Geometry of Banach spaces, absolute sums and Köthe-Bochner spaces

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Introduction

The purpose of this work is to study various geometric properties of Banach spaces, with particular emphasis on their stability with respect to the formation of (infinite) absolute sums and Köthe-Bochner spaces of vector-valued functions.

A large part of this thesis is devoted to the acs (alternatively convex or smooth) spaces and their uniform and local uniform versions (the uacs and luacs spaces).

For example, a real Banach space X is called an acs space if the following holds: whenever $x, y \in X$ are such that $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$, then x and y have the same supporting functionals. The property acs is a common generalisation of both rotundity (strict convexity) and smoothness (Gâteaux-differentiability of the norm). Likewise, luacs spaces are a generalisation of the notion of WLUR (weakly locally uniformly rotund) spaces and uacs spaces are a common generalisation of the well-known notions of uniform rotundity (UR) and uniform smoothness (US).

The properties acs, luacs and uacs were originally introduced in [72] (the definition of acs spaces in the context of finite-dimesional spaces was already introduced in [71]) in connection with the so called anti-Daugavet property.

In Chapter I we will study these notions in detail. First we will recall the basic definitions of rotundity and smoothness properties, then properly introduce the acs, luacs and uacs spaces and discuss their relations. The connection to the aforementioned anti-Daugavet property will also be briefly reviewed. We will also introduce two more related geometric notions, the sluacs and wuacs spaces, which fit naturally into the picture of rotundity and acs-properties.

Then we will obtain various general results on acs spaces and their relatives. For example, we will give some equivalent characterisations and obtain results on duality and quotient spaces.

Two midpoint versions of acs spaces, which are analogues of the well-known notions of midpoint locally uniformly rotund (MLUR) and weakly MLUR (WMLUR) spaces, will also be introduced and discussed. Finally, a directional version (the uacsed spaces, in analogy to the concept of spaces which are uniformly rotund in every direction (URED)) will also be defined and studied.

We also collect some examples and counterexamples illustrating the differences between all these geometric properties.

Chapter II is devoted to infinite absolute sums of acs spaces and their related versions. Absolute sums are a well-known and substantial generalisation of the classical concept of ℓ^p-sums and many results on the stability of different rotundity properties under absolute sums have been obtained in the literature. Let us just mention a few examples. M. Day proved in [25, Theorem 3] that the ℓ^p-sum of any family of UR spaces which have a common modulus of convexity is again UR (for $1 < p < \infty$) and later generalised his own result to infinite absolute sums (with respect to a UR space) in [26, Theorem 3]. In [96, Theorem 1.2] A. Lovaglia provides a result on absolute sums (products in his language) of locally uniformly rotund (LUR) spaces, which were introduced in the same paper. Results on further rotundity properties in ℓ^p-sums can be found for example in [130]. For a survey of results on geometric properties of finite absolute sums see for instance [34].

In the spirit of such works, we will prove stability results for acs-type properties under infinite absolute sums, for example the sum of any family of acs spaces with respect to an acs space with absolute norm is again an acs space (Proposition II.2.1). Stability results for luacs, sluacs and wuacs spaces will also be obtained.

One of the main results in Chapter II is Theorem II.6.3, which is an analogue of Day's aforementioned result on sums of UR spaces, stating that the absolute sum with respect to a UR space[1] of any family of Banach spaces which have a common uacs-modulus[2] is again a uacs space. The proof is a modification of Day's technique.

Finally, we will also study absolute sums of the aforementioned midpoint and directional versions of acs spaces.

In Chapter III we will consider acs-type properties in Köthe-Bochner spaces of vector-valued functions. These spaces are in a certain sense a nondiscrete analogue of absolute sums and form a substantial generalisation of the class of Lebesgue-Bochner spaces $L^p(\mu, X)$.

There is also an extensive literature on rotundity properties of Lebesgue- and Köthe-Bochner spaces. M. Day already observed that his results [25, 26] on sums of UR spaces can be carried over to corresponding spaces of vector-valued functions. The properties LUR, URED and WUR (weak uniform rotundity) in Köthe-Bochner spaces are studied, among other properties, in the paper [80] by A. Kamińska and B. Turett. The authors of [19] proved a more general result on LUR points in Köthe-Bochner spaces.

For more results on rotundity properties in Lebesgue-Bochner and Köthe-

[1] Actually, a space with property (u^+) is enough. This is a formal weakening of the property UR, which is introduced by the author. Unfortunately, it is not known whether (u^+) is strictly weaker than UR.

[2] This is introduced in Chapter I.

Bochner spaces, see [19, 40, 42, 67, 80, 128] and references therein. For more information on Köthe-Bochner spaces in general, see the book [93], which also contains many results on the geometry of these spaces.

G. Sirotkin already proved in [123] that for $1 < p < \infty$ the Lebesgue-Bochner space $L^p(\mu, X)$ is acs/luacs/uacs if X is acs/luacs/uacs (where μ is any measure). We will consider the more general case of Köthe-Bochner spaces $E(X)$. For example, we will prove that $E(X)$ is acs whenever X is an acs Banach space and E an order continuous acs Köthe function space (over a complete, σ-finite measure space; see Proposition III.2.1).

Results on the properties luacs, sluacs and wuacs in Köthe-Bochner spaces will also be obtained. In particular, we show in Theorem III.6.2 that $E(X)$ is sluacs provided that X is sluacs and E is LUR. The proof makes use of the technique from [80, Theorem 5]. In Theorem III.8.2 we prove that $E(X)$ is uacs if X is uacs and E is UR.[3] The proof is analogous to the one of Theorem II.6.3 (which used Day's techniques). Finally, the midpoint versions of the acs property in Köthe-Bochner spaces will also be considered.

The first three chapters of this thesis are based on the author's papers [57, 58, 60] (see the separate introductions to each chapter for more details). In the following three chapters, some further geometric properties of Banach spaces are considered.

Chapter IV deals, among other properties, with the so called Opial property. A Banach space X is said to have the Opial property provided that

$$\limsup_{n\to\infty} \|x_n\| < \limsup_{n\to\infty} \|x_n - x\|$$

for every weakly null sequence $(x_n)_{n\in\mathbb{N}}$ in X and every $x \in X \setminus \{0\}$. This property was first considered by Opial in [109] in connection with a result on iterative approximations of fixed points of nonexpansive mappings. Typical examples are the spaces ℓ^p for $1 \leq p < \infty$.

There are also two variants, the nonstrict Opial property (with "$\leq$" instead of "$<$") and the uniform Opial property, which will also be considered in Chapter IV. Furthermore, two other geometric notions, the WORTH property ([121]) and the García-Falset coefficient ([48]), will be studied as well. All these properties are connected to the important fixed point property for nonexpansive mappings.

We will prove a general result on the WORTH property in infinite absolute sums, study the García-Falset coefficient of infinite absolute sums with respect to a uniformly rotund space, and provide results on the different types of Opial propeties in infinite ℓ^p-sums.

It is known that the spaces $L^p[0,1]$ for $1 < p < \infty, p \neq 2$ do not have the Opial property, thus one cannot expect any positive results on the Opial property in Lebesgue-Bochner spaces. However, we will prove some Opial-like

[3] Again, the formally weaker property (u^+) is enough.

results for Lebesgue-Bochner spaces in which weak convergence is replaced by pointwise weak convergence.

Finally, we will also consider the Opial property in sums with respect to the so called Cesàro sequence spaces and obtain some Opial-type results for Cesàro spaces of vector-valued functions.

The fourth chapter is based on the author's preprint [61], which has recently been submitted (in a slightly revised form) to Commentationes Mathematicae (the results on Cesàro sums and Cesàro function spaces are not contained in this preprint, they have not been published before).

Chapter V, which is based on the author's recent preprint [62], deals with the ball generated property (BGP) of Banach spaces. A Banach space is said to have the BGP if each closed, bounded, convex subset can be written as an intersection of finite unions of closed balls. This property was introduced by Godefroy and Kalton in [51] but it implicitly appeared before in [23]. It is known that every reflexive space has the BGP, while for example c_0 does not have it.

S. Basu proved in [8] that the BGP is stable under infinite ℓ^p-sums for $1 < p < \infty$ and also under infinite c_0-sums. We will consider only sums of two spaces with the BGP here, but with respect to more general absolute norms. More precisely, we will prove that the sum of two BGP spaces with respect to an absolute norm on $\mathbb{R}^2$ which is Gâteaux-differentiable at $(0, 1)$ and $(1, 0)$ also has the BGP. In the proof we will make use of a description of absolute norms on $\mathbb{R}^2$ via the boundary curve of their unit ball (which is surely well known, but the necessary results and proofs are included in Chapter V, since the author was not able to find a reference).

The last chapter (based on the author's preprint [59], submitted to Studia Mathematica) concerns generalised lush spaces and the Mazur-Ulam property. By the classical Mazur-Ulam theorem, every surjective isometry between two real normed linear spaces must be affine. An old problem of Tingley ([134]) asks whether every surjective isometry between the unit spheres of two real Banach spaces X and Y can be extended to a linear isometry between the whole spaces. Surprisingly, this question is still open even in two dimensions, though many partial positive answers are known (see Section VI.1 for a more detailed account). According to [22], a space X is said to have the Mazur-Ulam property (MUP), if the answer to Tingley's question for this particular space X (and every target space Y) is affirmative.

The authors of [66] introduced the notion of generalised lush (GL) spaces as follows: the space X is GL if for every x of norm one and every $\varepsilon > 0$ there is some norm-one functional $x^* \in X^*$ such that $x \in S(x^*, \varepsilon)$ and

$$\operatorname{dist}(y, S(x^*, \varepsilon)) + \operatorname{dist}(y, -S(x^*, \varepsilon)) < 2 + \varepsilon$$

holds for every $y \in X$ of norm one, where $S(x^*, \varepsilon)$ denotes the slice of the unit ball determined by x^* and ε, i. e. $\{x \in X : \|x\| \leq 1, \, x^*(x) > 1 - \varepsilon\}$, and dist is the usual distance function.

Lush spaces were introduced before in [15] as those spaces X for which the following holds: for all points x, y of norm one and every $\varepsilon > 0$ there is some norm-one functional $x^* \in X^*$ such that $x \in S(x^*, \varepsilon)$ and

$$\operatorname{dist}(y, \operatorname{aco} S(x^*, \varepsilon)) < \varepsilon,$$

where aco stands for the absolutely convex hull. The original motivation for introducing lush spaces in [15] was a problem concerning the so called numerical index of Banach spaces (see Section VI.1 for details).

It was proved in [66] that every separable lush space is GL and that every GL-space has the MUP. Many stability results for the class of GL-spaces were also obtained in [66], for example, the property GL is stable under ℓ^1-, c_0- and ℓ^∞-sums.

We will establish some further stability results, for example: the property GL is preserved by ultraproducts and it is inherited by a certain class of geometric ideals, which includes in particular the important M-ideals (see [63], a corresponding result for M-ideals in lush spaces was obtained in [112]). We will also prove that a space has the MUP if its bidual is GL.

Finally, concerning rotundity properties in GL-spaces, we will also show that a GL-space with 1-unconditional basis cannot have any LUR points.

I would like to express my gratitude to my supervisor Dirk Werner, who introduced me to the field of Banach space theory and supported me with advice and encouragement during the work on my thesis. Furthermore, I would like to thank the state of Berlin for granting me an Elsa-Neumann-Stipendium.

Zusammenfassung

Die Dissertation befasst sich mit diversen geometrischen Eigenschaften von Banachräumen, mit besonderem Hinblick auf deren Stabilität unter der Bildung von (vorwiegend unendlichen) absoluten Summen und Köthe-Bochner-Räumen vektorwertiger Funktionen.

Ein Großteil der Arbeit ist den sogenannten acs-Räumen (von engl. "alternatively convex or smooth") und ihren lokal gleichmäßigen und gleichmäßigen Varianten (den luacs- und uacs-Räumen) gewidmet. Diese wurden in [72] eingeführt (im Zusammenhang mit der sogenannten Anti-Daugavet-Eigenschaft) und bilden gemeinsame Verallgemeinerungen gängiger Konvexitäts- und Glattheitsbegriffe für Banachräume. In Kapitel I werden diese Eigenschaften (und einige weitere, vom Autor selbst eingeführte Varianten der acs-Eigenschaft) definiert und anschließend im Detail analysiert. Wir beweisen äquivalente Charakterisierungen, studieren acs-Eigenschaften in Quotienten- und Dualräumen und diskutieren einige Beispiele und Gegenbeispiele.

Kapitel II enthält diverse Resultate betreffend die verschiedenen acs-Eigenschaften in (unendlichen) absoluten Summen (welche eine wesentliche Verallgemeinerung des klassischen Konzepts der ℓ^p-Summen bilden).

In Kapitel III werden acs-Eigenschaften in Köthe-Bochner-Räumen vektorwertiger Funktionen studiert. Diese Räume stellen eine weitreichende Verallgemeinerung der Lebesgue-Bochner-Räume dar (Stabilitätsresultate für die Eigenschaften acs, luacs und uacs in Lebesgue-Bochner-Räumen wurden bereits von G. Sirotkin in [123] bewiesen).

Die ersten drei Kapitel basieren auf den Artikeln [57, 58, 60] des Autors. In den folgenden Kapiteln werden weitere geometrische Eigenschaften von Banachräumen betrachtet.

Kapitel IV (basierend auf dem Preprint [61] des Autors) behandelt unter anderem die sogenannte Opial-Eigenschaft (zuerst eingeführt in [109]) und ihre Varianten, die nichtstrikte und die gleichmäßige Opial-Eigenschaft. Weiterhin werden auch die WORTH-Eigenschaft ([121]) und der García-Falset-Koeffizient von Banachräumen ([48]) betrachtet. Diese geometrischen Begriffe stehen sämtlich im Zusammenhang mit der bedeutenden Fixpunkteigenschaft für nichtexpansive Abbildungen. Wir beweisen einige Resultate betreffend die Stabilität dieser Eigenschaften unter gewissen unendlichen absoluten Summen und auch einige der Opial-Eigenschaft analoge Resultate in Lebesgue-Bochner-Räumen und Cesàro-Räumen vektorwertiger Funktionen.

Das kurze Kapitel V (basierend auf dem Preprint [62] des Autors) betrifft die sogenannte "ball generated property" (BGP) in Banachräumen. Wir beweisen die Stabilität dieser Eigenschaft für gewisse absolute Summen zweier Räume, in partieller Verallgemeinerung früherer Resultate von S. Basu ([8]).

Das letzte Kapitel schließlich, welches auf dem Preprint [59] des Autors basiert, behandelt die sogenannten GL-Räume (GL steht für "generalised lush"). Diese wurden in [66] zum Studium der Mazur-Ulam-Eigenschaft ("Mazur-Ulam property", MUP) eingeführt. Gemäß [66] bilden diese, zumindest für separable Räume, eine Verallgemeinerung der "lush spaces"[4], welche in [15] eingeführt wurden (im Zusammenhang mit einer Frage betreffend Banachräume mit numerischem Index 1). Wir beweisen einige Stabilitätseigenschaften für GL-Räume, insbesondere die Vererbung der Eigenschaft GL auf eine gewisse Klasse von Unterräumen, die speziell die M-Ideale (siehe [63]) umfasst (ein entsprechendes Resultat für M-Ideale in üppigen Räumen wurde in [112] bewiesen).

[4] Zu deutsch "üppige Räume."

Eidesstattliche Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe, dass alle Stellen der Arbeit, die wörtlich oder sinngemäß aus anderen Quellen übernommen wurden, als solche kenntlich gemacht sind und dass die Arbeit in gleicher oder ähnlicher Form noch keiner Prüfungsbehörde vorgelegt wurde.

Jan-David Hardtke Berlin, 14.April 2015

I Acs spaces and their relatives

In this chapter, we will first recall various rotundity and smoothness properties of Banach spaces, in particular the notions of acs, luacs and uacs spaces that were introduced in [72] (the definition of acs spaces in the context of finite-dimesional spaces was already introduced in [71]). Some further related notions, that were originally introduced by the author, will also be defined.

These properties are then studied from a general point of view. We will analyse the connections between them, give some examples and counterexamples, obtain equivalent characterisations, address topics such as duality and quotient spaces, and establish some other general facts.

The new notions and results presented in this chapter are contained in the author's papers [57] and [60][1], except for the Propositions I.10.3–I.10.7, which first appeared in the author's paper [58].

Before we begin, we will introduce some notation that will be used throughout the whole thesis (if not otherwise stated). By X, Y etc. we denote *real* Banach spaces.[2] The dual of X is denoted by X^*. For $x \in X$ and $r > 0$ we write $B_r(x)$ for the closed ball with center x and radius r. The closed unit ball $B_1(0)$ is simply denoted by B_X, while S_X stands for the unit sphere. By $L(X)$ we denote the space of all linear bounded operators from X into itself. The identity operator is denoted by id_X (or simply id if X is tacitly understood). For a subset $A \subseteq X$ we denote by $\operatorname{span} A$ resp. $\overline{\operatorname{span}} A$ its linear resp. closed linear hull. Further notations will be introduced in the text when they are needed. For any unexplained Banach space notions the reader is referred to standard books on functional analysis and Banach space theory, for example [41, 65, 69, 139].

I.1 Basic notions

We start by recalling the most important notions of rotundity for Banach spaces.

[1] [60] is the published version of the preprint [57], which was, however, shortened to the first three sections.

[2] We consider only real Banach spaces mainly for the sake of simplicity. Most of the definitions and results in this thesis could be generalised to complex spaces in a standard way.

Definition I.1.1. A Banach space X is called

(i) *rotund* (R in short) if for any two elements $x, y \in S_X$ the equality $\|x + y\| = 2$ implies $x = y$,

(ii) *locally uniformly rotund* (LUR in short) if for every $x \in S_X$ the implication

$$\|x_n + x\| \to 2 \;\Rightarrow\; \|x_n - x\| \to 0$$

holds for every sequence $(x_n)_{n\in\mathbb{N}}$ in S_X,

(iii) *weakly locally uniformly rotund* (WLUR in short) if for every $x \in S_X$ and every sequence $(x_n)_{n\in\mathbb{N}}$ in S_X we have

$$\|x_n + x\| \to 2 \;\Rightarrow\; x_n \to x \text{ weakly},$$

(iv) *uniformly rotund* (UR in short) if for any two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in S_X the implication

$$\|x_n + y_n\| \to 2 \;\Rightarrow\; \|x_n - y_n\| \to 0$$

holds,

(v) *weakly uniformly rotund* (WUR in short) if for any two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in S_X the following implication holds

$$\|x_n + y_n\| \to 2 \;\Rightarrow\; x_n - y_n \to 0 \text{ weakly}.$$

Note that often in the literature, the term "convexity" is used instead of "rotundity", e. g. rotund spaces are called strictly convex, LUR spaces are called locally uniformly convex, etc.

The obvious implications between the above properties are summarised in the diagram below and no other implications are valid in general, as is shown by the examples in [127].

$$\begin{tikzcd}
 & \text{WUR} \arrow[dr] & & \\
\text{UR} \arrow[ur] \arrow[dr] & & \text{WLUR} \arrow[r] & \text{R} \\
 & \text{LUR} \arrow[ur] & &
\end{tikzcd}$$

Fig. I.1

The standard examples of UR spaces are the Hilbert spaces (this follows easily from the parallelogram law) and, more generally, the spaces $L^p(\mu)$ for any $p \in (1, \infty)$ and any measure space $(\Omega, \mathcal{A}, \mu)$ (see for instance [41, Theorem 9.3.]).

For a finite-dimensional space X all the above notions coincide, as is easily proved using the compactness of the unit ball. Note also that, by standard normalisation arguments, X is UR if and only if for all bounded sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X which fulfil the conditions $\|x_n + y_n\| - \|x_n\| - \|y_n\| \to 0$ and $\|x_n\| - \|y_n\| \to 0$ we have that $\|x_n - y_n\| \to 0$ and further that the two conditions $\|x_n + y_n\| - \|x_n\| - \|y_n\| \to 0$ and $\|x_n\| - \|y_n\| \to 0$ can be replaced by the single equivalent condition $2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \to 0$ (cf. [41, Fact 9.5.]). Similar remarks also hold for LUR, WUR and WLUR spaces.

Let us also recall the definition of the modulus of convexity of X. It is the function δ_X on $(0, 2]$ defined by

$$\delta_X(\varepsilon) = \inf\{1 - 1/2\|x + y\| : x, y \in B_X \text{ and } \|x - y\| \geq \varepsilon\} \quad \forall \varepsilon \in (0, 2].$$

X is UR if and only if $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$.

For the local version one defines

$$\delta_X(x, \varepsilon) = \inf\{1 - 1/2\|x + y\| : y \in B_X \text{ and } \|x - y\| \geq \varepsilon\}$$

for every $x \in S_X$ and each $\varepsilon \in (0, 2]$. Then X is LUR if and only if $\delta_X(x, \varepsilon) > 0$ for all $x \in S_X$ and all $0 < \varepsilon \leq 2$.

Next we turn our attention to different notions of smoothness. First of all, the space X is called *smooth* (S in short) if its norm is Gâteaux-differentiable at every non-zero point (equivalently at every point of S_X), which is the case if and only if for every $x \in S_X$ there is a unique functional $x^* \in S_{X^*}$ with $x^*(x) = 1$ (cf. [41, Lemma 8.4 (ii)]).

X is called *Fréchet-smooth* (FS in short) if the norm is Fréchet-differentiable at every non-zero point and the norm of the space X is said to be *uniformly Gâteaux-differentiable* (UG in short) if for each $y \in S_X$ the limit $\lim_{\tau\to 0}(\|x + \tau y\| - 1)/\tau$ exists uniformly in $x \in S_X$.

Finally, X is called *uniformly smooth* (US in short) if $\lim_{\tau\to 0} \rho_X(\tau)/\tau = 0$, where ρ_X denotes the modulus of smoothness of X defined by

$$\rho_X(\tau) = \frac{1}{2} \sup\{\|x + \tau y\| + \|x - \tau y\| - 2 : x, y \in S_X\} \quad \forall \tau > 0.$$

Obviously, FS implies S and from [41, Fact 9.7] it follows that US implies FS. It is also well known that X is US if and only if X^* is UR and X is UR if and only if X^* is US (cf. [41, Theorem 9.10]).

The property UG lies between US and S. It is known (cf. [30, Theorem II.6.7]) that X^* is UG if and only if X is WUR and X is UG if and only if X^* is WUR* (which means that X^* fulfils the definition of WUR with weak- replaced by weak*-convergence).

Let us also recall the important result that every Banach space X which is UR or US is reflexive (cf. [41, Theorem 9.12])), even more, it is superreflexive (see [41, p.294]), i. e. every Banach space Y which is finitely representable in X is reflexive.

Finite representability of Y in X means that for every $\varepsilon > 0$ and every finite-dimensional subspace F of Y there is some finite-dimensional subspace E of X and an isomorphism $T : F \to E$ with $\|T\|\|T^{-1}\| < 1 + \varepsilon$.

Now we come to the definitions of acs, luacs and uacs spaces that were introduced in [72] and are the main subject of study in this and also in the following two chapters.

Definition I.1.2. A Banach space X is called

(i) *alternatively convex or smooth* (acs in short) if for every $x, y \in S_X$ with $\|x + y\| = 2$ and every $x^* \in S_{X^*}$ with $x^*(x) = 1$ we have $x^*(y) = 1$ as well,

(ii) *locally uniformly alternatively convex or smooth* (luacs in short) if for every $x \in S_X$, every sequence $(x_n)_{n\in\mathbb{N}}$ in S_X and every functional $x^* \in S_{X^*}$ we have

$$\|x_n + x\| \to 2 \text{ and } x^*(x_n) \to 1 \;\Rightarrow\; x^*(x) = 1,$$

(iii) *uniformly alternatively convex or smooth* (uacs in short) if for all sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in S_X and $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} we have

$$\|x_n + y_n\| \to 2 \text{ and } x_n^*(x_n) \to 1 \;\Rightarrow\; x_n^*(y_n) \to 1.$$

Clearly, R and S both imply acs, WLUR implies luacs, and UR and US both imply uacs. Again by standard normalisation arguments one can easily check that X is uacs if and only if for all bounded sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in X and $(x_n^*)_{n\in\mathbb{N}}$ in X^* with $x_n^*(x_n) - \|x_n^*\|\|x_n\| \to 0$, $\|x_n + y_n\| - \|x_n\| - \|y_n\| \to 0$ and $\|x_n\| - \|y_n\| \to 0$ (or equivalently $2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \to 0$) we also have $x_n^*(y_n) - \|x_n^*\|\|y_n\| \to 0$. A similar characterisation holds for luacs spaces. Note also that in the case $\dim X < \infty$, by compactness of the unit ball, the notions of acs, luacs and uacs spaces coincide.

The acs, luacs and uacs spaces were introduced in [72] to obtain geometric characterisations of the so called anti-Daugavet property, which was introduced in the same paper (acs spaces and the anti-Daugavet property in the context of finite-dimensional spaces were already introduced and studied in [71]). Let us briefly recall the related definitions and results.

First of all, an operator $T \in L(X)$ is said to satisfy the Daugavet equation (DE in short) if

$$\|\mathrm{id} + T\| = 1 + \|T\|.$$

The space X is said to have the Daugavet property (DP) if every rank-one operator satisfies the DE. Typical examples of such spaces are $C(K)$ (the space of continuous functions on K), where K is a compact Hausdorff space without isolated points, and $L^1(\mu)$ for an atomless measure μ (see the examples in [137]).

Suprisingly, it turns out that if X has the DP, then actually every weakly compact operator on X satisfies the DE (see [72, Theorem 2.3]).

On the other hand, it is well known and easy to see that, regardless of the underlying space X, if $T \in L(X)$ satisfies $\|T\| \in \sigma(T)$, then T also satisfies the DE (here, $\sigma(T)$ denotes the spectrum of T). In fact, the following more general statement holds (this is surely known as well, but a proof is included here since the author was not able to find it explicitly in the literature).

Lemma I.1.3. *For any Banach space X and every $T \in L(X)$ the inequality*

$$\|\mathrm{id} + T\| \geq 1 + \|T\| - \operatorname{dist}(\|T\|, \sigma(T) \cap \mathbb{R})$$

holds, where $\operatorname{dist}(\|T\|, \sigma(T) \cap \mathbb{R})$ *denotes the distance of* $\|T\|$ *to* $\sigma(T) \cap \mathbb{R}$*.*

If $\sigma(T) \cap \mathbb{R} = \emptyset$, then $\operatorname{dist}(\|T\|, \sigma(T) \cap \mathbb{R})$ is understood to be ∞, and the inequality holds trivially.

Proof. If the claim was not true, there would be $\lambda \in \sigma(T) \cap \mathbb{R}$ such that $|\|T\| - \lambda| < 1 + \|T\| - \|\mathrm{id} + T\|$, hence $\|\mathrm{id} + T\| < 1 + \lambda$. Consequently, the operator $S := (1 + \lambda)^{-1}(\mathrm{id} + T)$ has norm less than 1, so $\mathrm{id} - S$ is invertible. But then the operator $(1 + \lambda)(\mathrm{id} - S) = \lambda\mathrm{id} - T$ would be invertible as well, contradicting $\lambda \in \sigma(T)$. $\square$

Now the following terminology was introduced in [72]: X is said to have the anti-Daugavet property (anti-DP) with respect to some class $M \subseteq L(X)$ of operators, if for every $T \in M$ the implication

$$\|\mathrm{id} + T\| = 1 + \|T\| \;\Rightarrow\; \|T\| \in \sigma(T)$$

holds.

By results of [1] every UR and every US space has the anti-DP with respect to $L(X)$, and every LUR space has the anti-DP with respect to the class of compact operators. In [71] it was proved that a finite-dimensional space has the anti-DP (with respect to $L(X)$) if and only if it is acs.

In [72] the following generalisations of these results were proved: X has the anti-DP for rank-1-operators if and only if X has the anti-DP for compact operators if and only if X is luacs (see [72, Theorem 4.3]); if X is even uacs, then it has the anti-DP with respect to all operators (see [72, Theorem 4.5]), but it is not known whether the converse of this statement is true.

For more information about the Daugavet equation, the interested reader is referred to [1, 71, 72, 137] and references therein.

Let us now discuss the acs spaces and their relatives a little further. First we explicitly note the following reformulation of the definition of acs spaces, which was observed in [72] (in [71] it was used directly as the definition). A Banach space X is acs if and only if the following holds: whenever $x, y \in S_X$ such that $\|x + y\| = 2$ then the norm of $\operatorname{span}\{x, y\}$ is Gâteaux-differentiable at x and y. We will come back to this reformulation in Proposition I.2.2.

Next we recall that a Banach space X is said to be uniformly non-square if there is some $\delta > 0$ such that for all $x, y \in B_X$ we have $\|x + y\| \leq 2(1 - \delta)$ or $\|x - y\| \leq 2(1 - \delta)$. It is easily seen that uacs spaces are uniformly non-square and hence by a well-known theorem of James (cf. [9, p.261]) they are superreflexive, as was observed in [72, Lemma 4.4].

Actually, to prove the superreflexivity of uacs spaces it is not necessary to employ the rather deep theorem of James, as we will see in Section I.3.

In [123] it was shown by G. Sirotkin that for every $1 < p < \infty$ and every measure space $(\Omega, \mathcal{A}, \mu)$ the Lebesgue-Bochner space $L^p(\mu, X)$ is uacs (resp. luacs, resp. acs) whenever X is a uacs (resp. luacs, resp. acs) Banach space (in Chapter III we will extend these results to larger classes of Köthe-Bochner function spaces). To prove his main result on uacs spaces, Sirotkin first established the following equivalent characterisation.

Proposition I.1.4 (Sirotkin, cf. [123])**.** *A Banach space X is uacs if and only if for any two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in S_X and every sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} we have*

$$\|x_n + y_n\| \to 2 \text{ and } x_n^*(x_n) = 1 \ \forall n \in \mathbb{N} \ \Rightarrow \ x_n^*(y_n) \to 1.$$

Instead of repeating the proof from [123] here, we shall give a slightly different proof in Proposition I.2.1, which—unlike Sirotkin's proof—does not use any reflexivity arguments (but see also the proof of Lemma I.10.1).

Now with the help of this characterisation we can define a kind of "uacs-modulus" of a given Banach space.

Definition I.1.5. For a Banach space X we define

$$D_X(\varepsilon) = \{(x, y) \in S_X \times S_X : \exists x^* \in S_{X^*} \ x^*(x) = 1 \text{ and } x^*(y) \leq 1 - \varepsilon\}$$

and

$$\delta_{\mathrm{uacs}}^X(\varepsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : (x, y) \in D_X(\varepsilon)\right\} \ \forall \varepsilon \in (0, 2].$$

Then by Proposition I.1.4 X is uacs if and only if $\delta_{\mathrm{uacs}}^X(\varepsilon) > 0$ for every $\varepsilon \in (0, 2]$ and we clearly have $\delta_X(\varepsilon) \leq \delta_{\mathrm{uacs}}^X(\varepsilon)$ for each $\varepsilon \in (0, 2]$. For the connection to the modulus of smoothness see Lemma I.2.6.

The characterisation of uacs spaces given above coincides with the notion of U-spaces introduced by Lau in [86] and our modulus δ_{uacs}^X is the same as the modulus of u-convexity from [47]. It was also observed in [86] that U-spaces are uniformly non-square (and hence reflexive), and that $\mathbb{R}^2$ equipped with a norm whose unit ball is a hexagon provides an example of a uniformly non-square space which is not a U-space.

Further, the notion of u-spaces which was introduced in [33] coincides with the notion of acs spaces. The interested reader may also have a look at [39], where two notions of local U-convexity are introduced and studied quantitatively.

The U-spaces (= uacs spaces) are of particular interest, because they possess normal structure (cf. [46, Theorem 3.2] or [123, Theorem 3.1]) and hence (since they are also reflexive) they enjoy the fixed point property (we will briefly discuss these notions in Chapter IV, see [53, Section 2] for more background).

Now we introduce two more notions related to uacs spaces (as mentioned before, they were originally introduced by the author in [57, 60]).

Definition I.1.6. A Banach space X is called

(i) *strongly locally uniformly alternatively convex or smooth* (sluacs in short) if for every $x \in S_X$ and all sequences $(x_n)_{n\in\mathbb{N}}$ in S_X and $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} we have

$$\|x_n + x\| \to 2 \text{ and } x_n^*(x_n) \to 1 \;\Rightarrow\; x_n^*(x) \to 1,$$

(ii) *weakly uniformly alternatively convex or smooth* (wuacs in short) if for any two sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in S_X and every functional $x^* \in S_{X^*}$ we have

$$\|x_n + y_n\| \to 2 \text{ and } x^*(x_n) \to 1 \;\Rightarrow\; x^*(y_n) \to 1.$$

With these definitions we get the following implication chart.

$$\begin{tikzcd}
 & \text{wuacs} \arrow[dr] & & \\
\text{uacs} \arrow[ur] \arrow[dr] & & \text{luacs} \arrow[r] & \text{acs} \\
 & \text{sluacs} \arrow[ur] & &
\end{tikzcd}$$

Fig. I.2

Including the rotundity properties, we obtain the diagram below.

$$\begin{tikzcd}
 & \text{WUR} \arrow[d] \arrow[dr] & & \\
\text{UR} \arrow[ur] \arrow[d] \arrow[dr] & \text{wuacs} \arrow[dr] & \text{WLUR} \arrow[r] \arrow[d] & \text{R} \arrow[d] \\
\text{uacs} \arrow[ur] \arrow[dr] & \text{LUR} \arrow[ur] \arrow[d] & \text{luacs} \arrow[r] & \text{acs} \\
 & \text{sluacs} \arrow[ur] & &
\end{tikzcd}$$

Fig. I.3

In Section I.9 we will see some examples which show that no other implications are valid in general.

Finally, let us remark that every space whose norm is UG is also sluacs (see Proposition I.2.3 below), thus we have the following diagram illustrating the connection to smoothness properties.

$$\begin{tikzcd} \text{US} \arrow[r] \arrow[d] & \text{UG} \arrow[r] \arrow[d] & \text{S} \arrow[d] \\ \text{uacs} \arrow[r] & \text{sluacs} \arrow[r] & \text{acs} \end{tikzcd}$$

Fig. I.4

In the next section we will discuss some equivalent characterisations of the various types of acs spaces.

I.2 Equivalent characterisations

We start with the promised alternative proof of Proposition I.1.4 which does not rely on reflexivity. Instead, we shall employ the Bishop–Phelps–Bollobás theorem (cf. [12, Chap. 8, Theorem 11]), an argument that will also work for the case of sluacs spaces. This idea was suggested to the author by Dirk Werner.

Proposition I.2.1. *A Banach space X is uacs if and only if for any two sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in S_X and every sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} we have*

$$\|x_n + y_n\| \to 2 \text{ and } x_n^*(x_n) = 1 \ \forall n \in \mathbb{N} \ \Rightarrow \ x_n^*(y_n) \to 1. \qquad \text{(I.2.1)}$$

X is sluacs if and only if for every $x \in S_X$ and all sequences $(x_n)_{n\in\mathbb{N}}$ and $(x_n^*)_{n\in\mathbb{N}}$ in S_X resp. S_{X^*} we have

$$\|x_n + x\| \to 2 \text{ and } x_n^*(x_n) = 1 \ \forall n \in \mathbb{N} \ \Rightarrow \ x_n^*(x) \to 1. \qquad \text{(I.2.2)}$$

Proof. We only prove the statement for uacs spaces, the proof for the sluacs case is completely analogous. Furthermore, only the "if" part of the stated equivalence requires proof. So suppose (I.2.1) holds for any two sequences in S_X and all sequences in S_{X^*}.
Now if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are sequences in S_X and $(x_n^*)_{n\in\mathbb{N}}$ is a sequence in S_{X^*} such that $\|x_n + y_n\| \to 2$ and $x_n^*(x_n) \to 1$ we can choose a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $x_{n_k}^*(x_{n_k}) > 1 - 2^{-2k-2}$ holds for all $k \in \mathbb{N}$. By the already cited Bishop–Phelps–Bollobás theorem we can find sequences $(\tilde{x}_k)_{k\in\mathbb{N}}$ in S_X and $(\tilde{x}_k^*)_{k\in\mathbb{N}}$ in S_{X^*} such that $\tilde{x}_k^*(\tilde{x}_k) = 1$, $\|\tilde{x}_k - x_{n_k}\| \leq 2^{-k}$ and $\left\|\tilde{x}_k^* - x_{n_k}^*\right\| \leq 2^{-k}$ for all $k \in \mathbb{N}$.
It follows that $\|\tilde{x}_k - x_{n_k}\| \to 0$ and $\left\|\tilde{x}_k^* - x_{n_k}^*\right\| \to 0$ and since $\|x_n + y_n\| \to 2$ we get that $\|\tilde{x}_k + y_{n_k}\| \to 2$.
But then we also have $\tilde{x}_k^*(y_{n_k}) \to 1$, by our assumption, which in turn implies $x_{n_k}^*(y_{n_k}) \to 1$.
In the same way we can show that every subsequence of $(x_n^*(y_n))_{n\in\mathbb{N}}$ has another subsequence that tends to one and hence $x_n^*(y_n) \to 1$ which completes the proof. $\square$

Next we would like to give characterisations of acs/sluacs/uacs spaces that do not explicitly involve the dual space. As mentioned in the last section, a Banach space X is acs if and only if x and y are smooth points of the unit ball of the two-dimensional subspace $\operatorname{span}\{x, y\}$ whenever $x, y \in S_X$ are such that $\|x + y\| = 2$.

It is possible to reformulate and refine this statement in the following way.

Proposition I.2.2. *For any Banach space X the following assertions are equivalent:*

(i) *X is acs.*

(ii) *For all $x, y \in S_X$ with $\|x + y\| = 2$ we have*

$$\lim_{t\to 0^+} \frac{\|x + ty\| + \|x - ty\| - 2}{t} = 0.$$

(iii) *For all $x, y \in S_X$ with $\|x + y\| = 2$ we have*

$$\lim_{t\to 0^+} \frac{\|x - ty\| - 1}{t} = -1.$$

(iv) *For all $x, y \in S_X$ with $\|x + y\| = 2$ there is some $1 \le p < \infty$ such that*

$$\lim_{t\to 0^+} \frac{\|x + ty\|^p + \|x - ty\|^p - 2}{t^p} = 0.$$

(v) *For all $x, y \in S_X$ with $\|x + y\| = 2$ there is some $1 \le p < \infty$ such that*

$$\lim_{t\to 0^+} \frac{(1 + t)^p + \|x - ty\|^p - 2}{t^p} = 0.$$

The analogous characterisation for sluacs spaces reads as follows.

Proposition I.2.3. *For any Banach space X the following assertions are equivalent:*

(i) *X is sluacs.*

(ii) *For every $\varepsilon > 0$ and every $y \in S_X$ there is some $\delta > 0$ such that for all $t \in [0, \delta]$ and each $x \in S_X$ with $\|x + y\| \ge 2(1 - t)$ we have*

$$\|x + ty\| + \|x - ty\| \le 2 + \varepsilon t.$$

(iii) *For every $\varepsilon > 0$ and every $y \in S_X$ there is some $\delta > 0$ such that for all $t \in [0, \delta]$ and each $x \in S_X$ with $\|x + y\| \ge 2 - t\delta$ we have*

$$\|x - ty\| \le 1 + t(\varepsilon - 1).$$

(iv) *For every $y \in S_X$ there is some $1 \leq p < \infty$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t \in [0, \delta]$ and each $x \in S_X$ with $\|x + y\| \geq 2(1 - t)$ we have*

$$\|x + ty\|^p + \|x - ty\|^p \leq 2 + \varepsilon t^p.$$

(v) *For every $y \in S_X$ there is some $1 \leq p < \infty$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t \in [0, \delta]$ and each $x \in S_X$ with $\|x + y\| \geq 2 - t\delta$ we have*

$$(1 + t)^p + \|x - ty\|^p \leq 2 + \varepsilon t^p.$$

Finally, we have the following characterisation for uacs spaces.

