Chapter 1
INTRODUCTION

One cannot accomplish things simply with cleverness. One must take a broad view. It will not do to make rash judgements concerning good or evil. However, one should not be sluggish. It is said that one is not truly a samurai if he does not make his decisions quickly and break right through to completion.

From 'The Book of the Samurai, Hagakure'

1.1 Perspective and motivation

Since the first steps towards a relativistic wave mechanics in the early 1930s up to the "gauge revolution" in the 1970s and 1980s leading to the formulation of the Standard Model, quantum field theory (QFT) has developed to the most successful physical theory. Its computational power in comparison with experimental data as well as its conceptual scope in explaining the subatomic world are met by no other model in theoretical physics. Yet despite this undeniable success there remains a discrepancy between the glorious predictive power on the one hand and the scarce knowledge about the underlying intricate mathematical structure on the other hand. Indeed, perturbative expansions together with a thicket of detailed renormalization prescriptions are in most cases the only possible access to the theory failing to reveal its true nature.

The search for a deeper understanding of the basic physical principles underlying QFT and for concise mathematical formulations took its main starting point in the 1950s after the observation that even the applicability of perturbation theory breaks down in the range of strong-interactions. The problem of quark confinement is one of the subtle up-to-day unsolved problems, which impressively demonstrates that non-perturbative aspects can become dominant and are inevitable in order to understand the theory to its depth. This initiated the search for alternative and entirely new techniques going beyond the conventional approach of calculating numerous Feynman diagrams. Clearly, the exact solution of *any* non-trivial QFT would yield profound insight and valuable help in this task, irrespective whether the model under consideration is linked directly to a concrete physical problem or not. Solving a QFT exactly is understood as the explicit calculation of all its n-point or correlation functions, since from the latter the field content and the physical state space can be recovered by means of **Wightman's reconstruction theorem** [1, 2]. It will become important below that this holds true for Minkowski as well as Euclidean space. In fact, the correlation functions in Minkowski space can be recovered from those in Euclidean space as was worked out by Osterwalder and Schrader [3]. Especially important is this connection in the context of **constructive field theory** which by use of functional methods achieved the mathematical exact construction of a number of field theories in two and three dimensions, see [4] and references therein. However, explicit and closed expressions for the relevant physical quantities are missing in this approach. It remains the ambitious aim to obtain these in order to check general assumptions about the structure of QFT for their consistency, prove certain non-perturbative approaches workable, test new concepts and learn about concrete physical problems.

Coleman-Mandula theorem

Naturally, the most likely candidates of QFT's for which the ambitious aim of an exact solution seems conceivable are those with powerful symmetries giving rise to a large number of conservation laws. The latter might ease the actual computations or even impose such severe constraints that exact solutions can be constructed. Unfortunately, in 3+1 dimensions* any progress in this direction is blocked by the **Coleman-Mandula theorem** [5], which states that additional symmetries besides Poincaré invariance and an internal gauge group describing the degeneracy of the particle spectrum render the scattering matrix of each massive QFT to be trivial, i.e. it describes non-interacting quantum particles. With only theses symmetries present it has not been achieved so far to overcome the conceptual difficulties which one encounters when looking for exact expressions.

1+1 dimensional integrable field theories

The situation greatly improves in 1+1 dimensional systems. Here the blockade of the Coleman-Mandula theorem is lifted due to the fact that one of the crucial assumptions in its proof ceases to hold true, the scattering amplitude does not any longer depend analytically on the scattering angle. (In one space dimension only forward and backward scattering are possible, whence the scattering angle can only assume the discrete values 0 or π.) One might therefore look for theories with higher conservation laws, in fact with infinitely many of them, in which case these theories are called **integrable**. The latter term originates in classical mechanics, where a system is said to be integrable if there are as many conserved quantities as degrees of freedom allowing to solve the equations of motion by integration. In case of a field theory *infinitely* many degrees of freedom are present, since the field configuration at each point in space-time has to be specified. In a loose sense one then refers to a field theory as integrable if it gives rise to an *infinite* set of conservation laws.

At first sight the specialization to 1+1 dimensions and a class of field theories with infinitely many conserved charges might appear quite restrictive. However, the study of low-dimensional QFT has turned out to reproduce many features which are of interest in higher dimensions, such as confinement mentioned earlier, duality or gauge

*We follow here the standard convention in splitting the number of dimensions into a sum with the first summand refering to the space and the second to the time dimension.

anomalies, see e.g. [6] and references therein for further details. Furthermore, integrable models have found direct physical applications in the off-critical description of statistical mechanics and condensed matter systems reduced to two space-dimensions as will be outlined below. In this context integrability appears naturally as a relict of broken conformal symmetry.

Exact scattering matrices

The presence of an infinite set of conservation laws imposes severe restrictions on the dynamics. In particular, it enforces that the particle number as well as the individual particle momenta are asymptotically conserved in a scattering process. Moreover, each scattering process can be decomposed into two-particle ones, reducing the task to determine the full scattering matrix (also referred to as S-matrix) to the calculation of the two-particle amplitude. These powerful constraints together with the idea of **minimal analyticity** originating in the **S-matrix theory** of the sixties [7, 8, 9] allow to construct fully *exact scattering amplitudes.* The basic idea is to regard the scattering amplitudes as boundary values of analytic functions in the complex plane. Setting up a set of functional relations reflecting general requirements like unitarity or crossing symmetry the general form of a two-particle scattering amplitude can already be written down without relying on a classical Lagrangian of the field theory. The actual dependence on the integrable model at hand is then invoked by the famous **Yang-Baxter** [10] or the **bootstrap equation**. The former is tightly linked to the integrability property and describes equivalent ways to factorize a scattering matrix in two-particle ones, while the latter reflects the principle of **nuclear democracy** which states that each bound state in the theory should also appear in the asymptotic particle spectrum. This method for constructing exact scattering matrices was pioneered in the articles [11] and has become known as the **bootstrap approach** in the literature. It has proven extremely successful over the years. In particular, it lead to the implicit definition of entirely new integrable models by writing down exact scattering matrices which satisfy all physical requirements. In a sense this realizes partially the ambitious program of the S-matrix theory of the sixties in low dimensions.

