{aligned}\frac{(m-\mathrm{rk}(A))-(|Q|-r_Q)}{m-|Q|} &= \frac{(m-\mathrm{rk}(A)-1)-(|Q|-r_Q-1)}{(m-1)-(|Q|-1)}\\ &< \frac{(m-\mathrm{rk}(A)-1)-(|Q|-1)/m_1(A)}{(m-1)-(|Q|-1)}\\ &= \frac{1}{m_1(A)}\,\frac{1-(|Q|-1)/(m-1)}{1-(|Q|-1)(m-1)} = \frac{1}{m_1(A)}.\end{aligned}$$

Using this, it follows that there exists some $\epsilon = \epsilon(A) > 0$ so that Equation (3.28) gives us

$$\Delta_\ell(\mathcal{S}_0(A,\mathbf{b},n))^{\frac{1}{m-\ell}} \leq \left(m^\ell \max_{Q\subseteq[m],\, |Q|=\ell} n^{(m-\mathrm{rk}(A))-(|Q|-r_Q)} \right)^{\frac{1}{m-\ell}} \leq m^{m-1}\, n^{1/m_1(A)-2\epsilon}$$

for $2 \leq \ell \leq m-1$. We know that there exists $c_0 = c_0(A,\mathbf{b}) > 0$ such that $d(\mathcal{S}_0(A,\mathbf{b},n)) \geq c_0/m!\; n^{m-\mathrm{rk}(A)-1}$ due to Equation (3.27) and therefore $n^{1/m_1(A)} \leq m!/c_0\; d(\mathcal{S}_0(A,\mathbf{b},n))^{1/(m-1)}$. We trivially also have $\Delta_1(\mathcal{S}_0(A,\mathbf{b},n)) \geq m\, d(\mathcal{S}_0(A,\mathbf{b},n))$. Taken together it follows that

$$\begin{aligned}\Delta_\ell(\mathcal{S}_0(A,\mathbf{b},n))^{\frac{1}{m-\ell}} &\le m^{m-1}\left(\frac{m!}{c_0}\right)^{\frac{1}{m-1}} d(\mathcal{S}_0(A,\mathbf{b},n))^{\frac{1}{m-1}}\, n^{-2\epsilon}\\ &\le m^{m-1}\left(\frac{m!}{c_0\, m}\right)^{\frac{1}{m-1}} \Delta_1(\mathcal{S}_0(A,\mathbf{b},n))^{\frac{1}{m-1}}\, n^{-2\epsilon}.\end{aligned}$$

Observe that for $2 \le \ell \le m-1$ there exists $n_{0,\ell} \in \mathbb{N}$ such that due to the previous inequality we have $\Delta_\ell(\mathcal{S}_0(A,\mathbf{b},n))^{1/(m-\ell)}\, n^\epsilon \le \Delta_1(\mathcal{S}_0(A,\mathbf{b},n))^{1/(m-1)}$ for all $n \ge n_{0,\ell}$. Let $C_1 = C_1(m)$ and $v_0 = v_0(m,\epsilon)$ be as given by Theorem 1.2.9. It follows now that for $n \ge n_{0,\mathrm{B}} = \max(v_0, \max_{2\le\ell\le m-1} n_{0,\ell})$ and $C_1' = C_1'(m) = C_1\, m^{\frac{1}{m-1}}$ Breaker has a winning strategy in $\mathcal{S}_0(A,\mathbf{b},n)$ since

$$\begin{aligned}q \ge C_1'\, n^{1/m_1(A)} &\ge C_1\, \Delta_1(\mathcal{S}_0(A,\mathbf{b},n))^{\frac{1}{m-1}}\\ &= C_1 \max\left(\Delta_1(\mathcal{S}_0(A,\mathbf{b},n))^{\frac{1}{m-1}}, \max_{2\le\ell\le m-1}\left(\Delta_\ell(\mathcal{S}_0(A,\mathbf{b},n))^{\frac{1}{m-\ell}}\right) n^\epsilon\right).\end{aligned}$$

□

3.7 Concluding remarks and open problems

In this chapter, we have established general criteria for hypergraphs $\mathcal{H}$, which guarantee that Maker's random strategy is essentially optimal in the biased Maker-Breaker game on $\mathcal{H}$. We have proved that several natural games fall into this category. These included the Maker-Breaker $\mathcal{G}$ games, for any fixed uniform hypergraph $\mathcal{G}$, as well as generalized van der Waerden games for solutions of linear systems of inhomogeneous equations.

3.7.1 Obtaining constants In our main theorems we determine the right order of magnitude of the threshold biases for the games considered. Hence one might rightfully be interested in obtaining more precise statements, involving the constant factors. For the triangle-building game Chvátal and Erdős [20] established upper and lower bounds that are tight up to a constant factor $\sqrt{2}$. Their upper bound was slightly improved by Balogh and Samotij [6], however the value of the right constant factor is still unknown.

We state here an upper and a lower bound for the 3-AP game, where we already established that the threshold bias is of the order $\sqrt{n}$.

Proposition 3.7.1. *For the threshold bias $q(n)$ of the 3-AP game played on $[n]$ we have*

$$\sqrt{\frac{n}{12}-\frac{1}{6}} \le q(n) \le \sqrt{3n}.$$

Proof. Let us first prove the upper bound by providing a winning strategy for Breaker if he is given a bias of $q \geq \sqrt{3n}$. The strategy will simply consist of blocking all possible 3-APs containing Maker's last choice and one of its previous choices. As for each fixed pair of integers there are at most three 3-APs containing them and Maker occupies at most $M = \lceil n/(q+1) \rceil$ integers during the course of the whole game, the number of 3-APs to be blocked is never more than $3\,(M-1)$. Since

$$3\ (M-1) \leq q$$

for $q \geq \sqrt{3n}$, Breaker has enough moves in each round to occupy the (at most) one unoccupied element in each of the dangerous 3-APs.

For the lower bound we use Becks biased Maker win criterion Theorem 1.2.7, with $X = v(\mathcal{H})$. For the hypergraph $\mathcal{H}_n$ of 3-APs in $[n]$ we observe that $v(\mathcal{H}_n) = n$, $e(\mathcal{H}_n) \geq n^2/4 - n/2$, and $\Delta_2(\mathcal{H}_n) \leq 3$. Consequently with a bias of $q < \sqrt{\frac{n}{12} - \frac{1}{6}}$ the condition (1.4) holds for $\mathcal{H}_n$ and Theorem 1.2.7 provides the winning strategy for Maker. □

Observe that the constants $\sqrt{1/12}$ and $\sqrt{3}$ are only a factor 6 apart and it would be interesting to close this gap.

Question 3.7.2. *Determine/prove the existence of a constant $C > 0$, such that the threshold bias of the 3-AP game is $(C + o(1))\sqrt{n}$.*

It should be noted that we already applied Theorem 1.2.7 in Section 3.1.1 to the k-AP game and obtained a lower bound of the right order of magnitude on the the threshold bias for every $k \geq 3$. The ad-hoc argument for Breaker's win does not seem to generalize immediately.

Conjecture 3.7.3. *For every positive and abundant matrix $A \in \mathbb{Z}^{r\times m}$ and vector $\boldsymbol{b} \in \mathbb{Z}^r$, there exists a constant $C = C(A, \boldsymbol{b}) > 0$ such that $q(\mathcal{S}_0(A, \boldsymbol{b}, n)) = (C + o(1))\, n^{1/m_1(A)}$.*

The analogous question for graph-building games has been posed by Bednarska and Łuczak [13]. For hypergraph-building games the same question can of course also be asked.

3.7.2 A corollary for strictly balanced structures From our general winning criteria, Theorem 1.2.8 and Theorem 1.2.9, we can deduce the following corollary.

Corollary 3.7.4. *For every integer $k \geq 2$ the following holds. If $\mathcal{H} = (\mathcal{H}_n)_{n\in\mathbb{N}}$ is a sequence of k-uniform hypergraphs satisfying*

$$\text{(i)}\ \Delta_1(\mathcal{H}_n) = O(d(\mathcal{H}_n)) \qquad \text{and} \qquad \text{(ii)}\ \omega(1) = f(\mathcal{H}_n) = o(v(\mathcal{H}_n))$$

$$\text{and} \qquad \text{(iii)}\ \Delta_\ell(\mathcal{H}_n)^{\frac{1}{k-\ell}}\, v(\mathcal{H}_n)^{o(1)} \leq \Delta_1(\mathcal{H}_n)^{\frac{1}{k-1}}\ \text{ for all } 2 \leq \ell \leq k-1$$

then the threshold bias of the game on $\mathcal{H}_n$ satisfies $q(\mathcal{H}_n) = \Theta(d(\mathcal{H}_n)^{1/(k-1)}) = \Theta(f(\mathcal{H}_n))$.

Note that this can only be applied directly in case that the target hypergraph or linear system is strictly balanced, as should be clear from the proofs of Breaker's part of Theorem 1.2.2 and Theorem 3.5.5.

3.7.3 A remark about repeated entries In general one may wonder whether it is necessary to compare games with repeated solutions to the proper game hypergraph $\mathcal{S}_0(A, \mathbf{b}, n)$. In the following we will argue that this is not the case. For each family $\mathcal{P}$ of non-vacant partitions of $[n]$, one can define the game hypergraph $\mathcal{S}(A, \mathbf{b}, n, \mathcal{P})$ containing all subsets that consist of the distinct components of such a solution to $A \cdot \mathbf{x}^T = \mathbf{b}^T$ for which $p(x) \in \mathcal{P}$, that is

$$\mathcal{S}(A, \mathbf{b}, n, \mathcal{P}) = \{\{x_1, \ldots, x_m\} : (x_1, \ldots, x_m) = \mathbf{x} \in S(A, \mathbf{b}) \cap [n]^m \text{ and } p(\mathbf{x}) \in \mathcal{P}\}. \quad (3.31)$$

In Theorem 3.5.5 we considered the game where only proper solutions were allowed. This of course deals with the case where $\mathcal{P}$ consists just of the partition $\{\{1\}, \ldots, \{m\}\}$. In Corollary 3.5.9 we studied the game where non-degenerate partitions were allowed, which corresponds to the case where $\mathcal{P} = \{p(\mathbf{x}) : \mathbf{x} \in S_1(A, \mathbf{b})\}$ consists of all non-degenerate partitions.

Considering the proof of Corollary 3.5.9 it should be clear that playing on the hypergraph of all solutions to A with repetitions indicated by some given partition p is the same as playing on the hypergraph $\mathcal{S}_0(A_p, \mathbf{b})$. The following statement follows immediately and can therefore be seen both as a generalisation of but also an easy corollary to Theorem 3.5.5.

Theorem 3.7.5. *For every matrix $A \in \mathbb{Z}^{r \times m}$, vector $\boldsymbol{b} \in \mathbb{Z}^r$ and family $\mathcal{P}$ of set partitions of $[m]$ the corresponding threshold bias satisfies $q(\mathcal{S}(A, \boldsymbol{b}, n, \mathcal{P})) = \Theta\big(\max_{p \in \mathcal{P}} q(A_p, \boldsymbol{b}, n)\big)$.*

This of course implies that if there exists $p \in \mathcal{P}$ such that A_p is positive and abundant and $p' \in \mathcal{P}$ is the one of those partitions that minimises the parameter $m_1(A_{p'})$, then we have $q(\mathcal{S}(A, \mathbf{b}, n, \mathcal{P})) = \Theta\big(n^{1/m_1(A_{p'})}\big)$. This result also establishes that the notion of non-degeneracy as defined in the introduction is the broadest possible notion that does not change the character of the game.

3.7.4 The probabilistic intuition Recall from the introduction that it was Chvatál and Erdős who first pointed out some surprising similarities between certain positional games and results in random graphs. Given some hypergraph $\mathcal{H}$ let the *appearance threshold* $p(\mathcal{H})$ be the threshold probability for the property that the random set $V(\mathcal{H})_p$ contains an edge. The *probabilistic intuition* states that the appearance threshold $p(\mathcal{H})$ hints at the bias threshold $q(\mathcal{H})$ of the game $\mathbf{G}(\mathcal{H}; q)$, namely that $q(\mathcal{H}) \sim p(\mathcal{H})^{-1}$. This intuition holds true for several 'global' properties such as hamiltonicity or connectivity. We have previously remarked that the biased Erdős-Selfridge strategy states that Breaker can do at least as well as when both players act randomly. Despite strong interest, a matching counterpart for Maker has never been found.

