Chapter 3
EXACT S-MATRICES

When meeting calamities or difficult situations, it is not enough to simply say that one is not at all flustered. When meeting difficult situations, one should dash forward bravely and with joy. It is the crossing of a single barrier and is like the saying, "The more the water, the higher the boat."

From 'The Book of the Samurai, Hagakure'

One of the central objects in quantum field theory is the scattering matrix, henceforth also referred to as S-matrix, which determines the on-shell structure of the model and describes the collision of quantum particles. While in general the S-matrix can only be calculated in the framework of perturbation theory by means of Feynman diagrams, one might pursue in case of integrable models an alternative approach originating in the early works of Heisenberg [7] and Chew [8]. The basic idea is that the scattering matrix by itself has to obey a number of physically motivated constraints which might be restrictive enough to calculate it directly without relying on the field content of the theory. Some of the general properties an S-matrix should satisfy are:

(S1) conservation of probability

(S2) Lorentz invariance

(S3) analyticity in the energy variables

(S4) crossing symmetry

In higher dimensions 3+1 the application of this method has led to the famous dispersion relations yielding rigid constraints on cross sections. However, a systematic way to fully construct an S-matrix is not known. In contrast, the approach becomes extremely powerful in 1+1 dimensions in context of integrable field theories. Their key feature is the presence of an infinite set of conserved charges (a pair of higher spin charges is actually sufficient, see [70]) implying the following severe restrictions on a scattering process,

- absence of particle production
- conservation of the individual particle momenta
- factorization of the scattering matrix into two-particle amplitudes

Especially the last property is of importance since it reduces the problem to determining the two-particle S-matrix. Furthermore, it gives rise to two sets of consistency conditions, the Yang-Baxter and the bootstrap equations. While the former describe equivalent ways to factorize a three particle scattering process when reflection is present, the latter ensure consistency when an intermediate bound state occurs. Provided the particle content and the bound state structure of the model are known the two-particle S-matrix can then be systematically constructed by exploiting the above restrictions (S1)-(S4) and the results obtained ought to be "exact", i.e. they should hold to all orders of perturbation theory.

Besides giving an exact answer this "bootstrap" approach, as it is called in the literature, supersedes the conventional perturbative one in two aspects. First it elegantly circumvents the messy and tedious computations of higher order perturbation and renormalization theory. Second, since the bootstrap does not rely on the field content, it might be performed whether or not an underlying classical Lagrangian formulation of the theory is known. Nevertheless, it is of advantage to have a classical Lagrangian at one's disposal since it serves physical intuition and also enables perturbative consistency checks of the bootstrap approach which are useful from time to time. In the subsequent chapter we will, however, encounter a different method to check the scattering matrices for consistency which also does not use the field quantities and yields valuable insight in the high energy regime of the associated quantum field theories.

This chapter starts with a review of the ideas underlying the bootstrap approach in 1+1 dimensional integrable quantum field theories in Section 3.1. The factorization of the scattering matrix is motivated followed by a discussion of the analytic structure of the two-particle S-matrix which becomes the central object of interest. The physical constraints on the scattering matrix stated above are translated to concrete functional relations for the two-particle amplitude and the general form of the solutions satisfying the properties (S1)-(S4) is stated. In order to extract from the possible solutions the S-matrix of a concrete quantum field theory one has to relate them to the particle content in a second step by invoking the bootstrap equation.

In Section 3.2 this is done for affine Toda field theory. As a starting point the classical Lagrangian, the classical mass spectrum and the classical fusing processes present in this class of theories are recalled. This serves as motivation for the subsequent discussion of the corresponding structures on the quantum level and the universal formulation of the S-matrix. This discussion will be given for all Toda models at once by exploiting the Lie algebraic techniques introduced in the previous chapter. In particular, we will see that the q-deformation of Chapter 2 has a concrete physical meaning in describing the renormalized quantum mass flow dependent on the coupling constant and it will become apparent how the weak and strong coupling regime of affine Toda theories is governed by different Lie algebraic structures. After having stated the quantum particle content of affine Toda theories two universal expressions of the two-particle scattering matrix are given which in their most elegant form only involve the q-deformed Cartan matrices discussed before.

One of the main observations we will recover from the universal treatment of affine Toda theory is the natural splitting of the models in two subclasses corresponding to simply-laced and non-simply laced algebras. The former are known to have simpler renormalization properties and the associated two-particle S-matrix can be separated in two factors one containing all the physical information about the particle content of the model and the other displaying the coupling dependence. This separation of the affine Toda S-matrix will then be used to define new integrable models by constructing new solutions to the functional equations of the bootstrap approach in Section 3.3.

3.1 Analyticity, crossing and the bootstrap equations

Since the S-matrix ought to describe a scattering experiment the forces are assumed to be sufficiently short ranged and the particles should become free at sufficiently large times ($t \to \pm\infty$). The space of physical states should therefore asymptotically be spanned by incoming or outgoing momentum eigenstates obeying the mass-shell condition of free particles, i.e. $p_\mu p^\mu = m^2$ with p being the relativistic two-momentum. This statement is called **asymptotic completeness** and it motivates the introduction of momentum creation and annihilation operators $a_{in}(p), a^*_{in}(p), a_{out}(p), a^*_{out}(p)$ which generate the Fock space of the *in* and *out*-states upon acting on the vacuum state $|0\rangle$,

$$|p_1,\dots,p_n\rangle_{\substack{in\\out}} = a^*_{\substack{in\\out}}(p_1)\cdots a^*_{\substack{in\\out}}(p_n)|0\rangle \;.$$

The asymptotic states might depend in addition on internal quantum numbers but for simplicity these are suppressed in the notation. Both sets of these states are assumed to be complete and orthonormal whence they have to be mapped onto each other by a unitary operator, $\mathcal{S}^*\mathcal{S} = \mathcal{S}\mathcal{S}^* = 1$, which is just the S-matrix,

$$\begin{aligned} {}_{out}\langle p'_1,\dots,p'_m|\,p_1,\dots,p_n\rangle_{in} &= {}_{in}\langle p'_1,\dots,p'_m|\,\mathcal{S}\,|\,p_1,\dots,p_n\rangle_{in} \\ &= {}_{out}\langle p'_1,\dots,p'_m|\,\mathcal{S}\,|\,p_1,\dots,p_n\rangle_{out} \;. \end{aligned} \tag{3.1}$$

The above equations show that we might drop the *in* and *out* labels. Thus, the first constraint (S1) is a direct consequence of postulating asymptotic completeness for the set of *in* and *out* states.

To satisfy the second condition of Lorentz invariance we require that the S-matrix elements are invariant under the action of the Lorentz group, e.g. the two particle amplitude should obey

$$\langle p_1,p_2|\,\mathcal{S}\,|\,p_3,p_4\rangle = \left\langle p_1,p_2\right|\Lambda\mathcal{S}\Lambda^{-1}\left|\,p_3,p_4\right\rangle \tag{3.2}$$

for every proper Lorentz transformation Λ. Here we followed the standard convention (see e.g. [9]) and specified Lorentz invariance for the whole matrix elements instead for their absolute values only. This fixes the phase uniquely. As a consequence the two-particle amplitude depends only on the so-called **Mandelstam variables**

$$s = (p_1+p_2)^2\,,\quad t = (p_1-p_3)^2 \quad \text{and} \quad u = (p_1-p_4)^2$$

up to an overall factor $\delta^{(4)}(p_1+p_2-p_3-p_4)$ reflecting energy-momentum conservation. The Mandelstam variables satisfy the relation $s+t+u = \sum_{i=1}^4 m_i^2$ from which it can be deduced that in 1+1 dimensions only one of them is independent. Before discussing the outstanding properties (S3) and (S4) we now turn to further restrictions imposed by integrability.

3.1.1 Conserved charges and factorization

As mentioned before integrable field theories are distinguished by the presence of an infinite set of conversation laws which are in involution and transform as higher rank tensors under the Lorentz group. Let $\mathcal{Q}^{(s)}$ denote such a conserved charge with Lorentz spin $\pm s > 1$ then we can choose the one-particle momentum eigenstates in the *in* and *out* basis such that they are simultaneously also eigenstates of $\mathcal{Q}^{(s)}$,

$$\mathcal{Q}^{(s)}\,|\,p\rangle = Q^{(s)}(p^{\pm})^s\,|\,p\rangle\ .$$

Here $p^{\pm} = p^0 \pm p^1$ are the light cone components of the two-momentum, $Q^{(s)}$ is a scalar and the upper or lower sign applies if s is either positive or negative, respectively. This particular form of the eigenvalue is required by the Lorentz transformation property of the conserved charge $\mathcal{Q}^{(s)}$. If we assume the charge to be local, i.e. to be an integral of a local charge density, then its action on a multiparticle state must be additive,

$$\mathcal{Q}^{(s)}\,|\,p_1,...,p_n\rangle = \left\{Q_1^{(s)}(p_1^{\pm})^s + \cdots + Q_n^{(s)}(p_n^{\pm})^s\right\}|\,p_1,...,p_n\rangle\ .$$

Now conservation of $\mathcal{Q}^{(s)}$ means that for a generic scattering process $|\,p_1,...,p_n\rangle^{in} \rightarrow |\,p'_1,...,p'_m\rangle^{out}$ the following sums must be equal

$$\sum_{i=1}^{n} Q_i^{(s)}(p_i^{\pm})^s = \sum_{i=1}^{m} Q_i^{(s)}(p_i'^{\pm})^s\ .$$

For an infinite set of higher spin charges this results in an infinite set of equations which allow only for the trivial solution namely that the sets of incoming and outgoing particle momenta are equal,

$$\{p_1,\ldots,p_n\} = \{p'_1,\ldots,p'_m\}\ . \tag{3.3}$$

Note that this in particular implies the absence of particle production. To see the third condition imposed by integrability, i.e. the factorization of the S-matrix, we now follow a line of reasoning given by Shankar and Witten [71].

Let us assume for simplicity that one of the higher spin conserved charges $\mathcal{Q}^{(s)}$ acts on the momentum eigenstates just as $(p^1)^s$ with p^1 being the spatial component of the two-momentum p. Then a localized one-particle state described by a wave function of the form

$$\psi(x) = N\int dp\, e^{-a(p-p^1)^2+ip(x-\xi)}$$

is transformed into

$$e^{i\lambda\mathcal{Q}^{(s)}}\psi(x) = N\int dp\, e^{-a(p-p^1)^2+ip(x-\xi)+i\lambda p^s}\ .$$

Here N is a normalization constant, λ an arbitrary real shift parameter and ξ the center of the localized wave packet. Thus, the action of the conserved charge has shifted the center of the wave packet dependent on the particle momentum,

$$\xi \to \xi' = \xi - s\lambda (p^1)^{s-1} \; .$$

Consider now a multiparticle state with wave packets of different momenta and localized in position just like the wave function above. Then the application of the operator $e^{i\lambda \mathcal{Q}^{(s)}}, s > 1$ will displace the wave-packets relative to each other. Explicitly, consider a three particle collision process with $p_1 < p_2 < p_3$ as depicted in Figure 3.1. By successive action with the conserved charge for different values of λ one might change the impact parameters of the colliding particles and produce all three different space-time diagrams shown. Clearly, the last two diagrams correspond to successive two-particle collisions showing that a three-particle scattering process factorizes in two-particle ones. Moreover, by conservation of $\mathcal{Q}^{(s)}$ both possible decompositions of the three-particle scattering amplitude must coincide,

$$S^{(2)}(p_2,p_3)S^{(2)}(p_3,p_1)S^{(2)}(p_1,p_2) = S^{(2)}(p_1,p_2)S^{(2)}(p_1,p_3)S^{(2)}(p_2,p_3) \; , \tag{3.4}$$

where $S^{(2)}(p_i,p_j) = \langle p_i, p_j | \, \mathcal{S} \, | \, p_i, p_j \rangle$ denotes the two-particle amplitude. This factorization identity is the celebrated **Yang-Baxter equation** [10]. Note that the two-particle amplitude might depend on internal quantum numbers whence the above equation is to be understood in a matrix notation. In the presence of such internal symmetries the Yang-Baxter equation imposes a surprisingly powerful constraint which often suffices to construct the S-matrix explicitly [11].

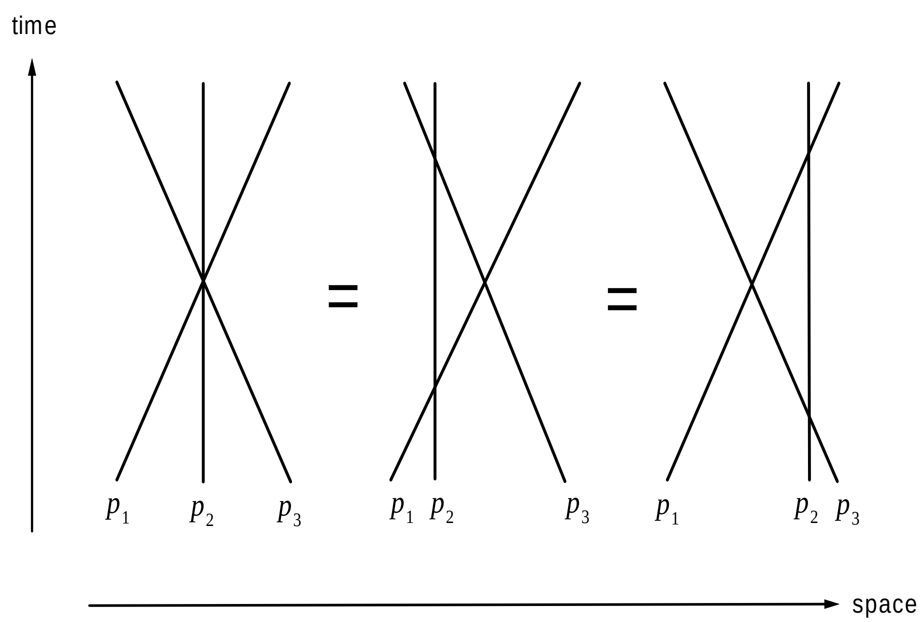

Figure 3.1: Depiction of the Yang-Baxter equation.

However, in due course we will only encounter theories where the particle spectrum is non-degenerate. Either because all the particle masses are different or since the particles can be distinguished by the eigenvalues of one of the conserved charges. This excludes a possible redistribution of the particles with the same quantum numbers in a scattering process and there is no reflection. One is left with a set of diagonal S-matrices, i.e. a set of phases, which trivially satisfy the Yang-Baxter equation. Thus, for the construction of diagonal scattering matrices (3.4) does not impose an additional constraint to (S1)-(S4). However, by the same argument one now motivates more generally that the n-particle scattering amplitude should factorize into two-particle ones,

$$S^{(n)}(p_1,\ldots,p_n) := {}^{out}\langle p_1,\ldots,p_n|\, p_1,\ldots,p_n\rangle^{in} = \prod_{i<j} S^{(2)}(p_i,p_j) \tag{3.5}$$

meaning that every scattering process can be decomposed into $n(n-1)/2$ subprocesses involving only a pair of particles. Note that we have omitted a pair of δ-function in the above definition of the scattering amplitude. The properties (3.3) and (3.5) are characteristic for integrable field theories and constitute the cornerstones for the successful application of the bootstrap approach in constructing exact S-matrices.

It should be mentioned that the assumption of an infinite number of conserved charges is actually too strong. As was shown by Parke [70] a pair of higher spin charges is already sufficient to deduce the crucial properties (3.3) and (3.5). For theories invariant under a parity transformation even *one* charge of spin $s > 1$ is sufficient because upon a parity transformation one obtains another one of spin $-s$. However, for the integrable models we are going to consider it is believed that an infinite set of higher spin charges is present.