The bootstrap construction of scattering matrices will play a central role in this thesis. Its powerful structure will be in particular exploited when constructing a class of hitherto unknown scattering matrices giving rise to new integrable quantum field theories.

Off-shell investigations and form factors

The exact construction of the scattering matrix is not only of interest in order to obtain a complete knowledge about the on-shell structure of a QFT, but it also serves as a preliminary step towards the calculation of the measurable quantities of the system, the n-point or correlation functions. Calculating the latter is the ultimate goal of each theory, since as mentioned above it amounts to solving the whole model completely via the reconstruction theorem. One of the most promising approaches in this context is the **form factor program** [12]. Writing down the correlation function of a local operator $\mathcal{O}$ one inserts a complete set of states, usually depending on the momentum, giving in a simplifying notation

$$\langle 0|\mathcal{O}(x)\mathcal{O}(0)|0\rangle = \sum_n \int d^n p \mid \langle 0|\mathcal{O}(0)|\mathbf{p}\rangle \mid^2 e^{i\sum_{k=1}^n p_k \cdot x} \,.$$

Here the sum runs over all possible particle numbers n, the vector $\mathbf{p} = (p_1, ..., p_n)$ consists of the individual particle momenta and the matrix elements $\langle 0|\mathcal{O}(0)|\mathbf{p}\rangle$ are referred to as form factors (actually a slightly modified version of them). Proceeding conceptually very similar as in the case of the scattering matrix one then continues analytically the form factors in the momentum variables. The purpose is to set up recursive functional equations, which require the exact scattering matrix as prerequisite input. In principle these equations enable one to derive all n-particle form factors and to calculate the correlation functions by evaluating the above infinite sum of integrals. Being a highly non-trivial step this remains an open challenge for almost all theories except the Ising model. Also the calculation of the complete set of form factors has so far only been achieved in a few cases, e.g. [13, 14].

Even though the final and complete construction of correlation functions in form of explicit analytic expressions is yet outstanding the form factor program gives reasonable hope that this might be achieved in the near future. Already now one might exploit the fact that the sum over the particle number n is rapidly convergent. Hence for many practical purposes it is sufficient to determine only the first few particle form factors, which correctly capture the low energy behaviour. The approximative description of the correlations obtained this way is of high accuracy and due to the non-perturbative input of the scattering matrix more precise than any calculation in perturbation theory.

Although the calculation of form factors will not be performed in this thesis, it has been mentioned since it constitutes nowadays one of the most interesting techniques in the study of integrable field theory and is tightly linked to the bootstrap construction of scattering matrices. In fact, in our article [14] the full set of form factors for the $su(3)_2$-Homogeneous Sine-Gordon model has recently been obtained. The latter belongs to a class of integrable models which are studied in some detail in this thesis.

In conclusion, one might say that from the field theoretic point of view the motivation to study integrable systems in 1+1 dimensions is to learn about the structure of QFT by constructing exact solutions and finding explicit expressions for the relevant physical quantities.

Integrability and broken scale invariance

The interest in integrable models not only resides in their role as excellent "testing laboratories" for exact non-perturbative methods of QFT, but more recently also in their interpretation as deformed **conformal field theories** (CFT) [15]. The latter form a particular class of integrable field theories but in contrast to the cases mentioned before they are associated with *massless* particles. In this particular case the infinite set of conversation laws is linked to conformal space-time symmetry in two-dimensions. An intense research activity in this area was initiated by the seminal paper of Belavin, Polyakov and Zamolodchikov [16]. They combined the representation theory of the Virasoro algebra describing the infinitesimal quantum generators of conformal transformations with the concept of a local operator algebra in order to show that a certain class of conformal theories, the so-called minimal models, constitute particular examples of solvable massless QFT's. Motivated by the observation of Polyakov that for physical systems with local interactions conformal symmetry is an immediate extension of scale invariance [17], the techniques of Euclidean CFT have been applied to study statistical mechanics and condensed matter systems in two *space*-dimensions, which undergo a second order phase transition. The latter become scale invariant at the critical point and fall into different universality classes fixed by the critical exponents, which describe the power law behaviour of the correlations in the system. One of the motivations to study CFT is the classification of all these universality classes.

A simple example for this picture is provided by the two-dimensional Ising model. In the continuum limit it can be described by a Euclidean field theory of free Majorana fermions whose mass is proportional to $|1/T - 1/T_c|$ with T being the temperature and T_c its special value at the critical point. Away from criticality the fermions in the system are massive and the correlations fall off over a finite length scale fixed by the Compton wave length. If the system approaches the critical point at $T = T_c$ the particles become massless and the associated correlation length diverges. In particular, the theory loses its dependence on the only dimensionful parameter and becomes therefore scale invariant.