As already discussed in the introduction, for the games studied by Bednarska and Łuczak [13] as well as the two types of Maker-Breaker games studied in this thesis, 'this' probabilistic intuition fails. The appearance threshold for a given fixed r-uniform hypergraph $\mathcal{G}$ in the r-uniform random graph $G^{(r)}_{n,p}$ occurs around $n^{-1/m(\mathcal{G})}$ where $m(\mathcal{G}) \neq m_r(\mathcal{G})$ is the density of $\mathcal{G}$ maximized over all subgraphs. Likewise, the appearance threshold of solutions to a given positive and abundant matrix $A \in \mathbb{Z}^{r\times m}$ occurs around $n^{-1/m(A)}$ where $m(A) \neq m_1(A)$ is a parameter of A maximized over all subsystems, see Rué et al. [67]. As an example, k-term arithmetic progressions start to appear around $n^{-2/k}$ whereas we have shown the threshold bias to satisfy $n^{1/(k-1)}$.

However, the random approach to the proof of Maker's criterion indicates that a different type of random intuition still plays an important role. Indeed, our results show a strong connection to sparse Turán- and Szemerédi-type statements. Given an r-uniform hypergraph $\mathcal{G}$, let $\mathrm{ex}(n, \mathcal{G})$ be the largest number of edges in a $\mathcal{G}$-free subgraph of $K^{(r)}_n$ and define the *Turán density*

$$\pi_r(\mathcal{G}) = \lim_{n\to\infty} \mathrm{ex}(n, \mathcal{G}) / \binom{n}{r}.$$

For $\epsilon > 0$ an r-uniform hypergraph $\mathcal{F}$ is called $(\mathcal{G}, \epsilon)$-*Turán* if every subgraph of $\mathcal{F}$ with at least $(\pi_r(\mathcal{G}) + \epsilon)\, e(\mathcal{F})$ edges contains a copy of $\mathcal{G}$. Conlon and Gowers (in the strictly balanced case) [22] and independently Schacht [70] showed that the threshold probability of the event that $G^{(r)}_{n,p}$ is $(\mathcal{G}, \epsilon)$-Turán is $\Theta(n^{-1/m_r(\mathcal{F})})$. Compare this to our result that the threshold bias of the Maker-Breaker $\mathcal{G}$-game is $\Theta(n^{1/m_r(\mathcal{G})})$. Similarly, Schacht [70] showed that for a given density regular matrix $A \in \mathbb{Z}^{r\times m}$ the threshold probability for the event that $[n]_p$ is (δ, A)-stable is $\Theta(n^{-1/m_1(A)})$. Spiegel [74] as well as independently Hancock, Staden and Treglown [41] extended this result to abundant matrices. Our result shows that the threshold bias of the Maker-Breaker A-game lies around $\Theta(n^{1/m_1(A)})$ for the much broader class of positive and abundant matrices. We call the intuition one might infer from this *probabilistic Turán intuition* for biased Maker-Breaker games. We have proven two criteria for Breaker as well as Maker that provide some criteria to verify if this intuition indeed holds true for a given game.

3.8 Appendix to Chapter 3

Recall that the *median* $\mu_{1/2}$ of a discrete random variable X satisfies $\mathbb{P}\left(X \leq \mu_{1/2}\right) \geq 1/2$ and $\mathbb{P}\left(X \geq \mu_{1/2}\right) \geq 1/2$. Furthermore recall that any median of the binomial distribution $\mathcal{B}(n,p)$ lies between $\lfloor np \rfloor$ and $\lceil np \rceil$, that is $\lfloor np \rfloor \leq \mu_{1/2}\left(\mathcal{B}(n,p)\right) \leq \lceil np \rceil$. A family of subsets $\mathcal{P} \subseteq 2^{[n]}$ is called *monotone decreasing* if $A \subseteq B$ and $B \in \mathcal{P}$ implies $A \in \mathcal{P}$. It is called *monotone increasing* if its complement in $2^{[n]}$ is monotone decreasing. As usual one identifies properties of subsets of $[n]$ with the corresponding family of subsets having the property. The purpose of this appendix is to prove the following lemma:

Lemma 3.8.1. *Let $X \sim \mathcal{B}(n,p)$ and let $\mathcal{P}$ be a monotone decreasing family of subsets of $[n]$. Then there exists a constant $C > 0$ such that if $\sqrt{np(1-p)} > C$, then*

$$\mathbb{P}\left([n]_{\lfloor np \rfloor} \in \mathcal{P}\right) \leq 3\,\mathbb{P}\left([n]_p \in \mathcal{P}\right).$$

Proof. Note that as a consequence of the (local) LYM inequality (see below), we have that

$$\mathbb{P}\left([n]_K \in \mathcal{P}\right) \geq \mathbb{P}\left([n]_L \in \mathcal{P}\right), \tag{3.32}$$

whenever $K \leq L$, since $\mathcal{P}$ is monotone decreasing. Thus

$$\begin{aligned}\mathbb{P}\left([n]_p \in \mathcal{P}\right) &= \sum_{M=0}^{n} \mathbb{P}\left([n]_p \in \mathcal{P} \mid |[n]_p| = M\right) \mathbb{P}\left(|[n]_p| = M\right) = \sum_{M=0}^{n} \mathbb{P}\left([n]_M \in \mathcal{P}\right) \mathbb{P}\left(|[n]_p| = M\right) \\ &\geq \sum_{M=0}^{\lfloor np \rfloor} \mathbb{P}\left([n]_M \in \mathcal{P}\right) \mathbb{P}\left(|[n]_p| = M\right) \geq \mathbb{P}\left([n]_{\lfloor np \rfloor} \in \mathcal{P}\right) \sum_{M=0}^{\lfloor np \rfloor} \mathbb{P}\left(|[n]_p| = M\right).\end{aligned}$$

Note that $\sum_{M=0}^{\lfloor np \rfloor} \mathbb{P}\left(|[n]_p| = M\right) = \mathbb{P}\left(X \leq \lfloor np \rfloor\right)$. Let $\mu_{1/2}$ be the median of X and assume first that $\lfloor np \rfloor \leq \mu_{1/2} < \lceil np \rceil$. Then $\mathbb{P}\left(X \leq \lfloor np \rfloor\right) = \mathbb{P}\left(X \leq \mu_{1/2}\right) \geq \frac{1}{2}$ and hence

$$\mathbb{P}([n]_{\lfloor np \rfloor} \in \mathcal{P}) \leq 2\,\mathbb{P}\left([n]_p \in \mathcal{P}\right).$$

It remains to be shown that the assertion follows as well if $\mu_{1/2} = \lceil np \rceil$. Note that

$$\mathbb{P}\left(X \leq \lfloor np \rfloor\right) = \mathbb{P}\left(X \leq \lceil np \rceil\right) - \mathbb{P}\left(X = \lceil np \rceil\right) \geq \frac{1}{2} - \mathbb{P}\left(X = \lceil np \rceil\right).$$

We will show that $\mathbb{P}\left(X = \lceil np \rceil\right) \leq 1/6$ which then implies $\mathbb{P}\left([n]_{\lfloor np \rfloor} \in \mathcal{P}\right) \leq 3\,\mathbb{P}\left([n]_p \in \mathcal{P}\right)$. To do so, we will upper bound the probability that $X = \lceil np \rceil$ and use the inequalities $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e$ as follows:

$$\mathbb{P}\left(X = \lceil np \rceil\right) = \binom{n}{\lceil np \rceil} p^{\lceil np \rceil}\,(1-p)^{n-\lceil np \rceil} = \frac{n!\,p^{\lceil np \rceil}\,(1-p)^{n-\lceil np \rceil}}{\lceil np \rceil!(n-\lceil np \rceil)!}$$

$$\leq \frac{\sqrt{n}\, n^n\, e\, p^{\lceil np \rceil}\, (1-p)^{n-\lceil np \rceil}}{\sqrt{2\pi \lceil np \rceil}\, (\lceil np \rceil)^{\lceil np \rceil}\, \sqrt{n-\lceil np \rceil}\, (n-\lceil np \rceil)^{n-\lceil np \rceil}}$$
$$= \frac{\sqrt{n}}{\sqrt{n-\lceil np \rceil}} \frac{(np)^{\lceil np \rceil}}{\lceil np \rceil^{\lceil np \rceil}} \frac{(n-np)^{n-\lceil np \rceil}}{(n-\lceil np \rceil)^{n-\lceil np \rceil}} \frac{e}{\sqrt{2\pi \lceil np \rceil}},$$

Clearly we have $(np)^{\lceil np \rceil}/\lceil np \rceil^{\lceil np \rceil} \leq 1$ as well as $(n-np)^{n-\lceil np \rceil}/(n-\lceil np \rceil)^{n-\lceil np \rceil} \leq e$. Hence we get

$$\mathbb{P}\left(X \leq \lceil np \rceil\right) \leq \frac{e^2}{\sqrt{2\pi}} \sqrt{\frac{n}{n-np-1}\frac{1}{np}} \leq \frac{3}{\sqrt{(1-p)np-p}} < \frac{3}{\sqrt{C^2-1}}.$$

Choosing $C > 0$ large enough such that $\mathbb{P}\left(X \leq \lceil np \rceil\right) \leq 1/6$ gives the desired property. □

To show that Equation 3.32 is valid, we will need the well-known (local) LYM inequality, which was proven independently by Lubell [56], Yamamoto [76] and Meshalkin [57]. Given a set system $\mathcal{A} \subseteq \binom{X}{r}$, where X is some set of size n, define its *shadow* as

$$\partial\mathcal{A} = \left\{ B \in \binom{X}{r-1} : B \subseteq A \text{ for some } A \in \mathcal{A} \right\}.$$

The local LYM inequality states that the proportion of elements of the shadow $\partial\mathcal{A}$ of a family $\mathcal{A} \subseteq \binom{X}{r}$ within $\binom{X}{r-1}$ is at least as big as the proportion of elements of $\mathcal{A}$ within $\binom{X}{r}$.

Theorem 3.8.2 (Local LYM)**.** *Let* $0 < r \leq n$, $\mathcal{A} \subseteq \binom{X}{r}$. *Then*

$$\frac{|\partial\mathcal{A}|}{\binom{n}{r-1}} \geq \frac{|\mathcal{A}|}{\binom{n}{r}}.$$

Equation 3.32 now readily follows from the local LYM inequality and the fact that $\mathcal{P}$ is monotone decreasing.

Chapter 4

Hypergraphs with Property O

This chapter is organised as follows. Firstly, we show that $7 \leq f(3) \leq 10$. The proof of the upper bound will then be generalised to prove that $f(k) \leq \left(\lfloor \frac{k}{2} \rfloor + 1\right) k! - \lfloor \frac{k}{2} \rfloor (k-1)!$, i.e. Theorem 1.3.3. This will be followed by constructing two oriented 3-graphs on 6 vertices that have property O. Finally, we discuss some open problems.

4.1 Proof of the upper bound

To motivate the construction, we will first consider the case $k = 3$. It nicely highlights the idea of the general construction while notationally being much more oversee-able.

Claim 4.1.1. *There exists an oriented* 3*-graph with* 10 *edges having* Property O*, i.e.* $f(3) \leq 10$*. Furthermore* $f(3) \geq 7$*.*

The lower bound $f(3) \geq 7$ was already mentioned in [24]. We will include its simple proof for the convenience of the reader. It is worth noting that this is the only non-trivial case in which a lower bound of the form $k! + 1$ is known.

Before proving this claim, let us describe the idea of the proof of the upper bound. We start by defining two edges (x, y, a) and (y, x, b). Any ordering is consistent with the relative order of exactly one of these edges with respect to the positions of x and y. If it happens to be $x < y$, but the edge (x, y, a) is not consistent with the ordering, then there are two possibilities for the position of a with respect to both x and y. For each possibility we introduce one new vertex and two edges, such that at least one of them is consistent with the ordering. Below are the details.

