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On associahedra and related topics

by

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Dedicated to my first child

Preface

The aim of this thesis is to present several developments regarding the associahedron and its relatives. The associahedron, as pointed out in a manuscript by Mark Haiman from 1984 [34], is a mythical polytope with a beautiful combinatorial structure. It first appeared 1951 in Dov Tamari's unpublished thesis [83], and since then, together with its generalizations, keeps showing unexpected connections as well as leading to fascinating results.

The present work contains a detailed study of the associahedron and some of its generalizations from a geometric and combinatorial point of view. The contents are subdivided into three chapters. The first chapter is the result of a joint work with Francisco Santos and Günter M. Ziegler [15]. It describes many different construction methods for the associahedron, which surprisingly produce substantially different geometric realizations. The second chapter, which is joint work with Jean-Philippe Labbé and Christian Stump [14], introduces and studies a new family of simplicial complexes called multi-cluster complexes; these complexes generalize the concept of cluster complexes and extend the notion of multi-associahedra to arbitrary finite Coxeter groups. The third chapter shows a new point of view on the problem of polytopality of multi-associahedra and spherical subword complexes, and presents two computational methods, which were implemented in joint work with Jean-Philippe Labbé, to produce polytopal realizations for small explicit examples.

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Contents

Notation

Ass_n	n-dimensional associahedron
$\mathrm{Ass}_n^{\mathrm{I}}$	associahedra of type I, by Hohlweg–Lange–Thomas
$\mathrm{Ass}_n^{\mathrm{II}}$	associahedra of type II, by Santos
c	a Coxeter element
$\mathbf{c}$	a word for a Coxeter element c
CFZ_n	Chapoton–Fomin–Zelevinsky associahedron
GKZ_n	Gelfand–Kapranov–Zelevinsky associahedron
h	the Coxeter number
$\mathrm{inv}(\pi)$	inversion set of π
Lr_c	bijection between letters of $\mathbf{c}\mathbf{w}_\circ(\mathbf{c})$ and almost positive roots
P_{n+3}	a convex $(n+3)$-gon
Post_n	Postnikov associahedron
$Q_{\overset{\circlearrowleft}{s}}$	rotation of the word Q along the letter s
r_F	root function associated to a facet F
RSS_n	Rote–Santos–Streinu associahedron
$\mathcal{S}_{n+1}$	symmetric group $\mathcal{S}_{n+1}$
T_0	the seed triangulation
$w_\circ$	longest element in a Coxeter group
$\mathbf{w}_\circ$	a word representing a reduced expression for $w_\circ$
$\mathbf{w}_\circ(\mathbf{c})$	the $\mathbf{c}$-sorting word of $w_\circ$
(W,S)	finite Coxeter system
$W_{\langle s \rangle}$	parabolic subgroup generated by $S \setminus \{s\}$
$\|\|_c$	c-compatibility relation on almost positive roots
$[1]$	the shift operation
Δ_m	boundary complex of the dual associahedron
$\Delta_{m,k}$	simplicial complex of k-triangulations
$\Delta(Q,\pi)$	subword complex associated to the word Q and the element π
$\Delta_c^k(W)$	the multi-cluster complex of type W
$\delta(Q)$	Demazure product of Q

Γ_Ω	Auslander-Reiten quiver of a quiver Ω
Ω	a quiver
Ω_c	the quiver associated to a Coxeter element c
Φ	roots in a root system
Φ^+	positive roots in a root system
$\Phi_{\geq -1}$	almost positive roots in a root system
$\Phi_{\langle s \rangle}$	root system associated to $W_{\langle s \rangle}$
Π	simple roots in a root system
π	an element of a Coxeter group
τ	the Auslander-Reiten translate
Θ	cyclic action on the vertices and facets of the multi-cluster complex
$\mathbb{Z}\Omega$	the repetition quiver of a quiver Ω

Chapter 1

Many non-equivalent realizations of the associahedron

1.1 Introduction

The n-dimensional associahedron is a simple polytope with $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$ (the Catalan number) vertices, corresponding to the triangulations of a convex $(n+3)$-gon, and $n(n+3)/2$ facets, in bijection with the diagonals of the $(n+3)$-gon. It appears in Dov Tamari's unpublished 1951 thesis [83], and was described as a combinatorial object and realized as a cellular ball by Jim Stasheff in 1963 in his work on the associativity of H-spaces [77, 78]. A realization as a polytope by John Milnor from the 1960s is lost; Huguet and Tamari claimed in 1978 that the associahedron can be realized as a convex polytope [37]. The first such construction, via an explicit inequality system, was provided in a manuscript by Mark Haiman from 1984 that remained unpublished, but is available as [34]. The first construction in print, which used stellar subdivisions in order to obtain the dual of the associahedron, is due to Carl Lee, from 1989 [48].

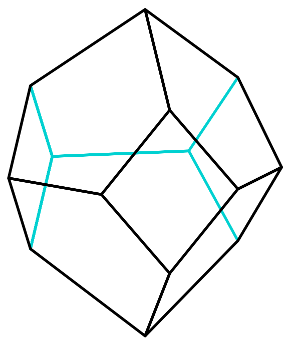


Figure 1.1: The 3-dimensional associahedron, realized as the secondary polytope of a regular hexagon.

Subsequently three systematic approaches were developed that produce realizations of the associahedra in more general frameworks and suggest generalizations:

- the associahedron as a secondary polytope due to Gelfand, Kapranov and Zelevinsky [32] [33] (see also [31, Chap. 7]),
- the associahedron associated to the cluster complex of type A_n, conjectured by Fomin and Zelevinsky [24] and constructed by Chapoton, Fomin and Zelevinsky [16], and
- the associahedron as a Minkowski sum of simplices introduced by Postnikov in [59]. Essentially the same associahedron, but described quite differently, had been constructed independently by Shnider and Sternberg [72] (compare Stasheff and Shnider [79, Appendix B]), Loday [49], Rote, Santos and Streinu [66], and most recently Buchstaber [12]. Following [35] we reference it as the "Loday realization", as Loday obtained explicit vertex coordinates that were used subsequently.

The last two approaches were generalized by Hohlweg and Lange [35] and by Santos [69], who showed that they are particular cases of *exponentially many* constructions of the associahedron. The Hohlweg–Lange construction produces roughly 2^{n-3} distinct realizations, while the Santos construction produces about $\frac{1}{2(n+3)}C_{n+1} \approx 2^{2n+1}/\sqrt{\pi n^5}$ different ones; exact counts are in Sections 1.4 and 1.5. The construction by Santos appears in print for the first time here by Ceballos, Santos and Ziegler [15], where we prove in detail that it actually works. For the others we rely on the original papers for most of the details.

This chapter contains the results with Francisco Santos and Günter M. Ziegler in [15]. The goal is to compare the constructions, showing that they produce essentially different realizations for the associahedron. Let us explain what we exactly mean by *different* (see more details in Section 1.2). Since the associahedron is simple, its realizations form an open subset in the space of $\frac{(n+3)n}{2}$-tuples of half-spaces in $\mathbb{R}^n$. Hence, classifying them by affine or projective equivalence does not seem the right thing to do. But most of the constructions of the associahedron (all the ones in this chapter except for the secondary polytope construction) happen to have facet normals with very small integer coordinates. This suggests that one natural classification is by *linear isomorphism of their normal fans* or, as we call it, *normal isomorphism.*

The secondary polytope construction has a completely different flavor from the others. Coordinates for its vertices are computed from the actual coordinates of the $(n+3)$-gon used, which can be arbitrary, and a continuous deformation of the polygon produces a continuous deformation of the associahedron obtained. The rest of the constructions are more combinatorial in nature, with no need to give coordinates for the polygon. This is apparent comparing Figures 1.1 and 1.2. The first one shows the secondary polytope of a regular hexagon, and the second shows (affine images of) other constructions of the 3-associahedron.

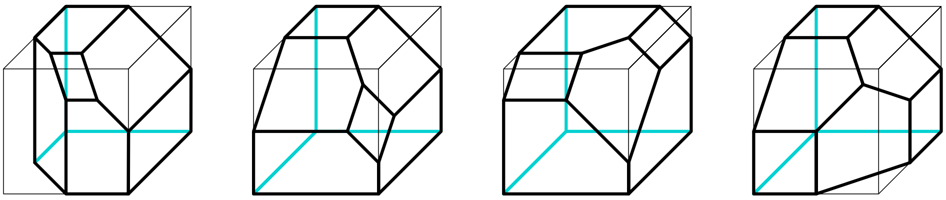


FIGURE 1.2: Four normally non-isomorphic realizations of the 3-dimensional associahedron. From left to right: The Postnikov associahedron (which is a special case of the Hohlweg–Lange associahedron), the Chapoton-Fomin-Zelevinsky associahedron (a special case of both Hohlweg–Lange and Santos) and the other two Santos associahedra. Since they all have three pairs of parallel facets, we draw them inscribed in a cube.

One way of pinning down this difference (and of testing, for example, whether two associahedra are normally isomorphic) is to look at which parallel facets arise, if any. We start doing this in Section 1.3, where we show that secondary polytope associahedra never have parallel facets (Theorem 1.8, but see Remark 1.9) while the Chapoton-Fomin-Zelevinsky and the Postnikov ones have n pairs of parallel facets each (Theorems 1.14 and 1.25).

In Sections 1.4 and 1.5 we present the families of realizations by Hohlweg–Lange and by Santos. The first one produces one n-associahedron for each sequence in $\{+,-\}^{n-1}$. The second one constructs one n-associahedron from each triangulation of the $(n+3)$-gon. We call them associahedra of *type* I and *type* II.

Apart of reviewing the two constructions, we show they both provide exponentially-many normally non-isomorphic realizations of the n-dimensional associahedron with the following common features:

- They all have n pairs of parallel facets.
- In the basis given by the normals to those n pairs, all facet normals have coordinates in $\{0, \pm 1\}$.

For the Santos construction both properties follow from the definition, for Hohlweg–Lange we prove them in Sections 1.4.2 and 1.4.3. All these constructions are (normally isomorphic to) polytopes obtained from the regular n-cube by cutting certain $\binom{n}{2}$ faces according to specified rules. Note that the last example of Figure 1.2 cannot be obtained by cutting faces one after the other; the three faces, edges in this case, need to be cut at about the same depth.

In Section 1.5.4 we use the Santos construction to present a simple combinatorial description of c-cluster complexes for Coxeter groups of type A as defined by Reading in [61]. We show that in the particular cases where the seed triangulation has a path as a dual tree, we obtain polytopal realizations of the associahedron for which the normal fans coincide with c-cluster fans.

In Section 1.6 we put together results from Sections 1.4 and 1.5, and show that there is a single associahedron that can be obtained both with the Hohlweg–Lange and the Santos construction, namely the one by Chapoton–Fomin–Zelevinsky.

We also note that Hohlweg–Lange–Thomas [36] provided a generalization of the Hohlweg–Lange construction to general finite Coxeter groups; Bergeron–Hohlweg–Lange–Thomas [7] have provided a classification of the Hohlweg–Lange–Thomas c-generalized associahedra in Coxeter group theoretic language up to isometry, and also up to normal isomorphism [7, Cor. 2.6]. For type A, this specializes to a classification of the Hohlweg–Lange associahedra, which we obtain in Theorem 1.32 in a different, more combinatorial, setting. Besides the isometries of c-generalized associahedra presented in [7], normal isomorphisms of these polytopes are discussed earlier by Reading–Speyer [63] in the context of c-Cambrian fans. In particular, they obtained combinatorial isomorphisms of the normal fans, which are in general only piecewise-linear [63, Thm. 1.1 and Sec. 5].

One of the questions that remains is whether there is a common generalization of the Hohlweg–Lange and the Santos construction, which may perhaps produce even more examples of "combinatorial" associahedra. It has to be noted that the associahedron seems to be quite versatile as a polytope. For example, besides the four 3-associahedra of Figure 1.2 we have found another four 3-associahedra that arise by cutting three faces of a 3-cube (see Figure 1.3). Do these admit a natural combinatorial interpretation as well?

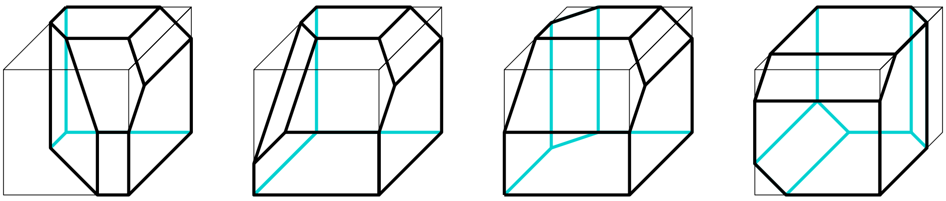


FIGURE 1.3: More 3-associahedra inscribed in a 3-cube. The 3-associahedron is the only simple 3-polytope with nine facets all of which are quadrilaterals or pentagons.

1.2 Some preliminaries

We start by recalling the definition of an n-dimensional associahedron in terms of polyhedral subdivisions of an $(n+3)$-gon.

Definition 1.1. Let P_{n+3} be a convex $(n+3)$-gon, whose vertices we label cyclically with the symbols 1 through $n+3$.

An *associahedron* Ass_n is an n-dimensional simple polytope whose poset of non-empty faces is isomorphic to the poset of non-crossing sets of diagonals of P_{n+3}, ordered by reverse inclusion.

Equivalently, the poset of non-empty faces of the associahedron is isomorphic to the set of polyhedral subdivisions of P_{n+3} (without new vertices), ordered by coarsening. The minimal elements (vertices of the associahedron) correspond to the *triangulations* of P_{n+3}.

For example, for the associahedron of dimension two we look at which diagonals of the pentagon cross each other. There are five diagonals, with five of the $\binom{5}{2}$ pairs of them crossing and the other five non-crossing. Thus, the poset of non-empty faces of the two-dimensional associahedron is isomorphic to the Hasse diagram of Figure 1.4, in which the five bottom elements correspond to the five triangulations of the pentagon and the top element corresponds to the "trivial" subdivision into a single cell, the pentagon itself.

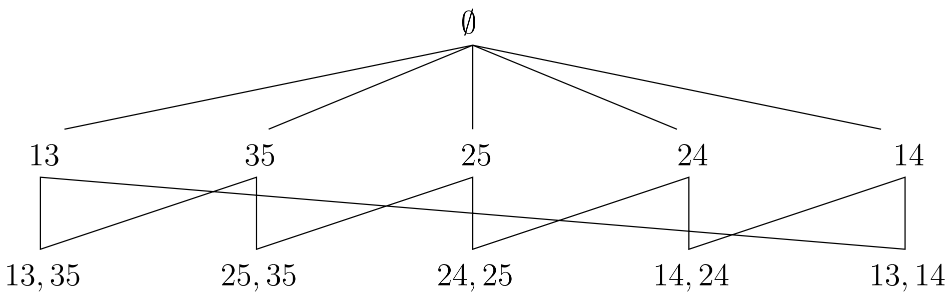


FIGURE 1.4: The Hasse diagram of the 2-dimensional associahedron.

This is also the Hasse diagram of the poset of non-empty faces of a pentagon, so the 2-dimensional associahedron is a pentagon. Figure 1.5 shows the associahedra of dimensions 0, 1, and 2.

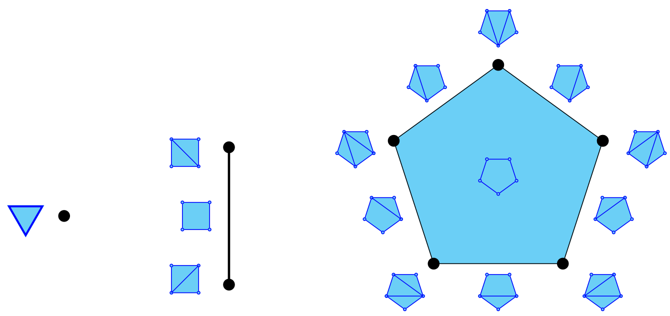


FIGURE 1.5: The associahedron Ass_n for $n = 0, 1$ and 2.

The goal of this chapter is to compare different types of constructions of the associahedron, saying which ones produce equivalent polytopes, in a suitable sense. The following notion reflects the fact that the main constructions that we are going to discuss produce associahedra whose normal vectors have small integer coordinates, usually 0 or ± 1. In these constructions the normal fan of the associahedron can be considered canonical, while there is still freedom in the right-hand sides of the inequalities. (See [85, Sec. 7.1] for a discussion of fans and of normal fans.) This leads us to use the following notion of equivalence.

Definition 1.2. Two pointed complete fans in real vector spaces V and V' of the same dimension are *linearly isomorphic* if there is a linear isomorphism $V \to V'$ sending each cone of one to a cone of the other. Two polytopes P and P' are *normally isomorphic* if they have linearly isomorphic normal fans.

Normal isomorphism is weaker than the usual notion of *normal equivalence*, in which the two polytopes P and P' are assumed embedded *in the same space* and their normal fans are required to be exactly the same, not only linearly isomorphic.

The following lemma is very useful in order to prove (or disprove) that two associahedra are normally isomorphic. It implies that all linear (or combinatorial, for that matter) isomorphisms between associahedra come from isomorphisms between the $(n+3)$-gons defining them.

Lemma 1.3. *The automorphism group of the face lattice of the associahedron* Ass_n *is the dihedral group of the* $(n+3)$*-gon: All automorphisms are induced by symmetries of the* $(n+3)$*-gon.*

Proof. Suppose φ is an automorphism of the face lattice of the associahedron Ass_n, and let D be the set of all diagonals of a convex $(n+3)$-gon. φ induces a natural bijection

$$\widetilde{\varphi} : D \longrightarrow D$$

such that for any two diagonals $\delta, \delta' \in D$ we have:

$$\delta \text{ cross } \delta' \quad \Longleftrightarrow \quad \widetilde{\varphi}(\delta) \text{ cross } \widetilde{\varphi}(\delta').$$

For a diagonal $\delta \in D$ denote by $\mathrm{length}(\delta)$ the minimum between the lengths of the two paths that connect the two end points of δ on the boundary of the $(n+3)$-gon. Then

$$\mathrm{length}(\delta) = \mathrm{length}(\widetilde{\varphi}(\delta)).$$

The reason is that the length of δ is determined by the number of diagonals that cross δ, and this property is invariant under the map $\widetilde{\varphi}$.

Let δ_0 be a diagonal of length 2, and $\widetilde{\varphi}(\delta_0)$ its image under $\widetilde{\varphi}$. The diagonals that cross δ_0 have a common intersection vertex v_0; from this vertex we label these diagonals

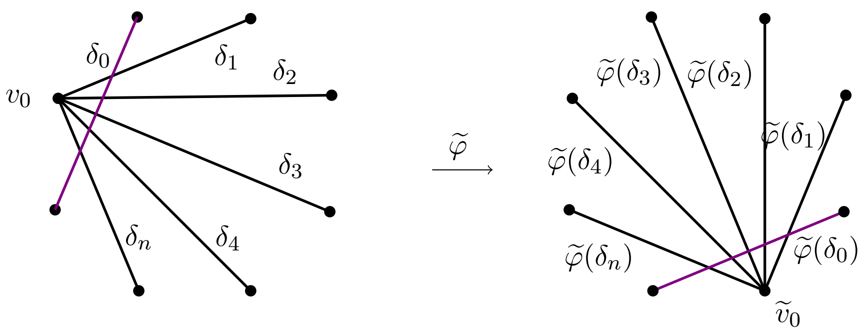


FIGURE 1.6: The situation in the proof of Lemma 1.3.

in clockwise direction by $\delta_1, \ldots, \delta_n$. Similarly, the diagonals that cross $\widetilde{\varphi}(\delta_0)$ have a common intersection vertex $\widetilde{v}_0$, and they are labeled by $\widetilde{\varphi}(\delta_1), \ldots, \widetilde{\varphi}(\delta_n)$. For any non-empty interval $I \subset [n]$ there is an unique diagonal δ_I that intersects the diagonal δ_i if and only if $i \in I$. Applying the map $\widetilde{\varphi}$ we obtain diagonals $\widetilde{\varphi}(\delta_I)$ that intersect $\widetilde{\varphi}(\delta_i)$ if and only if $i \in I$. This task is possible only if the labelings $\widetilde{\varphi}(\delta_1), \ldots, \widetilde{\varphi}(\delta_n)$ appear in either clockwise or counterclockwise direction. From this, we deduce that $\widetilde{\varphi}$ restricted to $\{\delta_1, \ldots, \delta_n\}$ is equivalent to a reflection-rotation map. Moreover, this map coincides with $\widetilde{\varphi}$ for all other diagonals δ_I. $\square$

1.3 Three realizations of the associahedron

1.3.1 The Gelfand–Kapranov–Zelevinsky associahedron

The secondary polytope is an ingenious construction motivated by the theory of hypergeometric functions as developed by I.M. Gelfand, M. Kapranov and A. Zelevinsky [31]. In this section we recall the basic definitions and main results related to this topic, which yield in particular that the secondary polytope of any convex $(n+3)$-gon is an n-dimensional associahedron. For more detailed presentations we refer to [17, Sec. 5] and [85, Lect. 9]. All the subdivisions and triangulations of polytopes that appear in the following are understood to be without new vertices.

The secondary polytope construction

Definition 1.4 (GKZ vector/secondary polytope)**.** Let Q be a d-dimensional convex polytope with $n+d+1$ vertices. The *GKZ vector* $v(t) \in \mathbb{R}^{n+d+1}$ of a triangulation t of Q is

$$v(t) \;:=\; \sum_{i=1}^{n+d+1} \mathrm{vol}(\mathrm{star}_t(i))e_i \;=\; \sum_{i=1}^{n+d+1} \sum_{\sigma\in t\,:\, i\in\sigma} \mathrm{vol}(\sigma)e_i$$

The *secondary polytope* of Q is defined as

$$\Sigma(Q) \ := \ \mathrm{conv}\{v(t) : t \text{ is a triangulation of } Q\}.$$

Theorem 1.5 (Gelfand–Kapranov–Zelevinsky [32])**.** *Let Q be a d-dimensional convex polytope with $m = n+d+1$ vertices. Then the secondary polytope $\Sigma(Q)$ has the following properties:*

(i) *$\Sigma(Q)$ is an n-dimensional polytope.*
(ii) *The vertices of $\Sigma(Q)$ are in bijection with the regular triangulations of Q.*
(iii) *The faces of $\Sigma(Q)$ are in bijection with the regular subdivisions of Q.*
(iv) *The face lattice of $\Sigma(Q)$ is isomorphic to the lattice of regular subdivisions of Q, ordered by refinement.*

The associahedron as the secondary polytope of a convex $(n+3)$-gon

Definition 1.6. The *Gelfand–Kapranov–Zelevinsky associahedron* $\mathrm{GKZ}_n(P_{n+3}) \subset \mathbb{R}^{n+3}$ is defined as the (n-dimensional) secondary polytope of a convex $(n+3)$-gon $P_{n+3} \subset \mathbb{R}^2$:

$$\mathrm{GKZ}_n(P_{n+3}) := \Sigma(P_{n+3}).$$

Corollary 1.7 ([32])**.** *$\mathrm{GKZ}_n(P_{n+3})$ is an n-dimensional associahedron.*

There is one feature that distinguishes the associahedron as a secondary polytope from all the other constructions that we mention in this chapter: the absence of parallel facets. This property, in particular, will imply that the GKZ–associahedra are not normally isomorphic to the associahedra produced by the other constructions:

Theorem 1.8. *The Gelfand-Kapranov-Zelevinsky associahedron $\mathrm{GKZ}_n(P_{n+3})$ has no parallel facets for $n \geq 2$.*

Our proof is based on the understanding of the facet normals in secondary polytopes. Let Q be an arbitrary d-polytope with $n+d+1$ vertices $\{q_1, \ldots, q_{n+d+1}\}$, so that $\mathrm{GKZ}_n(Q) := \Sigma(Q)$ lives in $\mathbb{R}^{n+d+1}$, although it has dimension n. In the theory of secondary polytopes one thinks of each linear functional $\mathbb{R}^{n+d+1} \to \mathbb{R}$ as a function $\omega :$ vertices$(Q) \to \mathbb{R}$ assigning a value $\omega(q_i)$ to each vertex q_i. In turn, to each triangulation t of Q (with no additional vertices) and any such ω one associates the function $g_{\omega,t} : Q \to \mathbb{R}$ which takes the value $\omega(q_i)$ at each q_i and is affine linear on each simplex of t. That is, we use t to piecewise linearly interpolate a function whose values $(\omega(q_1), \ldots, \omega(q_n))$ we know on the vertices of Q. The main result we need is the following equality for every

ω and every triangulation t (see, e.g., [17, Thm. 5.2.16]):

$$\langle \omega, v(t) \rangle = (d+1) \int_Q g_{\omega,t}(x)dx.$$

In particular:

- If ω is affine-linear (that is, if the points $\{(q_1, \omega_1), \ldots, (q_{n+d+1}, \omega_{n+d+1})\} \subset \mathbb{R}^{n+d+1} \times \mathbb{R}$ lie in a hyperplane) then $\langle \omega, v(t) \rangle$ is the same for all t. Moreover, the converse is also true: The affine-linear ω's form the lineality space of the normal fan of $\mathrm{GKZ}_n(Q)$.
- An ω lies in the linear cone of the (inner) normal fan of $\mathrm{GKZ}_n(Q)$ corresponding to a certain triangulation t (that is, $\langle \omega, v(t) \rangle \leq \langle \omega, v(t') \rangle$ for every other triangulation t') if and only if the function $g_{\omega,t}$ is convex; that is to say, if its graph is a convex hypersurface.

Proof of Theorem 1.8. With the previous description in mind we can identify the facet normals of the secondary polytope of a polygon P_{n+3}. For this we use the correspondence:

$$\begin{array}{rcl} \text{vertices} & \longleftrightarrow & \text{triangulations of } P_{n+3} \\ \text{facets} & \longleftrightarrow & \text{diagonals of } P_{n+3} \end{array}$$

For a given diagonal δ of P_{n+3}, denote by F_δ the facet of $\mathrm{GKZ}_n(P_{n+3})$ corresponding to δ. The vector normal to F_δ is not unique, since adding to any vector normal to F_δ an affine-linear ω_0 we get another one. One natural choice is

$$\omega_\delta(q_i) := \mathrm{dist}(q_i, l_\delta),$$

where l_δ is the line containing δ and $\mathrm{dist}(\cdot,\cdot)$ is the Euclidean distance. Indeed, ω_δ lifts the vertices of P_{n+3} on the same side of δ to lie in a half-plane in $\mathbb{R}^3$, with both half-planes having δ as their common intersection. That is, $g_{\omega_\delta,t}$ is convex for every t that uses δ. But another choice of normal vector is better for our purposes: choose one side of l_δ to be called positive and take

$$\omega_\delta^+(q_i) := \begin{cases} \mathrm{dist}(q_i, l_\delta) & \text{if } q_i \in l_\delta^+ \\ 0 & \text{if } q_i \in l_\delta^- \end{cases}.$$

For the end-points of δ, which lie in both l_δ^+ and l_δ^-, there is no ambiguity since both definitions give the value 0. Again, ω_δ^+ is a normal vector to F_δ since it lifts points on either side of l_δ to lie in a plane.