This scenario can be generalized to more complicated cases. Given a conformal field theory at the critical point one might in particular ask what happens to the infinite conservation laws linked to conformal invariance when the system becomes off-critical and scale invariance is lost. As Zamolodchikov pointed out [15] an infinite set of these conserved charges – even though they get deformed – might survive the breaking of conformal symmetry and render also the perturbed theory integrable. Provided the particle spectrum of the perturbed theory is purely massive one might now exploit the above non-perturbative techniques of the bootstrap program to obtain information about the off-critical behaviour of the system. This point of view has been supported by numerous concrete examples, starting with the study of the Ising model in an external magnetic field [18]. In fact, this interplay between field theoretic considerations and phase transitions in statistical mechanics renewed the interest in integrable field theories and made the subject flourish in the last years. To name a few examples in the area of condensed matter theory, non-perturbative methods of integrable field theory have been discussed in the context of quantum impurity problems, see [19] and references therein. Other possible applications have been investigated in the context of two leg Hubbard ladders and Carbon nanotubes [20]. There is also a series of papers which apply the theory of so called W-algebras, closely connected to affine or Kac-Moody algebras, to the quantum Hall effect [21].

Recovering conformal invariance

In contrast to the picture just described one might proceed in reverse order and start with a massive integrable field theory and ask how to recover the associated conformal model. Scale and therefore conformal invariance will be approximately restored in the high-energy regime, where the masses of the particles become negligible. The technique which will be applied in this thesis to investigate the high-energy limit of integrable quantum field theories is the **thermodynamic Bethe ansatz** (TBA) [22, 23]. Using the exact scattering matrix as the only input, this approach allows to calculate the free energy of the integrable field theory on an infinite cylinder after performing a Wick rotation and interpreting the imaginary time axis as temperature. When the latter reaches an energy scale which is large compared to the one set by the particle masses in the spectrum, numerous characteristic quantities of the CFT governing the ultraviolet behaviour can be extracted from the free energy, as for instance the (effective) **central charge** c playing the role of a Casimir energy. In this way the TBA forms an important interface between the conformal and the massive integrable model. In particular, it can be used to test scattering matrices constructed via the bootstrap approach for consistency and to relate massive to conformal spectra with the ultimate goal of obtaining a deeper understanding of the origin of mass. Moreover, regarding the applications to the off-critical behaviour of statistical mechanics or condensed matter systems one might assign by means of the TBA to each scattering matrix a conformal field theory or a universality class.

Additional motivation for the investigation of the intimate relation between integrable and conformal models also comes from **string theory**, whose objective is the unification of all forces in nature. Here the time evolution of a one-dimensional object, the string, sweeps a two-dimensional world-sheet in space-time and the shape of the string is described by fields which live on this world-sheet. In case of the vacuum state the world-sheet is assumed to be invariant under reparametrizations, what implies conformal symmetry for the field content. Another concept which is reminiscent of string theory is the notion of dual models or duality, which in an elementary and simple form will also be encountered in the context of the integrable models investigated in this thesis.

After this outline of the general perspective and techniques of two-dimensional integrable quantum field theory, the different aspects mentioned will now be elaborated with particular hindsight to **affine Toda field theories** (ATFT) [24], which constitute the most prominent, best studied and largest class of integrable models.

1.2 Affine Toda field theory

The simplest and best known examples of affine Toda field theories (ATFT) are the Sine-Gordon and the Sinh-Gordon model for imaginary and real values of the coupling constant, respectively. In general, they are associated with the following classical Lagrangian

$$\mathcal{L}_{\text{ATFT}}(\mathfrak{g}) = \frac{1}{2}\left\langle \partial_\mu \phi, \partial^\mu \phi \right\rangle - \frac{m^2}{\beta^2} \sum_{i=0}^{n} n_i e^{\beta \langle \alpha_i, \phi \rangle} \ . \qquad (1.1)$$

Here $\phi = (\phi_1, ..., \phi_n)$ are n-component fields transforming as scalars under the Lorentz group and m, β define a classical mass scale and coupling constant, respectively. The latter are classically unimportant but enter the quantum theory associated with the above Lagrangian. The integer constants n_i and the constant vectors $\alpha_i \in \mathbb{R}^n$ are restricted to special values in order to guarantee that the resulting field theory is integrable. In fact, it turns out that the allowed set of external parameters is in general linked to an **affine Lie algebra** $\hat{\mathfrak{g}}$ of rank n [25]. In many cases, however, it turns out that the structure of a simple finite-dimensional Lie algebra $\mathfrak{g}$, whose affine extension is $\hat{\mathfrak{g}}$, is sufficient for the description. The integers n_i are then interpreted as its Kac labels, the α_i's constitute its simple roots with $i = 1, ...n$ and $-\alpha_0$ is the highest root with $n_0 = 1$. The latter mathematical objects will be explained in more detail in course of the thesis. Choosing $\mathfrak{g} = su(2)$ we recover the Sine-Gordon or Sinh-Gordon Lagrangian upon noting that then one has $n = n_0 = n_1 = 1$ and $\alpha_1 = -\alpha_0$, whence the above potential acquires the form of a cos or a cosh-function for β purely imaginary or real, respectively.

In this thesis the discussion will exclusively deal with the models associated with a real coupling constant, i.e. with the generalizations of the Sinh-Gordon model. While in the latter case the above Lagrangian is manifestly real, this property is in general lost for imaginary β. However, due to the soliton solutions to which they give rise, also the latter models starting with the Sine-Gordon theory have been studied in detail in the literature. (For a relatively recent review of ATFT and references see [26].)

The outstanding property common to all these models is the rich underlying Lie algebraic structure encoded in $\mathfrak{g}$, which allows for the application of powerful mathematical concepts. Classically it can be used to show integrability by a Lax pair construction and to determine the solutions to the classical equations of motion [24]. Remarkably, also on the quantum level physical quantities like the mass spectrum, the fusing processes of particles as well as the S-matrix reflect the Lie algebraic structure.

One of the central aims of this thesis is to exploit this Lie algebraic structure in order to obtain generic and concise formulas for all relevant quantities, such that all models are encompassed at once. These universal expressions will in particular allow to separate model dependent features from more general ones and unify numerous case-by-case discussions found in the literature.