Proof of Claim 4.1.1. We will first show that $f(3) \leq 10$. To do so, let $\mathcal{H}$ be an oriented 3-graph with vertex set $V = \{x, y, a, b, c, d, e, f\}$, and edge set

$$\mathcal{E} = \{(x, y, a), (a, x, c), (c, x, y), (x, a, d), (d, a, y), (y, x, b), (b, y, e), (e, y, x), (y, b, f), (f, b, x)\}$$

We have to show that $\mathcal{H}$ has Property O. Let $<$ be an arbitrary ordering of V. Since either $x < y$ or $y < x$ at most one of the edges (x, y, a) and (y, x, b) can be consistent with $<$. Let us first assume $x < y$. If (x, y, a) is not consistent with $<$, then we either have $a < x < y$ or $x < a < y$. If $a < x < y$, then at least one of (a, x, c) or (c, x, y) is $<$-consistent. On the other hand, if $x < a < y$ then at least one of (x, a, d) or (d, a, y) is $<$-consistent.
Now, if $y < x$ but (y, x, b) is not $<$-consistent, then either $b < y < x$ or $y < b < x$. In the first case at least one of (b, y, e) or (e, y, x) is $<$-consistent and, in the latter at least one of (y, b, f) or (f, b, x) is $<$-consistent. Hence $\mathcal{H}$ has Property O, proving $f(3) \leq 10$.

We will now prove the lower bound. Suppose for a contradiction that $\mathcal{H} = (V, \mathcal{E})$ is a 3-graph with 6 edges on n vertices that has Property O. As already mentioned in the introduction, every edge is consistent with $\frac{n!}{3!}$ linear orders and so $|\mathcal{E}| \cdot \frac{n!}{3!} = n!$. This means that no collection of two edges is consistent with the same linear order. This forces every pair of edges of $\mathcal{H}$ to intersect in exactly two vertices: if there were a pair of edges $\overline{e_1}, \overline{e_2}$ with $|e_1 \cap e_2| \leq 1$, then clearly these two edges are both consistent with at least 2 linear orders. As we have at least 6 vertices, this means that $\mathcal{H}$ is a sunflower with core x, y, say, and at least 3 pedals. Therefore there is a pair of edges for which the relative order of x, y is equal. It is not hard to see that these two edges are consistent with at least 2 linear orders giving the desired contradiction. Hence $f(3) \geq 7 = 3! + 1$. □

Remark 4.1.2. *To simplify the notation of the generalisation, we used more vertices than actually needed. Indeed the number of vertices in the above construction can be reduced (see Section 4.2).*

Before proving Theorem 1.3.3 we will introduce some notation. Given a finite set V, set $\mathbf{x} := \{x_1, ..., x_{k-1}\} \in V^{k-1}$. Enumerate all permutations of $[k-1]$ by $\pi_1 = \mathrm{id}, ..., \pi_{(k-1)!}$ arbitrarily. Furthermore, for $j \in [(k-1)!]$ and $i \in [k-1]$, write $\pi_j(\mathbf{x}, i(y))$ to denote the k-tuple arising from the $(k-1)$-tuple $\pi_j(\mathbf{x})$ by putting y between $\pi_j(x_{i-1})$ and $\pi_j(x_i)$ (when we write "between $\pi_j(x_0)$ and $\pi_j(x_1)$", we mean "before $\pi_j(x_1)$").

The construction is a fairly straightforward generalization of the $k = 3$ construction. We start with $(k-1)!$ edges $(\pi_1(\mathbf{x}), a_1), ..., (\pi_{(k-1)!}(\mathbf{x}), a_{(k-1)!})$. Any permutation will be consistent with the relative order of the x_i for exactly one of the above k-tuples. Say it is with $\pi_1 = \mathrm{id}$ (in any other case we proceed in the same way), but the k-tuple $(\pi_1(\mathbf{x}), a_1)$ is not π_1-consistent. Then there are $(k-1)$ places for a_1 and for each possibility we introduce one new vertex and $\lfloor \frac{k}{2} \rfloor + 1$ edges such that at least one of them will be π_1-consistent.

Proof of Theorem 1.3.3. We begin with the construction of the desired hypergraph $\mathcal{H} = (V, \mathcal{E})$. Let the set of vertices be

$$V = \left\{ x_1, x_2 ..., x_{k-1}, a_1, ..., a_{(k-1)!}, a_1^{(1)}, a_1^{(2)}, ..., a_1^{(k-1)}, ..., a_{(k-1)!}^{(1)}, a_{(k-1)!}^{(2)}, ..., a_{(k-1)!}^{(k-1)} \right\}.$$

and let $\mathcal{E}$ be the set of ordered hyperedges (k-tuples) constructed as follows (compare with the example given below):

(1) Start with the $(k-1)!$ k-tuples of the form $(\pi_j(\mathbf{x}), a_j)$, for $j = 1, 2, ..., (k-1)!$, and put them into $\mathcal{E}$.

(2) For each fixed $j \in [(k-1)!]$ put the following $(k-1)\left(\lfloor \frac{k}{2} \rfloor + 1\right)$ k-tuples into $\mathcal{E}$: For every $i \in [k-1]$, consider the k-tuple $\pi_j(\mathbf{x}, i(a_j))$.

Now, for every odd position $l \in [k-1]$, replace the l-th element of $\pi_j(\mathbf{x}, i(a_j))$ by $a_i^{(j)}$ and put it into $\mathcal{E}$. Also, replace the k-th element with $a_i^{(j)}$ and put the resulting k-tuple into $\mathcal{E}$.

To prove that $\mathcal{H}$ has Property O, one proceeds precisely as in the proof of Claim 4.1.1. To finish the proof of Theorem 1.3.3, note that, using the description just above the proof, we have

$$|\mathcal{E}| = \left((k-1)\left(\lfloor \tfrac{k}{2} \rfloor + 1\right) + 1\right)(k-1)! = \left(\lfloor \tfrac{k}{2} \rfloor + 1\right) k! - \lfloor \tfrac{k}{2} \rfloor (k-1)!,$$

completing the proof. □

Example: Let us illustrate the construction in the proof of Theorem 1.3.3 for $k = 4$. We start with the edges $(\pi_j(\mathrm{x}), a_j)$ for $j = 1, 2, ..., 6$. Now suppose $j = 1$, i.e. $\pi_j = \mathrm{id}$ and define the following edges:

$\underline{i = 1}$ we have (a_1, x_1, x_2, x_3) and so we put

$$(a_1^{(1)}, x_1, x_2, x_3) \text{ and } (a_1, x_1, a_1^{(1)}, x_3) \text{ and } (a_1, x_1, x_2, a_1^{(1)}) \text{ into } \mathcal{E};$$

$\underline{i = 2}$ we have (x_1, a_1, x_2, x_3) and so we put

$$(a_2^{(1)}, a_1, x_2, x_3) \text{ and } (x_1, a_1, a_2^{(1)}, x_3) \text{ and } (x_1, a_1, x_2, a_2^{(1)}) \text{ into } \mathcal{E};$$

$\underline{i = 3}$ we have (x_1, x_2, a_1, x_3) and so we put

$$(a_3^{(1)}, x_2, a_1, x_3) \text{ and } (x_1, x_2, a_3^{(1)}, x_3) \text{ and } (x_1, x_2, a_1, a_3^{(1)}) \text{ into } \mathcal{E}.$$

For $j = 2, ..., 6$ one proceeds similarly.

Remark 4.1.3. *Note that the number of vertices in the construction above is $k! + k - 1$.*

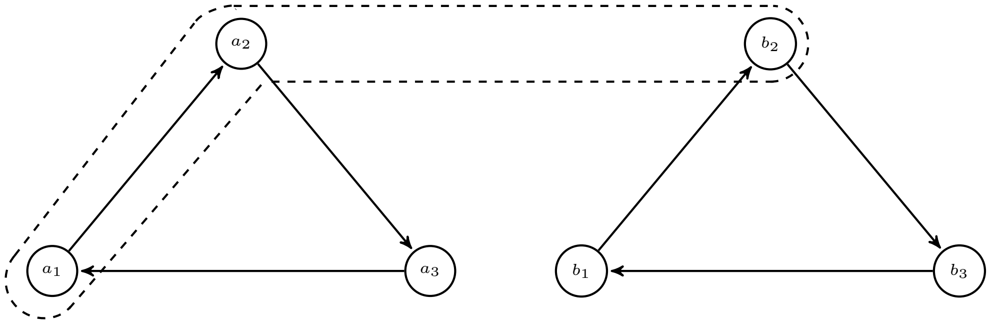

Figure 4.1: The 3-graph $\mathcal{H}$ with the edge (a_1, a_2, b_2) depicted.

4.2 3-uniform hypergraphs on 6 vertices having property O

In this short section we will construct two oriented 3-uniform hypergraphs on 6 vertices having Property O. Combined with the lower bound given in [24], this shows $n(3) = 6$.
The first one is obtained from a graph having Property O. Firstly, recall that a cyclicly ordered triangle is a graph having Property O.
Now take two disjoint copies of an oriented triangle, say $(a_0, a_1), (a_1, a_2)(a_2, a_0)$ and (b_0, b_1), $(b_1, b_2), (b_2, b_0)$ and define the following oriented edge set (where we take the indices mod 3):

$$\mathcal{E}_0 = \{(a_i, a_{i+1}, b_j), (b_i, b_{i+1}, a_j) : 0 \leq i, j \leq 2\}.$$

Claim 4.2.1. *The oriented 3-graph $\mathcal{H}$ with vertex set $V = \{a_0, a_1, a_2, b_0, b_1, b_2\}$ and edge set $\mathcal{E}_0$ has Property O. Thus $n(3) \leq 6$.*

Proof. Let π be an arbitrary ordering of V. We have to show that there is an edge $\overline{e} \in \mathcal{E}_0$ that is π-consistent.
Now, if there is some b_i such that b_i is greater than every a_j (with respect to π), then, since the 2-graph induced by the vertices a_0, a_1, a_2 has Property O, some edge of the form (a_j, a_{j+1}, b_i) is π-consistent.
If not, then there exists an a_i that is greater than every b_j. So by symmetry, the same argument as above shows that there exists an edge of the form (b_j, b_{j+1}, a_i) that is π-consistent. Hence $n(3) \leq 6$. □

The second example is a simple modification of the construction given in Section 4.1. Indeed, instead of using the vertices e and f, we could have used c and d again: Simply replace e by d and f by c. So we get $V = \{x, y, a, b, c, d\}$ and

$$\mathcal{E} = \{(x, y, a), (a, x, c), (c, x, y), (x, a, d), (d, a, y), (y, x, b), (b, y, d), (d, y, x), (y, b, c), (c, b, x)\}.$$

One can use the same proof to show that this oriented 3-graph has Property O. Similar modifications can be made in the construction in Theorem 1.3.3 to lower the number of vertices. However this is slightly more tedious and not the aim of the construction.

4.3 Concluding Remarks and open problems

In this chapter we showed $f(k) \leq \left(\lfloor \frac{k}{2} \rfloor + 1\right) k! - \lfloor \frac{k}{2} \rfloor (k-1)!$ for every $k \geq 3$, thus improving the bound of [24] by a $k \ln k$ factor. The main problem regarding hypergraphs having Property O is the following:

Problem 4.3.1. *Is it true that* $\frac{f(k)}{k!} \to \infty$ *as* $k \to \infty$*?*

In Section 2 we saw that $f(3) \geq 3! + 1$. Improving the lower bound seems to be the main task. A first step would be to answer the following question.

Problem 4.3.2. *Is it true that* $f(k) \geq k! + 1$ *for every* $k \geq 3$*?*

We believe that the answer should be yes. Of course, an improvement of the upper bound would be interesting as well.

Another natural question is whether there always exists an oriented k-graph $\mathcal{H}$ that has both the minimum possible number of edges and vertices.

Problem 4.3.3. *Let* $k \geq 3$*. Is there a* k*-uniform hypergraph with* $n(k)$ *vertices and* $f(k)$ *edges having* Property O*?*

The second construction in Section 4.2 has fairly few edges, namely 10, and $n(3) = 6$ vertices.