Proposition I.2.4. *For any Banach space X the following assertions are equivalent:*

(i) *X is uacs.*

(ii) *For every $\varepsilon > 0$ there exists some $\delta > 0$ such that for every $t \in [0, \delta]$ and all $x, y \in S_X$ with $\|x + y\| \geq 2(1 - t)$ we have*

$$\|x + ty\| + \|x - ty\| \leq 2 + \varepsilon t.$$

(iii) *For every $\varepsilon > 0$ there exists some $\delta > 0$ such that for every $t \in [0, \delta]$ and all $x, y \in S_X$ with $\|x + y\| \geq 2 - \delta t$ we have*

$$\|x - ty\| \leq 1 + t(\varepsilon - 1).$$

(iv) *There exists some $1 \leq p < \infty$ such that for every $\varepsilon > 0$ there is some $\delta > 0$ such that for all $t \in [0, \delta]$ and all $x, y \in S_X$ with $\|x+y\| \geq 2(1-t)$ we have*

$$\|x + ty\|^p + \|x - ty\|^p \leq 2 + \varepsilon t^p.$$

(v) *There exists some $1 \leq p < \infty$ such that for every $\varepsilon > 0$ there is some $\delta > 0$ such that for all $t \in [0, \delta]$ and all $x, y \in S_X$ with $\|x + y\| \geq 2 - t\delta$ we have*

$$(1 + t)^p + \|x - ty\|^p \leq 2 + \varepsilon t^p.$$

Proof. We will only explicitly prove the characterisation for uacs spaces (the results for acs and sluacs spaces are proved analogously). First we show (i) $\Rightarrow$ (ii). So suppose X is uacs and fix $\varepsilon > 0$. Then there exists some $\tilde{\delta} > 0$ such that for all $x, y \in S_X$ and $x^* \in S_{X^*}$ we have

$$\|x + y\| \geq 2(1 - \tilde{\delta}) \text{ and } x^*(x) \geq 1 - \tilde{\delta} \;\Rightarrow\; x^*(y) \geq 1 - \varepsilon.$$

Now if we put $\delta = \tilde{\delta}/2$ and take $t \in [0, \delta]$ and $x, y \in S_X$ such that $\|x + y\| \geq 2(1-t)$, then we can find a functional $x^* \in S_{X^*}$ such that $x^*(x-ty) = \|x-ty\|$ and conclude that

$$x^*(x) = \|x - ty\| + tx^*(y) \geq 1 - t - t = 1 - 2t \geq 1 - \tilde{\delta}.$$

By the choice of $\tilde{\delta}$ this implies $x^*(y) \geq 1 - \varepsilon$ and hence

$$\|x + ty\| + \|x - ty\| = \|x + ty\| + x^*(x - ty) \leq 1 + t + 1 - tx^*(y) \leq 2 + t\varepsilon.$$

Now let us prove (ii) $\Rightarrow$ (iii). For a given $\varepsilon > 0$ choose $\delta > 0$ to the value $\varepsilon/2$ according to (ii). We may assume $\delta \leq \min\{1, \varepsilon/2\}$.
Then, if $t \in [0, \delta]$ and $x, y \in S_X$ such that $\|x + y\| \geq 2 - \delta t$, we in particular have $\|x + y\| \geq 2(1 - t)$ and hence

$$\|x + ty\| + \|x - ty\| \leq 2 + t\frac{\varepsilon}{2}.$$

But on the other hand

$$\|x + ty\| \geq \|x + y\| - (1 - t)\|y\| \geq 2 - \delta t - 1 + t = 1 - \delta t + t \geq 1 - \frac{\varepsilon}{2}t + t.$$

It follows that $\|x - ty\| \leq 1 + t(\varepsilon - 1)$.
Next we prove that (iii) $\Rightarrow$ (i). Fix sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in S_X such that $\|x_n + y_n\| \to 2$ and a sequence $(x_n^*)_{n\in\mathbb{N}}$ of norm-one functionals with $x_n^*(x_n) \to 1$. Also, for every $n \in \mathbb{N}$ we fix $y_n^* \in S_{X^*}$ such that $y_n^*(y_n) = 1$.
For given $\varepsilon > 0$ we choose $\delta > 0$ according to (iii). For sufficiently large n we have $\|x_n + y_n\| \geq 2 - \delta^2$ and $x_n^*(x_n) \geq 1 - \varepsilon\delta$ and hence

$$\begin{aligned}&(y_n^* - x_n^*)(\delta y_n) = x_n^*(x_n - \delta y_n) - x_n^*(x_n) + \delta \leq \|x_n - \delta y_n\| + \delta - x_n^*(x_n)\\ &\leq \|x_n - \delta y_n\| + \delta - 1 + \varepsilon\delta \leq 1 + \delta(\varepsilon - 1) + \delta - 1 + \varepsilon\delta = 2\delta\varepsilon,\end{aligned}$$

where the last inequality holds because of $\|x_n + y_n\| \geq 2 - \delta^2$ and the choice of δ.
It follows that $x_n^*(y_n) \geq y_n^*(y_n) - 2\varepsilon = 1 - 2\varepsilon$ for sufficiently large n.
The implications (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (v) are clear. To prove (iv) $\Rightarrow$ (ii) recall the inequalities

$$\begin{aligned}&(a + b)^p \leq 2^{p-1}(a^p + b^p) \quad \forall a, b \geq 0, \forall p \in [1, \infty)\\ &(a + b)^\alpha \leq a^\alpha + b^\alpha \quad \forall a, b \geq 0, \forall \alpha \in (0, 1].\end{aligned}$$

They imply that for all $x, y \in S_X$, every $t > 0$ and each $1 \leq p < \infty$ one has

$$\begin{aligned}&\frac{\|x + ty\| + \|x - ty\| - 2}{t} \leq \frac{\left(2^{p-1}(\|x + ty\|^p + \|x - ty\|^p)\right)^{1/p} - 2}{t}\\ &\leq \left(\frac{2^{p-1}(\|x + ty\|^p + \|x - ty\|^p) - 2^p}{t^p}\right)^{1/p}\\ &= 2^{1-1/p}\left(\frac{\|x + ty\|^p + \|x - ty\|^p - 2}{t^p}\right)^{1/p},\end{aligned}$$

which shows (iv) $\Rightarrow$ (ii). If we replace $\|x+ty\|$ by $1+t$ in the above calculation, we also obtain a proof for (v) $\Rightarrow$ (iii). $\square$

If we define the modulus ρ_{uacs}^X by

$$\rho_{\mathrm{uacs}}^X(\tau) = \sup\{1/2(\|x+\tau y\| + \|x-\tau y\|) - 1 : (x,y) \in S_X(\tau)\},$$

where $\tau > 0$ and $S_X(\tau) = \{(x,y) \in S_X \times S_X : \|x+y\| \geq 2(1-\tau)\}$, then because of the equivalence of (i) and (ii) in Proposition I.2.4 X is uacs if and only if $\lim_{\tau\to 0} \rho_{\mathrm{uacs}}^X(\tau)/\tau = 0$. We obviously have $\rho_{\mathrm{uacs}}^X(\tau) \leq \rho_X(\tau)$.

Let us also define

$$\tilde{\delta}_{\mathrm{uacs}}^X(\varepsilon) = \inf\left\{\max\left\{1 - \frac{1}{2}\|x+y\|, 1 - x^*(x)\right\} : x, y \in S_X, x^* \in A_\varepsilon(y)\right\},$$

where $0 < \varepsilon \leq 2$ and $A_\varepsilon(y) = \{x^* \in S_{X^*} : x^*(y) \leq 1 - \varepsilon\}$.

From the very definition of the uacs spaces it follows that X is uacs if and only if $\tilde{\delta}_{\mathrm{uacs}}^X(\varepsilon) > 0$ for every $0 < \varepsilon \leq 2$.

Examining the proof of the implication (i) $\Rightarrow$ (ii) in Proposition I.2.4 we see that the following holds.

Lemma I.2.5. *If X is a Banach space and $0 < \varepsilon \leq 2$ such that $\tilde{\delta}_{\mathrm{uacs}}^X(\varepsilon) > 0$, then for every $\tau > 0$ with $2\tau < \tilde{\delta}_{\mathrm{uacs}}^X(\varepsilon)$ we have $2\rho_{\mathrm{uacs}}^X(\tau) \leq \tau\varepsilon$.*

The reverse connection between ρ_{uacs}^X and δ_{uacs}^X is given by the following lemma.

Lemma I.2.6. *Let X be any Banach space and $\tau > 0$ as well as $0 < \varepsilon \leq 2$. Then the inequality*

$$\delta_{\mathrm{uacs}}^X(\varepsilon) \geq \frac{\varepsilon\tau - 2\rho_{\mathrm{uacs}}^X(\tau)}{2(\tau+1)}.$$

holds.

Proof. We may assume $\varepsilon\tau - 2\rho_{\mathrm{uacs}}^X(\tau) > 0$, because otherwise the inequality is trivially satisfied. Let us put $R = (\varepsilon\tau - 2\rho_{\mathrm{uacs}}^X(\tau))(2(\tau+1))^{-1}$ and take $x, y \in S_X$ and $x^* \in S_{X^*}$ such that $x^*(x) = 1$ and $\|x+y\| > 2(1-R)$.
Then we can find $z^* \in S_{X^*}$ with $z^*(x+y) > 2(1-R)$ and hence $z^*(x) > 1-2R$ and $z^*(y) > 1-2R$.
It follows that

$$\begin{aligned}
&(z^* - x^*)(\tau y) = z^*(x+\tau y) + x^*(x-\tau y) - x^*(x) - z^*(x) \\
&\leq \|x+\tau y\| + \|x-\tau y\| - 1 - z^*(x) \leq 2\rho_{\mathrm{uacs}}^X(\tau) + 1 - z^*(x) \\
&\leq 2(\rho_{\mathrm{uacs}}^X(\tau) + R).
\end{aligned}$$

Hence

$$x^*(y) \geq z^*(y) - \frac{2}{\tau}\big(\rho_{\mathrm{uacs}}^X(\tau) + R\big) > 1 - 2R - \frac{2}{\tau}\big(\rho_{\mathrm{uacs}}^X(\tau) + R\big) = 1 - \varepsilon$$

and we are done. $\square$

The next characterisation of acs spaces is quite easy, but it readily implies that X is acs whenever X^* is acs. Further duality results will be discussed in Section I.4.

Proposition I.2.7. *A Banach space X is acs if and only if for all $x^*, y^* \in S_{X^*}$ and all $x, y \in S_X$ the implication*

$$(x^* + y^*)(x) = 2 \text{ and } x^*(y) = 1 \ \Rightarrow\ y^*(y) = 1 \tag{I.2.3}$$

holds. In particular, if X^* is acs then so is X and the converse is true if X is reflexive.

Proof. Suppose that X is acs. If $x \in S_X, x^*, y^* \in S_{X^*}$ with $(x^* + y^*)(x) = 2$, then $x^*(x) = y^*(x) = 1$. So if in addition $y \in S_X$ with $x^*(y) = 1$, then $\|x + y\| = 2$ and the fact that X is acs implies $y^*(y) = 1$.
Conversely, suppose that (I.2.3) holds. If $x, y \in S_X$ such that $\|x + y\| = 2$, we can choose $x^* \in S_{X^*}$ with $x^*(x + y) = 2$, hence $x^*(x) = x^*(y) = 1$.
Now let $y^* \in S_{X^*}$ with $y^*(x) = 1$. Then $(x^* + y^*)(x) = 2 = 2x^*(y)$ and (I.2.3) implies $y^*(y) = 1$. □

Now we will discuss some further characterisations of acs, luacs and sluacs spaces by apparently stronger properties.

Proposition I.2.8. *For a Banach space X, the following assertions are equivalent:*

(i) *X is acs.*

(ii) *For all sequences $(x_n^*)_{n\in\mathbb{N}}, (y_n^*)_{n\in\mathbb{N}}$ in B_{X^*} and all $x, y \in S_X$ the implication*
$$(x_n^* + y_n^*)(x) \to 2 \text{ and } y_n^*(y) \to 1 \ \Rightarrow\ x_n^*(y) \to 1$$
holds.

(iii) *For every sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} and all $x, y \in S_X$ the implication*
$$\|x + y\| = 2 \text{ and } x_n^*(x) \to 1 \ \Rightarrow\ x_n^*(y) \to 1$$
holds.

Proof. (i) $\Rightarrow$ (ii) follows easily from Proposition I.2.7 together with the fact that B_{X^*} is weak*-compact, the implication (iii) $\Rightarrow$ (i) is trivial and (iii) follows from (ii) by taking $y^* \in S_{X^*}$ with $y^*(x + y) = 2$ and $y_n^* = y^*$ for each n. □

Let us denote by $X^{(k)}$ the k-th dual of X. Then X resp. X^* naturally embeds into $X^{(2k)}$ resp. $X^{(2k+1)}$ for each k. For sluacs spaces we have the following stronger result.

Proposition I.2.9. *A Banach space X is sluacs if and only if for every $k \in \mathbb{N}$, for every sequence $(z_n)_{n\in\mathbb{N}}$ in $B_{X^{(2k)}}$, every $x \in S_X$ and each sequence $(z_n^*)_{n\in\mathbb{N}}$ in $B_{X^{(2k+1)}}$ the implication*

$$\|z_n + x\| \to 2 \text{ and } z_n^*(z_n) \to 1 \;\Rightarrow\; z_n^*(x) \to 1$$

holds.

Proof. The sufficiency is obvious. To prove the necessity, we first take sequences $(x_n^{**})_{n\in\mathbb{N}}$ in $B_{X^{**}}$ and $(x_n^{***})_{n\in\mathbb{N}}$ in $B_{X^{***}}$ as well as an element $x \in S_X$ such that $\|x_n^{**} + x\| \to 2$ and $x_n^{***}(x_n^{**}) \to 1$. Then we can find a sequence $(y_n^*)_{n\in\mathbb{N}}$ in S_{X^*} such that $x_n^{**}(y_n^*) \to 1$ and $y_n^*(x) \to 1$.
By Goldstine's theorem (applied to X^*) there is a sequence $(x_n^*)_{n\in\mathbb{N}}$ in B_{X^*} such that $x_n^{***}(x_n^{**}) - x_n^{**}(x_n^*) \to 0$ and $x_n^{***}(x) - x_n^*(x) \to 0$. Hence $x_n^{**}(x_n^*) \to 1$.
Again by Goldstine's theorem (now applied to X) there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in B_X such that $x_n^{**}(x_n^*) - x_n^*(x_n) \to 0$ and $x_n^{**}(y_n^*) - y_n^*(x_n) \to 0$. It follows that $x_n^*(x_n) \to 1$ and $y_n^*(x_n) \to 1$.
Taking into account that $y_n^*(x) \to 1$ we get $\|x_n + x\| \to 2$. Since X is sluacs it follows $x_n^*(x) \to 1$ and hence $x_n^{***}(x) \to 1$.
Thus we have proved our claim for $k = 1$. Continuing by induction with the above argument we can show it for all $k \in \mathbb{N}$. □

If we use the preceding proposition and the technique from the proof of Proposition I.2.4 we see that the following holds.

Proposition I.2.10. *For a Banach space X the following assertions are equivalent:*

(i) *X is sluacs.*

(ii) *For every $k \in \mathbb{N}$, every $\varepsilon > 0$ and every $y \in S_X$ there is some $\delta > 0$ such that for all $t \in [0, \delta]$ and each $z \in S_{X^{(2k)}}$ with $\|z + y\| \geq 2(1 - t)$ we have*

$$\|z + ty\| + \|z - ty\| \leq 2 + \varepsilon t.$$

(iii) *For every $k \in \mathbb{N}$, every $\varepsilon > 0$ and every $y \in S_X$ there is some $\delta > 0$ such that for all $t \in [0, \delta]$ and each $z \in S_{X^{(2k)}}$ with $\|z + y\| \geq 2 - t\delta$ we have*

$$\|z - ty\| \leq 1 + t(\varepsilon - 1).$$

By means of Goldstine's theorem one can also prove the following characterisation of luacs spaces (we omit the details).

Proposition I.2.11. *A Banach space X is luacs if and only if for every sequence $(x_n^{**})_{n\in\mathbb{N}}$ in $S_{X^{**}}$, every $x \in S_X$ and each $x^* \in S_{X^*}$ the implication*

$$\|x_n^{**} + x\| \to 2 \text{ and } x_n^{**}(x^*) \to 1 \;\Rightarrow\; x^*(x) = 1.$$

holds.

I.3 Reflexivity of uacs spaces revisited

In this section we give a proof of the superreflexivity of uacs spaces without using James' result on uniformly non-square Banach spaces. A key ingredient to James' proof is the following lemma of his, which may be found in [9, p.51].

Lemma I.3.1. *A Banach space X is* not *reflexive if and only if for every $0 < \theta < 1$ there is a sequence $(x_k)_{k\in\mathbb{N}}$ in B_X and a sequence $(x_n^*)_{n\in\mathbb{N}}$ in B_{X^*} such that for every $n \in \mathbb{N}$ we have*

$$x_n^*(x_k) = \begin{cases} \theta & \text{if } n \le k \\ 0 & \text{if } n > k. \end{cases}$$

Even with this lemma it is still difficult to prove the superreflexivity of uniformly non-square Banach spaces (cf. the proof in [9, p.261]), but it easily yields the result for uacs spaces. We can even prove a stronger result: it is a well-known fact that a Banach space X is reflexive if it satisfies $\liminf_{t\to 0^+} \rho_X(t)/t < 1/2$ (cf. [125, Theorem 2]).[3] We will see that the same holds if we replace ρ_X by ρ_{uacs}^X.

Proposition I.3.2. *If there is some $t > 0$ such that $\rho_{\mathrm{uacs}}^X(t) < t/2$, then X is superreflexive (actually, it is uniformly non-square).*

Proof. Put $\theta = 2\rho_{\mathrm{uacs}}^X(t)/t < 1$ and choose $\varepsilon > 0$ such that $\theta + \varepsilon < 1$. Also, put $\eta = \min\{t\varepsilon/5, \varepsilon/5\}$.
If $x, y \in S_X$ such that $\|x + y\| \ge 2(1 - \eta)$ and $x^* \in S_{X^*}$ with $x^*(x) \ge 1 - \eta$ fix $y^* \in S_{X^*}$ such that $y^*(x + y) \ge 2(1 - \eta)$. Then $y^*(x) \ge 1 - 2\eta$ and $y^*(y) \ge 1 - 2\eta$ and hence

$$\begin{aligned}&(y^* - x^*)(ty) = y^*(x + ty) + x^*(x - ty) - x^*(x) - y^*(x)\\ &\le \|x + ty\| + \|x - ty\| - 2 + 3\eta \le 2\rho_{\mathrm{uacs}}^X(t) + 3\eta = t\theta + 3\eta \le (\theta + \frac{3}{5}\varepsilon)t.\end{aligned}$$

Consequently, $x^*(y) \ge y^*(y) - \theta - \frac{3}{5}\varepsilon \ge 1 - 2\eta - \theta - \frac{3}{5}\varepsilon \ge 1 - \frac{2}{5}\varepsilon - \theta - \frac{3}{5}\varepsilon = 1 - (\theta + \varepsilon)$.
Next we fix $0 < \tau < 1/2$ such that $\tau(1 + (1 - 2\tau)^{-1}) \le \eta$ and put $\beta = 1 - (1 - \tau)(1 - 2\tau)(1 - \theta - \varepsilon)$. Then $0 < \beta < 1$.

Claim. If $x, y \in B_X$ such that $\|x + y\| \ge 2(1 - \tau)$ and $x^* \in B_{X^*}$ such that $x^*(x) \ge 1 - \tau$, then $x^*(y) \ge 1 - \beta$.

To see this, take x, y and x^* as above and observe that $\|x\|, \|y\| \ge 1 - 2\tau$. Hence

$$\begin{aligned}\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| &\ge \frac{\|x + y\|}{\|x\|} - \left|\frac{1}{\|x\|} - \frac{1}{\|y\|}\right|\|y\|\\ &\ge \|x + y\| - \left|\frac{1}{\|x\|} - \frac{1}{\|y\|}\right| \ge 2(1 - \tau) - \frac{2\tau}{1 - 2\tau} \ge 2(1 - \eta)\end{aligned}$$

[3]Note that the definition of ρ_X given there differs from our definition by a factor 1/2.

and moreover, since $\|x^*\|, \|x\| \leq 1$,

$$\frac{x^*}{\|x^*\|}\left(\frac{x}{\|x\|}\right) \geq 1-\tau \geq 1-\eta.$$

Thus by our previous considerations we must have

$$x^*(y) \geq \|x^*\|\|y\|(1-\theta-\varepsilon) \geq (1-\tau)(1-2\tau)(1-\theta-\varepsilon) = 1-\beta.$$

From the above claim together with the fact that $\beta < 1$ it could be easily deduced that X is uniformly non-square and hence superreflexive, but if we just want to prove the superreflexivity an application of Lemma I.3.1 is enough. For if X was not reflexive then by said Lemma we could find sequences $(x_k)_{k\in\mathbb{N}}$ in B_X and $(x_n^*)_{n\in\mathbb{N}}$ in B_{X^*} such that $x_n^*(x_k) = 0$ for $n > k$ and $x_n^*(x_k) = 1-\tau$ for $n \leq k$.
We only need the first two members of the sequences to derive a contradiction, namely we have $\|x_1 + x_2\| \geq x_1^*(x_1) + x_1^*(x_2) = 2(1-\tau)$ and $x_2^*(x_2) = 1-\tau$, but $x_2^*(x_1) = 0 < 1-\beta$ contradicting our just established claim.
Thus X must be reflexive and to prove the superreflexivity it only remains to show that for every Banach space Y which is finitely representable in X there exists $t' > 0$ such that $\rho_{\mathrm{uacs}}^{Y}(t') < t'/2$ which we will do in the next Lemma. ∎

Lemma I.3.3. *If there is some $t > 0$ such that $\rho_{\mathrm{uacs}}^{X}(t) < t/2$ and Y is finitely representable in X then there is $t' > 0$ such that $\rho_{\mathrm{uacs}}^{Y}(t') < t'/2$.*

Proof. Let $\theta, \varepsilon, \eta, \tau$ and β be as in the previous proof. Put $\nu = \tau/4$.
Claim. If $x, y \in B_X$ such that $\|x+y\| \geq 2(1-\nu)$ then $\|x+\nu y\| + \|x-\nu y\| \leq 2+\nu\beta$.
To establish this, take $x, y \in B_X$ as above and also fix $x^* \in S_{X^*}$ such that $x^*(x-\nu y) = \|x-\nu y\|$. Observe as before that $\|x\|, \|y\| \geq 1-\tau/2$. Hence we have

$$x^*(x) = \|x-\nu y\| + x^*(\nu y) \geq \|x\| - \nu\|y\| + \nu x^*(y) \geq \|x\| - 2\nu \geq 1-\tau.$$

The claim we established in the previous proof now gives us $x^*(y) \geq 1-\beta$. It follows that

$$\|x+\nu y\| + \|x-\nu y\| = \|x+\nu y\| + x^*(x-\nu y) \leq 2+\nu(1-x^*(y)) \leq 2+\nu\beta.$$

Next fix $\beta < \alpha < 1$ and $0 < \tilde{\eta} < \nu$ such that $(\beta\nu + 3\tilde{\eta})(\nu-\tilde{\eta})^{-1} < \alpha$. Put $t' = \nu - \tilde{\eta}$. Finally, choose $\tilde{\varepsilon} > 0$ such that $(1-t')(1+\tilde{\varepsilon})^{-1} > 1-\nu$ and $(1+\tilde{\varepsilon})(2+\nu\beta) \leq 2+\nu\beta+\tilde{\eta}$.
Now take $y_1, y_2 \in S_Y$ with $\|y_1+y_2\| \geq 2(1-t')$ and put $F = \mathrm{span}\{y_1, y_2\}$. Since Y is finitely representable in X there is a subspace $E \subseteq X$ and an isomorphism $T : F \to E$ such that $\|T\| = 1$ and $\|T^{-1}\| \leq 1+\tilde{\varepsilon}$. Let $x_i = Ty_i$ for $i = 1, 2$.

It easily follows that $\|x_1 + x_2\| \geq 2(1-t')(1+\tilde{\varepsilon})^{-1} > 2(1-\nu)$, whence $\|x_1 + \nu x_2\| + \|x_1 - \nu x_2\| \leq 2 + \nu\beta$, which implies $\|y_1 + \nu y_2\| + \|y_1 - \nu y_2\| \leq (1+\tilde{\varepsilon})(2+\nu\beta)$. Thus we have

$$\begin{aligned}
&\|y_1 + t' y_2\| + \|y_1 - t' y_2\| \leq \|y_1 + \nu y_2\| + \|y_1 - \nu y_2\| + 2\big|\nu - t'\big| \\
&\leq (1+\tilde{\varepsilon})(2+\nu\beta) + 2\tilde{\eta} \leq 2 + \nu\beta + 3\tilde{\eta} \leq 2 + \alpha(\nu - \tilde{\eta}) = 2 + \alpha t'.
\end{aligned}$$

So we have proved $2\rho^Y_{\mathrm{uacs}}(t')/t' \leq \alpha < 1$. □

We remark that the uniform non-squareness of a space X satisfying $2\rho^X_{\mathrm{uacs}}(t) < t$ for some $t > 0$ could also be deduced from our Lemma I.2.6 and [47, Theorem 2], where it is observed that $\delta^X_{\mathrm{uacs}}(1) > 0$ is sufficient to ensure that X is uniformly non-square.

I.4 Duality results

As mentioned in Section I.1, the properties UR and US are dual to each other (cf. [41, Theorem 9.10]). Since uacs is a common generalisation of both of them, it seems natural that uacs should be a self-dual property.

Indeed, in [86, Theorem 2.4] a proof of the fact that a Banach space X is a U-space if and only if its dual X^* is a U-space is proposed and in [39, Theorem 2.6] the stronger statement that for every U-space X the moduli of u-convexity of X and X^* coincide is claimed.[4] Both proofs make use of the following claim from [86, Remark after Definition 2.2]:

Claim. X is a U-space if and only if for every $\varepsilon > 0$ there is some $\delta > 0$ such that whenever $x, y \in S_X$ and $x^*, y^* \in S_{X^*}$ with $x^*(x) = 1 = y^*(y)$ and $\|x+y\| > 2(1-\delta)$, then $\|x^* + y^*\| > 2(1-\varepsilon)$.

A U-space certainly has the above property. However, the converse need not be true, not even in a two-dimensional space.

To see this, first note that if X is finite-dimensional, then by an easy compactness argument the condition of the claim is equivalent to the following one: whenever $x, y \in S_X$ and $x^*, y^* \in S_{X^*}$ with $x^*(x) = 1 = y^*(y)$ and $\|x+y\| = 2$ we also have $\|x^* + y^*\| = 2$.

Therefore, if X is finite-dimensional it fulfils the condition of the claim if for each $x, y \in S_X$ with $\|x+y\| = 2$ at least one of the two points x and y is a smooth point of the unit ball (for example, if x is a smooth point of B_X and we have $x^*, y^* \in S_{X^*}$ with $x^*(x) = 1 = y^*(y)$, then because of $\|x+y\| = 2$ we can find $z^* \in S_{X^*}$ such that $z^*(x) = 1 = z^*(y)$, the smoothness of B_X at x then implies $x^* = z^*$, and hence $(x^* + y^*)(y) = 2$, thus $\|x^* + y^*\| = 2$).

But as we have mentioned before, a two-dimensional space is acs (equivalently a U-space) if and only if whenever $x, y \in S_X$ with $\|x+y\| = 2$, then *both* points x and y are smooth points of the unit ball.

[4] Recall that the notion of U-spaces is equivalent to that of uacs spaces and the modulus of u-convexity coincides with δ^X_{uacs} (see the remarks after Definition I.1.5).

Taking all this into account, we see that the space $\mathbb{R}^2$ endowed with the norm whose unit ball is sketched below will be an example of a space which fulfils the condition of the claim but is not a U-space (this example could be easily made precise using the description of absolute, normalised norms[5] on $\mathbb{R}^2$ via the boundary curve of their unit ball that will be discussed in Chapter V; we skip the details).

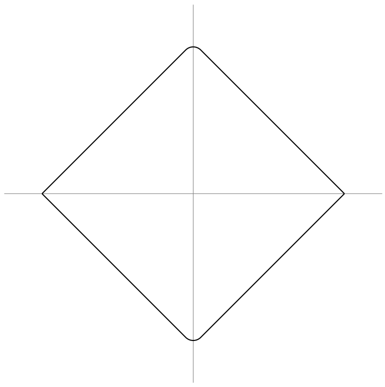

Fig. I.5

Unfortunately, both the proof of [86, Theorem 2.4] and the one of [39, Theorem 2.6] make use of the false implication in the above claim. However, it is possible to modify the proof from [86, Theorem 2.4] to show that the desired self-duality result is true nonetheless.

Proposition I.4.1. *Let X be a Banach space whose dual X^* is uacs. Then we have*

$$\delta_{\text{uacs}}^{X}(\varepsilon) \geq \delta_{\text{uacs}}^{X^*}\left(\delta_{\text{uacs}}^{X^*}(\varepsilon)\right) \ \forall \varepsilon \in (0,2]. \tag{I.4.1}$$

In particular, X is also uacs.

Proof. Take any $\varepsilon \in (0,2]$ and put $\delta = \delta_{\text{uacs}}^{X^*}(\varepsilon)$ and $\tilde{\delta} = \delta_{\text{uacs}}^{X^*}(\delta)$.
Now if $x, y \in S_X$ and $x^* \in S_{X^*}$ with $x^*(x) = 1$ and $\|x+y\| > 2(1-\tilde{\delta})$, choose $y^*, z^* \in S_{X^*}$ such that $y^*(y) = 1$ and $z^*(x+y) = \|x+y\|$.
Then we must have $z^*(x) > 1 - 2\tilde{\delta}$ and $z^*(y) > 1 - 2\tilde{\delta}$. It follows that $(z^* + x^*)(x) > 2 - 2\tilde{\delta}$ and $(z^* + y^*)(y) > 2 - 2\tilde{\delta}$ and hence

$$\left\|\frac{z^* + x^*}{2}\right\| > 1 - \tilde{\delta} \text{ and } \left\|\frac{z^* + y^*}{2}\right\| > 1 - \tilde{\delta}. \tag{I.4.2}$$

Next we pick any $z^{**} \in S_{X^{**}}$ with $z^{**}(z^*) = 1$. Then from (I.4.2) and the definition of $\tilde{\delta}$ we get that $z^{**}(x^*) > 1 - \delta$ and $z^{**}(y^*) > 1 - \delta$.
It follows that $\|x^* + y^*\| > 2(1-\delta)$ and because of $y^*(y) = 1$ and the definition of δ this implies $x^*(y) > 1 - \varepsilon$ and thus we have shown $\delta_{\text{uacs}}^{X}(\varepsilon) \geq \tilde{\delta} = \delta_{\text{uacs}}^{X^*}\big(\delta_{\text{uacs}}^{X^*}(\varepsilon)\big)$. □

[5] See Section II.1

Taking into account that uacs spaces are reflexive we finally get that being uacs is a self-dual property.

Corollary I.4.2. *A Banach space X is uacs if and only if X^* is uacs.*

The author does not know whether the equality $\delta_{\text{uacs}}^X = \delta_{\text{uacs}}^{X^*}$ for uacs spaces X that was claimed in [39, Theorem 2.6] is actually true.

Alternatively, we could also derive the self-duality from the following lemma (cf. the proof of [41, Lemma 9.9]). The modulus $\tilde{\rho}_{\text{uacs}}^X$ is defined exactly as ρ_{uacs}^X except that one replaces S_X by B_X. The argument that X is uacs if and only if $\lim_{\tau\to 0} \tilde{\rho}_{\text{uacs}}^X(\tau)/\tau = 0$ is analogous to the one for ρ_{uacs}^X.

Lemma I.4.3. *If X is any Banach space then for every $\tau > 0$ and every $0 < \varepsilon \leq 2$ the following inequalities hold:*

(i) $\delta_{\text{uacs}}^X(\varepsilon) + \rho_{\text{uacs}}^{X^*}(\tau) \geq \tau\frac{\varepsilon}{2}$,

(ii) $\delta_{\text{uacs}}^{X^*}(\varepsilon) + \tilde{\rho}_{\text{uacs}}^X(\tau) \geq \tau\frac{\varepsilon}{2}$.

Proof. We will only give an explicit proof for the slightly more difficult inequality (ii). To this end, fix $x^*, y^* \in S_{X^*}$ and $x^{**} \in S_{X^{**}}$ such that $x^{**}(x^*) = 1$ and $x^{**}(y^*) \leq 1 - \varepsilon$.
If $\|x^* + y^*\| \leq 2(1-\tau)$, then we certainly have $2 - \|x^* + y^*\| \geq \tau\varepsilon - 2\tilde{\rho}_{\text{uacs}}^X(\tau)$.
If $\|x^* + y^*\| > 2(1-\tau)$, then take an arbitrary $0 < \alpha < \|x^* + y^*\| - 2(1-\tau)$.
By Goldstine's theorem there is some $x \in B_X$ such that

$$|x^{**}(x^*) - x^*(x)| \leq \frac{\alpha}{2} \text{ and } |x^{**}(y^*) - y^*(x)| \leq \frac{\alpha}{2}.$$

Now choose $y \in S_X$ such that $(x^* + y^*)(y) > \|x^* + y^*\| - \alpha/2$. It follows that $(x^* + y^*)(y) > 2(1-\tau) + \alpha/2$ and hence $x^*(y), y^*(y) > 1 - 2\tau + \alpha/2$.
Thus we have

$$\|x + y\| \geq x^*(x + y) \geq x^{**}(x^*) - \frac{\alpha}{2} + 1 - 2\tau + \frac{\alpha}{2} = 2(1-\tau)$$

and hence

$$\begin{aligned}
&2\tilde{\rho}_{\text{uacs}}^X(\tau) \geq \|y + \tau x\| + \|y - \tau x\| - 2 \geq x^*(y + \tau x) + y^*(y - \tau x) - 2 \\
&= (x^* + y^*)(y) + \tau(x^*(x) - y^*(x)) - 2 \\
&\geq \|x^* + y^*\| - \frac{\alpha}{2} + \tau(x^{**}(x^*) - x^{**}(y^*) - \alpha) - 2 \\
&\geq \|x^* + y^*\| - \frac{\alpha}{2} + \tau(\varepsilon - \alpha) - 2.
\end{aligned}$$

For $\alpha \to 0$ we get $2 - \|x^* + y^*\| \geq \tau\varepsilon - 2\tilde{\rho}_{\text{uacs}}^X(\tau)$ and we are done. □

Now we turn to some duality results for the weaker versions of uacs spaces. It was already observed in Proposition I.2.7 that X is acs if X^* is acs (and the converse is true if X is reflexive). Concerning luacs, sluacs and wuacs spaces, the Proposition below is valid, in which we use the following terminology: a dual space X^* is said to be luacs* resp. wuacs* if it fulfils the definition of an luacs resp. wuacs space for all weak*-continuous functionals on X^*.

Proposition I.4.4. *For any Banach space X we have the following equivalences.*

(i) X^* luacs* $\iff$ X luacs

(ii) X^* wuacs* $\iff$ X sluacs

(iii) X^* sluacs $\iff$ X wuacs

In particular, if X is reflexive then X^* is luacs (resp. wuacs) if and only if X is luacs (resp. sluacs).

Proof. Since the arguments for (i), (ii) and (iii) are all similar, we will only prove (iii) explicitly. So let us first assume that X^* is sluacs and take sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in S_X and a functional $x^* \in S_{X^*}$ such that $\|x_n + y_n\| \to 2$ and $x^*(x_n) \to 1$.
Choose a sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} with $x_n^*(x_n + y_n) = \|x_n + y_n\|$ for every n. It follows that $x_n^*(x_n) \to 1$ and $x_n^*(y_n) \to 1$.
From $x^*(x_n) \to 1$ and $x_n^*(x_n) \to 1$ we get $\|x_n^* + x^*\| \to 2$. Together with $x_n^*(y_n) \to 1$ and the fact that X^* is sluacs this implies $x^*(y_n) \to 1$ and we are done.
Now assume X is wuacs and fix a sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} and $x^* \in S_{X^*}$ such that $\|x_n^* + x^*\| \to 2$ as well as a sequence $(x_n^{**})_{n\in\mathbb{N}}$ in $S_{X^{**}}$ with $x_n^{**}(x_n^*) \to 1$.
Because of $\|x_n^* + x^*\| \to 2$ we can find a sequence $(x_n)_{n\in\mathbb{N}}$ in S_X such that $x_n^*(x_n) \to 1$ and $x^*(x_n) \to 1$.
By Goldstine's theorem we can also find a sequence $(y_n)_{n\in\mathbb{N}}$ in B_X which satisfies

$$|x_n^*(y_n) - x_n^{**}(x_n^*)| \leq \frac{1}{n} \text{ and } |x^*(y_n) - x_n^{**}(x^*)| \leq \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

So we have $x_n^*(x_n + y_n) \to 2$ and hence $\|x_n + y_n\| \to 2$. Since X is wuacs and $x^*(x_n) \to 1$ we must also have $x^*(y_n) \to 1$ and consequently $x_n^{**}(x^*) \to 1$. $\square$

Finally, we would like to give necessary and sufficient conditions for a dual space to be acs resp. luacs resp. wuacs that do not explicitly involve the bidual space. We start with the acs case. The characterisation is inspired by [138, Proposition 3].

Proposition I.4.5. *Let X be any Banach space. The dual space X^* is acs if and only if for all sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in B_X and all functionals $x^*, y^* \in S_{X^*}$ the implication*

$$x^*(x_n + y_n) \to 2 \text{ and } y^*(x_n) \to 1 \;\Rightarrow\; y^*(y_n) \to 1$$

holds.

Proof. To prove the necessity, assume that X^* is acs and take sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ and functionals x^*, y^* as above. It follows that $\|x^*+y^*\| = 2$. By the weak*-compactness of $B_{X^{**}}$ we can find for an arbitrary subsequence $(y_{n_k})_{k\in\mathbb{N}}$ a subnet $(y_{n_{\phi(i)}})_{i\in I}$ that weak*-converges to some $y^{**} \in B_{X^{**}}$. It follows that $y^{**}(x^*) = 1$ and since X^* is acs we must also have $y^{**}(y^*) = 1$. Thus $y^*(y_{n_{\phi(i)}}) \to 1$ and the proof of the necessity is finished.
Now assume that X^* fulfils the above condition and take $x^*, y^* \in S_{X^*}$ and $x^{**} \in S_{X^{**}}$ such that $\|x^* + y^*\| = 2$ and $x^{**}(x^*) = 1$. Then we can find a sequence $(x_n)_{n\in\mathbb{N}}$ in B_X such that $x^*(x_n) \to 1$ and $y^*(x_n) \to 1$.
By Goldstine's theorem there is a sequence $(y_n)_{n\in\mathbb{N}}$ in B_X such that $x^*(y_n) \to x^{**}(x^*) = 1$ and $y^*(y_n) \to x^{**}(y^*)$.
Thus we have $x^*(x_n + y_n) \to 2$ and $y^*(x_n) \to 1$ and hence by our assumption we get $y^*(y_n) \to 1$, so $x^{**}(y^*) = 1$. □

The characterisations for the dual space to be luacs resp. wuacs are a bit more complicated. They read as follows.

Proposition I.4.6. *Let X be a Banach space.*

(i) *X^* is luacs if and only if for every $x^* \in S_{X^*}$ and all sequences $(x_n^*)_{n\in\mathbb{N}}$ and $(x_k)_{k\in\mathbb{N}}$ in S_{X^*} and B_X, respectively, the implication*

$$\|x^* + x_n^*\| \to 2 \text{ and } x_n^*(x_k) \xrightarrow[k\geq n]{k,n\to\infty} 1 \ \Rightarrow\ x^*(x_k) \to 1$$

holds.

(ii) *X^* is wuacs if and only if for all sequences $(x_n^*)_{n\in\mathbb{N}}, (y_n^*)_{n\in\mathbb{N}}$ in S_{X^*} and $(x_k)_{k\in\mathbb{N}}$ in B_X the implication*

$$\|x_n^* + y_n^*\| \to 2 \text{ and } x_n^*(x_k) \xrightarrow[k\geq n]{k,n\to\infty} 1 \ \Rightarrow\ \lim_{n\to\infty} \sup_{k\geq n} y_n^*(x_k) = 1.$$

holds.

Proof. To prove (ii) we first assume that X^* is wuacs and fix sequences $(x_n^*)_{n\in\mathbb{N}}, (y_n^*)_{n\in\mathbb{N}}$ in S_{X^*} and $(x_k)_{k\in\mathbb{N}}$ in B_X as above. Since $B_{X^{**}}$ is weak*-compact there is a subnet $(x_{\phi(i)})_{i\in I}$ that is weak*-convergent to some $x^{**} \in B_{X^{**}}$. We will show that $x^{**}(x_n^*) \to 1$.
Given any $\varepsilon > 0$ by our assumption on $(x_n^*)_{n\in\mathbb{N}}$ and $(x_k)_{k\in\mathbb{N}}$ we can find an $N \in \mathbb{N}$ such that

$$|x_n^*(x_k) - 1| \leq \varepsilon \quad \forall k \geq n \geq N.$$

For every $n \geq N$ it is possible to find an index $i \in I$ with $\phi(i) \geq n$ and $\left|x_n^*(x_{\phi(i)}) - x^{**}(x_n^*)\right| \leq \varepsilon$. It follows that $|x^{**}(y_n^*) - 1| \leq 2\varepsilon$ and the convergence is proved.
So we have $\|x_n^* + y_n^*\| \to 2$ and $x^{**}(x_n^*) \to 1$. Since X^* is wuacs this implies $x^{**}(y_n^*) \to 1$. Thus for any $\delta > 0$ there is some $n_0 \in \mathbb{N}$ such that

$|x^{**}(y_n^*) - 1| \le \delta$ for all $n \ge n_0$ and for any such n we find $j \in I$ with $\phi(j) \ge n$ and $\left|y_n^*(x_{\phi(i)}) - x^{**}(y_n^*)\right| \le \delta$. Hence $\left|y_n^*(x_{\phi(i)}) - 1\right| \le 2\delta$ and we have shown $\sup_{k\ge n} y_n^*(x_k) \ge 1 - 2\delta$ for all $n \ge n_0$.
Now let us prove the converse. We take sequences $(x_n^*)_{n\in\mathbb{N}}, (y_n^*)_{n\in\mathbb{N}}$ in S_{X^*} such that $\|x_n^* + y_n^*\| \to 2$ and a functional $x^{**} \in S_{X^{**}}$ with $x^{**}(x_n^*) \to 1$.
By means of Goldstine's theorem we find a sequence $(x_k)_{k\in\mathbb{N}}$ in B_X that satisfies

$$|x_n^*(x_k) - x^{**}(x_n^*)| \le \frac{1}{k} \text{ and } |y_n^*(x_k) - x^{**}(y_n^*)| \le \frac{1}{k} \quad \forall n \le k.$$

It is then easy to see that $(x_n^*(x_k))_{k\ge n}$ tends to 1 and hence our assumption gives us $\lim_{n\to\infty} \sup_{k\ge n} y_n^*(x_k) = 1$.
Thus for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ with $\sup_{k\ge n} y_n^*(x_k) > 1 - \varepsilon$ and $1/n \le \varepsilon$ for each $n \ge N$.
If we fix $n \ge N$ we find $k \ge n$ with $y_n^*(x_k) \ge 1 - \varepsilon$ and because of $|x^{**}(y_n^*) - y_n^*(x_k)| \le 1/k \le \varepsilon$ it follows that $x^{**}(y_n^*) \ge 1 - 2\varepsilon$ and the proof is finished. Part (i) is proved similarly. □

I.5 Quotient spaces

This section is devoted to the study of quotients of acs-type spaces.