3.1.2 The analytic structure of the two-particle S-matrix

So far we have discussed the restrictions on the scattering matrix imposed by the general constraints (S1), (S2) and integrability. From the latter we have learned that for integrable models one only needs to regard the two-particle amplitude which must be a function of one of the Mandelstam variables, say s. For the following discussion of the remaining general constraints (S3), (S4) and the formulation of functional equations reflecting them, it is convenient to introduce a different variable, the rapidity θ. The latter is defined by parametrizing the two-momentum in the following way,

$$p_i = m_i(\cosh\theta_i, \sinh\theta_i)\ . \tag{3.6}$$

Clearly, rewriting p in the above manner has the advantage that the on-shell condition $p_\mu p^\mu = m^2$ is satisfied automatically. The parametrization of the Mandelstam variable is then determined by

$$s = m_i^2 + m_j^2 + 2m_i m_j \cosh\theta_{ij} \tag{3.7}$$

where $\theta_{ij} = \theta_i - \theta_j$ is the rapidity difference. Henceforth, the two-particle scattering amplitude will therefore be written in either one of the following variants,

$$S^{(2)}(p_i,p_j) =: S_{ij}(s) \quad \text{or} \quad S^{(2)}(p_i,p_j) =: S_{ij}(\theta_{ij})\ . \tag{3.8}$$

The sole dependence either on the Mandelstam variable s or the rapidity difference θ_{ij} incorporates relativistic invariance according to requirement (S2). The ultimate reason for introducing the rapidity variable is that it simplifies the discussion of the analytic structure (S3) of the two-particle S-matrix to which we now turn. However, for sake of completeness we will relate the analytic properties in the variable θ to the ones in the variable s and vice versa.

The analytic domains

At the heart of the bootstrap principle lies the idea to view the physical scattering amplitudes as real-boundary values of analytic functions defined on domains of the complex plane. This property has been motivated to be linked to macroscopic causality properties, see e.g. [9]. We therefore interpret the two-particle amplitude (3.8) now as function of complex variables $s, \theta \in \mathbb{C}$ defined on the domains specified in Figure 3.2. Starting with $S_{ij}(s)$ one can deduce from unitarity arguments that the two-particle amplitude must have a square root branch point in the s-channel at the two-particle threshold $s = (m_i + m_j)^2$. This branching point signals the least energy required that inelastic processes can take place like the production of extra particles. A second one arises by crossing symmetry – which will be explained below – in the t-channel and is located at $s = (m_i - m_j)^2$. Since in the context of integrable models one only deals with two-particle unitarity, these are assumed to be the only branching points implying that the two particle scattering amplitude is meromorphic. The resulting left and right branch cuts in the s-plane are defined to lie in the ranges $s \leq (m_i - m_j)^2$ and $(m_i + m_j)^2 \leq s$ .Via the parametrization (3.7) these branch cuts are unfolded in the complex θ-plane and mapped onto the axes $\operatorname{Im}\theta = \pi$ and $\operatorname{Im}\theta = 0$, respectively. They enclose the strip $0 < \operatorname{Im}\theta < \pi$ which is referred to as **physical sheet**, since only the analytic continuation into this region is assumed to be related to concrete physical processes.

Hermitian analyticity and unitarity

Having introduced real branch cuts in the s-plane we need to specify how to obtain the physical values of the scattering amplitude, since for on-shell processes the Mandelstam variable s is real. Comparison with perturbation theory shows that these are recovered by the limit onto the branch cut from the upper-half plane [9],

$$\text{physical values:} \quad S_{ij}(s) \equiv \lim_{\varepsilon \to 0^+} S_{ij}(s + i\varepsilon)\ , \quad s \in \mathbb{R}\ .$$

This limit reflects Feynman's prescription in perturbation theory to add to each particle mass a small *negative* imaginary part $-i\varepsilon$ in order to make the integration over internal lines of the relevant Feynman diagrams well defined [9]. However, one might also consider the *unphysical* limit on the branch cut from the lower-half plane. The complex conjugate of this value can be linked to the parity transformed *physical* scattering amplitude

$$\lim_{\varepsilon \to 0^+} S_{ij}(s - i\varepsilon)^* = \lim_{\varepsilon \to 0^+} S_{ji}(s + i\varepsilon)\ , \quad s \in \mathbb{R}\ .$$

Thus, two different scattering processes are linked to each other across the branch cut as boundary values of the same analytic function. This property is known as **Hermitian analyticity** [72]. Analytic continuation to complex arguments then results in the following functional equation in terms of the rapidity variable,

$$S_{ij}(\theta) = S_{ji}(-\theta^*)^* \; . \tag{3.9}$$

Note that for parity invariant theories one has $S_{ij} = S_{ji}$, whence Hermitian analyticity then amounts to **real analyticity**. Combining (3.9) with the unitarity requirement $S_{ij}(\theta)S_{ij}(\theta)^* = 1,\ \theta \in \mathbb{R}$ for the two-particle amplitude yields upon analytic continuation the functional equation

$$S_{ij}(\theta)S_{ji}(-\theta) = 1 \quad , \tag{3.10}$$

which is assumed to hold for all complex arguments $\theta \in \mathbb{C}$. The concept that different boundary values of one analytic function relate different scattering processes is also exploited in the context of crossing symmetry.

Crossing symmetry

In a relativistic theory a crossing transformation amounts to the replacement of an incoming particle of momentum p by an outgoing antiparticle with momentum $-p$ and has its origin in the Lehmann-Symanzik-Zimmermann formalism, see any standard textbook on quantum field theory. In terms of the Mandelstam variables this corresponds to the transition from the s-channel to the t-channel

$$s = (p_i + p_j)^2 \;\to\; t = (p_i - p_j)^2 = 2m_i^2 + 2m_j^2 - s \; . \tag{3.11}$$

Note that we have used here the conservation of the individual particle momenta in integrable theories. Crossing symmetry now requires that the scattering amplitudes corresponding to the two collision processes

$$|\,p_i, p_j\rangle^{in} \to |\,p_i, p_j\rangle^{out} \quad \text{and} \quad |\,p_j, -p_{\bar{\imath}}\rangle^{in} \to |\,p_j, -p_{\bar{\imath}}\rangle^{out}$$

can be obtained from each other by analytic continuation. Here $\bar{\imath}$ denotes the antiparticle of the particles species i. The path of analytic continuation is depicted in Figure 3.2 and connects the upper side of the right branch cut with the lower side of the left branch cut according to (3.11). Upon noting that the right and left cut in the s-plane correspond to the axes $\mathrm{Im}\,\theta = 0$ and $\mathrm{Im}\,\theta = \pi$ in terms of the rapidity variable, crossing symmetry then simplifies to the functional equation

$$S_{ij}(i\pi - \theta) = S_{j\bar{\imath}}(\theta) \; . \tag{3.12}$$

The requirements of unitarity, Hermitian analyticity and crossing symmetry form powerful constraints on the possible form of a solution to the functional equations (3.10) and (3.12). It was shown in [74] that the most general form of a scattering amplitude obeying the stated restrictions must consist of ratios of hyperbolic functions†,

$$S_{ij}(\theta) = \prod_{x \in X_{ij}} \frac{\sinh \frac{1}{2}(\theta + i\pi x)}{\sinh \frac{1}{2}(\theta - i\pi x)} \tag{3.13}$$

where the set of real numbers X_{ij} incorporates the information characteristic to the specific quantum field theory under consideration. To determine the latter one needs an additional equation for the S-matrix displaying the particle content and the structure of the interaction.

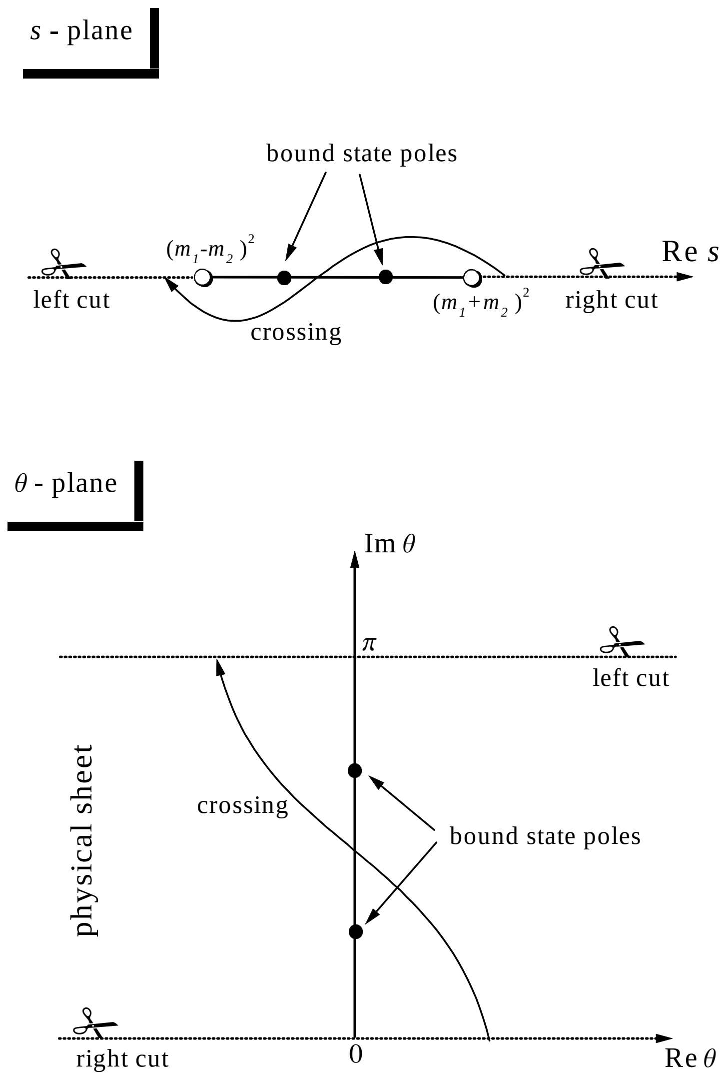

Figure 3.2: The analytic domains of the two-particle S-matrix.

†In non-diagonal theories one might encounter als infinite products of Γ-functions or elliptic functions.

Bound state poles

The general solution (3.13) resulting from (3.10) and (3.12) is obviously meromorphic and exhibits poles in the complex rapidity plane whenever the denominator vanishes. The unspecified numbers $x \in X_{ij}$ determine therefore the singularity structure of the S-matrix. The general discussion of this singularity structure is a delicate issue, here we will restrict ourselves to simple poles which lie in the physical strip and are associated with stable bound states. Explicitly, consider a simple pole in the two-particle amplitude of the particles i, j which are assumed to form a bound state labelled by $\bar{k}$. In the vicinity of the singularity the S-matrix must be of the form

$$S_{ij}(\theta) \sim \frac{iR_{ij}^k}{(\theta - iu_{ij}^k)} \ , \quad 0 < u_{ij}^k < \pi \ . \tag{3.14}$$

Comparison with perturbation theory motivates that these kind of poles are linked to the propagation of an intermediate bound state in the s-channel for $R_{ij}^k > 0$ and in the t-channel for $R_{ij}^k < 0$. That is, the two-particle state of a scattering process becomes dominated by a one particle state at purely imaginary rapidity difference $\theta_{ij} = iu_{ij}^k$ and the Mandelstam variable s ought to satisfy

$$s = m_{\bar{k}}^2 = m_i^2 + m_j^2 + 2m_i m_j \cos u_{ij}^k \ . \tag{3.15}$$

Such a process is called **fusing** $i + j \to \bar{k}$ in the literature and the rapidity difference u_{ij}^k fixing the singularity is referred to as **fusing angle**.

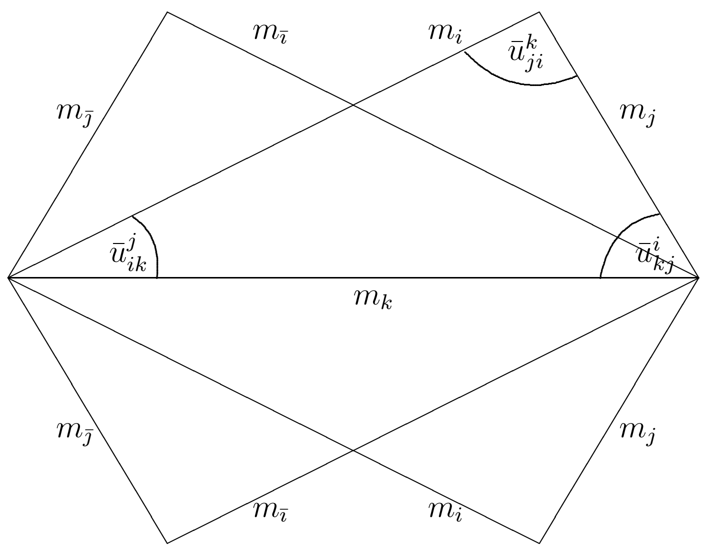

Figure 3.3: Mass triangles in the complex velocity plane. The fusing angles are defined as $\bar{u}_{ij}^k = \pi - u_{ij}^k$.

Assuming **nuclear democracy** the intermediate bound state labelled by $\bar{k}$ is supposed to belong to the asymptotic particle spectrum and compatibility with crossing symmetry then requires that also the fusing processes $j + k \to \bar{\imath}$ and $i + k \to \bar{\jmath}$ are present in the theory. Hence, one must have

$$u_{ij}^k + u_{ki}^j + u_{jk}^i = 2\pi \ . \tag{3.16}$$

The fusing angles and the fusing condition (3.15) can be geometrically visualized by drawing a triangle whose sides have length equal to the masses of the three particles, see Figure 3.3.

The crucial assumption is now that the intermediate state is present at macroscopic times which allows to formulate the following consistency equation known as **bootstrap identity**,

$$S_{li}(\theta - i\bar{u}_{ki}^{j})S_{lj}(\theta + i\bar{u}_{jk}^{i}) = S_{l\bar{k}}(\theta) \ , \quad \bar{u}_{ij}^{k} = \pi - u_{ij}^{k} \ . \tag{3.17}$$

This functional relation for the S-matrix states the equivalence of the two possibilities that either the scattering with a particle l takes place before the fusing $i + j \to \bar{k}$ occurs or afterwards. For a graphical depiction see Figure 3.4. Note that analytic continuation is also here crucial, since the S-matrices are evaluated at unphysical values. For later purposes we write the bootstrap equation in a more symmetric variant exploiting (3.10) and (3.12),

$$S_{li}(\theta)S_{lj}(\theta + iu_{ij}^{k})S_{lk}(\theta + iu_{ij}^{k} + iu_{jk}^{i}) = 1 \ . \tag{3.18}$$

The bootstrap identity plays the key role in the construction of diagonal S-matrices. It connects the on-shell scattering processes with the bound state structure characteristic for the field theory under consideration by treating the bound states at the same footing as the asymptotic particle states. The construction of an S-matrix can now be summarized in the following steps.

- Write down the general solution (3.13) satisfying unitarity, Hermitian analyticity and crossing symmetry.
- Given some information about the bound state structure, i.e. either the mass spectrum or the fusing processes and angles, introduce a minimum number of physical poles in the general solution (3.13) by fixing the numbers $x \in X_{ij}$.
- Check this solution for consistency by means of the bootstrap identity (3.17).
- Interpret the complete singularity structure inside the physical sheet.

Even though equation (3.17) is extremely powerful in the construction of exact S-matrices it turns out, that it only provides a consistency check and does not determine the solution uniquely. Once a "minimal" S-matrix obeying the functional equations is constructed one might multiply it by a factor possessing the same properties except that its poles lie exclusively outside of the physical sheet. Such a solution is called a **CDD factor** [39] and does not alter the bound state structure of the S-matrix. In order to remove this ambiguity an additional consistency check is required, either by perturbation theory or by the thermodynamic Bethe ansatz, which we will study in Chapter 4.

