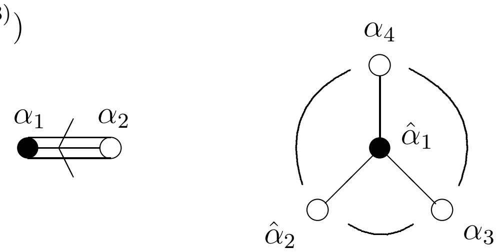
Appendix A

THE AFFINE TODA S-MATRIX CASE-BY-CASE

In order to illustrate the working of the general formulae derived for the affine Toda S-matrix it is useful to work them out explicitly for some concrete examples. We concentrate here on the non-simply laced case, since the simply laced case is covered extensively in the literature [27]. We will be most detailed for the $(G_2^{(1)}, D_4^{(3)})$-case. The conventions with regard to numbering and colouring may be read off from the Dynkin diagrams. As usual the arrow points towards the short roots. A black and white vertex corresponds to the colour value $c_i = -1$ and $c_i = 1$, respectively.

A.0.1 $(G_2^{(1)}, D_4^{(3)})$



The S-matrices of the theory read [77]

$$S_{11}(\theta) = \{\overbrace{1,1}^{2};\overbrace{3,5_2}^{1};5,11\}_\theta \tag{A.1}$$

$$S_{12}(\theta) = \{2,2_3;\overbrace{4,6_3}^{1}\}_\theta \tag{A.2}$$

$$S_{22}(\theta) = \{1,1_3;\overbrace{3,3_3}^{2};3,5_3;5,7_3\}_\theta\ . \tag{A.3}$$

Here we indicated which block is responsible for which type of fusing process. We have $h = 6$ and $H = 12$ for the Coxeter numbers. With the help of (3.61), we easily verify that for (A.1), (A.2) and (A.3) the following bootstrap identities hold

$$S_{1l}(\theta+\theta_h+\theta_H)\,S_{1l}(\theta-\theta_h-\theta_H) = S_{2l}(\theta) \qquad l=1,2 \tag{A.4}$$

$$S_{1l}(\theta+2\theta_h+4\theta_H)\,S_{1l}(\theta-2\theta_h-4\theta_H) = S_{1l}(\theta) \qquad l=1,2 \tag{A.5}$$

$$S_{2l}(\theta+2\theta_h+4\theta_H)\,S_{2l}(\theta-2\theta_h-4\theta_H) = S_{2l}(\theta) \qquad l=1,2\ . \tag{A.6}$$

As an example for the working of the generalized bootstrap and our criterion (3.79), (3.81) provided in Section 3.2.8, we plotted the imaginary part of the residues of $S_{22}(\theta)$ in Figure A.1 for several poles. We observe that the sign changes throughout the range for poles resulting from $\{1,1_3\}$ and $\{3,5_3\}$. Only the poles responsible for the self-coupling of particle 2 has a positive imaginary part of the residue throughout the range of the coupling constant β. Except at $B = 4/3$ where it is zero, such that this fusing process decouples.

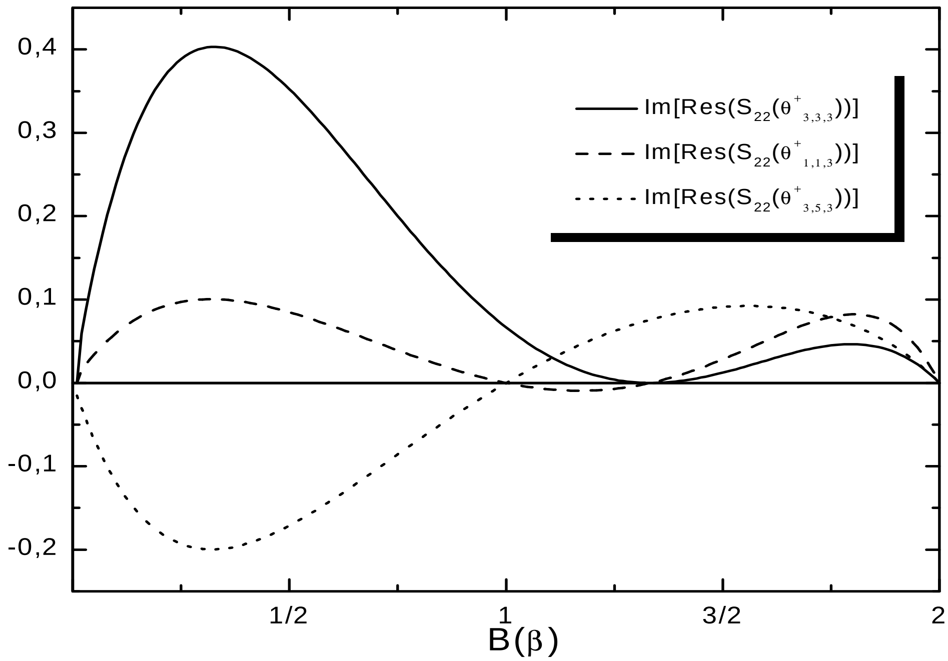


Figure A.1: The imaginary part of several residues of $S_{22}(\theta)$ as a function of the effective coupling constant.

Besides (A.4) the combined bootstrap identities (3.71) also yield

$$S_{l2}\left(\theta+\theta_h+3\theta_H\right) S_{l2}\left(\theta-\theta_h-3\theta_H\right) = S_{l1}\left(\theta\right) S_{l1}\left(\theta+2\theta_H\right) S_{l1}\left(\theta-2\theta_H\right), \qquad \text{(A.7)}$$

for $l = 1, 2$. These equations may be derived from (A.4) and (A.5) or verified directly for (A.1), (A.2) and (A.3), with the help of (3.61). The process corresponding to the combined bootstrap identity (A.7) is depicted in Figure A.2.
Reading off the fusing angles from the bootstrap equations we obtain the mass ratios according to (3.49)

$$\frac{m_1}{m_2} = \frac{\sinh\left(\theta_h+\theta_H\right)}{\sinh\left(2\theta_h+2\theta_H\right)} . \qquad \text{(A.8)}$$

We may construct all these formulae from the Lie algebraic data in alternative ways.

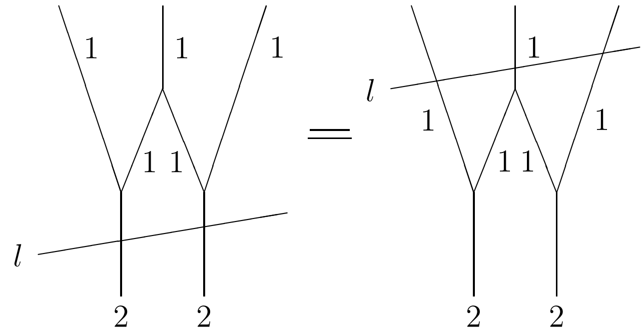


Figure A.2: $(G_2^{(1)}, D_4^{(3)})$-combined bootstrap identities (A.7).

$S_{ij}(\theta)$ from $G_2^{(1)}$

We start by exploiting the properties of $G_2^{(1)}$. The non-vanishing entries of the incidence matrix are $I_{12} = 1$ and $I_{21} = 3$. Consequently equation (2.55) yields $t_1 = 1$ and $t_2 = 3$. As indicated in the Dynkin diagram we choose $c_1 = -1$ and $c_2 = 1$, such that the q-deformed Coxeter element reads $\sigma_q = \sigma_1^q \tau \sigma_2^q \tau$. The result of successive actions of this element on the simple roots is reported in Table A.1. Here and in all further tables we choose the following conventions: To each γ_i we associate a column in which we report the powers of the q of the coefficients of the simple roots. We abbreviate

$$\pm(q^{\mu_1^1}+\ldots+q^{\mu_1^{l_1}})\alpha_1 \pm \ldots \pm (q^{\mu_n^1}+\ldots+q^{\mu_n^{l_n}})\alpha_n \to \pm \mu_1^1, \ldots, \mu_{l_1}^1; \ldots; \mu_n^1, \ldots, \mu_{l_n}^1 \ , \quad \text{(A.9)}$$

with $n = \operatorname{rank} \mathfrak{g}$. When q^μ occurs x-times we denote this by μ^x. Like in the undeformed case the overall sign of any element in Ω_i^q is definite. Therefore it suffices to report the sign only once as stated in (A.9). In the complete orbit we always have an equal number of plus and minus signs. When we do not report any signs in the column at all, the signs of the column to the left are adopted. In case the coefficient of the root is zero, we indicate this by a $*$. For instance from Table A.1 we read off : $\sigma_q \gamma_1 = -(q^4 + q^6)\alpha_1 - q^4 \alpha_2$.