The search for the universal scattering matrix of ATFT

The scattering matrices of affine Toda models associated with simply-laced Lie algebras, the so-called ADE series, were the first to be studied by standard perturbative methods as well as by the bootstrap approach, beginning with the paper by Arinshtein et al. about the $\mathfrak{g} = A_n \equiv su(n+1)$ theories and followed by articles from Mussardo and Christe as well as Braden et al., who considered the remaining cases [27]. Their results were put into a universal form in [28, 29] describing the scattering matrices of all ADE models in a unique and generic formula. The key feature they exploited is the Coxeter geometry naturally assigned to each simple Lie algebra $\mathfrak{g}$. This had been noticed to be crucial in the description of the three-point couplings of the theory

[30]. The latter determine the bound state structure (so-called fusing processes of the particles), an information required to perform the bootstrap construction of the scattering matrix.

Similar attempts failed for the remaining ATFT involving non simply-laced Lie algebras (the $BCFG$ series) due to the fact that their renormalization behaviour turned out to be quite different form the ADE series, where the classical mass ratios survive quantization up to one loop order in perturbation theory. For several years different proposals for the scattering matrices were put forward. They were plagued by mysterious higher-order poles which were coupling dependent and resisted a consistent physical interpretation.

The breakthrough in the understanding of the non simply-laced models started with the paper of Delius et al. [31]. Based on their perturbative calculations it was suggested [32] that these theories are governed by two classical Lagrangians belonging to a pair of "dual" algebras, one describing the system in the weak and the other in the strong coupling regime. Another crucial step made by Corrigan et al. was the formulation of the generalized bootstrap principle [33] giving a consistent prescription how to identify those poles in the physical sheet which are relevant to the bootstrap approach. However, the construction of the various non simply-laced scattering matrices was performed separately for the different models and a concise Lie algebraic formulation was lacking.

First steps towards this direction were made in the work by Chari and Pressley [34], who managed to reproduce the allowed fusing processes in terms of Coxeter geometry associated with the two dual algebras, and the article by Khastgir [35], who employed the idea of folding to reproduce the scattering matrices found in [33]. However, it was Oota who finally succeeded in writing down a closed universal expression for the scattering matrices by introducing q-deformed Coxeter elements [36]. The latter allow to link the fusing rules directly to the scattering matrices and to accommodate the coupling dependence of the theories by a special choice of the deformation parameter.

Various formulas found by Oota, which were until then only claimed on the base of a case-by-case analysis, have been rigorously derived in our paper [37] together with numerous entirely new identities. In particular, the precise relation between the different versions of fusing rules has been obtained therein and their consistency with the formulation of the mass spectrum as null vector of a q-deformed Cartan matrix demonstrated. Additional results of our work [37] include the systematic discussion of the bootstrap properties, the rigorous derivation of a generic integral representation for the scattering matrix found in [36] as well as the proof of new S-matrix equations, so-called combined bootstrap equations, which are intimately linked to the underlying Lie algebraic structure. The discussion of ATFT in this thesis will closely follow the arguments provided in [37].

The motivation for extracting as much of the Lie algebraic structures underlying ATFT as possible is twofold. First they provide a concise mathematical framework which eases the investigation of quantum integrable models. Having universal expressions for the characteristic physical quantities of the theory at hand, general claims about the structure of QFT might be checked for the infinite class of affine Toda models at once instead of only a few cases. Moreover, in future applications, e.g. the form factor program, calculations might be performed in a generic Lie algebraic framework avoiding tedious case-by-case studies. Second, once the interplay between the powerful mathematical structures and physical quantities has been understood, the Lie algebraic concepts might be employed to construct entirely new integrable models with similar features.

Colour valued scattering matrices and a new class of integrable models

In our article [38] a general construction principle has been suggested leading to new scattering matrices associated with integrable quantum models. Given the mass spectrum of an integrable model one might multiply it by assigning to each particle additional quantum numbers, so-called colours. Provided the scattering matrix of the original theory is explicitly coupling dependent one might then let particles of different colours act at different values of the coupling. A slightly more complicated version of this principle can be employed if the coupling dependence of the original scattering matrix can be absorbed in a separate factor. This is the case for the ATFT S-matrix S^{ADE} associated with simply-laced algebras. Explicitly one has a decomposition of the form

$$S^{ADE} = S^{\min} S^{\mathrm{CDD}} . \tag{1.2}$$

The so-called minimal factor $S^{\min}$ incorporates all the physical relevant information about the bound state structure and is independent of the coupling, while S^{CDD} is a so called CDD-factor [39], which only introduces poles outside the physical sheet and displays the full coupling dependence. Note that the particular feature (1.2) is characteristic of the ADE series and ceases to hold for non-simply-laced algebras [31]. Letting particles of the same colour interact through $S^{\min}$ and those of different colours through S^{CDD} one might invoke the same Lie algebraic concepts as in the case of ATFT leading to a class of $\mathfrak{g}|\tilde{\mathfrak{g}}$-theories [38] associated with two simply-laced simple Lie algebras. One describes the bound state structure and the other the colour degrees of freedom. In total this yields a class containing as many exact S-matrices as possible pairs $(\mathfrak{g}, \tilde{\mathfrak{g}}) \in ADE \times ADE$. There are certain scattering matrices obtained earlier in the literature which are contained as a subclass in the $\mathfrak{g}|\tilde{\mathfrak{g}}$-theories, namely the scaling models or minimal ATFT associated with $S^{\min}$ (see e.g. [18]) and the Homogeneous Sine-Gordon models [40]. (Both theories are explained in more detail below.) Additional motivation for the definition of this particular class of scattering matrices becomes apparent when one discusses the high-energy behaviour of the associated integrable quantum field theories by means of the TBA and determines the underlying conformal field theories as explained above.