If U is a closed subspace of X then $(X/U)^*$ is isometrically isomorphic to $U^\perp$ (the annihilator of U in X^*). Using this together with the self-duality of uacs spaces (Corollary I.4.2) and the obvious fact that closed subspaces of uacs spaces are again uacs, one immediately gets that quotients of uacs spaces are uacs as well.[6] An analogous argument using part (iii) of Proposition I.4.4 works for wuacs spaces, so in summary we have the following Proposition.

Proposition I.5.1. *Let U be a closed subspace of the Banach space X. If X is uacs (resp. wuacs) then X/U is also uacs (resp. wuacs).*

As for quotients of acs, luacs and sluacs spaces we have the following result which is an analogue of [84, Proposition 3.2].

Proposition I.5.2. *If U is a reflexive subspace of the Banach space X then the properties acs, luacs and sluacs pass from X to X/U.*

Proof. Let $\omega : X \to X/U$ be the canonical quotient map. As was observed in the proof of [84, Proposition 3.2] the reflexivity of U implies $\omega(B_X) = B_{X/U}$. Now suppose that X is sluacs and take a sequence $(z_n)_{n\in\mathbb{N}}$ in $S_{X/U}$ and an element $z \in S_{X/U}$ such that $\|z_n + z\| \to 2$. Further, take a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $S_{(X/U)^*}$ with $\psi_n(z_n) \to 1$.

[6] This is a standard type of argument, similar to the well-known proof that quotients of UR spaces are again UR (here the duality of UR and US is used).

Since $\omega(B_X) = B_{X/U}$ we can find a sequence $(x_n)_{n\in\mathbb{N}}$ in S_X and a point $x \in S_X$ such that $z_n = \omega(x_n)$ for every n and $z = \omega(x)$.
It easily follows from $\|z_n + z\| \to 2$ that we also have $\|x_n + x\| \to 2$.
We put $x_n^* := \psi_n \circ \omega \in S_{U^\perp}$ for every n and observe that $x_n^*(x_n) = \psi_n(z_n) \to 1$.
Since X is sluacs this implies $x_n^*(x) = \psi_n(z) \to 1$.
The proofs for acs and luacs spaces are analogous. □

Using again the relation $(X/U)^* \cong U^\perp$ for every closed subspace U of X we can derive the following from Propositions I.2.7 and I.4.4.

Proposition I.5.3. *If U is a closed subspace of the Banach space X the following implications hold.*

(i) X^* acs $\Rightarrow$ X/U acs

(ii) X^* luacs $\Rightarrow$ X/U luacs

(iii) X^* wuacs $\Rightarrow$ X/U sluacs

It is known (cf. [29, p.145]) that for any Banach space X the dual X^* is R (resp. S) if and only if every quotient space of X is S (resp. R) if and only if every two-dimensional quotient space of X is S (resp. R). By an analogous argument we can get the following result.

Proposition I.5.4. *For a Banach space X the following assertions are equivalent.*

(i) *X^* is acs.*

(ii) *X/U is acs for every closed subspace U of X.*

(iii) *X/U is acs for every closed subspace U of X with $\dim X/U = 2$.*

Proof. (i) $\Rightarrow$ (ii) holds according to Proposition I.5.3 and (ii) $\Rightarrow$ (iii) is trivial, so it only remains to prove (iii) $\Rightarrow$ (i). Obviously it suffices to show that every two-dimensional subspace of X^* is acs, so let us take such a subspace $V = \operatorname{span}\{x^*, y^*\}$. Then $V = U^\perp = (X/U)^*$, where $U = \ker x^* \cap \ker y^*$. The quotient space X/U is two-dimensional and hence by our assumption it is acs. Since X/U is in particular reflexive it follows from Proposition I.4.4 that $(X/U)^* = V$ is also acs. □

By [84, Proposition 3.4] there is an equivalent norm $|||\,.\,|||$ on ℓ^1 such that $(\ell^1, |||\,.\,|||)$ is R and every separable Banach space is isometrically isomorphic to a quotient space of $(\ell^1, |||\,.\,|||)$, so in particular ℓ^1 is a quotient of $(\ell^1, |||\,.\,|||)$. Thus quotients of acs spaces are in general not acs and it also follows (in view of Proposition I.5.4) that the fact that X is acs is not sufficient to ensure that X^* is acs.

There is also an analogue of Proposition I.5.4 for uacs spaces which reads as follows. (The corresponding result for UR spaces was proved by Day (cf. [27, Theorem 5.5]).)

Proposition I.5.5. *For a Banach space X let $\mathcal{S}(X)$ denote the set of all closed subspaces of X and $\mathcal{S}_2(X)$ the set of all closed subspaces U of X such that* $\dim X/U \leq 2$. *Then the following assertions are equivalent:*

(i) *X is uacs.*

(ii) $\inf\Big\{\delta_{\mathrm{uacs}}^{X/U}(\varepsilon) : U \in \mathcal{S}(X)\Big\} > 0 \quad \forall \varepsilon \in (0,2]$.

(iii) $\inf\Big\{\delta_{\mathrm{uacs}}^{X/U}(\varepsilon) : U \in \mathcal{S}_2(X)\Big\} > 0 \quad \forall \varepsilon \in (0,2]$.

Proof. (i) $\Rightarrow$ (ii) Let X be uacs. If $U \in \mathcal{S}(X)$ then $(X/U)^* \cong U^{\perp}$, hence $\delta_{\mathrm{uacs}}^{(X/U)^*}(\varepsilon) \geq \delta_{\mathrm{uacs}}^{X^*}(\varepsilon) \geq \delta_{\mathrm{uacs}}^{X}\big(\delta_{\mathrm{uacs}}^{X}(\varepsilon)\big)$ by Proposition I.4.1 and the reflexivity of X.
Using again Proposition I.4.1 (now applied to X/U) and the monotonicity of the uacs modulus we obtain

$$\delta_{\mathrm{uacs}}^{X/U}(\varepsilon) \geq \delta_{\mathrm{uacs}}^{X}\big(\delta_{\mathrm{uacs}}^{X}\big(\delta_{\mathrm{uacs}}^{X}\big(\delta_{\mathrm{uacs}}^{X}(\varepsilon)\big)\big)\big) > 0,$$

which finishes our argument.
Since (ii) $\Rightarrow$ (iii) is obvious it only remains to prove (iii) $\Rightarrow$ (i). Denote the infimum in (iii) by $\delta(\varepsilon)$ and take sequence $(x_n^*)_{n\in\mathbb{N}}, (y_n^*)_{n\in\mathbb{N}}$ in S_{X^*} such that $\|x_n^* + y_n^*\| \to 2$ and a sequence $(x_n^{**})_{n\in\mathbb{N}}$ in $S_{X^{**}}$ with $x_n^{**}(x_n^*) \to 1$.
We put $V_n = \operatorname{span}\{x_n^*, y_n^*\}$ and $U_n = \ker x_n^* \cap \ker y_n^*$ for every n. Then $V_n = U_n^{\perp} = (X/U_n)^*$. Again by Proposition I.4.1 (and reflexivity of X/U_n) we get that $\delta_{\mathrm{uacs}}^{V_n}(\varepsilon) \geq \delta_{\mathrm{uacs}}^{X/U_n}\Big(\delta_{\mathrm{ucas}}^{X/U_n}(\varepsilon)\Big) \geq \delta(\delta(\varepsilon))$.
Let φ_n denote the restriction of x_n^{**} to V_n and fix any $\varepsilon_0 > 0$. Because of $\|x_n^* + y_n^*\| \to 2$ we have $1 - 2^{-1}\|x_n^* + y_n^*\| < \delta(\delta(\varepsilon_0)) \leq \delta_{\mathrm{uacs}}^{V_n}(\varepsilon_0)$ for sufficiently large n.
Since $\varphi_n(x_n^*) = 1$ this implies that we eventually have $\varphi_n(y_n^*) = x_n^{**}(y_n^*) \geq 1 - \varepsilon_0$.
Thus we have shown that X^* is uacs and by Proposition I.4.1 X is uacs as well. $\square$

I.6 Symmetric versions: luacs$^+$ and sluacs$^+$ spaces

In this section, we will introduce a kind of symmetrised versions of the notions of luacs and sluacs spaces. These will be needed later in Chapters II and III, when we study absolute sums and Köthe-Bochner spaces of various acs-type spaces. Here is the definition.

Definition I.6.1. A Banach space X is called

(i) an luacs$^+$ space if for every $x \in S_X$, every sequence $(x_n)_{n\in\mathbb{N}}$ in S_X with $\|x_n + x\| \to 2$ and all $x^* \in S_{X^*}$ we have

$$x^*(x_n) \to 1 \iff x^*(x) = 1,$$

(ii) an sluacs$^+$ space if for every $x \in S_X$, every sequence $(x_n)_{n\in\mathbb{N}}$ in S_X with $\|x_n + x\| \to 2$ and all sequences $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} we have

$$x_n^*(x_n) \to 1 \iff x_n^*(x) \to 1.$$

If we include these two properties in our implication chart we get the following.

$$\begin{tikzcd}
 & \mathrm{WUR} \arrow[d] \arrow[dr] & & \\
\mathrm{UR} \arrow[ur] \arrow[d] \arrow[dr] & \mathrm{wuacs} \arrow[dr] & \mathrm{WLUR} \arrow[r] \arrow[d] & \mathrm{R} \arrow[dd] \\
\mathrm{uacs} \arrow[ur] \arrow[dr] & \mathrm{LUR} \arrow[ur] \arrow[d] & \mathrm{luacs}^+ \arrow[d] & \\
 & \mathrm{sluacs}^+ \arrow[ur] \arrow[d] & \mathrm{luacs} \arrow[r] & \mathrm{acs} \\
 & \mathrm{sluacs} \arrow[ur] & &
\end{tikzcd}$$

Fig. I.6

Let us mention that Proposition I.5.2 also holds for luacs$^+$ and sluacs$^+$ spaces (with the same argument). Also, Proposition I.2.11 resp. I.2.9 holds accordingly for luacs$^+$ resp. sluacs$^+$ spaces.

In analogy to Proposition I.2.4 one can prove that for any Banach space X the following conditions are equivalent:

(i) For all sequences $(x_n)_{n\in\mathbb{N}}$ in S_X, $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} and every $x \in S_X$ with $\|x_n + x\| \to 2$ and $x_n^*(x) \to 1$ one has $x_n^*(x_n) \to 1$.

(ii) For every $x \in S_X$ and every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|x + ty\| + \|x - ty\| \le 2 + \varepsilon t$$

whenever $t \in [0, \delta]$ and $y \in S_X$ with $\|x + y\| \ge 2(1 - t)$.

Since every FS space fulfils (i)[7], it follows that a space which is FS and sluacs (resp. luacs) is sluacs$^+$ (resp. luacs$^+$). In the context of FS spaces we also have the following result.

Proposition I.6.2. *If X is FS and X^* is acs then X is luacs$^+$. In particular, every reflexive FS space is luacs$^+$.*

[7] Take sequences $(x_n)_{n\in\mathbb{N}}$ in S_X, $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} and $x \in S_X$ with $\|x_n + x\| \to 2$ and $x_n^*(x) \to 1$. Find a sequence $(y_n^*)_{n\in\mathbb{N}}$ in S_{X^*} such that $y_n^*(x_n) \to 1$ and $y_n^*(x) \to 1$. Since X is FS it follows that $\|y_n^* - x_n^*\| \to 0$ (see for example [41, Lemma 8.4]) and hence $x_n^*(x_n) \to 1$. (In [110] the property (WM) was introduced, which is equivalent to the special case of a constant sequence of functionals in (i), in other words, to the reverse implication in the definition of luacs$^+$ spaces. It was already proved in [110, Theorem 3.7] that every strongly smooth(=FS) space has property (WM).)

Proof. By our previous considerations we only have to show that X is luacs. Take a sequence $(x_n)_{n\in\mathbb{N}}$ in S_X and a point $x \in S_X$ with $\|x_n + x\| \to 2$ as well as a functional $x^* \in S_{X^*}$ with $x^*(x_n) \to 1$. Choose a sequence $(y_n^*)_{n\in\mathbb{N}}$ in S_{X^*} such that $y_n^*(x_n + x) = \|x_n + x\|$ for every $n \in \mathbb{N}$. It follows that $y_n^*(x_n) \to 1$ and $y_n^*(x) \to 1$.
Because of $\|y_n^* + x^*\| \geq y_n^*(x_n) + x^*(x_n)$ for every n it follows that $\|y_n^* + x^*\| \to 2$. If $y^* \in S_{X^*}$ is the Fréchet-derivative of $\|\,.\,\|$ at x then $y_n^*(x) \to 1$ implies $\|y_n^* - y^*\| \to 0$ (see for instance [41, Lemma 8.4]). Hence we get $\|x^* + y^*\| = 2$ and $y^*(x) = 1$.
Since X^* is acs we can conclude that $x^*(x) = 1$. $\square$

I.7 Midpoint versions

In this section, we define further variants of acs spaces in analogy to MLUR and WMLUR spaces.

First recall that a Banach space X is said to be *midpoint locally uniformly rotund* (MLUR in short) if for any two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in S_X and every $x \in S_X$ we have

$$\left\| x - \frac{x_n + y_n}{2} \right\| \to 0 \;\Rightarrow\; \|x_n - y_n\| \to 0.$$

This notion was originally introduced in [5].

Also, X is called *weakly midpoint locally uniformly rotund* (WMLUR in short) if it satisfies the above condition with $\|x_n - y_n\| \to 0$ replaced by $x_n - y_n \xrightarrow{\sigma} 0$, where the symbol $\xrightarrow{\sigma}$ denotes the convergence in the weak topology of X.

We now introduce in an analogous way midpoint versions of luacs and sluacs spaces.

Definition I.7.1. Let X be a Banach space.

(i) The space X is said to be *midpoint locally uniformly alternatively convex or smooth* (mluacs in short) if for any two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in S_X, every $x \in S_X$ and every $x^* \in S_{X^*}$ we have that

$$\left\| x - \frac{x_n + y_n}{2} \right\| \to 0 \text{ and } x^*(x_n) \to 1 \;\Rightarrow\; x^*(y_n) \to 1.$$

(ii) The space X is called *midpoint strongly locally uniformly alternatively convex or smooth* (msluacs in short) if for any two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in S_X, every $x \in S_X$ and every sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} we have that

$$\left\| x - \frac{x_n + y_n}{2} \right\| \to 0 \text{ and } x_n^*(x_n) \to 1 \;\Rightarrow\; x_n^*(y_n) \to 1.$$

We then get the following implication chart.

$$\begin{tikzcd}
 & \text{MLUR} \arrow[d] \arrow[dr] & & \\
\text{LUR} \arrow[ur] \arrow[d] \arrow[dr] & \text{msluacs} \arrow[dr] & \text{WMLUR} \arrow[r] \arrow[d] & \text{R} \arrow[d] \\
\text{sluacs} \arrow[ur] \arrow[dr] & \text{WLUR} \arrow[ur] \arrow[d] & \text{mluacs} \arrow[r] & \text{acs} \\
 & \text{luacs} \arrow[ur] & &
\end{tikzcd}$$

Fig. I.7

No other implications are valid in general, as is shown by the examples in Section I.9.

Note that in the definition of msluacs spaces we can replace the condition $x_n^*(x_n) \to 1$ by $x_n^*(x_n) = 1$ for every $n \in \mathbb{N}$ and obtain an equivalent definition, by the same argument used in the proof of Proposition I.2.1.

Also, it is well known (and not hard to see) that a Banach space X is MLUR (resp. WMLUR) if and only if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X and each element $x \in X$ the condition $\|x \pm x_n\| \to \|x\|$ implies $\|x_n\| \to 0$ (resp. $x_n \xrightarrow{\sigma} 0$). In much the same way one can prove that X is msluacs if and only if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X, each $x \in X$ and every bounded sequence $(x_n^*)_{n\in\mathbb{N}}$ in X^* the two conditions $\|x \pm x_n\| \to \|x\|$ and $x_n^*(x + x_n) - \|x_n^*\|\|x\| \to 0$ imply $x_n^*(x_n) \to 0$. An analogous characterisation holds for mluacs spaces.

It was noted in [129, p.663] that by using the principle of local reflexivity one can easily check that X is WMLUR if and only if every point $x \in S_X$ is an extreme point of $B_{X^{**}}$[8](in particular, WMLUR and R coincide in reflexive spaces). In analogy to this result we can prove the following characterisation of mluacs spaces, which especially yields that mluacs and acs coincide in reflexive spaces.

Proposition I.7.2. *A Banach space X is mluacs if and only if the following holds: for any two elements $x^{**}, y^{**} \in S_{X^{**}}$ with $x^{**} + y^{**} \in 2S_X$ and every $x^* \in S_{X^*}$ with $x^{**}(x^*) = 1$ we also have $y^{**}(x^*) = 1$.*

Proof. The sufficiency is straightforwardly proved using the weak*-compactness of the bidual unit ball.
To prove the necessity, fix $x^{**}, y^{**} \in S_{X^{**}}$ such that $x^{**} + y^{**} \in 2S_X$ and $x^* \in S_{X^*}$ with $x^{**}(x^*) = 1$. Put $F = \operatorname{span}\{x^{**}, y^{**}\}$. By the principle of local reflexivity (cf. [3, Theorem 11.2.4]) we can find for each $n \in \mathbb{N}$ a finite-dimensional subspace $E_n \subseteq X$ and an isomorphism $T_n : F \to E_n$ such that $\|T_n\| \leq 1 + 2^{-n}$, $\|T_n^{-1}\| \leq 1 + 2^{-n}$, $T_n z = z$ for every $z \in X \cap F$ and $x^*(T_n z^{**}) = z^{**}(x^*)$ for all $z^{**} \in F$.
If we put $x_n = T_n x^{**}$ and $y_n = T_n y^{**}$ for every $n \in \mathbb{N}$, then we have $x_n + y_n = 2(x^{**} + y^{**})$ and $x^*(x_n) = 1$ as well as $\|x_n\|, \|y_n\| \to 1$. Since X is

[8] As before, we consider X canonically embedded into its second dual.

mluacs it follows that $x^*(y_n) \to 1$. But $x^*(y_n) = y^{**}(x^*)$ for every n, thus $y^{**}(x^*) = 1$. □

We can also prove a characterisation of msluacs spaces that is analogous to the results on acs, sluacs and uacs spaces given in the Propositions I.2.2, I.2.3 and I.2.4 (the proof is completely analogous as well).

Proposition I.7.3. *For a Banach space X the following assertions are equivalent:*

(i) *X is msluacs.*

(ii) *For every $z \in S_X$ and every $\varepsilon > 0$ there exists some $\delta > 0$ such that for every $t \in [0, \delta]$ and all $x, y \in S_X$ with $\|x + y - 2z\| \leq 2t$ we have*

$$\|x + ty\| + \|x - ty\| \leq 2 + \varepsilon t.$$

(iii) *For every $z \in S_X$ and every $\varepsilon > 0$ there exists some $\delta > 0$ such that for every $t \in [0, \delta]$ and all $x, y \in S_X$ with $\|x + y - 2z\| \leq \delta t$ we have*

$$\|x - ty\| \leq 1 + t(\varepsilon - 1).$$

(iv) *For every $z \in S_X$ there exists some $1 \leq p < \infty$ such that for every $\varepsilon > 0$ there is some $\delta > 0$ such that for all $t \in [0, \delta]$ and all $x, y \in S_X$ with $\|x + y - 2z\| \leq 2t$ we have*

$$\|x + ty\|^p + \|x - ty\|^p \leq 2 + \varepsilon t^p.$$

(v) *For every $z \in S_X$ there exists some $1 \leq p < \infty$ such that for every $\varepsilon > 0$ there is some $\delta > 0$ such that for all $t \in [0, \delta]$ and all $x, y \in S_X$ with $\|x + y - 2z\| \leq t\delta$ we have*

$$(1 + t)^p + \|x - ty\|^p \leq 2 + \varepsilon t^p.$$

Recall that the space X is said to have the Kadets-Klee property (also known as property (H)) if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X and each $x \in X$ the implication

$$x_n \xrightarrow{\sigma} x \text{ and } \|x_n\| \to \|x\| \;\Rightarrow\; \|x_n - x\| \to 0$$

holds. For example, it is easy to see that every LUR space has the Kadets-Klee property.

It was proved in [70] that a Banach space which is R, has the Kadets-Klee property and does not contain an isomorpic copy of ℓ^1 is actually MLUR. We can adopt the proof from [70] to show the analogous result for acs spaces.

Proposition I.7.4. *Let X be an acs space which has the Kadets-Klee property and does not contain an isomorphic copy of ℓ^1. Then X is msluacs.*

Proof. Take a sequence $(x_n)_{n\in\mathbb{N}}$ in X and $x \in X$ with $\|x_n \pm x\| \to \|x\|$ and a sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} such that $x_n^*(x_n + x) \to \|x\|$. If $(x_n^*(x_n))_{n\in\mathbb{N}}$ was not convergent to zero, then by passing to an appropriate subsequence we could assume $|x_n^*(x_n)| \geq \varepsilon$ for all n and some $\varepsilon > 0$.
Since X does not contain ℓ^1 we can, by Rosenthal's theorem (cf. [3, Theorem 10.2.1]), pass to a further subsequence such that $(x_n)_{n\in\mathbb{N}}$ is weakly Cauchy. But then the double-sequence $(x_n - x_m)_{n,m\in\mathbb{N}}$ is weakly null, so $\liminf\|2x + x_n - x_m\| \geq 2\|x\|$.
On the other hand, because of

$$\|2x + x_n - x_m\| \leq \|x + x_n\| + \|x - x_m\| \quad \forall n, m \in \mathbb{N},$$

we also have $\limsup\|2x + x_n - x_m\| \leq 2\|x\|$ and hence $\lim\|2x + x_n - x_m\| = 2\|x\|$. Since $2x + x_n - x_m \xrightarrow{\sigma} 2x$ the Kadets-Klee property of X implies that $\lim\|x_n - x_m\| = 0$, i. e. $(x_n)_{n\in\mathbb{N}}$ is norm Cauchy. Let y be the limit of $(x_n)_{n\in\mathbb{N}}$. It follows that $\|x \pm y\| = \|x\|$, so if we put $z_1 = (x + y)/\|x\|$ and $z_2 = (x - y)/\|x\|$ then $\|z_1\| = \|z_2\| = 1$ and $\|z_1 + z_2\| = 2$.
Since B_{X^*} is weak*-compact we can pass to a subnet $(x^*_{\varphi(i)})_{i\in I}$ that is weak*-convergent to some $x^* \in B_{X^*}$. Because of $x^*_{\varphi(i)}(x_{\varphi(i)} + x) \to \|x\|$ this implies $x^*_{\varphi(i)}(x_{\varphi(i)}) \to \|x\| - x^*(x)$. But we also have $x^*_{\varphi(i)}(y) \to x^*(y)$ and $\|x_{\varphi(i)} - y\| \to 0$, thus $x^*_{\varphi(i)}(x_{\varphi(i)}) \to x^*(y)$ and hence $x^*(z_1) = 1$.
Because of $|x^*_{\varphi(i)}(x_{\varphi(i)})| \geq \varepsilon$ for every $i \in I$ we have $x^*(y) \neq 0$. It follows that $x^*(z_2) \neq 1$ and hence X cannot be acs, contradicting our hypothesis. □

In the spirit of the duality results from Section I.4 it is also not difficult to prove the following assertions.

Proposition I.7.5. *Let X be a Banach space such that for all functionals $x^*, y^* \in S_{X^*}$ and every $x \in S_X$ the implication*

$$\|x^* + y^*\| = 2 \text{ and } x^*(x) = 1 \;\Rightarrow\; y^*(x) = 1$$

is valid. Then X is mluacs. In particular, X is mluacs whenever X^* is acs.

Proposition I.7.6. *Let X be a Banach space such that for every sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*}, every $x^* \in S_{X^*}$ and each $x \in S_X$ the implication*

$$\|x_n^* + x^*\| \to 2 \text{ and } x^*(x) = 1 \;\Rightarrow\; x_n^*(x) \to 1$$

is valid. Then X is msluacs. In particular, X is msluacs whenever X^* is luacs$^+$.

Proof. We will only prove Proposition I.7.6, since the proof of Proposition I.7.5 is analogous. So let us take sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in S_X and an element $x \in S_X$ such that $\|x_n + y_n - 2x\| \to 0$, as well as a sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} with $x_n^*(x_n) \to 1$.
Fix a functional $x^* \in S_{X^*}$ with $x^*(x) = 1$. Then $x^*(x_n + y_n) \to 2$ and hence $x^*(x_n) \to 1$ and $x^*(y_n) \to 1$. Because of $x_n^*(x_n) \to 1$ it follows that $\|x_n^* + x^*\| \to 2$. Thus our assumption implies $x_n^*(x) \to 1$.
But then $x_n^*(x_n+y_n) \to 2$ and hence $x_n^*(y_n) \to 1$, which finishes the proof. $\square$

I.8 A directional version

The purpose of this section is to define yet another variant of acs spaces, namely an analogue of the notion of URED spaces.

Recall that a Banach space X is said to be *uniformly rotund in every direction* (URED in short) if for any $z \in X \setminus \{0\}$ and all sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in S_X such that $\|x_n + y_n\| \to 2$ and $x_n - y_n \in \operatorname{span}\{z\}$ for every n one already has $\|x_n - y_n\| \to 0$.

This notion was first introduced by Garkavi in [50] and further studied by the authors of [28].

In this spirit, we define the following directionalisation of uacs spaces.

Definition I.8.1. A Banach space X is called *uniformly alternatively convex or smooth in every direction* (uacsed in short) if for every $z \in X \setminus \{0\}$ and all sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in S_X and $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} such that $\|x_n + y_n\| \to 2$, $x_n^*(x_n) \to 1$ and $x_n - y_n \in \operatorname{span}\{z\}$ for every n one also has $x_n^*(y_n) \to 1$.

Obviously, the following implications hold.

$$\begin{tikzcd} \mathrm{UR} \arrow[r] \arrow[d] & \mathrm{URED} \arrow[r] \arrow[d] & \mathrm{R} \arrow[d] \\ \mathrm{uacs} \arrow[r] & \mathrm{uacsed} \arrow[r] & \mathrm{acs} \end{tikzcd}$$

Fig. I.8

We first give some equivalent characterisations for a Banach space to be uacsed in analogy to the characterisations for a Banach space to be URED given in [28, Theorem 1].

Proposition I.8.2. *Let X be a Banach space and $2 \le p < \infty$. Then the following assertions are equivalent.*

(i) *X is uacsed.*

(ii) *For all sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in B_X, $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} and every $z \in X$ the implication*

$$\|x_n + y_n\| \to 2,\ x_n^*(x_n) \to 1 \text{ and } x_n - y_n \to z \ \Rightarrow\ x_n^*(z) \to 0$$

holds.

(iii) *For all sequences* $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ *in* S_X*,* $(x_n^*)_{n\in\mathbb{N}}$ *in* S_{X^*} *and every* $z \in X$ *the implication*

$$\|x_n + y_n\| \to 2,\ x_n^*(x_n) \to 1 \text{ and } x_n - y_n \to z \ \Rightarrow\ x_n^*(z) \to 0$$

holds.

(iv) *For all sequences* $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ *in* B_X*,* $(x_n^*)_{n\in\mathbb{N}}$ *in* S_{X^*} *and every* $z \in X \setminus \{0\}$ *the conditions*

$$\|x_n + y_n\| \to 2,\ x_n^*(x_n) \to 1 \text{ and } x_n - y_n \in \operatorname{span}\{z\}\ \forall n \in \mathbb{N}$$

imply $x_n^*(y_n) \to 1$*.*

(v) *For all sequences* $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ *in* S_X*,* $(x_n^*)_{n\in\mathbb{N}}$ *in* S_{X^*} *and every* $z \in X$ *the implication*

$$\|x_n + y_n\| \to 2,\ x_n^*(x_n) = 1\ \forall n \in \mathbb{N} \text{ and } x_n - y_n \to z \ \Rightarrow\ x_n^*(z) \to 0$$

holds.

(vi) *For all sequences* $(x_n)_{n\in\mathbb{N}}$ *in* B_X*,* $(x_n^*)_{n\in\mathbb{N}}$ *in* S_{X^*} *and every* $z \in X$ *the two conditions*

$$2^{p-1}(\|x_n + z\|^p + \|x_n\|^p) - \|2x_n + z\|^p \to 0 \text{ and } x_n^*(x_n) \to 1$$

imply that $x_n^*(z) \to 0$*.*

(vii) *For all sequences* $(x_n)_{n\in\mathbb{N}}$ *in* S_X*,* $(x_n^*)_{n\in\mathbb{N}}$ *in* S_{X^*} *and every* $z \in X$ *the two conditions*

$$\|2x_n + z\|^p - 2^{p-1}\|x_n + z\|^p \to 2^{p-1} \text{ and } x_n^*(x_n) = 1\ \forall n \in \mathbb{N}$$

imply that $x_n^*(z) \to 0$*.*

(viii) *For all sequences* $(x_n)_{n\in\mathbb{N}}$ *in* B_X*,* $(y_n)_{n\in\mathbb{N}}$ *in* X*,* $(x_n^*)_{n\in\mathbb{N}}$ *in* S_{X^*} *and every* $z \in X$ *the conditions*

$$\|x_n + y_n\| \to 2, \|y_n\| \to 1, x_n^*(x_n) \to 1 \text{ and } x_n - y_n = z\ \forall n \in \mathbb{N}$$

imply that $x_n^*(z) \to 0$*.*

(ix) *For all sequences* $(x_n)_{n\in\mathbb{N}}$ *in* S_X*,* $(y_n)_{n\in\mathbb{N}}$ *in* X*,* $(x_n^*)_{n\in\mathbb{N}}$ *in* S_{X^*} *and every* $z \in X$ *the conditions*

$$\|x_n + y_n\| \to 2, \|y_n\| \to 1, x_n^*(x_n) = 1\ \forall n \in \mathbb{N} \text{ and } x_n - y_n = z\ \forall n \in \mathbb{N}$$

imply that $x_n^*(z) \to 0$*.*

Proof. We first prove (i) $\Rightarrow$ (ii). So let us fix two sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in B_X, a sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} and a $z \in X\setminus\{0\}$ such that $\|x_n+y_n\| \to 2$, $x_n^*(x_n) \to 1$ and $x_n - y_n \to z$.
If there is a subsequence $(x_{n_k}^*(z))_{k\in\mathbb{N}}$ such that $x_{n_k}^*(z) \le 0$ for every $k \in \mathbb{N}$ then because of $x_n - y_n \to z$ and $x_n^*(x_n) \to 1$ we have

$$0 \ge \limsup_{k\to\infty} x_{n_k}^*(z) = \limsup_{k\to\infty} x_{n_k}^*(x_{n_k} - y_{n_k}) = 1 - \liminf_{k\to\infty} x_{n_k}^*(y_{n_k}) \ge 0,$$

so $\limsup_{k\to\infty} x_{n_k}^*(z) = 0$ and analogously $\liminf_{k\to\infty} x_{n_k}^*(z) = 0$, hence $\lim_{k\to\infty} x_{n_k}^*(z) = 0$.
Otherwise the sequence $(x_n^*(z))_{n\in\mathbb{N}}$ is eventually positive. Since $\|x_n+y_n\| \to 2$ and $\|x_n\|, \|y_n\| \le 1$ for each n it follows that $\|x_n\|, \|y_n\| \to 1$. Because of $x_n - y_n \to z$ this implies $\|x_n - z\| \to 1$.
It further follows from $\|x_n + y_n\| \to 2$ and $x_n - y_n \to z$ that $\|2x_n - z\| \to 2$.
Now as in the proof of [28, Theorem 1] we put

$$\omega_n = \min\Big\{1, \|x_n - z\|^{-1}\Big\},\ a_n = \omega_n x_n,\ b_n = \omega_n(x_n - z) \quad \forall n \in \mathbb{N}$$

and observe that $\|a_n\|, \|b_n\| \le 1$ and $a_n - b_n = \omega_n z$ for each n as well as $\omega_n \to 1$, $\|a_n + b_n\| \to 2$ and $x_n^*(a_n) \to 1$.
Also as in the proof of [28, Theorem 1] we fix to sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ of non-negative real numbers such that

$$u_n := a_n + \alpha_n z \in S_X \text{ and } v_n := b_n - \beta_n z \in S_X \quad \forall n \in \mathbb{N}.$$

Then $u_n - v_n = (\omega_n + \alpha_n + \beta_n)z$ for all $n \in \mathbb{N}$ and again as in the proof of [28, Theorem 1] one can show that $\|u_n + v_n\| \to 2$, namely:

$$\begin{aligned} u_n + v_n &= a_n + b_n + (\alpha_n - \beta_n)z = \omega_n x_n + \omega_n(x_n - z) + (\alpha_n - \beta_n)z \\ &= x_n + y_n + (\alpha_n - \beta_n)(x_n - y_n) + R_n, \end{aligned}$$

where $R_n := (\omega_n - 1)x_n + \omega_n(x_n - z) - y_n + (\alpha_n - \beta_n)(z - x_n + y_n)$ for every $n \in \mathbb{N}$.
Because of $\omega_n \to 1$ and $x_n - y_n \to z$ we have $R_n \to 0$. Furthermore, if $\alpha_n \ge \beta_n$, then

$$\begin{aligned} &\|x_n + y_n + (\alpha_n - \beta_n)(x_n - y_n)\| = \|(1 + \alpha_n - \beta_n)(x_n + y_n) - 2(\alpha_n - \beta_n)y_n\| \\ &\ge (1 + \alpha_n - \beta_n)\|x_n + y_n\| - 2(\alpha_n - \beta_n) = 2 - (2 - \|x_n + y_n\|)(1 + \alpha_n - \beta_n) \end{aligned}$$

and a similar inequality holds if $\alpha_n < \beta_n$. Since $\|x_n + y_n\| \to 2$ it follows that $\|u_n + v_n\| \to 2$.
Because of $x_n^*(z) > 0$ for sufficiently large n it follows from $x_n^*(a_n) \to 1$ that $x_n^*(u_n) \to 1$. Since X is a uacsed space it follows that $x_n^*(v_n) \to 1$. But $\omega_n + \alpha_n + \beta_n \ge \omega_n \to 1$, so we must have $x_n^*(z) \to 0$.

Thus we have shown that in any case there is a subsequence of $(x_n^*(z))_{n\in\mathbb{N}}$ that converges to one and the same argument works if we start with an arbitrary subsequence of $(x_n^*(z))_{n\in\mathbb{N}}$. Hence the whole sequence must be convergent to one.
The implication (ii) $\Rightarrow$ (iii) is trivial. To prove (iii) $\Rightarrow$ (i) take sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in S_X and $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ as well as $z \in X \setminus \{0\}$ such that $x_n - y_n = \alpha_n z$ for all n and $\|x_n + y_n\| \to 2$. Also, take a sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} with $x_n^*(x_n) \to 1$.
Since $(\alpha_n)_{n\in\mathbb{N}}$ is bounded by $2/\|z\|$, by passing to subsequence we may assume that $\alpha_n \to \alpha$ for some $\alpha \in \mathbb{R}$.
Hence $x_n - y_n \to \alpha z$ and thus (iii) implies $x_n^*(y_n) \to 1$.
(iv) $\Rightarrow$ (i) is trivial and (ii) $\Rightarrow$ (iv) is proved exactly as we have just proved (iii) $\Rightarrow$ (i). Thus the equivalence of (i)—(iv) is established.
(iii) $\Rightarrow$ (v) is trivial as well and (v) $\Rightarrow$ (iii) can be proved using the Bishop–Phelps–Bollobás theorem like in the proof of Proposition I.2.1.
Next we prove (ii) $\Rightarrow$ (vi). Take a sequence $(x_n)_{n\in\mathbb{N}}$ in B_X and an element $z \in X$ as well as a sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} such that

$$2^{p-1}(\|x_n + z\|^p + \|x_n\|^p) - \|2x_n + z\|^p \to 0 \text{ and } x_n^*(x_n) \to 1. \tag{I.8.1}$$

It follows that $\|x_n\| \to 1$. As in the proof of [28, Theorem 1] we can make use of the inequality

$$(a+b)^p + (a-b)^p \leq 2^{p-1}(a^p + b^p) \quad \forall a \geq b \geq 0, \forall p \geq 2$$

to infer that $\|x_n + z\| \to 1$ and $\|2x_n + z\| \to 2$, namely:

$$\begin{aligned}&2^{p-1}(\|x_n + z\|^p + \|x_n\|^p) - \|2x_n + z\|^p \\ &\geq 2^{p-1}(\|x_n + z\|^p + \|x_n\|^p) - (\|x_n + z\| + \|x_n\|)^p \geq |\|x_n + z\| - \|x_n\||^p.\end{aligned}$$

It follows that $\|x_n + z\| - \|x_n\| \to 0$ and hence $\|x_n + z\| \to 1$. From (I.8.1) it now follows that $\|2x_n + z\| \to 2$.
If we put $y_n = (x_n + z)/\|x_n + z\|$ then $y_n \in S_X$, $\|x_n + y_n\| \to 2$ and $x_n - y_n \to -z$, so (ii) implies $x_n^*(z) \to 0$ and we are done.
For the prove of (vi) $\Rightarrow$ (ii) fix two sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in B_X such that $\|x_n + y_n\| \to 2$ and $x_n - y_n \to z \in X$ as well as a sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} with $x_n^*(x_n) \to 1$.
It follows that $\|x_n\|, \|y_n\| \to 1$, $\|x_n - z\| \to 1$ and $\|2x_n - z\| \to 2$. Hence

$$2^{p-1}(\|x_n - z\|^p + \|x_n\|^p) - \|2x_n - z\|^p \to 0$$

and (vi) implies $x_n^*(z) \to 0$.
The equivalence of (v) and (vii) can be proved analogously.
(viii) $\Rightarrow$ (ix) is trivial and (vi) $\Rightarrow$ (viii) is also obvious. Let us finally prove (ix) $\Rightarrow$ (vii). If $(x_n)_{n\in\mathbb{N}}$ is a sequence in S_X, $(x_n^*)_{n\in\mathbb{N}}$ a sequence S_{X^*} and $z \in X$ such that

$$\|2x_n + z\|^p - 2^{p-1}\|x_n + z\|^p \to 2^{p-1} \text{ and } x_n^*(x_n) = 1 \ \forall n \in \mathbb{N}$$

then as before we can deduce that $\|x_n + z\| \to 1$ and $\|2x_n + z\| \to 2$. Thus $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}} := (x_n + z)_{n\in\mathbb{N}}$, $(x_n^*)_{n\in\mathbb{N}}$ and $-z$ meet the conditions of (ix) and hence $x_n^*(z) \to 0$. □

Let us also mention the following characterisation of the property uacsed in terms of the space X itself only. The proof is completely analogous to the one for Proposition I.2.4.