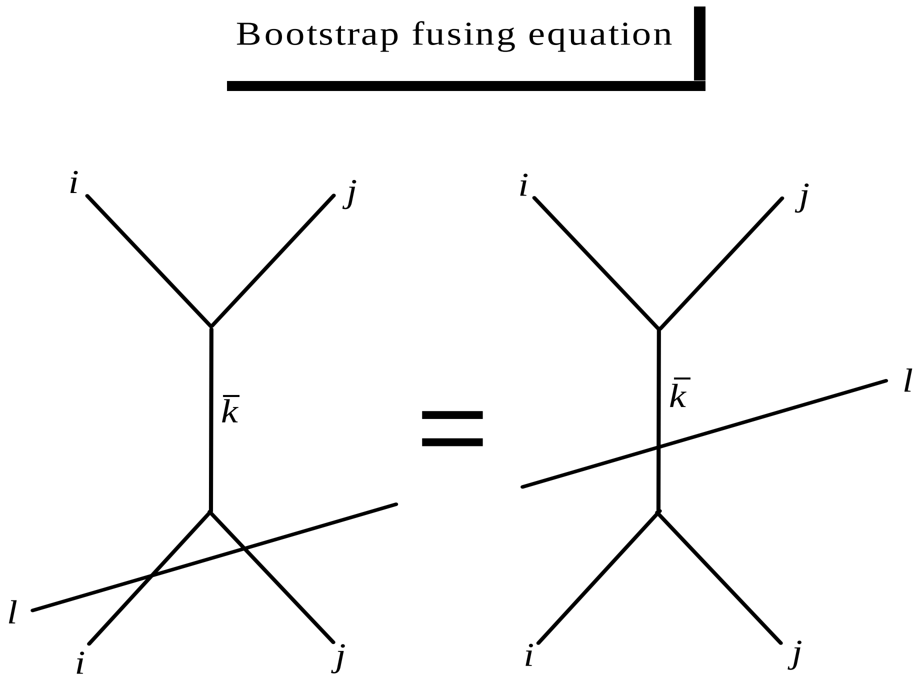

Figure 3.4: Depiction of the bootstrap equation.

3.2 The S-matrix of affine Toda theory

After the comments on the general structure of the two-particle S-matrix in 1+1 dimensional integrable field theories and the introduction of the bootstrap principle we now start to consider a class of concrete examples, affine Toda field theories. The latter have been under investigation for a long time and because of their appealing Lie algebraic structure belong to the most prominent and best studied examples by now. Due to their simple renormalization properties the set of S-matrices associated with simply-laced algebras was the first one to be completely constructed on the level of a case-by-case study [27]. Exploiting the Lie algebraic symmetry present in ATFT these S-matrices were put into a universal form [28, 29]. in particular, Fring and Olive demonstrated how the bootstrap properties of the S-matrix could be derived from generic Lie algebraic expressions involving Coxeter geometry [29]. *To push a theory to this level of universality is not only for economic and aesthetic reasons, since all models can be treated at once, but it also allows to distinguish model specific properties from more general ones. In this manner, it is the first step towards the possible discovery of more powerful mathematical concepts in constructing field theoretic models.*

The aim of this section is to achieve a similar formulation for the set of affine Toda field theories connected with non simply-laced Lie algebras by applying the method of q-deformation discussed at the end of the previous chapter. In fact, it will turn out that the latter provides the appropriate framework to give a concise treatment not only for non simply-laced affine Toda models but *all* of them. In order to stress this point the generic case is discussed first and it is then shown at the end how the special case of simply-laced affine Toda models can be extracted from the general formulas.

Following the spirit of the bootstrap program we could directly write down the S-matrix of affine Toda field theory and then discuss its bootstrap properties. However, it turns out to be more instructive to discuss some aspects of the underlying classical theory in advance for two reasons. First, many of the classical properties of ATFT survive the quantization when the underlying Lie algebra is simply-laced. Second, the classical discussion will prepare the formulation of universal formulas at the quantum level by showing that the transition *classical→ quantum* can in many cases achieved by replacing ordinary Lie algebraic objects by q-deformed ones.

3.2.1 The classical regime

The study of classical affine Toda field theory has been performed over years and is a subject on its own [24] not to mention the huge area of associated discrete models, so-called Toda chains [73]. Therefore, the following presentation is limited to those classical formulas which form the counterparts of the quantum formulas presented in due course. In particular, emphasis will be given to the mass spectrum and the fusing rules, since these are the only informations needed as an input for constructing the S-matrix via the bootstrap equations (3.17). Comparison between classical and quantum results will then demonstrate how q-deformation naturally fits into the subject.

We start with the classical affine Toda equations (2.2). The first step in making contact to a field theory is to find a classical action functional from which the equations follow under a small variation of the fields. For this purpose it is convenient to introduce new field variables $\phi = (\phi_1, ..., \phi_n)$ defined through the relation,

$$\varphi_i = \langle \alpha_i, \phi \rangle + \beta^{-1} \ln \frac{2n_i}{\langle \alpha_i, \alpha_i \rangle} \;, \quad i = 0, 1, ..., n \;.$$

The equations of motion are then rewritten in the variant

$$\partial_\mu \partial^\mu \phi + \frac{m^2}{\beta} \sum_{i=0}^{n} n_i \alpha_i e^{\beta \langle \alpha_i, \phi \rangle} = 0 \;. \tag{3.19}$$

The latter can be shown to coincide with the Euler-Lagrange equations w.r.t. the following action functional

$$S_{\text{ATFT}}(\mathfrak{g}) = \int \frac{1}{2} \langle \partial_\mu \phi, \partial^\mu \phi \rangle - \frac{m^2}{\beta^2} \sum_{i=0}^{n} n_i e^{\beta \langle \alpha_i, \phi \rangle} d^2 x \;. \tag{3.20}$$

Here $\mathfrak{g}$ denotes the simple Lie algebra of rank n associated with the Cartan matrix in (2.2), m the bare mass scale, β the classical coupling constant and $\{\alpha_1, ..., \alpha_n\}$ a set of simple roots, n_i the Coxeter labels and $\alpha_0 = -\theta$ the negative highest root with $n_0 = 1$. All these Lie algebraic quantities have been discussed in the previous chapter and determine the Lie algebraic structure of affine Toda field theory. In order to extract the mass spectrum and the possible fusing processes from the above functional we expand the potential in powers of the field variables,

$$\begin{aligned} V(\phi) &= \frac{m^2}{\beta^2}\sum_{i=0}^{n} n_i e^{\beta\langle\alpha_i,\phi\rangle} \\ &= \frac{m^2}{\beta^2}\sum_{i=0}^{n} n_i + \frac{1}{2}(M^2)_{ij}\phi^i\phi^j + \frac{1}{3!}\,C_{ijk}\phi^i\phi^j\phi^k + ... \end{aligned}$$

The constant term just shifts the vacuum energy and can be disregarded. The coefficient of the quadratic term is interpreted as the mass matrix

$$(M^2)_{ij} = m^2\sum_{k=0}^{n} n_k(\alpha_k)^i(\alpha_k)^j$$

whose eigenvalues $\mathbf{m} = (m_1, ..., m_n)$ give the classical masses of the fundamental particles in the spectrum. One of the striking results in classical ATFT is that for *untwisted* affine algebras these coincide with the components of the Perron-Frobenius eigenvector of the Cartan matrix A, i.e. the eigenvector to the smallest eigenvalue, associated with the Lie algebra $\mathfrak{g}$ [75, 30],

$$A\cdot\mathbf{m} = 4\sin^2\tfrac{\pi}{2h}\,\mathbf{m}\;, \quad \mathbf{m} = (m_1, ..., m_n)\;. \tag{3.21}$$

Note that this definition of the masses is compatible with physical requirements since the components of the Perron-Frobenius vector can be shown to be always positive. Furthermore, (3.21) motivates a one-to-one correspondence between particle labels and vertices in the Dynkin diagram $\Gamma(\mathfrak{g})$ of the associated Lie algebra. For later purposes we rewrite the above eigenvector equation in terms of the incidence matrix,

$$\left(2\cos\tfrac{\pi}{h} - I\right)\cdot\mathbf{m} = 0\;, \quad I = 2 - A\;. \tag{3.22}$$

The observation (3.21) can be generalized to all eigenvectors of the Cartan matrix whose components yield the eigenvalues of higher spin conserved quantities

$$A\cdot\mathbf{Q}^{(s)} = 4\sin^2\tfrac{\pi s}{2h}\,\mathbf{Q}^{(s)}\;, \quad \mathbf{Q}^{(s)} = (Q_1^{(s)}, ..., Q_n^{(s)})\;, \tag{3.23}$$

$$\left(2\cos\tfrac{\pi s}{h} - I\right)\cdot\mathbf{Q}^{(s)} = 0\;, \tag{3.24}$$

where s runs over the exponents of the algebra specified in Section 2.3.1. For $s = 1$ we recover the masses. For $s > 1$ the physical interpretation of the charges (3.23) is less direct. Their relevance lies in the preservation of the fusing relations to which we now turn.

A fusing process of the kind $i + j \to \bar{k}$ in terms of particles is related to a non-vanishing three point coupling of the associated fields defined in (3.20),

$$C_{ijk} \neq 0 \quad \Rightarrow \quad i + j \to \bar{k}\;. \tag{3.25}$$

Here $\bar{k}$ labels the anti-particle of k. From the above expansion of the potential the three-point couplings are read off as

$$C_{ijk} = \beta\, m^2\sum_{l=0}^{n} n_l(\alpha_l)^i(\alpha_l)^j(\alpha_l)^k\;.$$

Exploiting this explicit form the fusing condition (3.25) can be translated in terms of Coxeter geometry (again we have to restrict ourselves to untwisted algebras). This was first observed by Dorey [28] on a case-by-case basis and later rigorously derived by Fring, Liao and Olive [30] making use of the classical Lagrangian (3.20).

The three-point coupling C_{ijk} does not vanish if and only if there exist three representatives in the orbits $\Omega_i, \Omega_j, \Omega_k$ of the Coxeter element (2.37) which sum up to zero.

More explicitly, whenever $C_{ijk} \neq 0$ there is a triple of integers (ξ_i, ξ_j, ξ_k) such that

$$\sum_{l=i,j,k} \sigma^{\xi_l} \gamma_l = 0 \; . \tag{3.26}$$

Vice versa the existence of such a triple implies that particles labelled by i, j, k couple to each other. Equation (3.26) is known as **fusing rule** and has been the starting point for numerous attempts to derive more general constraints in terms of representation theory of the associated algebra being analogues of the well-known Clebsch-Gordon rules, e.g. [76]. However, so far such formulations turned out to be less restrictive than the ones presented here.

Notice that the particular form of the fusing rule allows for a whole set of solutions, simply by multiplying with an arbitrary power σ^x of the Coxeter element from the left. Thus, the integers are only defined up to equivalence w.r.t. the transformation $\xi_l \to \xi_l + x$. However, it is important to note that besides these equivalent solutions there exits always one which can not be obtained by a simultaneous shift in the integers. This second version of the fusing rule reads [29],

$$\sum_{l=i,j,k} \sigma^{\xi'_l} \gamma_l = 0 \quad \text{with} \quad \xi'_l = -\xi_l + \frac{c_l - 1}{2} \; . \tag{3.27}$$

The existence of a second solution is important for two reasons. Classically it is needed for proving what is known as **area law** in the literature. The latter states that the magnitude of the three-point coupling $|C_{ijk}|$ is proportional to the area of the fusing triangle depicted in Figure 3.3. In formulas [30],

$$|C_{ijk}| = \frac{2\beta}{\sqrt{h}} m_i m_j \sin u_{ij}^k \; .$$

On the quantum level, however, we will need two non-equivalent solutions of the fusing rule in order to solve the bootstrap equation (3.17).

Having stated the fusing rule, one might ask how to relate the fusing angles defined in (3.15) to it. Exploiting that the eigenvalues of the Cartan matrix and the Coxeter element are related to each other by (2.43) it was derived in [29] that the components of the eigenvectors (3.23) satisfy the relations

$$\sum_{l=i,j,k} e^{i\frac{\pi s}{h}\eta_l} Q_l^{(s)} = 0 \; , \quad \eta_l = -2\xi_l + \frac{c_l - 1}{2} \tag{3.28}$$

We will see in a subsequent sections how to derive this formula as a special case from the q-deformed Cartan matrices. Complex conjugation in (3.28) gives the relation w.r.t. to the second solution of the fusing rule, $\eta_l \to -\eta_l = -2\xi_l' + \frac{c_l-1}{2}$. Relation (3.28) can be interpreted geometrically as fixing a triangle in the euclidean plane upon identifying the complex numbers as euclidean vectors whose sum is zero. Specializing to $s=1$ the length of the sides of the triangle are just proportional to the masses and the triangle is the one depicted in Figure 3.3. We then find from the above equation with the help of (3.15) that the fusing angles are given by

$$u_{ij}^k = \frac{\pi}{h}\left(\eta_j - \eta_i\right) \tag{3.29}$$

Note that there are two possible triangles corresponding to the two non-equivalent solutions of the fusing rule. At the same time it can be seen on geometrical grounds that these are the only two possible solutions [29]. Analogously one might construct similar triangles for the higher spin conserved charges $Q_i^{(s)}$ in (3.23) which shows that they are preserved in the fusing processes determined by (3.26).

3.2.2 Renormalization properties

Having stated the mass spectrum and the fusing processes of classical ATFT in a universal form for all models associated with untwisted algebras, we are now prepared to turn to the quantization of them. This usually requires renormalization theory and in the following two paragraphs a short and qualitative summary of the renormalization properties of ATFT is presented. For details the reader is referred to the original literature.

In general it is to be expected that each of the classical masses will renormalize differently w.r.t. the coupling constant β. However, in case of ATFT the following important distinction can be made. Those theories which belong to simply-laced algebras, i.e. the ADE series, were shown to renormalize up to one loop order in a uniform manner: The quantum masses are the same as the classical ones up to an overall "re"-normalization factor [27],

$$\frac{\delta m_i}{m_i} = \frac{\beta^2}{2h}\cot\frac{\pi}{h}\ . \tag{3.30}$$

That this factor can be expressed by the above universal formula was derived by Braden et al. This simple renormalization behaviour has several important consequences. We immediately infer from (3.30) that the classical mass ratios are preserved in the quantum theory and therefore also the classical fusing angles (3.29) hold true on the quantum level. In view of the analytic structure of the two-particle S-matrix this fixes the bound state poles and one might start to derive generic Lie algebraic expressions for the scattering amplitude using (3.26) and (3.29) as it was done in [29].

In sharp contrast to these properties the mass spectra of the $BCFG$ series of ATFT models renormalize in a totally different way. By use of perturbation theory it was realized that for these kind of theories the classical mass ratios and fusing angles are not preserved in the quantum theory. First trial S-matrices with poles at the classical values were shown to incorporate extra singularities which could not be backed up by perturbation theory. In particular, it was shown to low orders in the perturbation expansion that the simple relation (3.30) ceases in general to be valid and that the renormalized mass ratios flow between different classical values dependent on the coupling constant [31, 77]. This lead to the suggestion [32] that the quantum field theories are dominated by different classical ATFT in the weak and strong coupling regime and one might have to consider pairs of Lie algebras $(\mathfrak{g}, \mathfrak{g}^\vee)$ – as introduced in Chapter 2, Table 2.2 – to formulate a consistent quantum field theory. To be more explicit consider the following example. Let $(G_2^{(1)}, D_4^{(3)})$ be the pair of algebras then the classical mass ratios of the associated ATFT (3.20) can be derived to

$$\left(\frac{m_1}{m_2}\right)_{G_2^{(1)}} = \frac{\sin\frac{\pi}{6}}{\sin\frac{\pi}{3}} \quad \text{and} \quad \left(\frac{m_1}{m_2}\right)_{D_4^{(3)}} = \frac{\sin\frac{\pi}{12}}{\sin\frac{\pi}{6}} \tag{3.31}$$

Based on the above assumption the corresponding quantum mass ratios are expected to asymptotically approximate these different classical values in the weak ($\beta \to 0$) and in the strong ($\beta \to \infty$) coupling regime, respectively. A transformation from small coupling values to large ones should therefore amount to an exchange of the dual Lie algebras and the associated classical ATFT. In formulas this can be summarized in a loose sense by the following equivalence of Olive-Montonen [78] and Langlands duality,

$$\beta \to 4\pi/\beta \quad \simeq \quad \alpha \to \alpha^\vee = \frac{2\,\alpha}{\langle\alpha,\alpha\rangle}\ . \tag{3.32}$$

This off course only describes the qualitative picture and one now has to make the quantitative relation of the mass flow explicit. Note that this also effects the fusing angles and the statement of the fusing rules in the corresponding theories. In fact, these look quite different in the dual classical theories. In the above example $(G_2^{(1)}, D_4^{(3)})$ the non-vanishing three point couplings can be derived to be

$$G_2^{(1)}: \quad C_{111}, C_{112}, C_{222} \quad \text{and} \quad D_4^{(3)}: \quad C_{111}, C_{112}, C_{222}, C_{221}\ . \tag{3.33}$$

Therefore, one needs also a criterion to select those fusing processes which survive quantization and are consistent with the mass flow. Below it is demonstrated how this can be achieved in a generic and universal manner by means of q-deformation.