Note that a trivial lower bound on $n(k)$ is $\left(\frac{k}{e}\right)^2$ (for every $k \geq 2$), since the number of edges is larger than $k!$ and smaller than $\binom{n}{k}$. On the other hand, Duffus et al. [24, pp. 3–4] showed that a k-tournament $\mathcal{T}_{n,k}$ on $n = \left(\frac{k}{e}\right)^2 (\pi \cdot \exp(e^2/2) \cdot k^3 \ln k)^{1/k}$ vertices with all the $\binom{n}{k}$ edges ordered randomly has Property O with positive probability. Hence

$$n(k) \leq \left(\frac{k}{e}\right)^2 (\pi \cdot \exp(e^2/2) \cdot k^3 \ln k)^{1/k} = (1 + o(1)) \left(\frac{k}{e}\right)^2,$$

and so $n(k) = (1 + o(1)) \left(\frac{k}{e}\right)^2$.

Chapter 5

Shattering extremal families

This chapter is organised as follows. We start by proving Proposition 1.4.8. We then motivate our new approach to the elimination conjecture and prove Theorem 1.4.9. This will be followed by introducing an equivalent version of the elimination conjecture which we use to prove the conjecture for small Sperner families. The last section is dedicated to the study of the connection between s-extremal families and Gröbner bases.

In order to simplify notation, set

$$\mathcal{H}(\mathcal{S}) = 2^{[n]} \setminus \mathrm{Up}(\mathcal{S}) = 2^{[n]} \setminus \bigcup_{S \in \mathcal{S}} \mathcal{P}_S,$$

and recall that

$$\mathcal{F}(\mathcal{S}, h) = 2^{[n]} \setminus \bigcup_{S \in \mathcal{S}} \mathcal{Q}_{S,h(S)}$$

Instead of $\mathcal{F}(\mathcal{S}, h)$, we sometimes simply write $\mathcal{F}$ when there is no danger of confusion. With this notation, Proposition 1.4.8 states the following:

Proposition 5.0.1. *Let $\mathcal{S} \subseteq 2^{[n]}$ be a Sperner family and let $h : \mathcal{S} \to 2^{[n]}$ be a function such that $h(S) \subseteq S$ for every $S \in \mathcal{S}$. Then $\mathcal{F} = \mathcal{F}(\mathcal{S}, h)$ is s-extremal with $Sh(\mathcal{F}) = \mathcal{H}(\mathcal{S})$ if and only if*

$$|\mathcal{H}(\mathcal{S})| = |\mathcal{F}(\mathcal{S}, h)|. \tag{5.1}$$

Proof. Suppose $\mathcal{F} = \mathcal{F}(\mathcal{S}, h)$ is s-extremal with $\mathrm{Sh}(\mathcal{F}) = \mathcal{H}(\mathcal{S})$. Then, since $\mathrm{Sh}(\mathcal{F}) = \mathcal{H}(\mathcal{S})$, we have $|\mathcal{F}| = |\mathrm{Sh}(\mathcal{F})| = |\mathcal{H}(\mathcal{S})|$ as claimed.

To see the other direction, suppose that $|\mathcal{H}(\mathcal{S})| = |\mathcal{F}(\mathcal{S}, h)|$. Note that by definition of $\mathcal{F}(\mathcal{S}, h)$, for every $S \in \mathcal{S}$ there does not exist $F \in \mathcal{F}$ such that $F \cap S = h(S)$ and so $S \notin \mathrm{Sh}(\mathcal{F})$. In particular, no superset of S is shattered by $\mathcal{F}$. Therefore, $\mathrm{Sh}(\mathcal{F}) \subseteq 2^{[n]} \setminus \mathrm{Up}(\mathcal{S}) = \mathcal{H}(\mathcal{S})$. Hence $|\mathrm{Sh}(\mathcal{F})| \leq |\mathcal{H}(\mathcal{S})| = |\mathcal{F}|$. However, the reverse inequality always holds by the Sauer-Shelah lemma and so $\mathcal{F}$ is s-extremal and $\mathrm{Sh}(\mathcal{F}) = \mathcal{H}(\mathcal{S})$ as claimed. □

Although this result is rather easy to state and prove, it does offer a new perspective to s-extremal set systems because it allows one to construct an s-extremal set system from a Sperner family with an appropriately defined function h. Given a Sperner family and a function h, one checks whether Equation (5.1) is satisfied and if it is, the resulting set system is s-extremal.

5.1 An approach to the elimination conjecture

In order to further justify our approach given by Proposition 5.0.1, we remark that it has a connection to the following generalisation of the Sauer inequality which was implicitly proved in the proof of Proposition 5.0.1. To emphasise it we shortly repeat the argument.

Proposition 5.1.1 (Generalised Sauer Inequality). *Let $\mathcal{S} \subseteq 2^{[n]}$ be a Sperner family and $\mathcal{F} \subseteq 2^{[n]}$ a set system that shatters no element of $\mathcal{S}$. Then*

$$|\mathcal{F}| \leq |\mathcal{H}(\mathcal{S})|.$$

Proof. For the proof just note that if $\mathcal{F}$ shatters no element of $\mathcal{S}$, then it shatters no set from $\mathrm{Up}(\mathcal{S})$ either, and so $\mathrm{Sh}(\mathcal{F}) \subseteq 2^{[n]} \setminus \mathrm{Up}(\mathcal{S})$. Accordingly, using the Sauer-Shelah Lemma, we get

$$|\mathcal{F}| \leq |\mathrm{Sh}(\mathcal{F})| \leq |2^{[n]} \setminus \mathrm{Up}(\mathcal{S})| = |\mathcal{H}(\mathcal{S})|$$

as wanted. □

For a Sperner family $\mathcal{S} \subseteq 2^{[n]}$ let us define a family $\mathcal{F} \subseteq 2^{[n]}$ shattering no element of $\mathcal{S}$ and satisfying $|\mathcal{F}| = |\mathcal{H}(\mathcal{S})|$ to be *$\mathcal{S}$-extremal.* Note that the original Sauer inequality can be recovered by setting $\mathcal{S} = \binom{[n]}{k}$, and $\binom{[n]}{k}$-extremal families are just the maximum classes. An interesting property here is that if we let $\mathcal{S}$ to vary, then we end up with s-extremality.

Proposition 5.1.2. *$\mathcal{F} \subseteq 2^{[n]}$ is s-extremal if and only if there exists a Sperner family $\mathcal{S}$ such that $\mathcal{F}$ is $\mathcal{S}$-extremal.*

Proof. First suppose that $\mathcal{F} \subseteq 2^{[n]}$ is s-extremal, i.e $|\mathcal{F}| = |\mathrm{Sh}(\mathcal{F})|$. As already discussed in the introduction, we know that if we let $\mathcal{S}$ to be the collection of all minimal sets not shattered by $\mathcal{F}$, then $\mathcal{S}$ is Sperner and $\mathrm{Sh}(\mathcal{F}) = 2^{[n]} \setminus \mathrm{Up}(\mathcal{S}) = \mathcal{H}(\mathcal{S})$. This implies $|\mathcal{F}| = |\mathcal{H}(\mathcal{S})|$ which together with the fact that the elements of $\mathcal{S}$ are not shattered gives that $\mathcal{F}$ is $\mathcal{S}$-extremal.

Now suppose that $\mathcal{F}$ is $\mathcal{S}$-extremal for some Sperner family $\mathcal{S}$, i.e. $\mathcal{F}$ shatters no element of $\mathcal{S}$ and $|\mathcal{F}| = |\mathcal{H}(\mathcal{S})|$. From the proof of Proposition 5.1.1 it follows that this is possible only if $\mathrm{Sh}(\mathcal{F}) = \mathcal{H}(\mathcal{S})$. However this means that $|\mathrm{Sh}(\mathcal{F})| = |\mathcal{H}(\mathcal{S})| = |\mathcal{F}|$ and so $\mathcal{F}$ is s-extremal. □

5.1.1 Analysing the equation $|\mathcal{H}(\mathcal{S})| = |\mathcal{F}(\mathcal{S}, h)|$ Let us now get back to families of the form $\mathcal{F}(\mathcal{S}, h)$. For a Sperner family $\mathcal{S} = \{S_1, \dots, S_N\}$, we set $h(S_i) = H_i$ to simplify notation. To analyse (5.1) in Proposition 5.0.1 first note that it holds if and only if $\text{Up}(\mathcal{S}) = \bigcup_{i=1}^N \mathcal{P}_{S_i}$ and $2^{[n]} \setminus \mathcal{F}(\mathcal{S}, h) = \bigcup_{i=1}^N \mathcal{Q}_{S_i,H_i}$ have the same size. To study this we will use the inclusion-exclusion formula. For this note that for every $1 \le i < j \le N$

(i) $\mathcal{P}_{S_i} \cap \mathcal{P}_{S_j} = \mathcal{P}_{S_i \cup S_j}$, and

(ii) $\mathcal{Q}_{S_i,H_i} \cap \mathcal{Q}_{S_j,H_j} = \begin{cases} \mathcal{Q}_{S_i \cup S_j, H_i \cup H_j} & \text{, if } S_i \cap H_j = S_j \cap H_i \\ \emptyset & \text{, otherwise} \end{cases}$

In particular this means that for $I \subseteq [N]$ we have that

$$\left|\bigcap_{i\in I} \mathcal{P}_{S_i}\right| = \left|\mathcal{P}_{\bigcup_{i\in I} S_i}\right| = \left|\mathcal{Q}_{\bigcup_{i\in I} S_i, \bigcup_{i\in I} H_i}\right| = \left|\bigcap_{i\in I} \mathcal{Q}_{S_i,H_i}\right|$$

whenever $\bigcap_{i\in I} \mathcal{Q}_{S_i,H_i}$ is non-empty, which happens exactly if for every $i \neq j \in I$ we have $S_i \cap H_j = S_j \cap H_i$. Let $\mathbb{I}_{i,j}$ be the indicator of $S_i \cap H_j = S_j \cap H_i$, i.e. it is 1 if the equality is satisfied and 0 otherwise. As $\text{Up}(\mathcal{S}) = \bigcup_{i=1}^N \mathcal{P}_{S_i}$ and $2^{[n]} \setminus \mathcal{F}(\mathcal{S}, h) = \bigcup_{i=1}^N \mathcal{Q}_{S_i,H_i}$, the inclusion-exclusion formula gives that we have $|\mathcal{H}(\mathcal{S})| = |\mathcal{F}(\mathcal{S}, h)|$ if and only if

$$\sum_{I\subseteq[N]} (-1)^{|I|+1} \left|\bigcap_{i\in I} \mathcal{P}_{S_i}\right| = \sum_{I\subseteq[N]} (-1)^{|I|+1} \left|\bigcap_{i\in I} \mathcal{Q}_{S_i,H_i}\right| = \sum_{I\subseteq[N]} (-1)^{|I|+1} \left(\prod_{i\neq j\in I} \mathbb{I}_{i,j}\right) \left|\bigcap_{i\in I} \mathcal{P}_{S_i}\right|$$

This latter equation can also be rewritten as

$$\sum_{I\subseteq[N]} (-1)^{|I|+1} \left(1 - \prod_{i\neq j\in I} \mathbb{I}_{i,j}\right) \left|\bigcap_{i\in I} \mathcal{P}_{S_i}\right| = 0.$$

5.1.2 Outline of the new approach Before we prove Theorem 1.4.9 we would like to begin with a high overview of our approach to the elimination conjecture.

(i) Start with a Sperner family $\mathcal{S} \subseteq 2^{[n]}$ and a function $h : \mathcal{S} \to 2^{[n]}$ such that for the resulting set system $\mathcal{F} = \mathcal{F}(\mathcal{S}, h)$ we have $|\mathcal{H}(\mathcal{S})| = |\mathcal{F}(\mathcal{S}, h)|$.

(ii) Choose a set $S_0 \in \mathcal{S}$ and replace it with sets from $\{S_0 \cup \{v\} : v \in [n] \setminus S_0\}$ to obtain a Sperner family $\mathcal{S}'$ with $\mathcal{H}(\mathcal{S}') = \mathcal{H}(\mathcal{S}) \cup \{S_0\}$. Note that $\mathcal{S}' = \mathcal{S} \setminus \{S_0\}$ is possible.