Proposition I.8.3. *For a Banach space X the following assertions are equivalent.*

(i) *X is uacsed.*

(ii) *For every $z \in X \setminus \{0\}$ and every $\varepsilon > 0$ there exists some $\delta > 0$ such that for every $t \in [0, \delta]$ and all $x, y \in S_X$ with $\|x + y\| \geq 2(1 - t)$ and $x - y \in \operatorname{span}\{z\}$ we have*

$$\|x + ty\| + \|x - ty\| \leq 2 + \varepsilon t.$$

(iii) *For every $z \in X \setminus \{0\}$ and every $\varepsilon > 0$ there exists some $\delta > 0$ such that for every $t \in [0, \delta]$ and all $x, y \in S_X$ with $\|x + y\| \geq 2 - \delta t$ and $x - y \in \operatorname{span}\{z\}$ we have*

$$\|x - ty\| \leq 1 + t(\varepsilon - 1).$$

(iv) *For every $z \in X \setminus \{0\}$ there exists some $1 \leq p < \infty$ such that for every $\varepsilon > 0$ there is some $\delta > 0$ such that for all $t \in [0, \delta]$ and all $x, y \in S_X$ with $\|x + y\| \geq 2(1 - t)$ and $x - y \in \operatorname{span}\{z\}$ we have*

$$\|x + ty\|^p + \|x - ty\|^p \leq 2 + \varepsilon t^p.$$

(v) *For every $z \in X \setminus \{0\}$ there exists some $1 \leq p < \infty$ such that for every $\varepsilon > 0$ there is some $\delta > 0$ such that for all $t \in [0, \delta]$ and all $x, y \in S_X$ with $\|x + y\| \geq 2 - t\delta$ and $x - y \in \operatorname{span}\{z\}$ we have*

$$(1 + t)^p + \|x - ty\|^p \leq 2 + \varepsilon t^p.$$

If we use the characterisation for uacsed spaces with the condition $x_n - y_n \to z$ instead of $x_n - y_n \in \operatorname{span}\{z\}$ that was given above, we also see that Proposition I.8.3 still holds true if we replace the condition $x - y \in \operatorname{span}\{z\}$ by $\|x - y - z\| \leq \delta$ in the assertions (ii)–(v).

Finally, let us consider quotient spaces. It is known that the quotient of a Banach space which is URED by a finite-dimensional subspace is again URED (cf. [124, Remark before Problem 2]). By the same method of proof we can obtain the analogous result for the property uacsed.

Proposition I.8.4. *Let X be a Banach space which is uacsed and $U \subseteq X$ a finite-dimensional subspace. Then X/U is also uacsed.*

Proof. As was implicitly mentioned in [124], if $W \subseteq X/U$ is compact and U is finite-dimensional then $\omega^{-1}(W) \cap (2B_X)$ is also compact, where ω is the canonical quotient map (the analogue of [124, Lemma 2.11] for norm-compactness). For, if $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\omega^{-1}(W) \cap (2B_X)$ then by compactness of W we can pass to a subsequence such that $\omega(x_n) \to \omega(x)$ for some $x \in X$. Next fix a sequence $(y_n)_{n\in\mathbb{N}}$ in U such that $\|x_n - x - y_n\| \to 0$. It follows that $(y_n)_{n\in\mathbb{N}}$ is bounded and hence we can pass to a further subsequence such that $y_n \to y \in U$. Then $x_n \to x + y$ and since the set $\omega^{-1}(W) \cap (2B_X)$ is closed we must have $x + y \in \omega^{-1}(W) \cap (2B_X)$, so $\omega^{-1}(W) \cap (2B_X)$ is compact.
Now let us take two sequences $(z_n)_{n\in\mathbb{N}}$, $(w_n)_{n\in\mathbb{N}}$ in $S_{X/U}$ and an element $z \in X/U$ such that $\|z_n + w_n\| \to 2$ and $z_n - w_n \to z \in X/U$. Also, fix a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $S_{(X/U)^*}$ with $\varphi_n(z_n) \to 1$.
Then $x_n^* := \varphi_n \circ \omega \in S_{U^\perp}$ for all $n \in \mathbb{N}$. Since U is in particular reflexive, we can find sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in B_X such that $\omega(x_n) = z_n$ and $\omega(y_n) = w_n$ for each n (cf. the proof of Proposition I.5.2).
It follows that $\|x_n + y_n\| \to 2$ and $x_n^*(x_n) \to 1$.
The set $W := \{z_n - w_n : n \in \mathbb{N}\} \cup \{z\}$ is compact, hence by the introductory observation $\omega^{-1}(W) \cap (2B_X)$ is also compact and so we can find a subsequence $(x_{n_k} - y_{n_k})_{k\in\mathbb{N}}$ that is convergent in X.
Since X is uacsed Proposition I.8.2 implies $x_{n_k}^*(y_{n_k}) \to 1$. Hence $\varphi_{n_k}(z) \to 0$. The same argument shows that every subsequence of $(\varphi_n(z))_{n\in\mathbb{N}}$ possesses a subsubsequence which converges to zero, so we have $\varphi_n(z) \to 0$ and hence by Proposition I.8.2 the quotient space X/U is also uacsed. $\square$

I.9 Examples

In this section we collect some examples showing that no other arrows can be drawn in the Figures I.3, I.4 and I.7 (except, of course, for combinations of two or more existing arrows).

In what follows, we use the notation $x' = (0, x(2), x(3), \dots)$ for every $x \in \mathbb{R}^{\mathbb{N}}$. Further, e_n denotes the sequence whose n-th entry is 1 and all other entries are 0.

Example I.9.1. *A uacs space which is neither R nor S.*
We can simply take a norm on $\mathbb{R}^2$ which is neither R nor S but still acs (since the space is finite-dimensional it will then be even uacs). The unit ball of one such norm is sketched below.

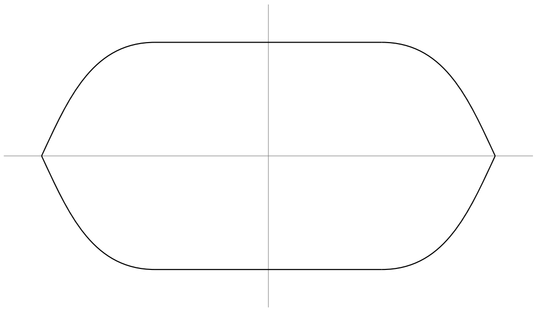

Fig. I.9

The norm is not rotund, because the unit sphere contains line segments, but the endpoints of these line segments are smooth points of the unit ball. On the other hand, the norm has also non-smooth points, but they are not the endpoints of any line segments on the unit sphere, so on the whole the space is acs but neither R nor S (again, it is of course possible to make this example precise, for instance by using the description of absolute, normalised norms[9] on $\mathbb{R}^2$ via the boundary curve of their unit ball that will be discussed in Chapter V; we skip the details here for the sake of brevity).

Example I.9.2. *A space which is LUR and URED but not wuacs.*
In [127, Example 6] Smith defines an equivalent norm on ℓ^1 as follows:

$$|||x|||^2 = \|x\|_1^2 + \|x\|_2^2 \quad \forall x \in \ell^1.$$

He shows that $(\ell^1, |||\,.\,|||)$ is LUR and URED but not WUR. In fact, $(\ell^1, |||\,.\,|||)$ is not even wuacs. To see this, put

$$\begin{aligned} \beta_n &= \frac{2}{\sqrt{4n^2+2n}}, \\ x_n &= (\underbrace{\beta_n, 0, \beta_n, 0, \dots, \beta_n, 0}_{2n}, 0, 0, \dots), \\ y_n &= (\underbrace{0, \beta_n, 0, \beta_n, \dots, 0, \beta_n}_{2n}, 0, 0, \dots) \end{aligned}$$

for every $n \in \mathbb{N}$. Then it is easily checked that $|||x_n + y_n||| = 2$ for every $n \in \mathbb{N}$ and $|||x_n||| = |||y_n||| \to 1$.

Now let x^* be the functional on ℓ^1 represented by $(1, 0, 1, 0, \dots) \in \ell^\infty$. Then $|||x^*|||^* \leq 1$ (where $|||\,.\,|||^*$ denotes the dual norm of $|||\,.\,|||$) and it is easy to see that $x^*(x_n) \to 1$. On the other hand, $x^*(y_n) = 0$ for every $n \in \mathbb{N}$ thus $(\ell^1, |||\,.\,|||)$ is not wuacs.

Example I.9.3. *A space which is WUR and URED but not msluacs.*
The following equivalent norm $|||\,.\,|||$ on ℓ^2 was also defined in [127, Example 2]. Fix a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $(0, 1]$ which decreases to 0 and define $T : \ell^2 \to \ell^2$ by $Tx = (x(1), \alpha_2 x(2), \alpha_3 x(3), \dots)$ for every $x \in \ell^2$. Then put

$$|||x|||^2 = \max\{|x(1)|, \|x'\|_2\}^2 + \|Tx\|_2^2 \quad \forall x \in \ell^2.$$

[9] See Section II.1

It is shown in [127, Example 2] that $(\ell^2, |||\,.\,|||)$ is WUR and URED but not MLUR.

To prove the latter, Smith defines $\alpha = 1/\sqrt{2}$, $x = \alpha e_1$, $x_n = \alpha(e_1 + e_n)$ and $y_n = \alpha(e_1 - e_n)$ for every n and observes that $|||x||| = 1$, $|||x_n||| = |||y_n||| \to 1$ and $x_n + y_n = 2x$ for every n, but $|||x_n - y_n||| \to \sqrt{2}$.

If x_n^* denotes the functional on ℓ^2 represented by $\alpha(e_1 + e_n) \in \ell^2$ for every $n \in \mathbb{N}$, then we have $|||x_n^*|||^* \leq 1$ and $x_n^*(x_n) = 1$ for every n, but $x_n^*(y_n) = 0$ for every n, hence $(\ell^2, |||\,.\,|||)$ is not even msluacs.

Example I.9.4. *A Banach space which is R but not mluacs.*
This example is a slight modification of [41, Exercise 8.52]. We define an equivalent norm on ℓ^1 by

$$|||x||| = \max\{|x(1)|, \|x'\|_1\} + \|x\|_2 \quad \forall x \in \ell^1.$$

Using the fact that $(\ell^2, \|\,.\,\|_2)$ is R it is easy to see that $(\ell^1, |||\,.\,|||)$ is also R. To see that $(\ell^1, |||\,.\,|||)$ is not mluacs, put $x = e_1$ and

$$x_n = \Big(1, \underbrace{\frac{1}{n}, \dots, \frac{1}{n}}_{n}, 0, 0, \dots\Big),$$
$$y_n = \Big(1, \underbrace{-\frac{1}{n}, \dots, -\frac{1}{n}}_{n}, 0, 0, \dots\Big)$$

for every $n \in \mathbb{N}$. Then it is easy to see that $|||x||| = 2$, $|||x_n||| = |||y_n||| \to 2$ and $x_n + y_n = 2x$ for every n.

If x^* is the functional on ℓ^1 represented by $(1, 1, \dots) \in \ell^\infty$ then $|||x^*|||^* \leq 1$ and $x^*(x_n) = 2$ for every n, but $x^*(y_n) = 0$ for every n, hence $(\ell^1, |||\,.\,|||)$ is not mluacs.

Example I.9.5. *A Banach space which is MLUR but not luacs.*
We define an equivalent norm $\|\,.\,\|_M$ on ℓ^1 as follows:

$$|||x||| = \|x'\|_1 + \|x\|_2 \text{ and}$$
$$\|x\|_M^2 = \|x\|_1^2 + \|x'\|_2^2 + |||x|||^2 \quad \forall x \in \ell^1.$$

Then we have $\sqrt{2}\|x\|_1 \leq \|x\|_M \leq \sqrt{6}\|x\|_1$ for every $x \in \ell^1$.

We first show that $(\ell^1, \|\,.\,\|_M)$ is not luacs. To do so, we put $x = e_1$ and

$$x_n = \Big(0, \underbrace{\frac{1}{n}, \dots, \frac{1}{n}}_{n}, 0, 0, \dots\Big) \quad \forall n \in \mathbb{N}.$$

Then it is easy to calculate $\|x\|_M = \sqrt{2}$, $\|x_n\|_M \to \sqrt{2}$ and $\|x_n + x\|_M \to 2\sqrt{2}$. Let x^* be the functional on ℓ^1 represented by $(0, \sqrt{2}, \sqrt{2}, \dots) \in \ell^\infty$. Then

$\|x^*\|_M^* \leq 1$ and $x^*(x_n) = \sqrt{2}$ for every $n \in \mathbb{N}$, but $x^*(x) = 0$, thus $(\ell^1, \|\,.\,\|_M)$ is not luacs.

Now we prove that $(\ell^1, \|\,.\,\|_M)$ is MLUR. So let us fix two sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in ℓ^1 and $x \in \ell^1$ such that $\|x_n\|_M = \|x\|_M = \|y_n\|_M = 1$ for every $n \in \mathbb{N}$ and $\|x_n + y_n - 2x\|_M \to 0$.

Then we also have

$$\|x_n + y_n - 2x\|_1, \|x_n' + y_n' - 2x'\|_2, |||x_n + y_n - 2x||| \to 0 \tag{I.9.1}$$

and hence

$$\|x_n + y_n\|_1 \to 2\|x\|_1, \|x_n' + y_n'\|_2 \to 2\|x'\|_2, |||x_n + y_n||| \to 2\,|||x|||\,. \tag{I.9.2}$$

We further have

$$\begin{aligned}\|x_n + y_n\|_M &\leq \left((\|x_n\|_1 + \|y_n\|_1)^2 + (\|x_n'\|_2 + \|y_n'\|_2)^2 + (|||x_n||| + |||y_n|||)^2\right)^{\frac{1}{2}} \\ &\leq \|x_n\|_M + \|y_n\|_M = 2\end{aligned}$$

and because of $\|x_n + y_n\|_M \to 2$ and the uniform rotundity of the Euclidean norm on $\mathbb{R}^3$ this implies

$$(\|x_n\|_1 + \|y_n\|_1)^2 - \|x_n + y_n\|_1^2 \to 0, \tag{I.9.3}$$

$$\left(\|x_n'\|_2 + \|y_n'\|_2\right)^2 - \|x_n' + y_n'\|_2^2 \to 0, \tag{I.9.4}$$

$$(|||x_n||| + |||y_n|||)^2 - |||x_n + y_n|||^2 \to 0 \tag{I.9.5}$$

and

$$\|x_n\|_1 - \|y_n\|_1, \|x_n'\|_2 - \|y_n'\|_2, |||x_n||| - |||y_n||| \to 0. \tag{I.9.6}$$

Combining (I.9.3), (I.9.4), (I.9.5), (I.9.2) and (I.9.6) we get that

$$\|x_n\|_1, \|y_n\|_1 \to \|x\|_1, \|x_n'\|_2, \|y_n'\|_2 \to \|x'\|_2, |||x_n|||\,, |||y_n||| \to |||x|||\,. \tag{I.9.7}$$

Since $(\ell^2, \|\,.\,\|_2)$ is UR we can deduce from (I.9.1) and (I.9.6) that

$$\|x_n' - x'\|_2, \|y_n' - x'\|_2 \to 0. \tag{I.9.8}$$

By (I.9.7) and the definition of $|||\,.\,|||$ we have $\|x_n'\|_1 + \|x_n\|_2 \to \|x'\|_1 + \|x\|_2$, which together with $\|x_n\|_1 \to \|x\|_1$ implies

$$|x_n(1)| - \|x_n\|_2 \to |x(1)| - \|x\|_2. \tag{I.9.9}$$

Let us put $a_n = |x_n(1)|$ and $b_n = \|x_n'\|_2$ for every n, as well as $a = |x(1)|$ and $b = \|x'\|_2$.

Then (I.9.9) reads

$$a_n - \sqrt{a_n^2 + b_n^2} \to a - \sqrt{a^2 + b^2} \tag{I.9.10}$$

and by (I.9.7) we have $b_n \to b$.

If $b \neq 0$ this easily implies $a_n \to a$. If $b = 0$ then $x = x(1)e_1$ and because of $\|x\|_M = 1$ it follows $|x(1)| = 1/\sqrt{2}$. But by (I.9.1) we have $|x_n(1) + y_n(1)| \to 2|x(1)| = \sqrt{2}$ and since $|x_n(1)| \leq \|x_n\|_1 \leq \|x_n\|_M/\sqrt{2} = 1/\sqrt{2}$ (and likewise $|y_n(1)| \leq 1/\sqrt{2}$) for every n it follows that $|x_n(1)|, |y_n(1)| \to 1/\sqrt{2}$.

Thus we have $|x_n(1)| \to |x(1)|$ in any case and analogously we can show that we always have $|y_n(1)| \to |x(1)|$.

Because of $x_n(1) + y_n(1) \to 2x(1)$ this implies $x_n(1) \to x(1)$ and $y_n(1) \to y(1)$. Taking into account (I.9.8) it follows

$$x_n(i) \to x(i) \text{ and } y_n(i) \to x(i) \quad \forall i \in \mathbb{N}. \tag{I.9.11}$$

By (I.9.7) we also have $\|x_n\|_1, \|y_n\|_1 \to \|x\|_1$ and it is well known that these two conditions together imply $\|x_n - x\|_1, \|y_n - x\|_1 \to 0$. Hence we have $\|x_n - y_n\|_M \to 0$, as desired.

I.10 Miscellaneous

In this last section we will collect some further facts on acs spaces and their relatives which might be of interest. First let us have a look at the quantitative connection between the uacs-modulus δ^X_{uacs} and the modulus $\tilde{\delta}^X_{\text{uacs}}$ that was introduced in Section I.2 and treated in Lemma I.2.5.

Lemma I.10.1. *If X is uacs then*

$$\tilde{\delta}^X_{\text{uacs}}(\varepsilon) \geq \delta^X_{\text{uacs}}\big(\delta^X_{\text{uacs}}(\varepsilon)\big)$$

for every $0 < \varepsilon \leq 2$.

Proof. Here we can adopt Sirotkin's idea from the proof of Proposition I.1.4 in [123]. Put $\delta = \delta^X_{\text{uacs}}\big(\delta^X_{\text{uacs}}(\varepsilon)\big)$ and take $x, y \in S_X$ and $x^* \in S_{X^*}$ such that $\|x + y\| > 2(1 - \delta)$ and $x^*(x) > 1 - \delta$.
Since X is reflexive, there is some $z \in S_X$ with $x^*(z) = 1$. It follows that $\|x + z\| \geq x^*(x + z) > 2(1 - \delta)$.
Now fix $y^* \in S_{X^*}$ such that $y^*(x) = 1$. Then by the definition of δ we must have $y^*(z) > 1 - \delta^X_{\text{uacs}}(\varepsilon)$ and $y^*(y) > 1 - \delta^X_{\text{uacs}}(\varepsilon)$ and hence $\|y + z\| > 2(1 - \delta^X_{\text{uacs}}(\varepsilon))$.
Because of $x^*(z) = 1$ this implies $x^*(y) > 1 - \varepsilon$ and the proof is finished. □

Next we will deal with continuity of the uacs-modulus. It is claimed in [31, Lemma 3.10] that the modulus of U-convexity, which coincides with our modulus δ^X_{uacs}, is continuous on $(0, 2)$, but it seems that the proof given there only works in the case of arguments $\varepsilon < 1$ (this is not a major drawback since one is usually interested in small values of ε). We wish to point out that for values between 0 and 1 even more is true, namely δ^X_{uacs} is Lipschitz continuous on $[a, 1)$ for every $0 < a < 1$.

Lemma I.10.2. *For every Banach space X and all $0 < \varepsilon, \varepsilon' < 1$ we have*

$$\left|\delta^X_{\mathrm{uacs}}(\varepsilon) - \delta^X_{\mathrm{uacs}}(\varepsilon')\right| \leq \frac{|\varepsilon - \varepsilon'|}{\min\{\varepsilon, \varepsilon'\}}.$$

In particular, δ^X_{uacs} is Lipschitz continuous on $[a, 1)$ for all $0 < a < 1$.

Proof. Let $0 < \varepsilon < 1$ and $0 < \beta < 1 - \varepsilon$. Put $\tau = \beta/(\varepsilon + \beta)$ and take $x, y \in S_X$ and $x^* \in S_{X^*}$ such that $x^*(x) = 1$ and $x^*(y) \leq 1 - \varepsilon$. Let $z = (y - \tau x)/\|y - \tau x\|$. Note that, since $\|y - \tau x\| \geq 1 - \tau$ and $\varepsilon + \tau < 1$, we have

$$x^*(z) \leq \frac{1 - \varepsilon - \tau}{1 - \tau} = 1 - \varepsilon\left(1 + \frac{\tau}{1 - \tau}\right) = 1 - (\varepsilon + \beta)$$

and hence

$$1 - \left\|\frac{x + z}{2}\right\| \geq \delta^X_{\mathrm{uacs}}(\varepsilon + \beta).$$

Furthermore, we have

$$\|y - z\| \leq \frac{\|(\|y - \tau x\| - 1)y + \tau x\|}{1 - \tau} \leq \frac{2\tau}{1 - \tau} = \frac{2\beta}{\varepsilon}.$$

It follows that

$$1 - \left\|\frac{x + y}{2}\right\| \geq \delta^X_{\mathrm{uacs}}(\varepsilon + \beta) - \frac{\beta}{\varepsilon}.$$

Thus we have

$$\delta^X_{\mathrm{uacs}}(\varepsilon + \beta) \geq \delta^X_{\mathrm{uacs}}(\varepsilon) \geq \delta^X_{\mathrm{uacs}}(\varepsilon + \beta) - \frac{\beta}{\varepsilon}$$

for all $0 < \varepsilon < 1$ and every $0 < \beta < 1 - \varepsilon$, which finishes the proof. □

In the next results (Propositions I.10.3–I.10.7) we collect some connections between the various versions of rotundity, smoothness and acs-type properties. These results originally appeared in the author's paper [58] in the section "Miscellaneous" (only assertion (i) of Proposition I.10.7 has been improved here).

First we have to recall one more definition: A. Lovaglia ([96]) called a Banach space X weakly locally uniformly rotund if for every sequence $(x_n)_{n\in\mathbb{N}}$ in S_X, every $x \in S_X$ and each $x^* \in S_{X^*}$ the implication

$$\|x_n + x\| \to 2 \ \text{ and } \ x^*(x) = 1 \quad \Rightarrow \quad x^*(x_n) \to 1$$

holds. Since this notion of weak local uniform rotundity is strictly weaker than the notion of WLUR spaces that is nowadays commonly used, we will call such spaces WLUR in the sense of Lovaglia.[10] By definition, a Banach space is luacs$^+$ if and only if it is luacs and WLUR in the sense of Lovaglia.

[10] A dual Banach space will be called WLUR* in the sense of Lovaglia if it fulfils Lovaglia's definition for all evaluation functionals (in [96] this is phrased as "the dual is WLUC as a set of linear functionals").

As already mentioned in a footnote in Section I.6 the property (WM) introduced in [110] is equivalent to the fact that X is WLUR in the sense of Lovaglia and it was proved in [110, Theorem 3.7] that every FS space has this property (in fact, FS spaces fulfil an even stronger condition, as we have seen in said footnote). We also have the following result.

Proposition I.10.3. *A Banach space X is luacs$^+$ if and only if X is WLUR in the sense of Lovaglia and for all $x^*, y^* \in S_{X^*}$ with $\|x^* + y^*\| = 2$ and every $x \in S_X$ with $x^*(x) = 1$ one also has $y^*(x) = 1$.*

Proof. The necessity is clear because of part (i) of Proposition I.4.4. For the sufficiency we only have to prove that X is luacs, so let us take a sequence $(x_n)_{n\in\mathbb{N}}$ in S_X and $x \in S_X$ such that $\|x_n + x\| \to 2$ as well as $x^* \in S_{X^*}$ with $x^*(x_n) \to 1$. Since $B_{X^{**}}$ is weak*-compact we can find $x^{**} \in B_{X^{**}}$ and a subnet $(x_{\varphi(i)})_{i\in I}$ which is weak*-convergent to x^{**}. It follows that $x^{**}(x^*) = 1 = \|x^{**}\|$.
Now fix a sequence $(y_n^*)_{n\in\mathbb{N}}$ in S_{X^*} such that $y_n^*(x_n+x) \to 2$. Then $y_n^*(x_n) \to 1$ and $y_n^*(x) \to 1$. There is $y^* \in B_{X^*}$ and a subnet $(y^*_{\psi(j)})_{j\in J}$ which is weak*-convergent to y^*. It follows that $y^*(x) = 1 = \|y^*\|$. Since X is WLUR in the sense of Lovaglia we conclude $y^*(x_n) \to 1$. It follows that $x^{**}(y^*) = 1 = x^{**}(x^*)$, hence $\|x^* + y^*\| = 2$.
Because of $y^*(x) = 1$ our assumption implies $x^*(x) = 1$ and we are done. $\square$

We can also make the following easy observation.

Proposition I.10.4. *If X is a Banach space which is WLUR in the sense of Lovaglia and such that X^* is WLUR* in the sense of Lovaglia then X is sluacs.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in S_X and $x \in S_X$ with $\|x_n + x\| \to 2$. Let $(x_n^*)_{n\in\mathbb{N}}$ be a sequence in S_{X^*} such that $x_n^*(x_n) \to 1$. Then take a functional $x^* \in S_{X^*}$ such that $x^*(x) = 1$. Since X is WLUR in the sense of Lovaglia it follows that $x^*(x_n) \to 1$.
But then we must have $\|x_n^* + x^*\| \to 2$ and since X^* is WLUR* in the sense of Lovaglia we obtain $x_n^*(x) \to 1$. $\square$

Under additional assumptions on the space X it is possible to prove some more results.

Proposition I.10.5. *Let X be a reflexive Banach space.*

(i) *If X is WLUR in the sense of Lovaglia then X is luacs$^+$.*

(ii) *If X is sluacs and luacs$^+$ then X is wuacs.*

(iii) *If X is wuacs and R then X is WLUR.*

Proof. (i) follows directly from Proposition I.10.3 and Proposition I.2.7.
(ii) Suppose X is sluacs and luacs$^+$. Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be sequences in S_X with $\|x_n + y_n\| \to 2$ and let $x^* \in S_{X^*}$ such that $x^*(x_n) \to 1$.
Find a sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} such that $x_n^*(x_n+y_n) \to 2$. Hence $x_n^*(x_n) \to 1$ and $x_n^*(y_n) \to 1$.
Since X is reflexive we can assume that $(x_n)_{n\in\mathbb{N}}$ is weakly convergent to some $x \in B_X$. It then follows that $x^*(x) = 1 = \|x\|$ and $\|x_n + x\| \to 2$ (since $x^*(x_n) \to 1$). Because X is sluacs we can conclude $x_n^*(x) \to 1$. Since $x_n^*(y_n) \to 1$ this implies $\|y_n + x\| \to 2$. But X is luacs$^+$, so since $x^*(x) = 1$ it follows that $x^*(y_n) \to 1$, as desired.
(iii) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in S_X and $x \in S_X$ such that $\|x_n + x\| \to 2$. Again we can find a sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} such that $x_n^*(x_n + x) \to 2$ and hence $x_n^*(x_n) \to 1$ and $x_n^*(x) \to 1$.
Since X is reflexive we may assume that $(x_n^*)_{n\in\mathbb{N}}$ is weak*-convergent to some $y^* \in B_{X^*}$ and $(x_n)_{n\in\mathbb{N}}$ is weakly convergent to some $y \in B_X$. It follows that $y^*(x) = 1$ and hence $\|x_n^* + y^*\| \to 2$.
Since X is wuacs the dual space X^* is sluacs (by Proposition I.4.4) and thus (because of $x_n^*(x_n) \to 1$) we can conclude $y^*(x_n) \to 1$, whence $y^*(y) = 1 = y^*(x)$, which implies $\|x + y\| = 2$, which by the rotundity of X implies $x = y$. □

Proposition I.10.6. *Let X be a reflexive Banach space with the Kadets-Klee property.*[11]

(i) *If X is acs then X is luacs.*

(ii) *If X is WLUR in the sense of Lovaglia then X is wuacs and sluacs$^+$.*

(iii) *If X is WLUR in the sense of Lovaglia and R then X is wuacs and LUR.*

Proof. (i) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in S_X and $x \in S_X$ such that $\|x_n+x\| \to 2$. Also, let $x^* \in S_{X^*}$ with $x^*(x_n) \to 1$. We may assume that $(x_n)_{n\in\mathbb{N}}$ weakly converges to some $y \in B_X$. Then $x^*(y) = 1$ and hence $\|y\| = 1$. Since X has the Kadets-Klee property it follows that $\|x_n - y\| \to 0$ and thus $\|x + y\| = 2$. Because X is acs we obtain $x^*(x) = 1$, as desired.
(ii) We first show that X is wuacs. Take two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in S_X such that $\|x_n + y_n\| \to 2$ and a functional $x^* \in S_{X^*}$ with $x^*(x_n) \to 1$. By the reflexivity of X we may assume that $(x_n)_{n\in\mathbb{N}}$ is weakly convergent to some $x \in B_X$. Then $x^*(x) = 1$, hence $\|x\| = 1$.
But X has the Kadets-Klee property, so this implies $\|x_n - x\| \to 0$.
Now fix a sequence $(y_n^*)_{n\in\mathbb{N}}$ in S_{X^*} such that $y_n^*(x_n) \to 1$ and $y_n^*(y_n) \to 1$. It follows that $y_n^*(x) \to 1$ and consequently $\|y_n + x\| \to 2$.

[11] See the definition in Section I.7.

Since $x^*(x) = 1$ and X is WLUR in the sense of Lovaglia we get $x^*(y_n) \to 1$, proving that X is wuacs.
Now we will show that X is sluacs. Take $(x_n)_{n\in\mathbb{N}}$ and x in S_X with $\|x_n+x\| \to 2$ and a sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} such that $x_n^*(x_n) \to 1$. Also, fix a sequence $(y_n^*)_{n\in\mathbb{N}}$ in S_{X^*} with $y_n^*(x_n) \to 1$ and $y_n^*(x) \to 1$.
We may assume that $(x_n)_{n\in\mathbb{N}}$ is weakly convergent to some $y \in B_X$ and $(y_n^*)_{n\in\mathbb{N}}$ is weak*-convergent to some $y^* \in B_{X^*}$. It follows that $y^*(x) = 1$ and hence $\|y^* + y_n^*\| \to 2$.
Since X is wuacs X^* is sluacs (Proposition I.4.4) and thus we get $y^*(x_n) \to 1$. It follows that $y^*(y) = 1$, hence $\|y\| = 1$ and $\|x + y\| = 2$. The Kadets-Klee property of X gives us $\|x_n - y\| \to 0$.
Because of $x_n^*(x_n) \to 1$ we can now infer $x_n^*(y) \to 1$. Since X is in particular acs this implies $x_n^*(x) \to 1$ (cf. Proposition I.2.8).
We will skip the last part of the proof, the reverse implication in the definition of sluacs$^+$, since it is similar to previous arguments.
(iii) By (ii) X is wuacs and sluacs$^+$. Let us take a sequence $(x_n)_{n\in\mathbb{N}}$ in S_X and an element $x \in S_X$ such that $\|x_n + x\| \to 2$. Fix a sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} such that $x_n^*(x_n) = 1$ for every $n \in \mathbb{N}$. Since X is sluacs it follows that $x_n^*(x) \to 1$.
Assume that $(x_n)_{n\in\mathbb{N}}$ is weakly convergent to $y \in B_X$ and that $(x_n^*)_{n\in\mathbb{N}}$ is weak*-convergent to $x^* \in B_{X^*}$. It follows that $x^*(x) = 1$ and hence $x^* \in S_{X^*}$. Moreover, since X is WLUR in the sense of Lovaglia we get that $x^*(x_n) \to 1$. Since $(x_n)_{n\in\mathbb{N}}$ converges weakly to y this implies $x^*(y) = 1$ and hence $\|y\| = 1$. Now the Kadets-Klee property of X allows us to conclude $\|x_n - y\| \to 0$.
Because of $x^*(x) = x^*(y) = 1$ we must have $\|x + y\| = 2$ and thus the rotundity of X implies $x = y$. $\square$

We remark that the implication "reflexive, Kadets-Klee property, R and WLUR in the sense of Lovaglia $\Rightarrow$ LUR" in the above Proposition is probably known via the following alternative argument. A Banach space X is called strongly convex if for every $x \in S_X$, every $x^* \in S_{X^*}$ with $x^*(x) = 1$ and all sequences $(x_n)_{n\in\mathbb{N}}$ in B_X with $x^*(x_n) \to 1$ one has $\|x_n - x\| \to 0$ (see [55, Definition 8]). Obviously, every LUR space is strongly convex and every strongly convex space is rotund and has the Kadets-Klee property. Moreover, by similar arguments as we have used in the preceding proofs, it is easy to see that every reflexive rotund space which has the Kadets-Klee property is strongly convex. Finally, it is also easy to check that every strongly convex space which is WLUR in the sense of Lovaglia is actually LUR,[12] establishing

[12] Suppose that X is strongly convex and WLUR in the sense of Lovaglia. Let $x_n, x \in S_X$ such that $\|x_n + x\| \to 2$. Take $x_n^* \in S_{X^*}$ with $x_n^*(x_n) \to 1$ and $x_n^*(x) \to 1$. Since B_{X^*} is weak*-compact, there is a subnet $(x^*_{\phi(i)})_{i\in I}$ which is weak*-convergent to some $x^* \in B_{X^*}$. It follows that $x^*(x) = 1 = \|x^*\|$. Since X is WLUR in the sense of Lovaglia it follows that $x^*(x_{\phi(i)}) \to 1$ and since X is strongly convex this implies $\|x_{\phi(i)} - x\| \to 0$, which is enough to show that X is LUR.

the desired implication.

For the next result recall that a dual Banach space X^* is said to have the Kadets-Klee* property if it fulfils the definition of the Kadets-Klee property with weak- replaced by weak*-convergence.

Proposition I.10.7. *Let X be a Banach space such that X^* has the Kadets-Klee* property and B_{X^*} is weak*-sequentially compact.*[13]

(i) *If X is S then it is actually FS and hence for all sequences $(x_n)_{n\in\mathbb{N}}$ in S_X, $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} and every $x \in S_X$ with $\|x_n + x\| \to 2$ and $x_n^*(x) \to 1$ one has $x_n^*(x_n) \to 1$.*

(ii) *If X^* is acs then X is luacs$^+$ and for all sequences $(x_n)_{n\in\mathbb{N}}$ in S_X, $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} and every $x \in S_X$ with $\|x_n + x\| \to 2$ and $x_n^*(x) \to 1$ one has $x_n^*(x_n) \to 1$.*

(iii) *If X^* is WLUR* in the sense of Lovaglia then X is sluacs.*

Proof. (i) Every FS space satisfies the second condition in (i), as was noted in Section I.6 before Proposition I.6.2. So it suffices to show that X is FS. This is quite probably known, but since the author could not find a reference, the proof will be given here for the sake of completeness. So let $x \in S_X$ and let $x^* \in S_{X^*}$ be the Gâteaux derivative of $\|\cdot\|$ at x. Let $(x_n^*)_{n\in\mathbb{N}}$ be a sequence in S_{X^*} such that $x_n^*(x) \to 1$. By [41, Lemma 8.4] it is enough to show that $\|x_n^* - x^*\| \to 0$.
Since B_{X^*} is weak*-sequentially compact we may without loss of generality assume that $(x_n^*)_{n\in\mathbb{N}}$ is weak*-convergent to some $y^* \in B_{X^*}$. It follows that $y^*(x) = 1$ and hence $x^* = y^*$. Since X^* has the Kadets-Klee* property it follows that $\|x_n^* - x^*\| \to 0$.
Of the two remaining assertions, we will only prove (iii) explicitly (the arguments are all similar). So let $(x_n)_{n\in\mathbb{N}}$ and x be in S_X with $\|x_n + x\| \to 2$ and $(x_n^*)_{n\in\mathbb{N}}$ a sequence in S_{X^*} such that $x_n^*(x_n) \to 1$. Let $(y_n^*)_{n\in\mathbb{N}}$ be a sequence in S_{X^*} with $y_n^*(x_n) \to 1$ and $y_n^*(x) \to 1$.
By assumption, we may suppose that $(y_n^*)_{n\in\mathbb{N}}$ is weak*-convergent to some $y^* \in B_{X^*}$. Then $y^*(x) = 1$, hence $y^* \in S_{X^*}$. By the Kadets-Klee* property of X^* we must have $\|y_n^* - y^*\| \to 0$.
It follows that $y^*(x_n) \to 1$, hence $\|x_n^* + y^*\| \to 2$. Since X^* is WLUR* in the sense of Lovaglia we obtain $x_n^*(x) \to 1$. □

Now let us come back once more to the area of the Daugavet property that was briefly discussed in Section I.1. There is another version, the so called alternative Daugavet property (aDP in short). A Banach space X is said to have the aDP if the so called alternative Daugavet equation (aDE)

$$\max_{\omega\in\mathbb{T}} \|\mathrm{id} + \omega T\| = 1 + \|T\|$$

[13] For example, if X is separable or reflexive, or more generally a so called WCG space, see [41, Theorem 11.16].

holds for every rank-1-operator $T \in L(X)$, where $\mathbb{T}$ denotes the set of all scalars of modulus one. Since we consider only real Banach spaces, we have $\mathbb{T} = \{-1, 1\}$. This notion was originally introduced in [100]. For more information and background on the aDP and its relation to the concept of numerical index of Banach spaces we refer the reader to [73, 100] and references therein (see also the introductory section of Chapter VI).

In [73] it is shown that a Banach space with the aDP (and its dual) cannot have certain rotundity or smoothness properties, more precisely it is shown that if X has the aDP and dimension greater than one, then X^* is neither R nor S (cf. [73, Theorem 2.1]) and the unit ball of X has no WLUR points (cf. [73, Proposition 2.4]). A point $x \in S_X$ is called a WLUR point of B_X if for every sequence $(x_n)_{n\in\mathbb{N}}$ in S_X the condition $\|x_n + x\| \to 2$ implies that $(x_n)_{n\in\mathbb{N}}$ converges weakly to x. One can easily generalise these results to the acs resp. luacs case (for real Banach spaces). Since the proofs stay almost exactly the same as in [73], they will be omitted.

Proposition I.10.8. *Let X be a real Banach space with the aDP of dimension at least two. Then X^* is not acs.*

We call a point $x \in S_X$ an luacs point of B_X if for every sequence $(x_n)_{n\in\mathbb{N}}$ in S_X and every $x^* \in S_{X^*}$ the two conditions $\|x_n + x\| \to 2$ and $x^*(x_n) \to 1$ imply $x^*(x) = 1$. Then we have the following result.

Proposition I.10.9. *Let X be a real Banach space with the aDP of dimension at least two. Then B_X has no luacs points.*

Let us now conclude this chapter with a simple Lemma that will be frequently used in the sequel. It is the generalisation of [1, Lemma 2.1] to sequences, while the proof remains virtually the same.

Lemma I.10.10. *Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in the (real or complex) normed space X such that $\|x_n + y_n\| - \|x_n\| - \|y_n\| \to 0$.*

Then for any two bounded sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ of non-negative real numbers we also have $\|\alpha_n x_n + \beta_n y_n\| - \alpha_n\|x_n\| - \beta_n\|y_n\| \to 0$.

Proof. Let $n \in \mathbb{N}$ be arbitrary. If $\alpha_n \geq \beta_n$ then

$$\begin{aligned}\|\alpha_n x_n + \beta_n y_n\| &\geq \alpha_n\|x_n + y_n\| - (\alpha_n - \beta_n)\|y_n\| \\ &= \alpha_n(\|x_n + y_n\| - \|x_n\| - \|y_n\|) + \alpha_n\|x_n\| + \beta_n\|y_n\|\end{aligned}$$

and hence

$$0 \geq \|\alpha_n x_n + \beta_n y_n\| - \alpha_n\|x_n\| - \beta_n\|y_n\| \geq \alpha_n(\|x_n + y_n\| - \|x_n\| - \|y_n\|).$$

Analogously one can show that

$$0 \geq \|\alpha_n x_n + \beta_n y_n\| - \alpha_n\|x_n\| - \beta_n\|y_n\| \geq \beta_n(\|x_n + y_n\| - \|x_n\| - \|y_n\|)$$

if $\alpha_n < \beta_n$. Since $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are bounded we obtain the desired conclusion. □

II Absolute sums of acs-type spaces

This chapter is devoted to the study of (mainly infinite) absolute sums of the various versions of acs-type spaces. These results were first published in the author's papers [57, 60] (but some of the proofs presented here have been simplified a little using a trick from the proof of [19, Theorem 4]).

First we have to recall some definitions and results around absolute sums of Banach spaces.

II.1 Preliminaries on absolute sums

In the following I always denotes a non-empty set and E a subspace of $\mathbb{R}^I$ with $e_i \in E$ for all $i \in I$, where e_i denotes the characteristic function of $\{i\}$. We consider a complete norm $\|\,.\,\|_E$ on E which is *absolute*, i. e.

$$\begin{aligned}&(a_i)_{i\in I} \in E,\ (b_i)_{i\in I} \in \mathbb{R}^I \text{ and } |a_i| = |b_i|\ \forall i \in I\\ &\Rightarrow\ (b_i)_{i\in I} \in E \text{ and } \|(a_i)_{i\in I}\|_E = \|(b_i)_{i\in I}\|_E.\end{aligned}$$

The norm is further assumed to be *normalised*, i. e. $\|e_i\|_E = 1$ for every $i \in I$.

Standard examples of subspaces of $\mathbb{R}^I$ with absolute, normalised norm are of course the spaces $\ell^p(I)$ for $1 \le p \le \infty$ and $c_0(I)$. A more general class of examples is provided, for instance, by the Orlicz sequence spaces ℓ^φ equipped with the Luxemburg norm (for definitions and background, and in particular for results concerning rotundity properties of such spaces, see for example [76, 79] and references therein).

We have the following important Lemma on absolute, normalised norms, whose proof can be found for example in [87, Remark 2.1].