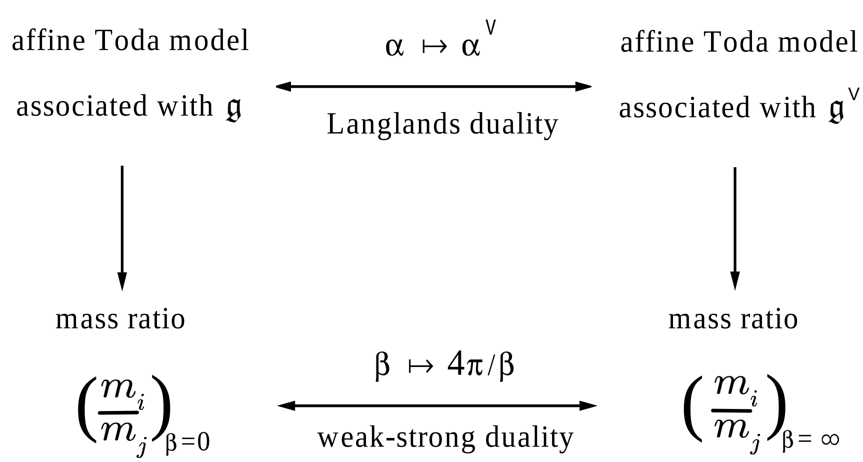

Figure 3.5: Schematic description of the renormalization flow in affine Toda models.

We conclude by pointing out that as far as this qualitative renormalization picture is concerned also the simply-laced algebras fit into the scheme expressed by (3.32). In the *ADE* case an interchange of roots and coroots does not alter the Lie algebra, i.e. both dual algebras coincide, $\mathfrak{g} = \mathfrak{g}^\vee$. According to (3.32) this amounts to a self-duality in the coupling constant and the coupling dependent mass flow is then just the trivial one, namely the mass ratios stay constant at their classical values.

3.2.3 Quantum fusing rules

Looking at (3.33) giving the classical fusing rules of two dual algebras one immediately infers that the classical fusing rule (3.26) has to be modified on the quantum level in order to accommodate the renormalization flow of ATFT. The first solution which comes to mind is to take the intersection of the non-vanishing three-point couplings in both dual theories upon identifying the particles in a suitable manner as explained in 2.4.1. In fact, this remedy to the problem of obviously different fusing structures of the dual partners was first suggested by Chari and Pressley [34].

Fusing rule in $\Omega, \hat{\Omega}$. *The* quantum *three-point coupling does not vanish if and only if there exist three representatives in the orbits $\Omega_i, \Omega_j, \Omega_k$ of the Coxeter element (2.37) of the Lie algebra $\mathfrak{g}$ and three representatives in the orbits $\hat{\Omega}_i, \hat{\Omega}_j, \hat{\Omega}_k$ of the twisted Coxeter element (2.47) of $\mathfrak{g}^\vee$ which separately sum up to zero.*

In formulas this means that in contrast to the classical case (3.26) there exist now *two* triples of integers (ξ_i, ξ_j, ξ_k) and $(\hat{\xi}_i, \hat{\xi}_j, \hat{\xi}_k)$ such that

$$\sum\nolimits_{l=i,j,k} \sigma^{\xi_l}\,\gamma_l = 0 \quad \text{and} \quad \sum\nolimits_{l=i,j,k} \hat{\sigma}^{\hat{\xi}_l}\,\hat{\gamma}_l^\omega = 0 \tag{3.34}$$

holds. Vice versa the existence of such triples implies that the quantum particles labelled by i, j, k couple to each other. Like before there always exists a second solution which is non-equivalent in the sense that it cannot be obtained by a simple shift. Namely, there are integers

$$\xi_l' = -\xi_l + \frac{c_l - 1}{2} \quad \text{and} \quad \hat{\xi}_l' = -\hat{\xi}_l + \frac{1 - c_l}{2}\ell_l + c_l + 1, \quad l = i, j, k. \tag{3.35}$$

such that (3.34) holds with the replacement $\xi_l \to \xi_l'$ and $\hat{\xi}_l \to \hat{\xi}_l'$. The argument why this is the only non-equivalent solution is similar to the one in the classical case and will be given below. The identification of the particles in the dual theories follows the prescription of Chapter 2, i.e. we identify the orbits of the first n roots in the twisted algebra with the roots of the untwisted algebra without relabelling. Note that there is a slight change in this fusing rule to the one stated in [34] due to differently chosen conventions for the twisted Coxeter element (2.47) as explained in Section 2.3.2.

Even though (3.34) correctly displays the fusing processes for the quantum ATFT it has the disadvantage to treat the dual algebras separately. In order to describe the coupling dependent mass flow and to derive a universal expression of the S-matrix it will become apparent below that it is essential to combine the information of both algebras in one setting. It was Oota who first observed that this might be done by means of q-deformation [36].

Using the definitions of Chapter 2 we state now two other fusing rules in terms of ordinary and twisted q-deformed Coxeter elements. Afterwards the precise relation among them and to (3.34) will be derived and all three fusing rules will turn out to be equivalent. Let us start with the non-twisted algebra $\mathfrak{g}$.

Fusing rule in Ω^q. *The* quantum *three-point coupling does not vanish if and only if there exist three representatives in the orbits $\Omega_i^q, \Omega_j^q, \Omega_k^q$ of the q-deformed Coxeter element (2.58) of the Lie algebra $\mathfrak{g}$ which sum up to zero.*

In concrete terms this fusing rule is expressed as follows. The existence of the mentioned three representatives implies for the q-deformed case that there are two triples of integers (ξ_i, ξ_j, ξ_k), $(\zeta_i, \zeta_j, \zeta_k)$ such that

$$\sum\nolimits_{l=i,j,k} q^{\zeta_l}\, \sigma_q^{\xi_l}\, \gamma_l = 0 \quad . \tag{3.36}$$

As we will see below the powers of the deformation parameter q now incorporate the information of the dual algebra. Again, we find equivalent solutions to (3.36) by acting with $q^y \sigma_q^x$ from the right on the above equation. A second *non*-equivalent solution, however, is obtained when replacing

$$\zeta_l \to \zeta_l' = -\zeta_l - (1+c_l)t_l \quad \text{and} \quad \xi_l \to \xi_l' = -\xi_l + \frac{c_l - 1}{2}, \qquad l = i,j,k \; . \tag{3.37}$$

This second solution can be constructed directly from (3.36), see [37], and in addition it is the only one. This will be proven below when rewriting the fusing structure in terms of q-deformed matrices.

The third and last fusing rule to be stated in terms of Coxeter geometry involves the data of the twisted or dual algebra $\mathfrak{g}^\vee$ only.

Fusing rule in $\hat{\Omega}^q$. *The* quantum *three-point coupling does not vanish if and only if there exist three representatives in the orbits $\hat{\Omega}_i^q, \hat{\Omega}_j^q, \hat{\Omega}_k^q$ of the q-deformed twisted Coxeter element (2.76) of the Lie algebra $\mathfrak{g}^\vee$ which sum up to zero.*

This fusing rule implies that there are again two triples of integers $(\hat{\xi}_i, \hat{\xi}_j, \hat{\xi}_k)$, $(\hat{\zeta}_i, \hat{\zeta}_j, \hat{\zeta}_k)$ such that the following equation holds,

$$\sum\nolimits_{l=i,j,k} q^{\hat{\zeta}_l} \hat{\sigma}_q^{\hat{\xi}_l}\, \hat{\gamma}_l^\omega = 0 \; . \tag{3.38}$$

Analogously to the cases considered before this fusing rule is complemented by a second solution when replacing

$$\hat{\zeta}_l \to \hat{\zeta}_l' = -\hat{\zeta}_l + 1 - c_l \quad \text{and} \quad \hat{\xi}_l \to \hat{\xi}_l' = -\hat{\xi}_l + \frac{1-c_l}{2}\ell_l + c_l + 1 \; , \quad l = i,j,k \; . \tag{3.39}$$

This second solution can be derived from the given one (3.38) and as in the untwisted case it is unique, whence the statement of only one is sufficient as existence criterion.

Having stated three versions of fusing rules we need to clarify whether they are equivalent and how they are related to each other in order to obtain a consistent picture. One conclusion can be drawn immediately. From the definition of the q-deformed Coxeter and twisted Coxeter element we see that in the "classical" limit

$q \to 1$ one recovers from the two q-deformed versions (3.36) and (3.38) the non-deformed fusing rule (3.34). This is a first hint that the three versions are in some sense compatible to each other. It remains, however, to give the precise relation between the integers appearing in the different versions and to prove that one of them is sufficient to imply all the other. This discussion is postponed after the next subsection in which the connection between the fusing rules and the conserved quantities, in particular the masses and their fusing angles, will be worked out.

3.2.4 Quantum mass spectrum and conserved charges

For the moment let us assume as a working hypothesis that the three variants (3.34), (3.36) and (3.38) of the fusing rule are equivalent. Then the next step towards the construction of the S-matrix is the determination of the mass spectrum and the corresponding fusing angles. As the classical relations hold true for the ADE subclass of ATFT one expects to find close analogues for the eigenvalue equations (3.21) and more generally (3.23). After comparing the classical fusing rule with the q-deformed ones a possible conjecture which comes to mind is to replace the ordinary Cartan matrix by the q-deformed one (2.67) or equivalently (2.85) defined in Chapter 2. However, the relation turns out to be slightly different and in particular one has to specify the coupling dependence of the masses first by a specific choice of the deformation parameters $q, \hat{q}$, which hitherto have been kept completely generic.

Quantum charges. *The quantum analogues to the charges (3.28) preserved by the fusing processes of the theory are given by the following null vector of the q-deformed Cartan matrix as specified in (2.67),*

$$\sum_{j=1}^{n} A_{ij}(q = e^{i\pi s\vartheta_h}, \hat{q} = e^{i\pi s\vartheta_H}) Q_j^{(s)} = 0 \; . \tag{3.40}$$

Here the deformation parameters are chosen in terms of the angles

$$\vartheta_h := \frac{2-B}{2h} \qquad \vartheta_H := \frac{B}{2H} \tag{3.41}$$

which determine the coupling dependence of the charges via the function

$$B(\beta) = \frac{2\beta^2}{4\pi h/h^\vee + \beta^2} \; . \tag{3.42}$$

As in the classical case the spin s runs over the exponents of the Lie algebra $\mathfrak{g}$.

Admittedly, this definition of the quantum conserved charges and their coupling dependence appears to be a bit *ad hoc*. The particular parametrization of the **effective coupling constant** B was first suggested in [79] and will allow to express the coupling dependence of the fusing angles in linear terms. Note that its range lies in the interval $0 \leq B \leq 2$ where the boundary values correspond to the weak and strong coupling limit, respectively. Under a duality transformation in the coupling constant $\beta \to 4\pi/\beta$ *and* a simultaneous exchange of the Coxeter numbers one has $B \to 2 - B$. In this sense it reflects the renormalization behaviour (3.32).

The null vector equation (3.40) can be motivated by the following arguments. According to the renormalization behaviour (3.30) we ought to recover the classical relation (3.21) for the ADE series of ATFT. If the algebras are simply-laced then $\mathfrak{g} = \mathfrak{g}^\vee$ and all the roots have same length implying the integers (2.55) to be all one. Since also the entries of the incidence matrix $I = 2 - A$ are zero or one the q-deformed Cartan matrix (2.67) reduces to

$$ADE: \quad A(q,\hat{q}) = q\hat{q} + q^{-1}\hat{q}^{-1} - I \tag{3.43}$$

which upon inserting in (3.40) and noting that $h = h^\vee$ reduces to the classical relation (3.22) which can be transformed into the eigenvalue equation (3.21). In the generic case, however, a genuine eigenvalue equation cannot be regained from (3.40). In particular for $s = 1$ giving the quantum masses it leads to

$$\sum_{j=1}^{n} [I_{ij}]_{\hat{q}}\, m_j = 2\cos\pi\,(\vartheta_h + t_i\vartheta_H)\; m_i\;, \quad \hat{q} = e^{i\pi s\vartheta_H}\;. \tag{3.44}$$

The scalar factor corresponding to the eigenvalue in the classical equation (3.22) now depends via the symmetrizer t_i on the particle index spoiling the eigenvalue property. Nevertheless, (3.44) exhibits nicely the Lie algebraic structure present and provides us with a universal formula of the quantum mass spectrum for *all* models of ATFT. Moreover, in the limit $\beta \to 0$ the classical equation is recovered once more, which demonstrates compatibility with the renormalization properties outlined in Subsection 3.2.2.

Despite these compatibility checks which leave little doubt about the correctness of the above assertion a generic Lie algebraic proof is still missing and requires more profound insight in the structure of the associated quantum field theory. On a case-by-case study, however, it has been established in [36] and [37], see also the appendix.

3.2.5 The quantum fusing angles

After having stated how to obtain the charges preserved by a fusing process we need to check it for consistency with the three fusing rules (3.36), (3.38) and (3.34). This will give additional support to the mass spectrum formula (3.44) and provide us not only with the fusing angles needed for the bootstrap equation but also yield an intrinsic proof of the equivalence of all three versions of the fusing rule [37]. From the discussion of the classical regime we infer that an equation analogous to (3.28) for the quantum conserved charges is required in order to determine the fusing triangle (Figure 3.3) and, thus, the fusing angles.

Fusing angles. *The quantities defined through the null vector (3.40) are conserved in the fusing processes specified by the rules (3.36), (3.38) and (3.34). That is, they satisfy the equation*

$$\sum\nolimits_{l=i,j,k} e^{i\pi s(\eta_l\vartheta_h + \hat{\eta}_l\vartheta_H)}\, Q_l^{(s)} = 0 \tag{3.45}$$

whenever the particles labelled by i, j, k couple to each other. In particular, the integers $\eta_l, \hat{\eta}_l$ determine the (quantum) fusing angles to be

$$u_{ij}^k = (\eta_j - \eta_i)\pi\vartheta_h + (\hat{\eta}_j - \hat{\eta}_i)\pi\vartheta_H \tag{3.46}$$

The angle coefficients $\eta_l, \hat{\eta}_l$ are related to the integers ζ_l, ξ_l and $\hat{\zeta}_l, \hat{\xi}_l$ defined in (3.36) and (3.38) via the equations

$$\eta_l = -2\xi_l + \frac{c_l - 1}{2} \qquad \hat{\eta}_l = \zeta_l + \frac{1 + c_l}{2}\, t_l \,, \tag{3.47}$$

$$\eta_l = \hat{\zeta}_l + \frac{c_l - 1}{2} \qquad \hat{\eta}_l = -2\hat{\xi}_l + \frac{1 - c_l}{2}\, \ell_l + c_l + 1 \,, \tag{3.48}$$

for $l = i, j, k$. Moreover, the identity (3.45) is equivalent to the fusing rules (3.36) and (3.38).