We are now ready to prove the theorem. If two diagonals δ and δ' of P_{n+3} do not cross, then they can simultaneously be used in a triangulation. Hence, the corresponding facets

F_δ and $F_{\delta'}$ meet, and they cannot be parallel. So, assume in what follows that δ and δ' are two crossing diagonals. Let $\delta = pr$ and $\delta' = qs$, with $pqrs$ being cyclically ordered along P_{n+3}. Since $n \geq 2$ there is at least another vertex a in P_{n+3}. Without loss of generality suppose a lies between s and p. Now, we call negative the side of l_δ and the side of $l_{\delta'}$ containing a, and consider the normal vectors ω_δ^+ and $\omega_{\delta'}^+$ as defined above. They take the following values on the five points of interest:

$$\omega_\delta^+(a) = 0, \quad \omega_\delta^+(p) = 0, \quad \omega_\delta^+(q) > 0, \quad \omega_\delta^+(r) = 0, \quad \omega_\delta^+(s) = 0,$$
$$\omega_{\delta'}^+(a) = 0, \quad \omega_{\delta'}^+(p) = 0, \quad \omega_{\delta'}^+(q) = 0, \quad \omega_{\delta'}^+(r) > 0, \quad \omega_{\delta'}^+(s) = 0.$$

Suppose that F_δ and $F_{\delta'}$ were parallel. This would imply that δ and δ' are linearly dependent or, more precisely, that there is a linear combination of them that gives an affine-linear ω (in the lineality space of the normal fan). But any (non-trivial) linear combination ω of ω_δ^+ and $\omega_{\delta'}^+$ necessarily takes the following values on our five points, which implies that ω is not affine-linear:

$$\omega(a) = 0, \quad \omega(p) = 0, \quad \omega(q) \neq 0, \quad \omega(r) \neq 0, \quad \omega(s) = 0.$$

□

Remark 1.9. The secondary polytope can be defined for any set of points $\{q_1, \ldots, q_{n+3}\}$ in the plane, not necessarily the vertices of a convex polygon. In general this does not produce an associahedron, but there is a case in which it does: if the points are cyclically placed on the boundary of an m-gon with $m \leq n+3$ in such a way that no four of them lie on a boundary edge. By the arguments in the proof above, a necessary condition for the associahedron obtained to have parallel facets is that $m \leq 4$. For $m = 4$ we can obtain associahedra up to dimension 4 with exactly one pair of parallel facets (those corresponding to the main diagonals of the quadrilateral). For $m = 3$, we can obtain 2-dimensional associahedra with two pairs of parallel facets, and 3-dimensional associahedra with three pairs of parallel facets. The latter is obtained for six points $\{p, q, r, a, b, c\}$ with p, q and r being the vertices of a triangle and $a \in pq$, $b \in qr$ and $c \in ps$ intermediate points in the three sides. The associahedron obtained has the following three pairs of parallel facets:

$$F_{pq} || F_{ar}, \quad F_{qr} || F_{bs}, \quad F_{ps} || F_{cq}.$$

Remark 1.10. Rote, Santos and Streinu [66] introduce a *polytope of pseudo-triangulations* associated to each finite set A of m points (in general position) in the plane. This polytope lives in $\mathbb{R}^{2m}$ and has dimension $m + 3 + i$, where i is the number of points interior to $\mathrm{conv}(A)$. They show that for points in convex position their polytope is affinely isomorphic to the secondary polytope for the same point set. Their constructions uses rigidity theoretic ideas: the edge-direction joining two neighboring triangulations t and t' is the vector of velocities of the (unique, modulo translation and rotation) infinitesimal flex of the embedded graph of $t \cap t'$.

1.3.2 The Postnikov associahedron

We now review two further realizations of the associahedron: one by Postnikov [59] and one by Rote–Santos–Streinu [66] (different from the one in Remark 1.10). The main goal of this section is to prove that these two constructions produce affinely equivalent results. As special cases of these constructions one obtains, respectively, the realizations by Loday [49] and Buchstaber [12], which turn out to be affinely equivalent as well.

1.3.2.1 The Postnikov associahedron

Definition 1.11. For any vector $\mathbf{a} = \{\mathrm{a}_{ij} > 0 : 1 \leq i \leq j \leq n+1\}$ of positive parameters we define the *Postnikov associahedron* as the polytope

$$\mathrm{Post}_n(\mathbf{a}) := \sum_{1 \leq i \leq j \leq n+1} \mathrm{a}_{ij} \Delta_{[i,\ldots,j]},$$

where $\Delta_{[i,\ldots,j]}$ denotes the simplex $\mathrm{conv}\{e_i, e_{i+1}, \ldots, e_j\}$ in $\mathbb{R}^{n+1}$.

Proposition 1.12 (Postnikov [59, Sec. 8.2])**.** $\mathrm{Post}_n(\mathbf{a})$ *is an n-dimensional associahedron. In particular, for $a_{ij} \equiv 1$ this yields the realization of Loday [49].*

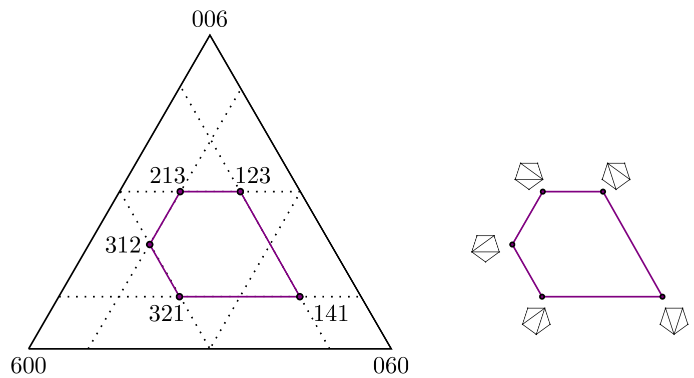


FIGURE 1.7: The Postnikov associahedron $\mathrm{Post}_n(\mathbf{1})$ with the coordinates of the vertices. This coincides with the realization of Loday.

In terms of inequalities the Postnikov associahedron is given as follows.

Lemma 1.13.

$$\mathrm{Post}_n(\mathbf{a}) = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{p<i<q} x_i \geq f_{p,q} \quad \text{for } 0 \leq p < q \leq n+2,$$
$$x_1 + \cdots + x_{n+1} = f_{0,n+2}\},$$

where $f_{p,q} = \sum_{p<i\leq j<q} \mathrm{a}_{i,j}$.

The facet of $\mathrm{Post}_n(\mathbf{a})$ determined by the hyperplane with right hand side parameter $f_{p,q}$ corresponds to the diagonal pq of an $(n+3)$-gon with vertices labeled in counterclockwise direction from 0 to $n+2$. In particular:

Theorem 1.14. $\mathrm{Post}_n(\mathbf{a})$ *has exactly* n *pairs of parallel facets. These correspond to the pairs of diagonals* $(\{0, i+1\}, \{i, n+2\})$ *for* $1 \leq i \leq n$*, as illustrated in* Figure 1.8.

Proof. Two hyperplanes of the form $\sum_{i\in S_1} x_i \geq c_1$ and $\sum_{i\in S_2} x_i \geq c_2$ for $S_1, S_2 \subseteq [n+1]$, intersected with an affine hyperplane $x_1 + \cdots + x_{n+1} = c$ are parallel if and only if $S_1 \cup S_2 = [n+1]$ and $S_1 \cap S_2 = \emptyset$. Therefore two diagonals pq and rs correspond to parallel facets if and only if $pq = \{0, i+1\}$ and $qr = \{i, n+2\}$. □

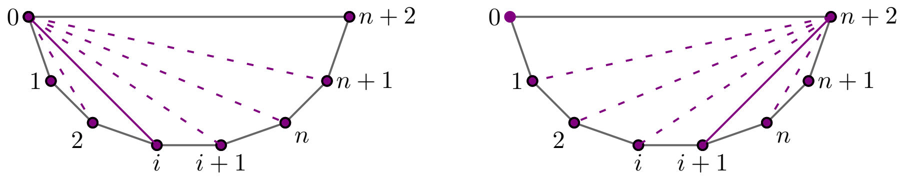


FIGURE 1.8: Diagonals of the $(n+3)$-gon that correspond to the pairs of parallel facets of both $\mathrm{Post}_n(\mathbf{a})$ and $\mathrm{RSS}_n(\mathbf{g})$.

1.3.2.2 The Rote–Santos–Streinu associahedron

By "generalizing" the construction of Remark 1.10 to sets of points along a line, Rote, Santos and Streinu [66] obtain a second realization of the associahedron.

Definition 1.15. The *Rote–Santos–Streinu associahedron* is the polytope

$$\mathrm{RSS}_n(\mathbf{g}) = \{(y_0, y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+2} : y_j - y_i \geq g_{i,j} \text{ for } j > i,\ y_0 = 0,\ y_{n+1} = g_{0,n+1}\},$$

where $\mathbf{g} = (g_{i,j})_{0\leq i<j\leq n+1}$ is any vector with real coordinates satisfying

$$g_{i,l} + g_{j,k} > g_{i,k} + g_{j,l} \quad \text{for all} \quad i < j \leq k < l,$$
$$g_{i,l} > g_{i,k} + g_{k,l} \quad \text{for all} \quad i < k < l.$$

Proposition 1.16 (Rote–Santos–Streinu [66, Sec. 5.3])**.** *If the vector* $\mathbf{g}$ *satisfies the previous inequalities then* $\mathrm{RSS}_n(\mathbf{g})$ *is an* n*-dimensional associahedron.*

A particular example of valid parameters $\mathbf{g}$ is given by $\mathbf{g_0}$: $g_{i,j} = i(i-j)$. In this case we get the realization of the associahedron introduced by Buchstaber in [12, Lect. II Sec. 5].

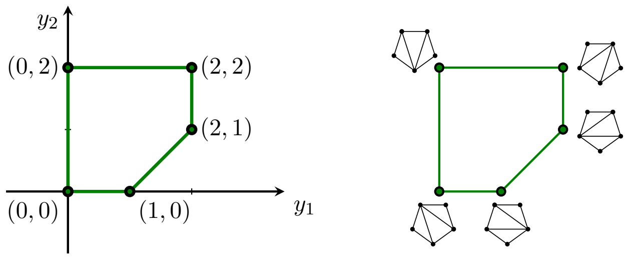


FIGURE 1.9: The Rote–Santos–Streinu associahedron $\mathrm{RSS}_2(\mathbf{g_0})$ with the coordinates of the vertices. This coincides with the realization of Buchstaber.

The facet of $\mathrm{RSS}_n(\mathbf{g})$ related to $y_j - y_i \geq g_{i,j}$ corresponds to the diagonal $\{i, j+1\}$ of an $(n+3)$-gon with vertices labeled in counterclockwise direction from 0 to $n+2$. One can also see that with this specified combinatorics of the facets, the conditions on the vector $\mathbf{g}$ are also necessary for the proposition to hold.

Theorem 1.17. $\mathrm{RSS}_n(\mathbf{g})$ *has exactly* n *pairs of parallel facets. They correspond to the pairs of diagonals* $(\{0, i+1\}, \{i, n+2\})$ *for* $1 \leq i \leq n$*, as illustrated in* Figure 1.8*.*

Rote, Santos and Streinu stated in [66, Sec. 5.3] that $\mathrm{RSS}_n(\mathbf{g})$ is not affinely equivalent to neither the associahedron as a secondary polytope nor the associahedron from the cluster complex of type A. Next we prove that $\mathrm{RSS}_n(\mathbf{g})$ is affinely isomorphic to $\mathrm{Post}_n(\mathbf{a})$. Furthermore, we prove, in Corollary 1.33 and Theorem 1.47, that these two polytopes are not normally isomorphic to the associahedron as a secondary polytope or the associahedron from the cluster complex of type A.

1.3.2.3 Affine equivalence

Theorem 1.18. *Let* φ *be the affine transformation*

$$\begin{array}{cccc} \varphi : & \mathbb{R}^{n+1} & \to & \mathbb{R}^n \\ & (x_1, \ldots, x_{n+1}) & \to & (y_1, \ldots, y_n) \end{array}$$

defined by $y_k = \sum_{i=1}^{k}(x_i - i)$*. Then* φ *maps* $\mathrm{Post}_n(\mathbf{a})$ *bijectively to* $\mathrm{RSS}_n(\mathbf{g})$*, for* $\mathbf{g}$ *given by* $g_{i,j} - \frac{(i+j+1)(j-i)}{2} = f_{i,j+1}(\mathbf{a})$*. In particular,* φ *maps the Loday associahedron* $\mathrm{Post}_n(\mathbf{1})$ *to the Buchstaber associahedron* $\mathrm{RSS}_n(\mathbf{g_0})$*.*

Proof.

$$\begin{aligned} y_j - y_i &\geq g_{i,j} \\ (x_{i+1} + \cdots + x_j) + ((i+1) + \cdots + j) &\geq g_{i,j} \\ x_{i+1} + \cdots + x_j &\geq g_{i,j} - \tfrac{(i+j+1)(j-i)}{2}. \end{aligned}$$

□

Corollary 1.19 (Minkowski sum decomposition of $\mathrm{RSS}_n(\mathbf{g})$)**.** *Every Rote–Santos–Streinu associahedron can be written as*

$$\mathrm{RSS}_n(\mathbf{g}) = \sum_{1 \leq i \leq j \leq n} b_{i,j} \widetilde{\Delta}_{i,j},$$

for certain $(b_{i,j})$ *with* $b_{i,j} > 0$ *whenever* $i < j$*, and* $b_{i,i}$ *possibly negative. Here* $\widetilde{\Delta}_{i,j} = \mathrm{conv}\{u_i, u_{i+1}, \ldots, u_j\}$ *and* $u_i = (0, \ldots, 0, 1, \ldots, 1) \in \mathbb{R}^n$ *is a* $0/1$*-vector with* i *zeros.*

1.3.3 The Chapoton–Fomin–Zelevinsky associahedron

1.3.3.1 The associahedron associated to a cluster complex

Cluster complexes are combinatorial objects that arose in the theory of cluster algebras [25] [26] initiated by Fomin and Zelevinsky. They correspond to the normal fans of polytopes known as generalized associahedra because the particular case of type A_n yields to the classical associahedron. This polytope was constructed by Chapoton, Fomin and Zelevinsky in [16]. We refer to [24], [23] and [16] for more detailed presentations.

1.3.3.2 The cluster complex of type A_n

The *root system of type* A_n is the set $\Phi := \Phi(A_n) = \{e_i - e_j,\ 1 \leq i \neq j \leq n+1\} \subset \mathbb{R}^{n+1}$. The *simple roots* of type A_n are the elements of the set $\Pi = \{\alpha_i = e_i - e_{i+1}, i \in [n]\}$, the set of *positive roots* is $\Phi^+ = \{e_i - e_j : i < j\}$, and the set of *almost positive roots* is $\Phi_{\geq -1} := \Phi^+ \cup -\Pi$.

There is a natural correspondence between the set $\Phi_{\geq -1}$ and the diagonals of the $(n+3)$-gon P_{n+3}: We identify the negative simple roots $-\alpha_i$ with the diagonals on the *snake* of P_{n+3} illustrated in Figure 1.10.

Each positive root is a consecutive sum

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \qquad 1 \leq i \leq j \leq n,$$

and thus is identified with the unique diagonal of P_{n+3} crossing the (consecutive) diagonals that correspond to $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$.

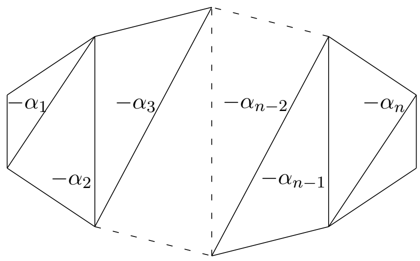


FIGURE 1.10: Snake and negative roots of type A_n.

Definition 1.20 (Cluster complex of type A_n)**.** Two roots α and β in $\Phi_{\geq -1}$ are *compatible* if their corresponding diagonals do not cross. The *cluster complex* $\Delta(\Phi)$ of type A_n is the clique complex of the compatibility relation on $\Phi_{\geq -1}$, i.e., the complex whose simplices correspond to the sets of almost positive roots that are pairwise compatible. Maximal simplices of $\Delta(\Phi)$ are called *clusters*.

In this case, the cluster complex satisfies the following correspondence, which is dual to the complex of the associahedron:

vertices	$\longleftrightarrow$	diagonals of a convex $(n+3)$-gon
simplices	$\longleftrightarrow$	polyhedral subdivisions of the $(n+3)$-gon (viewed as collections of non-crossing diagonals)
maximal simplices	$\longleftrightarrow$	triangulations of the $(n+3)$-gon (viewed as collections of n non-crossing diagonals)

Theorem 1.21 ([24, Thms. 1.8, 1.10])**.** *The simplicial cones $\mathbb{R}_{\geq 0}C$ generated by all clusters C of type A_n form a complete simplicial fan in the ambient space*

$$\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 + \cdots + x_{n+1} = 0\}.$$

Theorem 1.22 ([16, Thm. 1.4])**.** *The simplicial fan in* Theorem 1.21 *is the normal fan of a simple n-dimensional polytope P.*

Theorem 1.21 is the case of type A_n of [24, Thm. 1.10]. It allows us to think of the cluster complex as the complex of a complete simplicial fan. Theorem 1.22 was conjectured by Fomin and Zelevinsky [24, Conj. 1.12] and subsequently proved by Chapoton, Fomin, and Zelevinsky [16]. For an explicit description by inequalities see [16, Cor. 1.9]. These two theorems are special cases of Theorems 1.37 and 1.38, proved in Section 1.5.

1.3.3.3 The Chapoton–Fomin–Zelevinsky associahedron $\mathrm{CFZ}_n(A_n)$

Definition 1.23. The *Chapoton–Fomin–Zelevinsky associahedron* $\mathrm{CFZ}_n(A_n)$ is any polytope whose normal fan is the fan with maximal cones $\mathbb{R}_{\geq 0}C$ generated by all clusters C of type A_n.

Proposition 1.24 ([16, 24])**.** *$\mathrm{CFZ}_n(A_n)$ is an n-dimensional associahedron.*

A polytopal realization of the associahedron $\mathrm{CFZ}_2(A_2)$ is illustrated in Figure 1.11; note how the facet normals correspond to the almost positive roots of A_2.

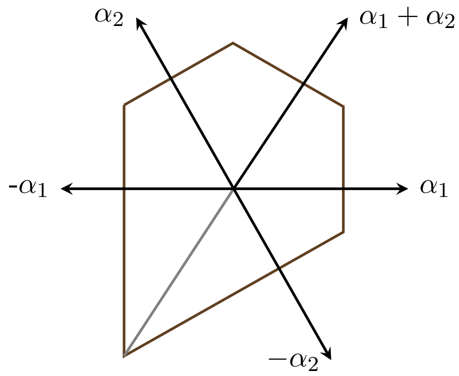


FIGURE 1.11: The complete simplicial fan of the cluster complex of type A_2 and an associahedron $\mathrm{CFZ}_2(A_2)$.

Theorem 1.25. *$\mathrm{CFZ}_n(A_n)$ has exactly n pairs of parallel facets. These correspond to the pairs of roots $\{\alpha_i, -\alpha_i\}$, for $i = 1, \ldots, n$, or, equivalently, to the pairs of diagonals $\{\alpha_i, -\alpha_i\}$ as indicated in* Figure 1.12. ☐

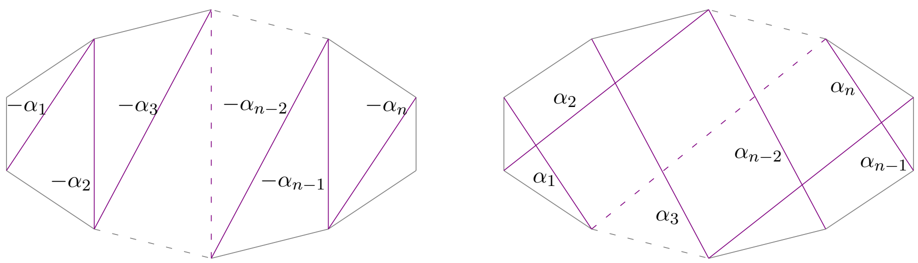


FIGURE 1.12: The diagonals of the $(n+3)$-gon that correspond to the pairs of parallel facets of $\mathrm{CFZ}_n(A_n)$.

1.4 Exponentially many realizations, by Hohlweg–Lange

1.4.1 The Hohlweg–Lange construction

In this section we give a short description of the first, "type I", exponential family of realizations of the associahedron, as obtained by Hohlweg and Lange in [35]. We prove that the number of normally non-isomorphic realizations obtained this way is equal to the number of sequences $\{+,-\}^{n-1}$ modulo reflection and reversal. This number is equal to $2^{n-3} + 2^{\lfloor \frac{n-3}{2} \rfloor}$ for $n \geq 3$ (see [74, Sequence A005418]).

Let $\sigma \in \{+,-\}^{n-1}$ be a sequence of signs on the edges of an horizontal path on n nodes. We identify $n+3$ vertices $\{0, 1, \ldots, n+1, n+2\}$ with the signs of the sequence $\widetilde{\sigma} = \{+,-,\sigma,-,+\}$, and place them in convex position from left to right so that all positive vertices are above the horizontal path, and all negative vertices are below it. These vertices form a convex $(n+3)$-gon that we call $P_{n+3}(\sigma)$. Figure 1.13 illustrates the example $P_7(\{+,-,+\})$, where $n = 4$.

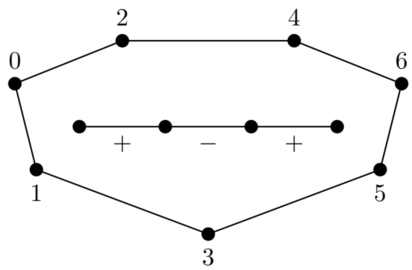


FIGURE 1.13: $P_7(\{+,-,+\})$.

Definition 1.26. For a diagonal ij $(i < j)$ of $P_{n+3}(\sigma)$, we denote by $R_{ij}(\sigma)$ the set of vertices strictly below it. We define the set $S_{ij}(\sigma)$ as the result of replacing 0 by i in $R_{ij}(\sigma)$ if $0 \in R_{ij}(\sigma)$, and replacing $n+2$ by j if $n+2 \in R_{ij}(\sigma)$.

The *Hohlweg–Lange associahedron* $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ is the polytope

$$\mathrm{Ass}_n^{\mathrm{I}}(\sigma) = \Big\{(x_1, \ldots x_{n+1}) \in \mathbb{R}^{n+1} : \quad \sum_{i \in S_\delta(\sigma)} x_i \geq \tfrac{1}{2}|S_\delta(\sigma)|(|S_\delta(\sigma)|+1) \text{ for all diagonals } \delta, \quad x_1 + \cdots + x_{n+1} = \tfrac{(n+1)(n+2)}{2}\Big\}.$$

Remark 1.27. If in $\widetilde{\sigma} = \{+,-,\sigma,-,+\}$ we interchange the first two signs and/or the last two signs, the sets $S_\delta(\sigma)$ do not change and the construction will produce the same associahedron $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$.

Proposition 1.28 ([35, Thm. 1.1])**.** $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ *is an n-dimensional associahedron.*

Proposition 1.29 ([35, Remarks 1.2 and 4.3])**.** $\mathrm{Ass}_n^{\mathrm{I}}(\{-,-,\ldots,-\})$ *produces the Postnikov (Loday) associahedron* $\mathrm{Post}_n(\mathbf{1})$*, and* $\mathrm{Ass}_n^{\mathrm{I}}(\{+,-,+,-,\ldots\})$ *is normally isomorphic to the Chapoton–Fomin–Zelevinsky associahedron* $\mathrm{CFZ}_n(A_n)$*.*

Proof. For the first part we note that for $\sigma = \{-,-,\cdots-\}$, the set $S_{p,q}(\sigma)$ of a diagonal pq is given by $S_{p,q} = \{i : p < i < q\}$, and that the description of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ coincides with that of $\mathrm{Post}_n(\mathbf{a})$ in Lemma 1.13 for $\mathbf{a} = 1$. For the second part let $\sigma = \{+,-,+,-,\ldots\}$. We write S_δ instead of $S_\delta(\sigma)$ for simplicity, and denote by $I_S \in \mathbb{R}^{n+1}$ the 0/1 vector with ones in the positions of a set $S \subseteq [n+1]$. The snake triangulation is given by the set of diagonals of the form $i, i+1$, for $1 \leq i \leq n$ (in the case where $n, n+1$ is not a diagonal we interchange vertices $n+1$ and $n+2$; this doesn't change the associahedron we get, see Remark 1.27). We denote by $-\alpha_i = I_{S_{i,i+1}}$ the normal vector associated to the diagonal $i, i+1$, and by $n_{i,j} = I_{S_{i-1,j+2}}$ $(i \leq j)$ the normal vector associated to the diagonal crossing $\{-\alpha_i, -\alpha_2, \dots, -\alpha_j\}$. We need to prove that

$$n_{i,j} \equiv \alpha_i + \alpha_{i+1} + \cdots + \alpha_j \mod (1, \dots, 1).$$

The reason is that our polytope lies in an affine hyperplane orthogonal to the vector $(1, \dots, 1)$, and so we must consider the normal vectors modulo $(1, \dots, 1)$. To this end, note that

$$n_{i,i} = \alpha_i + (1, \dots, 1)$$

and

$$n_{i,j+1} = \begin{cases} n_{i,j} + (1, \dots, 1) + \alpha_{j+1} & \text{if } j \text{ is odd,} \\ n_{i,j} + \alpha_{j+1} & \text{if } j \text{ is even.} \end{cases}$$

□

Remark 1.30. The Postnikov associahedron was defined as a Minkowski sum of certain faces Δ_S of the standard simplex $\Delta_{[n+1]}$. The question arises whether such Minkowski sum descriptions exist for $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ in general. A partial answer is as follows. Postnikov introduced the family of generalized permutahedra in [59]. A *generalized permutahedron* is a polytope with facet normals contained in those of the standard permutahedron such that the collection of right hand side parameters of the defining inequalities belongs to the deformation cone of the standard permutahedron (compare with Postnikov et al. [58]). This includes all the Minkowski sums $\sum_{S \subseteq [n+1]} a_S \Delta_S$ for which the coefficients a_S are non-negative. Ardila et al. [1] have shown that every generalized permutahedron admits a (unique) expression as a *Minkowski sum and difference* of faces of the standard simplex. These decompositions, for the case of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$, are studied by Lange in [47]. A different decomposition arises from the work of Pilaud and Santos [56], who show that the associahedra $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ are the "brick polytopes" of certain sorting networks. As such, they admit a decomposition as the Minkowski sum of the $\binom{n}{2}$ polytopes associated to the individual "bricks". However, these summands need not be simplices.

1.4.2 Parallel facets

Theorem 1.31. $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ *has exactly* n *pairs of parallel facets. They correspond to the diagonals of the quadrilaterals with vertices* $\{i, j, j+1, k\}$ *for* $j = 1, \ldots, n$*, where*

$$i = \max\{0 \leq r < j : \mathrm{sign(r)} \cdot \mathrm{sign(j)} = -\}$$

$$k = \min\{j+1 < r \leq n+2 : \mathrm{sign(r)} \cdot \mathrm{sign(j+1)} = -\}$$

Proof. Two diagonals δ and δ' correspond to two parallel facets of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ if and only if the sets S_δ and $S_{\delta'}$ satisfy $S_\delta \cup S_{\delta'} = [n+1]$ and $S_\delta \cap S_{\delta'} = \emptyset$. These two properties hold if and only if δ and δ' are the diagonals of the quadrilateral $\{i, j, j+1, k\}$ for $j = 1, \ldots, n$, and i and k satisfying the conditions of the theorem. □

Associated to a sequence σ we define two operations, reflection and reversal. The *reflection* of σ is the sequence $-\sigma$, and the *reversal* σ^t is the result of reversing the order of the signs in σ.