(iii) Extend the function h from $\mathcal{S}$ to $\mathcal{S}'$ and consider the resulting set system $\mathcal{F}' = \mathcal{F}(\mathcal{S}', h)$. Note that $|\mathcal{F}'| \le |\mathcal{F}| + 1$, by the generalised Sauer Inequality.

(iv) Prove that $\mathcal{F} \subseteq \mathcal{F}'$ and $|\mathcal{F}'| = |\mathcal{F}| + 1$.

As we will see after the proof of Theorem 1.4.9, one cannot simply take any $S_0 \in \mathcal{S}$. Another issue is that it is not clear how to extend the function h. Natural choices would be to set $h(S_0 \cup \{v\})$ equal to either $h(S_0)$ or to $h(S_0) \cup \{v\}$.

5.1.3 Proof of Theorem 1.4.9 The following proposition establishes that given any Sperner family $\mathcal{S}$, if we define the corresponding function h via a fixed set $A \subseteq [n]$, then the resulting set system is s-extremal.

Proposition 5.1.3. *Let $\mathcal{S} = \{S_1, ..., S_N\} \subseteq 2^{[n]}$ be a Sperner family and $A \subseteq [n]$ be a fixed set. Furthermore let $h_A : \mathcal{S} \to 2^{[n]}$ be defined as $h_A(S) = S \cap A$, i.e. $H_i = h_A(S_i) = S_i \cap A$ for $i \in [N]$. Then $\mathcal{F}(\mathcal{S}, h_A)$ is s-extremal and Sh$(\mathcal{F}(\mathcal{S}, h_A)) = \mathcal{H}(\mathcal{S})$.*

Proof. For the proof only note that in this case, for every $1 \leq i < j \leq N$ we have

$$S_j \cap H_i = S_j \cap S_i \cap A = S_i \cap S_j \cap A = S_i \cap H_j,$$

i.e. $\mathbb{I}_{i,j} = 1$. In this case $1 - \prod_{i \neq j \in I} \mathbb{I}_{i,j} = 0$ for every $I \subseteq [N]$, and so

$$\sum_{I \subseteq [N]} (-1)^{|I|+1} \left(1 - \prod_{i \neq j \in I} \mathbb{I}_{i,j}\right) \left|\bigcap_{i \in I} \mathcal{P}_{S_i}\right| = 0.$$

Equivalently this means that $|\mathcal{H}(\mathcal{S})| = |\mathcal{F}(\mathcal{S}, h)|$, and so by Proposition 5.0.1 $\mathcal{F}(\mathcal{S}, h_A)$ is s-extremal and Sh$(\mathcal{F}(\mathcal{S}, h_A)) = \mathcal{H}(\mathcal{S})$. □

A natural first question is that perhaps the converse is also true. Unfortunately, this is not the case, i.e. if $\mathcal{F}(\mathcal{S}, h)$ is extremal and Sh$(\mathcal{F}(\mathcal{S}, h)) = \mathcal{H}(\mathcal{S})$ then there does not necessarily exist a set $A \subseteq [n]$ such that $h = h_A$, as shown by the following example.

Example 5.1.4. *Let $n = 3$ and $\mathcal{S} = \{S_1, S_2, S_3\}$, where $S_1 = \{1, 2\}$, $S_2 = \{1, 3\}$ and $S_3 = \{2, 3\}$. Furthermore take h such that $H_1 = \{1\}$, $H_2 = \emptyset$ and $H_3 = \emptyset$. Then*

$$\mathcal{P}_1 = \mathcal{P}_{S_1} = \{\{1, 2\}, \{1, 2, 3\}\},\ \mathcal{P}_2 = \mathcal{P}_{S_2} = \{\{1, 3\}, \{1, 2, 3\}\},\ \mathcal{P}_3 = \mathcal{P}_{S_3} = \{\{2, 3\}, \{1, 2, 3\}\}$$

$$\mathcal{Q}_1 = \mathcal{Q}_{S_1,H_1} = \{\{1\}, \{1, 3\}\},\ \mathcal{Q}_2 = \mathcal{Q}_{S_2,H_2} = \{\emptyset, \{2\}\},\ \mathcal{Q}_3 = \mathcal{Q}_{S_3,H_3} = \{\emptyset, \{1\}\},$$

and so

$$\mathcal{F}(\mathcal{S}, h) = 2^{[3]} \setminus (\mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \mathcal{Q}_3) = \{\{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}.$$

On the other hand

$$\mathcal{H}(\mathcal{S}) = 2^{[3]} \setminus (\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3) = \{\emptyset, \{1\}, \{2\}, \{3\}\}$$

and so as both have size 4, by Proposition 5.0.1 $\mathcal{F}(\mathcal{S}, h)$ is s-extremal and Sh$(\mathcal{F}(\mathcal{S}, h)) = \mathcal{H}(\mathcal{S})$. However it is easily seen that there is no $A \subseteq [3]$ such that $h = h_A$ would hold.

We are now in a position to show that families of the form $\mathcal{F}(\mathcal{S}, h_A)$ satisfy Conjecture 1.4.6, i.e. to prove Theorem 1.4.9.

Proof of Theorem 1.4.9. To shorten notation put $\mathcal{F} = \mathcal{F}(\mathcal{S}, h_A)$. Recall that by Proposition 5.1.3 $\mathcal{F}$ is s-extremal, i.e. $|\mathcal{F}| = |\mathrm{Sh}(\mathcal{F})|$ and $\mathrm{Sh}(\mathcal{F}) = \mathcal{H}(\mathcal{S})$. Pick an arbitrary $S_0 \in \mathcal{S}$ with $H_0 = S_0 \cap A$. Then there exists a unique (possibly empty) family $\{S'_1, ..., S'_k\} \subseteq \{S_0 \cup \{v\} \ : \ v \in [n] \setminus S_0\}$ such that $\mathcal{S}' = \mathcal{S} \setminus \{S_0\} \cup \{S'_1, ..., S'_k\}$ is again a Sperner familiy and

$$\mathcal{H}(\mathcal{S}') = 2^{[n]} \setminus \mathrm{Up}(\mathcal{S}') = \left(2^{[n]} \setminus \mathrm{Up}(\mathcal{S})\right) \cup \{S_0\} = \mathcal{H}(\mathcal{S}) \cup \{S_0\}.$$

For $i \in [k]$ let $H'_i = h_A(S'_i) = S'_i \cap A$ and let $\mathcal{F}'$ be the shorthand notation for $\mathcal{F}(\mathcal{S}', h_A)$. Again, by Proposition 5.1.3, $\mathcal{F}'$ is s-extremal and $\mathrm{Sh}(\mathcal{F}') = \mathcal{H}(\mathcal{S}')$. In particular, since $|\mathrm{Sh}(\mathcal{F}')| = |\mathcal{H}(\mathcal{S}')| = |\mathcal{H}(\mathcal{S}) \cup \{S_0\}| = |\mathcal{H}(\mathcal{S})| + 1 = |\mathrm{Sh}(\mathcal{F})| + 1$, we have $|\mathcal{F}'| = |\mathcal{F}| + 1$. Accordingly all that remains to be shown to prove the theorem is that $\mathcal{F} \subseteq \mathcal{F}'$, since in that case the unique set F in $\mathcal{F}' \setminus \mathcal{F}$ is a good choice, i.e. it is such that $\mathcal{F} \cup \{F\}$ is s-extremal. To see this, first note that $\mathcal{Q}_{S'_i, H'_i} \subseteq \mathcal{Q}_{S_0, H_0}$ since $S_0 \subseteq S'_i$ for every $i \in [k]$, and hence

$$\bigcup_{i=1}^{k} \mathcal{Q}_{S'_i, H'_i} \subseteq \mathcal{Q}_{S_0, H_0}.$$

However in this case

$$\begin{aligned}
\mathcal{F} &= \left(2^{[n]} \setminus \bigcup_{\substack{S \in \mathcal{S} \\ S \neq S_0}} \mathcal{Q}_{S, S\cap A} \right) \setminus \mathcal{Q}_{S_0, H_0} \\
&\subseteq \left(2^{[n]} \setminus \bigcup_{\substack{S \in \mathcal{S} \\ S \neq S_0}} \mathcal{Q}_{S, S\cap A} \right) \setminus \bigcup_{i=1}^{k} \mathcal{Q}_{S'_i, H'_i} = 2^{[n]} \setminus \bigcup_{S \in \mathcal{S}'} \mathcal{Q}_{S, S\cap A} = \mathcal{F}',
\end{aligned}$$

as desired. □

Theorem 1.4.9 solves only a further special case of Conjecture 1.4.6, so the conjecture remains open in general. However the approach presented offers a possible way to tackle it.

As already mentioned after the outline of the approach, one cannot take any $S_0 \in \mathcal{S}$. Indeed consider Example 5.1.4. If we take any $S_0 \in \mathcal{S}$, then one does not need to add any set to $\mathcal{S} \setminus \{S_0\}$, as we already have $\mathcal{H}(\mathcal{S}') = \mathcal{H}(\mathcal{S} \setminus \{S_0\}) = \mathcal{H}(\mathcal{S}) \cup \{S_0\}$. However if we were to choose $S_0 = S_3 = \{2, 3\}$, then the resulting $\mathcal{F}'$ is the the same as $\mathcal{F}$. In the special case, when $h = h_A$ for some $A \subseteq [n]$, this was not possible by the extremality of $\mathcal{F}'$, which was guaranteed by Proposition 5.1.3. Here we remark, that $\mathcal{F} = \mathcal{F}'$ does not contradict with the uniqueness of $\mathcal{S}$ and h, as for $\mathcal{S}'$ we have that $\mathrm{Sh}(\mathcal{F}) \subsetneq \mathcal{H}(\mathcal{S}')$. In the above example for instance $\mathrm{Sh}(\mathcal{F}) = \mathcal{H}(\mathcal{S}) = \{\emptyset, \{1\}, \{2\}, \{3\}\} \subsetneq \{\emptyset, \{1\}, \{2\}, \{3\}, \{2,3\}\} = \mathcal{H}(\mathcal{S}')$. Accordingly the main issue here is to rule out the possibility $\mathcal{F} = \mathcal{F}'$ by choosing S_0 and the new values for h carefully. Let us mention that in the above example S_1 and S_2 are good choices for S_0. Note that to prove the conjecture we need only one good instance. A possible step in this direction would be to characterise for a given Sperner family $\mathcal{S}$ the possible functions h such that $\mathcal{F}(\mathcal{S}, h)$ is s-extremal.

5.2 Small Sperner systems

In this section we will prove that Conjecture 1.4.6 holds for set systems $\mathcal{F}$ whose corresponding Sperner family has size at most 4. To do so, we will state an equivalent version of Conjecture 1.4.6. Before doing so, let us consider an example of a set system $\mathcal{F}$ which is s-extremal of the form $\mathcal{F}(\mathcal{S}, h)$ where $|\mathcal{S}| = 4$. Start with the following Sperner family $\mathcal{S} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}\}$ and let $h : \mathcal{S} \to 2^{[n]}$ be defined as $h(S) = \emptyset$ for every $S \in \mathcal{S}$. The resulting family $\mathcal{F}(\mathcal{S}, h)$ is s-extremal using Proposition 5.1.3, with $A = \emptyset$. Note that, for example, $\mathcal{Q}_{\{1,2\},\emptyset} = \{F \in 2^{[n]} : F \cap \{1,2\} = \emptyset\}$. Hence

$$\mathcal{F}(\mathcal{S}, h) = 2^{[n]} \setminus \bigcup_{S \in \mathcal{S}} Q_{S,h(S)} = \{F \in 2^{[n]} : F \cap \{1,2,3,4,5\} \neq \emptyset\}.$$

Similarly one can define lots of examples of s-extremal families where the corresponding Sperner family has size (at most) four. Also note that for families like this, we already knew that they satisfies Conjecture 1.4.6 due to Theorem 1.4.9, by choice of the function h. In general, given a Sperner family $\mathcal{S}$, it is not easy to construct a suitable function h for which the resulting set system $\mathcal{F}(\mathcal{S}, h)$ is s-extremal, see Section 5.4. Another example of an s-extremal family whose Sperner family has size three (with $n = 3$) was given in Example 5.1.4.