The proof of these assertion is given momentarily. First notice that the fusing angles (3.46) are now coupling dependent via the defining relations (3.41) and (3.42). Moreover, while the first summand $(\eta_j - \eta_i)$ determines the *classical* fusing angle of the non-twisted algebra via (3.29), the second $(\hat{\eta}_j - \hat{\eta}_i)$ yields the classical value for the dual, twisted algebra. This is in accordance with the renormalization picture outlined in 3.2.2. Specializing to $s = 1$ we deduce from (3.45) the following "floating" mass ratios

$$\frac{m_i}{m_j} = \frac{\sin\left[(\eta_k - \eta_j)\pi\vartheta_h + (\hat{\eta}_k - \hat{\eta}_j)\pi\vartheta_H\right]}{\sin\left[(\eta_i - \eta_k)\pi\vartheta_h + (\hat{\eta}_i - \hat{\eta}_k)\pi\vartheta_H\right]} \,. \tag{3.49}$$

This especially implies that the fusing triangle in Figure 3.3 which is "static" in the classical case now starts to vary its shape when the coupling constant β is tuned between the weak and strong coupling regime. It then interpolates between the classical values at $\beta \to 0$ and $\beta \to \infty$. More generally we have the relation

$$Q_i^{(s)}/Q_j^{(s)} = \frac{\sin\left[(\eta_k - \eta_j)\pi s\vartheta_h + (\hat{\eta}_k - \hat{\eta}_j)\pi s\vartheta_H\right]}{\sin\left[(\eta_i - \eta_k)\pi s\vartheta_h + (\hat{\eta}_i - \hat{\eta}_k)\pi s\vartheta_H\right]} \,. \tag{3.50}$$

Both relations can be interpreted in the complex velocity plane as explained in the *ADE* case [29]. Together with the last two equations all the structures presented in the context of classical ATFT have been "translated" to the quantum level yielding all the necessary ingredients for the bootstrap construction of the two-particle S-matrix. However, before constructing the latter, we need to prove the above assertions (3.45),(3.47),(3.48) and to verify the equivalence between the fusing rules. To do this we exploit the matrix structure present in the theory.

3.2.6 The relation between the fusing rules

We start by identifying the matrix elements of the M-matrix defined in (2.64) as the quantum conserved charges (3.40). In the determining equation (2.65) for the M-matrix it is immediate to see that

$$A(q,\hat{q})M(q,\hat{q}) = 0$$

whenever $q^{2h}\hat{q}^{2H} = 1$. The last property is easily verified for the special choice of the deformation parameters in (3.40). Moreover, from the discussion of Chapter 2, and the particular form of the determinant (2.69) we infer that for $q = e^{i\pi s\vartheta_h}, \hat{q} = e^{i\pi s\vartheta_H}$ the matrix M is only non-zero when s is an exponent of the Lie algebra $\mathfrak{g}$. Recalling in addition that M is symmetric we are led to the conclusion

$$M_{ij}(e^{i\pi s\vartheta_h}, e^{i\pi s\vartheta_H}) \propto Q_i^{(s)} Q_j^{(s)} \tag{3.51}$$

by comparison with (3.40). Note that the proportionality factor in (3.51) does not depend on the particle indices i or j. The expected fusing equation (3.45) in terms of conserved quantities can therefore be translated into a fusing equation of the matrix M,

$$\sum\nolimits_{l=i,j,k} q^{\eta_l}\hat{q}^{\hat{\eta}_l} M_{ml}(q,\hat{q}) = 0 \ , \quad q = e^{i\pi s\vartheta_h}, \hat{q} = e^{i\pi s\vartheta_H} \ , \tag{3.52}$$

for all indices $1 \leq m \leq n$. The crucial step is now to relate (3.52) to one of the "quantum" fusing rules, say (3.36), and to derive the explicit expressions for $\eta_l, \hat{\eta}_l$ in terms of the integers ζ_l, ξ_l.

Let us assume that the fusing rule (3.36) in terms of the q-deformed Coxeter element holds for some integers $(\zeta_i, \zeta_j, \zeta_k)$ and (ξ_i, ξ_j, ξ_k). Since $\hat{q}^{\,-2H}\sigma_{\hat{q}}^h = 1$, we may assume $0 < \xi_l < h$ without loss of generality. Acting with $[t_m]_{\hat{q}}/2\ \hat{q}^{\frac{1-c_m}{2}\,t_m} q^{2x-\frac{c_m}{2}-1}\sigma_{\hat{q}}^x$ on the above equation and summing over x afterwards yields,

$$0 = -\frac{[t_m]_{\hat{q}}}{2} \sum_{l=i,j,k} \hat{q}^{\zeta_l + \frac{1-c_m}{2}\,t_m} \sum_{x=1}^{h} \left\langle \lambda_m^\vee, \sigma_{\hat{q}}^{\;x+\xi_l}\,\gamma_l \right\rangle\, q^{2x-\frac{c_m}{2}-1}.$$

Now splitting up the sum into two parts,

$$\sum_{x=1}^{h} \left\langle \lambda_m^\vee, \sigma_{\hat{q}}^{\;x+\xi_l}\,\gamma_l \right\rangle\, q^{2x-\frac{c_m}{2}-1} = q^{-2\xi_l} \sum_{x=\xi_l+1}^{h+\frac{c_l-1}{2}} \left\langle \lambda_m^\vee, \sigma_{\hat{q}}^{\;x}\,\gamma_l \right\rangle\, q^{2x-\frac{c_m}{2}-1}$$
$$+\, q^{-2\xi_l} \sum_{x=h+\frac{c_l+1}{2}}^{h+\xi_l} \left\langle \lambda_m^\vee, \sigma_{\hat{q}}^{\;x}\,\gamma_l \right\rangle\, q^{2x-\frac{c_m}{2}-1}$$

and remembering that $q^{2h}\hat{q}^{2H} = 1$ we have by means of equation (2.64) that

$$\sum_{l=i,j,k} q^{-2\xi_l + \frac{c_l-1}{2}}\, \hat{q}^{\zeta_l + \frac{1+c_l}{2}\,t_l}\, M_{lm}(q,\hat{q}) = 0,$$

which is the desired reformulation of the fusing rule with $\eta_l = -2\xi_l + \frac{c_l-1}{2}$ and $\hat{\eta}_l = \zeta_l + \frac{1+c_l}{2}\,t_l$. To prove the complementary assertion we assume that (3.52) holds and define $\eta_l =: -2\xi_l + \frac{c_l-1}{2}$. Then by use of (2.64) we see that

$$0 = \sum_{l=i,j,k} \hat{q}^{\bar{\eta}_l} \sum_{n=1}^{2h} M_{lm}(\tau^n,\hat{q})\tau^{n(\eta_l+\frac{c_m+1}{2})} = -\frac{[t_m]_{\hat{q}}}{2}\hat{q}^{\frac{1-c_m}{2}\,t_m} \sum_{l=i,j,k} \hat{q}^{\hat{\eta}_l - \frac{1+c_l}{2}\,t_l} \left\langle \lambda_m^\vee, \sigma_{\hat{q}}^{\xi_l}\,\gamma_l \right\rangle$$

for all $1 \leq m \leq n$. But since the fundamental co-weights form a basis, this implies the fusing rule (3.36) with $\eta_l = -2\xi_l + \frac{c_l-1}{2}$ and $\hat{\eta}_l = \zeta_l + \frac{1+c_l}{2}\,t_l$.

Completely, analogous one proves the equivalence between (3.38) and (3.52) for the twisted algebra by means of the N-matrix and its definition in terms of the q-deformed twisted Coxeter element (2.82) and the identity (2.86). Note that the identification of the N and the M-matrix is crucial in this step. In particular, one has as an immediate consequence also the equivalence of the two fusing rules (3.36) and (3.38) in terms of the q-deformed Coxeter elements [37]. Having established the equivalence between the latter fusing rules the one in terms of the non-deformed Coxeter elements now also follows in the classical limit $q \to 1$ as we already have seen. A Lie algebraic proof how to derive the q-deformed versions from the non-deformed ones is outstanding, but it can be verified case-by-case. This shows that all three different formulations are consistent and that any of them implies all the others. In particular, we also proved that the definition of the quantum charges (3.45) is consistent with the fusing rules [37], which makes their definition as null-vector of the q-deformed Cartan matrix very suggestive. To close the picture we summarize the relations between the various integers used in the different equations.

Summary. *The fusing rules involving the q-deformed Coxeter and twisted Coxeter element are linked to each other by [37]*

$$-2\xi_l = \hat{\zeta}_l, \quad \text{and} \quad \zeta_l = -2\hat{\xi}_l + \frac{1-c_l}{2}\ell_l - \frac{1+c_l}{2}t_l + c_l + 1\,, \quad l = i,j,k\,. \tag{3.53}$$

Notice that as in the classical case the non-equivalent solutions (3.37) and (3.39) are reflected in (3.45) by complex conjugation, i.e. $\eta_l \to -\eta_l$, $\hat{\eta}_l \to -\hat{\eta}_l$ *when either* $\xi_l \to \xi_l'$, $\zeta_l \to \zeta_l'$ *or* $\hat{\xi}_l \to \hat{\xi}_l'$, $\hat{\zeta}_l \to \hat{\zeta}'$. *In particular, this proves the existence of the non-equivalent solutions. In terms of them the relations between the powers appearing in the fusing rules can be simplified [37],*

$$\eta_l = \xi_l' - \xi_l = \frac{\hat{\zeta}_l - \hat{\zeta}_l'}{2} \quad \text{and} \quad \hat{\eta}_l = \frac{\zeta_l - \zeta_l'}{2} = \hat{\xi}_l' - \hat{\xi}_l\,, \quad l = i,j,k\,. \tag{3.54}$$

Furthermore, interpreting (3.45) geometrically we conclude by the same argumentation as in the classical or ADE case [29] that only two non-equivalent solutions corresponding to the two fusing triangles shown in Figure 3.3 exist.

3.2.7 The S-matrix in blocks of meromorphic functions

The universal expressions for the fusing rules, the mass spectrum and the fusing angles allow to write down a generic formula for the ATFT S-matrix instead of constructing the scattering amplitudes for each model separately,

$$S_{ij}(\theta) = \prod_{x=1}^{2h}\prod_{y=1}^{2H} \{x,y\}_\theta^{\mu_{ij}(x,y)} = \prod_{x=1}^{h}\prod_{y=1}^{H} \{x,y\}_\theta^{2\mu_{ij}(x,y)}\ . \tag{3.55}$$

Here $\{x,y\}_\theta$ denotes a ratio of hyperbolic functions similar to (3.13) mentioned in the context of the functional equations and which will be specified momentarily. The power function $\mu = \mu(x,y)$ takes its values in the half integers $\frac{1}{2}\mathbb{Z}$ and is positive for the range of arguments stated in (3.55). In the course of the argumentation both writings of the product will be used and their equivalence derived. Moreover, it will be demonstrated that μ can be expressed in universal Lie algebraic terms and the structure developed in the preceding paragraphs will then serve to prove the correct bootstrap properties of (3.55).

Blocks of meromorphic functions

We already have seen that the analytic requirements on the two-particle scattering amplitude together with unitarity (3.10) restricted severely the form of solutions to the functional equations. For the discussion of the ATFT matrix it was first observed by Dorey [79] that the combination of hyperbolic functions into the following building blocks is most suitable,

$$\{x,y\}_\theta := \frac{[x,y]_\theta}{[x,y]_{-\theta}} \quad [x,y]_\theta := \frac{\langle x-1,y-1\rangle_\theta \, \langle x+1,y+1\rangle_\theta}{\langle x-1,y+1\rangle_\theta \, \langle x+1,y-1\rangle_\theta} \tag{3.56}$$

and

$$\langle x,y\rangle_\theta := \sinh \tfrac{1}{2}\left(\theta + x\theta_h + y\theta_H\right) . \tag{3.57}$$

This is of the general form (3.13) discussed in 3.1.2. The shifts depending on the integer entries x, y are defined in terms of the angles (3.41) introduced in context of the fusing rule and the conserved charges,

$$\theta_h := i\pi\vartheta_h \qquad \text{and} \qquad \theta_H := i\pi\vartheta_H .$$

It should be emphasized that this definition of the building blocks as innocent as it might look at first sight is essential in displaying the Lie algebraic structure. Indeed, as it will turn out below the integer entries will be related to the powers of the q-deformed Coxeter elements and the deformation parameters. Already from the defining relation involving the Coxeter numbers h, H we can deduce that the first entry x will be related to the structure of the untwisted algebra $\mathfrak{g}$ while the second is connected to the twisted one $\mathfrak{g}^\vee$. Moreover, the block structure together with the effective coupling constant B defined in (3.42) nicely incorporates the renormalization properties: Upon a strong-weak duality transformation in the classical coupling constant, $\beta \to 4\pi/\beta$, and a simultaneous exchange of the algebras, $\mathfrak{g} \leftrightarrow \mathfrak{g}^\vee$, the effective coupling transforms as $B \to 2 - B$ and the S-matrix stays invariant.

For later purposes it is important to note that the blocks (3.56) can alternatively be expressed in terms of Fourier integrals of the form

$$\{x,y\}_\theta = \exp \int_0^\infty \frac{dt}{t\sinh t} \; f_{x,y}^{h,H}(t,B) \sinh\left(\frac{\theta t}{i\pi}\right) , \tag{3.58}$$
$$f_{x,y}^{h,H}(t,B) := 8 \sinh t\vartheta_h \sinh t\vartheta_H \sinh t(1 - x\vartheta_h - y\vartheta_H) .$$

However, for proving the bootstrap properties of the S-matrix (3.55) the block form in terms of hyperbolic functions is more convenient. Especially, easy to derive are the subsequent functional relations which will become important in due course,

$$\{x,y\}_\theta = \{x+2h, y+2H\}_\theta = \{-x,-y\}_\theta^{-1} \tag{3.59}$$
$$\{x,y\}_{\theta+x'\theta_h+y'\theta_H} = \frac{[x+x', y+y']_\theta}{[x-x', y-y']_{-\theta}} \tag{3.60}$$
$$\{x,y\}_{\theta+p\theta_h+q\theta_H} \, \{x,y\}_{\theta-p\theta_h-q\theta_H} = \{x+p, y+q\}_\theta \, \{x-p, y-q\}_\theta . \tag{3.61}$$

The equivalent integral expression requires some comments about convergence when one analytically continues into the complex plane w.r.t. the rapidity. Performing the shift $\theta \to \theta + x'\theta_h + y'\theta_H$ convergence is maintained if

$$0 \leq (x - x' - 1)\vartheta_h + (y - y' - 1)\vartheta_H \leq 2(1 - (1 + x')\vartheta_h - (1 + y')\vartheta_H) \ .$$

An additional aspect which deserves careful attention is the non-commutativity of certain limits when θ tends to zero. While we infer that in general $\{x, y\}_{\theta=0} = 1$, we take the convention to set $\{1, 1\}_{\theta=0} = -1$ meaning that one first should set $x = y = 1$ and then take the limit $\theta \to 0$. Similarly, the integral representation (3.58) requires also to set $x = y = 1$ first, to integrate thereafter and finally to take the limit $\theta \to 0$.

Having specified the building blocks and their analytic properties the second step in the derivation of the universal expression (3.55) is the definition of the power function. There are three equivalent ways to define $\mu(x, y)$ and all of them have different advantages for displaying the various Lie algebraic structures connected to the pair of dual algebras $(\mathfrak{g}, \mathfrak{g}^\vee)$.