Theorem 1.32. *Let* $\sigma_1, \sigma_2 \in \{+,-\}^{n-1}$. *Then the two realizations* $\mathrm{Ass}_n^{\mathrm{I}}(\sigma_1)$ *and* $\mathrm{Ass}_n^{\mathrm{I}}(\sigma_2)$ *are normally isomorphic if and only if* σ_2 *can be obtained from* σ_1 *by reflections and reversals.*

Proof. Suppose there is a linear isomorphism between the normal fans of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma_1)$ and $\mathrm{Ass}_n^{\mathrm{I}}(\sigma_2)$. It induces an automorphism of the face lattice of the associahedron that, by Lemma 1.3, corresponds to a certain reflection-rotation of the polygon. We denote this reflection-rotation by $\varphi : P_{n+3}(\sigma_1) \to P_{n+3}(\sigma_2)$. Any linear isomorphism of the normal fans preserves the property of a pair of facets being parallel, so φ maps the "parallel" pairs of diagonals of $P_{n+3}(\sigma_1)$, to the "parallel" pairs of diagonals of $P_{n+3}(\sigma_2)$. Furthermore, for both realizations there are exactly four diagonals that cross at least one diagonal of every parallel pair; they are $\{0, n+1\}, \{0, n+2\}, \{1, n+1\}$ and $\{1, n+2\}$. The set of these four diagonals is also preserved under φ. This is possible only if φ is a reflection-rotation that corresponds to a composition of reflections and reversals of the sequence $\widetilde{\sigma_1} = \{+, -, \sigma_1, -, +\}$.

It remains to prove that $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ is normally-isomorphic to both $\mathrm{Ass}_n^{\mathrm{I}}(-\sigma)$ and $\mathrm{Ass}_n^{\mathrm{I}}(\sigma^t)$. The isomorphism between the normal fans of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ and $\mathrm{Ass}_n^{\mathrm{I}}(-\sigma)$ is given by multiplication by -1, since $S_\delta(-\sigma) = [n] - S_\delta(\sigma)$. The isomorphism between the normal fans of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ and $\mathrm{Ass}_n^{\mathrm{I}}(\sigma^t)$ is given by the permutation of coordinates $\tau(i) = n+1-i$, as $S_\delta(\sigma^t) = \tau(S_\delta(\sigma))$. □

Corollary 1.33. *The Postnikov associahedron is not normally isomorphic to the Chapoton–Fomin–Zelevinsky associahedron.*

Proof. The Postnikov associahedron is produced by the sequence $\sigma_1 = \{-, -, \dots, -\}$, and the Chapoton–Fomin–Zelevinsky associahedron is normally isomorphic to the one produced by the sequence $\sigma_2 = \{+, -, +, -, \dots\}$. The two sequences are not equivalent under reflections and reversals. □

1.4.3 Facet vectors

We now show that $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ can (modulo normal isomorphism) be embedded in $\mathbb{R}^n$ so that its facet normals are a subset of $\{0, -1, +1\}^n$ and contain the n standard basis vectors and their negatives among them. That is, it can be obtained from a cube by cutting certain faces, as in Figures 1.2 and 1.3.

Obviously, the basis vectors and their negatives will correspond to the n pairs of parallel facets that we identified in Theorem 1.31. Each such pair consists of a diagonal with positive slope and one with negative slope. We choose as "positive basis vector" the one with positive slope, which can be characterized as follows:

Lemma 1.34. *Let* $\{i, j, j+1, k\}$ *for* $j = 1, \dots, n$ *be as in* Theorem 1.31*. Let*

$$\begin{aligned} a &:= \max\{0 \le r \le j : \mathrm{sign(r)} = -\}, \\ b &:= \min\{j+1 \le r \le n+2 : \mathrm{sign(r)} = +\}. \end{aligned}$$

Then ab *is one of the diagonals of the quadrilateral with vertices* $\{i, j, j+1, k\}$ *and it has positive slope.*

Proof. By construction, $\{i, j, j+1, k\}$ has two positive points and two negative points (i and j have opposite sign, as have $j+1$ and k). Our definition of a and b is equivalent to: a is the negative point in $\{i, j\}$ and b is the positive point in $\{j+1, k\}$. □

As customary, we call *characteristic vector* of a set $S \subset [n+1]$ the vector in $\{0, 1\}^{n+1}$ with 1's in the coordinates of the elements of S. We denote it e_S. In particular, the i-th standard basis vector is $e_i = e_{\{i\}}$.

For each $j = 1, \dots, n$, let $X_j = e_{S_{ab}(\sigma)}$, where a and b are as in Lemma 1.34 and $S_{ab}(\sigma)$ is from Definition 1.26. Then X_j is normal to the facet of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ corresponding to the diagonal ab, one of the facets in the j-th parallel pair. By convention, let $X_{n+1} = e_\emptyset = (0, \dots, 0)$ and $X_0 = e_{[n+1]} = (1, \dots, 1)$.

Theorem 1.35. *For every $S \subset [n+1]$, the characteristic vector of S is a linear combination of $\{X_0, \dots, X_{n+1}\}$ with coefficients in $\{0, +1, -1\}$.*

Proof. Since

$$e_S = \sum_{j \in S} e_j,$$

the statement follows from the formula

$$e_j = X_{j-1} - X_j, \qquad \forall j \in [n],$$

which we prove distinguishing the case of j being positive or negative (the cases $j = 1$ and $j = n+1$ need separate treatment, but the formula holds for them too). Let a and b be as in Lemma 1.34 and let a' and b' be the same, but computed for $j-1$ instead of j. That is, let X_{j-1} be the characteristic vector of $S_{a'b'}$. If j is positive, then $a = a'$, $b' = j$ and b is the next positive point after j. If j is negative, then $b = b'$, $a = j$ and a' is the previous negative point before j. $\square$

Definition 1.26 says that the characteristic vector of $S_\delta(\sigma)$ is a normal vector to the facet of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ corresponding to a certain diagonal δ. Since $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ is not full-dimensional, the normal to each facet is not unique. Others are obtained adding multiples of $e_{[n+1]} = (1, \dots, 1)$ to it. Put differently, the normal fan of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ lives naturally in $(\mathbb{R}^{n+1})^* / \langle X_0 \rangle$. For the basis $\{X_1, \dots, X_n\}$ in this space, Theorem 1.35 yields the following.

Corollary 1.36. *With respect to the basis $\{X_1, \dots, X_n\}$, the normal vectors of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ are all in $\{0, +1, -1\}^n$ and include the $2n$ vectors $\{\pm X_1, \dots, \pm X_n\}$.*

1.5 Catalan many realizations, by Santos

In this section we describe a generalization of the Chapoton–Fomin–Zelevinsky construction of the associahedron (Section 1.3.3), originally presented at a conference in 2004 [69]. We prove that the number of normally non-isomorphic realizations obtained this way, our "type II exponential family", is equal to the number of triangulations of an $(n+3)$-gon modulo reflections and rotations. Interest in this number goes back to Motzkin (1948) [51]. An explicit formula for it is

$$\tfrac{1}{2(n+3)} C_{n+1} + \tfrac{1}{4} C_{(n+1)/2} + \tfrac{1}{2} C_{\lfloor (n+1)/2 \rfloor} + \tfrac{1}{3} C_{n/3},$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n \in \mathbb{Z}$ and $C_n = 0$ otherwise [74, Sequence A000207].

Let $\alpha_1, \ldots, \alpha_n$ denote a linear basis of an n-dimensional real vector space $V \cong \mathbb{R}^n$, and let T_0 be a certain triangulation of the $(n+3)$-gon, fixed once and for all throughout the construction. We call T_0 the *seed triangulation*. The CFZ associahedron will arise as the special case where $V = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum x_i = 0\}$, $\alpha_i = e_i - e_{i+1}$, and T_0 is the *snake triangulation* of Figure 1.10.

Let $\{\delta_1, \ldots, \delta_n\}$ denote the n diagonals present in the seed triangulation T_0. To each diagonal pq out of the $\frac{n(n+3)}{2}$ possible diagonals of the $(n+3)$-gon we associate a vector v_{pq} as follows:

- If $pq = \delta_i$ for some i (that is, if pq is used in T_0) then let $v_{pq} = -\alpha_i$.
- If $pq \notin T_0$ then let
$$v_{pq} := \sum_{pq \text{ crosses } \delta_i} \alpha_i.$$

As a running example, consider the triangulation $\{123, 345, 156, 135\}$ of a hexagon with its vertices labelled cyclically. Let $\delta_1 = 13$, $\delta_2 = 35$ and $\delta_3 = 15$. Written with respect to the basis $\{\alpha_1, \alpha_2, \alpha_3\}$ the nine vectors v_{pq} that we get are as follows (see Figure 1.14):

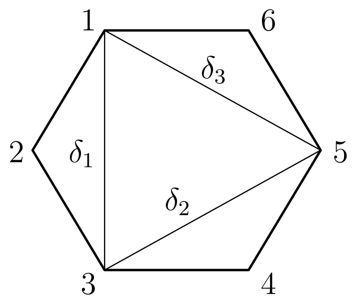


FIGURE 1.14: A seed triangulation for Santos' construction.

$$\begin{array}{lll}
v_{13} = -\alpha_1 = (-1,0,0), & v_{35} = -\alpha_2 = (0,-1,0), & v_{15} = -\alpha_3 = (0,0,-1), \\
v_{25} = \alpha_1 = (1,0,0), & v_{14} = \alpha_2 = (0,1,0), & v_{36} = \alpha_3 = (0,0,1), \\
v_{46} = \alpha_2 + \alpha_3 = (0,1,1), & v_{26} = \alpha_1 + \alpha_3 = (1,0,1), & v_{24} = \alpha_1 + \alpha_2 = (1,1,0).
\end{array}$$

With a slight abuse of notation, for each subset of diagonals of the polygon we denote with the same symbol the set of diagonals and the set of vectors associated with them. For example, $\mathbb{R}_{\geq 0} T_0 = \mathbb{R}_{\geq 0}\{-\alpha_1, \ldots, -\alpha_n\}$ is the negative orthant in V (with respect to the basis $[\alpha_i]_i$). More generally, for each triangulation T of the $(n+3)$-gon consider the cone $\mathbb{R}_{\geq 0} T$. We claim the following generalizations of Theorems 1.21 and 1.22, and Proposition 1.24:

Theorem 1.37. *The simplicial cones $\mathbb{R}_{\geq 0} T$ generated by all triangulations T of the $(n+3)$-gon form a complete simplicial fan $\mathcal{F}_{T_0}$ in the ambient space V.*

Theorem 1.38. *This fan $\mathcal{F}_{T_0}$ is the normal fan of an n-dimensional associahedron.*

1.5.1 Proof of Theorem 1.37

The statement follows from the following two claims:

(1) $\mathbb{R}_{\geq 0}T_0$ is a simplicial cone and is the only cone in $\mathcal{F}_{T_0}$ that intersects (the interior of) the negative orthant.

(2) If T_1 and T_2 are two triangulations that differ by a flip, let $v_1 \in T_1$ and $v_2 \in T_2$ be the diagonals removed and inserted by the flip. That is, $T_1 \setminus T_2 = \{v_1\}$ and $T_2 \setminus T_1 = \{v_2\}$. Then there is a linear dependence in $T_1 \cup T_2$ which has coefficients of the same sign (and different from zero) in the elements v_1 and v_2.

The first assertion is obvious, and the second one is Lemma 1.39 below. Before proving it let us argue why these two assertions imply Theorem 1.37. Suppose that we have two triangulations T_1 and T_2 related by a flip as in the second assertion, and suppose that we already know that one of them, say T_1, spans a full-dimensional cone (that is, we know that T_1 considered as a set of vectors is independent). Then assertion (2) implies that T_2 spans a full-dimensonal cone as well and that $\mathbb{R}_{\geq 0}T_1$ and $\mathbb{R}_{\geq 0}T_2$ lie in opposite sides of their common facet $\mathbb{R}_{\geq 0}(T_1 \cap T_2)$. This, together with the fact that there is some part of V covered by exactly one cone (which is why we need assertion (1)) implies that we have a complete fan. (See, for example, [17, Cor. 4.5.20], where assertion (2) is a special case of "property (ICoP)" and assertion (1) a special case of "property (IPP)".)

Lemma 1.39. *Let T_1 and T_2 be two triangulations that differ by a flip, and let v_1 and v_2 be the diagonals removed and inserted by the flip from T_1 to T_2, respectively (that is, $T_1 \setminus T_2 = \{v_1\}$ and $T_2 \setminus T_1 = \{v_2\}$). Then there is a linear dependence in $T_1 \cup T_2$ which has coefficients of the same sign in the elements v_1 and v_2.*

Proof. Let p, q, r and s be the four points involved by the two diagonals v_1 and v_2, in cyclic order. That is, the diagonals removed and inserted are pr and qs. We claim that one (and exactly one) of the following things occurs (see Figure 1.15):

(a) There is a diagonal in the seed triangulation T_0 that crosses two opposite edges of the quadrilateral $pqrs$.

(b) One of pr and qs is used in the seed triangulation T_0.

(c) There is a triangle abc in T_0 with a vertex in $pqrs$ and the opposite edge crossing two sides of $pqrs$ (that is, without loss of generality $p = a$ and bc crosses both qr and rs).

(d) There is a triangle abc in T_0 with an edge in common with $pqrs$ and with the other two edges of the triangle crossing the opposite edge of the quadrilateral (that is, without loss of generality, $p = a$, $q = b$ and rs crosses both ac and bc).

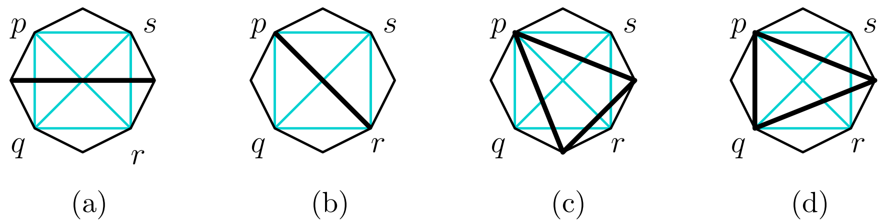


FIGURE 1.15: The four cases in the proof of Lemma 1.39.

To prove that one of the four things occurs we argue as follows. It is well-known that in any triangulation of a k-gon one can "contract a boundary edge" to get a triangulation of a $(k-1)$-gon. Doing that in all the boundary edges of the seed triangulation T_0 except those incident to either p, q, r or s we get a triangulation $\widetilde{T_0}$ of a polygon $\widetilde{P}$ with at most eight vertices: the four vertices p, q, r and s and at most one extra vertex between each two of them. We embed $\widetilde{P}$ having as vertex a subset of the vertices of a regular octagon, with $pqrs$ forming a square. We now look at the position of the center of the octagon $\widetilde{P}$ with respect to the triangulation $\widetilde{T_0}$: If it lies in the interior of an edge, then this edge is a diameter of the octagon and we are in cases (a) or (b). If it lies in the interior of a triangle of $\widetilde{T_0}$, then we are in cases (c) or (d). See Figure 1.15 again.

Now we show explicitly the linear dependences involved in $T_1 \cup T_2$ in each case.

(a) Suppose T_0 has a diagonal crossing pq and rs. Then

$$v_{pr} + v_{qs} = v_{pq} + v_{rs}, \tag{1.1}$$

because every diagonal of T_0 intersecting the two (respectively, one; respectively none) of pr and qs intersects also the two (respectively, one; respectively none) of pq and rs.

(b) If T_0 contains the diagonal pr, let a and b be vertices joined to pr in T_0, with a on the side of q and b on the side of s. We define the following vectors w_a and w_b:
   - w_a equals 0, v_{pq} or v_{qr} depending on whether a equals q, lies between p and q, or lies between q and r.
   - w_b equals 0, v_{ps} or v_{rs} depending on whether a equals s, lies between p and s, or lies between s and r.

   We claim that in the nine cases we have the equality

$$v_{pr} + v_{qs} = w_a + w_b. \tag{1.2}$$

This is so because $v_{pr} + v_{qs}$ now equals the sum of the α_i's corresponding to the diagonals of $T_0 \setminus \{pr\}$ crossing qs, and we have that:

- The diagonals of T_0 crossing qs in the q-side of pr are none, the same as those crossing pq, or the same as those crossing qr in the three cases of the definition of w_a, and
- The diagonals of T_0 crossing qs in the s-side of pr are none, the same as those crossing ps, or the same as those crossing rs in the three cases of the definition of w_b

(c) If T_0 contains a triangle pbc with bc crossing both qr and rs then we have the equality

$$2v_{pr} + v_{qs} = v_{qr} + v_{rs}, \tag{1.3}$$

because in this case the diagonals of T_0 crossing pr are the same as those crossing both qr and rs, while the ones crossing qs are those crossing one, but not both, of qr and rs.

(d) If T_0 contains a triangle pqc with rs crossing both pc and qc then we have the equality

$$v_{pr} + v_{qs} = v_{rs} \tag{1.4}$$

because the diagonals of T_0 crossing rs are the same as those crossing pr and the same as those crossing qs.

□

Observe that when T_0 is a snake triangulation (the CFZ case) or, more generally, when the dual tree of T_0 is a path, cases (c) and (d) do not occur.

1.5.2 Proof of Theorem 1.38

To prove that $\mathcal{F}_{T_0}$ is the normal fan of a polytope we use the following characterization.

Lemma 1.40. *Let $\mathcal{F}$ be a complete simplicial fan in a real vector space V and let A be the set of generators of $\mathcal{F}$ (more precisely, A has one generator of each ray of $\mathcal{F}$). Then the following conditions are equivalent:*

(1) *$\mathcal{F}$ is the normal fan of a polytope.*

(2) *There is a map $\omega : A \to \mathbb{R}_{>0}$ such that for every pair (C_1, C_2) of maximal adjacent cones of $\mathcal{F}$ the following happens: Let $\lambda : A \to \mathbb{R}$ be the (unique, up to a scalar multiple) linear dependence with support in $C_1 \cup C_2$, with its sign chosen so that λ is positive in the generators of $C_1 \setminus C_2$ and $C_2 \setminus C_1$. Then the scalar product $\lambda \cdot \omega = \sum_v \lambda(v)\omega(v)$ is strictly positive.*

Proof. One short proof of the lemma is that both conditions are equivalent to "$\mathcal{F}$ is a regular triangulation of the vector configuration A" [17]. But let us show a more explicit proof of the implication from (2) to (1), which is the one we need. What we are going to show is that if such an ω exists and if we consider the set of points

$$\widetilde{A} := \left\{ \tfrac{v}{\omega(v)} : v \in A \right\},$$

then the convex hull of $\widetilde{A}$ is a simplicial polytope with the same face lattice as the complete fan $\mathcal{F}$. (We think of $\widetilde{A}$ as points in an affine space, rather than as vectors in a vector space.) Hence $\mathcal{F}$ is the *central fan* of $\mathrm{conv}(\widetilde{A})$, which coincides with the normal fan of the polytope polar to $\mathrm{conv}(\widetilde{A})$.

To show the claim on $\mathrm{conv}(\widetilde{A})$ we argue as follows. Consider the simplicial complex Δ with vertex set $\widetilde{A}$ obtained by embedding the face lattice of $\mathcal{F}$ in it. That is, for each cone C of $\mathcal{F}$ we consider the simplex with vertex set in $\widetilde{A}$ corresponding to the generators of C. Since $\mathcal{F}$ is a complete fan and since the elements of $\widetilde{A}$ are generators for its rays (they are positive scalings of the elements of A), Δ is the boundary of a star-shaped polyhedron with the origin in its kernel. The only thing left to be shown is that this polyhedron is strictly convex, that is, that for any two adjacent maximal simplices σ_1 and σ_2 the origin lies in the same side of σ_1 as $\sigma_2 \setminus \sigma_1$ (or, equivalently, in the same side of σ_2 as $\sigma_1 \setminus \sigma_2$). Equivalently, if we understand σ_1 and σ_2 as subsets of $\widetilde{A}$, we have to show that the unique affine dependence between the points $\{O\} \cup \sigma_1 \cup \sigma_2$ has opposite sign in O than in σ_1 and σ_2.

Now the proof is easy. The coefficients in the *linear* dependence among the *vectors* in $\sigma_1 \cup \sigma_2$ are the vector

$$(\lambda(v)\omega(v))_{v \in A}.$$

To turn this into an *affine* dependence of points involving the origin we simply need to give the origin the coefficient $-\sum_v \lambda(v)\omega(v)$ which is, by hypothesis, negative. $\square$

So, in the light of Lemma 1.40, to prove Theorem 1.38 we simply need to choose weights ω_{ij} for the diagonals of the polygon with the property that, for each of the linear dependences exhibited in equations (1.1), (1.2), (1.3), and (1.4), the equation $\sum_{ij} \omega_{ij}\lambda_{ij} > 0$ holds.

As a first approximation, let $\omega_{ij} = 2$ if ij is in T_0 and $\omega_{ij} = 1$ otherwise. This is good enough for equations (1.3) and (1.4) in which all the ω's in the dependence are 1 and the sum of the coefficients in the left-hand side is greater than in the right-hand side. It also works for equations (1.2), in which we have

$$\omega_{pr} = 2, \qquad \omega_{qs} = 1, \qquad \lambda_{pr} = 1, \qquad \lambda_{qs} = 1,$$

so that the sum $\sum_{ij} \omega_{ij}\lambda_{ij}$ for the left-hand side is three, while that of the right-hand side can be 0, -1 or -2 depending on the cases for the points a and b.

The only (weak) failure is that in equation (1.1) we have

$$\lambda_{pr} = 1, \qquad \lambda_{qs} = 1, \qquad \lambda_{pq} = -1, \qquad \lambda_{rs} = -1$$

and all the ω's are 1, so we get $\sum_{ij} \omega_{ij}\lambda_{ij} = 0$. We solve this by slightly perturbing the ω's. A slight perturbation will not change the correct signs we got for equations (1.2), (1.3), and (1.4). For example, for each ij not in T_0 change ω_{ij} to

$$\omega_{ij} = 1 + \varepsilon g_{ij}$$

for a sufficiently small $\varepsilon > 0$ and for a vector $(g_{ij})_{ij}$ satisfying

$$g_{ik} + g_{jl} > \max\{g_{ij} + g_{kl}, g_{il} + g_{jk}\} \quad \text{for all } i, j, k, l,\ 1 \leq i < j < k < l \leq n + 3.$$

This holds (for example) for $g_{ij} := (j - i)(n + 3 + i - j)$.

1.5.3 Distinct seed triangulations produce distinct realizations

Let $\mathrm{Ass}_n^{\mathrm{II}}(T)$ denote the n-dimensional associahedron obtained with the construction of the previous section starting with a certain triangulation T. (This is a slight abuse of notation, since the associahedron depends also in the weight vector ω that gives the right-hand sides for an inequality definition of our associahedron. Put differently, by $\mathrm{Ass}_n^{\mathrm{II}}(T)$ we denote the normal fan rather than the associahedron itself.) We want to classify the associahedra $\mathrm{Ass}_n^{\mathrm{II}}(T)$ by normal isomorphism.

In principle, it looks like we have as many associahedra as there are triangulations (that is, Catalan-many) but that is not the case because, clearly, changing T by a rotation or a reflection does not change the associahedron obtained. The question is whether this is the only operation that preserves $\mathrm{Ass}_n^{\mathrm{II}}(T)$, modulo normal isomorphism. The answer is yes, as we show below.

Lemma 1.41. $\mathrm{Ass}_n^{\mathrm{II}}(T_0)$ *has exactly n pairs of parallel facets, each pair consisting of (the facet of) one diagonal in T_0 and the diagonal obtained from it by a flip in T_0.*

Proof. As always, a necessary condition for the facets corresponding to two diagonals to be parallel is that the diagonals cross; if the diagonals do not cross, they are present in some common triangulation which implies the corresponding facets intersect.

So, let pr and qs be two crossing diagonals. Since $\mathrm{Ass}_n^{\mathrm{II}}(T)$ is full-dimensional, their facets are parallel only if v_{pr} and v_{qs} are linearly dependent. By definition of the vectors v_{ij} this only happens when $\{v_{pr}, v_{qs}\} = \{\pm\alpha_i\}$ for some i, which is the case of the statement. □

Lemma 1.42. *Let P_{n+3} be an $(n+3)$-gon, with $n \geq 2$. For each triangulation T of P_{n+3} let B_T denote the set consisting of the n diagonals in T plus the n diagonals that can be introduced by a single flip from T. Then for every $T_1 \neq T_2$ we have $B_{T_1} \neq B_{T_2}$.*

Proof. Suppose that T_1 and T_2 had $B_{T_1} = B_{T_2}$. We claim that T_2 is obtained from T_1 by a set of "parallel flips". That is, by choosing a certain subset of diagonals of T_1 such that no two of them are incident to the same triangle and flipping them simultaneously. This is so because every diagonal pr in T_2 but not in T_1 intersects a single diagonal qs of T_1. If pqr and prs were not triangles in T_2, then let a be a vertex joined to pr in T_2, different from q or s. One of pa and ra intersects the diagonal qs of T_1 and one of the edges pq, qr, rs and pr of T_1.

Once we have proved this for T_2, the statement is obvious. For every T_2 different from T_1 but with all its diagonals in B_{T_1} there is a diagonal that we can flip to get one that is not in B_{T_1} (same argument, let pr be a diagonal in T_2 but not in T_1; let pq, qr, rs and pr be the other sides of the two triangles of T_2 containing pq. Flipping any of them, say pq, gives a diagonal that crosses pq and qs, which are both in T_1). □

Corollary 1.43. *Let T_1 and T_2 be two triangulations of a convex $(n+3)$-gon. Then $\mathrm{Ass}_n^{\mathrm{II}}(T_1)$ and $\mathrm{Ass}_n^{\mathrm{II}}(T_2)$ are normally isomorphic if and only if T_1 and T_2 are equivalent under rotation-reflection.*

Proof. If T_1 and T_2 are equivalent under rotation-reflection then the resulting associahedra are clearly the same. Now suppose that $\mathrm{Ass}_n^{\mathrm{II}}(T_1)$ and $\mathrm{Ass}_n^{\mathrm{II}}(T_2)$ are normally isomorphic. By Lemma 1.3 the automorphism of the associahedron face lattice induced by the isomorphism corresponds to a rotation-reflection of the polygon. Now, normal isomorphism preserves the property of a pair of facets being parallel, so using the previous lemma we get that this rotation-reflection sends T_1 to T_2. □

However, the same is not true if we only look at the set of normal vectors of $\mathrm{Ass}_n^{\mathrm{II}}(T)$:

Proposition 1.44. *Let T_1 and T_2 be two triangulations of the $(n+3)$-gon. Let $A(T_1)$ and $A(T_2)$ be the sets of normal vectors of $\mathrm{Ass}_n^{\mathrm{II}}(T_1)$ and $\mathrm{Ass}_n^{\mathrm{II}}(T_2)$. Then $A(T_1)$ and $A(T_2)$ are linearly equivalent if, and only if, T_1 and T_2 have isomorphic dual trees.*

Proof. Let $\mathcal{T}$ be the dual tree of a triangulation T. Observe that the edges of $\mathcal{T}$ correspond bijectively to the inner diagonals in T. Moreover, the diagonals of the polygon not used in T correspond bijectively to the possible paths in $\mathcal{T}$. More precisely: for every pair of nodes of $\mathcal{T}$, the two corresponding triangles of T have the property that one edge of each triangle "see each other". Let p and q be the vertices of the two triangles opposite (equivalently, not incident) to those two edges. Then the diagonals of T crossed by pq correspond to the path in $\mathcal{T}$ joining the two nodes.