5.2.1 An equivalent conjecture Let $\mathcal{F} \subsetneq 2^{[n]}$ be s-extremal. We have already mentioned that it must be of the form $\mathcal{F}(\mathcal{S}, h)$ for a unique Sperner family $\mathcal{S}$ and function h. Let $\mathcal{S} = \{S_1, ..., S_N\}$ and let us write $H_i = h(S_i)$ as well as $\mathcal{Q}_i = \mathcal{Q}_{S_i,H_i}$ for ease of notation.

Conjecture 5.2.1. *Let $\mathcal{F} \subsetneq 2^{[n]}$ be s-extremal of the form $\mathcal{F}(\mathcal{S}, h)$, where $\mathcal{S} = \{S_1, ..., S_N\}$. Then there exists $i \in [N]$ such that*

$$\mathcal{Q}_i \not\subseteq \bigcup_{j \in [N] \setminus \{i\}} \mathcal{Q}_j.$$

We will now show that Conjecture 1.4.6 and Conjecture 5.2.1 are indeed equivalent.

Lemma 5.2.2. *Let $\mathcal{F} \subsetneq 2^{[n]}$ be s-extremal of the form $\mathcal{F}(\mathcal{S}, h)$, where $\mathcal{S} = \{S_1, ..., S_N\}$. Then there exists $F \in 2^{[n]} \setminus \mathcal{F}$ such that $\mathcal{F}' = \mathcal{F} \cup \{F\}$ is s-extremal if and only if there exists $i \in [N]$ such that $\mathcal{Q}_i \not\subseteq \bigcup_{j \in [N] \setminus \{i\}} \mathcal{Q}_j$.*

Proof. Assume first that there exists $F \in 2^{[n]} \setminus \mathcal{F}$ such that $\mathcal{F}' = \mathcal{F} \cup \{F\}$ is s-extremal. In particular, we have $|\mathcal{F}| + 1 = |\mathcal{F}'| = |\mathrm{Sh}(\mathcal{F}')| = |\mathrm{Sh}(\mathcal{F})| + 1$. Therefore $\mathcal{F}'$ shatters one of the sets of $\mathcal{S}$: indeed, it cannot shatter a superset of one of the S_i's because then it would shatter S_i as well and so we would get $|\mathrm{Sh}(\mathcal{F}')| \geq |\mathrm{Sh}(\mathcal{F})| + 2$. So we may assume that $\mathcal{F}'$ shatters S_1. As shown in Lemma 5.3.5, $H_1 = h(S_1)$ was the unique subset of S_1 that could not be obtained as an intersection of elements of $\mathcal{F}$ with S_1, so now we have $F \cap S_1 = H_1$. This implies $\mathcal{Q}_1 \not\subseteq \bigcup_{j \in [N]\setminus\{1\}} \mathcal{Q}_j$: if there was an index $j \in [N] \setminus \{1\}$ with $F \in \mathcal{Q}_j$, then $\mathcal{F}'$ would shatter S_j too, contradicting $|\mathrm{Sh}(\mathcal{F}')| = |\mathrm{Sh}(\mathcal{F})| + 1$.

Conversely, suppose there exists an index $i \in [N]$ such that $\mathcal{Q}_i \not\subseteq \bigcup_{j \in [N]\setminus\{i\}} \mathcal{Q}_j$. Then there exists an $F \in \mathcal{Q}_i$ that is not contained in any other $\mathcal{Q}_j$, $j \in [N] \setminus \{i\}$.

Following the approach described in Subsection 5.1.2, we now replace S_i by an appropriate family $\{S'_1, ..., S'_k\}$, where each S'_ℓ is of the form $\{S_\ell \cup \{v_\ell\} \ : v_\ell \in [n] \setminus S_\ell\}$, so that

$$\mathcal{S}' = (\mathcal{S} \setminus \{S_i\}) \cup \{S'_1, ..., S'_k\}$$

is a Sperner family and $\mathcal{H}(\mathcal{S}') = \mathcal{H}(\mathcal{S}) \cup \{S_i\}$. For every $\ell = 1, .., k$ we now choose $h(S'_\ell)$ to be $h(S_i)$ or $h(S_i) \cup \{v_\ell\}$ such that

$$F \notin \mathcal{Q}_{S'_\ell, h(S'_\ell)}.$$

This can be done since the dimension of the cube $\mathcal{Q}_{S'_\ell, h(S'_\ell)}$ is one smaller than the dimension of the cube $\mathcal{Q}_i$. By construction, we have that $F \in \mathcal{F}' = \mathcal{F}(\mathcal{S}', h)$ but $F \notin \mathcal{F}$. Hence $|\mathcal{F}'| \geq |\mathcal{F}| + 1$ and $|\mathrm{Sh}(\mathcal{F}')| \geq |\mathrm{Sh}(\mathcal{F})| + 1$. On the other hand, by the generalised Sauer inequality (Proposition 5.1.1), we also know that

$$|\mathrm{Sh}(\mathcal{F}')| \leq |\mathcal{H}(\mathcal{S}')| = |\mathcal{H}(\mathcal{S})| + 1 = |\mathcal{F}| + 1 = |\mathrm{Sh}(\mathcal{F})| + 1,$$

where the last equality follows since $\mathcal{F}$ is s-extremal. Hence we have $|\mathrm{Sh}(\mathcal{F}')| = |\mathrm{Sh}(\mathcal{F})| + 1$. Therefore, using the Sauer-Shelah Lemma, we get that $|\mathcal{F}'| = |\mathcal{F}| + 1$, finishing the proof. □

The following lemma is useful when one works with the above version of the conjecture. It states that the family of cubes $\mathcal{Q}_i$ form a Sperner family.

Lemma 5.2.3. *Let $\mathcal{F} \subsetneq 2^{[n]}$ be of the form $\mathcal{F}(\mathcal{S}, h)$. Then, for every $i \neq j$, we have $\mathcal{Q}_i \not\subseteq \mathcal{Q}_j$.*

Proof. Suppose the claim is false. We may assume that $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$. By definition, this means $H_1 + 2^{[n]\setminus S_1} \subseteq H_2 + 2^{[n]\setminus S_2}$. As $H_1 \in \mathcal{Q}_1$, we have $H_1 \in H_2 + 2^{[n]\setminus S_2}$ and this implies that $H_1 \cap S_2 = H_2$. From this we get that $H_1 \supseteq H_2$, since $H_2 = H_1 \cap S_2 = H_1 \cap S_1 \cap S_2 \subseteq H_1 \cap S_1 = H_1$. But this implies that $H_1 \cup [n] \setminus S_1 = H_2 \cup (H_1 \setminus H_2) \cup [n] \setminus S_1 \in H_2 + 2^{[n]\setminus S_2}$, since $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$, and so $[n] \setminus S_1 \subseteq [n] \setminus S_2$. Hence $S_1 \supseteq S_2$ contradicting the fact that $\mathcal{S}$ is a Sperner family. □

5.2.2 Proof of Theorem 1.4.10 Given an s-extremal family $\mathcal{F} \subsetneq 2^{[n]}$ of the form $\mathcal{F}(\mathcal{S}, h)$, let us define the following auxiliary graph $G_{\mathcal{F}} = G_{\mathcal{F}(\mathcal{S},h)}$: Its vertex set is given by $V(G_{\mathcal{F}}) = \{(S_i, H_i) \ : i \in [N]\}$ and we join two vertices (S_i, H_i) and (S_j, H_j) if $S_i \cap H_j = S_j \cap H_i$. In this case, we refer to the pair of adjacent vertices as a *good pair*. We will now state and prove some basic facts about $G_{\mathcal{F}}$ with respect to Conjecture 1.4.6.

Firstly, Bollobás and Radcliffe proved that if a set system $\mathcal{F}$ is s-extremal then its so called *cover graph* graph is connected (see Theorem 3 in [17]). From this we immediately get the following lemma.

Lemma 5.2.4. *If $\mathcal{F}$ is s-extremal of the form $\mathcal{F}(\mathcal{S}, h)$, then $G_{\mathcal{F}}$ is connected.*

Next, we show that if the auxiliary graph of an s-extremal family has a vertex of degree one, then $\mathcal{F}$ satisfies the elimination conjecture.

Claim 5.2.5. *Suppose $\mathcal{F}$ is s-extremal of the form $\mathcal{F}(\mathcal{S}, h)$. If there is a vertex of degree 1 in $G_{\mathcal{F}}$, then $\mathcal{F}$ satisfies Conjecture 1.4.6, i.e. there is $F \notin \mathcal{F}$ such that $\mathcal{F} \cup F$ is again s-extremal.*

Proof. After possible relabelling, we may assume (S_1, H_1) has degree 1, and that (S_2, H_2) is its unique neighbour. By definition of $G_{\mathcal{F}}$ we know that $\mathcal{Q}_1 \cap \mathcal{Q}_j = \emptyset$ for every $3 \leq j \leq N$. We claim that

$$\mathcal{Q}_1 \not\subseteq \bigcup_{j \geq 2} \mathcal{Q}_j.$$

For the sake of contradiction, assume $\mathcal{Q}_1 \subseteq \bigcup_{j \geq 2} \mathcal{Q}_j$. Since $\mathcal{Q}_1$ is disjoint from all $\mathcal{Q}_j$, $j \geq 3$, this forces $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$. However, this contradicts Claim 5.2.3. Thus $\mathcal{F}$ satisfies Conjecture 5.2.1 and hence, by Lemma 5.2.2, it satisfies Conjecture 1.4.6 as well.

□

Claim 5.2.6. *Suppose $\mathcal{F}$ is s-extremal of the form $\mathcal{F}(\mathcal{S}, h)$. If $G_{\mathcal{F}}$ is the complete graph on N vertices, then $\mathcal{F}$ satisfies Conjecture 1.4.6.*

Proof. Since $G_{\mathcal{F}}$ is the complete graph on N vertices, we have $S_i \cap H_j = S_j \cap H_i$ for all $1 \leq i < j \leq N$. We will show that $h = h_A$, where $A = H_1 \cup ... \cup H_N \subseteq [n]$. The claim then follows from Theorem 1.4.9. Recall that $h_A(S_i) = S_i \cap A$. To prove the claim, just note that $S_i \cap A = S_i \cap (H_1 \cup ... \cup H_N) = (S_i \cap H_1) \cup ... \cup (S_i \cap H_N) = H_i$, since $S_i \cap H_i = H_i$ and for every $j \neq i$ we have $S_i \cap H_j = S_j \cap H_i \subseteq H_i$. □

Corollary 5.2.7. *Let $\mathcal{F} \subsetneq 2^{[n]}$ be s-extremal of the form $\mathcal{F}(\mathcal{S}, h)$, where $|\mathcal{S}| \leq 2$, then $\mathcal{F}$ satisfies Conjecture 1.4.6.*

Proof. Since $\mathcal{F} = \mathcal{F}(\mathcal{S}, h)$ is s-extremal, we get that $G_{\mathcal{F}}$ is a clique of size one (if $|\mathcal{S}| = 1$) or two (if $|\mathcal{S}| = 2$, since $G_{\mathcal{F}}$ is connected), and hence $\mathcal{F}$ satisfies Conjecture 1.4.6 by Claim 5.2.6. □

Lemma 5.2.8. *Let $\mathcal{F} \subsetneq 2^{[n]}$ be s-extremal of the form $\mathcal{F}(\mathcal{S}, h)$, where $|\mathcal{S}| = 3$, then $\mathcal{F}$ satisfies Conjecture 1.4.6.*

Proof. Here $G_{\mathcal{F}}$ is either a K_3, in which case $\mathcal{F}$ satisfies Conjecture 1.4.6 by Claim 5.2.6, or contains a vertex of degree 1, because $G_{\mathcal{F}}$ is connected, in which case $\mathcal{F}$ satisfies Conjecture 1.4.6 by Claim 5.2.5. □

Lemma 5.2.9. *Let $\mathcal{F} \subsetneq 2^{[n]}$ be s-extremal of the form $\mathcal{F}(\mathcal{S}, h)$, where $|\mathcal{S}| = 4$, then $\mathcal{F}$ satisfies Conjecture 1.4.6.*

Proof. The only cases not handled by Claim 5.2.5 and Claim 5.2.6 are when $G_{\mathcal{F}}$ is a C_4, i.e. a cycle of length four, or a K_4^-, i.e. a K_4 minus an edge.
Suppose first that $G_{\mathcal{F}} = C_4$ and that $\mathcal{Q}_1 \cap \mathcal{Q}_3 = \emptyset = \mathcal{Q}_2 \cap \mathcal{Q}_4$. If $\mathcal{F}$ does not satisfy Conjecture 5.2.1, then this means that $\mathcal{Q}_1, \mathcal{Q}_3 \subseteq \mathcal{Q}_2 \cup \mathcal{Q}_4$ and $\mathcal{Q}_2, \mathcal{Q}_4 \subseteq \mathcal{Q}_1 \cup \mathcal{Q}_3$. But this means that

$$\mathcal{Q}_1 \cup \mathcal{Q}_3 = \mathcal{Q}_2 \cup \mathcal{Q}_4.$$

Since the cubes on both sides are disjoints, this implies $\{\mathcal{Q}_1, \mathcal{Q}_3\} = \{\mathcal{Q}_2, \mathcal{Q}_4\}$ yielding $\{S_1, S_3\} = \{S_2, S_4\}$ - a contradiction.