The power function in terms of q-deformed matrices

In regard to the building blocks just discussed we start with the "matrix representation" of the power function μ since this is the most convenient one to show how the analytic properties (3.59) of a single block are reflected in the powers of the S-matrix (3.55). In Chapter 2 it was argued that the M-matrix (2.64) consists of a polynomial in the deformation parameters $q, \hat{q}$ and the power function $\mu(x, y)$ was implicitly defined in 2.4.3 as the coefficient of the monomials $q^x \hat{q}^y$ in the expansion (2.68). Formally, the latter equation might be inverted by discrete Fourier transformation,

$$\mu(x, y) = \frac{1}{2h} \sum_{r=1}^{2h} \frac{1}{2H} \sum_{s=1}^{2H} M(\tau, \hat{\tau}) \tau^{rx} \hat{\tau}^{sy} \ , \tag{3.62}$$

where $\tau, \hat{\tau}$ are roots of unity of order $2h$ and $2H$, respectively. This equality just states that up to a factor one half the power of the block $\{x, y\}$ appearing in (3.55) equals the number of times the monomial $q^x \hat{q}^y$ occurs in the expansion of $M(q, \hat{q})$. The factor $1/2$ originates in the defining relation (2.64). By the restriction of the deformation parameters to roots of unity, $q = \tau, \hat{q} = \hat{\tau}$, in (3.62) and the matrix identity (2.66) it is now straightforward to verify [37] the following properties of the power function ,

$$\mu(x, y) = \mu(x + 2h, y + 2H) = -\mu(2h - x, 2H - y) = \mu(x, y)^t \tag{3.63}$$

The first property is obvious since $\tau^{2h} = \hat{\tau}^{2H} = 1$, the second follows from (2.74) and the last one just reflects that the M-matrix is symmetric due to (2.71), i.e. $M(q, \hat{q}) = M(q, \hat{q})^t$. Recall that the manipulations of the M-matrix at roots of unity in terms of the identity (2.66) require some care. As discussed in Chapter 2 the determinant of the q-deformed Cartan matrix (2.69) has to be canceled against the prefactor in (2.66) first and then one might safely set $q = \tau, \hat{q} = \hat{\tau}$. Clearly, the first two properties (3.63) are analogues of the block behaviour (3.59). In particular we infer from (3.63) that the expansion of the M-matrix as polynomial has to be of the form

$$M(q,\hat{q}) = \sum_{x=1}^{h}\sum_{y=1}^{H} \mu(x,y) \left(q^x \hat{q}^y - q^{2h-x}\hat{q}^{2H-y} \right) \tag{3.64}$$

from which the second equality in (3.55) is deduced. The latter rewriting is important, because it ensures that the building blocks $\{x, y\}$ appear with integral powers in the S-matrix. This in turn guarantees that $S_{ij}(\theta)$ is a meromorphic function of the rapidity meeting the general requirement (S3).

There are two more important identities needed for the discussion of the bootstrap properties of the S-matrix (3.55). The first one is connected to crossing symmetry and is a direct consequence of the definition of the anti-particle (2.61),

$$\mu_{j\bar{\imath}}(x,y) = \mu_{ij}(h-x, H-y) \; . \tag{3.65}$$

The second one is linked to a fusing process $i + j \to \bar{k}$ and can be derived from the matrix fusing rule (3.52),

$$\sum_{l=i,j,k} \mu_{ml}\left(x \pm \eta_l, y \pm \hat{\eta}_l\right) = 0 \, . \tag{3.66}$$

An additional identity for the power function can be deduced from the determining equation for the M-matrix (2.65) which played a central role in the discussion of the fusing rules and the conserved quantities,

$$\mu_{ij}(x+1, y+t_i) + \mu_{ij}(x-1, y-t_i) = \sum_{n=1}^{I_{il}} \sum_{l\in\Delta} \mu_{lj}(x, y+2n-1-I_{il}) \tag{3.67}$$

where it is understood that the sum gives zero when $I_{il} = 0$. Its derivation is immediate when inverting (2.65) by discrete Fourier transformation. The identity (3.67) for the power function was first mentioned in [36] where it was used as recursive relation to generate the powers in (3.55). When discussing the corresponding identities for the S-matrix below we will see that (3.67) should, however, regarded as a consequence of the bootstrap equations (3.66) which turn out to be more fundamental [37].

Remark. *Notice that instead of the M-matrix one might also use the structure of the twisted algebra $\mathfrak{g}^\vee$ and define everything in terms of the N-matrix together with the identity (2.83). But since both matrices were shown to be equivalent in (2.86) the same structure emerges [37].*

Furthermore, it should be emphasized that the matrix representation of the power function requires only a minimum of Lie algebraic data and is certainly most convenient in deriving the relations (3.63) needed for verifying the bootstrap equations of the S-matrix in the next subsection. Despite these obvious advantages, however, it should kept in mind that the matrix identities (2.66) and (2.83) were derived by means of Coxeter geometry. Also the expansion of the M-matrix or N-matrix in a polynomial – necessary for determining μ – might turn out to be rather complicated for higher rank algebras, since it involves the inversion of the q-deformed Cartan matrix. Another point is the dual structure of the two Lie algebra $(\mathfrak{g}, \mathfrak{g}^\vee)$ which remains quite hidden in this formalism. Therefore, it is advisable to present the power function also in an alternative manner by use of Coxeter geometry.

The power function in terms of q-deformed Coxeter elements

In the preceding paragraphs the logic of the q-deformation as introduced in Chapter 2 was in a certain sense "reversed". Everything was expressed in terms of the most sophisticated structure derived in the end, the q-deformed Cartan matrices. We now return to Coxeter geometry and instead of calculating the power function by first extracting polynomials from the M or N-matrix we directly read it off from the orbit structure of σ_q and $\hat{\sigma}_q$ as defined in (2.58) and (2.76), respectively.

For the untwisted algebra X_n the powers in the representation (3.55) can be defined in terms of the q-deformed Coxeter element as the generating function

$$\sum_y \mu_{ij}\left(2x - \frac{c_i + c_j}{2}, y\right) q^y = -\frac{[t_j]_q}{2} q^{\frac{(1-c_j)t_j - (1+c_i)t_i}{2}} \left\langle \lambda_j^\vee, \sigma_q^x \gamma_i \right\rangle \ , \tag{3.68}$$

for fixed integer x. Taking x in the range $(3-c_i)/2 \leq x \leq h+(1-c_i)/2$ ensures that the first argument of μ is between 1 and $2h$. This definition of the power function makes it especially evident that the definition of the S-matrix (3.55) is a clear generalization of the one of the simply-laced case found in [29]. The q-deformation reflects the presence of the second dual algebra, since the powers of the deformation parameter determine the second entries for which $\mu(x, y)$ is non-vanishing. The properties (3.63) might now be derived directly from (2.60), (2.73) and (2.72) in Section 2.4.4. Also the crossing (3.65) and bootstrap property (3.66) can be verified by means of (2.61) and (3.36), respectively [37].

Alternatively, one might use the twisted algebra and the associated q-deformed twisted Coxeter element to define $\mu(x, y)$ as the generating function

$$\sum_x \mu_{ij}\left(x, 2y - c_i + \frac{c_i - 1}{2}\ell_i - \frac{c_j - 1}{2}\ell_j\right) q^x = -\frac{q^{-\frac{c_i+c_j}{2}}}{2} \sum_{k=1}^{\ell_j} \left\langle \lambda_{\omega^k(j)}, \hat{\sigma}_q^y \hat{\gamma}_i^\omega \right\rangle \ . \tag{3.69}$$

This time the second entry y is kept fixed while x varies and now the structure of the untwisted algebra is incorporated in the deformation parameter. Using the identities (2.78), (2.90), (2.89) together with the definition of the anti-particle (2.79) and the fusing rule (3.38) for the twisted algebra one again verifies all the desired properties of μ [37].

Remark. *The three definitions (3.62), (3.68) and (3.69) of the power function are equivalent by means of the identities (2.64), (2.82) and (2.86) proved in Chapter 2 [37]. This means in particular that the representation (3.55) of the ATFT S-matrix might be computed in three different ways, via the orbits of the q-deformed Coxeter element, the orbits of the q-deformed twisted Coxeter element and the q-deformed Cartan matrix upon inversion.*

Which of the mentioned methods is preferable in order to construct explicitly the powers of the building blocks might depend on the particular example at hand.

In the appendix all three of them are demonstrated for numerous examples. The next step is to establish that the scattering amplitudes (3.55) satisfy the bootstrap functional relations proving them to be consistent solutions.

3.2.8 The bootstrap properties

Since ATFT is assumed to be invariant under parity transformation we expect that the two-particle scattering amplitude is symmetric in the particle indices. Indeed, $S_{ij}(\theta) = S_{ji}(\theta)$ follows for our choice (3.55) from the last equality in (3.63). This in particular implies that Hermitian analyticity is replaced by the stronger requirement of real analyticity and the unitarity-analyticity equation (3.10) reduces to

$$S_{ij}(\theta) S_{ij}(-\theta) = 1 \ .$$

That the latter functional equation is fulfilled is immediate to verify from the property $\{x,y\}_\theta \{x,y\}_{-\theta} = 1$ of each individual building block in (3.55).

Crossing symmetry

In contrast, the crossing relation $S_{ij}(\theta) = S_{\bar{\imath}j}(i\pi - \theta)$ requires in general a little bit more effort,

$$\begin{aligned} S_{ij}(i\pi-\theta) &= \prod_{x=1}^{2h}\prod_{y=1}^{2H}\{h-x,H-y\}_\theta^{\mu_{ij}(x,y)} = \prod_{x=1}^{2h}\prod_{y=1}^{2H}\{x+h,y+H\}_\theta^{-\mu_{ij}(x,y)} \\ &= \prod_{x=1}^{h}\prod_{y=1}^{H}\{x+2h,y+2H\}_\theta^{-\mu_{ij}(x+h,y+H)} \prod_{x=h+1}^{2h}\prod_{y=H+1}^{2H}\{x,y\}_\theta^{-\mu_{ij}(x-h,y-H)} \\ &= \prod_{x=1}^{2h}\prod_{y=1}^{2H}\{x,y\}_\theta^{\mu_{ij}(h-x,H-y)} = S_{j\bar{\imath}}(\theta) \ . \end{aligned}$$

Here we have used first the shifting property (3.59) of the building blocks and second the corresponding relation for the power function (3.63) as well as the definition of the anti-particle (3.65).

Bootstrap equation

For our purposes it is convenient to use the second variant (3.18) of the bootstrap equations which in terms of the fusing angles (3.46) reads

$$\prod_{l=i,j,k} S_{ml}(\theta + \eta_l \theta_h + \hat{\eta}_l \theta_H) = 1 \, . \tag{3.70}$$

This might be verified by similar shifting techniques in the building blocks as in the case of crossing symmetry. In view of the property (3.60) it turns out to be necessary to separate the building blocks in two sub-blocks of hyperbolic functions,

$$S_{ml}(\theta + \eta_l \theta_h + \hat{\eta}_l \theta_H) = \prod_{x=1}^{2h}\prod_{y=1}^{2H}\left(\frac{[x+\eta_l, y+\hat{\eta}_l]_\theta}{[x-\eta_l, y-\hat{\eta}_l]_{-\theta}}\right)^{\mu_{ml}(x,y)} \ .$$

Now numerator and denominator may be treated separately. One of which requires the equation (3.66) for the upper and the other for the lower sign showing that both non-equivalent solutions to the fusing rules are needed as has been mentioned earlier. Proceeding similar like in the case of crossing symmetry one ends up with the expression

$$S_{ml}(\theta+\eta_l\theta_h+\hat{\eta}_l\theta_H)=\prod_{x=1}^{2h}\prod_{y=1}^{2H}\frac{[x,y]_{\theta}^{\mu_{ml}(x-\eta_l,y-\hat{\eta}_l)}}{[x,y]_{-\theta}^{\mu_{ml}(x+\eta_l,y+\hat{\eta}_l)}}$$

which upon using (3.66) gives the desired bootstrap identity (3.70). Thus, the claim of the universal ATFT S-matrix (3.55) is proven to be consistent with the formulated fusing rules of Subsection 3.2.3 [37].

Combined bootstrap identities

Besides the bootstrap identities describing the consistency between factorization of the S-matrix and the fusing structure, there are additional product identities special for the affine Toda models. In the course of the presented argumentation equation (2.64) turned out to be the backbone in analyzing the Lie algebraic structure, since it provides the link between q-deformed Coxeter geometry and q-deformed Cartan matrices. It is therefore remarkable that the same equation translated in terms of the power function (3.67) has also a direct interpretation in terms of S-matrices. Namely, using the shifting property (3.61) of the building blocks one obtains [37]

$$S_{ij}\left(\theta+\theta_h+t_i\theta_H\right)S_{ij}\left(\theta-\theta_h-t_i\theta_H\right)=\prod\nolimits_{l=1}^{n}\prod\nolimits_{k=1}^{I_{il}}S_{lj}\left(\theta+(2k-1-I_{il})\theta_H\right)\;, \tag{3.71}$$

where it is understood that the product contributes 1 if $I_{il}=0$. In context of simply-laced algebras such identities were first obtained in [80] but by different methods. Here they follow from a purely geometrical context (2.64) and it was first noted in [37] that they can be built up from the elementary bootstrap equations (3.70), whence they are referred to as “combined” bootstrap equations. Note that only direct Lie algebraic quantities like the length of the roots and the incidence matrix enter the above equation. Moreover, in contrast to the ADE case the r.h.s. in (3.71) involves multiple products of S-matrices for non simply-laced algebras. Explicit examples are worked out in the appendix.

Singularities and the generalized bootstrap criterion

From the bootstrap equation (3.70) for the ATFT S-matrix we infer that the bound state poles depend on the coupling via the fusing angles (3.46) which is in accordance with the renormalization properties of the mass spectrum. On general grounds [9] connected to the analytic structure of the amplitude $S_{ij}(\theta)$ the bound state poles ought to be of odd order or even simple and to move in the physical sheet when $0\leq\beta\leq\infty$ or $0\leq B\leq 2$ is varied,

$$\phi=\pm(\eta_i-\eta_j)\theta_h\pm(\hat{\eta}_i-\hat{\eta}_j)\theta_H\;. \tag{3.72}$$

The two signs correspond to the two non-equivalent solutions of the fusing rules. Furthermore, at a bound state pole the S-matrix is expected to have up to a factor $\sqrt{-1}$ a residue of definite sign, see the general discussion in Section 3.1.2 While in the case of simply-laced Lie algebras all simple poles in the physical sheet could be shown to have this kind of behaviour and could also be traced to the fusing processes described by (3.26), once more matters turn out to be more involved in the non simply-laced case. Here additional simple or odd order poles appear in the physical sheet which cannot consistently be interpreted in terms of bound states as was shown in perturbation theory. Corrigan et al. noted that these poles can be characterized by the property that their residue $R_{ij}^k(\beta)$ eventually changes its sign when β sweeps through the allowed range and suggested the following "generalized" bootstrap principle [33]:

Only odd order poles which have residue of definite sign over the whole range of the coupling constant participate in the bootstrap, while those of varying sign together with even order poles are excluded.

It are exactly the poles singled out by the above prescription, which are described by the fusing rules (3.34), (3.36), and (3.38). To see this directly for the constructed S-matrices in the appendix it is convenient to derive a simple criterion in terms of the building blocks (3.56) in (3.55) to decide whether or not an odd order pole has the physical interpretation of a fusing process.

For this purpose the building blocks (3.56) are not well suited since numerous cancellations of zeroes and poles take place when there occur specific combinations of them. This motivates the definition of the combined block

$$\begin{aligned} \{x, y_n\}_\theta &= \prod_{l=0}^{n-1} \{x, y+2l\}_\theta \\ &= \frac{\langle x-1, y-1\rangle_\theta \, \langle x+1, y-1+2n\rangle_\theta}{\langle x+1, y-1\rangle_\theta \, \langle x-1, y-1+2n\rangle_\theta} \times (\theta \to -\theta)^{-1} \ , \end{aligned} \tag{3.73}$$

together with the angles

$$\theta^{\pm}_{x,y,n} = (x \pm 1)\theta_h + (2n + y - 1)\theta_H \ . \tag{3.74}$$

One sees easily that there are four zeros of $\{x, y_n\}_\theta$ located at $\pm\theta^{\pm}_{x,y,0}, \mp\theta^{\pm}_{x,y,n}$ and four simple poles at $\pm\theta^{\pm}_{x,y,n}, \pm\theta^{\mp}_{x,y,0}$, respectively. Blocks of the above type can be seen to occur gradually in the S-matrix and the obvious advantage of the definition (3.73) is that it takes the cancellation of several hyperbolic functions in the product into account. Note that we recover the basic building blocks (3.56) for $n = 1$.