σ_q^x	$\alpha_1 = -\gamma_1$	$\alpha_2 = \gamma_2$
1	$4, 6; 4$	$-4, 6, 8; 6$
2	$10; 8$	$-8, 10, 12; 8, 10$
3	$-12; *$	$-*; 12$
4	$-16, 18; 16$	$16, 18, 20; 18$
5	$-22; 20$	$20, 22, 24; 20, 22$
6	$24; *$	$*; 24$

Table A.1: The orbits Ω_i^q created by the action of σ_q^x on γ_i

For the conventions chosen the generating functions (3.68) for the powers of the building blocks are obtainable from the generating functions

$$\begin{aligned} \sum_y \mu_{11}\,(2x+1,y)\,q^y &= -q^1(\,\lambda_1^\vee\cdot(\sigma_q)^x\gamma_1)/2 && \text{(A.10)}\\ \sum_y \mu_{21}\,(2x,y)\,q^y &= -q^{-2}(\,\lambda_1^\vee\cdot(\sigma_q)^x\gamma_2)/2 && \text{(A.11)}\\ \sum_y \mu_{22}\,(2x-1,y)\,q^y &= -q^{-3}\,[3]_q\,(\,\lambda_2^\vee\cdot(\sigma_q)^x\gamma_2)/2\;. && \text{(A.12)} \end{aligned}$$

We may now read off the Lie algebraic data from Table A.1 and we can construct the scattering matrices (A.1), (A.2) and (A.3) according to formula (3.55).

The two non-equivalent solutions to (3.36) corresponding to the S-matrix bootstrap equations (A.4), (A.5) and (A.6) read

$$q\sigma_q^{-1}\gamma_1+q^{-1}\gamma_1=q^{-3}\gamma_2, \qquad q^{-1}\gamma_1+q\sigma_q^{-1}\gamma_1=q^{-3}\gamma_2\,, \tag{A.13}$$

$$q^3\sigma_q^{-1}\gamma_1+q^{-5}\sigma_q\gamma_1=q^{-1}\gamma_1, \qquad q^{-3}\gamma_1+q^5\sigma_q^{-2}\gamma_1=q\sigma_q^{-1}\gamma_1\;, \tag{A.14}$$

$$q^{16}\sigma_q\gamma_2+\sigma_q^5\gamma_2=q^{20}\gamma_2, \qquad q^4\sigma_q^4\gamma_2+q^{20}\gamma_2=\sigma_q^5\gamma_2\;, \tag{A.15}$$

respectively. These relations may be obtained either from (A.4), (A.5) and (A.6) together with the formulae (3.47) which relate the fusing angles to the solution of the fusing rules in terms of the q-deformed Coxeter element or alternatively they may be read off directly from Table A.1. For a direct comparison with (3.47) one should cross all term to one side of the equation by means of (2.61).

It is also instructive to consider explicitly the matrix representation and verify the general formulae (2.66), (2.67) and (3.62) of Section 2.4.3 and 3.2.7. The q–deformed Cartan matrix for generic q and $\hat{q}$ reads

$$A(q,\hat{q})=\begin{pmatrix} q\hat{q}+q^{-1}\hat{q}^{-1} & -1\\ -(1+\hat{q}^2+\hat{q}^{-2}) & q\hat{q}^3+q^{-1}\hat{q}^{-3}\end{pmatrix} \tag{A.16}$$

with determinant $\det A(q,\hat{q})=q^2\hat{q}^4+q^{-2}\hat{q}^{-4}-1$. The right nullvectors are evaluated to

$$\begin{aligned} y(1) &= (\sinh(\theta_h+\theta_H),\sinh(2\theta_h+2\theta_H)) && \text{(A.17)}\\ y(2) &= (\sinh(5\theta_h+5\theta_H),\sinh(10\theta_h+10\theta_H))\;. && \text{(A.18)}\end{aligned}$$

From (A.16) we compute the M-matrix according to (2.66)

$$M(q,\hat{q})=\frac{1-q^{12}\hat{q}^{24}}{2}\begin{pmatrix}\frac{q\hat{q}+q^3\hat{q}^7}{1-q^2\hat{q}^4+q^4\hat{q}^8} & \frac{1+\hat{q}^2+\hat{q}^{-2}}{q^2\hat{q}^4+q^{-2}\hat{q}^{-4}-1}\\ \frac{1+\hat{q}^2+\hat{q}^{-2}}{q^2\hat{q}^4+q^{-2}\hat{q}^{-4}-1} & \frac{(q\hat{q}+q^3\hat{q}^3)(1+\hat{q}^2+\hat{q}^4)}{1+q^2\hat{q}^4+q^4\hat{q}^8}\end{pmatrix}\;. \tag{A.19}$$

Now, careful cancellation against the prefactor produces the polynomials

$$\begin{aligned} M_{11}(q,\hat{q}) =&\ \tfrac{1}{2}(1+q^2\hat{q}^4-q^6\hat{q}^{12}-q^8\hat{q}^{16})(q\hat{q}+q^3\hat{q}^7)\\ M_{12}(q,\hat{q}) =&\ \tfrac{1}{2}(1+\hat{q}^2+\hat{q}^{-2})(q^2\hat{q}^4+q^4\hat{q}^8-q^{10}\hat{q}^{20}-q^8\hat{q}^{16})\\ M_{22}(q,\hat{q}) =&\ \tfrac{1}{2}(1+\hat{q}^2+\hat{q}^4)(q\hat{q}+q^3\hat{q}^3)(1+q^6\hat{q}^{12}-q^2\hat{q}^4-q^8\hat{q}^{16}) \end{aligned}$$

which after expansion give rise to the S-matrix elements via (3.55) and (3.68). Evaluating the M-matrix at roots of unity, $M(e^{s_k\theta_h}, e^{s_k\theta_H})$ with the exponents $s_1 = 1 = h - s_2$, leads to

$$M_{ij}(e^{\theta_h}, e^{\theta_H}) = \frac{2i\sqrt{3}(1+2\cosh\theta_H)}{\sinh(\theta_h+\theta_H)\sinh(2\theta_h+2\theta_H)}\, y_i(1)y_j(1) \qquad \text{(A.20)}$$

$$M_{ij}(e^{5\theta_h}, e^{5\theta_H}) = \frac{-2i\sqrt{3}(1+2\cosh(5\theta_H))}{\sinh(5\theta_h+5\theta_H)\sinh(10\theta_h+10\theta_H)}\, y_i(2)y_j(2)\,, \qquad \text{(A.21)}$$

which confirms equation (3.51) including also the precise factor of proportionality.

$S_{ij}(\theta)$ from $D_4^{(3)}$

Instead of using the data from $G_2^{(1)}$, we can also employ the properties of $D_4^{(3)}$. As indicated in the Dynkin diagram, we choose the values of the bi-colouration to be $c_1 = -1$ and $c_2 = c_3 = c_4 = 1$. Our conventions for the incidence matrix I, the action of $\hat{\tau}$ on the simple roots and the action of the automorphism ω on the simple roots are

$$I = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \hat{\tau}(\vec{\alpha}) = \begin{pmatrix} q^2\alpha_1 \\ q^2\alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}, \qquad \omega(\vec{\alpha}) = \begin{pmatrix} \alpha_1 \\ \alpha_4 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}. \qquad \text{(A.22)}$$

The lengths of the orbits are $l_1 = 1$, $l_2 = l_3 = l_4 = 3$ and the q-deformed twisted Coxeter element reads therefore $\hat{\sigma}_q = \omega^{-1}\hat{\sigma}_1^q\hat{\tau}\hat{\sigma}_2^q$. Successive actions of this element on the representatives of Ω_i^ω are reported in Table A.2.

$\hat{\sigma}_q^x$	$\hat{\alpha}_1 = -\hat{\gamma}_1^\omega$	$\hat{\alpha}_2 = \hat{\gamma}_2^\omega$
1	$*; *; 2; *$	$-2; *; 2; *$
2	$2; *; *; 2$	$-2; *; 4; 2$
3	$2; 2; 4; *$	$-2, 4; 2; 4; 4$
4	$*; *; *; 4$	$-4; 4; 6; 4$
5	$4; 4; *; *$	$-4; 4; *; 6$
6	$-6; *; *; *$	$-*; 6; *; *$
7	$-*; *; 8; *$	$8; *; 8; *$
8	$-8; *; *; 8$	$8; *; 10; 8$
9	$-8; 8; 10; *$	$8, 10; 8; 10; 10$
10	$-*; *; *; 10$	$10; 10; 12; 10$
11	$-10; 10; *; *$	$10; 10; *; 12$
12	$12; *; *; *$	$*; 12; *; *$

Table A.2: The orbits $\hat{\Omega}_i^q$ created by the action of $\hat{\sigma}_q^x$ on γ_i

For the generating functions (3.69) we obtain

$$\sum_x \mu_{11}(x, 2y+1)\, q^x \;=\; -q(\hat{\lambda}_1 \cdot (\hat{\sigma}_q)^y \hat{\gamma}_1^\omega)/2 \tag{A.23}$$

$$\sum_x \mu_{12}(x, 2y)\, q^x \;=\; -(\hat{\lambda}_2 \cdot (\hat{\sigma}_q)^y \hat{\gamma}_1^\omega)/2 \tag{A.24}$$

$$\sum_x \mu_{22}(x, 2y-1)\, q^x \;=\; -q^{-1}(\hat{\lambda}_2 \cdot (\hat{\sigma}_q)^y \hat{\gamma}_2^\omega)/2 \tag{A.25}$$

which yield the scattering matrices (A.1), (A.2) and (A.3) with the help of table 2.