Now assume that $G_{\mathcal{F}} = K_4^-$ and that $\mathcal{Q}_1 \cap \mathcal{Q}_3 = \emptyset$. If $\mathcal{F}$ does not satisfy Conjecture 5.2.1, then this in particular means that $\mathcal{Q}_1, \mathcal{Q}_3 \subseteq \mathcal{Q}_2 \cup \mathcal{Q}_4$. We will show that this yields a contradiction. To do so, let $\mathcal{Q}_2' = \mathcal{Q}_1 \cap \mathcal{Q}_2$ and let $\mathcal{Q}_4' = \mathcal{Q}_1 \cap \mathcal{Q}_4$. Then both $\mathcal{Q}_2'$ and $\mathcal{Q}_4'$ are subcubes of $\mathcal{Q}_1$ and $\mathcal{Q}_2' \cup \mathcal{Q}_4' = \mathcal{Q}_1$. There are two cases to consider.

Either we have $\mathcal{Q}_2' = \mathcal{Q}_1$ or $\mathcal{Q}_4' = \mathcal{Q}_1$, in which case we have $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$ or $\mathcal{Q}_1 \subseteq \mathcal{Q}_4$, contradicting Claim 5.2.3.

Otherwise $\mathcal{Q}_2'$ and $\mathcal{Q}_4'$ are two half-cubes of $\mathcal{Q}_1$, in which case $\mathcal{Q}_2' \cap \mathcal{Q}_4' = \emptyset$. In particular, there exists a direction $i \in [n]$ that distinguishes the two half-cubes. But the same direction is then distinguishing $\mathcal{Q}_2$ and $\mathcal{Q}_4$ implying $\mathcal{Q}_2 \cap \mathcal{Q}_4 = \emptyset$ - a contradiction.

□

5.3 Gröbner Bases

In this section we shall give a rather fast paced introduction to Gröbner bases that will enable us to state and prove our main result regarding Gröbner bases as well as to highlight some connections between them and shattering extremal families. For more information on the connections between Gröbner bases and shattering extremal families, the reader may consult [65].

5.3.1 A short introduction to Gröbner bases Let $\mathbb{F}$ be an arbitrary field and let $\mathbb{F}[x_1, ..., x_n] = \mathbb{F}[\mathbf{x}]$ be the polynomial ring over $\mathbb{F}$ with variables $x_1, ..., x_n$. Given some set $F \subseteq [n]$, let $v_F \in \{0,1\}^n$ be its *characteristic vector*, i.e. the i-th coordinate of v_F is 1 if $i \in F$ and 0 otherwise. Therefore we can identify a set system $\mathcal{F} \subseteq 2^{[n]}$ with the vector system

$$\mathcal{V}(\mathcal{F}) = \{v_F \ : \ F \in \mathcal{F}\} \subseteq \{0,1\}^n \subseteq \mathbb{F}^n.$$

One can then associate to $\mathcal{F}$ a polynomial ideal $I(\mathcal{V}(\mathcal{F})) \trianglelefteq \mathbb{F}[\mathbf{x}]$, where

$$I(\mathcal{F}) = I(\mathcal{V}(\mathcal{F})) = \{f \in \mathbb{F}[\mathbf{x}] \ : \ f(v_F) = 0 \ \forall \ F \in \mathcal{F}\}.$$

In words, $I(\mathcal{F})$ is the vanishing ideal of the set of characteristic vectors of the elements of $\mathcal{F}$. Note that we always have $\{x_i^2 - x_i \ : \ i \in [n]\} \subseteq I(\mathcal{F})$. For more details about vanishing ideals of finite point sets see e.g. [65].

If one works with polynomial ideals, it is useful to have a nice ideal basis. Such nice bases are given by the so-called *Gröbner bases*, which we will now briefly define. For more details the interested reader may consult e.g. [1]. A total order $\prec$ on the monomials in $\mathbb{F}[\mathbf{x}]$ is a *term order*, if 1 is the minimal element of $\prec$, and $\prec$ is compatible with multiplication with monomials. One well-known and important term order is the *lexicographic (lex) order*. Here one has $x_1^{w_1}...x_n^{w_n} \prec_{\text{lex}} x_1^{u_1}...x_n^{u_n}$ if and only if for the smallest index k with $w_k \neq u_k$ one has $w_k < u_k$. For example, for two variables we would get

$$1 \prec x_2 \prec x_2^2 \prec x_2^3 \prec \ldots \prec x_1 \prec x_1x_2 \prec x_1x_2^2 \prec \ldots \prec x_1^2 \prec \ldots$$

One can build a lex order based on other orderings of the variables as well, so altogether we have $n!$ different lex orders. Given some term order $\prec$ and $f \in \mathbb{F}[\mathbf{x}]$, the *leading monomial* $\text{Lm}(f)$ of f, is the largest monomial (with respect to $\prec$) appearing with non-zero coefficient in the canonical form of f. We are now in a position to define Gröbner bases.

Definition 5.3.1. *Let $I \trianglelefteq \mathbb{F}[\boldsymbol{x}]$ be an ideal and $\prec$ a term order. A finite subset $\mathbb{G} \subseteq I$ is called a* Gröbner basis of I *with respect to $\prec$ if for every $f \in I$ there exists a $g \in \mathbb{G}$ such that $Lm(g)$ divides $Lm(f)$.*

Note that a Gröbner basis is not unique, since one can always add polynomials to it. Gröbner bases were originally introduced to solve the *ideal membership problem*: Given an ideal $I \trianglelefteq \mathbb{F}[\mathbf{x}]$ and a polynomial $f \in \mathbb{F}[\mathbf{x}]$, then how can we decide whether $f \in I$? However, we will not deal with this problem here.
The following result guarantees the existence of Gröbner bases.

Theorem 5.3.2. *Every non-zero ideal $0 \neq I \trianglelefteq \mathbb{F}[\boldsymbol{x}]$ has a Gröbner basis for every term order.*

For a proof of this theorem, we refer the reader to [1].
Note that if $\mathbb{G}$ is a Gröbner basis of I for some term order, then $\mathbb{G}$ generates I as an ideal as well, i.e. $I = \langle \mathbb{G} \rangle$.

5.3.2 Gröbner bases and s-extremal families We now make the jump to shattering extremal families. First of all, there is a natural counterpart to the concept of leading monomials. Given an ideal $I \trianglelefteq \mathbb{F}[\mathbf{x}]$, a monomial is called a *standard monomial*, if it is not the leading monomial of any polynomial $f \in I$. We get the following correspondence:

$$\mathrm{Sm}(I(\mathcal{F})) \longleftrightarrow \{G \subseteq [n] \; : \mathbf{x}_G \in \mathrm{Sm}(I(\mathcal{F}))\} \subseteq 2^{[n]} \tag{5.2}$$

A particularly nice thing about standard monomials is, that with a little more work one can show that for a set system $\mathcal{F}$ we have $|\mathcal{F}| = |\mathrm{Sm}(I(\mathcal{F}))|$.

We can now state a result relating $\mathrm{Sh}(\mathcal{F})$ with standard monomials.

Theorem 5.3.3 (Rónyai-Mészáros [65])**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be a set system. Then*

$$Sh(\mathcal{F}) = \bigcup_{\text{lex term orders}} Sm(I(\mathcal{F})).$$

As an immediate consequence, we get the following.

Theorem 5.3.4. *Let $\mathcal{F} \subseteq 2^{[n]}$ be a set system. Then $\mathcal{F}$ is s-extremal if and only if $Sm(I(\mathcal{F}))$ are the same for all lex term orders.*

For a subset $H \subseteq [n]$, set $\mathbf{x}_H = \prod_{i \in H} x_i$. Given a pair of sets $H \subseteq S \subseteq [n]$ we then define the polynomial

$$f_{S,H}(\mathbf{x}) = \mathbf{x}_H \cdot \prod_{i \in S \setminus H} (x_i - 1).$$

Note that $\mathrm{Lm}(f_{S,H}) = x_S$ for every term order. A nice property of these polynomials is that for a set $F \subseteq [n]$ we have

$$f_{S,H}(v_F) \neq 0 \text{ if and only if } F \cap S = H. \tag{5.3}$$

We can now show that if S is a minimal set not shattered by an s-extremal set system then there is a unique whitness for that. We will include its short proof for the convenience of the reader.

Lemma 5.3.5 ([65])**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be a s-extremal set system. If S is a minimal set not shattered by $\mathcal{F}$, then there exists a unique $H \subseteq S$ for which there does not exist an $F \in \mathcal{F}$ with $F \cap S = H$.*

Proof. Assume for a contradiction that there are two distinct sets $H_1, H_2 \subseteq S$ such that there does not exist $F \in \mathcal{F}$ with $F \cap S = H$. By 5.3 this means that $f_{S,H_i} \in I(\mathcal{F})$, $i = 1, 2$. Then $g = f_{S,H_1} - f_{S,H_2} \in I(\mathcal{F})$. Now fix a term order.
For this term order we have that $\mathrm{Lm}(g) = x_{S'}$ for some $S' \subsetneq S$. Since $\mathcal{F}$ is s-extremal, we must have $\mathrm{Sm}(I(\mathcal{F})) = \mathrm{Sh}(\mathcal{F})$ (by Theorem 5.3.4). But $\mathbf{x}_{S'} \notin \mathrm{Sm}(I(\mathcal{F}))$ and hence S' is not shattered by $\mathcal{F}$, contradicting the minimality of S. □

Now we are in a position to state the connection between s-extremal families and the theory of Gröbner bases.

Theorem 5.3.6 ([65])**.** *$\mathcal{F} \subseteq 2^{[n]}$ is s-extremal if and only if there are polynomials of the form $f_{S,H}$, which together with $\{x_i^2 - x_i \ : \ i \in [n]\}$ form a Gröbner basis of $I(\mathcal{F})$ for all term orders.*

Remark 5.3.7. *From the proof of the theorem one can deduce that if there is a suitable Gröbner basis for one particular term order, then $\mathcal{F}$ is already extremal (see [65]).*

Given a Sperner family $\mathcal{S} \subseteq 2^{[n]}$ and a function $h : \mathcal{S} \to 2^{[n]}$, set

$$\mathbb{G}(\mathcal{S}, h) = \{f_{S,h(S)} \ : \ S \in \mathcal{S}\} \cup \{x_i^2 - x_i \ : \ i \in [n]\}.$$

Using the approach given by Proposition 5.0.1, we can now state and prove our main result of this section. The proof requires some basic knowledge of commutative algebra, in particular ideal theory (see the appendix to this chapter).