Affine Toda field theories as perturbed conformal models

As outlined in the first section of the introduction, integrable quantum field theories have a natural interpretation in terms of perturbed conformal field theories. In this context one decomposes the associated classical action functional of the integrable model into two parts,

$$S = S_{\mathrm{CFT}} + \lambda \int d^2x \; \Phi(x,t) \; ,$$

where S_{CFT} denotes the action of the conformal model describing the system at the critical point, Φ is a relevant spinless field operator of the unperturbed theory and λ is the coupling constant of the perturbation term. On dimensional grounds λ is proportional to the mass scale of the perturbed theory, $\lambda \propto m^{2-d_\Phi}$ with $d_\Phi < 2$ being the anomalous scaling dimension of Φ in the conformal limit. This writing of the classical action functional appeals to a renormalization group point of view originating in the study of critical phenomena. At $\lambda = 0$ the system is massless and constitutes a fixed point w.r.t. renormalization group transformations. Adding a relevant perturbation term, $\lambda \neq 0$, it is dragged away from the critical point giving rise to a renormalization group flow towards a massive theory.

The second part of this thesis is concerned with reversing this renormalization group flow. Employing the thermodynamic Bethe ansatz the high-energy regime of the mentioned integrable quantum field theories is analyzed in detail. In this approach the mass scale is eventually sent to zero, i.e. $\lambda \to 0$*, what implies in the above picture that the massive system floats back into the fixed point. This in particular will assign to each of the exact scattering matrices the central charge of the underlying ultraviolet conformal field theory and associate them with an off-critical model.*

The affine Toda models constitute particular examples for perturbed conformal field theories: Omitting the term associated with the affine or highest root α_0 in (1.1) one obtains the so-called **Toda field theories** (TFT), which are conformally invariant [41]. For instance the Sinh-Gordon theory can be viewed as perturbation of the ubiquitous Liouville theory, which is the simplest and best known example of TFT. In particular, it appears in applications related to string theory and two-dimensional quantum gravity. All Toda models have central charge $c \geq 25$ and belong to a class of conformal field theories about the structure of which not much is known. Having the scattering matrix of the off-critical theory at hand, namely the affine Toda S-matrix, one might investigate the high-energy regime by means of the TBA and perform certain consistency checks on conjectures concerning the structure of the unperturbed conformal model. In fact, this has first been done for the simplest examples, Liouville theory and the Sinh-Gordon model, see [42]. Therein a numerical TBA calculation was used to supplement semi-classical considerations giving support to a proposal for the explicit form of the three and four-point function of Liouville theory on a sphere.

Focussing on the massive side a detailed TBA analysis of all ATFT has been performed in our articles [43] and [44], the results of which are presented in this thesis. Combining extensive numerical investigations with approximate analytical considerations the off-critical Casimir energy is determined to leading order. Based on the Lie algebraic framework developed in the context of ATFT it is demonstrated that the latter can be obtained in a universal formula involving only basic Lie algebraic data like the rank of the algebra or the Coxeter number [44]. The behaviour found matches with the results in [42] and is also in agreement with the findings in [45, 46].

The latter articles rely also on semi-classical considerations and are similar in spirit to the work on the Liouville model by Zamolodchikov and Zamolodchikov, but treat general cases of TFT and ATFT.

In view of this comparison between data obtained from the conformal model and the perturbed massive theory the TBA analysis also yields a consistency check for the bootstrap construction of the ATFT S-matrix. This application of the TBA to test scattering matrices is of particular importance when mass spectra and the scattering amplitudes have been derived by semiclassical arguments which need to be verified on the quantum level as it is the case for the Homogeneous Sine-Gordon (HSG) models [47] mentioned above as particular subclass of the $\mathfrak{g}|\tilde{\mathfrak{g}}$-theories.

Colour valued scattering matrices, WZNW cosets and off-critical models

The interest in the class of colour valued scattering matrices described above stems from the fact that many of them can be related in the ultraviolet limit to Wess-Zumino-Novikov-Witten (WZNW) cosets [48]. WZNW theories constitute the best known and possibly best understood conformal models due to an underlying Lie algebraic structure similar to the case of ATFT. In addition, the coset construction of these models allows one to construct nearly every other conformal field theory. Hence, WZNW models can be viewed as the basic building blocks in constructing conformally invariant theories. Concrete examples are the minimal conformal models listed in Table 1.1. They can be described by cosets of the form

$$\text{coset:} \quad \mathfrak{g}_1 \otimes \mathfrak{g}_1 / \mathfrak{g}_2 \qquad \text{central charge:} \quad c = \frac{2 \dim \mathfrak{g}}{(h+1)(h+2)}$$

Here $h = \sum_{i=0}^{n} n_i$ denotes the Coxeter number of the Lie algebra $\mathfrak{g}$ and the lower index indicates the so called level of the representation of the associated affine Lie algebra [25]. These cosets are the most prominent representatives, since they can be directly linked to statistical mechanics systems in two-dimensions. Initiated by the paper of Zamolodchikov on the Ising model in an external magnetic field, it was realized by extensive TBA studies, that the off-critical behaviour of these systems under a perturbation by the relevant field operator with conformal weights $\Delta = \bar{\Delta} = 2/(h{+}2)$ is described by so-called **scaling models** or **minimal affine Toda theories** [18, 49, 23, 50], already mentioned above. The name indicates that the massive field theory describing the perturbed system away from criticality is associated with the minimal factor $S^{\min}$ of the ATFT S-matrix [27] only (compare the separation property (1.2) for simply-laced algebras). Historically, these were the first examples of exact scattering matrices considered in context of affine Toda field theory. They provide concrete examples for the application of integrable quantum field theory to statistical mechanics. This observation had important impact on the subsequent investigations of ATFT in the literature.