The two non-equivalent solutions to (3.38) corresponding to (A.4), (A.5) and (A.6) read

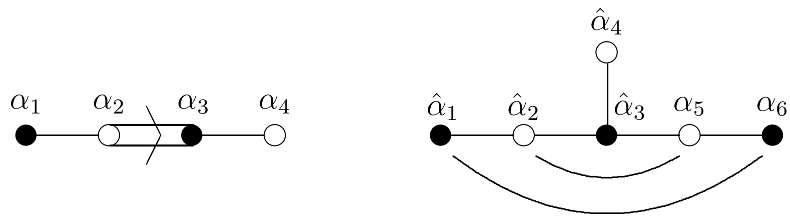
$$q^2\hat{\gamma}_1^\omega + \hat{\sigma}_q\hat{\gamma}_1^\omega = \hat{\sigma}_q\hat{\gamma}_2^\omega, \qquad \hat{\sigma}_q\hat{\gamma}_1^\omega + q^2\hat{\gamma}_1^\omega = \hat{\sigma}_q\hat{\gamma}_2^\omega, \tag{A.26}$$

$$q\hat{\sigma}_q^{-1}\hat{\gamma}_1^\omega + q^{-3}\hat{\sigma}_q^3\hat{\gamma}_1^\omega = q^{-1}\hat{\sigma}_q\hat{\gamma}_1^\omega, \qquad q^{-3}\hat{\sigma}_q^3\hat{\gamma}_1^\omega + q\hat{\sigma}_q^{-1}\hat{\gamma}_1^\omega = q^{-1}\hat{\sigma}_q\hat{\gamma}_1^\omega, \tag{A.27}$$

$$q^{-2}\hat{\sigma}_q^6\hat{\gamma}_2^\omega + q^2\hat{\sigma}_q^2\hat{\gamma}_2^\omega = \hat{\sigma}_q^4\hat{\gamma}_2^\omega, \qquad q^2\hat{\sigma}_q^2\hat{\gamma}_2^\omega + q^{-2}\hat{\sigma}_q^6\hat{\gamma}_2^\omega = \hat{\sigma}_q^4\hat{\gamma}_2^\omega, \tag{A.28}$$

respectively. These relations may be obtained either from (A.4), (A.5) and (A.6) together with the relation (3.48) which relates the fusing angles to the solution of the fusing rules in terms of the q-deformed twisted Coxeter element or alternatively they may be read off directly from Table A.2. Exploiting the relationship between the different versions of the fusing rules (3.53), we may also obtain (A.26), (A.27) and (A.28) from (A.13), (A.14) and (A.15).

A.0.2 $(F_4^{(1)}, E_6^{(2)})$



The S-matrices of the theory read [28]

$$
\begin{aligned}
S_{11}(\theta) &= \{1,1_2;5,7_2;7,9_2;11,15_2\}_\theta \\
S_{12}(\theta) &= \{2,3_2;4,5_2;6,7_2;6,9_2;8,11_2;10,13_2\}_\theta \\
S_{13}(\theta) &= \{3,4_2;5,6_2;7,10_2;9,12_2\}_\theta \\
S_{14}(\theta) &= \{4,5_2;8,11_2\}_\theta \\
S_{22}(\theta) &= \{1,1_2;3,3_2;3,5_2;5,5_2;5,7_2^2;7,9_2^2;7,11_2;9,11_2;9,13_2;11,15_2\}_\theta \\
S_{23}(\theta) &= \{2,2_2;4,4_2;4,6_2;6,8_2^2;8,10_2;8,12_2;10,14_2\}_\theta \\
S_{24}(\theta) &= \{3,3_2;5,7_2;7,9_2;9,13_2\}_\theta \\
S_{33}(\theta) &= \{1,1;3,3_2;5,7;5,7_2;7,9_2;7,11;9,13_2;11,17\}_\theta \\
S_{34}(\theta) &= \{2,2;4,6;6,8_2;8,12;10,16\}_\theta \\
S_{44}(\theta) &= \{1,1;5,7;7,11;11,17\}_\theta \,.
\end{aligned}
$$

We have $h = 12$ and $H = 18$ for the Coxeter numbers. We will not report here all boostrap identities, but we state the combined bootstrap identities (3.71)

$$S_{1l}(\theta + \theta_h + 2\theta_H)S_{1l}(\theta - \theta_h - 2\theta_H) = S_{l2}(\theta) \tag{A.29}$$
$$S_{2l}(\theta + \theta_h + 2\theta_H)S_{2l}(\theta - \theta_h - 2\theta_H) = S_{l1}(\theta)S_{l3}(\theta - \theta_H)S_{l3}(\theta + \theta_H) \tag{A.30}$$
$$S_{3l}(\theta + \theta_h + \theta_H)S_{3l}(\theta - \theta_h - \theta_H) = S_{l2}(\theta)S_{l4}(\theta) \tag{A.31}$$
$$S_{4l}(\theta + \theta_h + \theta_H)S_{4l}(\theta - \theta_h - \theta_H) = S_{l3}(\theta) \tag{A.32}$$

for $l = 1, 2, 3, 4$. Once again there occurs one equation which is more involved than the usual bootstrap which we depict in Figure A.3.

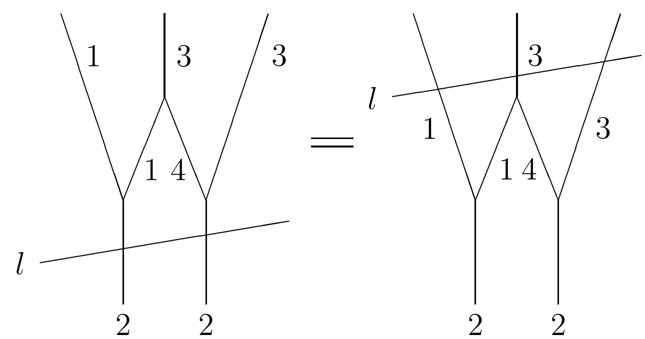


Figure A.3: $(F_4^{(1)}, E_6^{(2)})$-combined bootstrap identities (A.30).

Reading off the fusing angles from the bootstrap equations we obtain the mass ratios from (3.49)

$$\frac{m_1}{m_2} = \frac{\sinh(\theta_h + 2\theta_H)}{\sinh(10\theta_h + 14\theta_H)} \qquad \frac{m_1}{m_3} = \frac{\sinh(3\theta_h + 5\theta_H)}{\sinh(7\theta_h + 10\theta_H)} \tag{A.33}$$
$$\frac{m_1}{m_4} = \frac{\sinh(3\theta_h + 5\theta_H)}{\sinh(2\theta_h + 3\theta_H)} \qquad \frac{m_2}{m_3} = \frac{\sinh(9\theta_h + 15\theta_H)}{\sinh(2\theta_h + 2\theta_H)} \tag{A.34}$$
$$\frac{m_2}{m_4} = \frac{\sinh(9\theta_h + 15\theta_H)}{\sinh(\theta_h + \theta_H)} \qquad \frac{m_3}{m_4} = \frac{\sinh(2\theta_h + 2\theta_H)}{\sinh(\theta_h + \theta_H)}. \tag{A.35}$$

As in the previous case these formulae can be re-constructed from the twisted as well as the untwisted Lie algebra.

$S_{ij}(\theta)$ from $F_4^{(1)}$

According to our conventions the q-deformed Coxeter element reads in terms of simple Weyl reflections $\sigma_q = \sigma_1^q \sigma_3^q \tau \sigma_2^q \sigma_4^q \tau$. The result of successive actions of this element on the simple roots is reported in Table A.3.