This means that, if we label the edges of $\mathcal{T}$ with the numbers 1 through n in the same manner as we labelled the diagonals of T we have that

$$A(T) = \{-\alpha_i : i \in [n]\} \cup \{\sum_{i \in p} \alpha_i : p \text{ is a path in } \mathcal{T}\}.$$

In particular, $A(T)$ can be recovered knowing only $\mathcal{T}$ as an abstract graph. For the converse, observe that if two trees are not isomorphic then there is no bijection between their edges that sends paths to paths. For example, knowing only the sets of edges that form paths we can identify the (stars of) vertices of the tree as the sets of edges such that every two of them form a path. □

In particular, this gives us exponentially many ways of embedding the associahedron of dimension n with facet normals in the root system of A_n:

Corollary 1.45. *Let T_0 be a triangulation whose dual tree is a path. Let its diagonals be numbered from 1 to n in the order they appear in the path. Then, taking $\alpha_i = e_{i+1} - e_i$, we have that $A(T_0)$ is the set of almost positive roots in the root system A_n.*

The number of normally non-isomorphic classes of associahedra, for which the dual tree of the seed triangulation T_0 is a path, is equal to the number of sequences $\{+,-\}^{n-1}$ modulo reflection and reversal.

It is surprising that the number of realizations that we get in this way is exactly the same as we got in the previous section. Nevertheless, we prove in Theorem 1.48 that the two sets of realizations are (almost) disjoint.

1.5.4 Combinatorial description of c-cluster complexes in type A_n

In this section, we present a simple combinatorial description of c-cluster complexes in type A_n as introduced by Reading in [61]. These complexes are more general than the cluster complexes of Fomin and Zelevinsky [24], and have an extra parameter c corresponding to a *Coxeter element*. In type A_n, Coxeter elements can be represented by a sequence of signs $c \in \{+,-\}^{n-1}$; the corresponding Coxeter element is given by a product of generators $s_1, \ldots, s_n$ in some order such that s_{i+1} comes after s_i if the i-th sign in the sequence is positive, and s_{i+1} comes before s_i if the i-th sign is negative.

As in the description of the cluster complex of type A_n in Section 1.3.3.2 consider the root system of type A_n and the set of almost positive roots $\Phi_{\geq -1}$. In addition, consider a sequence of signs $c \in \{+,-\}^{n-1}$ and let T_c be a triangulation of an $(n+3)$-gon P_{n+3} whose dual tree is a path encoded by the sequence of signs c as illustrated in Figure 1.16.

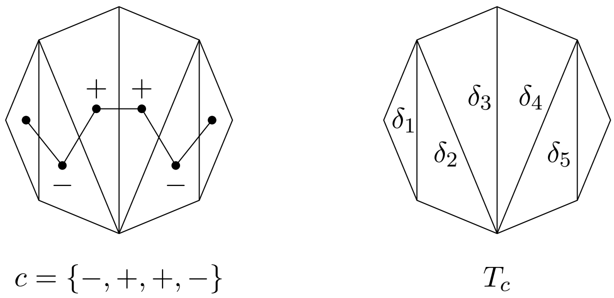


FIGURE 1.16: The triangulation T_c corresponding to the sequence of signs $c = \{-,+,+,-\}$.

The description of the c-cluster complex follows the same steps in the description of the cluster complex in Section 1.3.3.2 using the triangulation T_c. We label the diagonals of T_c by $\{\delta_1, \ldots, \delta_n\}$ in the order they appear in the dual path. There is a natural correspondence between the set $\Phi_{\geq -1}$ and the diagonals of P_{n+3}: We identify the negative simple roots $\{-\alpha_1, \ldots, -\alpha_n\}$ with the diagonals $\{\delta_1, \ldots, \delta_n\}$, and each positive root

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \qquad 1 \leq i \leq j \leq n,$$

with the unique diagonal of P_{n+3} crossing the (consecutive) diagonals $-\delta_i, -\delta_{i+1}, \ldots, -\delta_j$.

We say that two roots α and β in $\Phi_{\geq -1}$ are *c-compatible* if their corresponding diagonals do not cross. The *c-cluster complex* can then be described as the simplicial complex whose faces correspond to sets of almost positive roots that are pairwise c-compatible.

The maximal simplices in this simplicial complex, which naturally correspond to triangulations of the polygon, are called *c-clusters*. For instance, the set

$$\{\alpha_1 + \alpha_2 + \alpha_3,\ \alpha_2 + \alpha_3,\ \alpha_2 + \alpha_3 + \alpha_4,\ \alpha_3,\ -\alpha_5\}$$

is a c-cluster of type A_5 for $c = (-,+,+,-)$ corresponding to the Coxeter element $s_2s_1s_3s_5s_4$. The reason is that its corresponding diagonals in Figure 1.16 form a triangulation of the polygon. This algorithm gives a simple combinatorial way of computing c-cluster complexes in type A. The proof that this description of c-cluster complexes actually coincides with the original description by Reading follows the two steps (i) and (ii) in the definition of the c-compatibility relation in [64, Section 5]. As a consequence we obtain

Proposition 1.46. *The normal fan of the associahedron* $\mathrm{Ass}_n^{\mathrm{II}}(T_c)$ *coincides with the* c*-cluster fan of type* A_n.

1.6 How many associahedra?

We have presented several constructions of the associahedron. We call associahedra of types I and II the associahedra $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ and $\mathrm{Ass}_n^{\mathrm{II}}(T)$ studied in the previous two sections. Associahedra of type I include the Postnikov (or Rote–Santos–Streinu, or Loday, or Buchstaber) associahedron, and both types I and II include the Chapoton–Fomin–Zelevinsky associahedron. They all have pairs of parallel facets while the secondary polytope on an n-gon (according to Section 1.3.1) does not. This implies that:

Theorem 1.47. *The associahedron as a secondary polytope is never normally isomorphic to any associahedron of type I or type II. In particular, it is not normally isomorphic to the Postnikov associahedron or the Chapoton–Fomin–Zelevinsky associahedron.*

Both types I and II produce exponentially many normally non-isomorphic realizations. The number of normally non-equivalent associahedra of type I is asymptotically 2^{n-3}, while for type II is asymptotically $2^{2n+1}/\sqrt{\pi n^5}$. Explicit computations up to dimension 15 are given in Table 1.1.

$n=$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\mathrm{Ass}_n^{\mathrm{I}}$	1	1	1	2	3	6	10	20	36	72	136	272	528	1056	2080	4160
$\mathrm{Ass}_n^{\mathrm{II}}$	1	1	1	3	4	12	27	82	228	733	2282	7528	24834	83898	285357	983244

Table 1.1: The number of normally non-isomorphic realizations of the associahedron of types I and II up to dimension 15.

Surprisingly, the realizations of types I and II are (almost) disjoint:

Theorem 1.48. *The only associahedron that is normally isomorphic to both one of type I and one of type II is the Chapoton–Fomin–Zelevinsky associahedron.*

Proof. Suppose that a sequence $\sigma \in \{+,-\}^{n-1}$ and a triangulation T produce normally isomorphic associahedra $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ and $\mathrm{Ass}_n^{\mathrm{II}}(T)$. The induced automorphism between the face lattice of these two associahedra comes from a reflection-rotation map on the $(n+3)$-gon, by Lemma 1.3, so there is no loss of generality in assuming that this reflection-rotation is the identity.

Denote by B_σ and B_T the $2n$ diagonals corresponding to the n pairs of parallel facets in both constructions respectively. The diagonals of B_T consist of the diagonals of T together with its flips. Since normal isomorphisms preserve pairs of parallel facets, $B_T = B_\sigma$.

We consider the $(n+3)$-gon drawn in the Hohlweg–Lange fashion (with vertices placed along two x-monotone chains, the positive and the negative one, placed in the x-order indicated by σ). The crucial property we use is that B_σ contains only diagonals between vertices of opposite signs. Knowing this we conclude:

- *Every triangle in T contains a boundary edge in one of the chains. (That is, the dual tree of T is a path).* Suppose, in the contrary, that T has a triangle pqr with no boundary edge. Then the three diagonals pq, pr and qr lie in $B_T = B_\sigma$. This is impossible since at least two of p, q and r must have the same sign.
- *The third vertex of each triangle is in the opposite chain. (That is, the dual path of T separates the two chains).* Otherwise the three vertices of a certain triangle lie in the same chain. This is impossible, because (at least) one of the three edges of each triangle is a diagonal, hence it is in B_σ.
- *No two consecutive boundary edges in one chain are joined to the same vertex in the opposite chain. (That is, the dual tree of T alternates left and right turns).* Otherwise, let abp and bcp be two triangles in T with ab and bc consecutive boundary edges in one of the chains. Then the flip in bp inserts the edge ac, so that $ac \in B_\sigma$. This is impossible, since a and c are in the same chain.

These three properties imply that T is the snake triangulation, so $\mathrm{Ass}_n^{\mathrm{II}}(T)$ is the Chapoton–Fomin–Zelevinsky associahedron. □

Chapter 2

Subword complexes, cluster complexes, and generalized multi-associahedra

2.1 Introduction

Cluster complexes were introduced by S. Fomin and A. Zelevinsky to encode exchange graphs of cluster algebras [24]. N. Reading then showed that the definition of cluster complexes can be extended to all finite Coxeter groups [61, 62]. In this chapter, we present a new combinatorial description of cluster complexes using *subword complexes*. These were introduced by A. Knutson and E. Miller, first in type A to study the combinatorics of determinantal ideals and Schubert polynomials [45], and then for all Coxeter groups in [44]. We provide, for any finite Coxeter group W and any Coxeter element $c \in W$, a subword complex which is isomorphic to the c-cluster complex of the corresponding type, and we thus obtain an explicit type-free characterization of c-clusters. This characterization generalizes a description for crystallographic types obtained by K. Igusa and R. Schiffler in the context of cluster categories [39]. The present approach allows us to define a new family of simplicial complexes by introducing an additional parameter k, such that one obtains c-cluster complexes for $k = 1$. In type A, this simplicial complex turns out to be isomorphic to the simplicial complex of multi-triangulations of a convex polygon which was described by C. Stump in [82] (see also C. Stump and L. Serrano [70]), and, in a similar manner, by V. Pilaud and M. Pocchiola in the framework of sorting networks [54]. In type B, we obtain that this simplicial complex is isomorphic to the simplicial complex of centrally symmetric multi-triangulations of a regular convex polygon. Therefore, we call them *multi-cluster complexes*. They are different from *generalized cluster complexes* as defined by S. Fomin and N. Reading [22], and in some sense complementary. In the generalized cluster complex, the vertices are given by the simple negative roots together with several distinguished copies of the positive roots, while the vertices of the multi-cluster complex correspond to the positive roots together with several distinguished copies of the simple negative roots. Multi-cluster complexes turn out to be intimately related to Auslander-Reiten quivers and repetition quivers [28]. In particular, the Auslander-Reiten translate on facets of multi-cluster complexes in types A and B corresponds to cyclic rotation of (centrally symmetric) multi-triangulations. Furthermore, multi-cluster complexes uniformize questions about multi-triangulations, subword complexes, and cluster complexes. One important example concerns the open problem of realizing the simplicial complexes of (centrally symmetric) multi-triangulations and spherical subword complexes as boundary complexes of convex polytopes.

In this chapter, we present the results obtained with Jean-Philippe Labbé and Christian Stump in [14]. In Section 2.2, we recall the various objects in question, namely multi-triangulations, subword complexes, and cluster complexes. Moreover, the main results are presented and the multi-cluster complex is defined (Definition 2.5). In Section 2.3, we study flips on spherical subword complexes and present two natural isomorphisms. In Section 2.4, we prove that the multi-cluster complex is independent of the choice of the Coxeter element (Theorem 2.6). Section 2.5 contains a proof that the multi-cluster complex is isomorphic to the cluster complex for $k = 1$ (Theorem 2.2). In Section 2.6, we discuss possible generalizations of associahedra using subword complexes; we review known results about polytopal realizations, prove polytopality of multi-cluster cluster complexes of rank 2 (Theorem 2.36), and prove that the multi-cluster complex is universal in the sense that every spherical subword complex is the link of a face of a multi-cluster complex (Theorem 2.14). Section 2.7 contains a combinatorial description of the sorting words of the longest element of finite Coxeter groups (Theorem 2.39), and an alternative definition of multi-cluster complexes in terms of the strong intervening neighbors property (Theorem 2.7). In Section 2.8, we define a natural action on the vertices and facets of the multi-cluster complex (Definition 2.50) and use this action to relate multi-cluster complexes to Auslander-Reiten and repetition quivers (Proposition 2.49). Finally, in Section 2.9, we discuss open problems and questions arising in the context of multi-cluster complexes.

In [57], C. Stump and V. Pilaud study the geometry of subword complexes and use the theory presented in this chapter (developed in [14]) to describe the connections to *Coxeter-sortable elements*, and how to recover *Cambrian fans*, *Cambrian lattices*, and the *generalized associahedra* purely in terms of subword complexes.

2.2 Definitions and main results

In this section, we review the essential notions concerning multi-triangulations, subword complexes and cluster complexes of finite type and present the main results. Throughout the chapter, (W, S) denotes a *finite Coxeter system* of rank n, and c denotes a *Coxeter element*, i.e., the product of the generators in S in some order. The smallest integer h for which $c^h = \mathbf{1} \in W$ is called *Coxeter number*. Coxeter elements of W are in bijection with *(acyclic) orientations of the Coxeter graph* of W: a non-commuting pair $s, t \in S$ has the orientation $s \longrightarrow t$ if and only if *s comes before t in c*, i.e., s comes before t in any reduced expression for c; see Shi [71]. In the simply-laced types A, D, and E, this procedure yields a *quiver* Ω_c associated to a given Coxeter element c, where by quiver we mean a directed graph without loops or two-cycles. For two examples, see Figure 2.1 on page 60. The *length function* on W is given by $\ell(w) = \min\{r : w = a_1 \cdots a_r,\ a_i \in S\}$, an expression of minimal length is called *reduced*, and the unique *longest element* in W is denoted by $w_\circ$, its length is given by $\ell(w_\circ) = N := nh/2$. We refer the reader to Humphreys [38] for further definitions and a detailed introduction to finite Coxeter groups. Next, we adopt some writing conventions: in order to emphasize the distinction between words and group elements, we write words in the alphabet S as a sequence between brackets $(a_1, a_2, \ldots, a_r)$ and use square letters such as $\mathbf{w}$ to denote them, and we write group elements as a concatenation of letters $a_1 a_2 \cdots a_r$ using normal script such as w to denote them.

2.2.1 Multi-triangulations

Let Δ_m be the simplicial complex with vertices being diagonals of a convex m-gon and faces being subsets of non-crossing diagonals. Its facets correspond to *triangulations* (i.e., maximal subsets of diagonals which are mutually non-crossing). This simplicial complex is the boundary complex of the *dual associahedron* (see Chapter 1). It can be generalized using a positive integer k with $2k+1 \leq m$: define a *$(k+1)$-crossing* to be a set of $k+1$ diagonals which are pairwise crossing. A diagonal is called *k-relevant* if it is contained in some $(k+1)$-crossing, that is, if there are at least k vertices of the m-gon on each side of the diagonal. The complex $\Delta_{m,k}$ is the simplicial complex of $(k+1)$-crossing free sets of k-relevant diagonals. Its facets are given by *k-triangulations* (i.e., maximal subsets of diagonals which do not contain a $(k+1)$-crossing), without considering k-irrelevant diagonals. The reason for restricting the set of diagonals is that including all other diagonals would yield the join of $\Delta_{m,k}$ and an mk-simplex. This simplicial complex has been studied by several authors, see e.g. [18, 41, 42, 46, 52, 67, 70, 82]; an interesting recent treatment of k-triangulations using complexes of star polygons can be found in [55].

In [82], the following description of $\Delta_{m,k}$ is exhibited: let $\mathcal{S}_{n+1}$ be the symmetric group generated by the n simple transpositions $s_i = (i\ i{+}1)$ for $1 \leq i \leq n$, where $n = m-2k-1$. The k-relevant diagonals of a convex m-gon are in bijection with (positions of) letters in the word

$$Q = (\underbrace{s_n, \dots, s_1, \quad \cdots \quad s_n, \dots, s_1,}_{k \text{ times } s_n,\dots,s_1} \quad s_n, \dots, s_1, \quad s_n, \dots, s_2, \quad \cdots \quad s_n, s_{n-1}, \quad s_n)$$

of length $kn + \binom{n+1}{2} = \binom{m}{2} - mk$. If the vertices of the m-gon are cyclically labelled by the integers from 1 to m, the bijection sends the i-th letter of Q to the i-th k-relevant diagonal in lexicographic order. Under this bijection, a collection of diagonals forms a facet of $\Delta_{m,k}$ if and only if the complement of the corresponding subword in Q forms a reduced expression for the permutation $[n+1, \dots, 2, 1] \in \mathcal{S}_{n+1}$. A similar approach which admits various possibilities for the word Q was described in [54] in the context of sorting networks.

Example 2.1. For $m = 5$ and $k = 1$, we get $Q = (q_1, q_2, q_3, q_4, q_5) = (s_2, s_1, s_2, s_1, s_2)$. By cyclically labeling the vertices of the pentagon with the integers $\{1, \dots, 5\}$, the bijection sends the (position of the) letter q_i to the i-th entry of the list of ordered diagonals $[1,3], [1,4], [2,4], [2,5], [3,5]$. On one hand, two cyclically consecutive diagonals in the list form a triangulation of the pentagon. On the other hand, the complement of two cyclically consecutive letters of Q form a reduced expression for $[3,2,1] = s_1 s_2 s_1 = s_2 s_1 s_2 \in \mathcal{S}_3$.

The main objective of this chapter is to describe and study a natural generalization of multi-triangulations to finite Coxeter groups.

2.2.2 Subword complexes

Let $Q = (q_1, \dots, q_r)$ be a word in the generators S of W and let $\pi \in W$. The subword complex $\Delta(Q, \pi)$ was introduced by A. Knutson and E. Miller in order to study Gröbner geometry of Schubert varieties, see [45, Definition 1.8.1], and was further studied in [44]. It is defined as the simplicial complex whose faces are given by subwords P of Q for which the complement $Q \setminus P$ contains a reduced expression of π. Note that subwords come with their embedding into Q; two subwords P and P' representing the same word are considered to be different if they involve generators at different positions within Q. In Example 2.1, we have seen an instance of a subword complex with

$Q = (q_1, q_2, q_3, q_4, q_5) = (s_2, s_1, s_2, s_1, s_2)$ and $\pi = s_1s_2s_1 = s_2s_1s_2$. In this case, $\Delta(Q, \pi)$ has vertices $\{q_1, \ldots, q_5\}$ and facets

$$\{q_1, q_2\}, \{q_2, q_3\}, \{q_3, q_4\}, \{q_4, q_5\}, \{q_5, q_1\}.$$

Subword complexes are known to be vertex-decomposable and hence shellable [44, Theorem 2.5]. Moreover, they are topologically spheres or balls depending on the Demazure product of Q. Let Q' be the word obtained by adding $s \in S$ at the end of a word Q. The *Demazure product* $\delta(Q')$ is recursively defined by

$$\delta(Q') = \begin{cases} \mu s & \text{if } \ell(\mu s) > \ell(\mu), \\ \mu & \text{if } \ell(\mu s) < \ell(\mu), \end{cases}$$

where $\mu = \delta(Q)$ is the Demazure product of Q, and where the Demazure product of the empty word is defined to be the identity element in W. A subword complex $\Delta(Q, \pi)$ is a sphere if and only if $\delta(Q) = \pi$, and a ball otherwise [44, Corollary 3.8].

2.2.3 Cluster complexes

In [24], S. Fomin and A. Zelevinsky introduced *cluster complexes* associated to finite crystallographic root systems. This simplicial complex along with the *generalized associahedron* has become the object of intensive studies and generalizations in various contexts in mathematics, see for instance [16, 36, 50, 61]. A generator $s \in S$ is called *initial* or *final* in a Coxeter element c if $\ell(sc) < \ell(c)$ or $\ell(cs) < \ell(c)$, respectively. The group W acts naturally on the real vector space V with basis $\Pi = \{\alpha_s : s \in S\}$, its elements are called *simple roots*. Let $\Pi \subseteq \Phi^+ \subseteq \Phi \subset V$ be the set of *positive roots* and the set of *roots* for (W, S), respectively. Furthermore, let $\Phi_{\geq -1} = \Phi^+ \cup -\Pi$ be the set of *almost positive roots*. By convention, we denote the maximal standard parabolic subgroup generated by $S \setminus \{s\}$ by $W_{\langle s \rangle}$, and the associated subroot system by $\Phi_{\langle s \rangle}$. For $s \in S$, the involution $\sigma_s : \Phi_{\geq -1} \longrightarrow \Phi_{\geq -1}$ is given by

$$\sigma_s(\beta) = \begin{cases} \beta & \text{if } -\beta \in \Pi \setminus \{\alpha_s\}, \\ s(\beta) & \text{otherwise.} \end{cases}$$

N. Reading showed that the definition of cluster complexes can be extended to all finite root systems and enriched with a parameter c being a Coxeter element [61]. These c-cluster complexes are defined using a family $\|_c$ of c-compatibility relations on $\Phi_{\geq -1}$, see [64, Section 5]. This family is characterized by the following two properties:

(i) for $s \in S$ and $\beta \in \Phi_{\geq -1}$,

$$-\alpha_s \parallel_c \beta \Leftrightarrow \beta \in \left(\Phi_{\langle s \rangle}\right)_{\geq -1},$$

(ii) for $\beta_1, \beta_2 \in \Phi_{\geq -1}$ and s being initial in c,

$$\beta_1 \parallel_c \beta_2 \Leftrightarrow \sigma_s(\beta_1) \parallel_{scs} \sigma_s(\beta_2).$$

A maximal subset of pairwise c-compatible almost positive roots is called *c-cluster*. The *c-cluster complex* is the simplicial complex whose vertices are the almost positive roots and whose facets are c-clusters. It turns out that all c-cluster complexes for the various Coxeter elements are isomorphic, see [50, Proposition 4.10] and [61, Proposition 7.2]. In crystallographic types, they are moreover isomorphic to the cluster complex as defined in [24].

2.2.4 Main results

We are now in the position to state the main results of this chapter and to define the central object, the *multi-cluster complex*. Let $\mathbf{c} = (c_1, \ldots, c_n)$ be a word corresponding to a Coxeter element $c \in W$, and let $\mathbf{w}_\circ(\mathbf{c}) = (w_1, \ldots, w_N)$ be the lexicographically first subword of $\mathbf{c}^\infty$ which represents a reduced expression for the longest element $w_\circ \in W$. The word $\mathbf{w}_\circ(\mathbf{c})$ is called *c-sorting word* for $w_\circ$. The first theorem (proved in Section 2.5) gives a description of the cluster complex as a subword complex.

Theorem 2.2. *The subword complex $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$ is isomorphic to the c-cluster complex. The isomorphism is given by sending the letter c_i of $\mathbf{c}$ to the negative root $-\alpha_{c_i}$, and the letter w_i of $\mathbf{w}_\circ(\mathbf{c})$ to the positive root $w_1 \cdots w_{i-1}(\alpha_{w_i})$.*

As an equivalent statement, we obtain the following explicit description of the c-compatibility relation.

Corollary 2.3. *A subset C of $\Phi_{\geq -1}$ is a c-cluster if and only if the complement of the corresponding subword in $\mathbf{c}\mathbf{w}_\circ(\mathbf{c}) = (c_1, \ldots, c_n, w_1, \ldots, w_N)$ represents a reduced expression for $w_\circ$.*

This description was obtained independently by K. Igusa and R. Schiffler [39] for finite crystallographic root systems in the context of cluster categories [39, Theorem 2.5]. They use results of W. Crawley-Beovey and C.M. Ringel saying that the braid group acts transitively on isomorphism classes of exceptional sequences of modules over a hereditary algebra, see [39, Section 2]. K. Igusa and R. Schiffler then show combinatorially that the braid group acting on sequences of elements in any Coxeter group W of rank n acts as well transitively on all sequences of n reflections whose product is a given Coxeter element [39, Theorem 1.4]. They then deduce Corollary 2.3 in crystallographic types from these two results, see [39, Theorem 2.5]. The present approach holds uniformly for all finite Coxeter groups, and is developed purely in the context of Coxeter group theory. We study the connections to the work of K. Igusa and R. Schiffler more closely in Section 2.8.

Example 2.4. Let W be the Coxeter group of type B_2 generated by $S = \{s_1, s_2\}$ and let $c = c_1c_2 = s_1s_2$. Then the word $\mathbf{cw}_\circ(\mathbf{c})$ is $(c_1, c_2, w_1, w_2, w_3, w_4) = (s_1, s_2, s_1, s_2, s_1, s_2)$. The corresponding list of almost positive roots is

$$[-\alpha_1,\ -\alpha_2,\ \alpha_1,\ \alpha_1+\alpha_2,\ \alpha_1+2\alpha_2,\ \alpha_2].$$

The subword complex $\Delta(\mathbf{cw}_\circ(\mathbf{c}), w_\circ)$ is an hexagon with facets being any two cyclically consecutive letters. The corresponding c-clusters are

$$\{-\alpha_1, -\alpha_2\}, \{-\alpha_2, \alpha_1\}, \{\alpha_1, \alpha_1+\alpha_2\}, \{\alpha_1+\alpha_2, \alpha_1+2\alpha_2\}, \{\alpha_1+2\alpha_2, \alpha_2\}, \{\alpha_2, -\alpha_1\}.$$

Inspired by results in [82] and [54], we generalize the subword complex in Theorem 2.2 by considering the concatenation of k copies of the word $\mathbf{c}$. In type A, this generalization coincides with the description of the complex $\Delta_{m,k}$ given in [54].

Definition 2.5. The *multi-cluster complex* $\Delta_c^k(W)$ is the subword complex $\Delta(\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c}), w_\circ)$.

Multi-cluster complexes are in fact independent of the Coxeter element c. In particular, we reobtain that all c-cluster complexes are isomorphic (see Section 2.4 for the proof).

Theorem 2.6. *All multi-cluster complexes $\Delta_c^k(W)$ for the various Coxeter elements are isomorphic.*

A word $Q = (q_1, \ldots, q_r)$ in S has the *intervening neighbors property*, if all non-commuting pairs $s, t \in S$ alternate within Q, see [19, Section 3]. Let $\psi : S \to S$ be the involution given by $\psi(s) = w_\circ^{-1} s w_\circ$, and extend ψ to words as $\psi(Q) = (\psi(q_1), \ldots, \psi(q_r))$. We say that Q has the *strong intervening neighbors property* (*SIN-property*), if $Q\psi(Q) = (q_1, \ldots, q_r, \psi(q_1), \ldots, \psi(q_r))$ has the intervening neighbors property, and if in addition the Demazure product $\delta(Q)$ is $w_\circ$. Two words coincide *up to commutations* if they can be obtained from each other by a sequence of interchanges of consecutive commuting letters. The next theorem (proved in Section 2.7) characterizes all words that are equal to $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$ up to commutations. This gives an alternative definition of multi-cluster complexes not using the notion of sorting words.