Lemma II.1.1. *Let $(E, \|\,.\,\|_E)$ be a subspace of $\mathbb{R}^I$ with an absolute, normalised norm. Then the following is true.*

$$\begin{aligned}&(a_i)_{i\in I} \in E,\ (b_i)_{i\in I} \in \mathbb{R}^I \text{ and } |b_i| \le |a_i|\ \forall i \in I\\ &\Rightarrow\ (b_i)_{i\in I} \in E \text{ and } \|(b_i)_{i\in I}\|_E \le \|(a_i)_{i\in I}\|_E.\end{aligned}$$

Furthermore, the inclusions $\ell^1(I) \subseteq E \subseteq \ell^\infty(I)$ hold and the respective inclusion mappings are of norm one.

For a given subspace $(E, \|\,.\,\|_E)$ of $\mathbb{R}^I$ endowed with an absolute, normalised norm we put

$$E' := \left\{(a_i)_{i\in I} \in \mathbb{R}^I : \sup_{(b_i)_{i\in I}\in B_E} \sum_{i\in I} |a_i b_i| < \infty\right\}.$$

It is easy to check that E' is a subspace of $\mathbb{R}^I$ and that

$$\|(a_i)_{i\in I}\|_{E'} := \sup_{(b_i)_{i\in I}\in B_E} \sum_{i\in I} |a_i b_i| \quad \forall (a_i)_{i\in I} \in E'$$

defines an absolute, normalised norm on E'.

The map $T : E' \to E^*$ defined by

$$T((a_i)_{i\in I})((b_i)_{i\in I}) := \sum_{i\in I} a_i b_i \quad \forall (a_i)_{i\in I} \in E', \forall (b_i)_{i\in I} \in E$$

is easily seen to be an isometric embedding. Moreover, if $\operatorname{span}\{e_i : i \in I\}$ is dense in E then T is onto, so in this case we can identify E^* and E' (compare with the duality results for ℓ^p, in this case one has $(\ell^p)' = \ell^q$, where $p \in [1, \infty)$ and $q \in (1, \infty]$ with $1/p + 1/q = 1$).

We remark that if $\operatorname{span}\{e_i : i \in I\}$ is dense in E, then one must have $(a_i)_{i\in I} = \sum_{i\in I} a_i e_i$ (unconditional convergence with respect to $\|\cdot\|_E$) for every $(a_i)_{i\in I} \in E$, as is easily checked.

Now if $(X_i)_{i\in I}$ is a family of Banach spaces[1] we put

$$\Big[\bigoplus_{i\in I} X_i\Big]_E := \left\{(x_i)_{i\in I} \in \prod_{i\in I} X_i : (\|x_i\|)_{i\in I} \in E\right\}.$$

It is not hard to see that this defines a subspace of the product space $\prod_{i\in I} X_i$ which becomes a Banach space when endowed with the norm

$$\|(x_i)_{i\in I}\|_E := \|(\|x_i\|)_{i\in I}\|_E \quad \forall (x_i)_{i\in I} \in \Big[\bigoplus_{i\in I} X_i\Big]_E.$$

This Banach space is called the absolute sum of the family $(X_i)_{i\in I}$ with respect to E. For $E = \ell^p(I)$ resp. $E = c_0(I)$ one obtains the usual ℓ^p- resp. c_0-sum of $(X_i)_{i\in I}$.

Again it is not difficult to check that the map

$$S : \Big[\bigoplus_{i\in I} X_i^*\Big]_{E'} \to \Big[\bigoplus_{i\in I} X_i\Big]_E^*$$
$$S((x_i^*)_{i\in I})((x_i)_{i\in I}) := \sum_{i\in I} x_i^*(x_i)$$

[1] Remember that we consider only real Banach spaces, but the following definition of absolute sums would work for complex spaces as well.

is an isometric embedding and it is onto if $\operatorname{span}\{e_i : i \in I\}$ is dense in E.

Let us also mention the following well-known fact, which will be needed later.

Lemma II.1.2. *If E is a subspace of $\mathbb{R}^I$ endowed with an absolute normalised norm and* $\operatorname{span}\{e_i : i \in I\}$ *is dense in E, then E does not contain an isomorphic copy of ℓ^1 if and only if* $\operatorname{span}\{e_i : i \in I\}$ *is dense in E'.*

Finally, let us introduce the following convenient notation. For a fixed index set I, we denote by $\mathcal{E}_I$ the set of all subspaces $(E, \|\cdot\|_E)$ of $\mathbb{R}^I$ with an absolute, normalised norm such that $\operatorname{span}\{e_i : i \in I\}$ is dense in E.

II.2 Sums of acs spaces

We begin by considering (infinite) sums of acs spaces. The following result holds.

Proposition II.2.1. *If $(X_i)_{i\in I}$ is a family of acs spaces and $E \in \mathcal{E}_I$ is acs, then $\left[\bigoplus_{i\in I} X_i\right]_E$ is also acs.*

Of course, if the sum $\left[\bigoplus_{i\in I} X_i\right]_E$ is acs, then every summand X_i has to be acs as well, since the acs property is trivially inherited by subspaces.

Proof. Let $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ be elements of the unit sphere of $\left[\bigoplus_{i\in I} X_i\right]_E$ and $x^* = (x_i^*)_{i\in I}$ an element of the dual unit sphere such that $\|x + y\|_E = 2$ and $x^*(x) = 1$. We then have

$$1 = x^*(x) = \sum_{i\in I} x_i^*(x_i) \le \sum_{i\in I} \|x_i^*\| \|x_i\| \le \|x^*\|_{E'} \|x\|_E = 1$$

and hence

$$x_i^*(x_i) = \|x_i^*\| \|x_i\| \ \forall i \in I \text{ and } \sum_{i\in I} \|x_i^*\| \|x_i\| = 1. \tag{II.2.1}$$

Moreover, by Lemma II.1.1 we have

$$\begin{aligned} 2 = \|x + y\|_E &= \|(\|x_i + y_i\|)_{i\in I}\|_E \le \|(\|x_i\| + \|y_i\|)_{i\in I}\|_E \\ &\le \|x\|_E + \|y\|_E = 2 \end{aligned}$$

and thus

$$\|(\|x_i\| + \|y_i\|)_{i\in I}\|_E = 2. \tag{II.2.2}$$

Since E is acs, (II.2.2) and the second part of (II.2.1) imply that

$$\sum_{i\in I} \|x_i^*\| \|y_i\| = 1. \tag{II.2.3}$$

Another application of Lemma II.1.1 shows

$$4 = 2\|x+y\|_E \leq \|(\|x_i+y_i\| + \|x_i\| + \|y_i\|)_{i\in I}\|_E \leq 4$$

and hence

$$\|(\|x_i+y_i\| + \|x_i\| + \|y_i\|)_{i\in I}\|_E = 4. \tag{II.2.4}$$

Again, since E is acs we get from (II.2.4), (II.2.3) and the second part of (II.2.1) that

$$\sum_{i\in I} \|x_i^*\|\|x_i+y_i\| = 2,$$

which together with (II.2.1) and (II.2.3) implies

$$\|x_i^*\|(\|x_i\| + \|y_i\| - \|x_i+y_i\|) = 0 \quad \forall i \in I. \tag{II.2.5}$$

Next we claim that

$$x_i^*(y_i) = \|x_i^*\|\|y_i\| \quad \forall i \in I. \tag{II.2.6}$$

To see this, fix any $i_0 \in I$ with $x_{i_0}^* \neq 0$ and $y_{i_0} \neq 0$. Define $a_i = \|x_i^*\|$ for all $i \in I \setminus \{i_0\}$ and $a_{i_0} = 0$. Then $(a_i)_{i\in I} \in B_{E'}$, because of Lemma II.1.1.
If $x_{i_0} = 0$ it would follow that $\sum_{i\in I} a_i\|x_i\| = \sum_{i\in I} \|x_i^*\|\|x_i\| = 1$ and hence (because of (II.2.2) and since E is acs) we would also have $\sum_{i\in I} a_i\|y_i\| = 1$. But by (II.2.3) this would imply $\|y_{i_0}\|\|x_{i_0}^*\| = \sum_{i\in I}\|y_i\|(\|x_i^*\| - a_i) = 0$, a contradiction.
Thus $x_{i_0} \neq 0$. From (II.2.5) and Lemma I.10.10 we get that

$$\left\| \frac{x_{i_0}}{\|x_{i_0}\|} + \frac{y_{i_0}}{\|y_{i_0}\|} \right\| = 2.$$

Taking into account the first part of (II.2.1) and the fact that X_{i_0} is acs we get $x_{i_0}^*(y_{i_0}) = \|x_{i_0}^*\|\|y_{i_0}\|$, as desired.
Now from (II.2.6) and (II.2.3) it follows that $x^*(y) = 1$ and we are done. $\square$

We remark that the special case of finitely many summands in the above proposition has already been treated in [33] (for two summands) and [107] (for finitely many summands) in the context of u-spaces and the so called ψ-direct sums (an equivalent formulation of the concept of absolute sums of finitely many spaces).

Since for $p \in (1,\infty)$ the space $\ell^p(I)$ is even UR we get as a special case that p-sums of acs spaces are again acs.

Corollary II.2.2. *If $(X_i)_{i\in I}$ is a family of acs spaces, then $\left[\bigoplus_{i\in I} X_i\right]_p$ is also acs for every $p \in (1,\infty)$.*

II.3 Sums of luacs and sluacs spaces

In this section we study sums of luacs and sluacs spaces. First we have to introduce another technical definition.

Definition II.3.1. A space $E \in \mathcal{E}_I$ is said to have the property (P) if for every sequence $(a_n)_{n\in\mathbb{N}}$ in S_E and every $a \in S_E$ we have

$$\|a_n + a\|_E \to 2 \;\Rightarrow\; a_n \to a \text{ pointwise.}$$

If E is WLUR then it obviously has property (P). The converse is true if E does not contain an isomorphic copy of ℓ^1 by Lemma II.1.2.

With this notion we can formulate the following proposition.

Proposition II.3.2. *If $(X_i)_{i\in I}$ is a family of sluacs (resp. luacs) spaces and $E \in \mathcal{E}_I$ is sluacs (resp. luacs) and has the property (P) then $\left[\bigoplus_{i\in I} X_i\right]_E$ is sluacs (resp. luacs) as well.*

Proof. We only prove the sluacs case. The argument for luacs spaces is analogous.

So let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the unit sphere of $\left[\bigoplus_{i\in I} X_i\right]_E$ and $x = (x_i)_{i\in I}$ another element of norm one such that $\|x_n + x\|_E \to 2$ and let $(x_n^*)_{n\in\mathbb{N}}$ be a sequence in the dual unit sphere such that $x_n^*(x_n) \to 1$.

Write $x_n = (x_{n,i})_{i\in I}$ and $x_n^* = (x_{n,i}^*)_{i\in I}$ for each n. We then have

$$x_n^*(x_n) = \sum_{i\in I} x_{n,i}^*(x_{n,i}) \le \sum_{i\in I} \|x_{n,i}^*\|\|x_{n,i}\| \le \|x_n^*\|_{E'}\|x_n\|_E = 1,$$

which gives us

$$\lim_{n\to\infty} \sum_{i\in I} \|x_{n,i}^*\|\|x_{n,i}\| = 1 \tag{II.3.1}$$

and

$$\lim_{n\to\infty} \left(x_{n,i}^*(x_{n,i}) - \|x_{n,i}^*\|\|x_{n,i}\|\right) = 0 \quad \forall i \in I. \tag{II.3.2}$$

Applying Lemma II.1.1 we also get

$$\|x_n + x\|_E \le \|(\|x_{n,i}\| + \|x_i\|)_{i\in I}\|_E \le \|x_n\|_E + \|x\|_E = 2$$

and hence

$$\lim_{n\to\infty} \|(\|x_{n,i}\| + \|x_i\|)_{i\in I}\|_E = 2. \tag{II.3.3}$$

Since E has property (P) this implies

$$\lim_{n\to\infty} \|x_{n,i}\| = \|x_i\| \quad \forall i \in I. \tag{II.3.4}$$

Because E is sluacs we get from (II.3.1) and (II.3.3) that

$$\lim_{n\to\infty} \sum_{i\in I} \|x_{n,i}^*\|\|x_i\| = 1. \tag{II.3.5}$$

We further have, using again Lemma II.1.1,

$$3\|x_n+x\|_E-2 \leq \|x_n+3x\|_E \leq \|(2\|x_i\|+\|x_{n,i}+x_i\|)_{i\in I}\|_E \leq 2+\|x_n+x\|_E,$$

which implies

$$\lim_{n\to\infty} \|(2\|x_i\| + \|x_{n,i} + x_i\|)_{i\in I}\|_E = 4 \tag{II.3.6}$$

(this trick is taken from the proof of [19, Theorem 4], see also the proof of [67, Theorem 4]).
Since E has property (P) and $\lim_{n\to\infty}\|(\|x_{n,i} + x_i\|)_{i\in I}\|_E = 2$ it follows from (II.3.6) together with some standard normalisation arguments that

$$\lim_{n\to\infty} \|x_{n,i} + x_i\| = 2\|x_i\| \quad \forall i \in I. \tag{II.3.7}$$

Because each X_i is sluacs it follows from (II.3.4), (II.3.7) and (II.3.2) (and again some standard normalisation arguments) that

$$\lim_{n\to\infty} \left(x_{n,i}^*(x_i) - \|x_{n,i}^*\|\|x_i\|\right) = 0 \quad \forall i \in I. \tag{II.3.8}$$

Now take any $\varepsilon > 0$. Then there is a finite subset $J \subseteq I$ such that

$$\left\|\sum_{i\in J}\|x_i\|e_i - (\|x_i\|)_{i\in I}\right\|_E \leq \varepsilon. \tag{II.3.9}$$

By (II.3.8) we can find an index $n_0 \in \mathbb{N}$ such that

$$\left|\sum_{i\in J}\left(x_{n,i}^*(x_i) - \|x_{n,i}^*\|\|x_i\|\right)\right| \leq \varepsilon \quad \forall n \geq n_0. \tag{II.3.10}$$

Then for all $n \geq n_0$ we have

$$\begin{aligned}
\left|x_n^*(x) - \sum_{i\in I}\|x_{n,i}^*\|\|x_i\|\right| &= \left|\sum_{i\in I}\left(x_{n,i}^*(x_i) - \|x_{n,i}^*\|\|x_i\|\right)\right| \\
&\leq \left|\sum_{i\in J}\left(x_{n,i}^*(x_i) - \|x_{n,i}^*\|\|x_i\|\right)\right| + \left|\sum_{i\in I\setminus J}\left(x_{n,i}^*(x_i) - \|x_{n,i}^*\|\|x_i\|\right)\right| \\
&\overset{\text{(II.3.10)}}{\leq} \varepsilon + 2\sum_{i\in I\setminus J}\|x_{n,i}^*\|\|x_i\| \leq \varepsilon + 2\left\|\sum_{i\in J}\|x_i\|e_i - (\|x_i\|)_{i\in I}\right\|_E \overset{\text{(II.3.9)}}{\leq} 3\varepsilon.
\end{aligned}$$

Thus we have shown $x_n^*(x)-\sum_{i\in I}\|x_{n,i}^*\|\|x_i\| \to 0$ which together with (II.3.5) leads to $x_n^*(x) \to 1$ finishing the proof. □

The above result implies in particular that the properties sluacs and luacs are preserved by p-sums for $1 < p < \infty$.

Corollary II.3.3. *If $(X_i)_{i\in I}$ is a family of sluacs (resp. luacs) spaces, then $\left[\bigoplus_{i\in I} X_i\right]_p$ is also sluacs (resp. luacs) for every $p \in (1, \infty)$.*

In the next result we shall see that instead of supposing that E possesses the property (P) we can also assume that E is sluacs$^+$ (resp. luacs$^+$) to come to the same conclusion.

Proposition II.3.4. *If $(X_i)_{i\in I}$ is a family of sluacs (resp. luacs) spaces and $E \in \mathcal{E}_I$ is sluacs$^+$ (resp. luacs$^+$) then $\left[\bigoplus_{i\in I} X_i\right]_E$ is also sluacs (resp. luacs).*

Proof. Again we only show the sluacs case, the luacs case being analogous. So fix a sequence $(x_n)_{n\in\mathbb{N}}$, a point x and a sequence $(x_n^*)_{n\in\mathbb{N}}$ of functionals just like in the proof of the preceding Proposition.
As in this proof we can show

$$\lim_{n\to\infty} \sum_{i\in I} \|x_{n,i}^*\| \|x_{n,i}\| = 1 \tag{II.3.11}$$

and

$$\lim_{n\to\infty} \left(x_{n,i}^*(x_{n,i}) - \|x_{n,i}^*\| \|x_{n,i}\|\right) = 0 \;\; \forall i \in I, \tag{II.3.12}$$

as well as

$$\lim_{n\to\infty} \|(\|x_{n,i}\| + \|x_i\|)_{i\in I}\|_E = 2 \tag{II.3.13}$$

and

$$\lim_{n\to\infty} \sum_{i\in I} \|x_{n,i}^*\| \|x_i\| = 1. \tag{II.3.14}$$

Also as in the proof of Proposition II.3.2 we can see

$$\lim_{n\to\infty} \|(2\|x_i\| + \|x_{n,i} + x_i\|)_{i\in I}\|_E = 4. \tag{II.3.15}$$

Since E is sluacs$^+$ it follows from (II.3.15) and (II.3.14) (with the usual normalisation arguments) that

$$\lim_{n\to\infty} \sum_{i\in I} \|x_{n,i}^*\| \|x_{n,i} + x_i\| = 2.$$

Together with (II.3.11) we get

$$\lim_{n\to\infty} \sum_{i\in I} \|x_{n,i}^*\| (\|x_{n,i} + x_i\| - \|x_{n,i}\| - \|x_i\|) = 0$$

and hence

$$\lim_{n\to\infty} \|x_{n,i}^*\| (\|x_{n,i} + x_i\| - \|x_{n,i}\| - \|x_i\|) = 0 \;\; \forall i \in I. \tag{II.3.16}$$

Next we show that

$$\lim_{n\to\infty}\left(x_{n,i}^*(x_i)-\|x_{n,i}^*\|\|x_i\|\right)=0 \quad \forall i\in I. \tag{II.3.17}$$

To see this we fix $i_0 \in I$ with $x_{i_0}\neq 0$. If $\|x_{n,i_0}^*\|\to 0$ the statement is clear. Otherwise there is some $\varepsilon>0$ such that $\|x_{n,i_0}^*\|\geq\varepsilon$ for infinitely many n. Without loss of generality we may assume that this inequality holds for every $n\in\mathbb{N}$.
For each $n\in\mathbb{N}$ we put $a_{n,i}=\|x_{n,i}^*\|$ for $i\in I\setminus\{i_0\}$ and $a_{n,i_0}=0$. Then $(a_{n,i})_{i\in I}\in B_{E'}$ for every n.
We have $|\sum_{i\in I}(a_{n,i}-\|x_{n,i}^*\|)\|x_{n,i}\||=\|x_{n,i_0}^*\|\|x_{n,i_0}\|\leq\|x_{n,i_0}\|$.
So if $\|x_{n,i_0}\|\to 0$ then by (II.3.11) we would also have $\lim_{n\to\infty}\sum_{i\in I}a_{n,i}\|x_{n,i}\|=1$.
But since E is a sluacs$^+$ space this together with (II.3.13) would also imply $\lim_{n\to\infty}\sum_{i\in I}a_{n,i}\|x_i\|=1$, which in turn implies (because of (II.3.14)) $\|x_{n,i_0}^*\|\|x_{i_0}\|=|\sum_{i\in I}(a_{n,i}-\|x_{n,i}^*\|)\|x_i\||\to 0$, where on the other hand $\|x_{n,i_0}^*\|\|x_{i_0}\|\geq\varepsilon\|x_{i_0}\|>0$ for all $n\in\mathbb{N}$, a contradiction.
So we must have $\|x_{n,i_0}\|\not\to 0$ and hence there is some $\delta>0$ such that $\|x_{n,i_0}\|\geq\delta$ for infinitely many (say for all) $n\in\mathbb{N}$.
Now since $(\|x_{n,i_0}^*\|)_{n\in\mathbb{N}}$ is bounded away from zero, (II.3.16) gives us that $\lim_{n\to\infty}(\|x_{n,i_0}+x_{i_0}\|-\|x_{n,i_0}\|-\|x_{i_0}\|)=0$.
Because $(\|x_{n,i_0}\|)_{n\in\mathbb{N}}$ is bounded away from zero as well, this together with Lemma I.10.10 implies that

$$\lim_{n\to\infty}\left\|\frac{x_{n,i_0}}{\|x_{n,i_0}\|}+\frac{x_{i_0}}{\|x_{i_0}\|}\right\|=2.$$

Using (II.3.12) and the fact that X_{i_0} is sluacs we now get the desired conclusion.
Now that we have established (II.3.17), the rest of the proof can be carried out exactly as in Proposition II.3.2. □

II.4 Sums of luacs$^+$ and sluacs$^+$ spaces

Now we will consider sums of luacs$^+$ and sluacs$^+$ spaces. First we have a result on sums of luacs$^+$ spaces under the assumption of property (P).

Proposition II.4.1. *If $(X_i)_{i\in I}$ is a family of luacs$^+$ spaces and $E\in\mathcal{E}_I$ is luacs$^+$ and has the property (P) then $\left[\bigoplus_{i\in I}X_i\right]_E$ is also an luacs$^+$ space.*

Proof. By Proposition II.3.4 (or Proposition II.3.2) we already know that the space $\left[\bigoplus_{i\in I}X_i\right]_E$ is luacs.
Now take a sequence $(x_n)_{n\in\mathbb{N}}$ and an element $x=(x_i)_{i\in I}$ in the unit sphere of $\left[\bigoplus_{i\in I}X_i\right]_E$ such that $\|x_n+x\|\to 2$ and a functional $x^*=(x_i^*)_{i\in I}$ of norm one with $x^*(x)=1$. Write $x_n=(x_{n,i})_{i\in I}$ for all $n\in\mathbb{N}$.

As in the proof of Proposition II.2.1 it follows from $x^*(x) = 1$ that

$$x_i^*(x_i) = \|x_i^*\|\|x_i\| \ \forall i \in I \text{ and } \sum_{i \in I} \|x_i^*\|\|x_i\| = 1 \tag{II.4.1}$$

and as in the proof of Proposition II.3.2 one can show that

$$\lim_{n\to\infty} \|(\|x_{n,i}\| + \|x_i\|)_{i\in I}\|_E = 2. \tag{II.4.2}$$

Since E is luacs$^+$ it follows from (II.4.2) and the second part of (II.4.1) that we also have

$$\lim_{n\to\infty} \sum_{i\in I} \|x_i^*\|\|x_{n,i}\| = 1. \tag{II.4.3}$$

Because E has property (P) it also follows from (II.4.2) that

$$\lim_{n\to\infty} \|x_{n,i}\| = \|x_i\| \quad \forall i \in I. \tag{II.4.4}$$

Exactly as in the proof of Proposition II.3.2 we can see

$$\lim_{n\to\infty} \|x_{n,i} + x_i\| = 2\|x_i\| \quad \forall i \in I. \tag{II.4.5}$$

Since each X_i is luacs$^+$ we infer from (II.4.5), (II.4.4) and the first part of (II.4.1) that

$$\lim_{n\to\infty} x_i^*(x_{n,i}) = \|x_i^*\|\|x_i\| \quad \forall i \in I. \tag{II.4.6}$$

Now take an arbitrary $\varepsilon > 0$ and fix a finite subset $J \subseteq I$ such that

$$\left\| \sum_{i\in J} \|x_i\| e_i - (\|x_i\|)_{i\in I} \right\|_E \leq \varepsilon. \tag{II.4.7}$$

From (II.4.1), (II.4.3) and (II.4.4) it follows that

$$\lim_{n\to\infty} \sum_{i\in I\setminus J} \|x_i^*\|\|x_{n,i}\| = \sum_{i\in I\setminus J} \|x_i^*\|\|x_i\|$$

and by (II.4.6) we also have

$$\lim_{n\to\infty} \sum_{i\in J} x_i^*(x_{n,i}) = \sum_{i\in J} \|x_i^*\|\|x_i\|.$$

Hence there is some $n_0 \in \mathbb{N}$ such that

$$\left| \sum_{i\in J} (x_i^*(x_{n,i}) - \|x_i^*\|\|x_i\|) \right| \leq \varepsilon \text{ and } \tag{II.4.8}$$

$$\left| \sum_{i\in I\setminus J} \|x_i^*\|(\|x_{n,i}\| - \|x_i\|) \right| \leq \varepsilon \quad \forall n \geq n_0. \tag{II.4.9}$$

But then we have for every $n \geq n_0$

$$\begin{aligned}
|x^*(x_n) - 1| &\overset{\text{(II.4.1)}}{=} \left|\sum_{i\in I}(x_i^*(x_{n,i}) - \|x_i^*\|\|x_i\|)\right| \\
&\overset{\text{(II.4.8)}}{\leq} \varepsilon + \left|\sum_{i\in I\setminus J}(x_i^*(x_{n,i}) - \|x_i^*\|\|x_i\|)\right| \\
&\leq \varepsilon + \sum_{i\in I\setminus J}\|x_i^*\|(\|x_{n,i}\| + \|x_i\|) \\
&\overset{\text{(II.4.9)}}{\leq} 2\varepsilon + 2\sum_{i\in I\setminus J}\|x_i^*\|\|x_i\| \overset{\text{(II.4.7)}}{\leq} 4\varepsilon.
\end{aligned}$$

Thus we have $x^*(x_n) \to 1$ and the proof is finished. □

For another result concerning sums of luacs$^+$ spaces see also Proposition II.5.1. The next result deals primarily with sums of sluacs$^+$ spaces.

Proposition II.4.2. *If $(X_i)_{i\in I}$ is a family of sluacs$^+$ (resp. luacs$^+$) spaces and $E \in \mathcal{E}_I$ is sluacs$^+$ then $\left[\bigoplus_{i\in I} X_i\right]_E$ is sluacs$^+$ (resp. luacs$^+$) as well.*

Proof. Suppose all the X_i and E are sluacs$^+$. Then by Proposition II.3.4 $\left[\bigoplus_{i\in I} X_i\right]_E$ is sluacs.
Now take sequences $(x_n)_{n\in\mathbb{N}}$ and $(x_n^*)_{n\in\mathbb{N}}$ in the unit sphere and in the dual unit sphere of $\left[\bigoplus_{i\in I} X_i\right]_E$ respectively, as well as another element $x = (x_i)_{i\in I}$ in $\left[\bigoplus_{i\in I} X_i\right]_E$ of norm one such that $\|x_n + x\|_E \to 2$ and $x_n^*(x) \to 1$.
As usual we write $x_n = (x_{n,i})_{i\in I}$ and $x_n^* = (x_{n,i}^*)_{i\in I}$ for every $n \in \mathbb{N}$.
In much the same way as we have done before one can show that

$$\lim_{n\to\infty}\left(x_{n,i}^*(x_i) - \|x_{n,i}^*\|\|x_i\|\right) = 0 \ \forall i \in I \text{ and } \lim_{n\to\infty}\sum_{i\in I}\|x_{n,i}^*\|\|x_i\| = 1, \tag{II.4.10}$$

as well as

$$\lim_{n\to\infty}\|(\|x_{n,i}\| + \|x_i\|)_{i\in I}\|_E = 2. \tag{II.4.11}$$

It follows from (II.4.11), the second part of (II.4.10), and the fact that E is sluacs$^+$ that

$$\lim_{n\to\infty}\sum_{i\in I}\|x_{n,i}^*\|\|x_{n,i}\| = 1. \tag{II.4.12}$$

As in the proof of Proposition II.3.4 we see that

$$\lim_{n\to\infty}\|x_{n,i}^*\|(\|x_{n,i} + x_i\| - \|x_{n,i}\| - \|x_i\|) = 0 \quad \forall i \in I. \tag{II.4.13}$$

Now using an argument analogous to that in the proof of Proposition II.3.4 shows

$$\lim_{n\to\infty}\left(x_{n,i}^*(x_{n,i}) - \|x_{n,i}^*\|\|x_{n,i}\|\right) = 0 \quad \forall i \in I. \tag{II.4.14}$$

Put $b_J = (\|x_i\|)_{i\in I} - \sum_{i\in J}\|x_i\|e_i$ and $c_{n,J} = \sum_{i\in J}\|x^*_{n,i}\|e_i$ for every $n\in\mathbb{N}$ and every finite subset $J\subseteq I$. Then for every n and J we have

$$
\begin{aligned}
|c_{n,J}((\|x_i\|)_{i\in I}) - 1| &= \left|\sum_{i\in J}\|x^*_{n,i}\|\|x_i\| - 1\right| \\
&\le \left|\sum_{i\in I\setminus J}\|x^*_{n,i}\|\|x_i\|\right| + \left|\sum_{i\in I}\|x^*_{n,i}\|\|x_i\| - 1\right| \\
&\le \|b_J\|_E + \left|\sum_{i\in I}\|x^*_{n,i}\|\|x_i\| - 1\right|. \qquad \text{(II.4.15)}
\end{aligned}
$$

Now take any $\varepsilon > 0$. Because E is sluacs$^+$ there is some $\delta > 0$ such that

$$
\begin{aligned}
&a\in S_E,\ g\in B_{E^*} \text{ with } \|a + (\|x_i\|)_{i\in I}\|_E \ge 2-\delta \\
&\text{and } g((\|x_i\|)_{i\in I}) \ge 1-\delta \ \Rightarrow\ g(a)\ge 1-\varepsilon. \qquad \text{(II.4.16)}
\end{aligned}
$$

Fix a finite subset $J_0\subseteq I$ such that $\|b_{J_0}\|_E \le \delta/2$ and also fix an index n_0 such that $\left|\sum_{i\in I}\|x^*_{n,i}\|\|x_i\| - 1\right| \le \delta/2$ and $\|(\|x_{n,i}\| + \|x_i\|)_{i\in I}\|_E \ge 2-\delta$ for all $n\ge n_0$ (which is possible because of (II.4.10) and (II.4.11)).
Then (II.4.15) and (II.4.16) give us

$$
c_{n,J_0}((\|x_{n,i}\|)_{i\in I}) = \sum_{i\in J_0}\|x^*_{n,i}\|\|x_{n,i}\| \ge 1-\varepsilon \quad \forall n\ge n_0. \qquad \text{(II.4.17)}
$$

By (II.4.14) we may also assume that

$$
\left|\sum_{i\in J_0}\big(x^*_{n,i}(x_{n,i}) - \|x^*_{n,i}\|\|x_{n,i}\|\big)\right| \le \varepsilon \quad \forall n\ge n_0. \qquad \text{(II.4.18)}
$$

Then for every $n\ge n_0$ we have

$$
\begin{aligned}
\left|x^*_n(x_n) - \sum_{i\in I}\|x^*_{n,i}\|\|x_{n,i}\|\right| &= \left|\sum_{i\in I}\big(x^*_{n,i}(x_{n,i}) - \|x^*_{n,i}\|\|x_{n,i}\|\big)\right| \\
&\overset{\text{(II.4.18)}}{\le} \varepsilon + \left|\sum_{i\in I\setminus J_0}\big(x^*_{n,i}(x_{n,i}) - \|x^*_{n,i}\|\|x_{n,i}\|\big)\right| \\
&\le \varepsilon + 2\sum_{i\in I\setminus J_0}\|x^*_{n,i}\|\|x_{n,i}\| \overset{\text{(II.4.17)}}{\le} 3\varepsilon.
\end{aligned}
$$

Thus $x^*_n(x_n) - \sum_{i\in I}\|x^*_{n,i}\|\|x_{n,i}\| \to 0$ which together with (II.4.12) implies $x^*_n(x_n)\to 1$.
The proof for the luacs$^+$ case can be done in a very similar fashion. $\square$

Again we explicitly note the case of p-sums that follows from the preceding results.

Corollary II.4.3. *If $(X_i)_{i\in I}$ is a family of sluacs$^+$ (resp. luacs$^+$) spaces, then $\left[\bigoplus_{i\in I} X_i\right]_p$ is also sluacs$^+$ (resp. luacs$^+$) for every $p \in (1, \infty)$.*

II.5 Sums of wuacs spaces

This section concerns sums of wuacs (and luacs$^+$) spaces for the case that E does not contain an isomorphic copy of ℓ^1.

Proposition II.5.1. *If $(X_i)_{i\in I}$ is a family of wuacs (resp. luacs$^+$) spaces and if $E \in \mathcal{E}_I$ is wuacs (resp. luacs$^+$) and does not contain an isomorphic copy of ℓ^1, then $\left[\bigoplus_{i\in I} X_i\right]_E$ is also wuacs (resp. luacs$^+$).*

Proof. Let us suppose that E and all the X_i are wuacs and fix two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in the unit sphere of $\left[\bigoplus_{i\in I} X_i\right]_E$ as well as a norm one functional $x^* = (x_i^*)_{i\in I}$ on $\left[\bigoplus_{i\in I} X_i\right]_E$ such that $\|x_n + y_n\|_E \to 2$ and $x^*(x_n) \to 1$. Write $x_n = (x_{n,i})_{i\in I}$ and $y_n = (y_{n,i})_{i\in I}$ for each n.
Similar to what we have done before we can deduce

$$\lim_{n\to\infty} (x_i^*(x_{n,i}) - \|x_i^*\|\|x_{n,i}\|) = 0 \ \forall i \in I \text{ and } \lim_{n\to\infty} \sum_{i\in I} \|x_i^*\|\|x_{n,i}\| = 1 \tag{II.5.1}$$

and

$$\lim_{n\to\infty} \|(\|x_{n,i}\| + \|y_{n,i}\|)_{i\in I}\|_E = 2 \tag{II.5.2}$$

as well as

$$\lim_{n\to\infty} \|(\|x_{n,i} + y_{n,i}\| + \|x_{n,i}\| + \|y_{n,i}\|)_{i\in I}\|_E = 4. \tag{II.5.3}$$

Since E is wuacs (II.5.2) and the second part of (II.5.1) imply

$$\lim_{n\to\infty} \sum_{i\in I} \|x_i^*\|\|y_{n,i}\| = 1. \tag{II.5.4}$$

Applying again the fact that E is wuacs together with (II.5.3), (II.5.4) and the second part of (II.5.1) gives us

$$\lim_{n\to\infty} \sum_{i\in I} \|x_i^*\|(\|x_{n,i}\| + \|y_{n,i}\| - \|x_{n,i} + y_{n,i}\|) = 0$$

and hence

$$\lim_{n\to\infty} \|x_i^*\|(\|x_{n,i}\| + \|y_{n,i}\| - \|x_{n,i} + y_{n,i}\|) = 0 \ \ \forall i \in I. \tag{II.5.5}$$

Now we can show

$$\lim_{n\to\infty}(x_i^*(y_{n,i}) - \|x_i^*\|\|y_{n,i}\|) = 0 \quad \forall i \in I. \tag{II.5.6}$$

The argument for this is similiar to what we have done before but we state it here for the sake of completeness. Fix $i_0 \in I$ with $x_{i_0}^* \neq 0$ and $y_{n,i_0} \not\to 0$. Then there is $\tau > 0$ such that $\|y_{n,i_0}\| \geq \tau$ for infinitely many (without loss of generality for all) $n \in \mathbb{N}$.
Put $a_{i_0} = 0$ and $a_i = \|x_i^*\|$ for every $i \in I \setminus \{i_0\}$. If $\|x_{n,i_0}\| \to 0$ then because of the second part of (II.5.1) it would follow that $\lim_{n\to\infty}\sum_{i\in I} a_i\|x_{n,i}\| = 1$. Since E is wuacs this together with (II.5.2) would imply that we also have $\lim_{n\to\infty}\sum_{i\in I} a_i\|y_{n,i}\| = 1$ which because (II.5.4) would give us $\|x_{i_0}^*\|\|y_{n,i_0}\| \to 0$, a contradiction.
Hence there must be some $\delta > 0$ such that $\|x_{n,i_0}\| \geq \delta$ for infinitely many (say for every) $n \in \mathbb{N}$.
Now since the sequences $(\|x_{n,i_0}\|)_{n\in\mathbb{N}}$ and $(\|y_{n,i_0}\|)_{n\in\mathbb{N}}$ are bounded away from zero it follows from (II.5.1), (II.5.5) and Lemma I.10.10 that

$$\lim_{n\to\infty}\left\|\frac{x_{n,i_0}}{\|x_{n,i_0}\|} + \frac{y_{n,i_0}}{\|y_{n,i_0}\|}\right\| = 2 \text{ and } \lim_{n\to\infty}\frac{x_{i_0}^*}{\|x_{i_0}^*\|}\left(\frac{x_{n,i_0}}{\|x_{n,i_0}\|}\right) = 1.$$

Since X_{i_0} is wuacs this implies our desired conclusion.
Now we fix any $\varepsilon > 0$. Because $\ell^1 \not\subseteq E$ by Lemma II.1.2 there must be some finite set $J \subseteq I$ such that

$$\left\|(\|x_i^*\|)_{i\in I} - \sum_{i\in J}\|x_i^*\|e_i\right\|_{E'} \leq \varepsilon. \tag{II.5.7}$$

By (II.5.6) we can find some $n_0 \in \mathbb{N}$ such that

$$\left|\sum_{i\in J}(x_i^*(y_{n,i}) - \|x_i^*\|\|y_{n,i}\|)\right| \leq \varepsilon \quad \forall n \geq n_0. \tag{II.5.8}$$

We then have for every $n \geq n_0$

$$\begin{aligned}\left|x^*(y_n) - \sum_{i\in I}\|x_i^*\|\|y_{n,i}\|\right| &\overset{(II.5.8)}{\leq} \varepsilon + \left|\sum_{i\in I\setminus J}(x_i^*(y_{n,i}) - \|x_i^*\|\|y_{n,i}\|)\right| \\ &\leq \varepsilon + 2\sum_{i\in I\setminus J}\|x_i^*\|\|y_{n,i}\| \overset{(II.5.7)}{\leq} 3\varepsilon.\end{aligned}$$

So we have $x^*(y_n) - \sum_{i\in I}\|x_i^*\|\|y_{n,i}\| \to 0$. From (II.5.4) it now follows that $x^*(y_n) \to 1$.
The luacs$^+$ case is proved analogously. □

The above Proposition especially applies to the case that E is WUR because a WUR space cannot contain an isomorphic copy of ℓ^1 (cf. [140, Remark 4]). We note again the particular case of p-sums.

Corollary II.5.2. *If $(X_i)_{i\in I}$ is a family of wuacs spaces, then $\left[\bigoplus_{i\in I} X_i\right]_p$ is also wuacs for every $p \in (1,\infty)$.*

The author does not know whether a wuacs space can contain an isomorphic copy of ℓ^1 at all, but at least it cannot contain particularly "good" copies of ℓ^1 in the following sense (introduced in [37]).

Definition II.5.3. A Banach space X is said to contain an *asymptotically isometric copy of ℓ^1* if there is a sequence $(x_n)_{n\in\mathbb{N}}$ in B_X and a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $[0,1)$ with $\varepsilon_n \to 0$ such that for each $m \in \mathbb{N}$ and all scalars $a_1, \dots, a_m$ we have

$$\sum_{i=1}^{m}(1-\varepsilon_i)|a_i| \leq \left\|\sum_{i=1}^{m} a_i x_i\right\| \leq \sum_{i=1}^{m}|a_i|.$$

Likewise, X is said to contain an *asymptotically isomorphic copy of c_0* if there are two such sequences $(x_n)_{n\in\mathbb{N}}$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ which fulfil

$$\max_{i=1,\dots,m}(1-\varepsilon_i)|a_i| \leq \left\|\sum_{i=1}^{m} a_i x_i\right\| \leq \max_{i=1,\dots,m}|a_i|$$

for each $m \in \mathbb{N}$ and all scalars $a_1, \dots, a_m$.

The following observation can be made.

Proposition II.5.4. *If the Banach space X is wuacs then it does not contain an asymptotically isometric copy of ℓ^1.*

Proof. Suppose that X contains an asymptotically isometric copy of ℓ^1. Then fix two sequences $(x_n)_{n\in\mathbb{N}}$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ as in the above definition.
We can find $\alpha > 1$ such that $\alpha\varepsilon_n < 1$ for every $n \in \mathbb{N}$. Put $\tilde{x}_n = (1-\alpha\varepsilon_n)^{-1}x_n$ for each n. Then for every finite sequence $(a_i)_{i=1}^m$ of scalars we have

$$\left\|\sum_{i=1}^{m} a_i\tilde{x}_i\right\| = \left\|\sum_{i=1}^{m} \frac{a_i}{1-\alpha\varepsilon_i}x_i\right\| \geq \sum_{i=1}^{m}\frac{1-\varepsilon_i}{1-\alpha\varepsilon_i}|a_i| \geq \sum_{i=1}^{m}|a_i|. \qquad \text{(II.5.9)}$$

In other words, the operator $T: \ell^1 \to X$ defined by $T((a_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} a_n\tilde{x}_n$ is an isomorphism onto its range $U = \operatorname{ran} T$ with $\|T^{-1}\| \leq 1$.
Define $(b_n)_{n\in\mathbb{N}} \in \ell^\infty = (\ell^1)^*$ by $b_n = 1$ if n is even and $b_n = 0$ if n is odd. Then $u^* = (T^{-1})^*((b_n)_{n\in\mathbb{N}}) \in B_{U^*}$. Take a Hahn-Banach extension x^* of u^* to X.
Note that because of (II.5.9) we have in particular $\|\tilde{x}_n\| \geq 1$ for every n and on the other hand $\|\tilde{x}_n\| \leq (1-\alpha\varepsilon_n)^{-1}$ and $\varepsilon_n \to 0$, hence $\|\tilde{x}_n\| \to 1$.