σ_q^x	$\alpha_1=-\gamma_1$	$\alpha_3=-\gamma_3$	$\alpha_2=\gamma_2$	$\alpha_4=\gamma_4$
1	$*;4;3,5;*$	$3;3;2,4;2$	$-4;4;3,5;*$	$*;*;2;2$
2	$6;6;5,7;5,7$	$5;5,7;6^2,8;6$	$-6;6,8;5,7^2,9;5,7$	$5;5;6;*$
3	$8;8,10;9,11;*$	$9;9^2;8,10^2;8,10$	$-8,10;8,10^2;9^2,11^2;9,11$	$*;9;8,10;8$
4	$*;12;11,13;11,13$	$11;11,13;12,14;12$	$-12;12^2,14;11,13^2,15;11,13$	$11;11;12;12$
5	$14;14;*;*$	$*;15;16;16$	$-14;14,16;15,17;15,17$	$*;15;16;*$
6	$-18;*;*;*$	$*;*;18;*$	$-*;18;*;*$	$*;*;*;18$
7	$-*;22;21,23;*$	$21;21;20,22;20$	$22;22;21,23;*$	$*;*;20;20$
8	$-24;24;23,25;23,25$	$23;23,25;24^2,26;24$	$24;24,26;23,25^2,27;23,25$	$23;23;24;*$
9	$-26;26,28;27,29;*$	$27;27^2;26,28^2;26,28$	$26,28;26,28^2;27^2,29^2;27,29$	$*;27;26,28;26$
10	$-*;30;29,31;29,31$	$29;29,31;30,32;30$	$30;30^2,32;29,31^2,33;29,31$	$29;29;30;30$
11	$-32;32;*;*$	$*;33;34;34$	$32;32,34;33,35;33,35$	$*;33;34;*$
12	$36;*;*;*$	$*;*;36;*$	$*;36;*;*$	$*;*;*;36$

Table A.3: The orbits Ω_i^q created by the action of σ_q^x on γ_i.

By using Table A.3 we may recover the $(F_4^{(1)}, E_6^{(2)})-$S-matrices with the help of generating functions (3.68). The two non-equivalent solutions of the fusing rule in Ω^q are

$$\begin{aligned}
\gamma_l+q^{-12}\sigma_q^4\gamma_l &= q^{-6}\sigma_q^2\gamma_l, \quad \sigma_q^{-1}\gamma_l+q^{12}\sigma_q^{-5}\gamma_l=q^6\sigma_q^{-3}\gamma_l, \quad l=1,2,3,4\\
\gamma_1+q^{-4}\sigma_q\gamma_1 &= q^{-4}\sigma_q\gamma_2, \quad \sigma_q^{-1}\gamma_1+q^4\sigma_q^{-2}\gamma_1=\sigma_q^{-1}\gamma_2,\\
\gamma_2+q^{-14}\sigma_q^5\gamma_1 &= \gamma_1, \quad q^{-4}\gamma_2+q^{14}\sigma_q^{-6}\gamma_1=\sigma_q^{-1}\gamma_1,\\
\gamma_4+q^{-2}\sigma_q\gamma_4 &= \gamma_3, \quad q^{-2}\gamma_4+\sigma_q^{-1}\gamma_4=\sigma_q^{-1}\gamma_3,\\
\gamma_4+q^{-16}\sigma_q^5\gamma_3 &= q^{-16}\sigma_q^5\gamma_4, \quad q^{-2}\gamma_4+q^{16}\sigma_q^{-6}\gamma_3=q^{14}\sigma_q^{-5}\gamma_4,\\
\gamma_1+q^{-15}\sigma_q^5\gamma_3 &= q^{-11}\sigma_q^4\gamma_4, \quad \sigma_q^{-1}\gamma_1+q^{15}\sigma_q^{-6}\gamma_3=q^9\sigma_q^{-4}\gamma_4,\\
\gamma_1+q^{-9}\sigma_q^3\gamma_4 &= q^{-3}\sigma_q\gamma_3, \quad \sigma_q^{-1}\gamma_1+q^7\sigma_q^{-3}\gamma_4=q^3\sigma_q^{-2}\gamma_3,\\
\gamma_3+q^{-14}\sigma_q^5\gamma_4 &= q^{-3}\sigma_q\gamma_1, \quad \sigma_q^{-1}\gamma_3+q^{12}\sigma_q^{-5}\gamma_4=q^3\sigma_q^{-2}\gamma_1,\\
\gamma_2+q^{-15}\sigma_q^5\gamma_3 &= q^{-1}\sigma_q\gamma_4, \quad q^{-4}\gamma_2+q^{15}\sigma_q^{-6}\gamma_3=q^{-1}\sigma_q^{-1}\gamma_4,\\
\gamma_2+q^{-15}\sigma_q^5\gamma_4 &= q\gamma_3, \quad q^{-4}\gamma_2+q^{13}\sigma_q^{-5}\gamma_4=q^{-1}\sigma_q^{-1}\gamma_3,\\
\gamma_3+q^{-4}\sigma_q^2\gamma_4 &= q^{-3}\sigma_q\gamma_2, \quad \sigma_q^{-1}\gamma_3+q^2\sigma_q^{-2}\gamma_4=q^{-1}\sigma_q^{-1}\gamma_2,\\
\gamma_4+q^{-8}\sigma_q^3\gamma_4 &= q^{-3}\sigma_q\gamma_1, \quad q^{-2}\gamma_4+q^6\sigma_q^{-3}\gamma_4=q^3\sigma_q^{-2}\gamma_1,\\
\gamma_4+q^{-13}\sigma_q^4\gamma_1 &= q^{-10}\sigma_q^3\gamma_4, \quad q^{-2}\gamma_4+q^{13}\sigma_q^{-5}\gamma_1=q^8\sigma_q^{-3}\gamma_4.
\end{aligned}$$

Once again we can confirm from these solution the equivalence of the bootstrap equations and the fusing rules by means of (3.47) and also verify the relation for the mass ratios (3.49).

$S_{ij}(\theta)$ from $\hat{E}_6^{(2)}$

The q-deformed twisted Coxeter element in the conventions stated reads explicitly $\hat{\sigma}_q=\omega^{-1}\hat{\sigma}_1^q\hat{\sigma}_3^q\hat{\tau}\hat{\sigma}_2^q\hat{\sigma}_4^q$. The successive actions of this element on the representatives of Ω_i^ω are reported in Table A.4.

$\hat{\sigma}_q^x$	$\alpha_6 = -\hat{\gamma}_1^\omega$	$\hat{\alpha}_3 = -\hat{\gamma}_3^\omega$	$\hat{\alpha}_2 = \hat{\gamma}_2^\omega$	$\hat{\alpha}_4 = \hat{\gamma}_4^\omega$
1	$0; \ast; \ast; \ast; \ast; \ast$	$\ast; \ast; 2; 2; 2; 2$	$-\ast; \ast; 2; \ast; 2; 2$	$\ast; \ast; 2; 2; \ast; \ast$
2	$\ast; \ast; 2; \ast; 2; \ast$	$2; 2; 2; \ast; 4; 4$	$-2; 2; 2, 4; 4; 4; 4$	$\ast; \ast; \ast; \ast; 4; 4$
3	$\ast; 2; 2, 4; 4; 4; 4$	$4; 4; 4^2; 4; 4; \ast$	$-4; 4; 4^2; 4; 4, 6; 6$	$4; 4; 4; \ast; \ast; \ast$
4	$4; 4; 4; 4; 6; 6$	$\ast; 4; 4, 6; 6; 6^2; 6$	$-6; 4, 6; 4, 6^2; 6; 6^2; 6$	$\ast; \ast; 6; 6; 6; \ast$
5	$6; 6; 6; \ast; 6; \ast$	$6; 6^2; 6^2; 6; 8; 8$	$-6; 6^2; 6^2, 8; 6, 8; 8^2; 8$	$\ast; 6; 6; \ast; 8; 8$
6	$\ast; 6; 6, 8; 8; 8; \ast$	$8; 8; 8; 8; 8; \ast$	$-8; 8^2; 8^2; 8; 8, 10; 10$	$8; 8; 8; 8; \ast; \ast$
7	$\ast; 8; 8; 8; 10; 10$	$\ast; 8; 8; \ast; 10; \ast$	$-10; 8, 10; 8, 10; 10; 10; \ast$	$\ast; \ast; \ast; \ast; 10; \ast$
8	$10; 10; \ast; \ast; \ast; \ast$	$\ast; 10; 10; 10; \ast; \ast$	$-\ast; 10; 10; 10; 12; \ast$	$\ast; 10; 10; \ast; \ast; \ast$
9	$-\ast; \ast; \ast; \ast; \ast; 12$	$\ast; \ast; 12; \ast; \ast; \ast$	$-\ast; 12; \ast; \ast; \ast; \ast$	$\ast; \ast; \ast; 12; \ast; \ast$
10	$-12; \ast; \ast; \ast; \ast; \ast$	$\ast; \ast; 14; 14; 14; 14$	$\ast; \ast; 14; \ast; 14; 14$	$\ast; \ast; 14; 14\ast; \ast$
11	$-\ast; \ast; 14; \ast; 14; \ast$	$14; 14; 14; \ast; 16; 16$	$14; 14; 14, 16; 16; 16; 16$	$\ast; \ast; \ast; \ast; 16; 16$
12	$-\ast; 14;14,16;16;16;16$	$16; 16; 16^2; 16; 16; \ast$	$16; 16; 16^2; 16; 16, 18; 18$	$16; 16; 16; \ast; \ast; \ast$
13	$-16; 16; 16; 16; 18; 18$	$\ast;16;16,18;18;18^2;18$	$18;16,18;16,18^2;18;18^2;18$	$\ast; \ast; 18; 18; 18; \ast$
14	$-18; 18; 18; \ast; 18; \ast$	$18; 18^2; 18^2; 18; 20; 20$	$18;18^2;18^2,20;18,20;20^2;20$	$\ast; 18; 18; \ast; 20; 20$
15	$-\ast; 18; 18, 20; 20; 20; \ast$	$20; 20; 20; 20; 20; \ast$	$20; 20^2; 20^2; 20; 20, 22; 22$	$20; 20; 20; 20; \ast; \ast$
16	$-\ast; 20; 20; 20; 22; 22$	$\ast; 20; 20; \ast; 22; \ast$	$22; 20, 22; 20, 22; 22; 22; \ast$	$\ast; \ast; \ast; \ast; 22; \ast$
17	$-22; 22; \ast; \ast; \ast; \ast$	$\ast; 22; 22; 22; \ast; \ast$	$\ast; 22; 22; 22; 24; \ast$	$\ast; 22; 22; \ast; \ast; \ast$
18	$\ast; \ast; \ast; \ast; \ast; 24$	$\ast; \ast; 24; \ast; \ast; \ast$	$\ast; 24; \ast; \ast; \ast; \ast$	$\ast; \ast; \ast; 24; \ast; \ast$