Theorem 2.7. *A word in S has the SIN-property if and only if it is equal to $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$, up to commutations, for some Coxeter element c and some non-negative integer k.*

The next proposition generalizes [39, Lemma 3.2]. It gives a different description of the facets of multi-cluster complexes. Set $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c}) = (q_1, q_2, \dots, q_{kn+N})$. For an index $1 \le i \le kn+N$, set the reflection t_i to be $q_1 q_2 \dots q_{i-1} q_i q_{i-1} \dots q_2 q_1$. E.g., in Example 2.4, we obtain the sequence

$$(t_1, t_2, t_3, t_4, t_5, t_6) = (s_1, s_1 s_2 s_1, s_2 s_1 s_2, s_2, s_1, s_1 s_2 s_1).$$

Proposition 2.8. *A collection $\{q_{\ell_1}, \dots, q_{\ell_{kn}}\}$ of letters in $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$ forms a facet of $\Delta_c^k(W)$ if and only if*

$$t_{\ell_{kn}} \cdots t_{\ell_2} t_{\ell_1} = c^k.$$

Proof. The proof follows the lines of the proof of [39, Lemma 3.2]. A direct calculation shows that $t_{\ell_1} \cdots t_{\ell_{kn}} q_1 q_2 \cdots q_{kn+N}$ equals the product of all letters in $\mathbf{c}\mathbf{w}_\circ(\mathbf{c})$ not in $\{q_{\ell_1}, \dots, q_{\ell_{kn}}\}$. We get that $\{q_{\ell_1}, \dots, q_{\ell_{kn}}\}$ is a facet of $\Delta_c^k(W)$ if and only if $t_{\ell_1} \cdots t_{\ell_{kn}} q_1 q_2 \cdots q_{kn+N} = w_\circ$. As $q_1 q_2 \cdots q_{kn+N} = c^k w_\circ$, the statement follows. □

We have seen in Section 2.2.1 that the multi-cluster complex of type A_{m-2k-1} is isomorphic to the simplicial complex whose facets correspond to k-triangulations of a convex m-gon,

$$\Delta_c^k(A_{m-2k-1}) \cong \Delta_{m,k}.$$

Thus, the multi-cluster complex extends the concept of multi-triangulations to finite Coxeter groups and provides a unifying approach to multi-triangulations and cluster complexes. The dictionary for type A is presented in Table 2.1.

	$\Delta_{m,k}$	$\Delta_c^k(A_{m-2k-1})$
vertices:	k-relevant diagonals of a convex m-gon	letters of $Q = \mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$
facets:	maximal sets of k-relevant diagonals without $(k+1)$-crossings	$P \subset Q$ such that $\prod_{s \in Q \setminus P} s = w_\circ$
simplices:	sets of k-relevant diagonals without $(k+1)$-crossings	$P \subset Q$ such that $Q \setminus P$ contains a reduced expression for $w_\circ$
ridges:	flips between two k-triangulations	facet flips using Lemma 2.18

Table 2.1: The correspondence between the concepts of diagonals, multi-triangulations and flips of multi-triangulations in $\Delta_{m,k}$, and the multi-cluster complex $\Delta_c^k(A_{m-2k-1})$.

Also in type B, we obtain a previously known object, namely the simplicial complex $\Delta^{sym}_{m,k}$ of centrally symmetric k-triangulations of a regular convex $2m$-gon (see Section 2.6.3 for the proof). This simplicial complex was studied in algebraic and combinatorial contexts in [68, 75].

Theorem 2.9. *The multi-cluster complex $\Delta^k_c(B_{m-k})$ is isomorphic to the simplicial complex of centrally symmetric k-triangulations of a regular convex $2m$-gon.*

The description of the simplicial complex of centrally symmetric multi-triangulations as a subword complex provides straightforward proofs of non-trivial results about centrally symmetric multi-triangulations.

Corollary 2.10. *The following properties of centrally symmetric multi-triangulations of a regular convex $2m$-gon hold.*

(i) All centrally symmetric k-triangulations of a regular convex $2m$-gon contain exactly mk relevant (centrally) symmetric pairs of diagonals, of which k are diameters.

(ii) For any centrally symmetric k-triangulation T and any k-relevant symmetric pair of diagonals $d \in T$, there exists a unique k-relevant symmetric pair of diagonals d' not in T such that $T' = (T\backslash\{d\}) \cup \{d'\}$ is again a centrally symmetric k-triangulation. The operation of interchanging a symmetric pair of diagonals between T and T' is called symmetric flip.

(iii) All centrally symmetric k-triangulations of a $2m$-gon are connected by symmetric flips.

The dictionary between the type B multi-cluster complex and the simplicial complex of centrally symmetric k-triangulations of a regular convex $2m$-gon is presented in Table 2.2.

	$\Delta^{sym}_{m,k}$	$\Delta^k_c(B_{m-k})$
vertices:	k-relevant symmetric pairs of diagonals of a regular convex $2m$-gon	letters of $Q = \mathbf{c}^k\mathbf{w}_\circ(\mathbf{c}) = \mathbf{c}^m$
facets:	maximal sets of k-relevant centrally symmetric diagonals without $(k+1)$-crossings	$P \subset Q$ such that $\prod_{s\in Q\setminus P} s = w_\circ$
simplices:	sets of k-relevant symmetric pairs of diagonals without $(k+1)$-crossings	$P \subset Q$ such that $Q \setminus P$ contains a reduced expression for $w_\circ$
ridges:	symmetric flips between two centrally symmetric k-triangulations	facet flips using Lemma 2.18

Table 2.2: The generalization of the concept of diagonals, multi-triangulations and flips of multi-triangulations to the Coxeter group of type B_n.

In the particular case of $\mathbf{c} = (s_1, s_2, \ldots, s_n)$, where $n = m-k$ and $(s_1s_2)^4 = (s_is_{i+1})^3 = \mathbf{1}$ for $1 < i < n$, the bijection between the k-relevant symmetric pairs of diagonals of a regular convex $2m$-gon and letters of the word $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c}) = \mathbf{c}^m$ is given as follows. If the vertices of the $2m$-gon are labeled cyclically by the integers 1 to $2m$, the bijection sends the letter s_i in the j-th copy of $\mathbf{c}$ in $\mathbf{c}^m$ to the symmetric pair of diagonals $[m+j, i+j-1]_{\mathrm{sym}} := \big\{[m+j, i+j-1], [j, m+i+j-1]\big\}$ (observe that both diagonals coincide for $i = 1$). Under this bijection, a collection of k-relevant symmetric pairs of diagonals forms a facet of $\Delta^{sym}_{m,k}$ if and only if the complement of the corresponding subword in $\mathbf{c}^m$ forms a reduced expression for $w_\circ$.

Example 2.11. Let $m = 5$ and $k = 2$, and let W be the Coxeter group of type B_3 generated by $S = \{s_1, s_2, s_3\}$ where $(s_1s_2)^4 = (s_2s_3)^3 = (s_1s_3)^2 = \mathbf{1}$. The multi-cluster complex $\Delta^2_c(B_3)$ is isomorphic to the simplicial complex of centrally symmetric 2-triangulations of a regular convex 10-gon. In the particular case where the Coxeter element $c = c_1c_2c_3 = s_1s_2s_3$, the bijection between 2-relevant symmetric pairs and the letters of the word $Q = \mathbf{c}^2\mathbf{w}_\circ(\mathbf{c}) = (s_1, s_2, s_3)^5$ is given by

$$\begin{array}{ccccccccccccccc}
s_1 & s_2 & s_3 & s_1 & s_2 & s_3 & s_1 & s_2 & s_3 & s_1 & s_2 & s_3 & s_1 & s_2 & s_3 \\
[6,1] & [6,2] & [6,3] & [7,2] & [7,3] & [7,4] & [8,3] & [8,4] & [8,5] & [9,4] & [9,5] & [9,6] & [10,5] & [10,6] & [10,7] \\
 & [1,7] & [1,8] & & [2,8] & [2,9] & & [3,9] & [3,10] & & [4,10] & [4,1] & & [5,1] & [5,2]
\end{array}$$

For instance, the first appearance of the letter s_3 is mapped to the symmetric pair of diagonals $[6,3]_{\mathrm{sym}} = \big\{[6,3], [1,8]\big\}$, while the third appearance of s_1 is mapped to the symmetric pair of diagonals $[8,3]_{\mathrm{sym}} = \big\{[8,3]\big\}$. The centrally symmetric k-triangulations can be easily described using the subword complex approach. For example, the symmetric pairs of diagonals at positions $\{3, 5, 7, 9, 13, 15\}$ form a facet of $\Delta^{sym}_{m,k}$, and the symmetric flips are interpreted using Lemma 2.18.

Using algebraic techniques, D. Soll and V. Welker proved that $\Delta^{sym}_{m,k}$ is a (mod 2)-homology-sphere [75, Theorem 10]. Theorem 2.9 implies the following stronger result.

Corollary 2.12. *The simplicial complex of centrally symmetric k-triangulations of a regular convex $2m$-gon is a vertex-decomposable simplicial sphere.*

This result together with the proof of [75, Conjecture 13] given in [68][1] implies the following conjecture by Soll and Welker.

Corollary 2.13 ([75, Conjecture 17])**.** *For the term-order $\preceq$ defined in [75, Section 7], the initial ideal $\mathrm{in}_{\preceq}(I_{n,k})$ of the determinantal ideal $I_{n,k}$ defined in [75, Section 3] is spherical.*

We finish this section by describing all spherical subword complexes in terms of faces of multi-cluster complexes (see Section 2.6.5 for the proofs).

[1] The proof appeared in Section 7 in the arxiv version, see `http://arxiv.org/abs/0904.1097v2`.

Theorem 2.14. *A simplicial sphere can be realized as a subword complex of a given finite type W if and only if it is the link of a face of a multi-cluster complex $\Delta_c^k(W)$.*

The previous theorem can be obtained for any family of subword complexes, for which arbitrary large powers of $\mathbf{c}$ appear as subwords. However, computations seem to indicate that the multi-cluster complex maximizes the number of facets among subword complexes $\Delta(Q, w_\circ)$ with word Q of the same size. We conjecture that this is true in general, see Conjecture 2.63. We also obtain the following corollary.

Corollary 2.15. *The following two statements are equivalent.*

(i) Every spherical subword complex is polytopal.

(ii) Every multi-cluster complex is polytopal.

2.3 General results on spherical subword complexes

Before proving the main results, we discuss several properties of spherical subword complexes in general which are not specific to multi-cluster complexes. Throughout this section, we let $Q = (q_1, \ldots, q_r)$ be a word in S and $\pi = \delta(Q)$.

2.3.1 Flips in spherical subword complexes

Lemma 2.16 (Knutson–Miller)**.** *Let F be a facet of $\Delta(Q, \delta(Q))$. For any vertex $q \in F$, there exists a unique vertex $q' \in Q \setminus F$ such that $\big(F \setminus \{q\}\big) \cup \{q'\}$ is again a facet.*

Proof. This follows from the fact that $\Delta(Q, \delta(Q))$ is a simplicial sphere [44, Corollary 3.8]. See [44, Lemma 3.5] for an analogous reformulation. □

Such a move between two adjacent facets is called *flip*. Next, we describe how to find the unique vertex $q' \notin F$ corresponding to $q \in F$. For this, we introduce the notion of root functions.

Definition 2.17. The *root function* $r_F : Q \to \Phi$ associated to a facet F of $\Delta(Q, \pi)$ sends a letter $q \in Q$ to the root $\mathsf{r}_F(q) := w_q(\alpha_q) \in \Phi$, where $w_q \in W$ is given by the product of the letters in the prefix of $Q \setminus F = (q_{i_1}, \ldots, q_{i_\ell})$ that appears on the left of q in Q, and where α_q is the simple root associated to q.

Lemma 2.18. *Let F, q and q' be as in Lemma 2.16. The vertex q' is the unique vertex not in F for which $\mathsf{r}_F(q') \in \{\pm \mathsf{r}_F(q)\}$.*

Proof. Since $q_{i_1} \dots q_{i_\ell}$ is a reduced expression for $\pi = \delta(Q)$, the set $\{\mathsf{r}_F(q_{i_1}), \dots, \mathsf{r}_F(q_{i_\ell})\}$ is equal to the inversion set $\mathrm{inv}(\pi) = \{\alpha_{i_1}, q_{i_1}(\alpha_{i_2}), \dots, q_{i_1} \cdots q_{i_{\ell-1}}(\alpha_{i_\ell})\}$ of π, which only depends on π and not on the chosen reduced expression. In particular, any two elements in this set are distinct. Notice that the root $\mathsf{r}_F(q)$ for $q \in F$ is, up to sign, also contained in $\mathrm{inv}(\pi)$, otherwise it would contradict the fact that the Demazure product of Q is π. If we insert q into the reduced expression of π, we have to delete the unique letter q' that corresponds to the same root, with a positive sign if it appears on the right of q in Q, or with a negative sign otherwise. The resulting word is again a reduced expression for π. □

Remark 2.19. In the case of cluster complexes, this description can be found in [39, Lemma 2.7].

Example 2.20. As in Example 2.4, consider the Coxeter group of type B_2 generated by $S = \{s_1, s_2\}$ with $c = c_1c_2 = s_1s_2$ and $\mathbf{cw}_\circ(\mathbf{c}) = (c_1, c_2, w_1, w_2, w_3, w_4) = (s_1, s_2, s_1, s_2, s_1, s_2)$. Consider the facet $F = \{c_2, w_1\}$, we obtain

$$\begin{aligned}
\mathsf{r}_F(c_1) &= \alpha_1, & \mathsf{r}_F(w_2) &= s_1(\alpha_2) = \alpha_1 + \alpha_2, \\
\mathsf{r}_F(c_2) &= s_1(\alpha_2) = \alpha_1 + \alpha_2, & \mathsf{r}_F(w_3) &= s_1s_2(\alpha_1) = \alpha_1 + 2\alpha_2, \\
\mathsf{r}_F(w_1) &= s_1(\alpha_1) = -\alpha_1, & \mathsf{r}_F(w_4) &= s_1s_2s_1(\alpha_2) = \alpha_2.
\end{aligned}$$

Since $\mathsf{r}_F(c_2) = \mathsf{r}_F(w_2)$, the letter c_2 in F flips to w_2. As w_2 appears on the right of c_2, both roots have the same sign. Similarly, the letter w_1 flips to c_1, because $\mathsf{r}_F(c_1) = -\mathsf{r}_F(w_1)$. In this case, the roots have different signs because c_1 appear on the left of w_1.

The following lemma describes the relation between the root functions of two facets connected by a flip.

Lemma 2.21. *Let F and $F' = \big(F \setminus \{q\}\big) \cup \{q'\}$ be two adjacent facets of the subword complex $\Delta(Q, \delta(Q))$, and assume that q appears on the left of q' in Q. Then, for every letter $p \in Q$,*

$$\mathsf{r}_{F'}(p) = \begin{cases} t_q(\mathsf{r}_F(p)) & \text{if } p \text{ is between } q \text{ and } q', \text{ or } p = q', \\ \mathsf{r}_F(p) & \text{otherwise.} \end{cases}$$

Here, $t_q = w_q q w_q^{-1}$ where w_q is the product of the letters in the prefix of $Q \backslash F$ that appears on the left of q in Q. By construction, t_q is the reflection in W orthogonal to the root $r_F(q) = w_q(\alpha_q)$.

Proof. Let p be a letter in Q, and w_p, w'_p be the products of the letters in the prefixes of $Q\backslash F$ and $Q\backslash F'$ that appear on the left of p. Then, by definition $r_F(p) = w_p(\alpha_p)$ and $r_{F'}(p) = w'_p(\alpha_p)$. We consider the following three cases:

- If p is on the left of q or $p = q$, then $w_p = w'_p$ and $r_F(p) = r_{F'}(p)$.
- If p is between q and q' or $p = q'$, then w'_p can be obtained from w_p by adding the letter q at its corresponding position. This addition is the result of multiplying w_p by $t_q = w_q q w_q^{-1}$ on the left, i.e. $w'_p = t_q w_p$. Therefore, $\mathsf{r}_F(p) = t_q(\mathsf{r}_{F'}(p))$.
- If p is on the right of q', consider the reflection $t_{q'} = w_{q'} q' w_{q'}^{-1}$ where $w_{q'}$ is the product of the letters in the prefix of $Q\backslash F$ that appears on the left of q'. By the same argument, one obtains that $w'_p = t_q t_{q'} w_p$. In addition, $t_q = t_{q'}$ because they correspond to the unique reflection orthogonal to the roots $r_F(q)$ and $r_F(q')$, which are up to sign equal by Lemma 2.18. Therefore, $w'_p = w_p$ and $r_{F'}(p) = r_F(p)$. □

2.3.2 Isomorphic spherical subword complexes

We now reduce the study of spherical subword complexes in general to the case where $\delta(Q) = \pi = w_\circ$, and give two operations on the word Q giving isomorphic subword complexes.

Theorem 2.22. *Every spherical subword complex $\Delta(Q, \pi)$ is isomorphic to $\Delta(Q', w_\circ)$, for some word Q' such that $\delta(Q') = w_\circ$.*

Proof. Let $\mathbf{r}$ be a reduced word for $\pi^{-1}w_\circ = \delta(Q)^{-1}w_\circ \in W$. Moreover, define the word Q' as the concatenation of Q and $\mathbf{r}$. By construction, the Demazure product of Q' is $w_\circ$, and every reduced expression of $w_\circ$ in Q' must contain all the letters in $\mathbf{r}$. The reduced expressions of $w_\circ$ in Q' are given by reduced expressions of π in Q together with all the letters in $\mathbf{r}$. Therefore, the subword complexes $\Delta(Q, \pi)$ and $\Delta(Q', w_\circ)$ are isomorphic. □

Recall the involution $\psi : S \to S$ given by $\psi(s) = w_\circ^{-1} s w_\circ$. This involution was used in [7] to characterize isometry classes of the c-generalized associahedra. Define the *rotated word* $Q_{\circlearrowleft s}$ or the *rotation* of $Q = (s, q_2, \ldots, q_r)$ along the letter s as $(q_2, \ldots, q_r, \psi(s))$. The following two propositions are direct consequences of the definition of subword complexes.

Proposition 2.23. *If two words Q and Q' coincide up to commutations, then $\Delta(Q, \pi) \cong \Delta(Q', \pi)$.*

Proposition 2.24. *Let $Q = (s, q_2, \ldots, q_r)$. Then $\Delta(Q, w_\circ) \cong \Delta(Q_{\underset{s}{\circlearrowleft}}, w_\circ)$.*

Theorem 2.22 and Proposition 2.24 give an alternative viewpoint on spherical subword complexes. First, we can consider π to be the longest element $w_\circ \in W$. Second, $\Delta(Q, w_\circ)$ does not depend on the word Q but on the bi-infinite word

$$\begin{aligned}\widetilde{Q} &= \cdots \quad Q \qquad \psi(Q) \qquad Q \qquad \cdots \\ &= \ldots q_1, \ldots, q_r, \psi(q_1), \ldots, \psi(q_r), q_1, \ldots, q_r, \ldots.\end{aligned}$$

Taking any connected subword in $\widetilde{Q}$ of length r gives rise to an isomorphic spherical subword complex.

2.4 Proof of Theorem 2.6

In this section, we prove that all multi-cluster complexes for the various Coxeter elements are isomorphic. This result relies on the theory of *sorting words* and *sortable elements* introduced by N. Reading in [61]. The c-*sorting word* for $w \in W$ is the lexicographically first (as a sequence of positions) subword of $\mathbf{c}^\infty = \mathbf{ccc}\ldots$ which is a reduced word for w. We use the following result of D. Speyer.

Lemma 2.25 ([76, Corollary 4.1])**.** *The longest element $w_\circ \in W$ can be expressed as a reduced prefix of $\mathbf{c}^\infty$ up to commutations.*

The next lemma unifies previously known results; the first statement it trivial, the second statement can be found in [76, Section 4], and the third statement is equivalent to [36, Lemma 1.6].

Lemma 2.26. *Let s be initial in c and let $\mathbf{p} = (s, p_2, \ldots, p_r)$ be a prefix of $\mathbf{c}^\infty$ up to commutations. Then,*

(i) $(p_2, \ldots, p_r)$ is a prefix of $(\mathbf{scs})^\infty$ up to commutations, where $\mathbf{scs}$ denotes the word for the Coxeter element scs,
(ii) if $p = sp_2 \cdots p_r$ is reduced then $\mathbf{p}$ is the c-sorting word for p up to commutations,
(iii) if $sp_2 \cdots p_r s'$ is reduced for some $s' \in S$ then $\mathbf{p}$ is a prefix of the c-sorting word for ps' up to commutations.

Proposition 2.27. *Let s be initial in c and let $\mathbf{w}_\circ(\mathbf{c}) = (s, w_2, \ldots, w_N)$ be the c-sorting word of $w_\circ$ up to commutations. Then, $(w_2, \ldots, w_N, \psi(s))$ is the scs-sorting word of $w_\circ$ up to commutations.*

Proof. By Lemma 2.25, the element $w_\circ$ can be written as a prefix of $\mathbf{c}^\infty$. By Lemma 2.26, this prefix is equal to the c-sorting of $w_\circ$, which we denote by $\mathbf{w}_\circ(\mathbf{c})$. Let **scs** denote the word for the Coxeter element scs. By Lemma 2.26 (i), the word $(w_2, \dots, w_N)$ is a prefix of $(\mathbf{scs})^\infty$ and by (ii) it is the scs-sorting word for $w_2 \cdots w_N$. By definition of ψ, the word $(w_2, \dots, w_N, \psi(s))$ is a reduced expression for $w_\circ$. Lemma 2.26 (iii) with the word $(w_2, \dots, w_N)$ and $\psi(s)$ implies that $(w_2, \dots, w_N, \psi(s))$ is the scs-sorting word for $w_\circ$ up to commutations. $\square$

Remark 2.28. In [64], N. Reading and D. Speyer present a uniform approach to the theory of sorting words and sortable elements. This approach uses an anti-symmetric bilinear form which is used to extend many results to infinite Coxeter groups. In particular, the previous proposition can be easily deduced from [64, Lemma 3.8].

We are now in the position to prove that all multi-cluster complexes for the various Coxeter elements are isomorphic.

Proof of Theorem 2.6. Let c and c' be two Coxeter elements such that $c' = scs$ for some initial letter s of c. Moreover, let $Q_c = \mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$, and $Q_{scs} = (\mathbf{scs})^k\mathbf{w}_\circ(\mathbf{scs})$. By Proposition 2.23, we can assume that $Q_c = (s, c_2, \dots, c_n)^k \cdot (s, w_2, \dots, w_N)$, and by Proposition 2.27, we can also assume that $Q_{scs} = (c_2, \dots, c_n, s)^k \cdot (w_2, \dots, w_N, \psi(s))$. Therefore, $Q_{scs} = (Q_c)_{\underset{s}{\circlearrowleft}}$, and Proposition 2.24 implies that the subword complexes $\Delta(Q_c, w_\circ)$ and $\Delta(Q_{scs}, w_\circ)$ are isomorphic. Since any two Coxeter elements can be obtained from each other by conjugation of initial letters (see [30, Theorem 3.1.4]), the result follows. $\square$

2.5 Proof of Theorem 2.2

In this section, we prove that the subword complex $\Delta(\mathbf{cw}_\circ(\mathbf{c}), w_\circ)$ is isomorphic to the c-cluster complex. As in Theorem 2.2, we identify letters in $\mathbf{cw}_\circ(\mathbf{c}) = (c_1, \dots, c_n, w_1, \dots, w_N)$ with almost positive roots using the bijection $\mathsf{Lr}_c : \mathbf{cw}_\circ(\mathbf{c}) \xrightarrow{\sim} \Phi_{\geq -1}$ given by

$$\mathsf{Lr}_c(q) = \begin{cases} -\alpha_{c_i} & \text{if } q = c_i \text{ for some } 1 \leq i \leq n, \\ w_1 w_2 \cdots w_{i-1}(\alpha_{w_i}) & \text{if } q = w_i \text{ for some } 1 \leq i \leq N. \end{cases}$$

In [61] this map was used to establish a bijection between c-sortable elements and c-clusters. Note that under this bijection, letters of $\mathbf{cw}_\circ(\mathbf{c})$ correspond to almost positive roots and subwords of $\mathbf{cw}_\circ(\mathbf{c})$ correspond to subsets of almost positive roots. We use this identification to simplify several statements in this section. Observe, that in the particular case given by $F_0 = \mathbf{c}$ in $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$,

$$\mathsf{Lr}_c(q) = \mathsf{r}_{F_0}(q) \text{ for every } q \in \mathbf{w}_\circ(\mathbf{c}) \subset \mathbf{c}\mathbf{w}_\circ(\mathbf{c}),$$

where $\mathsf{r}_{F_0}(q)$ is the root function as defined in Definition 2.17. We interpret the two parts (i) and (ii) in the definition of c-compatibility (see Section 2.2.3), in Theorem 2.29 and Theorem 2.35. Proving these two conditions yields a proof of Theorem 2.2. The majority of this section is devoted to the proof of the initial condition. The proof of the recursive condition follows afterwards.

2.5.1 Proof of condition (i)

The following theorem implies that $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$ satisfies the initial condition.

Theorem 2.29. *$\{-\alpha_s, \beta\}$ is a face of the subword complex $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$ if and only if $\beta \in \big(\Phi_{\langle s\rangle}\big)_{\geq -1}$.*

We prove this theorem in several steps.

Lemma 2.30. *Let F be a facet of the subword complex $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$ such that $c_i \in F$. Then*

(i) for every $q \in F$ with $q \neq c_i$, $r_F(q) \in \Phi_{\langle c_i\rangle}$.
(ii) for every $q \in \mathbf{c}\mathbf{w}_\circ(\mathbf{c})$, $r_F(q) \in \Phi_{\langle c_i\rangle}$ if and only if $\mathsf{Lr}_c(q) \in (\Phi_{\langle c_i\rangle})_{\geq -1}$.

Proof. For the proof of (i) notice that if $F = \mathbf{c}$ then the result is clear. Now suppose the result is true for a given facet F with $c_i \in F$, and consider the facet $F' = \big(F \setminus \{p\}\big) \cup \{p'\}$ obtained by flipping a letter $p \neq c_i$ in F. Since all the facets containing c_i are connected by flips which do not involve the letter c_i, then it is enough to prove the result for the facet F'. By hypothesis, since $p \in F$ and $p \neq c_i$ then $r_F(p) \in \Phi_{\langle c_i\rangle}$. Then, the reflection t_p orthogonal to $r_F(p)$ defined in Lemma 2.21 satisfies $t_p \in W_{\langle c_i\rangle}$. Using Lemma 2.21 we obtain that for every $q \in \mathbf{c}\mathbf{w}_\circ(\mathbf{c})$,

$$r_{F'}(q) \in \Phi_{\langle c_i\rangle} \Leftrightarrow r_F(q) \in \Phi_{\langle c_i\rangle}.$$

If $q \in F'$ and $q \neq c_i$ then $(q \in F$ and $q \neq c_i)$ or $q = p'$. In the first case, $r_F(q)$ is contained in $\Phi_{\langle c_i\rangle}$ by hypothesis, and consequently $r_{F'}(q) \in \Phi_{\langle c_i\rangle}$. By Lemma 2.18, the second case $q = p'$ implies that $r_F(q) = \pm r_F(p)$. Again since $r_F(p)$ belongs to $\Phi_{\langle c_i\rangle}$ by hypothesis, the root $r_{F'}(q)$ belongs to $\Phi_{\langle c_i\rangle}$.