$\mathfrak{g}$	minimal model	c	perturbation
A_1	critical Ising	1/2	$\Delta = 1/2$
A_2	three state Potts	4/5	$\Delta = 2/5$
E_6	tricritical Potts	6/7	$\Delta = 1/7$
E_7	tricritical Ising	7/10	$\Delta = 1/10$
E_8	critical Ising	1/2	$\Delta = 1/16$

Table 1.1: Minimal conformal field theories as $\mathfrak{g}_1 \otimes \mathfrak{g}_1/\mathfrak{g}_2$ WZNW cosets and the conformal weights of the perturbing operator corresponding to minimal affine Toda field theory.

Notice that the Ising model can be realized in two different ways involving either the Lie algebra $A_1 \equiv su(2)$ or the exceptional Lie algebra E_8. This can also be seen from the Virasoro characters [51, 52]. While A_1 represents the simplest example of a simple Lie algebra, E_8 is the most complex and intricate one. The two different realizations also play an important role for the perturbed theories given by the different field operators in Table 1.1. In case of A_1 the system is disturbed thermally and the corresponding minimal ATFT S-matrix describes a single massive free Majorana fermion, i.e. it is almost trivial $S^{\min} = -1$. In contrast, the perturbation in case of E_8 corresponds to an external magnetic field and the associated off-critical field theory contains now 8 different species of quantum particles, which interact with each other in a highly non-trivial fashion. This is reflected by a complicated scattering matrix, which involves up to 12^{th} order poles. This shows that already for the easiest example of a conformal field theory, the free fermion, complex integrable structures might appear, when the system becomes off-critical. Moreover, it motivates the study of the general Lie algebraic structures, since a broad range of different simple Lie algebras might occur in the applications as the above list of examples impressively demonstrates.

The close connection between the Lie algebraic structures of ATFT and of WZNW models is further supported by the class of $\mathfrak{g}|\tilde{\mathfrak{g}}$-theories. In our paper [38] a universal formula for the central charge was obtained from the exact scattering matrices by means of a TBA analysis which is also presented in this thesis,

$$c = \frac{\tilde{h}}{h + \tilde{h}} \operatorname{rank} \mathfrak{g} \cdot \operatorname{rank} \tilde{\mathfrak{g}}$$

with $h, \tilde{h}$ being the Coxeter numbers of the algebra $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$, respectively. Remarkably, this formula has also been stated in a different context involving conformal parafermionic theories [53]. Specializing this general result to particular choices of the related simply-laced Lie algebras, as for instance keeping $\mathfrak{g} = ADE$ generic and restricting $\tilde{\mathfrak{g}}$ to $A_{k-1} \equiv su(k)$, $k > 1$ the above central charge matches with the one of the following series of WZNW coset models for simply-laced algebras,

$$\text{coset:} \quad \underbrace{\mathfrak{g}_1 \otimes \cdots \otimes \mathfrak{g}_1}_{k}/\mathfrak{g}_k \qquad \text{central charge:} \quad c = k\,\frac{\dim \mathfrak{g}}{h+1} - \frac{k \dim \mathfrak{g}}{h+k}\ .$$

Choosing $k = 2$, i.e. $\tilde{\mathfrak{g}} = A_1 \equiv su(2)$, one obtains in particular the minimal affine Toda theories in accordance with the considerations outlined above. From this point of view the latter class of scattering matrices can be viewed as an extension of the scaling models. Exchanging now the role of both algebras, i.e. choosing $\tilde{\mathfrak{g}} = ADE$ generic and imposing this time the restriction $\mathfrak{g} = A_{k-1}$, $k > 1$, one obtains the central charges belonging to WZNW cosets of the form

$$\text{coset:} \quad \tilde{\mathfrak{g}}_k/u(1)^{\times \operatorname{rank} \tilde{\mathfrak{g}}} \qquad \text{central charge:} \quad c = \frac{k \dim \tilde{\mathfrak{g}}}{k+h} - \operatorname{rank} \tilde{\mathfrak{g}} \; .$$

For this last choice of the Lie algebraic structure the associated exact scattering matrices coincide with the ones proposed in [40] in the context of the Homogeneous Sine-Gordon models [47]. These integrable models were explicitly constructed from the above WZNW cosets by perturbing with an operator of conformal weight $\Delta = \bar{\Delta} = h/(k+h)$. In [54] it was argued that the particular choice of the coset then ensures that these theories are purely massive and quantum integrable, ensuring that the scattering matrix can be constructed by the bootstrap approach. Additional distinguished features of these integrable models are parity violation and soliton solutions. The latter allow for a semi-classical quantization of the mass spectrum by means of the Bohr-Sommerfeld rule [55].