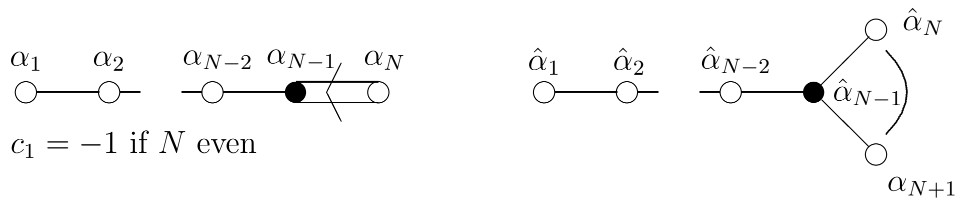
Table A.4: The orbits $\hat{\Omega}_i^q$ created by the action of $\hat{\sigma}_q^x$ on γ_i.

Using the orbits $\hat{\Omega}_i^q$ listed in Table A.4 we recover with help of the generating functions (3.69) the $(F_4^{(1)}, E_6^{(2)})$-S-matrices. The two non-equivalent solutions to the fusing rule in $\hat{\Omega}_q$ read

$$
\begin{aligned}
\hat{\gamma}_l^\omega + q^{-8}\hat{\sigma}_q^6\hat{\gamma}_l^\omega &= q^{-4}\hat{\sigma}_q^3\hat{\gamma}_l^\omega, \quad q^2\hat{\sigma}_q^2\hat{\gamma}_l^\omega + q^{10}\hat{\sigma}_q^{-4}\hat{\gamma}_l^\omega = q^6\hat{\sigma}_q^{-1}\hat{\gamma}_l^\omega, \quad l = 1,2,3,4 \\
\hat{\gamma}_1^\omega + q^{-2}\hat{\sigma}_q^2\hat{\gamma}_1^\omega &= q^{-2}\hat{\sigma}_q\hat{\gamma}_2^\omega, \quad q^2\hat{\sigma}_q^2\hat{\gamma}_1^\omega + q^4\hat{\gamma}_1^\omega = q^2\hat{\sigma}_q\hat{\gamma}_2^\omega, \\
\hat{\gamma}_2^\omega + q^{-10}\hat{\sigma}_q^8\hat{\gamma}_1^\omega &= \hat{\sigma}_q\hat{\gamma}_1^\omega, \quad \hat{\sigma}_q^2\hat{\gamma}_2^\omega + q^{12}\hat{\sigma}_q^{-6}\hat{\gamma}_1^\omega = q^2\hat{\sigma}_q\hat{\gamma}_1^\omega, \\
\hat{\gamma}_4^\omega + q^{-2}\hat{\sigma}_q\hat{\gamma}_4^\omega &= \hat{\gamma}_3^\omega, \quad \hat{\sigma}_q^2\hat{\gamma}_4^\omega + q^2\hat{\sigma}_q\hat{\gamma}_4^\omega = q^2\hat{\sigma}_q\hat{\gamma}_3^\omega, \\
\hat{\gamma}_4^\omega + q^{-10}\hat{\sigma}_q^8\hat{\gamma}_3^\omega &= q^{-10}\hat{\sigma}_q^8\hat{\gamma}_4^\omega, \quad \hat{\sigma}_q^2\hat{\gamma}_4^\omega + q^{12}\hat{\sigma}_q^{-7}\hat{\gamma}_3^\omega = q^{10}\hat{\sigma}_q^{-6}\hat{\gamma}_4^\omega, \\
\hat{\gamma}_1^\omega + q^{-10}\hat{\sigma}_q^7\hat{\gamma}_3^\omega &= q^{-8}\hat{\sigma}_q^5\hat{\gamma}_4^\omega, \quad q^2\hat{\sigma}_q^2\hat{\gamma}_1^\omega + q^{12}\hat{\sigma}_q^{-6}\hat{\gamma}_3^\omega = q^8\hat{\sigma}_q^{-3}\hat{\gamma}_4^\omega, \\
\hat{\gamma}_1^\omega + q^{-6}\hat{\sigma}_q^4\hat{\gamma}_4^\omega &= q^{-2}\hat{\sigma}_q\hat{\gamma}_3^\omega, \quad q^2\hat{\sigma}_q^2\hat{\gamma}_1^\omega + q^6\hat{\sigma}_q^{-2}\hat{\gamma}_4^\omega = q^4\hat{\gamma}_3^\omega, \\
\hat{\gamma}_3^\omega + q^{-10}\hat{\sigma}_q^7\hat{\gamma}_4^\omega &= q^{-2}\hat{\sigma}_q^2\hat{\gamma}_1^\omega, \quad q^2\hat{\sigma}_q\hat{\gamma}_3^\omega + q^{10}\hat{\sigma}_q^{-5}\hat{\gamma}_4^\omega = q^4\hat{\gamma}_1^\omega, \\
\hat{\gamma}_2^\omega + q^{-10}\hat{\sigma}_q^8\hat{\gamma}_3^\omega &= q^{-2}\hat{\sigma}_q\hat{\gamma}_4^\omega, \quad \hat{\sigma}_q^2\hat{\gamma}_2^\omega + q^{12}\hat{\sigma}_q^{-7}\hat{\gamma}_3^\omega = q^2\hat{\sigma}_q\hat{\gamma}_4^\omega, \\
\hat{\gamma}_2^\omega + q^{-10}\hat{\sigma}_q^8\hat{\gamma}_4^\omega &= \hat{\gamma}_3^\omega, \quad \hat{\sigma}_q^2\hat{\gamma}_2^\omega + q^{10}\hat{\sigma}_q^{-6}\hat{\gamma}_4^\omega = q^2\hat{\sigma}_q\hat{\gamma}_3^\omega, \\
\hat{\gamma}_3^\omega + q^{-4}\hat{\sigma}_q^2\hat{\gamma}_4^\omega &= q^{-2}\hat{\sigma}_q\hat{\gamma}_2^\omega, \quad q^2\hat{\sigma}_q\hat{\gamma}_3^\omega + q^4\hat{\gamma}_4^\omega = q^2\hat{\sigma}_q\hat{\gamma}_2^\omega, \\
\hat{\gamma}_4^\omega + q^{-6}\hat{\sigma}_q^4\hat{\gamma}_4^\omega &= q^{-2}\hat{\sigma}_q^2\hat{\gamma}_1^\omega, \quad \hat{\sigma}_q^2\hat{\gamma}_4^\omega + q^6\hat{\sigma}_q^{-2}\hat{\gamma}_4^\omega = q^4\hat{\gamma}_1^\omega, \\
\hat{\gamma}_4^\omega + q^{-8}\hat{\sigma}_q^7\hat{\gamma}_1^\omega &= q^{-6}\hat{\sigma}_q^5\hat{\gamma}_4^\omega, \quad \hat{\sigma}_q^2\hat{\gamma}_4^\omega + q^{10}\hat{\sigma}_q^{-5}\hat{\gamma}_1^\omega = q^6\hat{\sigma}_q^{-3}\hat{\gamma}_4^\omega.
\end{aligned}
$$

Again we confirm from these solution the equivalence between the bootstrap equations and the fusing rules by means of (3.48) and also verify the relation for the mass ratios (3.49).