Since only the poles in the physical strip are of interest, one needs a first criterion to restrict the imaginary part of the angles (3.74) to the range $0 \leq \operatorname{Im}\theta \leq \pi$. From the second equation in (3.55) we infer that the block entries are limited to the values $0 < x < h$, $0 < y < H$. In addition, the effective coupling lies in the interval $0 \leq B \leq 2$ whence one derives the condition

$$0 \leq \operatorname{Im}(\theta^{\pm}_{x,y,n}) \leq \pi \qquad \text{for } B \leq \frac{2H(h - x \mp 1)}{|h(2n + y - 1) - H(x \pm 1)|} \ . \tag{3.75}$$

In order to check the generalized bootstrap principle one needs to evaluate the corresponding residues [37],

$$\operatorname*{Res}_{\theta=\theta^-_{x,y,0}} \{x,y_n\} = \frac{-2\sinh\theta_h \sinh(n\theta_H)\sinh(x\theta_h+(n+y-1)\theta_H)\sinh\left(\theta^-_{x,y,0}\right)}{\sinh(\theta_h+n\theta_H)\sinh(x\theta_h+(y-1)\theta_H)\sinh((x-1)\theta_h+(y+n-1)\theta_H)} \tag{3.76}$$

$$\operatorname*{Res}_{\theta=\theta^+_{x,y,n}} \{x,y_n\} = \frac{2\sinh\theta_h \sinh(n\theta_H)\sinh(x\theta_h+(n-1+y)\theta_H)\sinh\left(\theta^+_{x,y,n}\right)}{\sinh(\theta_h+n\theta_H)\sinh((1+x)\theta_h+(n+y-1)\theta_H)\sinh(x\theta_h+(2n+y-1)\theta_H)} \ . \tag{3.77}$$

For the stated range of x, y, B, n together with the restriction (3.75) it is now tedious but straightforward to check that

$$\operatorname{Im}\left(\operatorname*{Res}_{\theta=\theta^-_{x,y,0}} \{x,y_n\}_\theta\right) < 0 \qquad \text{and} \qquad \operatorname{Im}\left(\operatorname*{Res}_{\theta=\theta^+_{x,y,n}} \{x,y_n\}_\theta\right) > 0 \ , \tag{3.78}$$

indicating that $\theta^+_{x,y,n}$ is a possible candidate for a direct channel pole obeying the generalized bootstrap prescription. However, there are additional contributions from other blocks which might change their sign when the coupling is tuned from the weak to the strong limit. (Note that this is in contrast to the simply-laced case where the above knowledge is sufficient to determine the sign of the total residue of the whole S-matrix, e.g. [29]). Thus, one has also to investigate the behaviour of a second independent block at the particular pole $\theta^+_{x,y,n}$.

Take this block to be $\{x', y'_{n'}\}_{\theta^+_{x,y,n}}$ then a clear indication for a sign change are different signs of the values in the extreme weak and extreme strong coupling regime. In general, $\lim_{\beta\to 0,\infty} \{x', y'_{n'}\}_{\theta^+_{x,y,n}} = 1$ with the exception when $x' = x$, where one has

$$\lim_{\beta\to 0} \{x, y'_{n'}\}_{\theta^\pm_{x,y,n}} = \left(\frac{y'-y-2n}{y'-y+2n'-2n}\right)^{\pm 1} \tag{3.79}$$

$$\lim_{\beta\to\infty} \{x, y'_{n'}\}_{\theta^\pm_{x,y,n}} = 1 \ . \tag{3.80}$$

Provided that the right hand side of (3.79) is negative, the imaginary parts of the possible additional blocks

$$\{x, y'_{n'}\}_{\theta^+_{x,y,n}} \qquad \text{and} \qquad \{x+2, y''_{n''}\}_{\theta^+_{x,y,n}} \tag{3.81}$$

both change their sign while β runs from zero to infinity. Therefore the pole $\theta^+_{x,y,n}$ is excluded from the bootstrap, whenever the scattering matrix contains in addition the blocks (3.81) to an odd power and these do not cross the real axis w.r.t. β at the same point. Given the S-matrix explicitly in block form (3.55) the condition on y, y', n, n' by which the l.h.s. of (3.79) becomes negative, together with the occurrence of blocks like (3.81) yields a simple criterion [37] which proves to be sufficient for practical purposes as can be checked at the examples listed in the appendix. Nevertheless, a direct Lie algebraic formula is desirable.

3.2.9 The S-matrix in integral form

After having established that the two-particle scattering amplitude possesses all the desired bootstrap properties one might look for alternative expressions of it. Although the block form (3.55) is most suitable for exhibiting the bootstrap properties in a simple manner as just demonstrated, it is convenient to rewrite the S-matrix in a different variant in regard of several applications, e.g. the thermodynamic Bethe ansatz (see Chapter 4) or the form factor program [12]. Explicitly, we are looking for an expression of the type

$$S_{ij}(\theta) = \exp \int_0^\infty \frac{dt}{t} \, \phi_{ij}(t) \sinh\left(\frac{\theta t}{i\pi}\right) , \tag{3.82}$$

where the matrix valued integral kernel ϕ has to be specified. In fact, as will be proven below for ATFT it is determined by

$$\phi\,(t) = 8 \sinh(\vartheta_h t) \sinh(t_j \vartheta_H t) A(e^{t\vartheta_h}, e^{t\vartheta_H})^{-1} \; . \tag{3.83}$$

Notice that only the inverse q-deformed Cartan matrix (2.67) and the length of the roots (2.55) enter the expression. The deformation parameters have been chosen to be $q = \exp(t\vartheta_h)$ and $\hat{q} = \exp(t\vartheta_H)$, where the angles ϑ_h, ϑ_H are the same as defined in (3.41). That the S-matrix can be cast into this neat and universal form with a minimum of Lie algebraic information was first noticed in [36] on a case-by-case basis. We now turn to the proof of this formula [37] which will be an immediate consequence of the Lie algebraic structure developed in Chapter 2 and the preceding sections.

Rewriting each of the building blocks occurring in (3.55) in the integral form (3.58) the product in (3.55) is transformed into a sum in the exponent. Then the natural question arises whether this sum may be performed to give a more compact expression. Specifying the deformation parameters $q, \hat{q}$ as stated above one infers the following identity by means of (3.64),

$$\begin{aligned} \phi(t) &= \frac{1}{\sinh t} \sum_{x=1}^{h} \sum_{y=1}^{H} 2\mu(x,y) \, f_{x,y}^{h,H}(t,B) \; , \\ &= -\frac{8 \sinh(\vartheta_h t) \sinh(\vartheta_H t)}{\sinh t} e^{-t} M(e^{t\vartheta_h}, e^{t\vartheta_H}) \end{aligned} \tag{3.84}$$

upon noting that $q^{2h}\hat{q}^{2H} = e^{2t}$. Applying now the matrix identity (2.66) and using some elementary identity for hyperbolic functions gives the integral representation. Notice that this easy derivation builds once more on the identification of the link between q-deformed Coxeter geometry and the q-deformed Cartan matrix. In [37] also an alternative derivation in terms of contour integrals in the complex rapidity plane can be found. For simply-laced algebras see also [43].

To conclude it should be emphasized that the q-deformed Cartan matrix now appears clearly to be the central object in the theory, since one can formulate in terms of it not only the quantum mass spectrum (3.44) and the fusing rules (3.45) but also the S-matrix. Each of these informations can be written down in a single formula for *all* affine Toda models, whose behaviour differs greatly depending on the algebras in question. This level of universality is rarely achieved for any other class of field theories except for conformal WZNW theories, which have similar underlying

Lie algebraic structures. *As was claimed before, such sophisticated mathematical structures do not only neatly reorganize known structures in a general setting, but they can also be extrapolated to find entirely new integrable models as is demonstrated in the next section.* Before coming to this point we extract from the block form (3.55) and the integral formula (3.82) the known universal S-matrix of simply-laced ATFT [29] as final consistency check. At the same time this will prepare the subsequent definition of colour valued S-matrices.

3.2.10 Reduction to simply-laced affine Toda

To make the difference between the two classes of affine Toda models of simply-laced and non simply-laced Lie algebras also explicit in terms of the S-matrix, the general formulas (3.55) and (3.82) are now evaluated for the ADE series. In particular, it will turn out that the simple renormalization properties (3.30) of the latter ATFT allow to separate the two-particle scattering amplitude in two factors, one containing the bound state poles and another one displaying the coupling dependence. This feature has consequences beyond ATFT, since it might be used to relate the techniques developed for constructing the ATFT S-matrix to other integrable models of similar type.

For the following we choose $\mathfrak{g} \equiv ADE$, which implies as mentioned before that both dual algebras coincide, $\mathfrak{g} = \mathfrak{g}^\vee$, as the associated root system of $\mathfrak{g}$ is self-dual under the exchange of roots and coroots. This has a number of consequences. First of all for simply-laced algebras all the roots are of the same length, whence the symmetrizer (2.55) is trivial $t_i = 1$ and the incidence matrix $I = 2 - A$ has only entries zero or one. Thus, the q-deformation of the simple Weyl reflections (2.52) vanishes and the action of the q-deformed Coxeter element (2.58) simplifies to

$$ADE: \quad \sigma_q = q^2 \sigma \ ,$$

where σ is just the ordinary non-deformed Coxeter transformation (2.37). To recover the universal formula for the ADE S-matrix [29] we express the power function μ in the variant (3.68) and find for this special case

$$ADE: \quad \sum_y \mu_{ij}\left(2x - \frac{c_i + c_j}{2}, y\right) q^y = -\frac{1}{2} q^{2x - \frac{c_i + c_j}{2}} \langle \lambda_j, \sigma^x \gamma_i \rangle \ . \qquad (3.85)$$

Hence $y = 2x - \frac{c_i + c_j}{2}$ and the first and second entry of the building blocks in (3.55) are the same, i.e. only blocks of the type $\{x, x\}$ occur*. The block form (3.55) of the scattering matrix then reduces to

$$ADE: \quad S_{ij}(\theta) = \prod_{x=1}^{h} \left\{ 2x - \frac{c_i + c_j}{2}, 2x - \frac{c_i + c_j}{2} \right\}_\theta^{-\frac{1}{2}\langle \lambda_j, \sigma^x \gamma_i \rangle} , \qquad (3.86)$$

which coincides with the expression found by Fring and Olive [29] upon noting that for the self-dual case the two relevant Coxeter numbers are the same, $h = H = \ell h^\vee$. In fact, since both dual algebras coincide the twist of $\mathfrak{g}^\vee$ is trivial, $\omega = 1, \ell = 1$,

*The block $\{x, x\}_\theta$ corresponds to the block $\{x\}_\theta$ as defined in [29] or [43].

meaning that all orbits have length $\ell_i = 1$ and expression (3.86) follows by analogous steps.

Reorganizing the building blocks (3.56) in the ADE S-matrix in the following manner

$$\{x,x\} = \frac{\langle x+1,x+1\rangle_\theta \langle x-1,x-1\rangle_\theta}{\langle x+1,x+1\rangle_{-\theta} \langle x-1,x-1\rangle_{-\theta}} \times \frac{\langle x+1,x-1\rangle_{-\theta} \langle x-1,x+1\rangle_{-\theta}}{\langle x+1,x-1\rangle_\theta \langle x-1,x+1\rangle_\theta} \tag{3.87}$$

one realizes that (3.86) can be decomposed into a so-called "minimal" and a coupling dependent part [27],

$$S_{ij}(\theta) = S_{ij}^{\min}(\theta) F_{ij}(\theta, B) \,, \tag{3.88}$$

where $S_{ij}^{\min}$ contains only the first and $F_{ij}(\theta, B)$ the second factor of the building block as decomposed in (3.87). Observing that for simply-laced algebras the definition of the building blocks (3.56) is simplified by the relations $\vartheta_h + \vartheta_H = h^{-1}$ and $\vartheta_h - \vartheta_H = (1-B)/h$ one immediately verifies that the coupling dependence in the first factor drops out giving rise to the factorization (3.88). Moreover, by the same techniques as before one verifies that each factor obeys the bootstrap equations separately. However, from the defining relation (3.56) and (3.87) one immediately infers that the poles of the coupling dependent factor lie outside of the physical strip. Thus, the bound state poles and therefore the associated fusing processes are incorporated exclusively in the minimal part. With regard to the general remarks at the beginning of this chapter, F_{ij} is therefore a particular example for a CDD factor, which can be multiplied to the minimal solution $S_{ij}^{\min}$ without violating the bootstrap properties.

It needs to be stressed, that this splitting property of the S-matrix is a characteristic feature of the ADE series and is not shared by the non simply-laced affine Toda models, as was first pointed out by Delius et al. [31].

For the sake of completeness and later purposes we now also discuss the separation of the ADE S-matrix (3.86) in the integral representation. We already saw in context of the conserved charges that the q-deformed Cartan matrix simplifies for ADE algebras, $A(q,\hat{q}) = q\hat{q} + q^{-1}\hat{q}^{-1} - I$ (compare Section 3.2.4). Thus, the universal integral kernel (3.83) specializes to

$$\begin{aligned} \phi(t) &= 8 \sinh \tfrac{Bt}{2h} \sinh \tfrac{(2-B)t}{2h} \left(2\cosh \tfrac{t}{h} - I\right)^{-1} \\ &= 4\left(\cosh \tfrac{t}{h} - \cosh \tfrac{t(1-B)}{h}\right)\left(2\cosh \tfrac{t}{h} - I\right)^{-1} \,. \end{aligned} \tag{3.89}$$

Here the equality of the Coxeter number h, H and the explicit definition of the angles (3.41) as well as a simple trigonometric identity has been used. Thus, the integral kernel ϕ can be written as the sum of two terms implying via (3.82) the factorization (3.88). Explicitly the expressions read

$$S_{ij}^{\min}(\theta) = \exp \int_0^\infty \frac{dt}{t}\, 4\cosh\frac{t}{h}\left(2\cosh\frac{t}{h} - I\right)^{-1}_{ij} \sinh\frac{t\theta}{i\pi} \tag{3.90}$$

and

$$F_{ij}(\theta, B) = \exp - \int_0^\infty \frac{dt}{t}\, 4\cosh\frac{t(1-B)}{h}\left(2\cosh\frac{t}{h} - I\right)^{-1}_{ij} \sinh\frac{t\theta}{i\pi} \,. \tag{3.91}$$

From the integral representation of the CDD factor several properties concerning the coupling dependence of the ADE S-matrix are immediate to verify. First of all, in the weak coupling limit one has that

$$F_{ij}(\theta, B=0) = S_{ij}^{\min}(\theta)^{-1} \ .$$

The last property is to be expected on physical grounds, since the particles should not interact, $S_{ij} = 1$, when $\beta \to 0$. The second property which can be easily deduced from (3.91) is the strong-weak self-duality in the coupling constant which is mirrored by the self-duality in terms of the root system, compare (3.32). Since the CDD factor F_{ij} contains the whole coupling dependence, it ought to obey the relation

$$F_{ij}(\theta, B) = F_{ij}(\theta, 2-B) \ , \quad B_{ADE} = \frac{2\beta^2}{4\pi + \beta^2}$$

implying the invariance of the S-matrix under the mapping $\beta \to 4\pi/\beta$. Notice that the effective coupling constant B is the one defined in (3.42) for the special case of simply-laced algebras and that the strong-weak self-duality then amounts to the invariance under the transformation $B \to 2-B$. This self-duality is obvious from (3.91) and we conclude that the S-matrix (3.86) derived from the general expression is in accordance with the known behaviour of simply-laced ATFT.