Theorem 5.3.8. *$\mathbb{G} = \mathbb{G}(\mathcal{S}, h)$ is a Gröbner basis (of $\langle \mathbb{G} \rangle$) for some term order $\prec$ if and only if*

$$|\mathcal{H}(\mathcal{S})| = |\mathcal{F}(\mathcal{S}, h)|.$$

Proof. Suppose first that $\mathbb{G}$ is a Gröbner basis for some term order $\prec$. We start by showing that the ideal generated by $\mathbb{G}$, denoted $\langle \mathbb{G} \rangle$, is a radical ideal, i.e. $\langle \mathbb{G} \rangle = \sqrt{\langle \mathbb{G} \rangle}$. To see this, first note that clearly $J = \langle x_i^2 - x_i \ : i \in [n] \rangle \subseteq \langle \mathbb{G} \rangle$. Now a basic fact from Algebra states that $\langle \mathbb{G} \rangle$ is a radical ideal in $\mathbb{F}[\mathbf{x}]$ if and only if $\langle \mathbb{G} \rangle / J$ is a radical ideal in $\mathbb{F}[\mathbf{x}]/J$ (see Lemma 5.5.5). However, $\mathbb{F}[\mathbf{x}]/J$ is isomorphic to $\mathbb{F}^{2^n}$, because both are isomorphic to the ring of all functions from $\{0, 1\}^n$ to $\mathbb{F}$. Using the fact that the only ideals of a field are the zero ideal and the field itself, and that the only ideals in a finite cartesian product of rings are products of ideals (see Lemma 5.5.6), one easily verifies that every ideal in $\mathbb{F}^{2^n}$ is the intersection of maximal ideals. This in turn implies that in $\mathbb{F}[\mathbf{x}]/J$ every ideal is a radical ideal and so in particular $\langle \mathbb{G} \rangle / J$ is. Hence $\langle \mathbb{G} \rangle$ is a radical ideal and is thus a vanishing ideal of some finite set in $\{0, 1\}^n$, since $J \subseteq \langle \mathbb{G} \rangle$. This means $\langle \mathbb{G} \rangle = I(\mathcal{F})$ where $\mathcal{F}$ (more precisely

$\mathcal{V}(\mathcal{F})$) is the set of common roots of the polynomials in $\mathbb{G}$. We will now show that by the earlier mentioned properties of the $f_{S,h(S)}$ polynomials (see (5.3)) we have that $\mathcal{F}$ is precisely $\mathcal{F}(\mathcal{S}, h)$

$$\mathcal{F} = \bigcap_{S\in\mathcal{S}} \{F \;:\; v_F \text{ is a root of } f_{S_h(S)}\} = \bigcap_{S\in\mathcal{S}} \{v_F \;:\; F\cap S \neq h(S)\}$$
$$= \{v_F \;:\; F\cap S \neq h(S)\ \forall S\in\mathcal{S}\} = 2^{[n]} \setminus \bigcup_{S\in\mathcal{S}} \mathcal{Q}_{S,h(S)} = \mathcal{F}(\mathcal{S}, h).$$

Thus $\langle\mathbb{G}\rangle = I(\mathcal{F}(\mathcal{S}, h))$ and so, by Theorem 5.3.6, $\mathcal{F}(\mathcal{S}, h)$ is s-extremal, i.e. $|\mathcal{F}(\mathcal{S}, h)| = |\mathrm{Sh}(\mathcal{F}(\mathcal{S}, h))|$. However, as noted above, in this case we have that $\mathrm{Sh}(\mathcal{F}(\mathcal{S}, h)) = \mathcal{H}(\mathcal{S})$, and so $|\mathcal{F}(\mathcal{S}, h)| = |\mathcal{H}(\mathcal{S})|$.

Now suppose $|\mathcal{H}(\mathcal{S})| = |\mathcal{F}(\mathcal{S}, h)|$. In terms of Theorem 5.3.6 it is enough to show that $\mathcal{F}(\mathcal{S}, h)$ is s-extremal. Note that by definition, for every $S\in\mathcal{S}$ there does not exist $F\in\mathcal{F}(\mathcal{S}, h)$ such that $F\cap S = h(S)$ and so $S\notin \mathrm{Sh}(\mathcal{F}(\mathcal{S}, h))$. In particular no superset of S is shattered by $\mathcal{F}(\mathcal{S}, h)$. Therefore $\mathrm{Sh}(\mathcal{F}(\mathcal{S}, h)) \subseteq 2^{[n]}\setminus\mathrm{Up}(\mathcal{S}) = \mathcal{H}(\mathcal{S})$ and hence $|\mathrm{Sh}(\mathcal{F}(\mathcal{S}, h))| \leq |\mathcal{H}(\mathcal{S})| = |\mathcal{F}(\mathcal{S}, h)|$. However the opposite inequality holds by the Sauer-Shelah Lemma for every set system and thus $\mathcal{F}(\mathcal{S}, h)$ is necessary s-extremal. □

5.4 Concluding remarks and open problems

In this chapter we proved the elimination conjecture for some special cases and presented a new approach that we hope should work to prove the conjecture in full generality.
The main task is of course to prove Conjecture 1.4.6. A first step would be to solve the following problem.

Problem 5.4.1. *For a given Sperner family $\mathcal{S}\subseteq 2^{[n]}$ determine all possible functions h such that the resulting set system $\mathcal{F}(\mathcal{S}, h)$ is s-extremal.*

If one wants to avoid this, a natural first step would be to prove the conjecture using our approach in case $\mathcal{S}'$ can be taken to be $\mathcal{S}\setminus\{S_0\}$. In this case, the advantage would be that one does not need to extend the function h.

Furthermore we would like to point out that while proving the conjecture for small Sperner families, we mostly did not use that the family we started with was s-extremal, except for the graph $G_{\mathcal{F}}$ being connected, which we used for simplicity. However, the same statements can be proven without the connectedness assumption. This leads to the following question.

Question 5.4.2. *Given a Sperner family $\mathcal{S} = \{S_1, ..., S_N\} \subseteq 2^{[n]}$ and a function h. Is it true that there exists $i\in[N]$ such that $\mathcal{Q}_i \not\subseteq \bigcup_{j\in[N]\setminus\{i\}} \mathcal{Q}_j$?*

Note that if the answer to this question is yes, then this implies the elimination conjecture.

We end with the following beautiful conjecture of Frankl from 1989, that we already mentioned in the introduction.

Conjecture 5.4.3 (Frankl [32]). *Suppose $n \geq 2k$ and let $\mathcal{F} \subseteq 2^{[n]}$ be a Sperner family with $\dim_{VC}(\mathcal{F}) < k$. Then*

$$|\mathcal{F}| \leq \binom{n}{k-1}.$$

Note that the case $n < 2k$ is excluded, because in this case one can take $\mathcal{F} = \binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ which is as large as a Sperner family can possible be, by Sperner's Theorem.

5.5 Appendix to Chapter 5

We collect some basic facts from ideal theory that were implicitly used in this chapter. For more information and proofs we refer the reader to classic book of Atiyah and MacDonald [5] or to the more recent book of Sharp [71].

Ideals constitute the most important substructure of a commutative ring. Throughout R denotes a commutative ring with a 1.

Definition 5.5.1. *A subset I of R is called an* ideal*, if it is an additive subgroup of R and if whenever $a \in I$ and $r \in R$, then $ra \in I$ also.*

As is usual we write $I \trianglelefteq R$, if I is an ideal of R. An ideal $\mathfrak{p} \trianglelefteq R$ is called a *prime ideal* if $1 \notin R$ and if $xy \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. An ideal $\mathfrak{m} \trianglelefteq R$ is called a *maximal ideal* if there is no ideal J with $I \subsetneq I \subsetneq R$. It is an elementary fact that every maximal ideal is prime. Using the Lemma of Zorn, one can show that every ideal is contained in a maximal ideal.

Definition 5.5.2. *Let $I \trianglelefteq R$ be an ideal. The* radical *of I is*

$$\sqrt{I} = \{r \in R \ : \ \text{there exists } n \in \mathbb{N} \text{ with } r^n \in I\}.$$

Using the binomial theorem it is not hard to show that $\sqrt{I}$ is an ideal. Note that $I \subseteq \sqrt{I}$ trivially holds for every ideal I. An ideal $I \trianglelefteq R$ is called a *radical ideal* if $I = \sqrt{I}$.

With these definitions, we can state the results we have used in this chapter.

Lemma 5.5.3. *Let $I \trianglelefteq R$ be an ideal. Then*

$$\sqrt{I} = \bigcap_{\text{prime ideals } \mathfrak{p} \supseteq I} \mathfrak{p}.$$

We get the following as an immediate consequence.

Corollary 5.5.4. *If R is such that every ideal is the intersection of maximal ideals, then every ideal is radical.*

We have also used special cases of the following results.

Lemma 5.5.5. *Let $I \subseteq J$ be ideals in R. Then J is radical/prime/maximal if and only if J/I is radical/prime/maximal in R/I.*

Lemma 5.5.6. *Let $R_1, ..., R_n$ be commutative rings. If $I_i \trianglelefteq R_i$ is an ideal for every $i \in [n]$, then $I_1 \times ... \times I_n$ is an ideal in $R_1 \times ... \times R_n$. Furthermore, each ideal of $R_1 \times ... \times R_n$ is of this form.*

Zusammenfassung

Diese Dissertation besteht aus 5 Kapiteln. Das erste Kapitel dient als Einleitung und stellt die in der Dissertation behandelten Themen und Ergebnisse vor.
Das zweite Kapitel befasst sich mit sogenannten "strong Ramsey games", bei denen zwei Spieler abwechselnd Kanten des vollständigen (Hyper-)Graphen beanspruchen. Gewinner des Spiels ist der erste Spieler, welcher einen zuvor fest gewählten (Hyper-)Graphen vollständig für sich beanspruchen kann. Hat der vollständige Graph hinreichend viele Knoten, so kann gezeigt werden, dass für den beginnenden Spieler stets eine Strategie existiert, die es ihm ermöglicht, das Spiel zu gewinnen. Entgegen der allgemeinen Vermutung, dass dies auch auf unendliche vollständige Graphen zutrifft, konstruieren wir einen 5-uniformen Hypergraphen, für den der zweite Spieler ein Unentschieden erzwingen kann.
Das dritte Kapitel befasst sich mit sogenannten 'biased $(1:q)$ Maker-Breaker' Spielen. Zwei Spieler, genannt *Maker* und *Breaker*, beanspruchen abwechselnd Knoten eines gegebenen Hypergraphen. Maker beansprucht einen Knoten pro Runde und Breaker q Knoten. Maker gewinnt, falls er eine Hyperkante vollständig für sich beanspruchen kann, andernfalls gewinnt Breaker. Eine der zentralen Fragen auf diesem Gebiet ist, die sogenannte *threshold bias* zu finden. Wir beweisen allgemeine Gewinnkriterien, eins für Maker und eins für Breaker, und wenden diese auf zwei Klassen von Spielen an. Zum Einen verallgemeinern wir ein bekanntes Resultat von Bednarska und Łuczak auf Hypergraphen. Zum Anderen bestimmen wir, bis auf eine Konstante, die threshold bias, wenn das Ziel des Spiels eine Lösung zu einem beliebigen, aber festen linearen System von inhomogenen Gleichungen ist.
Das vierte Kapitel beschäftigt sich mit der sogenannten Ordnungseigenschaft von geordneten Hypergraphen, welche kürzlich von Duffus, Kay und Rödl eingeführt worden ist. Wir verbessern die von jenen bewiesene obere Schranke um einen Faktor $k \ln k$.
Das letzte Kapitel befasst sich mit dem Begriff des *Zerschmetterns* (‚shatter'). Ein Mengensystem heißt s-extremal, wenn es die Ungleichung von Sauer und Shelah mit Gleichheit erfüllt. Eine Vermutung von Mészáros und Rónyai besagt, dass man zu einem s-extremalen Mengensystem stets eine Menge hinzufügen kann, sodass das resultierende Mengensystem wiederum s-extremal ist. Wir beweisen diese Vermutung für bestimmte Mengensysteme, welche durch sogenannte Sperner Familien definiert werden und für den Fall, dass die Sperner Familie aus höchstens vier Mengen besteht. Zusätzlich beweisen wir ein neues Resultat, welches den Zusammenhang mit Gröbnerbasen weiter beleuchtet.

Eidesstattliche Erklärung

Gemäß § 7 (4) der Promotionsordnung des Fachbereichs Mathematik und Informatik der Freien Universität Berlin versichere ich hiermit, dass ich alle Hilfsmittel und Hilfen angegeben und auf dieser Grundlage die Arbeit selbstständig verfasst habe. Des Weiteren versichere ich, dass ich diese Arbeit nicht schon einmal zu einem früheren Promotionsverfahren eingereicht habe.

Berlin, den

Christopher Kusch

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