A.0.3 $(C_2^{(1)}, D_3^{(2)})$



The S-matrices are given as

$$S_{11}(\theta) = \{1, 1; 3, 5\}_\theta \quad S_{12}(\theta) = \{2, 2_2\}_\theta \quad S_{22}(\theta) = \{1, 1_2; 3, 3_2\}_\theta.$$

We have $h = 4$ and $H = 6$ for the Coxeter numbers. The combined bootstrap equations (3.71) yield

$$S_{1l}(\theta + \theta_h + \theta_H)S_{1l}(\theta - \theta_h - \theta_H) = S_{l2}(\theta) \tag{A.36}$$
$$S_{2l}(\theta + \theta_h + 2\theta_H)S_{2l}(\theta - \theta_h - 2\theta_H) = S_{l1}(\theta - \theta_H)S_{l1}(\theta + \theta_H) \tag{A.37}$$

for $l = 1, 2$.

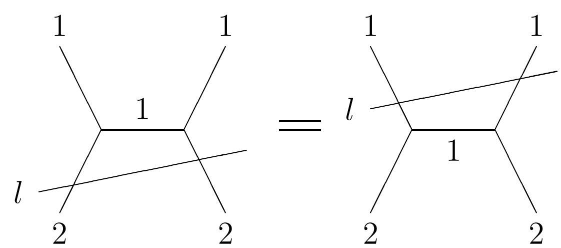


Figure A.4: $(C_2^{(1)}, D_3^{(2)})$-combined bootstrap identities (A.37).

The mass ratio according to (3.44) are

$$\frac{m_1}{m_2} = \frac{\sinh(\theta_h + \theta_H)}{\sinh(2\theta_h + 4\theta_H)}. \tag{A.38}$$

$S_{ij}(\theta)$ **from** $C_2^{(1)}$**:**

The result of successive actions of the q-deformed Coxeter element on the simple roots is reported in Table A.5.

σ_q^x	$\alpha_1 = -\gamma_1$	$\alpha_2 = \gamma_2$
1	$4; 3$	$-3, 5; 4$
2	$-6; *$	$-*; 6$
3	$-10; 9$	$9, 11; 10$
4	$12; *$	$*; 12$

Table A.5: The orbits Ω_i^q created by the action of σ_q^x on γ_i

The two non-equivalent solutions to the fusing rule in Ω_q read

$$\begin{aligned}\gamma_1 + q^{-2}\sigma_q\gamma_1 &= q^{-3}\sigma_q\gamma_2, \quad \sigma_q^{-1}\gamma_1 + q^2\sigma_q^{-2}\gamma_1 = q^{-1}\sigma_q^{-1}\gamma_2,\\ \gamma_1 + q^{-7}\sigma_q^2\gamma_2 &= q^{-4}\sigma_q\gamma_1, \quad \sigma_q^{-1}\gamma_1 + q^3\sigma_q^{-2}\gamma_2 = q^4\sigma_q^{-2}\gamma_1.\end{aligned}$$

$S_{ij}(\theta)$ **from** $\hat{D}_3^{(2)}$**:**

The result of successive actions of the q-deformed twisted Coxeter element on the simple roots is reported in Table A.6.

$\hat{\sigma}_q^x$	$\hat{\alpha}_1 = -\hat{\gamma}_1^\omega$	$\hat{\alpha}_2 = \hat{\gamma}_2^\omega$
1	$*;*;2$	$-2;*;2$
2	$2;2;*$	$-2;2;4$
3	$-4;*;*$	$-*;4;*$
4	$-*;*;6$	$6;*;6$
5	$-6;6;*$	$6;6;8$
6	$8;*;*$	$*;8;*$

Table A.6: The orbits $\hat{\Omega}_i^q$ created by the action of $\hat{\sigma}_q^x$ on $\hat{\gamma}_i^\omega$

The two non-equivalent solutions to the fusing rule in $\hat{\Omega}_q$ read

$$\begin{aligned}\hat{\gamma}_1^\omega + q^{-2}\hat{\sigma}_q\hat{\gamma}_1^\omega &= q^{-2}\hat{\sigma}_q\hat{\gamma}_2^\omega, \quad q^2\hat{\sigma}_q\hat{\gamma}_1^\omega + q^4\hat{\gamma}_1^\omega = q^2\hat{\sigma}_q\hat{\gamma}_2^\omega,\\ \hat{\gamma}_1^\omega + q^{-4}\hat{\sigma}_q^3\hat{\gamma}_2^\omega &= q^{-2}\hat{\sigma}_q^2\hat{\gamma}_1^\omega, \quad q^2\hat{\sigma}_q\hat{\gamma}_1^\omega + q^4\hat{\sigma}_q^{-1}\hat{\gamma}_2^\omega = q^4\hat{\sigma}_q^{-1}\hat{\gamma}_1^\omega.\end{aligned}$$

A.0.4 $(C_3^{(1)}, D_4^{(2)})$

The S-matrices are

$$\begin{aligned}S_{11}(\theta) &= \{1,1;5,7\}_\theta \quad S_{12}(\theta) = \{2,2;4,6\}_\theta \quad S_{33}(\theta) = \{1,1_2;3,3_2;5,5_2\}_\theta\\ S_{22}(\theta) &= \{1,1;3,3_2;5,7\}_\theta \quad S_{23}(\theta) = \{2,2_2;4,4_2\}_\theta \quad S_{13}(\theta) = \{3,3_2\}_\theta.\end{aligned}$$

We have $h = 6$ and $H = 8$ for the Coxeter numbers. The combined bootstrap identities read

$$S_{1l}(\theta+\theta_h+\theta_H)S_{1l}(\theta-\theta_h-\theta_H) = S_{l2}(\theta) \tag{A.39}$$
$$S_{2l}(\theta+\theta_h+\theta_H)S_{2l}(\theta-\theta_h-\theta_H) = S_{l1}(\theta)S_{l3}(\theta) \tag{A.40}$$
$$S_{3l}(\theta+\theta_h+2\theta_H)S_{3l}(\theta-\theta_h-2\theta_H) = S_{l2}(\theta-\theta_H)S_{l2}(\theta+\theta_H). \tag{A.41}$$

The mass ratios turn out to be

$$\frac{m_1}{m_2} = \frac{\sinh(\theta_h+\theta_H)}{\sinh(4\theta_h+6\theta_H)} \quad \frac{m_1}{m_3} = \frac{\sinh(\theta_h+\theta_H)}{\sinh(3\theta_h+5\theta_H)} \quad \frac{m_2}{m_3} = \frac{\sinh(2\theta_h+2\theta_H)}{\sinh(3\theta_h+5\theta_H)}. \tag{A.42}$$

$S_{ij}(\theta)$ from $C_3^{(1)}$

The result of successive actions of the q-deformed Coxeter element on the simple roots is reported in Table A.7.

σ_q^x	$\alpha_1 = \gamma_1$	$\alpha_3 = \gamma_3$	$\alpha_2 = -\gamma_2$
1	$-2; 2; *$	$*; 3, 5; 4$	$2; 2, 4; 3$
2	$-*; 6; 5$	$5, 7; 5, 7; 6$	$6; 6; 5$
3	$-8; *; *$	$*; *; 8$	$-*; 8; *$
4	$10; 10; *$	$*; 11, 13; 12$	$-10; 10, 12; 11$
5	$*; 14; 13$	$13, 15; 13, 15; 14$	$-14; 14; 13$
6	$16; *; *$	$*; *; 16$	$*; 16; *$

Table A.7 : The orbits Ω_i^q created by the action of σ_q^x on γ_i

The solutions of the fusing rule in Ω^q are

$$\begin{aligned}
\gamma_1 + q^{-2}\sigma_q\gamma_1 &= \gamma_2, \quad q^{-2}\gamma_1 + \sigma_q^{-1}\gamma_1 = \sigma_q^{-1}\gamma_2, \\
\gamma_1 + q^{-6}\sigma_q^2\gamma_2 &= q^{-6}\sigma_q^2\gamma_1, \quad q^{-2}\gamma_1 + q^6\sigma_q^{-3}\gamma_2 = q^4\sigma_q^{-2}\gamma_1, \\
\gamma_1 + q^{-2}\sigma_q\gamma_2 &= q^{-3}\sigma_q\gamma_3, \quad q^{-2}\gamma_1 + q^2\sigma_q^{-2}\gamma_2 = q^{-1}\sigma_q^{-1}\gamma_3, \\
\gamma_1 + q^{-7}\sigma_q^2\gamma_3 &= q^{-4}\sigma_q\gamma_2, \quad q^{-2}\gamma_1 + q^3\sigma_q^{-2}\gamma_3 = q^4\sigma_q^{-2}\gamma_2, \\
\gamma_2 + q^{-9}\sigma_q^3\gamma_3 &= q^{-6}\sigma_q^2\gamma_1, \quad \sigma_q^{-1}\gamma_2 + q^5\sigma_q^{-3}\gamma_3 = q^4\sigma_q^{-2}\gamma_1.
\end{aligned}$$

$S_{ij}(\theta)$ from $\hat{D}_4^{(2)}$

The result of successive actions of the q-deformed twisted Coxeter element on the simple roots is reported in Table A.8.