3.3 Colour valued S-matrices

In this section a general construction principle [38] is proposed to equip coupling dependent scattering matrices of integrable models with colour degrees of freedom. This means that for a given mass spectrum of an integrable model an additional quantum number is assigned to the associated particles and in this manner the original model is "multiplied". At the heart of this method lies the splitting of the original S-matrix into a minimal and coupling dependent part as we just encountered in the case of simply-laced ATFT. It will allow to generate in an easy manner new S-matrices with underlying Lie algebraic structure and which have the novel feature to violate parity invariance. To emphasize the usefulness of the presented construction method the class of generated S-matrices will be shown for special choices of the colour structure to contain examples already known in the literature.

To define new integrable models via writing down consistent S-matrices might at first seem a bit abstract, since in general the associated classical Lagrangian to these quantum field theories is not known. However, it is in full accordance with the spirit of the S-matrix theory and the bootstrap approach which declare the S-matrix as the central and determining object of each quantum field theory. The latter is supposed to be more fundamental than any classical description. Nevertheless, valuable insight is gained when determining the corresponding classical field theories. A first step towards this direction is postponed to the next chapter, where the new S-matrices will be linked to perturbed conformal field theories providing additional motivation for their definition.

3.3.1 Construction principle

Suppose that we are given an integrable model whose two-particle S-matrix obeys a separation property analogous to the one of simply-laced ATFT (3.88), i.e. it separates into one minimal factor containing the bound state poles and one of CDD type, for instance F_{ij} in equation (3.88), displaying the coupling dependence only, $S_{ij}(\theta) = S_{ij}^{\min}(\theta) S_{ij}^{\mathrm{CDD}}(\theta, B)$. Then one possible way to introduce a "colour" dependence in the S-matrix is given by choosing the coupling to be colour dependent, i.e. one performs the replacement [38]

$$S_{ij}(\theta) = S_{ij}^{\min}(\theta) S_{ij}^{\mathrm{CDD}}(\theta, B) \rightarrow \hat{S}_{ij}^{ab}(\theta) = S_{ij}^{\min}(\theta) S_{ij}^{\mathrm{CDD}}(\theta, B_{ab}) \ . \tag{3.92}$$

Here the indices i, j refer to the (original) particle masses and the indices a, b to the colours, i.e. each particle carries now two quantum numbers (i, a). The corresponding ranges may be chosen differently, $1 \leq i \leq n$ and $1 \leq a \leq \tilde{n}$, and consequently one obtains a new mass spectrum of $n \times \tilde{n}$ different particle types. Clearly, the bootstrap properties of the original S-matrix are not changed by this prescription, whence the relevant functional equations for the S-matrix are satisfied. In particular, the original fusing structure for particles of the *same* colour is preserved and particles of *different* colours only interact at different values of the effective coupling. This motivates to refer to the second type of indices as colour degrees of freedom. Alternatively, one might take the splitting of the S-matrix more seriously and even separate both factors to define an S-matrix of the type [38]

$$S_{ij}^{ab}(\theta) = \begin{cases} S_{ij}^{\min}(\theta) = (S_{ij}^{\mathrm{CDD}}(\theta, B_{aa} = 0))^{-1} & \text{for } i = j \\ S_{ij}^{\mathrm{CDD}}(\theta, B_{ab}) & \text{for } i \neq j \end{cases} \ . \tag{3.93}$$

This means whenever $a = b$ we simply have $\tilde{n}$ copies of theories which interact via a minimal scattering matrix and for $a \neq b$ the particles interact purely via a CDD-factor. Clearly by construction also (3.93) satisfies the bootstrap equations. It should be noted here that (3.92) and (3.93) still describe scattering processes for which backscattering is absent. Hence, these type of colour values play a different role as those which occur for instance in S-matrices related to affine Toda field theories [24] with purely imaginary coupling constant, e.g. [81]. Despite the fact that the relative mass spectra related to (3.93) are degenerate, this is consistent when we encounter $\tilde{n}$ different overall mass scales dependent on the colour or the particles have different charges. To provide a concrete realization for $S_{ij}^{ab}(\theta)$, which is of affine Toda field theory type, the concrete separation (3.88) for ATFT will now be exploited and main and colour quantum numbers be linked to a pair of simply-laced Lie algebras. It is, however, clear from the previous comments that the forms (3.92) and (3.93) are of a more general nature.

3.3.2 The $\mathfrak{g}|\tilde{\mathfrak{g}}$ *S-matrix in integral form*

Associate the main quantum numbers i, j as in the context of ATFT to the vertices of the Dynkin diagram of a simply-laced Lie algebra $\mathfrak{g}$ of rank n and the colour quantum numbers a, b to the vertices of the Dynkin diagram of a simply-laced Lie algebra $\tilde{\mathfrak{g}}$ of rank $\tilde{n}$. To each fixed colour quantum number a let there be a tower of particles whose mass ratios are the same as in the corresponding ATFT connected with $\mathfrak{g}$, i.e. $m_i^a/m_j^a := (m_i/m_j)_{\text{ATFT},\mathfrak{g}}$. To define the interaction between the particles we use now slightly modified version of the second scheme (3.93) outlined above. Let A and $\tilde{A}$ denote the Cartan matrices associated with $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$, then the S-matrix in its integral form is chosen to be [38]

$$S_{ij}^{ab}(\theta) = e^{i\pi\varepsilon_{ab}A_{ij}^{-1}} \exp \int_0^\infty \frac{dt}{t}\, 2\left(2\cosh\frac{t}{h} - \tilde{I}\right)_{ab} \left(2\cosh\frac{t}{h} - I\right)_{ij}^{-1} \sinh\frac{t\theta}{i\pi} \,. \tag{3.94}$$

Here $I = 2 - A, \tilde{I} = 2 - \tilde{A}$ are the corresponding incidence matrices, h is the Coxeter number of $\mathfrak{g}$ and ε_{ab} is the antisymmetric tensor, i.e. $\varepsilon_{ab} = -\varepsilon_{ba}$. Notice that due to the appearance of the latter the above expressions break parity invariance, $S_{ij}^{ab}(\theta) \neq S_{ji}^{ba}(\theta)$. Some comments are due to clarify how this definition fits into the general prescription (3.93). Choosing the colour quantum numbers to be equal we obtain in accordance with (3.93) the minimal ATFT S-matrix (3.90) associated with the first Lie algebra $\mathfrak{g}$, since the diagonal elements in the incidence matrix $\tilde{I} = 2 - \tilde{A}$ are zero,

$$S_{ij}^{aa}(\theta) = S_{ij}^{\min}(\theta) \,.$$

If on the other hand the colours are different, there are now two cases to distinguish. Whenever the vertices a and b are not linked on the $\tilde{\mathfrak{g}}$–Dynkin diagram the S-matrix becomes trivial, i.e. $S_{ij}^{ab} = 1$, and the particles do not interact. In contrast, when a and b are linked on the $\tilde{\mathfrak{g}}$–Dynkin diagram, we have $\tilde{I}_{ab} = 1$ from which in comparison with (3.91) it follows that

$$\tilde{I}_{ab} = 1 : \quad S_{ij}^{ab}(\theta) = e^{i\pi\varepsilon_{ab}A_{ij}^{-1}} F_{ij}(\theta, B=1)^{\frac{1}{2}} \,.$$

Analogously to (3.93) the interaction between particles of different colours is thus defined via the special CDD factor (3.91) with the slight change that the square root at effective coupling $B = 1$ is taken first. This, however, changes the bootstrap properties and explains the occurrence of the ominous looking phase factor as will be demonstrated in the next subsection when discussing (3.94) in its block form of meromorphic functions.

At the moment the neat Lie algebraic structure of (3.94) shall be sufficient as motivation for its definition. In the next chapter we will then see in retrospect that this particular combination of Lie algebraic structures can be traced back to WZNW models in the high energy limit.

Henceforth, the quantum field theories described by (3.94) are referred to as $\mathfrak{g}|\tilde{\mathfrak{g}}$ models[†] [38]. It should be clear from the discussion that this pairing of Lie algebras in the present context is conceptually not related to the pairing of Lie algebras encountered in the previous section. Furthermore, these new theories only involve simply-laced algebras.

[†]The notation should of course not be understood as a coset.

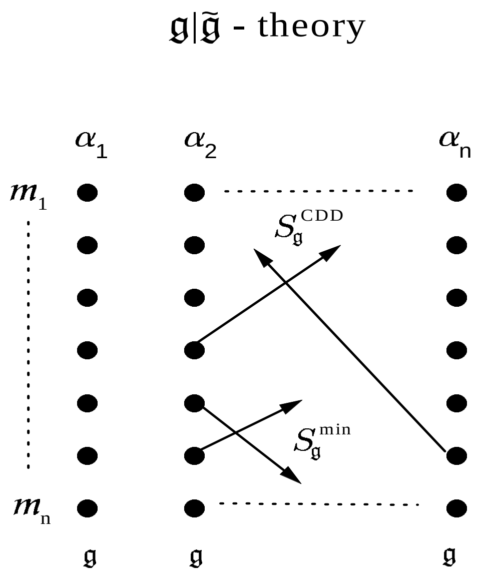

Figure 3.6: The structure of $\mathfrak{g}|\tilde{\mathfrak{g}}$-theories: To each simple root of α_a of $\tilde{\mathfrak{g}}$ there is an tower of $n = \text{rank}\,\mathfrak{g}$ particles whose mass ratios are identical to those of ATFT. Particles inside the same tower scatter via the minimal ATFT S-matrix associated with $\mathfrak{g}$, while particles in different towers interact via a CDD-factor.

3.3.3 *The $\mathfrak{g}|\tilde{\mathfrak{g}}$ S-matrix in block form*

In order to trace down the origin of the phase factor and for analyzing the pole structure it is convenient to rewrite (3.94) in blocks of meromorphic functions. Looking at the block expression for the CDD factor (3.91) at $B = 1$ one observes that $F_{ij}(\theta, B = 1)^{\frac{1}{2}}$ is built up in terms of meromorphic functions of the type

$$F_{ij}(\theta, B=1)^{\frac{1}{2}} = \prod_x \sqrt{\frac{\langle x+1, x-1\rangle_{-\theta} \langle x-1, x+1\rangle_{-\theta}}{\langle x+1, x-1\rangle_{\theta} \langle x-1, x+1\rangle_{\theta}}} = \prod_x \frac{\sinh\frac{1}{2}(\theta - i\frac{\pi}{h}x)}{\sinh\frac{1}{2}(\theta + i\frac{\pi}{h}x)} \tag{3.95}$$

from which one immediately deduces that taking the square root does not affect the desired analytic properties and minimizes the power of the poles in the unphysical sheet. This shows the operation of taking the square root in a more natural light. However, the behaviour of the CDD factor under a crossing transformation is changed by the possible appearance of a minus sign. This motivates the following definition of the new building blocks [38]

$$[x, B]_{\theta,ab} = e^{\frac{i\pi x}{h}\varepsilon_{ab}} \left(\frac{\sinh\frac{1}{2}(\theta + i\pi\frac{x-1+B}{h})\sinh\frac{1}{2}(\theta + i\pi\frac{x+1-B}{h})}{\sinh\frac{1}{2}(\theta - i\pi\frac{x-1+B}{h})\sinh\frac{1}{2}(\theta - i\pi\frac{x+1-B}{h})} \right)^{\frac{1}{2}} . \tag{3.96}$$

This block has the obvious properties

$$[x, B]_{\theta,ab}\,[x, B]_{-\theta,ba} = 1 \quad \text{and} \quad [h-x, B=1]_{\theta,ab} = [x, B=1]_{i\pi-\theta,ba} \ .$$

In a slightly loose notation it is understood that in the second equality one first takes the square root and thereafter performs the shifts in the arguments. Note further that the order of the colour values is relevant for crossing symmetry. The appearance of a minus sign in the original building blocks is now interpreted in terms of parity violation via a colour dependent phase factor. From (3.96) we can now construct the block form of the $\mathfrak{g}|\tilde{\mathfrak{g}}$–scattering matrix [38] by exploiting the equivalence between the analogous expressions of simply-laced ATFT in 3.2.10,

$$S_{ij}^{ab}(\theta) = \prod_{x=1}^{h} \left[2x - \tfrac{c_i + c_j}{2}, \tilde{I}_{ab} \right]_{\theta,ab}^{-\frac{\tilde{A}_{ab}}{2} \langle \lambda_j, \sigma^x \gamma_i \rangle} \quad . \tag{3.97}$$

Here the λ_i's are again the fundamental weights, the γ_i's are simple roots times a colour value $c_i = \pm 1$, h is the Coxeter number and σ is the Coxeter element related to the Lie algebra $\mathfrak{g}$. The remaining step to establish the equivalence between (3.97) and (3.94) is to prove that the parity breaking phase factor in (3.96) combines to the factor in front of the integral representation, i.e.

$$e^{i\pi A_{ij}^{-1}} = \prod_{x=1}^{h} \left[e^{i\frac{\pi}{h}\left(2x - \frac{c_i + c_j}{2}\right)} \right]^{-\frac{1}{2}\langle \lambda_j, \sigma^x \gamma_i \rangle} \quad .$$

However, this is immediate from the Lie algebraic identity (2.46) of Chapter 2.

Remark. *At first sight the power* $1/2$ *in the definition of the building block (3.96) seems to suggest the presence of square root branch cuts in the S-matrix (3.97). A careful analysis of the cases* $a = b$ *and* $a \neq b$ *shows, however, that one recovers the minimal S-matrix and the CDD factor in (3.88), respectively. Both are meromorphic functions, in particular for* $a = b$ *one has* $\tilde{A}_{ab} = 2$ *and for* $a \neq b$, $B = \tilde{I}_{ab} = 1$ *the square root can be taken directly in (3.96). The remaining power* $1/2$ *stemming from the exponent in (3.97) is compensated by the same mechanism as encountered in the context of ATFT, see Section 3.2.7, i.e. the combination of various equal blocks.*

It should be emphasized that there is no need to introduce the phase to satisfy the unitarity equation (3.10). It is further clear that (3.97) is Hermitian analytic. However, as already mentioned above the introduction of the phase factor is crucial in order to satisfy the crossing relation. *Whence the violation of parity invariance is a direct consequence of the functional equation (3.12) in the bootstrap approach.* Assuming the validity of the ADE-fusing rules (3.26) one may now verify by the same shifting arguments as in the previous section that the fusing bootstrap equations (3.17) are satisfied. This establishes (3.97) or equivalently (3.94) to be a consistent S-matrix and hence implicitly defines a new class of integrable quantum field theories.

Relation to known models

For special choices of the algebras one recovers from the general class of $\mathfrak{g}|\tilde{\mathfrak{g}}$-models two subclasses of already known S-matrices. Choose $\tilde{\mathfrak{g}}$ to be A_1 then the colour structure is removed (since there is only one possible colour value) and the system reduces to the one described by $S_{ij}^{\min}(\theta)$. The first examples of these S-matrices were constructed in [18] for $\mathfrak{g} = A_{n,} E_8$ and describe so-called scaling models mentioned in the introduction. This class is at the same time the only example for which (3.97), (3.94) do not violate parity invariance. Choosing instead $\mathfrak{g}$ to be A_n and $\tilde{\mathfrak{g}} = ADE$ we recover the S-matrices of the Homogeneous Sine-Gordon models for vanishing resonance parameter at level $(n+1)$ [40, 56]. The latter S-matrices were just recently formulated in context of massive perturbations of WZNW models. A detailed discussion of their properties and the possibility to include resonance poles in their definition will be postponed to the next chapter, where the high energy behaviour of the integrable models of this chapter is discussed.