Again because of (II.5.9) we have $\|\tilde{x}_n + \tilde{x}_{n+1}\| \geq 2$ for every n. It follows that $\|\tilde{x}_n + \tilde{x}_{n+1}\| \to 2$ and thus in particular $\|\tilde{x}_{2n} + \tilde{x}_{2n+1}\| \to 2$.
But we also have $x^*(\tilde{x}_{2n}) = u^*(\tilde{x}_{2n}) = b_{2n} = 1$ and likewise $x^*(\tilde{x}_{2n+1}) = b_{2n+1} = 0$ for every n and hence X cannot be a wuacs space. $\square$

If the space X contains an asymptotically isometric copy of c_0 then by [37, Theorem 2] X^* contains an asymptotically isometric copy of ℓ^1 and thus we get the following corollary.

Corollary II.5.5. *If X is a Banach space whose dual X^* is wuacs then X does not contain an asymptotically isometric copy of c_0.*

II.6 Sums of uacs spaces

In this section we treat sums of uacs spaces. We first consider the case of finitely many summands. In fact, this has been done before in [33] (for two summands) and in [107] (for finitely many summands) in the context of U-spaces and the so called ψ-direct sums. However, we include a sketch of our own slightly different proof here, for the sake of completeness.

Proposition II.6.1. *If I is a finite set, $(X_i)_{i\in I}$ a family of uacs Banach spaces and $\|\,.\,\|_E$ is an absolute normalised norm on $\mathbb{R}^I$ such that $E := (\mathbb{R}^I, \|\,.\,\|_E)$ is acs, then $\left[\bigoplus_{i\in I} X_i\right]_E$ is also a uacs space.*

Proof. First note that since E is finite-dimensional it is actually uacs. Now if we take two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in the unit sphere of $\left[\bigoplus_{i\in I} X_i\right]_E$ and a sequence $(x_n^*)_{n\in\mathbb{N}}$ in the dual unit sphere such that $\|x_n + y_n\|_E \to 2$ and $x_n^*(x_n) \to 1$ then we can show just as we have done before that

$$\lim_{n\to\infty} \sum_{i\in I} \|x_{n,i}^*\|\|x_{n,i}\| = 1 \tag{II.6.1}$$

and

$$\lim_{n\to\infty} \left(x_{n,i}^*(x_{n,i}) - \|x_{n,i}^*\|\|x_{n,i}\|\right) = 0 \quad \forall i \in I \tag{II.6.2}$$

as well as

$$\lim_{n\to\infty} \|(\|x_{n,i}\| + \|y_{n,i}\|)_{i\in I}\|_E = 2 \tag{II.6.3}$$

and

$$\lim_{n\to\infty} \|(\|x_{n,i} + y_{n,i}\| + \|x_{n,i}\| + \|y_{n,i}\|)_{i\in I}\|_E = 4. \tag{II.6.4}$$

Since E is uacs it follows from (II.6.1) and (II.6.3) that

$$\lim_{n\to\infty} \sum_{i\in I} \|x_{n,i}^*\|\|y_{n,i}\| = 1. \tag{II.6.5}$$

Again, since E is uacs it follows from (II.6.1), (II.6.5) and (II.6.4) that

$$\lim_{n\to\infty}\|x^*_{n,i}\|(\|x_{n,i}\|+\|y_{n,i}\|-\|x_{n,i}+y_{n,i}\|)=0 \quad \forall i\in I. \tag{II.6.6}$$

Now using (II.6.6), Lemma I.10.10, (II.6.1), (II.6.5), (II.6.2), the fact that each X_i is uacs and an argument similar the one used in the proof of Proposition II.5.1 we can infer that

$$\lim_{n\to\infty}\left(x^*_{n,i}(y_{n,i})-\|x^*_{n,i}\|\|y_{n,i}\|\right)=0 \quad \forall i\in I.$$

Since I is finite it follows that $x^*_n(y_n)-\sum_{i\in I}\|x^*_{n,i}\|\|y_{n,i}\|\to 0$ which together with (II.6.5) gives us $x^*_n(y_n)\to 1$ and the proof is over. $\square$

Before we can come to the study of absolute sums of infinitely many uacs spaces we have to introduce one more definition.

Definition II.6.2. The space $E\in\mathcal{E}_I$ is said to have the property (u^+) if for every $\varepsilon>0$ there is some $\delta>0$ such that for all $(a_i)_{i\in I},(b_i)_{i\in I}\in S_E$ and each $(c_i)_{i\in I}\in S_{E'}=S_{E^*}$ we have

$$\sum_{i\in I}a_ic_i=1 \text{ and } \|(a_i+b_i)_{i\in I}\|_E\geq 2(1-\delta) \ \Rightarrow \ \sum_{i\in I}|c_i||a_i-b_i|\leq\varepsilon.$$

Clearly, if E is UR then it has property (u^+) and the property (u^+) in turn implies that E is uacs. Unfortunately, the author does not know whether these implications are strict.

Now we can formulate and prove the following theorem, which is an analogue of Day's results on sums of UR spaces from [25, Theorem 3] (for the ℓ^p-case) and [26, Theorem 3] (for the general case). Also, its proof is just a slight modification of Day's technique.

Theorem II.6.3. *If $(X_i)_{i\in I}$ is a family of Banach spaces such that for every $0<\varepsilon\leq 2$ we have $\delta(\varepsilon):=\inf_{i\in I}\delta^{X_i}_{\text{uacs}}(\varepsilon)>0$ and if the space $E\in\mathcal{E}_I$ has the property (u^+) then $X:=\left[\bigoplus_{i\in I}X_i\right]_E$ is also uacs.*

The condition $\inf_{i\in I}\delta^{X_i}_{\text{uacs}}>0$ (analogous to the condition $\inf_{i\in I}\delta_{X_i}>0$ in Day's results) is clearly necessary since $\delta^{X_i}_{\text{uacs}}\geq\delta^{X}_{\text{uacs}}$ for every $i\in I$.

Proof. As in [25] and [26] the proof is divided into two steps. In the first step we show that for every $0<\varepsilon\leq 2$ there is some $\eta>0$ such that for any two elements $x=(x_i)_{i\in I}$ and $y=(y_i)_{i\in I}$ of the unit sphere of $\left[\bigoplus_{i\in I}X_i\right]_E$ with $\|x_i\|=\|y_i\|$ for every $i\in I$ and each functional $x^*=(x^*_i)_{i\in I}$ with $\|x^*\|_{E'}=x^*(x)=1$ and $x^*(y)<1-\varepsilon$ we have $\|x+y\|_E\leq 2(1-\eta)$.
So let $0<\varepsilon\leq 2$ be arbitrary. Since E is uacs there exists some $\eta>0$ such that

$$\begin{aligned}&a,b\in B_E, l\in B_{E^*}, l(a)=1 \text{ and } l(b)<1-\frac{\varepsilon}{4}\delta\Big(\frac{\varepsilon}{2}\Big)\\ &\quad\Rightarrow \ \|a+b\|_E\leq 2(1-\eta).\end{aligned} \tag{II.6.7}$$

We claim that this η fulfils our requirement. To show this, fix x, y and x^* as above and put $\beta_i = \|x_i\| = \|y_i\|$, $\nu_i = \|x_i^*\|$ and $\gamma_i = \nu_i\beta_i - x_i^*(y_i)$ for each $i \in I$. Then we have

$$0 \le \gamma_i \le 2\beta_i\nu_i \quad \forall i \in I. \tag{II.6.8}$$

From $x^*(x) = 1 = \|x^*\|_{E'} = \|x\|_E$ we get

$$\sum_{i\in I} \nu_i\beta_i = 1 \text{ and } x_i^*(x_i) = \nu_i\beta_i \ \forall i \in I. \tag{II.6.9}$$

Next we define

$$\alpha_i = \begin{cases} \frac{1}{2}\delta\Big(\frac{\gamma_i}{\nu_i\beta_i}\Big) & \text{if } \gamma_i > 0 \\ 0 & \text{if } \gamma_i = 0. \end{cases} \tag{II.6.10}$$

From the definition of the $\delta_{\text{uacs}}^{X_i}$ and the second part of (II.6.9) it easily follows that

$$\|x_i + y_i\| \le 2(1-\alpha_i)\beta_i \quad \forall i \in I. \tag{II.6.11}$$

By (II.6.8) and the first part of (II.6.9) we have $\sum_{i\in I}\gamma_i \le 2$ and further it is

$$\varepsilon < 1 - x^*(y) = x^*(x-y) = \sum_{i\in I} x_i^*(x_i - y_i) \le \sum_{i\in I}\gamma_i,$$

thus

$$\varepsilon < \sum_{i\in I}\gamma_i \le 2. \tag{II.6.12}$$

Now put $A = \{i \in I : 2\gamma_i > \varepsilon\nu_i\beta_i\}$ and $B = I \setminus A$. Then we get

$$\sum_{i\in B}\gamma_i \le \frac{\varepsilon}{2}\sum_{i\in B}\nu_i\beta_i \le \frac{\varepsilon}{2}\sum_{i\in I}\nu_i\beta_i \overset{\text{(II.6.9)}}{=} \frac{\varepsilon}{2}. \tag{II.6.13}$$

From (II.6.12) and (II.6.13) it follows that

$$\sum_{i\in A}\gamma_i = \sum_{i\in I}\gamma_i - \sum_{i\in B}\gamma_i > \frac{\varepsilon}{2}. \tag{II.6.14}$$

Using (II.6.8) and (II.6.14) we now get

$$\sum_{i\in A}\nu_i\beta_i > \frac{\varepsilon}{4}. \tag{II.6.15}$$

Write $t = (\beta_i\chi_B(i))_{i\in I}$ and $t' = (\beta_i\chi_A(i))_{i\in I}$, where χ_B and χ_A denote the characteristic function of B and A, respectively. Then $t, t' \in B_E$ (by Lemma II.1.1) and $t + t' = (\beta_i)_{i\in I}$. We also put $t'' = (1 - \delta(\varepsilon/2))t'$. Again by Lemma II.1.1 we have $\|t + t''\|_E \le \|t + t'\|_E = 1$. Further, $l = (\nu_i)_{i\in I}$ defines an element of S_{E^*} such that $l(t + t') = \sum_{i\in I}\nu_i\beta_i = 1$ (by (II.6.9)) and $l(t+t'') = 1-\delta(\varepsilon/2)l(t') = 1-\delta(\varepsilon/2)\sum_{i\in A}\nu_i\beta_i$ and hence (by (II.6.15))

$$l(t + t'') < 1 - \frac{\varepsilon}{4}\delta\Big(\frac{\varepsilon}{2}\Big).$$

Thus we can apply (II.6.7) to deduce

$$\frac{1}{2}\|2t+t'+t''\|_E = \left\| t + \left(1 - \frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right)\right) t' \right\|_E \leq 1 - \eta. \tag{II.6.16}$$

Since δ is obviously an increasing function we also have

$$\alpha_i \geq \frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right) \quad \forall i \in A. \tag{II.6.17}$$

Now we can conclude (with the aid of Lemma II.1.1)

$$\begin{aligned}
\|x+y\|_E &= \|(\|x_i+y_i\|)_{i\in I}\|_E \overset{(II.6.11)}{\leq} 2\|((1-\alpha_i)\beta_i)_{i\in I}\|_E \\
&\overset{(II.6.17)}{\leq} 2\left\| \left(\left(1 - \frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right)\right)\beta_i\chi_A(i) + \beta_i\chi_B(i) \right)_{i\in I} \right\|_E \\
&= 2\left\| \left(1 - \frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right)\right) t' + t \right\|_E \overset{(II.6.16)}{\leq} 2(1-\eta),
\end{aligned}$$

finishing the first step of the proof. Note that so far we have only used the fact that E is uacs and not the property (u^+).
Now for the second step we fix $0 < \varepsilon \leq 2$ and choose an $\eta > 0$ to the value $\varepsilon/2$ according to step one. Then we take $0 < \nu < 2\eta/3$. Since E is uacs we can find $\tau > 0$ such that

$$\begin{aligned}
&a, b \in B_E, l \in B_{E^*}, l(a) \geq 1 - \tau \text{ and } \|a+b\|_E \geq 2(1-\tau) \\
&\Rightarrow \; l(b) \geq 1 - \nu.
\end{aligned} \tag{II.6.18}$$

Next we fix $0 < \alpha < \min\{\varepsilon/2, 2\tau, \nu\}$. Now we can find a number $\tilde{\tau} > 0$ to the value α according to the definition of the property (u^+) (Definition II.6.2). Finally, we take $0 < \xi < \min\{\tau, \tilde{\tau}\}$.
Now suppose $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ are elements of the unit sphere of $\left[\bigoplus_{i\in I} X_i\right]_E$ and $x^* = (x_i^*)_{i\in I}$ is an element of the dual unit sphere such that $\|x+y\|_E \geq 2(1-\xi)$ and $x^*(x) = 1$. We will show that $x^*(y) > 1 - \varepsilon$.
To do so, we define

$$z_i = \begin{cases} \frac{\|x_i\|}{\|y_i\|} y_i & \text{if } y_i \neq 0 \\ x_i & \text{if } y_i = 0. \end{cases} \tag{II.6.19}$$

Then we have

$$\|z_i\| = \|x_i\| \text{ and } \|z_i - y_i\| = |\|x_i\| - \|y_i\|| \;\forall i \in I. \tag{II.6.20}$$

As before we can see that $\sum_{i\in I}\|x_i^*\|\|x_i\| = 1$ and further we have $2(1-\tilde{\tau}) \leq 2(1-\xi) \leq \|x+y\|_E \leq \|(\|x_i\| + \|y_i\|)_{i\in I}\|_E$.
Thus we get from the choice of $\tilde{\tau}$ that

$$\sum_{i\in I}\|x_i^*\|\|z_i - y_i\| \overset{(II.6.20)}{=} \sum_{i\in I}\|x_i^*\||\|x_i\| - \|y_i\|| \leq \alpha. \tag{II.6.21}$$

Further, we have

$$\|(\|x_i\|+\|y_i\|+\|x_i+y_i\|)_{i\in I}\|_E \geq 2\|x+y\|_E \geq 4(1-\xi) \geq 4(1-\tau)$$

and

$$\begin{aligned}
&\sum_{i\in I}\|x_i^*\|(\|x_i\|+\|y_i\|) = 1+\sum_{i\in I}\|x_i^*\|\|y_i\| \\
&\geq 1+\sum_{i\in I}\|x_i^*\|\|x_i\| - \sum_{i\in I}\|x_i^*\|\big|\|x_i\|-\|y_i\|\big| \\
&= 2-\sum_{i\in I}\|x_i^*\|\big|\|x_i\|-\|y_i\|\big| \overset{\text{(II.6.21)}}{\geq} 2-\alpha \geq 2(1-\tau).
\end{aligned}$$

Hence we can conclude from (II.6.18) that

$$\sum_{i\in I}\|x_i^*\|\|x_i+y_i\| \geq 2(1-\nu). \tag{II.6.22}$$

Using (II.6.21) and (II.6.22) we get

$$\begin{aligned}
&\|x+z\|_E \geq \sum_{i\in I}\|x_i^*\|\|x_i+z_i\| \\
&\geq \sum_{i\in I}\|x_i^*\|\|x_i+y_i\| - \sum_{i\in I}\|x_i^*\|\|y_i-z_i\| \\
&\geq 2(1-\nu)-\alpha > 2(1-\eta)
\end{aligned}$$

and thus the choice of η implies $x^*(z) \geq 1-\varepsilon/2$. But from (II.6.21) it also follows that $|x^*(y)-x^*(z)| \leq \alpha$ and hence $x^*(y) \geq 1-\varepsilon/2-\alpha > 1-\varepsilon$. □

Once again, because of the uniform rotundity of $\ell^p(I)$ for $1<p<\infty$ we have the following corollary.

Corollary II.6.4. *If $(X_i)_{i\in I}$ is a family of Banach spaces such that for every $0<\varepsilon\leq 2$ we have $\inf_{i\in I}\delta_{\mathrm{uacs}}^{X_i}(\varepsilon)>0$, then $\left[\bigoplus_{i\in I}X_i\right]_p$ is also uacs for every $1<p<\infty$.*

We can also get a more general corollary for a US space E.

Corollary II.6.5. *If $(X_i)_{i\in I}$ is a family of Banach spaces such that for every $0<\varepsilon\leq 2$ we have $\delta(\varepsilon):=\inf_{i\in I}\delta_{\mathrm{uacs}}^{X_i}(\varepsilon)>0$ and if $E\in\mathcal{E}_I$ is US then $\left[\bigoplus_{i\in I}X_i\right]_E$ is also a uacs space.*

Proof. Since E is US it is reflexive and hence it cannot contain an isomorphic copy of ℓ^1. Thus by Lemma II.1.2 $\operatorname{span}\{e_i : i\in I\}$ is dense in E'.
Further, since E is US the dual space $E^*=E'$ is UR, as already mentioned in the introduction. Because the spaces X_i are uacs they are also reflexive and hence Proposition I.4.1 and the monotonicity of the functions $\delta_{\mathrm{uacs}}^{X_i}$ gives us $\inf_{i\in I}\delta_{\mathrm{uacs}}^{X_i^*}(\varepsilon)\geq\delta(\delta(\varepsilon))>0$ for every $0<\varepsilon\leq 2$.
So by Theorem II.6.3 the space $\left[\bigoplus_{i\in I}X_i^*\right]_{E'}=\left[\bigoplus_{i\in I}X_i\right]_E^*$ is uacs and hence $\left[\bigoplus_{i\in I}X_i\right]_E$ is also uacs by Proposition I.4.1. □

II.7 Sums of mluacs and msluacs spaces

There are also some results on absolute sums of msluacs and mluacs spaces which we will prove in this section. The proof of the first one uses ideas from the proof of [38, Proposition 4].

Proposition II.7.1. *If $(X_i)_{i\in I}$ is a family of msluacs (resp. mluacs) Banach spaces and if $E \in \mathcal{E}_I$ is MLUR, then $\left[\bigoplus_{i\in I} X_i\right]_E$ is also msluacs (resp. mluacs).*

Proof. Suppose all the X_i are msluacs and take two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ as well as an element $x = (x_i)_{i\in I}$ in the unit sphere of $\left[\bigoplus_{i\in I} X_i\right]_E$ such that $\|x_n + y_n - 2x\|_E \to 0$. Also, fix a sequence $(x_n^*)_{n\in\mathbb{N}}$ of norm one functionals with $x_n^*(x_n) \to 1$. We write $x_n = (x_{n,i})_{i\in I}$, $y_n = (y_{n,i})_{i\in I}$ and $x_n^* = (x_{n,i}^*)_{i\in I}$ for each $n \in \mathbb{N}$.
As we have done many times before, we conclude

$$\lim_{n\to\infty} \left(x_{n,i}^*(x_{n,i}) - \|x_{n,i}^*\|\|x_{n,i}\|\right) = 0 \quad \forall i \in I \tag{II.7.1}$$

and

$$\lim_{n\to\infty} \sum_{i\in I} \|x_{n,i}^*\|\|x_{n,i}\| = 1, \tag{II.7.2}$$

as well as

$$\lim_{n\to\infty} \|(\|x_{n,i}\| + \|y_{n,i}\|)_{i\in I}\|_E = 2. \tag{II.7.3}$$

We also have

$$\lim_{n\to\infty} \|x_{n,i} + y_{n,i} - 2x_i\| = 0 \quad \forall i \in I. \tag{II.7.4}$$

Because of Lemma II.1.1 we get

$$\|(2\|x_i\| - \|x_{n,i} + y_{n,i}\|)_{i\in I}\|_E \le \|2x - x_n - y_n\|_E$$

and hence

$$\lim_{n\to\infty} \|(2\|x_i\| - \|x_{n,i} + y_{n,i}\|)_{i\in I}\|_E = 0. \tag{II.7.5}$$

Now we put for every $n \in \mathbb{N}$

$$a_n = (a_{n,i})_{i\in I} = \left(2\|x_i\| - \frac{1}{2}(\|x_{n,i}\| + \|y_{n,i}\|)\right)_{i\in I} \tag{II.7.6}$$

and

$$b_n = (b_{n,i})_{i\in I} = \left(\|x_i\| - \frac{1}{2}\|x_{n,i} + y_{n,i}\|\right)_{i\in I}. \tag{II.7.7}$$

We then have

$$\begin{aligned} \|x_i\| &\le \|x_i\| - \frac{1}{2}\|x_{n,i} + y_{n,i}\| + \frac{1}{2}(\|x_{n,i}\| + \|y_{n,i}\|) \\ &= b_{n,i} + \frac{1}{2}(\|x_{n,i}\| + \|y_{n,i}\|) \quad \forall i \in I. \end{aligned} \tag{II.7.8}$$

If $a_{n,i} \geq 0$ then

$$|a_{n,i}| = a_{n,i} = 2\|x_i\| - \frac{1}{2}(\|x_{n,i}\| + \|y_{n,i}\|) \overset{(II.7.8)}{\leq} 2|b_{n,i}| + \frac{1}{2}(\|x_{n,i}\| + \|y_{n,i}\|)$$

and if $a_{n,i} < 0$ then

$$|a_{n,i}| = -a_{n,i} = \frac{1}{2}(\|x_{n,i}\| + \|y_{n,i}\|) - 2\|x_i\| \leq 2|b_{n,i}| + \frac{1}{2}(\|x_{n,i}\| + \|y_{n,i}\|).$$

Thus we have

$$|a_{n,i}| \leq 2|b_{n,i}| + \frac{1}{2}(\|x_{n,i}\| + \|y_{n,i}\|) \quad \forall i \in I, \forall n \in \mathbb{N}. \tag{II.7.9}$$

Using (II.7.9) and Lemma II.1.1 we deduce that

$$\begin{aligned} &\frac{1}{2}\|(\|x_{n,i}\| + \|y_{n,i}\|)_{i\in I}\|_E + 2\|b_n\|_E \geq \left\| \left(2|b_{n,i}| + \frac{1}{2}(\|x_{n,i}\| + \|y_{n,i}\|) \right)_{i\in I} \right\|_E \\ &\geq \|a_n\|_E \geq 2 - \frac{1}{2}\|(\|x_{n,i}\| + \|y_{n,i}\|)_{i\in I}\|_E \end{aligned}$$

holds for every $n \in \mathbb{N}$, which together with (II.7.3) and (II.7.5) implies that $\|a_n\|_E \to 1$. Because of the definition of the sequence $(a_n)_{n\in\mathbb{N}}$ and the fact that E is MLUR this leads to $\|2a_n - (\|x_{n,i}\| + \|y_{n,i}\|)_{i\in I}\|_E \to 0$, in other words

$$\lim_{n\to\infty} \left\| \left(\|x_i\| - \frac{1}{2}(\|x_{n,i}\| + \|y_{n,i}\|) \right)_{i\in I} \right\|_E = 0. \tag{II.7.10}$$

Again, since E is MLUR it follows from (II.7.10) that

$$\lim_{n\to\infty} \|(\|x_{n,i}\| - \|y_{n,i}\|)_{i\in I}\|_E = 0. \tag{II.7.11}$$

From (II.7.10) resp. (II.7.11) it follows that $\|x_{n,i}\| + \|y_{n,i}\| \to 2\|x_i\|$ and $\|x_{n,i}\| - \|y_{n,i}\| \to 0$ for every $i \in I$ and hence

$$\lim_{n\to\infty} \|x_{n,i}\| = \|x_i\| = \lim_{n\to\infty} \|y_{n,i}\| \quad \forall i \in I. \tag{II.7.12}$$

Since each X_i is msluacs it follows from (II.7.1), (II.7.4) and (II.7.12) that

$$\lim_{n\to\infty} (x^*_{n,i}(x_i) - \|x^*_{n,i}\|\|x_i\|) = 0 \quad \forall i \in I. \tag{II.7.13}$$

From (II.7.10) and (II.7.11) we get $\|(\|x_i\| - \|x_{n,i}\|)_{i\in I}\|_E \to 0$, which together with (II.7.2) implies

$$\lim_{n\to\infty} \sum_{i\in I} \|x^*_{n,i}\|\|x_i\| = 1. \tag{II.7.14}$$

From (II.7.13) we can infer exactly as in the proof of Proposition II.3.2 that $x^*_n(x) - \sum_{i\in I}\|x^*_{n,i}\|\|x_i\| \to 0$. By (II.7.14) this implies $x^*_n(x) \to 1$. Together with $x^*_n(x_n) \to 1$ and $\|x_n + y_n - 2x\|_E \to 0$ it follows that $x^*_n(y_n) \to 1$ and the proof is finished. The case of mluacs spaces is proved analogously. $\square$

For the record, we explicitly note the case of p-sums.

Corollary II.7.2. *If $(X_i)_{i\in I}$ is a family of msluacs (resp. mluacs) Banach spaces, then $\left[\bigoplus_{i\in I} X_i\right]_p$ is also msluacs (resp. mluacs) for every $1 < p < \infty$.*

The second result on sums of msluacs (and mluacs) spaces reads as follows.

Proposition II.7.3. *If $(X_i)_{i\in I}$ is a family of msluacs (resp. mluacs) spaces and if $E \in \mathcal{E}_I$ is sluacs (resp. luacs) and has the property (P), then $\left[\bigoplus_{i\in I} X_i\right]_E$ is also msluacs (resp. mluacs).*

Proof. We suppose that every X_i is msluacs and that E is sluacs. Let us fix sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ and $(x_n^*)_{n\in\mathbb{N}}$ as well as an element x just as in the proof of Proposition II.7.1. Exactly as in this proof we can show that

$$\lim_{n\to\infty}\left(x_{n,i}^*(x_{n,i}) - \|x_{n,i}^*\|\|x_{n,i}\|\right) = 0 \ \ \forall i \in I, \tag{II.7.15}$$

$$\lim_{n\to\infty}\sum_{i\in I}\|x_{n,i}^*\|\|x_{n,i}\| = 1 \tag{II.7.16}$$

and

$$\lim_{n\to\infty}\|x_{n,i} + y_{n,i} - 2x_i\| = 0 \ \ \forall i \in I. \tag{II.7.17}$$

From $\|x_n + y_n - 2x\|_E \to 0$ and $\|x_n\|_E = \|x\|_E = \|y_n\|_E = 1$ for every n we can infer that $\|x + x_n\|_E \to 2$ and $\|x + y_n\|_E \to 2$, which in turn implies

$$\lim_{n\to\infty}\|(\|x_{n,i}\| + \|x_i\|)_{i\in I}\|_E = 2 = \lim_{n\to\infty}\|(\|y_{n,i}\| + \|x_i\|)_{i\in I}\|_E. \tag{II.7.18}$$

Since E is sluacs it follows from (II.7.16) and (II.7.18) that

$$\lim_{n\to\infty}\sum_{i\in I}\|x_{n,i}^*\|\|x_i\| = 1 \tag{II.7.19}$$

and since E has the property (P) we also get from (II.7.18) that

$$\lim_{n\to\infty}\|x_{n,i}\| = \|x_i\| = \lim_{n\to\infty}\|y_{n,i}\| \ \ \forall i \in I. \tag{II.7.20}$$

Now we can use (II.7.15), (II.7.17), (II.7.20) and the fact that each X_i is msluacs to get

$$\lim_{n\to\infty}\left(x_{n,i}^*(x_i) - \|x_{n,i}^*\|\|x_i\|\right) = 0 \ \ \forall i \in I \tag{II.7.21}$$

and then based on (II.7.19) and (II.7.21) the rest of the proof can be carried out exactly as the proof of Proposition II.7.1. Again, the mluacs case is proved analogously. □

II.8 Sums of uacsed spaces

This section addresses sums of uacsed spaces. In [126] Smith studied sums (products in his language) of URED spaces, proving in particular that (in our language) $\left[\bigoplus_{i\in I} X_i\right]_E$ is URED if E and each X_i are URED (as always, we assume here that $\operatorname{span}\{e_i : i \in I\}$ is dense in E). By an analogous proof one can establish the following result.

Theorem II.8.1. *If $E \in \mathcal{E}_I$ is URED and $(X_i)_{i\in I}$ is a family of uacsed spaces, then $\left[\bigoplus_{i\in I} X_i\right]_E$ is also uacsed.*

Proof. Fix a non-zero element z of $\left[\bigoplus_{i\in I} X_i\right]_E$ and two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in the unit sphere of $\left[\bigoplus_{i\in I} X_i\right]_E$ such that $\|x_n + y_n\|_E \to 2$ and $x_n - y_n \in \operatorname{span}\{z\}$, say $x_n - y_n = \alpha_n z$ for each $n \in \mathbb{N}$. Also, take a sequence $(x_n^*)_{n\in\mathbb{N}}$ in the dual unit sphere of $\left[\bigoplus_{i\in I} X_i\right]_E$ such that $x_n^*(x_n) \to 1$. As usual we write $x_n = (x_{n,i})_{i\in I}$, $y_n = (y_{n,i})_{i\in I}$ and $x_n^* = (x_{n,i}^*)_{i\in I}$ for each $n \in \mathbb{N}$, as well as $z = (z_i)_{i\in I}$ and as usual we conclude

$$\lim_{n\to\infty}\left(x_{n,i}^*(x_{n,i}) - \|x_{n,i}^*\|\|x_{n,i}\|\right) = 0 \quad \forall i \in I. \tag{II.8.1}$$

As in the proof from [126] we put $f_n(i) = \|x_{n,i}\|$ and $g_n^{(\beta)}(i) = \|x_{n,i} - \beta\alpha_n z_i\|$ for all $n \in \mathbb{N}$, all $i \in I$ and every $\beta \in \{1/2, 1\}$.
Then $\|f_n\|_E = \|g_n^{(1)}\|_E = 1$ for every n and

$$\|g_n^{(1/2)}\|_E = \frac{1}{2}\|x_n + y_n\|_E \to 1. \tag{II.8.2}$$

We also have

$$\|f_n + g_n^{(1)}\|_E = \big\|(\|x_{n,i}\| + \|y_{n,i}\|)_{i\in I}\big\|_E \to 2, \tag{II.8.3}$$

as we have shown many times before, and furthermore

$$\begin{aligned}
&1 + \|g_n^{(1/2)}\|_E \geq \|f_n + g_n^{(1/2)}\|_E = \frac{1}{2}\big\|(2\|x_{n,i}\| + \|x_{n,i} + y_{n,i}\|)_{i\in I}\big\|_E \\
&\geq \frac{1}{2}\Big(\big\|(2\|x_{n,i}\| + 2\|y_{n,i}\| + \|x_{n,i} + y_{n,i}\|)_{i\in I}\big\|_E - 2\Big) \\
&\geq \frac{1}{2}(3\|x_n + y_n\|_E - 2),
\end{aligned}$$

hence

$$\|f_n + g_n^{(1/2)}\|_E \to 2. \tag{II.8.4}$$

Note that $|f_n(i) - g_n^{(\beta)}(i)| \leq \beta|\alpha_n|\|z_i\| \leq 2\|z\|_E^{-1}\|z_i\|$ for all $n \in \mathbb{N}$, all $i \in I$ and every $\beta \in \{1/2, 1\}$.
The assumption that $\operatorname{span}\{e_i : i \in I\}$ is dense in E easily implies that for each $f \in E$ the set $\big\{g \in \mathbb{R}^I : |g(i)| \leq |f(i)| \ \forall i \in I\big\}$ is a compact subset of E.

Hence we can find a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$, two elements $h^{(1)}$ and $h^{(1/2)}$ of E and an $\alpha \in \mathbb{R}$ such that

$$\|f_{n_k} - g_{n_k}^{(\beta)} - h^{(\beta)}\|_E \to 0 \ \ \forall \beta \in \{1/2, 1\} \text{ and } \alpha_{n_k} \to \alpha. \tag{II.8.5}$$

Since E is URED it follows from (II.8.2), (II.8.3), (II.8.4) and (II.8.5) together with [28, Theorem 1] that $h^{(\beta)} = 0$ for $\beta \in \{1/2, 1\}$, thus

$$\lim_{k\to\infty} (\|x_{n_k,i}\| - \|x_{n_k,i} - \beta\alpha_{n_k} z_i\|) = 0 \ \ \forall i \in I, \forall \beta \in \{1/2, 1\}. \tag{II.8.6}$$

Now let us fix an arbitrary $i_0 \in I$. We can find a subsequence $(\|x_{n_{k_j},i_0}\|)_{j\in\mathbb{N}}$ that is convergent to some $a \in \mathbb{R}$. From (II.8.6) we get that

$$\|x_{n_{k_j},i_0} - \beta\alpha_{n_{k_j}} z_{i_0}\| \to a \ \ \forall \beta \in \{1/2, 1\},$$

hence

$$\|x_{n_{k_j},i_0}\|, \|y_{n_{k_j},i_0}\| \to a \text{ and } \|x_{n_{k_j},i_0} + y_{n_{k_j},i_0}\| \to 2a.$$

Together with (II.8.1), $x_{n_{k_j},i_0} - y_{n_{k_j},i_0} \to \alpha z_{i_0}$ and Proposition I.8.2 this easily implies $x^*_{n_{k_j},i_0}(\alpha z_{i_0}) \to 0$.
The same argument works if we start with an arbitrary subsequence of $(\|x_{n_k,i_0}\|)_{k\in\mathbb{N}}$ thus we have

$$\lim_{k\to\infty} x^*_{n_k,i}(\alpha z_i) = 0 \ \ \forall i \in I. \tag{II.8.7}$$

Now fix an arbitrary $\varepsilon > 0$ and a finite subset $J \subseteq I$ such that

$$\left\| \sum_{i\in J} \alpha\|z_i\| - \alpha(\|z_i\|)_{i\in I} \right\|_E \leq \varepsilon.$$

By (II.8.7) we have for all sufficiently large k

$$\left| \sum_{i\in J} x^*_{n_k,i}(\alpha z_i) \right| \leq \varepsilon.$$

These two inequalities together easily imply $x^*_{n_k}(\alpha z) \leq 2\varepsilon$ (for sufficiently large k). Thus we have $x^*_{n_k}(\alpha z) \to 0$ and hence $x^*_{n_k}(y_{n_k}) \to 1$.
Again, the same argument works if we start with an arbitrary subsequence of $(x^*_n(y_n))_{n\in\mathbb{N}}$, thus we must have $x^*_n(y_n) \to 1$ and the proof is finished. $\square$

As usual, we note explicitly the case of p-sums.

Corollary II.8.2. *If $(X_i)_{i\in I}$ is a family of uacsed spaces, then $\left[\bigoplus_{i\in I} X_i\right]_p$ is also uacsed for every $1 < p < \infty$.*

II.9 Summary of the results on absolute sums

We finish this chapter with a little table summarising the obtained results on absolute sums of the various acs-type spaces. We always assume that $E \in \mathcal{E}_I$.

Table II.1: *Summary of the results on absolute sums*

E	X_i	$\left[\bigoplus_{i\in I} X_i\right]_E$
acs	acs	acs
luacs + (P)	luacs	luacs
luacs$^+$	luacs	luacs
luacs$^+$ + (P)	luacs$^+$	luacs$^+$
luacs$^+$ + $\ell^1 \not\subseteq E$	luacs$^+$	luacs$^+$
sluacs + (P)	sluacs	sluacs
sluacs$^+$	sluacs	sluacs
sluacs$^+$	luacs$^+$	luacs$^+$
sluacs$^+$	sluacs$^+$	sluacs$^+$
wuacs + $\ell^1 \not\subseteq E$	wuacs	wuacs
acs + I finite	uacs	uacs
(u^+)	$\inf_{i\in I} \delta_{\mathrm{uacs}}^{X_i} > 0$	uacs
US	$\inf_{i\in I} \delta_{\mathrm{uacs}}^{X_i} > 0$	uacs
MLUR	mluacs	mluacs
luacs + (P)	mluacs	mluacs
MLUR	msluacs	msluacs
sluacs + (P)	msluacs	msluacs
URED	uacsed	uacsed

III Acs-type properties in Köthe-Bochner spaces

In this chapter we will study the different types of acs properties in Köthe-Bochner spaces of vector-valued functions. The theory is in a sense analogous to the one for absolute sums that we discussed in the previous chapter. However, since we now have to deal with functions on arbitrary (σ-finite) measure spaces (and values in a Banach space) instead of the "discrete" setting from the last chapter, we have to be more careful concerning, for example, questions of measurability and convergence. For some results we will need stronger assumptions than in the previous chapter. An essential tool will be the general (and highly nontrivial) description of the dual of a Köthe-Bochner function space from [18], see Theorem III.1.3 below.

The results presented here first appeared in the author's paper [58], but some of the proofs have been simplified here compared to the presentation in [58]. These simplifications consist in using the trick from the proof of [19, Theorem 4] that we already employed in Chapter II and making use of Lemma 2 from [67] (Lemma III.1.1 below) that was not known to the author before (it was also implicitly applied in the proof of [19, Theorem 4], its proof may be found in [2, Lemma 2 on p.97], as referenced in [67]). With this Lemma not only some proofs could be simplified, the assumptions of [58, Theorem 13] (Theorem III.4.1 below) could even be weakened a little.

Let us now begin with the necessary preliminaries on Köthe-Bochner spaces.

III.1 Preliminaries on Köthe-Bochner spaces

We consider a complete, σ-finite measure space $(S, \mathcal{A}, \mu)$. For $A \in \mathcal{A}$ we denote by χ_A the characteristic function of A.

A Köthe function space over $(S, \mathcal{A}, \mu)$ is a Banach space $(E, \|\cdot\|_E)$ of real-valued measurable (i. e. $\mathcal{A}$-Borel-measurable) functions on S modulo equality μ-almost everywhere[1] such that

(i) $\chi_A \in E$ for every $A \in \mathcal{A}$ with $\mu(A) < \infty$,

[1] We will henceforth abbreviate this by μ-a. e. or simply a. e. if μ is tacitly understood.

(ii) for every $f \in E$ and every set $A \in \mathcal{A}$ with $\mu(A) < \infty$ f is μ-integrable over A,

(iii) if g is measurable and $f \in E$ such that $|g(t)| \leq |f(t)|$ μ-a. e. then $g \in E$ and $\|g\|_E \leq \|f\|_E$.

The standard examples are of course the spaces $L^p(\mu)$ for $1 \leq p \leq \infty$. A wider class of examples is provided, for instance, by Orlicz function spaces equipped with the Luxemburg norm (see [77, 78, 115] for definitions and background and in particular for results on rotundity properties of such function spaces).

Note also that the spaces $(\mathbb{R}^I, \|\cdot\|_E)$ with absolute, normalised norms that we have considered in Chapter II can be viewed as Köthe function spaces over I endowed with the counting measure (which is, of course, not necessarily σ-finite (only if I is countable)).

The theory of Köthe function spaces is closely connected to the theory of Banach lattices. Recall that a Banach lattice is a Banach space X endowed with a partial order $\leq$ such that

(i) $x, y \in X$, $x \leq y \;\Rightarrow\; x + z \leq y + z \;\; \forall z \in X$.

(ii) $x \in X$, $x \geq 0 \;\Rightarrow\; \lambda x \geq 0 \;\; \forall \lambda \in [0, \infty)$.

(iii) For all $x, y \in X$ the supremum $x \vee y$ and the infimum $x \wedge y$ of $\{x, y\}$ exist.

(iv) For all $x, y \in X$: $|x| \leq |y| \;\Rightarrow\; \|x\| \leq \|y\|$, where for every $z \in X$ the absolute value is defined by $|z| := z_+ + z_-$ with $z_+ := z \vee 0$ and $z_- := -(z \wedge 0)$ being the positive respectively negative part of z.

Every Köthe function space E is a Banach lattice when endowed with the natural order $f \leq g \iff f(t) \leq g(t)$ μ-a. e.[2]

A Banach lattice X is said to be order complete (σ-order complete) if for every net (sequence) in X which is order bounded the supremum of said net (sequence) in X exists. X is called order continuous (σ-order continuous) provided that every decreasing net (sequence) in X whose infimum is zero is norm-convergent to zero.

It is easy to see that a Köthe function space E is always σ-order complete and thus by [93, Proposition 3.1.5] E is order continuous if and only if E is σ-order continuous if and only if E is order complete and order continuous. Also, reflexivity of E implies order continuity, for any σ-order complete Banach lattice which is not σ-order continuous contains an isomorphic copy of ℓ^∞ (cf. [93, Proposition 3.1.4]).

Now we state the aforementioned Lemma 2 from [67] ([2, Lemma 2 on p.97]), which will be used repeatedly in the sequel.

[2] $f \vee g$ and $f \wedge g$ are then just the usual pointwise maximum and minimum (up to equivalence a. e.) and thus also f_+, f_- and $|f|$ have the usual meaning for functions.