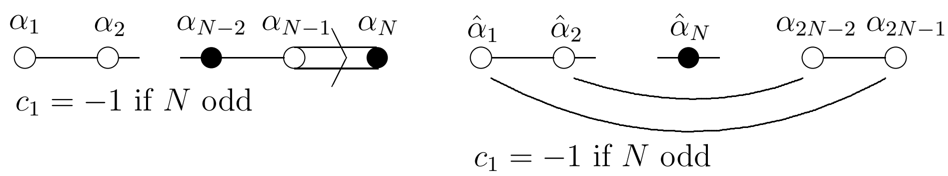
$\hat{\sigma}_q^x$	$\hat{\alpha}_1 = \hat{\gamma}_1^\omega$	$\hat{\alpha}_3 = \hat{\gamma}_3^\omega$	$\hat{\alpha}_2 = -\hat{\gamma}_2^\omega$
1	$-2; 2; *; *$	$*; 2; *; 2$	$2; 2; *; 2$
2	$-*; *; *; 4$	$4; 2, 4; 2; 4$	$*; 2; 2; 4$
3	$-*; 4; 4; *$	$4; 4; 4; 6$	$4; 4; 4; *$
4	$-6; *; *; *$	$*; *; 6; *$	$-*; 6; *; *$
5	$8; 8; *; *$	$*; 8; *; 8$	$-8; 8; *; 8$
6	$*; *; *; 10$	$10; 8, 10; 8; 10$	$-*; 8; 8; 10$
7	$*; 10; 10; *$	$10; 10; 10; 12$	$-10; 10; 10; *$
8	$12; *; *; *$	$*; *; 12; *$	$*; 12; *; *$

Table A.8 : The orbits $\hat{\Omega}_i^q$ created by the action of $\hat{\sigma}_q^x$ on $\hat{\gamma}_i^\omega$

The solutions of the fusing rule in $\hat{\Omega}^q$ are

$$\begin{aligned}
\hat{\gamma}_1^\omega + q^{-2}\hat{\sigma}_q\hat{\gamma}_1^\omega &= \hat{\gamma}_2^\omega, \quad \hat{\sigma}_q^2\hat{\gamma}_1^\omega + q^2\hat{\sigma}_q\hat{\gamma}_1^\omega = q^2\hat{\sigma}_q\hat{\gamma}_2^\omega, \\
\hat{\gamma}_1^\omega + q^{-4}\hat{\sigma}_q^3\hat{\gamma}_2^\omega &= q^{-4}\hat{\sigma}_q^3\hat{\gamma}_1^\omega, \quad \hat{\sigma}_q^2\hat{\gamma}_1^\omega + q^6\hat{\sigma}_q^{-2}\hat{\gamma}_2^\omega = q^4\hat{\sigma}_q^{-1}\hat{\gamma}_1^\omega, \\
\hat{\gamma}_1^\omega + q^{-2}\hat{\sigma}_q\hat{\gamma}_2^\omega &= q^{-2}\hat{\sigma}_q\hat{\gamma}_3^\omega, \quad \hat{\sigma}_q^2\hat{\gamma}_1^\omega + q^4\hat{\gamma}_2^\omega = q^2\hat{\sigma}_q\hat{\gamma}_3^\omega, \\
\hat{\gamma}_1^\omega + q^{-4}\hat{\sigma}_q^3\hat{\gamma}_3^\omega &= q^{-2}\hat{\sigma}_q^2\hat{\gamma}_2^\omega, \quad \hat{\sigma}_q^2\hat{\gamma}_1^\omega + q^4\hat{\sigma}_q^{-1}\hat{\gamma}_3^\omega = q^4\hat{\sigma}_q^{-1}\hat{\gamma}_2^\omega, \\
\hat{\gamma}_2^\omega + q^{-6}\hat{\sigma}_q^4\hat{\gamma}_3^\omega &= q^{-4}\hat{\sigma}_q^3\hat{\gamma}_1^\omega, \quad q^2\hat{\sigma}_q\hat{\gamma}_2^\omega + q^6\hat{\sigma}_q^{-2}\hat{\gamma}_3^\omega = q^4\hat{\sigma}_q^{-1}\hat{\gamma}_1^\omega.
\end{aligned}$$

A.0.5 $(B_2^{(1)}, A_3^{(2)})$



The S-matrices read

$$S_{11}(\theta) = \{1, 1_2; 3, 3_2\}_\theta \quad S_{12}(\theta) = \{2, 2_2\}_\theta \quad S_{22}(\theta) = \{1, 1; 3, 5\}_\theta.$$

We have $h = 4$ and $H = 6$ for the Coxeter numbers. The combined bootstrap identities are

$$S_{1l}(\theta + \theta_h + 2\theta_H)S_{1l}(\theta - \theta_h - 2\theta_H) = S_{l2}(\theta - \theta_H)S_{l2}(\theta + \theta_H) \qquad \text{(A.43)}$$

$$S_{2l}(\theta + \theta_h + \theta_H)S_{2l}(\theta - \theta_h - \theta_H) = S_{l1}(\theta). \qquad \text{(A.44)}$$

The mass ratio is

$$\frac{m_1}{m_2} = \frac{\sinh(2\theta_h + 4\theta_H)}{\sinh(\theta_h + \theta_H)}. \qquad \text{(A.45)}$$

$S_{ij}(\theta)$ **from** $B_2^{(1)}$

The result of successive actions of the q-deformed Coxeter element on the simple roots is reported in Table A.9.

σ_q^x	$\alpha_1 = \gamma_1$	$\alpha_2 = -\gamma_2$
1	$-4; 3, 5$	$3; 4$
2	$-6; *$	$-*; 6$
3	$10; 9, 11$	$-9; 10$
4	$12; *$	$*; 12$

Table A.9: The orbits Ω_i^q created by the action of σ_q^x on γ_i

Solutions of the fusing rule in Ω^q

$$\gamma_1 + q^{-3}\sigma_q\gamma_2 = q\gamma_2, \quad q^{-4}\gamma_1 + q^3\sigma_q^{-2}\gamma_2 = q^{-1}\sigma_q^{-1}\gamma_2,$$

$$\gamma_2 + q^{-2}\sigma_q\gamma_2 = q^{-3}\sigma_q\gamma_1, \quad \sigma_q^{-1}\gamma_2 + q^2\sigma_q^{-2}\gamma_2 = q^{-1}\sigma_q^{-1}\gamma_1.$$

$S_{ij}(\theta)$ **from** $\hat{A}_3^{(2)}$

The result of successive actions of the q-deformed twisted Coxeter element on the simple roots is reported in Table A.10.

$\hat{\sigma}_q^x$	$\hat{\alpha}_1 = \hat{\gamma}_1^\omega$	$\hat{\alpha}_2 = -\hat{\gamma}_2^\omega$
1	$-*;2;2$	$*;*;2$
2	$-2;2;4$	$2;2;*$
3	$-4;*;*$	$-*;4;*$
4	$*;6;6$	$-*;*;6$
5	$6;6;8$	$-6;6;*$
6	$8;*;*$	$*;8;*$

Table A.10 : The orbits $\hat{\Omega}_i^q$ created by the action of $\hat{\sigma}_q^x$ on $\hat{\gamma}_i^\omega$

The solutions to the fusing rule in $\hat{\Omega}^q$

$$\begin{aligned}\hat{\gamma}_1^\omega + q^{-2}\hat{\sigma}_q^2\hat{\gamma}_2^\omega &= \hat{\gamma}_2^\omega, \quad \hat{\sigma}_q^2\hat{\gamma}_1^\omega + q^4\hat{\sigma}_q^{-1}\hat{\gamma}_2^\omega = q^2\hat{\sigma}_q\hat{\gamma}_2^\omega, \\ \hat{\gamma}_2^\omega + q^{-2}\hat{\sigma}_q\hat{\gamma}_2^\omega &= q^{-2}\hat{\sigma}_q\hat{\gamma}_1^\omega, \quad q^2\hat{\sigma}_q\hat{\gamma}_2^\omega + q^4\hat{\gamma}_2^\omega = q^2\hat{\sigma}_q\hat{\gamma}_1^\omega.\end{aligned}$$