For the second part of the lemma, notice that the set $\{q \in \mathbf{c}\mathbf{w}_\circ(\mathbf{c}) : r_F(q) \in \Phi_{\langle c_i\rangle}\}$ is invariant for every facet F containing c_i. In particular, if $F = \mathbf{c}$ this set is equal to

$\{q \in \mathbf{cw}_\circ(\mathbf{c}) : \mathsf{Lr}_c(q) \in (\Phi_{\langle c_i \rangle})_{\geq -1}\}$. Therefore, $r_F(q) \in \Phi_{\langle c_i \rangle}$ if and only if $\mathsf{Lr}_c(q) \in (\Phi_{\langle c_i \rangle})_{\geq -1}$. □

Proposition 2.31. *If a facet F of $\Delta(\mathbf{cw}_\circ(\mathbf{c}), w_\circ)$ contains c_i and $q \neq c_i$, then $\mathsf{Lr}_c(q) \in (\Phi_{\langle c_i \rangle})_{\geq -1}$.*

Proof. This proposition is a direct consequence of Lemma 2.30. □

Next, we consider the parabolic subgroup $W_{\langle c_i \rangle}$ obtained by removing the generator c_i from S.

Lemma 2.32. *Let c' be the Coxeter element of the parabolic subgroup $W_{\langle c_i \rangle}$ obtained from c by removing the generator c_i. Consider the word $\widehat{Q} = \mathbf{c}'\mathbf{w}_\circ(\mathbf{c})$ obtained by deleting the letter c_i from $Q = \mathbf{cw}_\circ(\mathbf{c})$, and let $Q' = \mathbf{c}'\mathbf{w}_\circ(\mathbf{c}')$. Then, the subword complexes $\Delta(\widehat{Q}, w_\circ)$ and $\Delta(Q', w'_\circ)$ are isomorphic.*

Proof. Since every facet F of $\Delta(\widehat{Q}, w_\circ)$ can be seen as a facet $F \cup \{c_i\}$ of $\Delta(\mathbf{cw}_\circ(\mathbf{c}), w_\circ)$ which contains c_i, then for every $q \in F$ we have that $\mathsf{Lr}_c(q) \in (\Phi_{\langle c_i \rangle})_{\geq -1}$ by Proposition 2.31. This means that only the letters of $\widehat{Q}$ that correspond to roots in $(\Phi_{\langle c_i \rangle})_{\geq -1}$ appear in the subword complex $\Delta(\widehat{Q}, w_\circ)$. The letters in Q' are in bijection, under the map $\mathsf{Lr}_{c'}$, with the almost positive roots $(\Phi_{\langle c_i \rangle})_{\geq -1}$. Let φ be the map that sends a letter $q \in \widehat{Q}$ corresponding to a root in $(\Phi_{\langle c_i \rangle})_{\geq -1}$ to the letter in Q' corresponding to the same root. We will prove that φ induces an isomorphism between the subword complexes $\Delta(\widehat{Q}, w_\circ)$ and $\Delta(Q', w'_\circ)$. In other words, we show that F is a facet of $\Delta(\widehat{Q}, w_\circ)$ if and only if $\varphi(F)$ is a facet of $\Delta(Q', w'_\circ)$. Let $\widetilde{\mathsf{r}}_F$ and $\mathsf{r}'_{\varphi(F)}$ be the root functions associated to F and $\varphi(F)$ in $\widehat{Q}$ and Q' respectively. Then, for every $q \in \widehat{Q}$ such that $\mathsf{Lr}_c(q) \in (\Phi_{\langle c_i \rangle})_{\geq -1}$, we have

$$\widetilde{\mathsf{r}}_F(q) = \mathsf{r}'_{\varphi(F)}(\varphi(q)). \qquad (\star)$$

If $F = \mathbf{c}'$ then $\varphi(F) = \mathbf{c}'$ and the equality ($\star$) holds by the definition of φ. Moreover, if ($\star$) holds for a facet F then it is true for a facet F' obtained by flipping a letter in F. This follows by applying Lemma 2.21 and using the fact that the positive roots $(\Phi_{\langle c_i \rangle})_{\geq -1}$ in $\widehat{Q}$ and Q' appear in the same order, see [61, Prop. 3.2]. Finally, Lemma 2.18 and ($\star$) imply that the map φ sends flips to flips. Since $\mathbf{c}'$ and $\varphi(\mathbf{c}')$ are facets of $\Delta(\widehat{Q}, w_\circ)$ and $\Delta(Q', w'_\circ)$ respectively, and all facets are connected by flips, F is a facet of $\Delta(\widehat{Q}, w_\circ)$ if and only if $\varphi(F)$ is a facet of $\Delta(Q', w'_\circ)$. □

The next lemma states that every letter in $\mathbf{cw}_\circ(\mathbf{c})$ is indeed a vertex of $\Delta(\mathbf{cw}_\circ(\mathbf{c}), w_\circ)$.

Lemma 2.33. *Every letter in $\mathbf{cw}_\circ(\mathbf{c})$ is contained in some facet of $\Delta(\mathbf{cw}_\circ(\mathbf{c}), w_\circ)$.*

Proof. Write the word $Q = \mathbf{c}\mathbf{w}_\circ(\mathbf{c})$ as the concatenation of $\mathbf{c}$ and the c-factorization of $w_\circ$, i.e., $Q = \mathbf{c}\mathbf{c}_{K_1}\mathbf{c}_{K_2}\cdots\mathbf{c}_{K_r}$, where $K_i \subseteq S$ for $1 \leq i \leq r$ and c_I, with $I \subseteq S$, is the Coxeter element of W_I obtained from c by keeping only letters in I. Since $w_\circ$ is c-sortable, see [61, Corollary 4.4], the sets K_i form a decreasing chain of subsets of S, i.e., $K_r \subseteq K_{r-1} \subseteq \cdots \subseteq K_1 \subseteq S$. This implies that the word $\mathbf{c}\mathbf{c}_{K_1}\ldots\widehat{\mathbf{c}}_{K_i}\ldots\mathbf{c}_{K_r}$ contains a reduced expression for $w_\circ$ for any $1 \leq i \leq r$. Thus, all letters in $\mathbf{c}_{K_i}$ are indeed vertices. $\square$

Proposition 2.34. *For every $q \in \mathbf{c}\mathbf{w}_\circ(\mathbf{c})$ satisfying $\mathsf{Lr}_c(q) \in (\Phi_{\langle c_i\rangle})_{\geq -1}$, there exists a facet of $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$ that contains both c_i and q.*

Proof. Consider the parabolic subgroup $W_{\langle c_i\rangle}$ obtained by removing the letter c_i from S, and let $\widehat{Q}$ and Q' be the words as defined in Lemma 2.32. Since $\Delta(\widehat{Q}, w_\circ)$ and $\Delta(Q', w'_\circ)$ are isomorphic, applying Lemma 2.33 to $\Delta(Q', w'_\circ)$ completes the proof. $\square$

Proof of Theorem 2.29. Taking $c_i = s$, $-\alpha_s = \mathsf{Lr}_c(c_i)$ and $\beta = \mathsf{Lr}_c(q)$ the two directions of the equivalence follow from Propositions 2.31 and 2.34. $\square$

2.5.2 Proof of condition (ii)

The following theorem proves condition (ii).

Theorem 2.35. *Let $\beta_1, \beta_2 \in \Phi_{\geq -1}$ and s be an initial letter of a Coxeter element c. Then, $\{\beta_1, \beta_2\}$ is a face of the subword complex $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$ if and only if $\{\sigma_s(\beta_1), \sigma_s(\beta_2)\}$ is a face of the subword complex $\Delta(\mathbf{c}'\mathbf{w}_\circ(\mathbf{c}'), w_\circ)$, with $c' = scs$.*

Proof. Let $Q = \mathbf{c}\mathbf{w}_\circ(\mathbf{c})$, s be initial in c and $Q_{\circlearrowleft s}$ be the rotated word of Q, as defined in Section 2.3.2. By Proposition 2.27, the word $Q_{\circlearrowleft s}$ is equal to $\mathbf{c}'\mathbf{w}_\circ(\mathbf{c}')$ up to commutations, and by Proposition 2.24 the subword complexes $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$ and $\Delta(\mathbf{c}'\mathbf{w}_\circ(\mathbf{c}'), w_\circ)$ are isomorphic. For every letter $q \in \mathbf{c}\mathbf{w}_\circ(\mathbf{c})$, we denote by q' the corresponding letter in $\mathbf{c}'\mathbf{w}_\circ(\mathbf{c}')$ obtained from the previous isomorphism. We write $q_1 \sim_c q_2$ if and only if $\{q_1, q_2\}$ is a face of $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$. In terms of almost positive roots this is written as

$$\mathsf{Lr}_c(q_1) \sim_c \mathsf{Lr}_c(q_2) \Longleftrightarrow \mathsf{Lr}_{scs}(q'_1) \sim_{scs} \mathsf{Lr}_{scs}\left(q'_2\right).$$

Note that the bijection Lr_{scs} can be described using Lr_c. Indeed, it is not hard to check that $\mathsf{Lr}_{scs}(q') = \sigma_s(\mathsf{Lr}_c(q))$ for all $q \in Q$. Therefore,

$$\mathsf{Lr}_c(q_1) \sim_c \mathsf{Lr}_c(q_2) \Longleftrightarrow \sigma_s(\mathsf{Lr}_c(q_1)) \sim_{scs} \sigma_s(\mathsf{Lr}_c(q_2)).$$

Taking $\beta_1 = \mathsf{Lr}_c(q_1)$ and $\beta_2 = \mathsf{Lr}_c(q_2)$ we get the desired result. $\square$

2.6 Generalized multi-associahedra and polytopality of spherical subword complexes

In this section, we discuss the polytopality of spherical subword complexes and present what is known in the particular cases of cluster complexes, simplicial complexes of multi-triangulations, and simplicial complexes of centrally symmetric multi-triangulations. We then prove polytopality of multi-cluster complexes of rank 2. Finally, we show that every spherical subword complex is the link of a face of a multi-cluster complex, and consequently reduce the question of realizing spherical subword complexes to the question of realizing multi-cluster complexes. We use the term *generalized multi-associahedron* for the dual of a polytopal realization of a multi-cluster complex – but *the existence of such realizations remains open* in general, see Table 2.3. The subword complex approach provides new perspectives and methods for finding polytopal realizations. In [57] for example, C. Stump and V. Pilaud obtain a geometric construction of a class of subword complexes containing generalized associahedra purely in terms of subword complexes.

simplicial complex	polytopal realization of the dual
of triangulations (classical)	associahedron [34, 35, 48, 60]
of multi-triangulations [41, 46, 54, 55, 82]	multi-associahedron (existence conjectured)
of centrally symmetric multi-triangulations [68, 75]	multi-associahedron of type B (existence conjectured)
cluster complex [24, 60–62]	generalized associahedron [16, 35, 57, 81]
multi-cluster complex (present chapter)	generalized multi-associahedron (existence conjectured)

Table 2.3: Dictionary for generalized concepts of triangulations and associahedra.

2.6.1 Generalized associahedra

We have seen that for $k = 1$, the multi-cluster complex $\Delta_c^1(W)$ is isomorphic to the c-cluster complex. S. Fomin and A. Zelevinsky conjectured the existence of polytopal realizations of the cluster complex in [24, Conjecture 1.12]. F. Chapoton, S. Fomin, and A. Zelevinsky then proved this conjecture by providing explicit inequalities for the defining hyperplanes of generalized associahedra [16]. N. Reading constructed c-Cambrian fans, which are complete simplicial fans coarsening the Coxeter fan, see [60]. In [63], N. Reading and D. Speyer prove that these fans are combinatorially isomorphic to the normal fan of the polytopal realization in [16]. C. Hohlweg, C. Lange and H. Thomas then provided a family of c-generalized associahedra having c-Cambrian fans as normal fans by removing certain hyperplanes from the permutahedron [36]. V. Pilaud and C. Stump recovered c-generalized associahedra by giving explicit vertex and hyperplane descriptions purely in terms of the subword complex approach presented in this chapter [57].

2.6.2 Multi-associahedra of type A

In type A_n for $n = m - 2k - 1$, the multi-cluster complex $\Delta_c^k(A_n)$ is isomorphic to the simplicial complex $\Delta_{m,k}$ of k-triangulations of a convex m-gon. This simplicial complex is conjectured to be realizable as the boundary complex of a polytope[2]. It was studied in many different contexts, see [55, Section 1] for a detailed description of previous work on multi-triangulations. Apart from the most simple cases, very little is known about its polytopality. Nevertheless, this simplicial complex possesses very nice properties which makes this conjecture plausible. Indeed, the subword complex approach provides a simple descriptions of the 1-skeleton of a possible multi-associahedron (see Lemma 2.18), and gives a new and very simple proof that it is a vertex-decomposable triangulated sphere [82, Theorem 2.1]. Below, we survey the known polytopal realizations of $\Delta_{m,k}$ as boundary complexes of convex polytopes. The simplicial complex $\Delta_{m,k}$, or equivalently the multi-cluster complex $\Delta_c^k(A_n)$ for $n = m - 2k - 1$, is the boundary complex of

- a point, if $k = 0$,
- an n-dimensional dual associahedron, if $k = 1$,
- a k-dimensional simplex, if $n = 1$,
- a $2k$-dimensional cyclic polytope on $2k + 3$ vertices, if $n = 2$, see [55, Section 8],
- a 6-dimensional simplicial polytope, if $n = 3$ and $k = 2$, see [9].

The case $n = 2$ is also a direct consequence of the rank 2 description in Section 2.6.4. Further unsuccessful attempts to realize $\Delta_{m,k}$ come from various directions in discrete geometry.

(a) A generalized construction of the polytope of pseudo-triangulations using rigidity of pseudo-triangulations [53, Section 4.2 and Remark 4.82].

(b) A generalized construction of the secondary polytope. As presented in [31], the secondary polytope of a point configuration can be generalized using star polygons [53, Section 4.3].

(c) The brick polytope of a sorting network. This new approach brought up a large family of spherical subword complexes which are realizable as the boundary of a

[2] As far as we know, the first reference to this conjecture appears in [41, Section 1].

polytope. In particular it provides a new perspective on generalized associahedra [57, 57]. Unfortunately, this polytope fails to realize the multi-associahedron.

2.6.3 Multi-associahedra of type B

We start by proving Theorem 2.9 which says that the multi-cluster complex $\Delta_c^k(B_{m-k})$ is isomorphic to the simplicial complex of centrally symmetric k-triangulations of a regular convex $2m$-gon. This simplicial complex was studied in [68, 75]. We then present what is known about its polytopality. The new approach using subword complexes provides in particular very simple proofs of Corollaries 2.10, 2.12 and 2.13.

Proof of Theorem 2.9. Let $S = \{s_0, s_1, \ldots, s_{m-k-1}\}$ be the generators of B_{m-k}, where s_0 is the generator such that $(s_0 s_1)^4 = \mathbf{1} \in W$, and the other generators satisfy the same relations as in type A_{m-k-1}. Then, embed the group B_{m-k} in the group $A_{2(m-k)-1}$ by the standard folding technique: replace s_0 by s'_{m-k} and s_i by $s'_{m-k+i}s'_{m-k-i}$ for $1 \leq i \leq m-k-1$, where the set $S' = \{s'_1, \ldots, s'_{2(m-k)-1}\}$ generates the group $A_{2(m-k)-1}$. The multi-cluster complex $\Delta_c^k(B_{m-k})$ now has an embedding into the multi-cluster complex $\Delta_{c'}^k(A_{2(m-k)-1})$, where c' is the Coxeter element of type $A_{2(m-k)-1}$ corresponding to c in B_{m-k}; the corresponding subcomplex has the property that $2(m-k)$ generators (all of them except s'_{m-k}) always come in pairs. Using the correspondence between k-triangulations and the multi-cluster complex described in Section 2.2.1, the facets of $\Delta_c^k(B_{m-k})$ considered in $\Delta_{c'}^k(A_{2(m-k)-1})$ correspond to centrally symmetric multi-triangulations. □

Here, we present the few cases for which this simplicial complex is known to be polytopal. The multi-cluster complex $\Delta_c^k(B_{m-k})$ is the boundary complex of

- an $(m-1)$-dimensional dual cyclohedron (or type B associahedron), if $k = 1$, see [35, 73],
- an $(m-1)$-dimensional simplex, if $k = m-1$,
- a $(2m-4)$-dimensional cyclic polytope on $2m$ vertices, if $k = m-2$, see [75].

The case $k = m-2$ also follows from the rank 2 description in Section 2.6.4.

2.6.4 Generalized multi-associahedra of rank 2

We now prove that multi-cluster complexes of rank 2 can be realized as boundary complexes of cyclic polytopes. In other words, we show the existence of rank 2 multi-associahedra. This particular case was known independently by D. Armstrong[3].

Theorem 2.36 (Type $I_2(m)$ multi-associahedra)**.** *The multi-cluster complex $\Delta^k_c(I_2(m))$ is isomorphic to the boundary complex of a $2k$-dimensional cyclic polytope on $2k+m$ vertices. The multi-associahedron of type $I_2(m)$ is the simple polytope given by the dual of a $2k$-dimensional cyclic polytope on $2k+m$ vertices.*

Proof. This is obtained by Gale's evenness criterion on the word $Q = (a, b, a, b, a, \dots)$ of length $2k+m$: Let F be a facet of $\Delta^k_c(I_2(m))$, and take two consecutive letters x and y in the complement of F. Since the complement of F is a reduced expression of $w_\circ$, then x and y must represent different generators. Since the letters in Q are alternating, it implies that the number of letters between x and y is even. □

2.6.5 Generalized multi-associahedra

Recall from Section 2.2.2 that a subword complex $\Delta(Q, \pi)$ is homeomorphic to a sphere if and only if the Demazure product $\delta(Q) = \pi$, and to a ball otherwise. This motivates the question whether spherical subword complexes can be realized as boundary complexes of polytopes [44, Question 6.4.]. We show that it is enough to consider multi-cluster complexes to prove polytopality for all spherical subword complexes, and we characterize simplicial spheres that can be realized as subword complexes in terms of faces of multi-cluster complexes.

Lemma 2.37. *Every spherical subword complex $\Delta(Q, w_\circ)$ is the link of a face of a multi-cluster complex $\Delta(\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c}), w_\circ)$.*

Proof. Observe that any word Q in S can be embedded as a subword of $Q' = \mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$, for k less than or equal to the size of Q, by assigning the i-th letter of Q within the i-th copy of $\mathbf{c}$. Since the Demazure product $\delta(Q)$ is equal to $w_\circ$, the word Q contains a reduced expression for $w_\circ$. In other words, the set $Q' \setminus Q$ is a face of $\Delta(Q', w_\circ)$. The link of this face in $\Delta(Q', w_\circ)$ consists of subwords of Q – viewed as a subword of Q' – whose complements contain a reduced expression of $w_\circ$. This corresponds exactly to the subword complex $\Delta(Q, w_\circ)$. □

[3] Personal communication.

We now prove that simplicial spheres realizable as subword complexes are links of faces of multi-cluster complexes. This result extends the *universality of the multi-associahedron* presented in [53, Theorem 4.83] to finite Coxeter groups.

Proof of Theorem 2.14. For any spherical subword complex $\Delta(Q, \pi)$, we have that the Demazure product $\delta(Q)$ equals π. By Theorem 2.22, $\Delta(Q, \pi)$ is isomorphic to a subword complex of the form $\Delta(Q', w_\circ)$. Using the previous lemma we obtain that $\Delta(Q, \pi)$ is the link of a face of a multi-cluster complex. The other direction follows since the link of a subword (i.e., a face) of a multi-cluster complex is itself a subword complex, corresponding to the complement of this subword. □

Finally we prove that the question of polytopality of spherical subword complexes is equivalent to the question of polytopality of multi-cluster complexes.

Proof of Corollary 2.15. On one hand, if every spherical subword complex is polytopal then clearly every multi-cluster complex is polytopal. On the other hand, suppose that every spherical subword complex is polytopal. Every spherical subword complex is the link of a face of a multi-cluster complex. Since the link of a face of a polytope is also polytopal, Theorem 2.14 implies that every spherical subword complex is polytopal. □

2.7 Sorting words of the longest element and the SIN-property

In this section, we give a simple combinatorial description of the c-sorting words of $w_\circ$, and prove that a word Q coincides up to commutations with $\mathbf{c}^k \mathbf{w}_\circ(\mathbf{c})$ for some non-negative integer k if and only if Q has the SIN-property as defined in Section 2.2.4. This gives us an alternative way of defining multi-cluster complexes in terms of words having the SIN-property. Recall the involution $\psi : S \to S$ from Section 2.4 defined by $\psi(s) = w_\circ^{-1} s w_\circ$. The sorting word of $w_\circ$ has the following important property.

Proposition 2.38. *The sorting word $\mathbf{w}_\circ(\mathbf{c})$ is, up to commutations, equal to a word with suffix $(\psi(c_1), \ldots, \psi(c_n))$, where $c = c_1 \cdots c_n$.*

Proof. As $w_\circ$ has a c-sorting word having $\mathbf{c} = (c_1, \ldots, c_n)$ as a prefix, the corollary is obtained by applying Proposition 2.27 n times. □

Given a word $\mathbf{w}$ in S, define the function $\phi_{\mathbf{w}} : S \to \mathbb{N}$ given by $\phi_{\mathbf{w}}(s)$ being the number of occurrences of the letter s in $\mathbf{w}$.

Theorem 2.39. *Let $\mathbf{w}_\circ(\mathbf{c})$ be the c-sorting word of $w_\circ$ and let s,t be neighbors in the Coxeter graph such that s comes before t in c. Then*

$$\phi_{\mathbf{w}_\circ(\mathbf{c})}(s) = \begin{cases} \phi_{\mathbf{w}_\circ(\mathbf{c})}(t) & \text{if } \psi(s) \text{ comes before } \psi(t) \text{ in } c, \\ \phi_{\mathbf{w}_\circ(\mathbf{c})}(t)+1 & \text{if } \psi(s) \text{ comes after } \psi(t) \text{ in } c. \end{cases}$$

Proof. Sorting words of $w_\circ$ have intervening neighbors, see [76, Proposition 2.1] for an equivalent formulation. Therefore s and t alternate in $\mathbf{w}_\circ(\mathbf{c})$, with s coming first. Thus, $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s) = \phi_{\mathbf{w}_\circ(\mathbf{c})}(t)$ if and only if the last t comes after the last s. Using Proposition 2.38, this means that s appears before t in $\psi(\mathbf{c})$ or equivalently $\psi(s)$ appear before $\psi(t)$ in c. Otherwise, the last s will appear after the last t. □

It is known that if ψ is the identity on S, or equivalently if $w_\circ = -\mathbf{1}$, then the c-sorting word of $w_\circ$ is given by $\mathbf{w}_\circ(\mathbf{c}) = \mathbf{c}^{\frac{h}{2}}$, where h denotes the *Coxeter number* given by the order of any Coxeter element. In the case where ψ is not the identity on S (that is when W is of types A_n $(n \geq 2)$, D_n (n odd), E_6 and $I_2(m)$ (m odd), see [8, Exercise 10 of Chapter 4]), the previous theorem gives simple way to obtain the sorting words of $w_\circ$.

Algorithm 2.40. *Let W be an irreducible finite Coxeter group, and let $c = c_1c_2\cdots c_n$ be a Coxeter element.*

(i) Since the Coxeter diagram is connected, one can use Theorem 2.39 to compute $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s)$ for all s depending on $m := \phi_{\mathbf{w}_\circ(\mathbf{c})}(c_1)$;

(ii) using that the number of positive roots equals $nh/2$, one obtains m and thus all $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s)$ using

$$2\cdot\sum_{s\in S}\phi_{\mathbf{w}_\circ(\mathbf{c})}(s) = nh.$$

(iii) using that $\mathbf{w}_\circ(\mathbf{c}) = \mathbf{c}_{K_1}\mathbf{c}_{K_2}\cdots\mathbf{c}_{K_r}$ where $K_i \subseteq S$ for $1 \leq i \leq r$ and c_I, with $I \subseteq S$, is the Coxeter element of W_I obtained from c by keeping only letters in I, we obtain that $\mathbf{c}_{K_i}$ is the product of all s for which $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s) \geq i$.

This algorithm provides an explicit description of the sorting words of the longest element $w_\circ$ of any finite Coxeter group using nothing else than Coxeter group theory. This answers a question raised in [36, Remark 2.3] and simplifies a step in the construction of the c-generalized associahedron. We now give two examples of how to use this algorithm.

Example 2.41. Let $W = A_4$, $S = \{s_1, s_2, s_3, s_4\}$ and $c = s_1s_3s_2s_4$. Fix $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s_1) = m$. Since s_1 comes before s_2 in c and that $\psi(s_1) = s_4$ comes after $\psi(s_2) = s_3$, the letter s_1 appears one more time than the letter s_2 in $\mathbf{w}_\circ(\mathbf{c})$, i.e., $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s_2) = m-1$. Repeating the same argument gives $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s_3) = m$ and $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s_4) = m-1$. Summing up these values gives the equality $4m-2 = \frac{n \cdot h}{2} = \frac{4 \cdot 5}{2} = 10$, and thus $m = 3$. Finally, the c-sorting word is $\mathbf{w}_\circ(\mathbf{c}) = (s_1, s_3, s_2, s_4 | s_1, s_3, s_2, s_4 | s_1, s_3)$.

Example 2.42. Let $W = E_6$, $S = \{s_1, s_2, \dots, s_6\}$ with the labeling shown in Figure 2.1. Moreover, let $c = s_3 s_5 s_4 s_6 s_2 s_1$. Fix $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s_6) = m$. Repeating the same procedure from the previous example and using that $\psi(s_6) = s_6$, $\psi(s_3) = s_3$, $\psi(s_2) = s_5$, $\psi(s_1) = s_4$, one get $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s_1) = \phi_{\mathbf{w}_\circ(\mathbf{c})}(s_2) = m-1$, $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s_3) = \phi_{\mathbf{w}_\circ(\mathbf{c})}(s_6) = m$, $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s_4) = \phi_{\mathbf{w}_\circ(\mathbf{c})}(s_5) = m+1$. As the sum equals $\frac{nh}{2} = \frac{6 \cdot 12}{2} = 36$, we obtain $m = 6$. Finally, the c-sorting word is $(\mathbf{c}^5 | s_3, s_5, s_4, s_6 | s_5, s_4)$.

Remark 2.43. Propositions 2.27 and 2.38 have the following computational consequences. Denote by $\operatorname{rev}(\mathbf{w})$ the reverse of a word $\mathbf{w}$. First, up to commutations, we have

$$\mathbf{w}_\circ(\mathbf{c}) = \operatorname{rev}(\mathbf{w}_\circ(\psi(\operatorname{rev}(\mathbf{c})))).$$

Second, we also have, up to commutation,

$$\mathbf{c}^h = \mathbf{w}_\circ(\mathbf{c}) \operatorname{rev}(\mathbf{w}_\circ(\operatorname{rev}(\mathbf{c}))).$$

Third, for all $s \in S$,

$$\phi_{\mathbf{w}_\circ(\mathbf{c})}(s) + \phi_{\mathbf{w}_\circ(\operatorname{rev}(\mathbf{c}))}(s) = \phi_{\mathbf{w}_\circ(\mathbf{c})}(s) + \phi_{\mathbf{w}_\circ(\mathbf{c})}(\psi(s)) = h.$$

We are now in the position to prove Theorem 2.7.