With regard to these considerations a detailed TBA analysis of the HSG models has then been performed in our work [56], the results of which are presented in this thesis. The latter show in particular that the conjectured S-matrix is consistent with the semi-classical picture and in the ultraviolet limit gives rise to the correct coset central charge as mentioned above. Moreover, extensive numerical calculations for the $\tilde{\mathfrak{g}} = su(3)$ case together with analytical findings elucidate the role of resonance poles in the proposed HSG scattering matrix. The latter have been introduced in [40] in order to accommodate unstable particles in the semi-classical spectrum. In the TBA investigation the resonance poles give rise to a so-called **staircase pattern** in the free energy of the system. That is, the free energy exhibits plateaus of constant height corresponding to a flow of the perturbed theory towards different conformal field theories, which is controlled by the external parameters, such as the temperature and the mass scale of the unstable particles. The physical picture one then recovers for the $\tilde{\mathfrak{g}} = su(3)$ example is that depending on whether the formation of unstable particles has set in or not the HSG model in the ultraviolet regime approaches one of the two cosets [56],

$$\begin{aligned} \text{unstable particle formation} \quad &: \qquad \tilde{\mathfrak{g}}_k/u(1)^{\times \operatorname{rank} \tilde{\mathfrak{g}}} \\ \text{no unstable particle formation} \quad &: \qquad \operatorname{rank} \tilde{\mathfrak{g}} \times su(2)/u(1) \; . \end{aligned}$$

In the second case it is understood that one has $\operatorname{rank} \tilde{\mathfrak{g}}$ copies of the $su(2)/u(1)$ theory which do not interact with each other. In addition, the TBA analysis of the $\tilde{\mathfrak{g}} = su(3)$ case strongly suggests that in an alternative formulation one might describe the interpolation of the HSG theory between these two cosets by a massless ultraviolet-infrared flow. For the case at hand one recovers as a subsystem the flow between the tricritical and the critical Ising model [57].

Similar observations of resonance poles, staircase patterns and massless flows have occurred in the literature before [58, 59] in the context of the so-called **affine Toda resonance** or **staircase models**. Here the ATFT scattering matrix for simply-laced algebras is evaluated at complex values of the effective coupling constant. This introduces in particular resonance poles. Similarly to the situation above they generate a staircase pattern in the free energy of the system which interpolates between the WZNW coset models

$$\text{coset:} \quad \mathfrak{g}_1 \otimes \mathfrak{g}_k / \mathfrak{g}_{k+1} \quad \text{central charge:} \quad c = \dim \mathfrak{g} \left(\frac{1}{1+h} + \frac{k}{k+h} - \frac{k+1}{k+1+h} \right)$$

Also in this case the resonance poles of the scattering matrix have been suggested to belong to unstable particles. However, in contrast to the HSG models a concrete Lagrangian description of the resonance models, where this relation would become manifest is still lacking. It is this particular advantage of a classical Lagrangian allowing for a semi-classical approach together with the new feature of parity violation, which makes the HSG theories particularly interesting candidates for further investigations on the quantum level.

In summary, the above outline demonstrates that the Lie algebraic structures encountered in context of affine Toda models are relevant beyond these theories, since they appear at numerous other occasions in the discussion of integrable quantum field theories. In particular, this Lie algebraic framework is important for the connection between exact scattering matrices of affine Toda type and conformal field theories related to WZNW models. An overview of the different relations is given in Figure 1.1.

Starting form a conformal WZNW theory associated with a compact simple Lie group G and an associated simple Lie algebra $\mathfrak{g}$ upon suitable manipulation one might obtain different other conformal theories, as for example WZNW cosets or TFT. (The latter can be obtained by a decomposition of the WZNW fields and a suitable reductio procedure as was worked out in [60].) Perturbing these conformal theories by introducing a mass scale yields different integrable models, whose S-matrices can be obtained by the bootstrap approach. The hinge in linking the different models is the splitting of the ATFT scattering matrix in a minimal part and a CDD factor for simply-laced algebras. In contrast, the non-diagonal affine Toda models belonging to purely imaginary coupling, e.g. the Sine-Gordon theory, need to be understood in more detail. Here many scattering matrices have not even been constructed yet. It is to be expected that the study of the diagonal ATFT with real coupling will also be of great help in this study. For instance, it is well known that the scattering matrix of the breather spectrum in the Sine-Gordon model (i.e. the bound state spectrum of the fundamental particles) coincides with the minimal scattering matrix of $A_n^{(2)}$-ATFT for real coupling.