Lemma III.1.1. *Let E be a Köthe function space and $(f_n)_{n\in\mathbb{N}}$ a sequence in E as well as $f \in E$ such that $\|f_n - f\|_E \to 0$. Then there exists a function $g \in E$ with $g \geq 0$, a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ and a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ in $(0,\infty)$ which decreases to 0 such that $|f_{n_k}(t) - f(t)| \leq \varepsilon_k g(t)$ a. e. for every $k \in \mathbb{N}$. In particular, $f_{n_k}(t) \to f(t)$ a. e.*

For a Köthe function space E we denote by E' the space of all measurable functions $g : S \to \mathbb{R}$ (modulo equality μ-a. e.) such that

$$\|g\|_{E'} := \sup\left\{ \int_S |fg| \,\mathrm{d}\mu : f \in B_E \right\} < \infty.$$

Then $(E', \|\cdot\|_{E'})$ is again a Köthe function space, the so called Köthe dual of E. The operator $T : E' \to E^*$ defined by

$$(Tg)(f) = \int_S fg \,\mathrm{d}\mu \quad \forall f \in E, \forall g \in E'$$

is well-defined, linear and isometric. Moreover, T is onto if and only if E is order continuous (cf. [93, p.149]), thus for order continuous E we have $E^* = E'$. This is the analogue of the description of the dual of a subspace of $\mathbb{R}^I$ with absolute normalised norm (see Section II.1), but the proof is much more involved than in the discrete setting (it relies on the Radon-Nikodým theorem, generalising the usual duality result between L^p and L^q).

For more information on Banach lattices in general and Köthe function spaces in particular, the reader is referred to [94] and [93].

Now we come to Köthe-Bochner spaces of vector-valued functions. First recall that if X is any Banach space a function $f : S \to X$ is called simple if there are finitely many disjoint measurable sets $A_1, \ldots, A_n \in \mathcal{A}$ such that $\mu(A_i) < \infty$ for all $i = 1, \ldots, n$, f is constant on each A_i and $f(t) = 0$ for every $t \in S \setminus \bigcup_{i=1}^n A_i$. The function f is said to be Bochner-measurable if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions such that $\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0$ μ-a. e. and weakly measurable if $x^* \circ f$ is measurable for every functional $x^* \in X^*$. According to Pettis' measurability theorem (cf. [93, Theorem 3.2.2]) f is Bochner-measurable if and only if f is weakly measurable and almost everywhere separably valued (i. e. there is a separable subspace $Y \subseteq X$ such that $f(t) \in Y$ μ-a. e.).

A Bochner-measurable function f is called Bochner-integrable if there is a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions such that

$$\lim_{n\to\infty} \int_S \|f(t) - f_n(t)\| \,\mathrm{d}\mu(t) = 0.$$

For $A \in \mathcal{A}$ the Bochner-integral of f over A is then defined by

$$\int_A f \,\mathrm{d}\mu := \lim_{n\to\infty} \int_A f_n \,\mathrm{d}\mu,$$

where the integral of a simple function is defined in the usual way. It can be shown that this definition is independent of the choice of the approximating sequence $(f_n)_{n\in\mathbb{N}}$, and that f is Bochner-integrable if and only if $\int_S \|f(t)\| \,\mathrm{d}\mu(t) < \infty$.

For a Köthe function space E and a Banach space X we denote by $E(X)$ the space of all Bochner-measurable functions $f : S \to X$ (modulo equality a. e.) such that $\|f(\cdot)\| \in E$. Endowed with the norm $\|f\|_{E(X)} := \|\|f(\cdot)\|\|_E$, $E(X)$ becomes a Banach space, the so called Köthe-Bochner space induced by E and X.

For $E = L^p(\mu)$ one obtains the usual Lebesgue-Bochner spaces $L^p(\mu, X)$ for $1 \leq p \leq \infty$. The space $L^1(\mu, X)$ consists precisely of the Bochner-integrable functions.

Next we will recall the important duality results for Köthe-Bochner spaces. First of all, recall that a Banach space X is said to have the Radon-Nikodým property (RNP) if the Radon-Nikodým theorem holds for X-valued measures. More precisely, a closed, bounded, convex subset $K \subseteq X$ has the RNP if for every finite measure space (Ω, Σ, ν) and every vector-valued measure $m : \Sigma \to X$ which is absolutely continuous with respect to ν and satisfies $m(A)/\nu(A) \in K$ for all $A \in \Sigma$ with $\nu(A) > 0$, there exists a function $f \in L^1(\nu, X)$ such that

$$m(A) = \int_A f(t) \,\mathrm{d}\nu(t) \quad \forall A \in \mathcal{A},$$

where the integral on the right-hand side is a Bochner-integral. The space X itself is said to have the RNP if every closed, bounded, convex subset $K \subseteq X$ has the RNP. For equivalent characterisations of the RNP in terms of martingale convergence and in terms of geometric conditions (the so called dentability), see for example [93, Theorem 3.6.7].

It is well-known that every reflexive space and every separable dual space has the RNP, but for instance c_0 does not have the RNP (see for example [93, Corollary 3.6.12 and Example 3.6.2]).

Now the first duality result for Köthe-Bochner spaces that follows from the general representation theory in [18] reads as follows.

Theorem III.1.2 (cf. [18])**.** *If E is an order continuous Köthe function space over the complete, σ-finite measure space $(S, \mathcal{A}, \mu)$ and X is a Banach space such that X^* has the RNP, then the mapping $T : E'(X^*) \to E(X)^*$ given by*

$$T(F)(f) := \int_S F(t)(f(t)) \, d\mu(t) \quad \forall f \in E(X), \forall F \in E'(X^*)$$

is an isometric isomorphism.

Thus for order continuous E we can identify $E(X)^*$ with $E'(X^*)$ provided that X^* has the RNP.

For the general case (i. e. if X^* does not necessarily have the RNP), the description is more complicated and we have to introduce some more definitions: a function $F : S \to X^*$ is called weak*-measurable if $F(\cdot)(x)$ is measurable for every $x \in X$. We define an equivalence relation on the set of all weak*-measurable functions by setting $F \sim G$ if and only if for every $x \in X$ we have $F(t)(x) = G(t)(x)$ a. e. and we write $E'(X^*, w^*)$ for the space of all (equivalence classes of) weak*-measurable functions F such that there is some $g \in E'$ with $\|F(t)\| \leq g(t)$ a. e.

A norm on $E'(X^*, w^*)$ can be defined by

$$\|L\|_{E'(X^*,w^*)} := \inf\big\{\|g\|_{E'} : g \in E' \text{ and } \exists F \in L \ \|F(t)\| \leq g(t) \text{ a. e.}\big\}.$$

Then the following deep theorem holds (it comes from the general representation theory in [18], see also the exposition in [93, Theorem 3.2.4]).

Theorem III.1.3 (cf. [18])**.** *Let E be an order continuous Köthe function space over the complete, σ-finite measure space $(S, \mathcal{A}, \mu)$ and let X be a Banach space. Then the map $V : E'(X^*, w^*) \to E(X)^*$ defined by*

$$V([F])(f) := \int_S F(t)(f(t)) \,\mathrm{d}\mu(t) \quad \forall f \in E(X), \forall [F] \in E'(X^*, w^*)$$

is an isometric isomorphism and moreover every equivalence class L in $E'(X^*, w^*)$ has a representative F such that $\|F(\cdot)\| \in E'$ and $\|L\|_{E'(X^*,w^*)} = \|\|F(\cdot)\|\|_{E'}$.

For more information about the general theory of Köthe-Bochner spaces, the reader is referred to [93].

There are a number of papers on various rotundity and smoothness properties in Köthe-Bochner spaces in general and Lebesgues-Bochner spaces in particular, see for example [19, 40, 42, 67, 80, 128] and references therein.

As we have already mentioned in Section I.1, Sirotkin proved in [123] that for $1 < p < \infty$ the Lebesgue-Bochner space $L^p(\mu, X)$ is acs resp. luacs resp. uacs whenever X has the respective property. In the next section we will study the more general case of Köthe-Bochner spaces. A crucial tool for the proofs will be Theorem III.1.3.

III.2 The property acs in Köthe-Bochner spaces

In this section we consider the case of acs spaces. Here the following result holds, which is the analogue of Proposition II.2.1.

Proposition III.2.1. *If E is an order continuous acs Köthe function space over the complete, σ-finite measure space $(S, \mathcal{A}, \mu)$ and X is an acs Banach space, then $E(X)$ is acs as well.*

Proof. The proof is also similar to that of Proposition II.2.1. First we fix two elements $f, g \in S_{E(X)}$ such that $\|f+g\|_{E(X)} = 2$ and a functional $l \in S_{E(X)^*}$ with $l(f) = 1$.
Since E is order continuous, by Theorem III.1.3 l can be represented via an element $[F] \in E'(X^*, w^*)$ such that $\|F(\cdot)\| \in E'$ and $\|\|F(\cdot)\|\|_{E'} = \|[F]\|_{E'(X^*,w^*)} = \|l\| = 1$. It follows that

$$\begin{aligned} 1 = l(f) = \int_S F(t)(f(t))\,\mathrm{d}\mu(t) \le \int_S \|F(t)\|\|f(t)\|\,\mathrm{d}\mu(t) \\ \le \|\|F(\cdot)\|\|_{E'}\|\|f(\cdot)\|\|_E = \|l\|\|f\|_{E(X)} = 1 \end{aligned}$$

and hence

$$\int_S \|F(t)\|\|f(t)\|\,\mathrm{d}\mu(t) = 1 \tag{III.2.1}$$

and

$$F(t)(f(t)) = \|F(t)\|\|f(t)\| \quad \text{a. e.} \tag{III.2.2}$$

We also have

$$\begin{aligned} 2 = \|f+g\|_{E(X)} = \|\|f(\cdot)+g(\cdot)\|\|_E \le \|\|f(\cdot)\| + \|g(\cdot)\|\|_E \\ \le \|f\|_{E(X)} + \|g\|_{E(X)} = 2 \end{aligned}$$

and thus

$$\|\|f(\cdot)\| + \|g(\cdot)\|\|_E = 2. \tag{III.2.3}$$

Since E is acs it follows from (III.2.1) and (III.2.3) that

$$\int_S \|F(t)\|\|g(t)\|\,\mathrm{d}\mu(t) = 1. \tag{III.2.4}$$

We further have

$$4 = 2\|f+g\|_{E(X)} \le \|\|f(\cdot)+g(\cdot)\| + \|f(\cdot)\| + \|g(\cdot)\|\|_E \le 4,$$

thus

$$\|\|f(\cdot)+g(\cdot)\| + \|f(\cdot)\| + \|g(\cdot)\|\|_E = 4. \tag{III.2.5}$$

Because E is acs this together with (III.2.1), (III.2.3) and (III.2.4) implies

$$\int_S \|F(t)\|\|f(t)+g(t)\|\,\mathrm{d}\mu(t) = 2. \tag{III.2.6}$$

From (III.2.1), (III.2.4) and (III.2.6) we get

$$\|F(t)\|(\|f(t)\| + \|g(t)\| - \|f(t)+g(t)\|) = 0 \quad \text{a. e.} \tag{III.2.7}$$

Now we will show that

$$F(t)(g(t)) = \|F(t)\|\|g(t)\| \quad \text{a. e.} \tag{III.2.8}$$

To this end, let us denote by N_1 resp. N_2 the null sets on which the equality from (III.2.2) resp. (III.2.7) does not hold. Let $N = N_1 \cup N_2$.
Put $B = \{t \in S \setminus N : F(t) \neq 0 \text{ and } g(t) \neq 0\}$ and $C = \{t \in B : f(t) = 0\}$. We claim that C is a null set.
To see this, define $h : S \to \mathbb{R}$ by $h(t) = \|F(t)\|$ for $t \in S \setminus C$ and $h(t) = 0$ for $t \in C$. Then h is measurable and since $h(t) \leq \|F(t)\|$ for all $t \in S$ we have $h \in E'$ with $\|h\|_{E'} \leq 1$. We also have $h(t)\|f(t)\| = \|F(t)\|\|f(t)\|$ for every $t \in S$ and hence by (III.2.1)

$$\int_S h(t)\|f(t)\| \,\mathrm{d}\mu(t) = 1,$$

which also implies $\|h\|_{E'} = 1$. Together with (III.2.3) we now get

$$\int_S h(t)\|g(t)\| \,\mathrm{d}\mu(t) = 1,$$

since E is acs. Taking into account (III.2.4) we arrive at

$$\int_S (\|F(t)\| - h(t))\|g(t)\| \,\mathrm{d}\mu(t) = 0.$$

Hence $(\|F(t)\| - h(t))\|g(t)\| = 0$ a. e. and thus C must be a null set.
Now if $t \in (S \setminus C) \cap B$ then $F(t) \neq 0$, $f(t) \neq 0$ and $g(t) \neq 0$ and $\|F(t)\|\|f(t)\| = F(t)(f(t))$ as well as

$$\|f(t) + g(t)\| = \|f(t)\| + \|g(t)\|.$$

By Lemma I.10.10 this implies

$$\left\| \frac{f(t)}{\|f(t)\|} + \frac{g(t)}{\|g(t)\|} \right\| = 2.$$

Since X is acs it follows that $\|F(t)\|\|g(t)\| = F(t)(g(t))$.
So $M := N \cup C$ is a null set with $\|F(t)\|\|g(t)\| = F(t)(g(t))$ for every $t \in S \setminus M$ and (III.2.8) is proved.
Now combining (III.2.4) and (III.2.8) we obtain

$$l(g) = \int_S F(t)(g(t)) \,\mathrm{d}\mu(t) = 1,$$

which finishes the proof. □

As a corollary we obtain again Sirotkin's result from [123] on acs Lebesgue-Bochner spaces.

Corollary III.2.2. *If X is an acs space, then $L^p(\mu, X)$ is also acs for every $p \in (1, \infty)$.*

III.3 The property luacs in Köthe-Bochner spaces

In this section we will obtain sufficient conditions for a Köthe-Bochner space to be luacs.

First let us recall Egorov's theorem (cf. [56, Theorem A, p.88]), which states that for any finite measure space $(S, \mathcal{A}, \mu)$ and every sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions on S which converges to zero pointwise μ-a. e. and each $\varepsilon > 0$ there is a set $A \in \mathcal{A}$ with $\mu(S \setminus A) \leq \varepsilon$ such that $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent to zero on A.

Now we are ready to prove the following theorem.

Theorem III.3.1. *Let E be an order continuous Köthe function space over the complete, σ-finite measure space $(S, \mathcal{A}, \mu)$ and X an luacs Banach space. If*

(a) *E is WLUR or*

(b) *E is luacs$^+$ and E' is also order continuous,*

then $E(X)$ is also luacs.

Proof. Suppose that we are given a sequence $(f_n)_{n\in\mathbb{N}}$ in $S_{E(X)}$ and an element $f \in S_{E(X)}$ such that $\|f_n + f\|_{E(X)} \to 2$ as well as a functional $l \in S_{E(X)^*}$ such that $l(f_n) \to 1$. As before, we can represent l by an element $[F] \in E'(X^*, w^*)$. We then have

$$l(f_n) = \int_S F(t)(f_n(t))\,\mathrm{d}\mu(t) \leq \int_S \|F(t)\|\|f_n(t)\|\,\mathrm{d}\mu(t) \leq 1$$

and hence

$$\lim_{n\to\infty} \int_S \|F(t)\|\|f_n(t)\|\,\mathrm{d}\mu(t) = 1. \tag{III.3.1}$$

By passing to a subsequence we may also assume that

$$\lim_{n\to\infty} (\|F(t)\|\|f_n(t)\| - F(t)(f_n(t))) = 0 \ \text{ a. e.} \tag{III.3.2}$$

We further have

$$\|f_n + f\|_{E(X)} = \|\|f_n(\cdot) + f(\cdot)\|\|_E \leq \|\|f_n(\cdot)\| + \|f(\cdot)\|\|_E \leq 2$$

and thus

$$\lim_{n\to\infty} \|\|f_n(\cdot)\| + \|f(\cdot)\|\|_E = 2. \tag{III.3.3}$$

Using again the trick from the proof of [19, Theorem 4] (see also the proof of [67, Theorem 4]) we can conclude

$$\begin{aligned}
&3\|f_n + f\|_{E(X)} - 2 \leq \|f_n + 3f\|_{E(X)} \\
&\leq \|2\|f(\cdot)\| + \|f_n(\cdot) + f(\cdot)\|\|_E \leq 2 + \|f_n + f\|_{E(X)}
\end{aligned}$$

and hence

$$\lim_{n\to\infty}\|2\|f(\cdot)\| + \|f_n(\cdot) + f(\cdot)\|\|_E = 4. \tag{III.3.4}$$

Since E is in any case luacs$^+$ we first get from (III.3.1) and (III.3.3) that

$$\int_S \|F(t)\|\|f(t)\| \,\mathrm{d}\mu(t) = 1. \tag{III.3.5}$$

Then it follows from (III.3.4) and (III.3.5) that

$$\lim_{n\to\infty}\int_S \|F(t)\|\|f_n(t) + f(t)\| \,\mathrm{d}\mu(t) = 2$$

and thus

$$\lim_{n\to\infty}\int_S \|F(t)\|(\|f_n(t)\| + \|f(t)\| - \|f_n(t) + f(t)\|) \,\mathrm{d}\mu(t) = 0.$$

So by passing to a further subsequence we may assume

$$\lim_{n\to\infty}\|F(t)\|(\|f_n(t)\| + \|f(t)\| - \|f_n(t) + f(t)\|) = 0 \;\text{ a. e.} \tag{III.3.6}$$

Next we will show that

$$F(t)(f(t)) = \|F(t)\|\|f(t)\| \;\text{ a. e.} \tag{III.3.7}$$

Since $(S, \mathcal{A}, \mu)$ is σ-finite there is an increasing sequence $(A_m)_{m\in\mathbb{N}}$ in $\mathcal{A}$ such that $\mu(A_m) < \infty$ for every $m \in \mathbb{N}$ and $\bigcup_{m=1}^{\infty} A_m = S$.
Denote by N_1 resp. N_2 the null sets on which the convergence statement from (III.3.2) resp. (III.3.6) does not hold and let $N = N_1 \cup N_2$. Put $B = \{t \in S \setminus N : F(t) \neq 0 \text{ and } f(t) \neq 0\}$ and $C = \{t \in B : \|f_n(t)\| \to 0\}$. We shall see that C is a null set.
First we define for every $m \in \mathbb{N}$ a function $a_m : S \to \mathbb{R}$ by setting $a_m(t) = \|F(t)\|$ for $t \in S \setminus (C \cap A_m)$ and $a_m(t) = 0$ for $t \in C \cap A_m$. Note that each a_m is measurable and since $|a_m(t)| \leq \|F(t)\|$ for every $t \in S$ we have $a_m \in B_{E'}$.
We have $\lim_{k\to\infty}\|F(t)\|\|f_k(t)\|\chi_{C\cap A_m}(t) = 0$ for every $t \in S$ and every $m \in \mathbb{N}$, so by Egorov's theorem we can find for every $m \in \mathbb{N}$ an increasing sequence $(B_{n,m})_{n\in\mathbb{N}}$ in $\mathcal{A}|_{A_m}$ with $\mu(A_m \setminus B_{n,m}) \leq 1/n$ and such that $(\|F(\cdot)\|\|f_k(\cdot)\|\chi_{C\cap A_m})_{k\in\mathbb{N}}$ converges uniformly to zero on each $B_{n,m}$.
It follows that $M_m := \bigcap_{n=1}^{\infty} A_m \setminus B_{n,m}$ is a null set for every $m \in \mathbb{N}$.
Let us now first suppose that (b) holds, so E' is order continuous. We have

$$\lim_{n\to\infty}\|F(t)\|\chi_{C\cap(A_m\setminus B_{n,m})}(t) = 0 \;\; \forall t \in S \setminus M_m$$

and moreover this sequence is decreasing, so the order continuity of E' implies

$$\lim_{n\to\infty}\|\|F(\cdot)\|\chi_{C\cap(A_m\setminus B_{n,m})}\|_{E'} = 0.$$

So if $m \in \mathbb{N}$ and $\varepsilon > 0$ are given we can find an index $n \in \mathbb{N}$ such that $\| \|F(\cdot)\| \chi_{C\cap(A_m \setminus B_{n,m})}\|_{E'} \leq \varepsilon$ and then, by uniform convergence, an index $k_0 \in \mathbb{N}$ such that $\|F(t)\| \|f_k(t)\| \chi_{C\cap B_{n,m}}(t) \leq \varepsilon \mu(A_m)^{-1}$ for every $t \in S$ and every $k \geq k_0$.
Then we have

$$\begin{aligned}
&\int_{C\cap A_m} \|F(t)\| \|f_k(t)\| \,\mathrm{d}\mu(t) \\
&= \int_{C\cap B_{n,m}} \|F(t)\| \|f_k(t)\| \,\mathrm{d}\mu(t) + \int_{C\cap(A_m\setminus B_{n,m})} \|F(t)\| \|f_k(t)\| \,\mathrm{d}\mu(t) \\
&\leq \int_{C\cap B_{n,m}} \frac{\varepsilon}{\mu(A_m)} \,\mathrm{d}\mu(t) + \| \|F(\cdot)\| \chi_{C\cap(A_m\setminus B_{n,m})}\|_{E'} \leq 2\varepsilon
\end{aligned}$$

for each $k \geq k_0$.
In conclusion we have

$$\lim_{k\to\infty} \int_{C\cap A_m} \|F(t)\| \|f_k(t)\| \,\mathrm{d}\mu(t) = 0 \;\; \forall m \in \mathbb{N}. \tag{III.3.8}$$

Now if (a) holds, i. e. if E is WLUR, then by (III.3.3) the sequence $(\|f_k(\cdot)\|)_{k\in\mathbb{N}}$ must be weakly convergent to $\|f(\cdot)\|$ in E and hence

$$\lim_{k\to\infty} \int_{C\cap(A_m\setminus B_{n,m})} \|F(t)\| \|f_k(t)\| \,\mathrm{d}\mu(t) = \int_{C\cap(A_m\setminus B_{n,m})} \|F(t)\| \|f(t)\| \,\mathrm{d}\mu(t)$$

for all $n, m \in \mathbb{N}$. Since $(\|f(\cdot)\| \chi_{C\cap(A_m\setminus B_{n,m})})_{n\in\mathbb{N}}$ decreases to zero a. e. the order continuity of E gives us $\lim_{n\to\infty} \| \|f(\cdot)\| \chi_{C\cap(A_m\setminus B_{n,m})}\|_E = 0$ for every $m \in \mathbb{N}$.
A similiar argument as before now easily yields that (III.3.8) also holds in case (a). But (III.3.8) is nothing else than

$$\lim_{n\to\infty} \int_S (\|F(t)\| - a_m(t)) \|f_n(t)\| \,\mathrm{d}\mu(t) = 0 \;\; \forall m \in \mathbb{N}.$$

Combinig this with (III.3.1) leaves us with

$$\lim_{n\to\infty} \int_S a_m(t) \|f_n(t)\| \,\mathrm{d}\mu(t) = 1 \;\; \forall m \in \mathbb{N}.$$

Since E is luacs and because of (III.3.3) it follows that

$$\int_S a_m(t) \|f(t)\| \,\mathrm{d}\mu(t) = 1 \;\; \forall m \in \mathbb{N}.$$

Taking into account (III.3.5) we get

$$\int_S (\|F(t)\| - a_m(t)) \|f(t)\| \,\mathrm{d}\mu(t) = 0 \;\; \forall m \in \mathbb{N}$$

and hence for every $m \in \mathbb{N}$ we have $(\|F(t)\| - a_m(t))\|f(t)\| = 0$ a. e. Consequently, $C \cap A_m$ is a null set for every m and thus $C = \bigcup_{m=1}^{\infty} C \cap A_m$ is also a null set.
Now suppose that $t \in (S \setminus C) \cap B$. Then we have $F(t) \neq 0$, $f(t) \neq 0$ and $\|f_n(t)\| \not\to 0$, as well as $\|F(t)\|\|f_n(t)\| - F(t)(f_n(t)) \to 0$ and

$$\lim_{n\to\infty} (\|f_n(t)\| + \|f(t)\| - \|f_n(t) + f(t)\|) = 0.$$

We can pass to a subsequence $(n_k)_{k\in\mathbb{N}}$ of indices (depending on t) such that $(\|f_{n_k}(t)\|)_{k\in\mathbb{N}}$ is bounded away from zero. Then it follows from Lemma I.10.10 that

$$\lim_{k\to\infty} \left\| \frac{f_{n_k}(t)}{\|f_{n_k}(t)\|} + \frac{f(t)}{\|f(t)\|} \right\| = 2.$$

Also, we have

$$\lim_{k\to\infty} \frac{F(t)}{\|F(t)\|} \left(\frac{f_{n_k}(t)}{\|f_{n_k}(t)\|} \right) = 1.$$

Since X is luacs we can conclude that $F(t)(f(t)) = \|F(t)\|\|f(t)\|$.
So $M := N \cup C$ is a null set with $F(t)(f(t)) = \|F(t)\|\|f(t)\|$ for every $t \in S \setminus M$ and (III.3.7) is proved.
From (III.3.5) and (III.3.7) it follows that

$$l(f) = \int_S F(t)(f(t)) \, \mathrm{d}\mu(t) = 1$$

and we are done. □

Let us remark that in case (a) of the above Theorem, the assumption of order continuity is actually not necessary, since every WLUR Köthe function space is automatically order continuous (see the beginning of the proof of Theorem III.9.1).

We also remark that the above result contains in particular Sirotkin's result on luacs Lebesgue-Bochner spaces from [123].

Corollary III.3.2. *If X is an luacs space, then $L^p(\mu, X)$ is also luacs for every $p \in (1, \infty)$.*

III.4 The property luacs$^+$ in Köthe-Bochner spaces

Here we will discuss two results on luacs$^+$ Köthe-Bochner spaces. The first one is the following.

Theorem III.4.1. *If E is an LUR Köthe function space over the complete, σ-finite measure space $(S, \mathcal{A}, \mu)$ and X is an luacs$^+$ Banach space, then $E(X)$ is also an luacs$^+$ space.*

Proof. Since E is LUR it has the Kadets-Klee property (see Section I.7 for the definition) and hence it is order continuous (cf. [94, p.28]). Thus we can apply the representation Theorem III.1.3.
By Theorem III.3.1 we already know that $E(X)$ is luacs, so we only have to show the implication "$\Leftarrow$" in Definition I.6.1 (i). To this end, let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $S_{E(X)}$ and $f \in S_{E(X)}$ such that $\|f_n + f\|_{E(X)} \to 2$ and let $l \in S_{E(X)^*}$ such that $l(f) = 1$. It will be enough to show that a subsequence of $(l(f_n))_{n\in\mathbb{N}}$ converges to one.
Since E is order continuous we can as before represent l by some $[F] \in E'(X^*, w^*)$ and conclude

$$\int_S \|F(t)\|\|f(t)\| \,\mathrm{d}\mu(t) = 1 \tag{III.4.1}$$

and

$$\|F(t)\|\|f(t)\| = F(t)(f(t)) \quad \text{a.e.} \tag{III.4.2}$$

Also, just as we have done in the proof of Theorem III.3.1, we find that

$$\lim_{n\to\infty} \|\|f_n(\cdot)\| + \|f(\cdot)\|\|_E = 2 \tag{III.4.3}$$

and

$$\lim_{n\to\infty} \|2\|f(\cdot)\| + \|f_n(\cdot) + f(\cdot)\|\|_E = 4. \tag{III.4.4}$$

Since E is LUR it follows that

$$\lim_{n\to\infty} \|\|f_n(\cdot)\| - \|f(\cdot)\|\|_E = 0, \tag{III.4.5}$$

$$\lim_{n\to\infty} \|\|f_n(\cdot) + f(\cdot)\| - 2\|f(\cdot)\|\|_E = 0. \tag{III.4.6}$$

By (III.4.5) and (III.4.1) we also have

$$\lim_{n\to\infty} \int_S \|F(t)\|\|f_n(t)\| \,\mathrm{d}\mu(t) = 1. \tag{III.4.7}$$

It follows from (III.4.5) and Lemma III.1.1 that by passing to a subsequence we may assume that there exists a function $h \in E, h \geq 0$ such that

$$|\|f_n(t)\| - \|f(t)\|| \leq h(t) \quad \text{a.e. } \forall n \in \mathbb{N} \tag{III.4.8}$$

and

$$\lim_{n\to\infty} \|f_n(t)\| = \|f(t)\| \quad \text{a.e.} \tag{III.4.9}$$

Because of (III.4.6) we can pass to a further subsequence such that we also have

$$\lim_{n\to\infty} \|f_n(t) + f(t)\| = 2\|f(t)\| \quad \text{a.e.} \tag{III.4.10}$$

Since X is luacs$^+$ it follows from (III.4.2), (III.4.9) and (III.4.10) that

$$\lim_{n\to\infty}(\|F(t)\|\|f_n(t)\| - F(t)(f_n(t))) = 0 \quad \text{a. e.} \tag{III.4.11}$$

Because of (III.4.8) we have

$$|\|F(t)\|\|f_n(t)\| - F(t)(f_n(t))| \leq 2\|F(t)\|(h(t) + \|f(t)\|) \quad \text{a. e.}$$

and the function $\|F(\cdot)\|(h + \|f(\cdot)\|)$ lies in $L^1(\mu)$. Thus Lebesgue's theorem together with (III.4.11) implies

$$\lim_{n\to\infty}\int_S (\|F(t)\|\|f_n(t)\| - F(t)(f_n(t)))\,\mathrm{d}\mu(t) = 0$$

and because of (III.4.7) it follows that

$$l(f_n) = \int_S F(t)(f_n(t))\,\mathrm{d}\mu(t) \to 1,$$

finishing the proof. □

Let us note the special case of Lebesgue-Bochner spaces.

Corollary III.4.2. *If X is an luacs$^+$ space, then $L^p(\mu, X)$ is also luacs$^+$ for every $p \in (1, \infty)$.*

The second result on luacs$^+$ Köthe-Bochner spaces reads as follows.

Theorem III.4.3. *Let E be a Köthe function space over the complete, σ-finite measure space $(S, \mathcal{A}, \mu)$. If E is luacs$^+$, reflexive and has the Kadets-Klee property, then $E(X)$ is luacs$^+$ whenever X is luacs$^+$.*

By Theorem III.3.1 we already know that, under the above assumptions, $E(X)$ is luacs (note that, since E is reflexive, both E and E' are order continuous). Further note that the assumptions imply that E is actually wuacs, by Proposition I.10.6. With this in mind, the proof of the other implication in the definition of luacs$^+$ spaces could be carried out analogously to the proof of Theorem III.5.1 from the next section (it is the analogue of Theorem III.4.3 for wuacs spaces). Theorem III.5.1 will be proved in detail below, and thus, for the sake of brevity, we skip the detailed proof of Theorem III.4.3.

III.5 The property wuacs in Köthe-Bochner spaces

In this section we will prove the following result concerning wuacs Köthe-Bochner spaces that was already mentioned above.

Theorem III.5.1. *Let E be a Köthe function space over the complete, σ-finite measure space $(S, \mathcal{A}, \mu)$. If E is wuacs, reflexive and has the Kadets-Klee property, then $E(X)$ is wuacs whenever X is wuacs.*

Proof. Note that since E is reflexive (or since it has the Kadets-Klee property), it is order continuous.
Let us take two sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in the unit sphere of $E(X)$ such that $\|f_n + g_n\|_{E(X)} \to 2$ and a functional $l \in S_{E(X)^*}$, which can be represented as before by an element $[F] \in E'(X^*, w^*)$, with $l(f_n) \to 1$. Once again, it will be enough to show that a subsequence of $(l(g_n))_{n\in\mathbb{N}}$ converges to one.
As in the proof of Theorem III.3.1 we find

$$\lim_{n\to\infty} \int_S \|F(t)\|\|f_n(t)\| \,\mathrm{d}\mu(t) = 1 \tag{III.5.1}$$

and by passing to a subsequence also

$$\lim_{n\to\infty} (\|F(t)\|\|f_n(t)\| - F(t)(f_n(t))) = 0 \ \text{ a. e.} \tag{III.5.2}$$

Using similar arguments as, for example, in the proof of Proposition III.2.1, it is also easy to see that

$$\lim_{n\to\infty} \|\|f_n(\cdot)\| + \|g_n(\cdot)\|\|_E = 2 \tag{III.5.3}$$

and

$$\lim_{n\to\infty} \|\|f_n(\cdot)\| + \|g_n(\cdot)\| + \|f_n(\cdot) + g_n(\cdot)\|\|_E = 4. \tag{III.5.4}$$

Since E is wuacs it follows from (III.5.1) and (III.5.3) that

$$\lim_{n\to\infty} \int_S \|F(t)\|\|g_n(t)\| \,\mathrm{d}\mu(t) = 1. \tag{III.5.5}$$

Again since E is wuacs and because of (III.5.1), (III.5.3), (III.5.4) and (III.5.5) we can deduce that

$$\lim_{n\to\infty} \int_S \|F(t)\|(\|f_n(t)\| + \|g_n(t)\| - \|f_n(t) + g_n(t)\|) \,\mathrm{d}\mu(t) = 0 \tag{III.5.6}$$

and hence we can pass to a further subsequence such that

$$\lim_{n\to\infty} \|F(t)\|(\|f_n(t)\| + \|g_n(t)\| - \|f_n(t) + g_n(t)\|) = 0 \ \text{ a. e.} \tag{III.5.7}$$

By the reflexivity of E we can pass once more to a subsequence such that $(\|f_n(\cdot)\|)_{n\in\mathbb{N}}$ and $(\|g_n(\cdot)\|)_{n\in\mathbb{N}}$ are weakly convergent to $h_1 \in B_E$ resp. $h_2 \in B_E$. In view of (III.5.1) and (III.5.5) it follows that

$$\int_S \|F(t)\| h_i(t) \,\mathrm{d}\mu(t) = 1 \ \ \forall i \in \{1, 2\},$$

hence $\|h_1\|_E = \|h_2\|_E = 1$ and moreover

$$\|h_1 + h_2\|_E = 2. \tag{III.5.8}$$

The fact that E has the Kadets-Klee property implies that

$$\|\|f_n(\cdot)\| - h_1\|_E \to 0 \quad \text{and} \quad \|\|g_n(\cdot)\| - h_2\|_E \to 0.$$

Thus by Lemma III.1.1 we can, for the last time, pass to a subsequence such that there exist functions $\varphi, \psi \in E$, $\varphi, \psi \geq 0$ such that

$$|\|f_n(\cdot)\| - h_1| \leq \varphi \quad \text{and} \quad |\|g_n(\cdot)\| - h_2| \leq \psi \quad \text{a. e. } \forall n \in \mathbb{N} \tag{III.5.9}$$

as well as

$$\lim_{n\to\infty} \|f_n(t)\| = h_1(t) \quad \text{and} \quad \lim_{n\to\infty} \|g_n(t)\| = h_2(t) \quad \text{a. e.} \tag{III.5.10}$$

Now let N_1 resp. N_2 resp. N_3 denote the null sets on which the convergence statement from (III.5.2) resp. (III.5.7) resp. (III.5.10) does not hold and put $N = N_1 \cup N_2 \cup N_3$ as well as $B = \{t \in S \setminus N : F(t) \neq 0 \text{ and } h_2(t) \neq 0\}$ and $C = \{t \in B : h_1(t) = 0\}$.
Because of (III.5.8) and since E is in particular acs we can show just as in the proof of Proposition III.2.1 that C is a null set.
The fact that X is wuacs together with Lemma I.10.10 easily implies that

$$\lim_{n\to\infty} (\|F(t)\|\|g_n(t)\| - F(t)(g_n(t))) = 0 \quad \forall t \in S \setminus (N \cup C). \tag{III.5.11}$$

It follows from (III.5.9) that

$$|\|F(t)\|\|g_n(t)\| - F(t)(g_n(t))| \leq 2\|F(t)\|(\psi(t) + h_2(t)) \quad \text{a. e. } \forall n \in \mathbb{N}$$

and $\|F(\cdot)\|(\psi + h_2) \in L^1(\mu)$. Thus it follows from (III.5.11) via Lebesgue's Theorem that

$$\lim_{n\to\infty} \int_S (\|F(t)\|\|g_n(t)\| - F(t)(g_n(t)))\, \mathrm{d}\mu(t) = 0.$$

Because of (III.5.5) we get

$$\lim_{n\to\infty} l(g_n) = \lim_{n\to\infty} \int_S F(t)(g_n(t))\, \mathrm{d}\mu(t) = 1$$

and we are done. □

Again we explicitly note the important special case of Lebesgue-Bochner spaces.

Corollary III.5.2. *If X is a wuacs space, then $L^p(\mu, X)$ is also wuacs for every $p \in (1, \infty)$.*

III.6 The property sluacs in Köthe-Bochner spaces

This section is devoted to the study of sluacs Köthe-Bochner spaces. First we introduce an auxiliary modulus β_X that will be needed in the proof of Theorem III.6.2. An easy normalisation argument shows that a Banach space X is sluacs if and only if for every $x \in S_X$, every sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} and all sequences $(x_n)_{n\in\mathbb{N}}$ in X with $\|x_n+x\| \to 2$, $\|x_n\| \to 1$ and $x_n^*(x_n) \to 1$ we have $x_n^*(x) \to 1$ (we have already used such normalisation arguments before without mentioning them explicitly). In view of this characterisation, X is sluacs if and only if for every $x \in S_X$ and every $0 < \varepsilon \leq 2$ the number

$$\beta_X(x,\varepsilon) := \inf\left\{\max\left\{1-\left\|\frac{x+y}{2}\right\|, |\|y\|-1|, |x^*(y)-1|\right\} : (y,x^*) \in V_{x,\varepsilon}\right\}$$

is strictly positive, where

$$V_{x,\varepsilon} := \{(y,x^*) \in X \times S_{X^*} : x^*(y-x) \geq \varepsilon\}.$$

Next we will prove an easy Lemma on the continuity of β_X.

Lemma III.6.1. *For all $0 < \varepsilon, \tilde{\varepsilon}, \leq 2$ and all $x, \tilde{x} \in S_X$ we have*

$$|\beta_X(x,\varepsilon) - \beta_X(\tilde{x},\tilde{\varepsilon})| \leq \|x-\tilde{x}\| + |\varepsilon - \tilde{\varepsilon}|,$$

i. e. β_X is 1-Lipschitz continuous with respect to the norm of $X \oplus_1 \mathbb{R}$.

Proof. First we fix $0 < \varepsilon \leq 2$ and $x, \tilde{x} \in S_X$. Put $\delta = \|x - \tilde{x}\|$ and take $y \in X$, $x^* \in S_{X^*}$ such that $x^*(y-x) \geq \varepsilon$. It follows that $x^*(y-\tilde{x}) \geq \varepsilon - \delta$. Now let $0 < \tau < 1$ be arbitrary. We can find $z \in S_X$ with $x^*(z) \geq 1-\tau$. Define $\tilde{y} = y + \delta(1-\tau)^{-1}z$. Then

$$x^*(\tilde{y}-\tilde{x}) = \frac{\delta}{1-\tau}x^*(z) + x^*(y-\tilde{x}) \geq \delta + x^*(y-\tilde{x}) = \varepsilon$$

and hence

$$\max\left\{1-\left\|\frac{\tilde{x}+\tilde{y}}{2}\right\|, |\|\tilde{y}\|-1|, |x^*(\tilde{y})-1|\right\} \geq \beta_X(\tilde{x},\varepsilon).$$

But we have $|\|\tilde{y}\| - \|y\|| \leq \|y-\tilde{y}\| = \delta(1-\tau)^{-1}$ and $|x^*(\tilde{y}) - x^*(y)| \leq \|y-\tilde{y}\| = \delta(1-\tau)^{-1}$ as well as

$$\left|\left\|\frac{x+y}{2}\right\| - \left\|\frac{\tilde{x}+\tilde{y}}{2}\right\|\right| \leq \frac{1}{2}(\|x-\tilde{x}\| + \|y-\tilde{y}\|) = \frac{1}{2}\left(\delta + \frac{\delta}{1-\tau}\right) \leq \frac{\delta}{1-\tau}.$$

Thus we get

$$\max\left\{1-\left\|\frac{x+y}{2}\right\|, |\|y\|-1|, |x^*(y)-1|\right\} \geq \beta_X(\tilde{x},\varepsilon) - \frac{\delta}{1-\tau}$$

and since $0 < \tau < 1$ was arbitrary it follows that

$$\max\left\{1 - \left\|\frac{x+y}{2}\right\|, |\|y\| - 1|, |x^*(y) - 1|\right\} \geq \beta_X(\tilde{x}, \varepsilon) - \delta.$$

Again, since $(y, x^*) \in V_{x,\varepsilon}$ was arbitrary we can conclude that

$$\beta_X(\tilde{x}, \varepsilon) - \beta_X(x, \varepsilon) \leq \delta = \|x - \tilde{x}\|$$

and by symmetry it follows that

$$|\beta_X(\tilde{x}, \varepsilon) - \beta_X(x, \varepsilon)| \leq \|x - \tilde{x}\|.$$

Analogously one can prove that

$$|\beta_X(x, \tilde{\varepsilon}) - \beta_X(x, \varepsilon)| \leq |\varepsilon - \tilde{\varepsilon}|$$

for all $x \in S_X$ and all $0 < \varepsilon, \tilde{\varepsilon}, \leq 2$. An application of the triangle inequality then yields the result. □

In the paper [80] various theorems concerning different rotundity properties of Köthe-Bochner spaces are proved. For example, by [80, Theorem 5] if E has the so called Fatou property and is LUR then $E(X)$ is LUR whenever X is LUR. In [19, Theorem 4 and Corollary 1] it was proved that this result also holds without the assumption of the Fatou property. Also, the proof from [19] is much shorter than the one from [80].[3] However, this technique seems not to be applicable to the case that E is LUR and X is sluacs (at least not without imposing further assumptions on E and E^*, see also Theorem III.6.4 below). So we will adopt here the technique of proof from [80, Theorem 5] to show the following result.