Proof of Theorem 2.7. Suppose that a word Q has the SIN-property, then it has complete support by definition, and it contains, up to commutations, some word $\mathbf{c} = (c_1, \dots, c_n)$ for a Coxeter element c as a prefix. Moreover, the word $(\psi(c_1), \dots, \psi(c_n))$ is a suffix of Q, up to commutations. Observe that a word has intervening neighbors if and only if it is a prefix of $\mathbf{c}^\infty$ up to commutations, see [19, Section 3]. In view of Lemma 2.25 and the equality $\delta(Q) = w_\circ$, the word Q has, up to commutations, $\mathbf{w}_\circ(\mathbf{c})$ as a prefix. If the length of Q equals $w_\circ$ the proof ends here with $k = 0$. Otherwise, the analogous argument for $\operatorname{rev}(Q)$ gives that the word $\operatorname{rev}(Q)$ has, up to commutations, $\mathbf{w}_\circ(\psi(\operatorname{rev}(\mathbf{c})))$ as a prefix. By Remark 2.43, the word $\mathbf{w}_\circ(\psi(\operatorname{rev}(\mathbf{c})))$ is, up commutations, equal to the reverse of $\mathbf{w}_\circ(\mathbf{c})$. Therefore, Q has the word $\mathbf{w}_\circ(\mathbf{c})$ also as a suffix. Since $\mathbf{c} = (c_1, \dots, c_n)$ is a prefix of Q and of $\mathbf{w}_\circ(\mathbf{c})$, and Q has intervening neighbors, Q coincides with $\mathbf{c}^k \mathbf{w}_\circ(\mathbf{c})$ up to commutations. Moreover, if Q is equal to $\mathbf{c}^k \mathbf{w}_\circ(\mathbf{c})$ up to commutations, it has intervening neighbors, and a suffix $(\psi(c_1), \dots, \psi(c_n))$, up to commutations, by Proposition 2.38. This implies that the word Q has the SIN-property. $\square$

Remark 2.44. In light of Theorem 2.7 and Section 2.3.2, starting with a word Q having the SIN-property suffices to construct a multi-cluster complex, and choosing a particular connected subword in the bi-infinite word $\widetilde{Q}$, defined in Section 2.3.2, corresponds to choosing a particular Coxeter element.

We finish this section with a simple observation on the bi-infinite word $\widetilde{Q}$. For any letter q in the word $Q\psi(Q)$, let β_q be the root obtained by applying the prefix w_q of $Q\psi(Q)$ before q to the simple root α_q. To obtain roots for all letters in $\widetilde{Q}$, repeat this association periodically.

Proposition 2.45. *Let Q be a word in S having the SIN-property, and let q, q' be two consecutive occurrences of the same letter s in $\widetilde{Q}$. Then*

$$\beta_q + \beta_{q'} = \sum_p -a_{sp}\beta_p,$$

where the sum ranges over the collection of letters p in $\widetilde{Q}$ between q and q' corresponding to neighbors of s in the Coxeter graph, and where $(a_{st})_{s,t\in S}$ is the corresponding Cartan matrix.

Proof. Without loss of generality, we can assume that q is the first letter in some occurrence of Q, as otherwise, we can shift Q accordingly. Let $w_{\langle s\rangle}$ be the product of all neighbors of s in the Coxeter graph (in any order, as they all commute). The result follows from a direct calculation.

$$\beta_q + \beta_{q'} = \alpha_s + sw_{\langle s\rangle}(\alpha_s) = \alpha_s + s\big(\alpha_s + \sum_p -a_{sp}\alpha_p\big) = \sum_p -a_{sp}s(\alpha_p) = \sum_p -a_{sp}\beta_p,$$

where the first equality comes from the fact that Q has the SIN-property, the second comes from the fact that $p(\alpha_s) = \alpha_s - a_{sp}\alpha_p$, and that any two neighbors of s in the Coxeter graph commute, while the last two are trivial calculations. □

2.8 Multi-cluster complexes, Auslander-Reiten quivers, and repetition quivers

In this section, we connect multi-cluster complexes to Auslander-Reiten quivers and repetition quivers by introducing an action on vertices and facets in the multi-cluster complex. Auslander-Reiten and repetition quivers play a crucial role in *Auslander-Reiten theory* which studies the representation theory of Artinian rings and quivers. The Auslander-Reiten quiver Γ_Ω of a quiver Ω encodes the irreducible morphisms between isomorphism classes of indecomposable representations of right modules over $\mathbf{k}\Omega$. These were introduced by M. Auslander and I. Reiten in [3, 4]. We also refer to [5, 27] for further background.

2.8.1 The Auslander-Reiten quiver

In types A, D and E, sorting words of $w_\circ$ are intimately related to Auslander-Reiten quivers. Starting with a quiver Ω_c associated to a Coxeter element c (as described in Section 2.2), one can construct combinatorially the Auslander-Reiten quiver Γ_{Ω_c}, see [6, Section 2.6]. R. Bédard then shows how the Auslander-Reiten quiver provides all reduced expressions for $w_\circ$ adapted to Ω_c, i.e., the words equal to $\mathbf{w}_\circ(\mathbf{c})$ up to commutations. K. Igusa and R. Schiffler use these connections in order to obtain their description of $\mathbf{c}\mathbf{w}_\circ(\mathbf{c})$, see [39, Sections 2.1–2.3]. Conversely, given the c-sorting word $\mathbf{w}_\circ(\mathbf{c})$, one can recover the Auslander-Reiten quiver Γ_{Ω_c}, see [84, Proposition 1.2] and the discussion preceding it. Algorithm 2.40 thus provides a way to construct the Auslander-Reiten quiver in finite types using only Coxeter group theory; it uses results on admissible sequences [76] and words with intervening neighbors [20].

Algorithm 2.46. *The following fourth step added to Algorithm 2.40 yields the Auslander-Reiten quiver Γ_{Ω_c} of Ω_c.*

(iv) The vertices of Γ_{Ω_c} are the letters of $\mathbf{w}_\circ(\mathbf{c})$ and two letters q, q' of $\mathbf{w}_\circ(\mathbf{c})$ are linked by an arrow $q \longrightarrow q'$ in Γ_{Ω_c} if and only if q and q' are neighbors in the Coxeter graph and q comes directly before q' in $\mathbf{w}_\circ(\mathbf{c})$ when restricted to the letters q and q'.

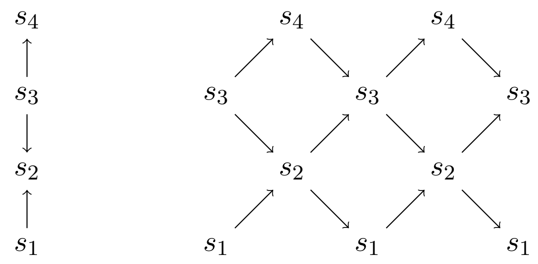
Figure 2.1 shows two examples of Auslander-Reiten quivers and how to obtain it using this algorithm.

2.8.2 The repetition quiver

Next, we define the repetition quiver.

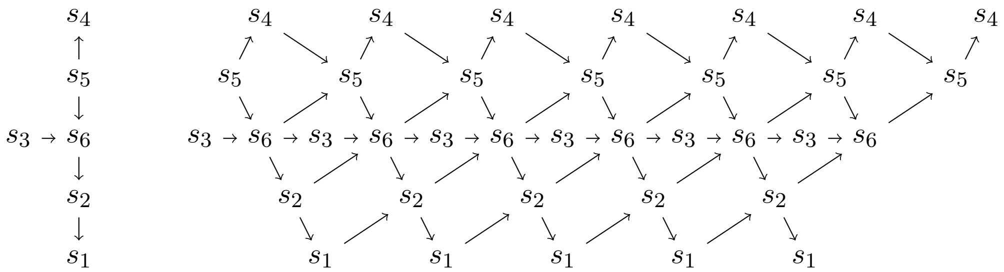
Definition 2.47 ([43, Section 2.2])**.** The *repetition quiver* $\mathbb{Z}\Omega$ of a quiver Ω consists of vertices (i, v) for a vertex v of Ω and $i \in \mathbb{Z}$. The arrows of $\mathbb{Z}\Omega$ are given by $(i, v) \longrightarrow (i, v')$ and $(i, v') \longrightarrow (i + 1, v)$, for any arrow $v \longrightarrow v'$ in Ω.

For a Coxeter element c, the repetition quiver $\mathbb{Z}\Omega_c$ turns out to be a bi-infinite sequence of Auslander-Reiten quivers Γ_{Ω_c} and $\Gamma_{\Omega_{\psi(\mathbf{c})}}$ linked at the initial $\mathbf{c}$ and the final $\psi(\mathbf{c})$. More



$\Omega_{s_1 s_3 s_2 s_4}$ $\quad$ Γ_{Ω_c}

$\mathbf{w}_\circ(\mathbf{c}) = (s_1, s_3, s_2, s_4 | s_1, s_3, s_2, s_4 | s_1, s_3)$



$\Omega_{s_3 s_5 s_6 s_2 s_4 s_1}$ $\quad$ Γ_{Ω_c}

$\mathbf{w}_\circ(\mathbf{c}) = (\mathbf{c}^5 | s_3, s_5, s_4, s_6 | s_5, s_4)$

FIGURE 2.1: Two examples of Auslander-Reiten quivers of types A_4 and E_6.

precisely, the repetition quiver $\mathbb{Z}\Omega_c$ can be obtained applying the procedure described in Algorithm 2.46 to the bi-infinite word

$$\widetilde{\mathbf{w}_\circ(\mathbf{c})} = \cdots \mathbf{w}_\circ(\mathbf{c}) \;\; \psi(\mathbf{w}_\circ(\mathbf{c})) \;\; \mathbf{w}_\circ(\mathbf{c}) \;\; \psi(\mathbf{w}_\circ(\mathbf{c})) \cdots .$$

As discussed in Remark 2.44, the word $\widetilde{\mathbf{w}_\circ(\mathbf{c})}$ does not depend on the choice of a Coxeter element. Therefore, the repetition quiver is independent of the choice of Coxeter element, as expected. The repetition quiver comes equipped with the *Auslander-Reiten translate* τ given by $\tau(i, v) = (i-1, v)$. A second natural map acts on the vertices of the repetition quiver: the *shift operation* $[1] : \mathbb{Z}\Omega_c \longrightarrow \mathbb{Z}\Omega_c$ which sends a vertex in $\mathbf{w}_\circ(\mathbf{c})$ or $\psi(\mathbf{w}_\circ(\mathbf{c}))$ to the corresponding vertex in the next (to the right) copy of $\psi(\mathbf{w}_\circ(\mathbf{c}))$ or of $\mathbf{w}_\circ(\mathbf{c})$ respectively. In Figure 2.2, we present an example of a repetition quiver of type A_4. Copies of the Auslander-Reiten quivers Γ_{Ω_c} and $\Gamma_{\Omega_{\psi(\mathbf{c})}}$ are separated by dashed arrows. The Auslander-Reiten translate τ sends a vertex to the one located directly to its left. One orbit of the shift operation shown in **bold**, we have $(4, s_1) = [1](1, s_4) = [2](-1, s_1)$.

Remark 2.48. Vertices in the Auslander-Reiten quiver correspond to (isomorphism classes of) indecomposable representations of Ω_c, and thus have a *dimension vector* attached. By the *knitting algorithm*, the dimension vector at a vertex $V = (i, v)$ of Γ_{Ω_c} plus the

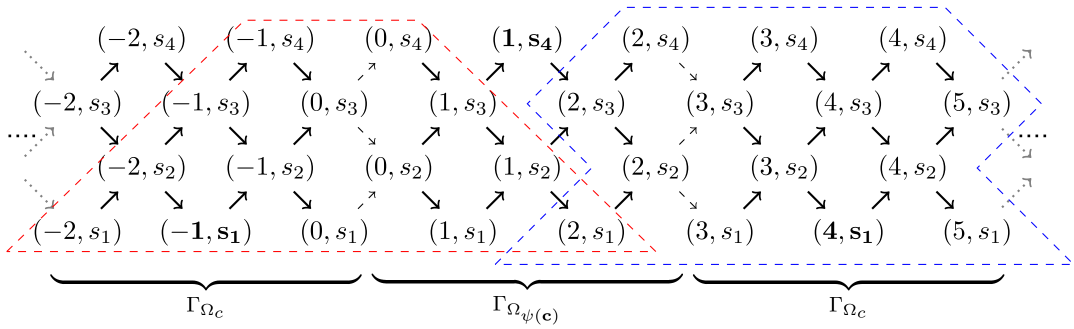


FIGURE 2.2: The repetition quiver of type A_4 with the quiver Ω_c associated to the Coxeter element $c = s_1s_3s_2s_4$.

dimension vector at the vertex $\tau(V)$ equals the sum of all dimension vectors at vertices V' for which $\tau(V) \longrightarrow V' \longrightarrow V$ are arrows in Γ_{Ω_c}, see [28, Section 10.2]. This procedure is intimately related to the SIN-property, which ensures that this sum is indeed over all neighbors of v. Moreover, Proposition 2.45 implies that this property holds as well for the root β_q attached to a letter q in the bi-infinite sequence $\widetilde{Q}$. This yields the well known property that the dimension vector and the corresponding root coincide.

The following proposition describes words for the multi-cluster complex using the repetition quiver, the Auslander-Reiten translate, and the shift operation.

Proposition 2.49. *Let Ω_c be a quiver corresponding to a Coxeter element c. Words for the multi-cluster complex are obtained from the bi-infinite word $\widetilde{\mathbf{w}_\circ(\mathbf{c})}$ by setting $\tau^k = [1]$. Choosing a particular fundamental domain for this identification corresponds to choosing a particular Coxeter element. In other words, words for multi-cluster complexes are obtained by a choice of linear extension of a fundamental domain of the identification $\tau^k = [1]$ in the repetition quiver.*

Proof. With the identification $[1]V = \tau^k V$ in the repetition quiver, a fundamental domain will consist of k copies of Ω_c and one copy of the Auslander-Reiten quiver Γ_{Ω_c}. This fundamental domain is exactly the quiver formed from the word $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$ using Algorithm 2.46. As linear extensions of this quiver correspond to words equal to $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$ up to commutations, the result follows. □

The red and the blue boxes in Figure 2.2 mark two particular choices of a fundamental domain for the multi-cluster complex of type A_4 with $k = 1$ corresponding to the Coxeter elements $s_1s_2s_3s_4$ and $s_1s_3s_2s_4$ respectively.

2.8.3 The Auslander-Reiten translate on multi-cluster complexes

The Auslander-Reiten translate gives a cyclic action on the vertices and facets of a multi-cluster complex. This action corresponds to natural actions on multi-triangulations in types A and B, and is well studied in the case of cluster complexes.

Definition 2.50. Let $Q = \mathbf{c}^k \mathbf{w}_\circ(\mathbf{c})$. The permutation $\Theta : Q \xrightarrow{\sim} Q$ is given by sending a letter $q_i = s$ to the next occurrence of s in Q, if possible, and to the first occurrence of $\psi(s)$ in Q otherwise.

Observe that in types ADE, the operation Θ corresponds to the inverse of the Auslander-Reiten translate, $\Theta = \tau^{-1}$ when considered within the repetition quiver.

Proposition 2.51. *The permutation* Θ *induces a cyclic action on the facets of* $\Delta(Q, w_\circ)$.

Proof. By Proposition 2.24, the subword complexes $\Delta(Q, w_\circ)$ and $\Delta(Q_{\circlearrowleft s}, w_\circ)$ are isomorphic for an initial letter s in Q. Proposition 2.27 asserts that $\mathbf{c}^k \mathbf{w}_\circ(\mathbf{c})$ and the rotated word obtained from $\mathbf{c}^k \mathbf{w}_\circ(\mathbf{c})$ by rotating n times are equal up to commutations. By construction, Θ is the automorphism of $\Delta(Q, w_\circ)$ given by inverse rotation of $\mathbf{c}$. $\square$

Example 2.52. As in Example 2.41, consider $c = s_1 s_3 s_2 s_4$ and $Q = \mathbf{c}\mathbf{w}_\circ(\mathbf{c}) = (q_i : 1 \leq i \leq 14) = (\mathbf{c}^2 | s_1 s_3 s_2 s_4 | s_1, s_3)$. After rotating along all letters in $\mathbf{c}$ from the right, we obtain the word $(s_3 s_1 s_4 s_2 | \mathbf{c}^2 | s_1, s_3)$, so we have to reorder the initial 4 letters using commutations to obtain again $(\mathbf{c}^3 | s_1, s_3)$. Therefore, Θ permutes the letter of Q along the permutation of the indices given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 2 & 1 & 4 & 3 \end{pmatrix}.$$

Here is an example of an orbit of Θ.

$$\begin{array}{l} \{q_1, q_2, q_3, q_4\} \mapsto_\Theta \{q_5, q_6, q_7, q_8\} \mapsto_\Theta \{q_9, q_{10}, q_{11}, q_{12}\} \mapsto_\Theta \{q_{13}, q_{14}, q_2, q_1\} \\ \mapsto_\Theta \{q_4, q_3, q_6, q_5\} \mapsto_\Theta \{q_8, q_7, q_{10}, q_9\} \mapsto_\Theta \{q_{12}, q_{11}, q_{14}, q_{13}\} \mapsto_\Theta \{q_1, q_2, q_3, q_4\}. \end{array}$$

To relate the permutation Θ to clusters, we recall the definition of bipartite Coxeter elements; consider a bipartition of the set $S = S_- \sqcup S_+$ such that any two generators in S_ϵ commute (this is possible since the graph of the Coxeter group is a tree), then form the Coxeter element $c^* = c_- c_+$, where $c_\epsilon = \prod_{s \in S_\epsilon} s$. Using the bijection Lr_{c^*} between letters in $\mathbf{c}^* \mathbf{w}_\circ(\mathbf{c}^*)$ and almost positive roots, the cyclic action induced by Θ is equal to the action induced by the *tropical Coxeter element*

$$\sigma_{c^*} := \prod_{s \in S_-} \sigma_s \prod_{s \in S_+} \sigma_s$$

on almost positive roots, see Section 2.2.3 for the definition of σ_s, and [2, Section 5.2] for more details about tropical Coxeter elements. In the case of cluster complexes, S. Fomin and N. Reading computed the order of Θ [23, Theorem 4.14]. Since the words $\mathbf{c}\mathbf{w}_\circ(\mathbf{c})$ are all connected via rotation along initial letters, the order of Θ does not depend on a specific choice of Coxeter element.

Theorem 2.53. *For $Q = \mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$, the order of Θ is given by*

$$\operatorname{ord}(\Theta) = \begin{cases} k + h/2 & \text{if } w_\circ = -\mathbf{1}, \\ 2k + h & \text{if } w_\circ \neq -\mathbf{1}. \end{cases}$$

Proof. To obtain the order of this action, we consider the length of Q divided by the length of $\mathbf{c}$ if $w_\circ \equiv -\mathbf{1}$, and twice the length of Q divided by the length of $\mathbf{c}$ otherwise. We have already seen in Algorithm 2.40 that the length of Q is given by $kn + nh/2$. As the length of $\mathbf{c}$ is given by n, the result follows. □

Remark 2.54. The action induced by the tropical Coxeter element on facets of the cluster complex was shown by S.-P. Eu and T.-S. Fu to exhibit a *cyclic sieving phenomenon* [21]. Therefore, the cyclic action induced by Θ exhibits a cyclic sieving phenomenon for facets of the cluster complex $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$ and *any* Coxeter element c.

Finally, for types A and B, the cyclic action $\Theta : Q \xrightarrow{\sim} Q$ corresponds to the cyclic action induced by rotation of the associated polygons.

Theorem 2.55. *Let $Q = \mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$. In type A_{m-2k-1}, the cyclic action Θ on letters in Q corresponds to the cyclic action induced by rotation on the set of k-relevant diagonals of a convex m-gon. In type B_{m-k}, the cyclic action Θ corresponds to the cyclic action induced by rotation on the set of k-relevant centrally symmetric diagonals of a regular convex $2m$-gon.*

Proof. The simplicial complex of k-triangulations of a convex m-gon is isomorphic to the multi-cluster complex of type A_{m-2k-1}, so the order of Θ is given by $2k + h = 2k + m - 2k = m$ as expected. The simplicial complex of centrally symmetric k-triangulations of a regular convex $2m$-gon is isomorphic to the multi-cluster complex of type B_{m-k}, so the order of Θ equals $k + h/2 = k + m - k = m$, as well. In type A, the result follows from the correspondence between letters in Q and k-relevant diagonals in the m-gon as described in Section 2.2.1. In type B, the result follows from the correspondence between letters in Q and k-relevant centrally symmetric diagonals in the $2m$-gon as described in Section 2.2.4. □

2.9 Open problems

We discuss open problems and present several conjectures. We start with two open problems concerning counting formulas for multi-cluster complexes.

Open Problem 2.56. Find *multi-Catalan numbers* counting the number of facets in the multi-cluster complex.

Although a formula in terms of invariants of the group for the number of facets of the generalized cluster complex defined by S. Fomin and N. Reading is known [22, Proposition 8.4], a general formula in terms of invariants of the group for the multi-cluster complex is yet to be found. An explicit formula for type A can be found in [41, Corollary 17]. In type B, a formula was conjectured in [75, Conjecture 13] and proved in [68][4]. In the dihedral type $I_2(m)$, the number of facets of the multi-cluster complex is equal to the number of facets of a $2k$-dimensional cyclic polytope on $2k+m$ vertices. These three formulas can be reformulated in terms of invariants of the Coxeter groups of type A, B and I_2 as follows,

$$\prod_{0\leq j<k}\prod_{1\leq i\leq n}\frac{d_i+h+2j}{d_i+2j},$$

where $d_1 \leq \ldots \leq d_n$ are the *degrees* of the corresponding group, and h is its Coxeter number. In general, this product is not an integer. The smallest example we are aware of is type D_6 with $k=5$. Therefore, this product cannot count facets of the multi-cluster complex in general. The cyclic action Θ (see Definition 2.50) on multi-cluster complexes might be useful to solve Open Problem 2.56, it gives rise to the following generalization.

Open Problem 2.57. Find *multi-Catalan polynomials* $f(q)$ such that the triple

$$\Big(\big\{\text{facets of } \Delta(\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c}), w_\circ)\big\}, f(q), \Theta\Big)$$

exhibits the cyclic sieving phenomenon as defined by V. Reiner, D. Stanton, and D. White in [65].

In types A, B, and I_2, there is actually a natural candidate for $f(q)$, namely

$$\prod_{0\leq j<k}\prod_{1\leq i\leq n}\frac{[d_i+h+2j]_q}{[d_i+2j]_q},$$

where $[m]_q = 1+q+\ldots+q^{m-1}$ is a q-analogue of the integer m. In the case of multi-triangulations and centrally symmetric multi-triangulations, this triple is conjectured

[4] The proof appeared in Section 7 in the arxiv version, see `http://arxiv.org/abs/0904.1097v2`.

to exhibit the cyclic sieving phenomenon.[5] The counting formula in types A, B and I_2 can be enriched with a parameter m such that it reduces for $k = 1$ to the Fuss-Catalan numbers counting the number of facets in the generalized cluster complexes. The next open problem raises the question of finding a family of simplicial complexes that includes the generalized cluster complexes of S. Fomin and N. Reading and the multi-cluster complexes.

Open Problem 2.58. Construct a family of simplicial complexes which simultaneously contains generalized cluster complexes and multi-cluster complexes.

The next open problem concerns a possible representation theoretic description of the multi-cluster complex in types ADE. For $k = 1$, one can describe the compatibility by saying that $V \parallel_c V'$ if and only $\dim(\operatorname{Ext}^1(V, V')) = 0$, see [11].

Open Problem 2.59. Describe the multi-cluster complex within the repetition quiver using similar methods.

The following problem extends the diameter problem of the associahedron to the family of multi-cluster complexes, see [53, Section 2.3.2] for further discussions in the case of multi-triangulations.

Open Problem 2.60. Find the diameter of the facet-adjacency graph of the multi-cluster complex $\Delta_c^k(W)$.

Finally, we present several combinatorial conjectures on the multi-cluster complexes. We start with a conjecture concerning minimal non-faces.

Conjecture 2.61. *Minimal non-faces of the multi-cluster complex $\Delta_c^k(W)$ have cardinality $k + 1$.*

Since $w_\circ$ is c-sortable, we have $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c}) = \mathbf{c}^k\mathbf{c}_{K_1}\mathbf{c}_{K_2}\cdots\mathbf{c}_{K_r}$ with $K_r \subseteq \ldots \subseteq K_2 \subseteq K_1$. This implies that the complement of any k letters still contains a reduced expression for $w_\circ$. In other words, minimal non-faces have at least cardinality $k + 1$. Moreover, using the connection to multi-triangulations and centrally symmetric triangulations, we see that the conjecture holds in types A and B. It also holds in the case of dihedral groups: it is not hard to see that the faces of the multi-cluster complex are given by subwords of $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c}) = (a, b, a, b, \ldots)$ which do not contain $k+1$ pairwise non-consecutive letters (considered cyclically). The conjecture was moreover tested for all multi-cluster complexes of rank 3 and 4 with $k = 2$.

[5]Personal communication with V. Reiner.

In types A and $I_2(m)$, there is a binary compatibility relation on the letters of $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$ such that the faces of the multi-cluster complex can be described as subsets avoiding $k+1$ pairwise not compatible elements. We remark that this is not possible in general: in type B_3 with $k=2$, as in Example 2.11, $\Delta_c^2(B_3)$ is isomorphic to the simplicial complex of centrally symmetric 2-triangulations of a regular convex 10-gon. Every pair of elements in the set $\mathcal{A} = \{[1,4]_{\text{sym}}, [4,7]_{\text{sym}}, [7,10]_{\text{sym}}\}$ is contained in a minimal non-face. But since $\mathcal{A}$ does not contain a 3-crossing, it forms a face of $\Delta_c^2(B_3)$.

Theorem 2.7 gives an alternative way of defining multi-cluster complexes as subword complexes $\Delta(Q, w_\circ)$ where the word Q has the SIN-property. It seems that this definition covers indeed all subword complexes isomorphic to multi-cluster complexes.

Conjecture 2.62. *Let Q be a word in S with complete support and $\pi \in W$. The subword complex $\Delta(Q,\pi)$ is isomorphic to a multi-cluster complex if and only if Q has the SIN-property and $\pi = \delta(Q) = w_\circ$.*

The fact that $\pi = \delta(Q)$ is indeed necessary so that the subword complex is a sphere. It remains to show that $\pi = w_\circ$ and that Q has the SIN-property. One reason for this conjecture is that if Q does not have the SIN-property then it seems that the subword complex $\Delta(Q, w_\circ)$ has fewer facets than required. Indeed, we conjecture that that multi-cluster complexes maximize the number of facets among all subword complexes with a word Q of a given size.

Conjecture 2.63. *Let Q be any word in S with $kn+N$ letters (where N denotes the length of $w_\circ$) and $\Delta(Q,w_\circ)$ be the corresponding subword complex. The number of facets of $\Delta(Q,w_\circ)$ is less than or equal to the number of facets of the multi-cluster complex $\Delta_c^k(W)$. Moreover, if both numbers are equal, then the word Q has the SIN-property.*

In fact, the last two conjectures hold for dihedral groups $I_2(m)$. In this case, the multi-cluster complex is isomorphic to the boundary complex of a cyclic polytope, which is a polytope that maximizes the number of facets among all polytopes in fixed dimension on a given number of vertices.