A.0.6 $(B_3^{(1)}, A_5^{(2)})$

The S-matrices read

$$\begin{aligned}S_{11}(\theta) &= \{1,1_2;5,7_2\}_\theta \quad S_{12}(\theta) = \{2,3_2;4,5_2\}_\theta \quad S_{33}(\theta) = \{1,1;3,5;5,9\}_\theta \\ S_{22}(\theta) &= \{1,1_2;3,3_2;3,5_2;5,7_2\}_\theta \quad S_{23}(\theta) = \{2,2_2;4,6_2\}_\theta \quad S_{13}(\theta) = \{3,4_2\}_\theta.\end{aligned}$$

We have $h = 6$ and $H = 10$ for the Coxeter numbers. The combined bootstrap identities read

$$\begin{aligned}S_{1l}(\theta+\theta_h+2\theta_H)S_{1l}(\theta-\theta_h-2\theta_H) &= S_{l2}(\theta) & \text{(A.46)}\\ S_{2l}(\theta+\theta_h+2\theta_H)S_{2l}(\theta-\theta_h-2\theta_H) &= S_{l1}(\theta)S_{l3}(\theta-\theta_H)S_{l3}(\theta+\theta_H) & \text{(A.47)}\\ S_{3l}(\theta+\theta_h+\theta_H)S_{3l}(\theta-\theta_h-\theta_H) &= S_{l2}(\theta). & \text{(A.48)}\end{aligned}$$

The mass ratios are

$$\frac{m_1}{m_2} = \frac{\sinh(\theta_h+2\theta_H)}{\sinh(4\theta_h+6\theta_H)} \quad \frac{m_1}{m_3} = \frac{\sinh(2\theta_h+4\theta_H)}{\sinh(2\theta_h+3\theta_H)} \quad \frac{m_2}{m_3} = \frac{\sinh(4\theta_h+8\theta_H)}{\sinh(\theta_h+\theta_H)}. \quad \text{(A.49)}$$

$S_{ij}(\theta)$ from $B_3^{(1)}$

The result of successive actions of the q-deformed Coxeter element on the simple roots is reported in Table A.11.

σ_q^x	$\alpha_1 = -\gamma_1$	$\alpha_3 = -\gamma_3$	$\alpha_2 = \gamma_2$
1	$*;4;3,5$	$3;3;4$	$-4;4;3,5$
2	$6;6;*$	$*;7;8$	$-6;6,8;7,9$
3	$-10;*;*$	$*;*;10$	$-*;10;*$
4	$-*;14;13,15$	$13;13;14$	$14;14;13,15$
5	$-16;16;*$	$*;17;18$	$16;16,18;17,19$
6	$20;*;*$	$*;*;20$	$*;20;*$

Table A.11: The orbits Ω_i^q created by the action of σ_q^x on γ_i

The solutions of the fusing rule in Ω^q are

$$\begin{aligned}
\gamma_1 + q^{-4}\sigma_q\gamma_1 &= q^{-4}\sigma_q\gamma_2, \quad \sigma_q^{-1}\gamma_1 + q^4\sigma_q^{-2}\gamma_1 = \sigma_q^{-1}\gamma_2, \\
\gamma_1 + q^{-10}\sigma_q^3\gamma_2 &= q^{-6}\sigma_q^2\gamma_1, \quad \sigma_q^{-1}\gamma_1 + q^6\sigma_q^{-3}\gamma_2 = q^6\sigma_q^{-3}\gamma_1, \\
\gamma_1 + q^{-7}\sigma_q^2\gamma_3 &= q^{-3}\sigma_q\gamma_3, \quad \sigma_q^{-1}\gamma_1 + q^7\sigma_q^{-3}\gamma_3 = q^3\sigma_q^{-2}\gamma_3, \\
\gamma_2 + q^{-7}\sigma_q^2\gamma_3 &= q\gamma_3, \quad q^{-4}\gamma_2 + q^7\sigma_q^{-3}\gamma_3 = q^{-1}\sigma_q^{-1}\gamma_3, \\
\gamma_3 + q^{-6}\sigma_q^2\gamma_3 &= q^{-3}\sigma_q\gamma_1, \quad \sigma_q^{-1}\gamma_3 + q^6\sigma_q^{-3}\gamma_3 = q^3\sigma_q^{-2}\gamma_1, \\
\gamma_3 + q^{-2}\sigma_q\gamma_3 &= q^{-3}\sigma_q\gamma_2, \quad \sigma_q^{-1}\gamma_3 + q^2\sigma_q^{-2}\gamma_3 = q^{-1}\sigma_q^{-1}\gamma_2.
\end{aligned}$$

$S_{ij}(\theta)$ from $\hat{A}_5^{(2)}$:

The result of successive actions of the q-deformed twisted Coxeter element on the simple roots is reported in Table A.12.

$\hat{\sigma}_q^x$	$\alpha_5 = -\hat{\gamma}_1^\omega$	$\hat{\alpha}_3 = -\hat{\gamma}_3^\omega$	$\hat{\alpha}_2 = \hat{\gamma}_2^\omega$
1	$0;*;*;*;*$	$*;*;*;2;2$	$-*;*;2;2;2$
2	$*;*;2;2;*$	$2;2;2;*;*$	$-2;2;2;4;4$
3	$*;2;2;4;4$	$*;*;*;4;*$	$-4;4;4;4;*$
4	$4;4;*;*;*$	$*;4;4;*;*$	$-*;4;4;6;*$
5	$-*;*;*;*;6$	$*;*;6;*;*$	$-*;6;*;*;*$
6	$-6;*;*;*;*$	$*;*;*;8;8$	$*;*;8;8;8$
7	$-*;*;8;8;*$	$8;8;8;*;*$	$8;8;8;10;10$
8	$-*;8;8;10;10$	$*;*;*;10;*$	$10;10;10;10;*$
9	$-10;10;*;*;*$	$*;10;10;*;*$	$*;10;10;12;*$
10	$*;*;*;*;12$	$*;*;12;*;*$	$*;12;*;*,*$

Table A.12 : The orbits $\hat{\Omega}_i^q$ created by the action of $\hat{\sigma}_q^x$ on $\hat{\gamma}_i^\omega$

The solutions of the fusing rule in $\hat{\Omega}^q$

$$\begin{aligned}
\hat{\gamma}_1^\omega + q^{-2}\hat{\sigma}_q^2\hat{\gamma}_1^\omega &= q^{-2}\hat{\sigma}_q\hat{\gamma}_2^\omega, \quad q^2\hat{\sigma}_q^2\hat{\gamma}_1^\omega + q^4\hat{\gamma}_1^\omega = q^2\hat{\sigma}_q\hat{\gamma}_2^\omega, \\
\hat{\gamma}_1^\omega + q^{-6}\hat{\sigma}_q^4\hat{\gamma}_2^\omega &= q^{-4}\hat{\sigma}_q^3\hat{\gamma}_1^\omega, \quad q^2\hat{\sigma}_q^2\hat{\gamma}_1^\omega + q^6\hat{\sigma}_q^{-2}\hat{\gamma}_2^\omega = q^6\hat{\sigma}_q^{-1}\hat{\gamma}_1^\omega, \\
\hat{\gamma}_1^\omega + q^{-4}\hat{\sigma}_q^3\hat{\gamma}_3^\omega &= q^{-2}\hat{\sigma}_q\hat{\gamma}_3^\omega, \quad q^2\hat{\sigma}_q^2\hat{\gamma}_1^\omega + q^6\hat{\sigma}_q^{-2}\hat{\gamma}_3^\omega = q^4\hat{\gamma}_3^\omega, \\
\hat{\gamma}_2^\omega + q^{-4}\hat{\sigma}_q^4\hat{\gamma}_3^\omega &= \hat{\gamma}_3^\omega, \quad \hat{\sigma}_q^2\hat{\gamma}_2^\omega + q^6\hat{\sigma}_q^{-3}\hat{\gamma}_3^\omega = q^2\hat{\sigma}_q\hat{\gamma}_3^\omega, \\
\hat{\gamma}_3^\omega + q^{-4}\hat{\sigma}_q^3\hat{\gamma}_3^\omega &= q^{-2}\hat{\sigma}_q^2\hat{\gamma}_1^\omega, \quad q^2\hat{\sigma}_q\hat{\gamma}_3^\omega + q^6\hat{\sigma}_q^{-2}\hat{\gamma}_3^\omega = q^4\hat{\gamma}_1^\omega, \\
\hat{\gamma}_3^\omega + q^{-2}\hat{\sigma}_q\hat{\gamma}_3^\omega &= q^{-2}\hat{\sigma}_q\hat{\gamma}_2^\omega, \quad q^2\hat{\sigma}_q\hat{\gamma}_3^\omega + q^4\hat{\gamma}_3^\omega = q^2\hat{\sigma}_q\hat{\gamma}_2^\omega.
\end{aligned}$$