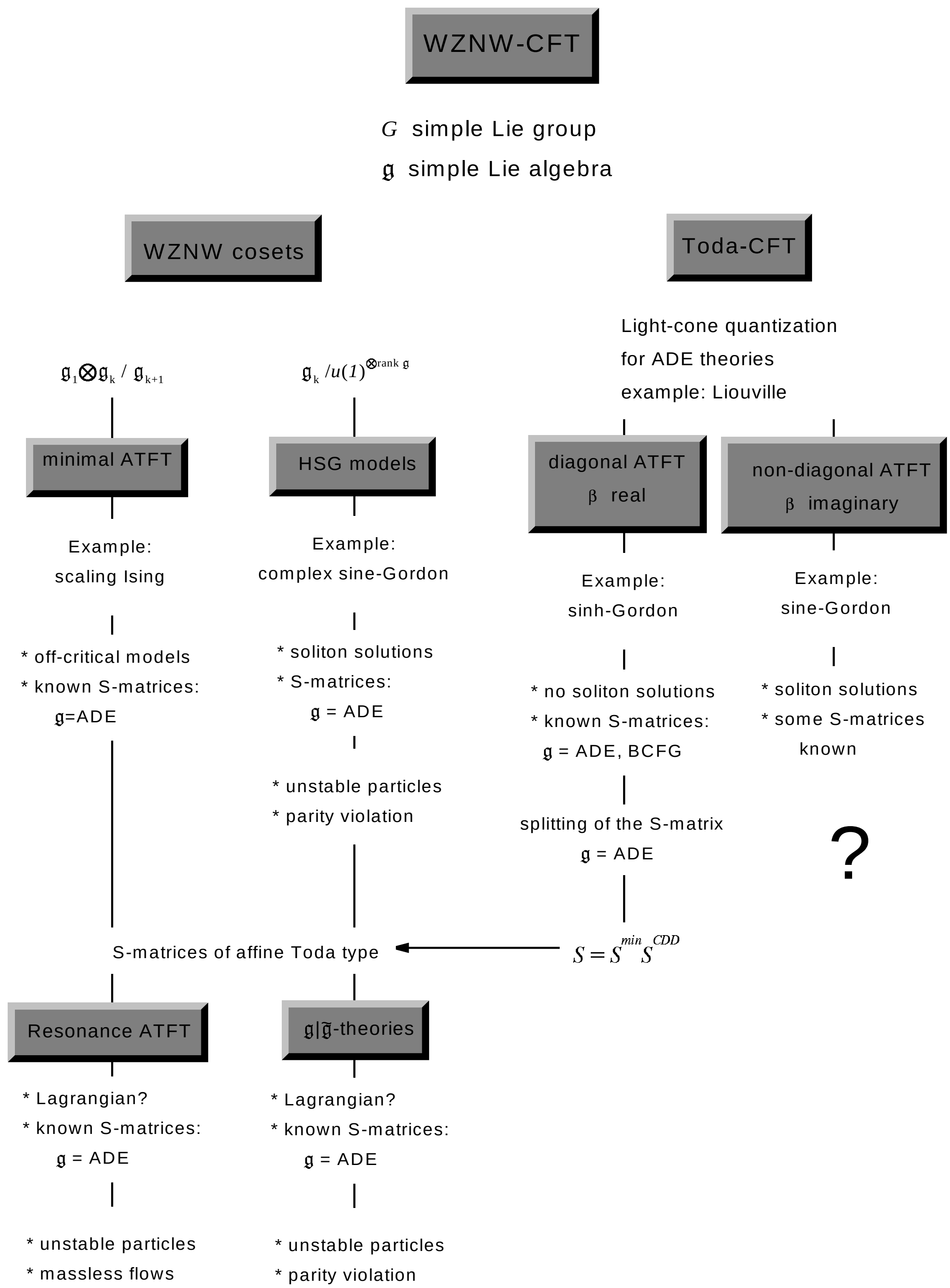

Figure 1.1: Overview of conformal and integrable field theories off affine Toda type.

1.3 Outline of the content

The description of the content is restricted to a few key-words, since a more detailed account is given at the beginning of each chapter. Introductory surveys in the first sections of the different chapters have the purpose to keep the discussion self-contained.

Chapter 2 is concerned with introducing those mathematical concepts which will be used subsequently in the thesis. The presentation starts with the theory of simple Lie algebras and their affine extensions focussing on those aspects which are important for the physical applications in due course. Of particular importance are the sections on Coxeter geometry, which prepare the subsequent introduction of q-deformed Coxeter elements and Cartan matrices needed to describe the renormalized mass-flow in ATFT.

Chapter 3 is devoted to the bootstrap analysis of exact scattering matrices, starting with a survey of the general analytic structure of the two-particle scattering amplitude in integrable models. Particular emphasis is given to ATFT and their Lie algebraic structure. Fusing rules, mass spectrum and scattering matrix are discussed in a generic Lie algebraic setting for all models at once and universal formulas are derived. Specializing the general case to simply-laced Lie algebras the separation property of the scattering matrix in a minimal and a CDD factor is presented. This serves as a preliminary step towards the definition of the colour valued S-matrices. The discussion of their bootstrap properties together with the feature of parity violation inherent to the $\mathfrak{g}|\tilde{\mathfrak{g}}$-theories closes this chapter.

Chapter 4 contains the thermodynamic Bethe ansatz of the integrable models discussed in the previous chapter. Since the link between conformal and integrable quantum field theories is at the center of interest in this presentation, some basic notions of conformal invariance are recalled. Different numerical and analytical methods in the context of the TBA are discussed in full generality and then applied to concrete examples, starting with ATFT. The central charge and the free energy up to leading order are determined, employing once more the generic Lie algebraic apparatus to find universal formulas. The discussion continues with a detailed analysis of the Homogeneous Sine-Gordon models with special focus on the semi-classical predictions made in the literature. Using the Lie algebraic structures underlying also these models, the methods of the TBA analysis are then extended to the more general class of $\mathfrak{g}|\tilde{\mathfrak{g}}$-theories and the associated colour valued scattering matrices. The main result is a generic formula depending only on the rank and the Coxeter numbers of the Lie algebras involved and which associates with each of these scattering matrices a central charge of the underlying conformal model in the ultraviolet limit.

Chapter 5 summarizes the results and presents the conclusions. In addition, an outlook is given on related problems and future investigations.

Throughout this thesis natural units will be used, i.e. Planck's constant and the velocity of light are set to one, $\hbar = c = 1$.

As has been indicated in the introduction, parts of this thesis have already been published. The main results presented here are contained in the following articles:

1. A. Fring, C. Korff and B.J. Schulz, *The ultraviolet behaviour of integrable field theory, affine Toda field theory*, Nucl. Phys. **B549** (1999) 579-612.

2. A. Fring, C. Korff and B.J. Schulz, *On the universal representation of the scattering matrix of affine Toda field theory*, Nucl. Phys. **B567** (2000) 409-453.

3. A. Fring and C. Korff, *Colour valued S-matrices*, Phys. Lett. **B477** (2000) 380-386.

4. O.A. Castro-Alvaredo, A. Fring, C. Korff and J.L. Miramontes, *Thermodynamic Bethe ansatz of the Homogeneous Sine-Gordon models*, Nucl. Phys. **B573** (2000) 535-560.

5. A. Fring and C. Korff, *Large and small density approximations to the thermodynamic Bethe ansatz*, Nucl. Phys. **B579** (2000) 617-631

6. O.A. Castro-Alvaredo, A. Fring and C. Korff, *Form factors of the homogeneous Sine-Gordon models*, Phys. Lett. **B484** (2000) 167-176.