In view of Corollary 2.15, the following conjecture restricts the study of [44, Question 6.4].

Conjecture 2.64. *The multi-cluster complex is the boundary complex of a simplicial polytope.*

In types A and B, this conjecture coincides with the conjecture on the existence of the corresponding multi-associahedra, see [41, 75], and Theorem 2.36 shows that this conjecture is true for dihedral groups.

Chapter 3

Computational realizations of multi-associahedra and spherical subword complexes

In this chapter we show a new point of view on the problem of polytopality of multi-associahedra and spherical subword complexes, and present two computational methods to find polytopal realizations for small explicit examples. The implementation of these methods was done in joint work with Jean-Philippe Labbé using the computer algebra system `Sage` [80]. Discussions with Vincent Pilaud also influenced a lot the results in this chapter.

We start by recalling the notion of *multi-associahedra*. Let $k \geq 1$ and $m \geq 2k+1$ be two positive integers. As introduced in Section 2.2.1, we denote by $\Delta_{m,k}$ the simplicial complex of k-triangulations of a convex m-gon. The vertices of this complex are given by *k-relevant diagonals* of the m-gon, and its faces are $(k+1)$-*crossing-free* sets of k-relevant diagonals. For $k = 1$, this complex coincides with the boundary complex of a dual associahedron. For this reason, we refer to $\Delta_{m,k}$ as the *simplicial multi-associahedron*.

The combinatorial structure of the multi-associahedron has been studied by several authors. Apparently, it first appeared in work of Capoyleas and Pach [13], who showed that the maximal number of diagonals in a $(k+1)$-crossing-free set is equal to $k(2m-2k-1)$. Nakamigawa [52] introduced the flip operation on k-triangulations and proved that the flip graph is connected. Dress, Koolen and Moulton [18] obtained a reformulation of the Capoyleas–Pach result, and in particular proved that all maximal $(k+1)$-crossing-free sets of diagonals have the same number of diagonals. The results of Nakamigawa and Dress–Koolen–Moulton imply that the multi-associahedron $\Delta_{m,k}$ is a pure simplicial complex of dimension $k(m-2k-1)-1$. A more recent approach for the study of

k-triangulations, using star polygons, was given by Pilaud and Santos [55]. In 2003, Jonsson [40] showed that the multi-associahedron is a piecewise linear sphere. He also found an explicit $k \times k$ determinantal formula of Catalan numbers counting the number of k-triangulations [41]. Additionally to the result of Jonsson about the multi-associahedron being a topological sphere, Stump [82] proved that it is a vertex-decomposable, and thus in particular shellable, simplicial sphere. See also the results by Serrano and Stump [70].

All these results suggest that the simplicial multi-associahedron $\Delta_{m,k}$ could be realized as the boundary complex of a simplicial polytope of dimension $k(m-2k-1)$. However, while for the classical associahedron we have many different construction methods (see Chapter 1), all the natural approaches seem to fail for the multi-associahedron. At the moment, very few cases of the multi-associahedron are known to be polytopal (see Section 2.6.2). Currently, the smallest open case is for $m = 9$ and $k = 2$. Is there a simplicial polytope of dimension 8 and f-vector $(18, 153, 732, 2115, 3762, 4026, 2376, 594)$ which realizes the simplicial multi-associahedron $\Delta_{9,2}$?

In Chapter 2, we have seen that the work of Ceballos, Labbé and Stump [14] provides a natural generalization of multi-associahedra to arbitrary finite Coxeter groups. The description of *generalized multi-associahedra* is based on the notion of subword complexes by Knutson and Miller [44]. We use this description to produce computational polytopal realizations of the simplicial multi-associahedron of type A_3 for $k = 2$, which corresponds to the simplicial complex $\Delta_{8,2}$. Different realizations of this polytope were given by Jürgen Bokowski and Vincent Pilaud in [9]. Our implementation can, in theory, be used to find polytopal realizations of spherical subword complexes in general. Of course this note is written with the hope that these methods will be applied with more advanced computer software and better implementations to find polytopal realizations for even more interesting bigger examples.

3.1 Primal problem and dual problem

In this section, (W, S) denotes a finite Coxeter system, Q is a word in the generators in S, while $w_\circ$ is the longest element of the group W, and $\Delta(Q, w_\circ)$ denotes the subword complex associated to Q and $w_\circ$ as described in Chapter 2. The objective of this section is to solve the following problem:

Primal Realization Problem 3.1. Find a complete simplicial fan which realizes a spherical subword complex $\Delta(Q, w_\circ)$ where

$$Q = (\ell_r, \dots \ell_2, \ell_1,\ w_1, \dots, w_N),$$

and $w_1, \ldots, w_N$ is a reduced expression of $w_\circ$.

In particular, we are interested in subword complexes giving rise to generalized multi-associahedra. A (simple) *generalized multi-associahedron* is a simple polytope whose normal fan is a solution of the Realization Problem 3.1, where the word Q is of the form $Q = \mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$ for some Coxeter element c. However, not every complete simplicial fan is the normal fan of a polytope, and so, after solving problem 3.1 we still need to check that the fan is regular.

In the same spirit of the proof of Theorem 1.37 in Section 1.5.1, the Primal Problem 3.1 is equivalent to find a matrix $M \in \mathbb{R}^{r\times m}$ for $m = r + N$, whose column vectors are associated to the letters of Q, and such that the following three conditions hold:

1. The vectors associated to a facet of $\Delta(Q, w_\circ)$ form a basis of $\mathbb{R}^r$.
2. If I and J are two adjacent facets that differ by a flip, that is $I\setminus\{i\} = J\setminus\{j\}$. Then the vectors associated to i and j lie in opposite sides of the hyperplane generated by the vectors associated to the intersection $I \cap J$.
3. There is a facet for which the interior of its associated cone is not intersected by any other cone.

We will show that conditions 1. and 2. are equivalent to a *dual realization problem* (Theorem 3.7). Condition 3. ensures that the fan is complete. Before we proceed to state the dual problem, we need to introduce the *sign function* for reduced expressions of $w_\circ$.

Definition 3.2. The *sign function* is a map

$$\text{sign}: \quad \text{reduced expressions of } w_\circ \quad \to \quad \{1, -1\}$$

such that if P and P' are two reduced expressions connected by a flip with $P\setminus p = P'\setminus p'$, then

$$\text{sign}(P') = (-1)^{\text{position of } p \text{ in } P - \text{ position of } p' \text{ in } P'} \cdot \text{sign}(P).$$

Remark 3.3. If such a sign function exists, it is uniquely defined up to a global multiplication by -1. Observe however that it is a priori not clear whether such a function always exists, but the next proposition shows that if this is not the case then there is a counterexample to the existence of generalized multi-associahedra. We also remark that the sign function seems to be well defined for reduced expressions of any element w in W, and that the results presented in this section are valid for general spherical subword complexes.

Proposition 3.4. *If the sign function in the previous definition is not well defined then there are examples of spherical subword complexes that are not polytopal.*

The proof of this proposition will be given in Section 3.2.

Dual Realization Problem 3.5. Find a matrix $M \in \mathbb{R}^{N\times m}$ for $m = r + N$, whose column vectors are associated to the letters of Q, and such that for every reduced expression $P \subset Q$ of $w_\circ$, we have that

$$\mathrm{sign}(P) \cdot \mathrm{Det}(P) > 0,$$

where $\mathrm{Det}(P)$ is the determinant of the matrix M restricted to P.

Remark 3.6. Notice that $\mathrm{sign}(P)$ only depends on the reduced expression represented by P and not on its explicit position within the word Q.

Given a matrix M, we say that M^G is a *Gale dual matrix* of M if the rows of M^G form a basis for the kernel of M [1]. Note that this dual matrix is determined up to linear transformation of the rows.

Theorem 3.7. *Let $M \in \mathbb{R}^{N\times m}$ and $M^G \in \mathbb{R}^{r\times m}$ be a Gale dual matrix of M. The following statements are equivalent:*

1. *M is a solution of the Dual Realization Problem 3.5.*
2. *M^G satisfies conditions 1. and 2. of the Primal Realization Problem 3.1.*

Example 3.8 (Realizations of multi-associahedra of type A_2)**.** Let $c = s_1s_2$ be a Coxeter element of type A_2, and let $Q = \mathbf{c}^k\mathbf{w}_\circ(\mathbf{c}) = (s_1, s_2, \dots s_1, s_2,\ s_1, s_2, s_1)$. The sign function for reduced expressions of $w_\circ$ in type A_2 is given by

$$\mathrm{sign}(s_1s_2s_1) = \mathrm{sign}(s_2s_1s_2) = 1.$$

Let $a_1, \dots, a_k$ and $b_1, \dots, b_k$ be real numbers such that $a_i < a_j < 0$ and $b_i > b_j > 0$ for every $1 \le i < j \le k$. Then, the matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & \dots & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 1 & 0 & 1 & 0 \\ a_1 & b_1 & a_2 & b_2 & \dots & a_k & b_k & 0 & 0 & 1 \end{pmatrix}$$

[1] The Gale dual matrix we use is different to the Gale transform, because we do not use an extra row filled with ones in our definition

is a solution for the Dual Realization Problem 3.5. Moreover, the Gale dual matrix

$$M^G = \left(\begin{array}{c|ccc} & 1 & 0 & a_1 \\ & 0 & 1 & b_1 \\ & 1 & 0 & a_2 \\ -I_{2k\times 2k} & 0 & 1 & b_2 \\ & \vdots & \vdots & \vdots \\ & 1 & 0 & a_k \\ & 0 & 1 & b_k \end{array} \right)$$

is a solution for the Primal Realization Problem 3.1. Conditions i. and ii. follow from Theorem 3.7, and condition iii. can be easily checked by inspection (one way of checking this is to see that the interior of the negative orthant is intersected only by the cone corresponding to facet $\mathbf{c}^k$).

3.2 Proof of Theorem 3.7 and Proposition 3.4

Proof of Theorem 3.7. Let $M \in \mathbb{R}^{N\times m}$ and $M^G \in \mathbb{R}^{r\times m}$ be a Gale dual matrix of M. By Gale duality, M^G satisfies conditions 1. and 2. of the Primal Realization Problem 3.1 if and only if M satisfies the following two conditions:

1. The vectors associated to the complement of a facet of $\Delta(Q, w_\circ)$ form a basis of $\mathbb{R}^N$.
2. If I and J are two adjacent facets that differ by a flip, that is $I\backslash\{i\} = J\backslash\{j\}$. Then the vectors associated to i and j lie in the same side of the hyperplane generated by the vectors associated to the complement of $I \cup J$.

Condition 1. implies that for every reduced expression $P \subset Q$ of $w_\circ$ the determinant $\mathrm{Det}(P)$ is different from zero. If we set the sign and the determinant of $w_1 \dots w_N \subset Q$ to be positive, then condition 2. implies that the sign of the determinant of P is determined by

$$\mathrm{sign}(P) \cdot \mathrm{Det}(P) > 0$$

Conversely, these inequalities imply both condition 1. and condition 2. □

Proof of Proposition 3.4. Suppose that the sign function is not well defined. That means that there is a reduced expression P of $w_\circ$ that can be obtained from $w_1 \dots w_N$ by using flips in two different ways such that one obtains two different signs for P. Pick a sufficiently big word Q such that the two chains of flips to obtain P correspond to flips in the subword complex $\Delta(Q, w_\circ)$. Then, if this subword complex was realizable as a polytope there would be a solution for the Dual Realization Problem 3.5, and we would have two subwords $\mathbf{P_1}$ and $\mathbf{P_2}$ of Q representing the same reduced expression of $w_\circ$ given by P, such that the determinants $\mathrm{Det}(\mathbf{P_1})$ and $\mathrm{Det}(\mathbf{P_2})$ have different signs. Let $\mathbf{P}$ be the left most reduced expression of $w_\circ$ in Q given by P. Both $\mathbf{P_1}$ and $\mathbf{P_2}$ can be connected by flips to $\mathbf{P}$ without changing the signs at any step: start by flipping the first letter of $\mathbf{P_1}$ to the first letter of $\mathbf{P}$, then the second to the first and so on. Thus, the signs of the determinants of $\mathbf{P_1}$ and $\mathbf{P_2}$ are both equal to the sign of the determinant of $\mathbf{P}$, which is a contradiction. □

3.3 Computational method 1

In this section we present a computational method to find solutions for the Primal Realization Problem 3.1. We applied this method to the particular case of the simplicial multi-associahedron of type A_3 for $k = 2$, which corresponds to the simplicial complex $\Delta_{8,2}$. The input of the method is a word $Q = (\ell_r, \dots, \ell_2, \ell_1,\ w_1, \dots, w_N)$ in a finite Coxeter group, where $w_1 \dots w_N$ is a reduced expression of $w_\circ$, and the output is a complete simplicial fan realizing the subword complex $\Delta(Q, w_\circ)$.

We start with the $N \times N$ identity matrix M_0, whose column vectors are associated to the letters $w_1, \dots, w_N$ of Q, and add column vectors on the left of this matrix, one at a time for each letter $\ell_r, \dots, \ell_1$, such that the following happens:

i. at the i-th step the matrix $M_i \in \mathbb{R}^{N\times(i+N)}$, obtained after adding i vectors, is a solution of the Dual Realization Problem 3.5 for $Q_i = (\ell_i, \dots, \ell_2, \ell_1,\ w_1, \dots, w_N)$, and

ii. the fan determined by the Gale dual matrix M_i^G, whose maximal cones correspond to the facets of $\Delta(Q_i, w_\circ)$, is complete.

After having found a solution for the matrix M_i, condition i. for the matrix M_{i+1} is equivalent to finding a solution of a system of linear inequalities, while condition ii. can be verified using for example `Polymake` [29] or `Sage` [80].

In joint work with Jean-Philippe Labbé, this method was implemented using the computer algebra system `Sage` [80]. The implementation finds several solutions for the matrix M_i at each step, and repeats the process for each of these solutions in the next step. Using this algorithm, we were able to find several solutions for the word $Q = (s_1, s_2, s_3, s_1, s_2, s_3,\ s_1, s_2, s_3, s_1, s_2, s_1)$ in type A_3. The subword complex $\Delta(Q, w_\circ)$ in this case is isomorphic to the simplicial multi-associahedron $\Delta_{8,2}$.

One of the solutions of the of the Dual Realization Problem 3.5, which we found for this word Q, is the matrix

$$M = \begin{pmatrix} 1/16 & -1/8 & -1/4 & 1/32 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1/8 & -1/8 & -1/2 & 1/32 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & -1/4 & 3/32 & 0 & -1/4 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1/2 & 9/16 & 1/2 & 3/8 & 1/2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3/8 & 1/2 & 1/2 & 5/8 & 0 & 1/4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 3/4 & -5/8 & 1/4 & 21/32 & -1/2 & 1/4 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Its Gale dual matrix

$$M^G = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1/16 & 1/8 & 1/4 & -1/2 & 3/8 & 3/4 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1/8 & -1/8 & 0 & 9/16 & 1/2 & -5/8 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1/4 & -1/2 & -1/4 & 1/2 & 1/2 & 1/4 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1/32 & 1/32 & 3/32 & 3/8 & 5/8 & 21/32 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1/4 & 0 & 1/4 & 1/4 \end{pmatrix}$$

determines a complete simplicial fan realizing the simplicial multi-associahedron $\Delta_{8,2}$. Its maximal cones have rays corresponding to the column vectors associated to facets of the subword complex $\Delta(Q, w_\circ)$.

A fan obtained as a solution of this method is not necessarily the normal fan of a polytope. For this particular solution, according to computations in `Sage` [80], the fan is not the normal fan of a polytope.

It remains the question whether the Dual Realization Problem 3.5 always has a solution for arbitrary spherical subword complexes. Or in other words, whether the system of polynomial inequalities

$$\operatorname{sign}(P) \cdot \operatorname{Det}(P) > 0$$

for reduced expressions $P \subset Q$ of $w_\circ$ always has a solution. In [10], Jürgen Bokowski and Bernd Sturmfels study the realizability problem of abstract geometric objects using computational methods. They showed the existence of *final polynomials* for every non-realizable case. Is there such a final polynomial for a particular case of the Dual Realization Problem 3.5?

3.4 Computational method 2

In this section we present another computational method to find polytopal realizations of spherical subword complexes. Again, we applied this method to the particular case of the simplicial multi-associahedron of type A_3 for $k = 2$, which corresponds to the simplicial complex $\Delta_{8,2}$. The input of the method is a word $Q = (\ell_r, \ldots, \ell_2, \ell_1,\ q_1, \ldots, q_m)$ in a finite Coxeter group, where $(q_1, \ldots, q_m) = \mathbf{c}\mathbf{w}_\circ(\mathbf{c})$ for some Coxeter element c, and a polytopal realization of the dual generalized associahedron $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$. The output is a polytope realizing the subword complex $\Delta(Q, w_\circ)$.

Our method is based on known polytopal realizations of generalized associahedra, and uses regular triangulations to construct polytopal realizations for bigger spherical subword complexes. The basic idea is the following. If P is a polytope realizing a subword complex $\Delta(Q, w_\circ)$, one can easily construct a polytopal realization of a subword complex $\Delta(\ell Q, w_\circ)$ under certain *geometric-regular-triangulation* assumption. In this procedure, we always assume that all the letters of Q actually appear as vertices of $\Delta(Q, w_\circ)$, and for this reason we start the process with the word $\mathbf{c}\mathbf{w}_\circ(\mathbf{c})$.

The facets of $\Delta(\ell Q, w_\circ)$ are subdivided into two different kinds, the ones that contain the letter ℓ and the ones that do not contain the letter ℓ. The facets that contain the letter ℓ are the joint of ℓ with facets of $\Delta(Q, w_\circ)$. The facets that do not contain the letter ℓ are exactly the facets of the subword complex $\Delta(Q, \ell w_\circ)$, which actually form a combinatorial triangulation T of the polytope P. If this combinatorial triangulation is indeed a geometric triangulation, and in addition it is regular, then one can lift the vertices of P in one dimension higher and add one extra vertex corresponding to the letter ℓ to obtain a new polytope realizing $\Delta(\ell Q, w_\circ)$. More explicitly, if $v_1, \ldots, v_n$ are the vertices of P then the new polytope can be obtained as the convex hull

$$\operatorname{conv}\{(0, h_0), (v_1, h_1), \ldots, (v_n, h_n)\}$$

for certain weights $h_0, h_1, \ldots, h_n \in R$. The vertex $(0, h_0)$ corresponds to the new letter ℓ for a sufficiently big negative value h_0, and $(v_1, h_1), \ldots, (v_n, h_n)$ are the lifted vertices giving rise to the regular triangulation T.

In joint work with Jean-Philippe Labbé, this idea was implemented using the computer algebra system `Sage` [80]. We start with a polytopal realization of the dual generalized associahedron $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$, and add the letters $\ell_r, \ldots, \ell_1$, one at time, making sure that the triangulations that appear at each step are both geometric and regular. These two conditions are equivalent to verify two systems of linear inequalities.

In type A_3, we were able to find several solutions for the subword complex $\Delta(Q, w_\circ)$ where the word $Q = \mathbf{c}^2\mathbf{w}_\circ(\mathbf{c}) = (s_1, s_2, s_3, s_1, s_2, s_3,\ s_1, s_2, s_3, s_1, s_2, s_1)$. In this case, $\Delta(Q, w_\circ)$ is isomorphic to the simplicial multi-associahedron $\Delta_{8,2}$.

We started with the polytopal realization of a dual 3-dimensional associahedron which was obtained by using the Santos' construction in Chapter 1 for the seed triangulation in Figure 1.14. The vertices of this realization are given by the columns of the following matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Using the method above, we obtained the first known polytopal realizations of the simplicial multi-associahedron $\Delta_{8,2}$ with rational coordinates. One of several solutions we found is given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -3/4 & 0 & -1/4 & 0 & -1/4 & 0 & 0 & 0 & 0 & 3/4 \\ 0 & -1/2 & 0 & -1/4 & -3/16 & -1/2 & 0 & -3/16 & 5/16 & 0 & 1/4 & -3/8 \\ -5/8 & 3/16 & 5/16 & 3/16 & 1/4 & 1/16 & 3/8 & 5/32 & 3/8 & -1/32 & -3/16 & 3/16 \end{pmatrix}.$$

The column vectors of this matrix are the vertices of a polytopal realization of $\Delta_{8,2}$, which is a 6-dimensional polytope with f-vector $(12, 66, 192, 306, 252, 84)$.

The method presented in this section is different from the one presented in Section 3.3. It has the advantage that the desired polytope is obtained directly without the need of constructing the normal fan first. The disadvantage is that even if the subword complex is polytopal, it is possible that the method has no solution. In practical terms, this method is good to find polytopal realizations for small explicit examples, but in order to prove or disprove polytopality, one should rather look at the first method in Section 3.3.

Appendix A

Summaries

A.1 English summary

This thesis presents several developments related to the associahedron. All results are motivated by two specific problems. The first one, which was completely solved in this work, concerns some polytopal realizations of associahedra (Chapter 1), while the second one is about the existence of polytopal realizations of multi-associahedra. Although this second problem was not solved in the thesis, it served as an starting point for very interesting results connecting subword complexes in the study of Gröbner geometry and cluster complexes in the theory of cluster algebras (Chapter 2). These results provide a new approach and new perspectives for problems related to multi-associahedra and, in a more general context, to generalized multi-associahedra. For example, we use this approach as a tool to produce polytopal realizations for small explicit examples (Chapter 3).

The thesis is subdivided into three chapters. The first chapter is focused on geometric realizations of the associahedron, and is joint work with Francisco Santos and Günter M. Ziegler [15]. We show that three systematic construction methods for the n-dimensional associahedron (as the secondary polytope of a convex $(n+3)$-gon by Gelfand, Kapranov and Zelevinsky, via cluster complexes of the root system A_n by Chapoton, Fomin and Zelevinsky, and as Minkowski sums of simplices by Postnikov) produce substantially different realizations, for any choice of the parameters for the constructions. The cluster complex and the Minkowski sum realizations were generalized by Hohlweg and Lange to produce exponentially many distinct realizations, all of them with normal vectors in $\{0, \pm 1\}^n$. We present another, even larger, exponential family, generalizing the cluster complex construction — and verify that this family is again disjoint

from the previous ones, with one single exception: The Chapoton–Fomin–Zelevinsky associahedron appears in both exponential families.

The second chapter is joint work with Jean-Philippe Labbé and Christian Stump [14]. We introduce, for any finite Coxeter group and any nonnegative integer k, a spherical subword complex called *multi-cluster complex*. This subword complex coincides with the cluster complex of the given type for $k = 1$, and extends the notion of multi-associahedra from types A and B to arbitrary finite Coxeter groups. We study combinatorial and geometric properties of multi-cluster complexes. In particular, we show that every spherical subword complex is the link of a face of a multi-cluster complex, and describe a natural cyclic action that yields a connection between multi-cluster complexes, Auslander-Reiten quivers and repetition quivers.

The third chapter shows a new point of view on the problem of polytopality of multi-associahedra and spherical subword complexes, and presents two computational methods to find polytopal realizations for small explicit examples. These methods were implemented in joint work with Jean-Philippe Labbé using the computer algebra system `Sage` [80].

A.2 Deutsche Zusammenfassung

Diese Arbeit präsentiert mehrere Entwicklungen im Zusammenhang mit Assoziaeder. Alle Resultate entstanden aus zwei spezifischen Fragestellungen. Die Erste, welche in dieser Arbeit vollständig gelöst wurde, befasst sich mit einigen polytopalen Realisierungen von Assoziaedern (Chapter 1), während es in der Zweiten um die Existenz polytopaler Realisierungen von Multiassoziaedern geht. Obwohl die zweite Fragestellung in dieser Arbeit nicht gelöst wurde, diente sie als Ausgangspunkt für sehr interessante Ergebnisse, welche Subwordkomplexe aus der Gröbner-Geometrie mit Clusterkomplexen aus der Clusteralgebra verbinden (Chapter 2). Diese Ergebnisse liefern einen neuen Ansatz und eine neue Perspektive für Fragestellungen, die mit Multiassoziaedern und - in einem allgemeineren Kontext - mit verallgemeinerten Multiassoziaedern zusammenhängen. Zum Beispiel nutzen wir diesen Ansatz als ein Hilfsmittel, um polytopale Realisierungen für kleine, explizite Beispiele zu erzeugen (Chapter 3).

Diese Arbeit gliedert sich in drei Teile. Der erste Teil konzentriert sich auf geometrische Realisierungen von Assoziaedern und entstand in Zusammenarbeit mit Francisco Santos und Günter M. Ziegler [15]. Wir zeigen, dass drei systematische Konstruktionsmethoden des n-dimensionale Assoziaeders (als Sekunddärpolytop eines konvexen $(n+3)$-gons von Gelfand, Kapranov und Zelevinsky, durch Clusterkomplexe des Wurzelsystems

A_n bei Chapoton, Fomin und Zelevinsky, und als Minkowskisumme von Simplizes bei Postnikov) grundlegend verschiedene Realisierungen für jede Wahl der Parameter der Konstruktionen erzeugen. Die Clusterkomplex- und Minkowski-Realisierung wurden von Hohlweg und Lange verallgemeinert, um exponentiell viele verschiedene Realisierungen zu erzeugen, deren Normalenvektoren in $\{0, \pm 1\}^n$ liegen. Wir stellen eine andere, sogar größere, exponentielle Familie vor, die die Clusterkomplexkonstruktion verallgemeinert - und weisen nach, dass diese Familie selbst schnittfremd mit den vorangegangenen ist, mit einer Außnahme: Das Chapoton-Fomin-Zelevinsky-Assoziaeder liegt in beiden exponentiellen Familien.

Der zweite Teil entstand in Zusammenarbeit mit Jean-Philippe Labbé und Christian Stump [14]. Für jede endliche Coxeter-Gruppe und jede nichtnegative ganze Zahl k führen wir einen sphärischen Subwordkomplex ein, den wir *Multiclusterkomplex* nennen. Dieser Subwordkomplex entspricht dem Clusterkomplex des gegebenen Types für $k = 1$ und erweitert den Begriff des Multiassoziaeders vom Typ A und B zu beliebigen endlichen Coxeter-Gruppen. Wir untersuchen kombinatorische und geometrische Eigenschaften der Multiclusterkomplexe. Insbesondere zeigen wir, dass jeder sphärische Teilwortkomplex der Link einer Seite eines Multiclusterkomplexes ist und beschreiben eine natürliche zyklische Verknüpfung, die einen Zusammenhang zwischen Multiclusterkomplexen, "Auslander-Reiten quivers" und "repetition quivers" herstellt.

Der dritte Teil zeigt eine neue Sichtweise auf die Frage nach der Polytopalität von Multiassoziaedern und sphärischen Subwordkomplexen und präsentiert zwei algorithmische Methoden, um Realisierungen von kleinen expliziten Beispielen zu finden. In Zusammenarbeit mit Jean-Philippe Labbé wurden diese Methoden unter Verwendung des Computer-Algebra-Systems `Sage` [80] implementiert.

Declaration of Authorship

I, Cesar Ceballos, declare that this thesis titled, "On associahedra and related topics" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: ______________________________

Date: ______________________________

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