Theorem III.6.2. *If E is an LUR Köthe function space over the complete, σ-finite measure space $(S, \mathcal{A}, \mu)$ and X is an sluacs Banach space, then $E(X)$ is also sluacs.*

Proof. Since E is LUR it is order continuous (see the beginning of the proof of Theorem III.4.1).
Let $0 < \varepsilon \leq 2$ and $f \in S_{E(X)}$ be arbitrary and let

$$A_n := \left\{t \in S : f(t) \neq 0 \text{ and } \beta_X\left(\frac{f(t)}{\|f(t)\|}, \frac{\varepsilon}{8}\right) \geq \frac{1}{n}\right\}$$

for every $n \in \mathbb{N}$. Since by Lemma III.6.1 $\beta_X(\cdot, \varepsilon/8)$ is continuous it follows that the sets A_n are measurable. Also, the sequence $(A_n)_{n\in\mathbb{N}}$ is increasing and because X is sluacs we have $\bigcup_{n=1}^{\infty} A_n = \{t \in S : f(t) \neq 0\}$, hence

[3] It implicitly uses Lemma 2 from [67] ([2, Lemma 2 on p.97], Lemma III.1.1 in our notation).

$(\|f(\cdot)\|\chi_{S\setminus A_n})_{n\in\mathbb{N}}$ decreases pointwise to zero. The order continuity of E implies $\|\|f(\cdot)\|\chi_{S\setminus A_n}\|_E \to 0$ and thus we can find $n_0 \in \mathbb{N}$ with

$$\|\|f(\cdot)\|\chi_{S\setminus A_{n_0}}\|_E \leq \frac{\varepsilon}{64}. \tag{III.6.1}$$

Now let us take $g \in S_{E(X)}$ and $l \in S_{E(X)^*}$ with $l(g) = 1$ and $l(f) \leq 1 - \varepsilon$. Let l be represented by $[F] \in E'(X^*, w^*)$. As in the proof of Proposition III.2.1 we can conclude

$$\int_S \|F(t)\|\|g(t)\| \,\mathrm{d}\mu(t) = 1 \tag{III.6.2}$$

and

$$\|F(t)\|\|g(t)\| = F(t)(g(t)) \text{ a. e.} \tag{III.6.3}$$

Next we define

$$\begin{aligned} &C := \{t \in S : F(t) \neq 0\} \text{ and} \\ &B := \Big\{t \in C : F(t)(g(t) - f(t)) \geq \frac{\varepsilon}{4}\|F(t)\| \max\{\|f(t)\|, \|g(t)\|\}\Big\}. \end{aligned}$$

Then B is measurable and

$$\begin{aligned} &\int_{S\setminus B} F(t)(g(t) - f(t)) \,\mathrm{d}\mu(t) \leq \frac{\varepsilon}{4}\int_{S\setminus B} \|F(t)\| \max\{\|f(t)\|, \|g(t)\|\} \,\mathrm{d}\mu(t) \\ &\leq \frac{\varepsilon}{4}\int_{S\setminus B} \|F(t)\|(\|f(t)\| + \|g(t)\|) \,\mathrm{d}\mu(t) \leq \frac{\varepsilon}{4}2 = \frac{\varepsilon}{2}. \end{aligned}$$

Since $l(g - f) \geq \varepsilon$ it follows that

$$\int_B F(t)(g(t) - f(t)) \,\mathrm{d}\mu(t) \geq \frac{\varepsilon}{2}. \tag{III.6.4}$$

Let us fix $0 < \eta < \min\{\varepsilon/16, 1/2n_0\}$ such that

$$\frac{\eta}{1-\eta} < \frac{2}{n_0}. \tag{III.6.5}$$

Now consider the sets

$$\begin{aligned} B_1 &:= \{t \in B : \|g(t)\| < (1-\eta)\|f(t)\|\}, \\ B_2 &:= \{t \in B : (1-\eta)\|f(t)\| \leq \|g(t)\| \leq \|f(t)\|\}, \\ B_3 &:= \{t \in B : (1-\eta)\|g(t)\| \leq \|f(t)\| < \|g(t)\|\}, \\ B_4 &:= \{t \in B : (1-\eta)\|g(t)\| > \|f(t)\|\}. \end{aligned}$$

Then $B_1, \ldots, B_4$ are measurable, pairwise disjoint and $\bigcup_{i=1}^4 B_i = B$. Thus by (III.6.4) there exists some $i \in \{1, \ldots, 4\}$ such that

$$\int_{B_i} F(t)(g(t) - f(t)) \,\mathrm{d}\mu(t) \geq \frac{\varepsilon}{8}.$$

If $i = 1$ then, since $\|g(t)\| \leq \|f(t)\|$ for $t \in B_1$, it follows that

$$\int_{B_1} \|F(t)\| \|f(t)\| \,\mathrm{d}\mu(t) \geq \frac{\varepsilon}{16}$$

and again by the definition of B_1 we obtain

$$\begin{aligned} &\| \|g(\cdot)\| - \|f(\cdot)\| \|_E = \| | \|g(\cdot)\| - \|f(\cdot)\| | \|_E \\ &\geq \int_{B_1} \|F(t)\| (\|f(t)\| - \|g(t)\|) \,\mathrm{d}\mu(t) \geq \eta \int_{B_1} \|F(t)\| \|f(t)\| \,\mathrm{d}\mu(t) \geq \eta \frac{\varepsilon}{16} \end{aligned}$$

and hence

$$\left\| \frac{f+g}{2} \right\|_{E(X)} \leq \left\| \frac{\|f(\cdot)\| + \|g(\cdot)\|}{2} \right\|_E \leq 1 - \delta_E\Big(\|f(\cdot)\|, \eta \frac{\varepsilon}{16}\Big),$$

where δ_E denotes the modulus of local uniform rotundity of E (see the definition in Section I.1).

In the case $i = 4$ one can obtain the same statement by an analogous argument.

To treat the remaining cases we need some preliminary considerations.

Let us denote by N the null set on which the equality from (III.6.3) does not hold and suppose that $t \in B_2 \cap A_{n_0} \cap (S \setminus N)$. Then in particular $t \in B$ and $\|f(t)\| \geq \|g(t)\|$ and hence

$$\frac{F(t)}{\|F(t)\|} \left(\frac{g(t)}{\|f(t)\|} - \frac{f(t)}{\|f(t)\|} \right) \geq \frac{\varepsilon}{4}.$$

Moreover, by the definitions of B_2 and A_{n_0} and the choice of η we have

$$\begin{aligned} \left| \left\| \frac{g(t)}{\|f(t)\|} \right\| - 1 \right| &= \left| \frac{\|g(t)\|}{\|f(t)\|} - 1 \right| \leq \eta < \frac{1}{n_0} \\ &\leq \beta_X\left(\frac{f(t)}{\|f(t)\|}, \frac{\varepsilon}{8} \right) \leq \beta_X\left(\frac{f(t)}{\|f(t)\|}, \frac{\varepsilon}{4} \right). \end{aligned}$$

Since $t \in (S \setminus N)$ we also have

$$\left| \frac{F(t)}{\|F(t)\|} \left(\frac{g(t)}{\|f(t)\|} \right) - 1 \right| = \left| \frac{\|g(t)\|}{\|f(t)\|} - 1 \right| < \beta_X\left(\frac{f(t)}{\|f(t)\|}, \frac{\varepsilon}{4} \right).$$

So by the definition of β_X we must have

$$\frac{1}{2} \left\| \frac{f(t)}{\|f(t)\|} + \frac{g(t)}{\|f(t)\|} \right\| \leq 1 - \beta_X\left(\frac{f(t)}{\|f(t)\|}, \frac{\varepsilon}{4} \right) \leq 1 - \frac{1}{n_0}.$$

Once more by the definition of B_1 this implies

$$\begin{aligned} &\left\| \frac{f(t) + g(t)}{2} \right\| \leq \left(1 - \frac{1}{n_0} \right) \|f(t)\| \leq \frac{1 - 1/n_0}{2(1-\eta)} (\|f(t)\| + \|g(t)\|) \\ &= \frac{1}{2}(1 - \alpha_1)(\|f(t)\| + \|g(t)\|), \end{aligned}$$

where $\alpha_1 := (1/n_0 - \eta)(1-\eta)^{-1} > 0$.
Now suppose that $t \in B_3 \cap A_{n_0} \cap (S \setminus N)$. Then

$$\frac{F(t)}{\|F(t)\|}\left(\frac{g(t)}{\|g(t)\|} - \frac{f(t)}{\|g(t)\|}\right) \geq \frac{\varepsilon}{4},$$

consequently

$$\begin{aligned}\frac{F(t)}{\|F(t)\|}\left(\frac{g(t)}{\|g(t)\|} - \frac{f(t)}{\|f(t)\|}\right) &\geq \frac{\varepsilon}{4} + \frac{F(t)}{\|F(t)\|}\left(\frac{f(t)}{\|g(t)\|} - \frac{f(t)}{\|f(t)\|}\right)\\ \geq \frac{\varepsilon}{4} - \left\|\frac{f(t)}{\|g(t)\|} - \frac{f(t)}{\|f(t)\|}\right\| &= \frac{\varepsilon}{4} - \left|\frac{\|f(t)\|}{\|g(t)\|} - 1\right| \geq \frac{\varepsilon}{4} - \eta \geq \frac{\varepsilon}{8}.\end{aligned}$$

Since $\|F(t)\|\|g(t)\| = F(t)(g(t))$ the definition of β_X implies that

$$\frac{1}{2}\left\|\frac{f(t)}{\|f(t)\|} + \frac{g(t)}{\|g(t)\|}\right\| \leq 1 - \beta_X\left(\frac{f(t)}{\|f(t)\|}, \frac{\varepsilon}{8}\right) \leq 1 - \frac{1}{n_0},$$

where the latter inequality holds because of $t \in A_{n_0}$. It follows that

$$\begin{aligned}&\frac{1}{2}\left\|\frac{f(t)}{\|f(t)\|} + \frac{g(t)}{\|f(t)\|}\right\| \leq 1 - \frac{1}{n_0} + \frac{1}{2}\left\|\frac{g(t)}{\|f(t)\|} - \frac{g(t)}{\|g(t)\|}\right\|\\ &= 1 - \frac{1}{n_0} + \frac{1}{2}\left|\frac{\|g(t)\|}{\|f(t)\|} - 1\right| \leq 1 - \frac{1}{n_0} + \frac{1}{2}\left(\frac{1}{1-\eta} - 1\right) = 1 - \alpha_2,\end{aligned}$$

where $\alpha_2 := 1/n_0 - \eta(2-2\eta)^{-1}$, which by (III.6.5) is greater than zero. Because of $\|f(t)\| \leq \|g(t)\|$ it follows that

$$\|f(t) + g(t)\| \leq (1 - \alpha_2)(\|f(t)\| + \|g(t)\|).$$

So if we put $\alpha = \min\{\alpha_1, \alpha_2\}$ and $P = B_2 \cap A_{n_0} \cap (S \setminus N)$, $Q = B_3 \cap A_{n_0} \cap (S \setminus N)$ then

$$\|f(t) + g(t)\| \leq (1 - \alpha)(\|f(t)\| + \|g(t)\|) \quad \forall t \in P \cup Q. \tag{III.6.6}$$

Now we will show that if $i = 2$ resp. $i = 3$ then

$$\int_P \|F(t)\|\|f(t)\| \,\mathrm{d}\mu(t) \geq \frac{\varepsilon}{64} \quad \text{resp.} \quad \int_Q \|F(t)\|\|f(t)\| \,\mathrm{d}\mu(t) \geq \frac{\varepsilon}{64}.$$

Let us first assume $i = 2$, i. e.

$$\int_{B_2} F(t)(g(t) - f(t)) \,\mathrm{d}\mu(t) \geq \frac{\varepsilon}{8}.$$

Since $\|f(t)\| \geq \|g(t)\|$ for $t \in B_2$ it follows that

$$\int_{B_2} \|F(t)\|\|f(t)\| \,\mathrm{d}\mu(t) \geq \frac{\varepsilon}{16}.$$

Because N is a null set we have

$$\int_P \|F(t)\|\|f(t)\| \,\mathrm{d}\mu(t) = \int_{B_2\cap A_{n_0}} \|F(t)\|\|f(t)\| \,\mathrm{d}\mu(t)$$
$$= \int_{B_2} \|F(t)\|\|f(t)\| \,\mathrm{d}\mu(t) - \int_{B_2\setminus A_{n_0}} \|F(t)\|\|f(t)\| \,\mathrm{d}\mu(t)$$
$$\geq \frac{\varepsilon}{16} - \int_{S\setminus A_{n_0}} \|F(t)\|\|f(t)\| \,\mathrm{d}\mu(t) \geq \frac{\varepsilon}{16} - \|\|f(\cdot)\|\chi_{S\setminus A_{n_0}}\|_E$$
$$\geq \frac{\varepsilon}{16} - \frac{\varepsilon}{64} \geq \frac{\varepsilon}{64},$$

where the second last inequality holds because of (III.6.1).
Now assume that $i = 3$, i. e.

$$\int_{B_3} F(t)(g(t) - f(t)) \,\mathrm{d}\mu(t) \geq \frac{\varepsilon}{8}.$$

It follows that

$$\frac{\varepsilon}{8} \leq \int_{B_3} \|F(t)\|(\|g(t)\| + \|f(t)\|) \,\mathrm{d}\mu(t)$$
$$\leq \int_{B_3} \|F(t)\|\Big(1 + \frac{1}{1-\eta}\Big)\|f(t)\| \,\mathrm{d}\mu(t) \leq 4\int_{B_3} \|F(t)\|\|f(t)\| \,\mathrm{d}\mu(t)$$

and hence as before we get

$$\int_Q \|F(t)\|\|f(t)\| \,\mathrm{d}\mu(t) \geq \frac{\varepsilon}{32} - \frac{\varepsilon}{64} = \frac{\varepsilon}{64}.$$

So if $i = 2$ or $i = 3$ then there is $R \in \{P, Q\}$ such that

$$\|\|f(\cdot)\|\chi_R\|_E \geq \int_R \|F(t)\|\|f(t)\| \,\mathrm{d}\mu(t) \geq \frac{\varepsilon}{64}.$$

Put $h = \|f(\cdot)\|(1 - 2\alpha\chi_R)$. Then $h \in B_E$ and moreover $\|\|f(\cdot)\| - h\|_E = 2\alpha\|\|f(\cdot)\|\chi_R\|_E \geq \alpha\varepsilon/32$, hence

$$\|\|f(\cdot)\|(1 - \alpha\chi_R)\|_E = \frac{1}{2}\|\|f(\cdot)\| + h\|_E \leq 1 - \delta_E\Big(\|f(\cdot)\|, \frac{\varepsilon\alpha}{32}\Big).$$

We further have

$$\Big\|\frac{f+g}{2}\Big\|_{E(X)} \leq \frac{1}{2}\|(\|f(\cdot)\| + \|g(\cdot)\|)\chi_{S\setminus R} + \|f(\cdot) + g(\cdot)\|\chi_R\|_E$$
$$\overset{\text{(III.6.6)}}{\leq} \frac{1}{2}\|(\|f(\cdot)\| + \|g(\cdot)\|)\chi_{S\setminus R} + (1-\alpha)(\|f(\cdot)\| + \|g(\cdot)\|)\chi_R\|_E$$
$$\leq \frac{1}{2}\|\|g(\cdot)\| + \|f(\cdot)\| - \alpha\|f(\cdot)\|\chi_R\|_E \leq \frac{1}{2} + \frac{1}{2}\|\|f(\cdot)\|(1 - \alpha\chi_R)\|_E$$
$$\leq \frac{1}{2} + \frac{1}{2}\Big(1 - \delta_E\Big(\|f(\cdot)\|, \frac{\varepsilon\alpha}{32}\Big)\Big) = 1 - \frac{1}{2}\delta_E\Big(\|f(\cdot)\|, \frac{\varepsilon\alpha}{32}\Big).$$

Altogether we have shown that for

$$\delta := \min\left\{\frac{1}{2}\delta_E\Big(\|f(\cdot)\|, \frac{\varepsilon\alpha}{32}\Big), \delta_E\Big(\|f(\cdot)\|, \frac{\varepsilon\eta}{16}\Big)\right\} > 0$$

we have for every $g \in S_{E(X)}$ and every $l \in S_{E(X)^*}$ with $l(g) = 1$ and $l(f) \leq 1 - \varepsilon$

$$\left\|\frac{f+g}{2}\right\|_{E(X)} \leq 1 - \delta.$$

By the characterisation of sluacs spaces from Proposition I.2.1 this implies that $E(X)$ is sluacs. $\square$

This result covers in particular the L^p-case.

Corollary III.6.3. *If X is an sluacs space, then $L^p(\mu, X)$ is also sluacs for every $p \in (1, \infty)$.*

Now we come to a second result concerning sufficient conditions for a Köthe-Bochner space to be sluacs.

Theorem III.6.4. *If E is a Köthe function space over the complete, σ-finite measure space $(S, \mathcal{A}, \mu)$ which is sluacs$^+$, reflexive and such that both E and E^* have the Kadets-Klee property, then $E(X)$ is sluacs whenever X is sluacs.*

Proof. Since E is reflexive it is order continuous and thus we can apply Theorem III.1.3.
Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $S_{E(X)}$ and $f \in S_{E(X)}$ such that we have $\|f_n + f\|_{E(X)} \to 2$. Also, let $(l_n)_{n\in\mathbb{N}}$ be a sequence in $S_{E(X)^*}$ such that $l_n(f_n) \to 1$. If we represent each l_n by $[F_n] \in E'(X^*, w^*)$ we can obtain by the usual arguments that

$$\lim_{n\to\infty} \int_S \|F_n(t)\|\|f_n(t)\| \,\mathrm{d}\mu(t) = 1 \tag{III.6.7}$$

and by passing to a subsequence also

$$\lim_{n\to\infty} (\|F_n(t)\|\|f_n(t)\| - F_n(t)(f_n(t))) = 0 \ \text{ a. e.} \tag{III.6.8}$$

as well as

$$\lim_{n\to\infty} \|\|f_n(\cdot)\| + \|f(\cdot)\|\|_E = 2. \tag{III.6.9}$$

As in the proof of Theorem III.3.1 we can also obtain

$$\lim_{n\to\infty} \|2\|f(\cdot)\| + \|f_n(\cdot) + f(\cdot)\|\|_E = 4. \tag{III.6.10}$$

Using the fact that E is sluacs$^+$ we can conclude that

$$\lim_{n\to\infty} \int_S \|F_n(t)\|\|f(t)\| \,\mathrm{d}\mu(t) = 1 \tag{III.6.11}$$

and

$$\lim_{n\to\infty}\int_S \|F_n(t)\|(\|f_n(t)\|+\|f(t)\|-\|f_n(t)+f(t)\|)\,\mathrm{d}\mu(t)=0. \qquad \text{(III.6.12)}$$

So we can pass to another subsequence such that

$$\lim_{n\to\infty}\|F_n(t)\|(\|f_n(t)\|+\|f(t)\|-\|f_n(t)+f(t)\|)=0 \ \text{ a. e.} \qquad \text{(III.6.13)}$$

Since E (and hence also E^*) is reflexive we may assume without loss of generality that $(\|f_n(\cdot)\|)_{n\in\mathbb{N}}$ is weakly convergent to some $h\in B_E$ and that $(\|F_n(\cdot)\|)_{n\in\mathbb{N}}$ is weakly convergent to some $g\in B_{E^*}=B_{E'}$.
It follows from (III.6.11) that

$$\int_S g(t)\|f(t)\|\,\mathrm{d}\mu(t)=1 \qquad \text{(III.6.14)}$$

and hence $g\in S_{E^*}$. Because of (III.6.14), (III.6.9) and the fact that E is sluacs$^+$ we get that

$$\lim_{n\to\infty}\int_S g(t)\|f_n(t)\|\,\mathrm{d}\mu(t)=1$$

and consequently

$$\int_S g(t)h(t)\,\mathrm{d}\mu(t)=1, \qquad \text{(III.6.15)}$$

whence $h\in S_E$. Since both E and E^* have the Kadets-Klee property it follows that

$$\big\|\|f_n(\cdot)\|-h\big\|_E\to 0 \ \text{ and } \ \big\|\|F_n(\cdot)\|-g\big\|_{E'}\to 0. \qquad \text{(III.6.16)}$$

Because of Lemma III.1.1 we can pass once more to subsequences and assume that there exist functions $\varphi\in E$, $\psi\in E'$, $\varphi,\psi\geq 0$ such that

$$\big|\|f_n(\cdot)\|-h\big|\leq\varphi \ \text{ and } \ \big|\|F_n(\cdot)\|-g\big|\leq\psi \ \text{ a. e. } \forall n\in\mathbb{N} \qquad \text{(III.6.17)}$$

as well as

$$\lim_{n\to\infty}\|f_n(t)\|=h(t) \ \text{ and } \ \lim_{n\to\infty}\|F_n(t)\|=g(t) \ \text{ a. e.} \qquad \text{(III.6.18)}$$

Combining (III.6.15) and (III.6.14) we also obtain

$$\big\|h+\|f(\cdot)\|\big\|_E=2. \qquad \text{(III.6.19)}$$

Let N be a null set such that the convergence statements of (III.6.8), (III.6.13) and (III.6.18) hold for every $t\in S\setminus N$.
Put $B=\{t\in S\setminus N: g(t)\neq 0 \text{ and } f(t)\neq 0\}$ and $C=\{t\in B: h(t)=0\}$.

Similar to the arguments in the proof of Theorem III.5.1 one can see that C is a null set and then, using the fact that X is sluacs, deduce that

$$\lim_{n\to\infty} (\|F_n(t)\|\|f(t)\| - F_n(t)(f(t))) = 0 \quad \text{a. e.} \tag{III.6.20}$$

We have $\|f(\cdot)\|(\psi + g) \in L^1(\mu)$ and (because of (III.6.17))

$$|\|F_n(t)\|\|f(t)\| - F_n(t)(f(t))| \le 2\|f(t)\|(\psi(t) + g(t)) \quad \text{a. e. } \forall n \in \mathbb{N}.$$

Thus it follows from (III.6.20) and Lebesgue's Theorem that

$$\lim_{n\to\infty} \int_S (\|F_n(t)\|\|f(t)\| - F_n(t)(f(t)))\,\mathrm{d}\mu(t) = 0.$$

Taking into account (III.6.11) we arrive at

$$\lim_{n\to\infty} l_n(f) = \lim_{n\to\infty} \int_S F_n(t)(f(t))\,\mathrm{d}\mu(t) = 1$$

and the proof is finished. □

III.7 The property sluacs$^+$ in Köthe-Bochner spaces

In this section we will consider sufficient conditions for a Köthe-Bochner function space to be sluacs$^+$. The following Theorem holds (recall that a dual Banach space X^* is said to have the Kadets-Klee* property if it fulfils the definition of the Kadets-Klee property with weak- replaced by weak*-convergence).

Theorem III.7.1. *Let E be a Köthe function space over the complete, σ-finite measure space $(S, \mathcal{A}, \mu)$ and let X be an sluacs$^+$ Banach space. If E^* has the Kadets-Klee* property and in addition*

(a) *E is sluacs$^+$, reflexive and has the Kadets-Klee property or*

(b) *E is LUR and B_{E^*} is weak*-sequentially compact,*[4]

then $E(X)$ is sluacs$^+$.

Proof. The proof is similar to the previous ones. First note that by Theorems III.6.2 and III.6.4 we already know that $E(X)$ is in both cases sluacs. Note also that in both cases E is order continuous. Now take a sequence $(f_n)_{n\in\mathbb{N}}$ in $S_{E(X)}$ and $f \in S_{E(X)}$ such that $\|f_n + f\|_{E(X)} \to 2$ and let $(l_n)_{n\in\mathbb{N}}$ be a sequence in $S_{E(X)^*}$ such that $l_n(f) \to 1$. If we represent each l_n by $[F_n] \in E'(X^*, w^*)$ we can obtain as usual

$$\lim_{n\to\infty} \int_S \|F_n(t)\|\|f(t)\|\,\mathrm{d}\mu(t) = 1 \tag{III.7.1}$$

[4]For example, if E is separable.

and by passing to a subsequence also

$$\lim_{n\to\infty} (\|F_n(t)\| \|f(t)\| - F_n(t)(f(t))) = 0 \quad \text{a. e.} \tag{III.7.2}$$

as well as

$$\lim_{n\to\infty} \| \|f_n(\cdot)\| + \|f(\cdot)\| \|_E = 2. \tag{III.7.3}$$

As in the proof of Theorem III.3.1 we can also obtain

$$\lim_{n\to\infty} \| 2\|f(\cdot)\| + \|f_n(\cdot) + f(\cdot)\| \|_E = 4. \tag{III.7.4}$$

Since E is sluacs$^+$ it follows that

$$\lim_{n\to\infty} \int_S \|F_n(t)\| \|f_n(t)\| \, \mathrm{d}\mu(t) = 1 \tag{III.7.5}$$

and

$$\lim_{n\to\infty} \int_S \|F_n(t)\| (\|f_n(t)\| + \|f(t)\| - \|f_n(t) + f(t)\|) \, \mathrm{d}\mu(t) = 0, \tag{III.7.6}$$

so that by passing to another subsequence we can assume

$$\lim_{n\to\infty} \|F_n(t)\| (\|f_n(t)\| + \|f(t)\| - \|f_n(t) + f(t)\|) = 0 \quad \text{a. e.} \tag{III.7.7}$$

In both cases (a) and (b) the dual unit ball B_{E^*} is weak*-sequentially compact so that we can also assume the weak*-convergence of $(\|F_n(\cdot)\|)_{n\in\mathbb{N}}$ to some $g \in B_{E^*}$. It follows from (III.7.1) that

$$\int_S g(t) \|f(t)\| \, \mathrm{d}\mu(t) = 1 \tag{III.7.8}$$

and hence $\|g\|_{E'} = 1$. Since E^* has the Kadets-Klee* property we get that

$$\| \|F_n(\cdot)\| - g \|_{E'} \to 0. \tag{III.7.9}$$

Next we claim that there is an $h \in S_E$ such that

$$\int_S g(t) h(t) \, \mathrm{d}\mu(t) = 1 \tag{III.7.10}$$

and, possibly after passing to a subsequence once more, also

$$\| \|f_n(\cdot)\| - h \|_E \to 0. \tag{III.7.11}$$

For in the case (b) E is LUR and thus by (III.7.3) and (III.7.8) we can take $h = \|f(\cdot)\|$. In the case (a) E is reflexive and hence we can assume that $(\|f_n(\cdot)\|)_{n\in\mathbb{N}}$ is weakly convergent to some $h \in B_E$. Then (III.7.10)

follows from (III.7.9) and (III.7.5). This also implies $\|h\|_E = 1$ and by the Kadets-Klee property of E we have (III.7.11).
Because of (III.7.9), (III.7.11) and Lemma III.1.1 we can also assume that there exist $\varphi \in E$, $\psi \in E'$ with $\varphi, \psi \geq 0$ such that

$$|\|f_n(\cdot)\| - h| \leq \varphi \quad \text{and} \quad |\|F_n(\cdot)\| - g| \leq \psi \quad \text{a.e.}\ \forall n \in \mathbb{N} \tag{III.7.12}$$

as well as

$$\lim_{n\to\infty} \|f_n(t)\| = h(t) \quad \text{and} \quad \lim_{n\to\infty} \|F_n(t)\| = g(t) \quad \text{a.e.} \tag{III.7.13}$$

Note that (III.7.8) and (III.7.10) imply that $\|\|f(\cdot)\| + h\|_E = 2$. Using all this and the fact that X is sluacs$^+$ one can prove, analogously to the arguments in the proof of Theorem III.6.4, that

$$\lim_{n\to\infty} (\|F_n(t)\|\|f_n(t)\| - F_n(t)(f_n(t))) = 0 \quad \text{a.e.} \tag{III.7.14}$$

By (III.7.12) we have

$$|\|F_n(t)\|\|f_n(t)\| - F_n(t)(f_n(t))| \leq 2(\psi(t) + g(t))(\varphi(t) + h(t)) \quad \text{a.e.}$$

and $(\psi+g)(\varphi+h) \in L^1(\mu)$. Thus we can deduce from (III.7.14) (by Lebesgue's Theorem) that

$$\lim_{n\to\infty} \int_S (\|F_n(t)\|\|f_n(t)\| - F_n(t)(f_n(t)))\, \mathrm{d}\mu(t) = 0.$$

Together with (III.7.5) it follows that $l_n(f_n) \to 1$, as desired. □

As a corollary we get, as usual, the special case of Lebesgue-Bochner spaces.

Corollary III.7.2. *If X is an sluacs$^+$ space, then $L^p(\mu, X)$ is also sluacs$^+$ for every $p \in (1, \infty)$.*

III.8 The property uacs in Köthe-Bochner spaces

Now we will treat the case of uacs spaces. In complete analogy to Definition II.6.2 we introduce the following terminology.

Definition III.8.1. Let E be an order continuous Köthe function space over the complete, σ-finite measure space $(S, \mathcal{A}, \mu)$. E has property (u^+) if for every $\varepsilon > 0$ there is some $\delta > 0$ such that for all $f, g \in S_E$ and every $h \in S_{E'}$ we have

$$\|f + g\|_E \geq 2(1 - \delta) \quad \text{and} \quad \int_S fh \, \mathrm{d}\mu = 1 \quad \Rightarrow \quad \int_S |h||f - g| \, \mathrm{d}\mu \leq \varepsilon.$$

As in the discrete case, this property implies that E is uacs and every UR space has property (u^+), but the author does not know whether these implications are strict.

The following analogue of Theorem II.6.3 holds. Its proof is also analogous to the one of Theorem II.6.3 (which was a modification of Day's proofs from [25, Theorem 3] and [26, Theorem 3] on sums of UR spaces; Day also already observed that his techniques could be carried over to show analogous results on uniform rotundity of Lebesgue-Bochner resp. Köthe-Bochner spaces), but we will explicitly give it here, for the readers' convenience.

Theorem III.8.2. *If E is a Köthe function space over the complete, σ-finite measure space $(S, \mathcal{A}, \mu)$ and E has property (u^+) (in particular, if E is UR) and X is a uacs Banach space, then $E(X)$ is also uacs.*

Proof. Since E is in particular uacs, it is reflexive and hence order continuous. Now let $0 < \varepsilon \leq 2$ be arbitrary. Again since E is uacs there is a number $\eta > 0$ such that for all functions $a, b \in B_E$ and every functional $l \in B_{E^*}$ with $l(a) = 1$ one has

$$l(b) < 1 - \frac{\varepsilon}{4}\delta^X_{\mathrm{uacs}}(\varepsilon/2) \;\Rightarrow\; \|a + b\|_E \leq 2(1 - \eta). \tag{III.8.1}$$

First we are going to prove that for all $f, g \in S_{E(X)}$ such that $\|f(t)\| = \|g(t)\|$ a. e. and all $L \in E(X)^*$ such that $L(f) = 1$ and $L(g) < 1 - \varepsilon$, we also have $\|f + g\|_{E(X)} \leq 2(1 - \eta)$.
Let L be represented by $[F] \in E'(X^*, w^*)$ and put $\beta = \|g(\cdot)\|$, $\nu = \|F(\cdot)\|$. Define γ by $\gamma(t) = \nu(t)\beta(t) - F(t)(g(t))$. Note that γ is measurable and

$$0 \leq \gamma(t) \leq 2\nu(t)\beta(t) \;\; \forall t \in S. \tag{III.8.2}$$

As before we can deduce from $L(f) = 1$ that

$$\int_S \|F(t)\|\|f(t)\| \,\mathrm{d}\mu(t) = 1 \tag{III.8.3}$$

and $F(t)(f(t)) = \|F(t)\|\|f(t)\|$ a. e., hence

$$F(t)(f(t)) = \nu(t)\beta(t) \;\; \text{a. e.} \tag{III.8.4}$$

Next we define

$$\alpha(t) = \begin{cases} \frac{1}{2}\delta^X_{\mathrm{uacs}}\left(\frac{\gamma(t)}{\nu(t)\beta(t)}\right) \text{ if } 0 < \gamma(t) < \nu(t)\beta(t) \\ 0 \text{ if } \gamma(t) = 0 \\ \frac{1}{2}\delta^X_{\mathrm{uacs}}(1) \text{ otherwise.} \end{cases}$$

Note that since δ^X_{uacs} is continuous on $(0, 1)$ (see Lemma I.10.2 or [31, Lemma 3.10]), the function α is measurable. Using (III.8.4) it is easy to see that

$$\|f(t) + g(t)\| \leq 2(1 - \alpha(t))\beta(t) \;\; \text{a. e.} \tag{III.8.5}$$

By (III.8.2) and (III.8.3) we have $\int_S \gamma(t)\,\mathrm{d}\mu(t) \le 2$. Furthermore, we also have

$$\varepsilon < 1 - L(g) = L(f-g) = \int_S F(t)(f(t)-g(t))\,\mathrm{d}\mu(t) \le \int_S \gamma(t)\,\mathrm{d}\mu(t),$$

thus

$$\varepsilon < \int_S \gamma(t)\,\mathrm{d}\mu(t) \le 2. \tag{III.8.6}$$

Now put $A = \{t \in S : 2\gamma(t) > \varepsilon\nu(t)\beta(t)\}$ and $B = S \setminus A$. We then have (because of (III.8.3))

$$\int_B \gamma(t)\,\mathrm{d}\mu(t) \le \frac{\varepsilon}{2}\int_B \nu(t)\beta(t)\,\mathrm{d}\mu(t) \le \frac{\varepsilon}{2}\int_S \nu(t)\beta(t)\,\mathrm{d}\mu(t) = \frac{\varepsilon}{2}.$$

Together with (III.8.6) it follows that

$$\int_A \gamma(t)\,\mathrm{d}\mu(t) > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Taking into account (III.8.2) we get

$$\int_A \nu(t)\beta(t)\,\mathrm{d}\mu(t) > \frac{\varepsilon}{4}. \tag{III.8.7}$$

Next we define $h = \beta\chi_B$ and $h' = \beta\chi_A$, as well as $h'' = (1 - \delta^X_{\mathrm{uacs}}(\varepsilon/2))h'$. Then $\|h + h''\|_E \le \|h + h'\|_E = \|\beta\|_E = 1$. Let l be the functional on E represented by $\nu = \|F(\cdot)\|$. We have $l(h+h') = l(\beta) = 1$ (by (III.8.3)) and further, by (III.8.7),

$$l(h+h'') = 1 - \delta^X_{\mathrm{uacs}}(\varepsilon/2)l(h') = 1 - \int_A \nu(t)\beta(t)\,\mathrm{d}\mu(t) < 1 - \frac{\varepsilon}{4}\delta^X_{\mathrm{uacs}}(\varepsilon/2).$$

So by our choice of η we get $\|2h + h' + h''\|_E \le 2(1-\eta)$, i. e.

$$\left\| h + \left(1 - \frac{1}{2}\delta^X_{\mathrm{uacs}}(\varepsilon/2)\right)h' \right\|_E \le 1 - \eta. \tag{III.8.8}$$

By the monotonicity of δ^X_{uacs} we have

$$\alpha(t) \ge \frac{1}{2}\delta^X_{\mathrm{uacs}}(\varepsilon/2) \quad \forall t \in A. \tag{III.8.9}$$

Using (III.8.5), (III.8.9) and (III.8.8) we obtain

$$\begin{aligned} \|f+g\|_{E(X)} &= \|\|f(\cdot)+g(\cdot)\|\|_E \le 2\|(1-\alpha)\beta\|_E \\ &\le 2\|(1 - 2^{-1}\delta^X_{\mathrm{uacs}}(\varepsilon/2))h' + h\|_E \le 2(1-\eta). \end{aligned}$$

The first step of the proof is completed (as in the proof of Theorem II.6.3, so far we have only used that E is uacs and not the property (u^+)). Next we

wish to remove the restriction $\|f(\cdot)\| = \|g(\cdot)\|$ a. e. So let again $0 < \varepsilon \leq 2$ be arbitrary and choose η as above but corresponding to the value $\varepsilon/2$. Take $0 < \omega < 2\eta/3$.
Since E is uacs we may find $\tau > 0$ such that for all $a, b \in B_E$ and every $l \in B_{E^*}$ we have

$$l(a) \geq 1 - \tau \text{ and } \|a + b\|_E \geq 2(1 - \tau) \;\Rightarrow\; l(b) \geq 1 - \omega. \qquad \text{(III.8.10)}$$

Next we fix $0 < \rho < \min\{\varepsilon/2, 2\tau, \omega\}$ and find a number $\tilde{\tau}$ to the value ρ according to the definition of the property (u^+) of E. Finally, let $0 < \xi < \min\{\tau, \tilde{\tau}\}$.
Let $f, g \in S_{E(X)}$ be arbitrary and $L \in S_{E(X)^*}$ (as usually represented by F) such that $L(f) = 1$ and $\|f + g\|_{E(X)} \geq 2(1 - \xi)$. We are going to prove that $L(g) > 1 - \varepsilon$, thus showing that $E(X)$ is uacs.
To this end, we define $z : S \to X$ by

$$z(t) = \begin{cases} \frac{\|f(t)\|}{\|g(t)\|} g(t) \text{ if } g(t) \neq 0 \\ f(t) \text{ if } g(t) = 0. \end{cases}$$

Then z is Bochner-measurable and $\|z(t)\| = \|f(t)\|$ for all $t \in S$ (hence $z \in E(X)$). Furthermore,

$$\|z(t) - g(t)\| = |\|f(t)\| - \|g(t)\|| \quad \forall t \in S. \qquad \text{(III.8.11)}$$

As before we have

$$\int_S \|F(t)\| \|f(t)\| \, \mathrm{d}\mu(t) = 1. \qquad \text{(III.8.12)}$$

Also,

$$2(1 - \tilde{\tau}) \leq 2(1 - \xi) \leq \|f + g\|_{E(X)} \leq \|\|f(\cdot)\| + \|g(\cdot)\|\|_E,$$

so the choice of $\tilde{\tau}$ together with (III.8.11) implies

$$\int_S \|F(t)\| \|z(t) - g(t)\| \, \mathrm{d}\mu(t) \leq \rho. \qquad \text{(III.8.13)}$$

Next we observe that

$$\|\|f(\cdot)\| + \|g(\cdot)\| + \|f(\cdot) + g(\cdot)\|\|_E \geq 2\|f + g\|_{E(X)} \geq 4(1 - \xi) \geq 4(1 - \tau)$$

and (because of (III.8.12) and (III.8.13))

$$\begin{aligned} &\int_S \|F(t)\| (\|f(t)\| + \|g(t)\|) \, \mathrm{d}\mu(t) = 1 + \int_S \|F(t)\| \|g(t)\| \, \mathrm{d}\mu(t) \\ &\geq 1 + \int_S \|F(t)\| \|f(t)\| \, \mathrm{d}\mu(t) - \int_S \|F(t)\| |\|f(t)\| - \|g(t)\|| \, \mathrm{d}\mu(t) \\ &= 2 - \int_S \|F(t)\| |\|f(t)\| - \|g(t)\|| \, \mathrm{d}\mu(t) \geq 2 - \rho \geq 2(1 - \tau). \end{aligned}$$

So (III.8.10) implies

$$\int_S \|F(t)\|\|f(t) + g(t)\| \,\mathrm{d}\mu(t) \geq 2(1-\omega). \tag{III.8.14}$$

Using (III.8.13) and (III.8.14) we can conclude

$$\begin{aligned}
&\|f + z\|_{E(X)} \geq \int_S \|F(t)\|\|f(t) + z(t)\| \,\mathrm{d}\mu(t) \\
&\geq \int_S \|F(t)\|\|f(t) + g(t)\| \,\mathrm{d}\mu(t) - \int_S \|F(t)\|\|g(t) - z(t)\| \,\mathrm{d}\mu(t) \\
&\geq 2(1-\omega) - \rho > 2(1-\eta).
\end{aligned}$$

By the choice of η this implies $L(z) \geq 1 - \varepsilon/2$. But by (III.8.13) we also have $|L(g) - L(z)| \leq \rho$, hence $L(g) \geq L(z) - \rho \geq 1 - \varepsilon/2 - \rho > 1 - \varepsilon$. □

As a corollary we obtain again Sirotkin's result from [123] on uacs Lebesgue-Bochner spaces.

Corollary III.8.3. *If X is a uacs space, then $L^p(\mu, X)$ is also uacs for every $p \in (1, \infty)$.*

As in the case of sums, we can also get a more general corollary for US spaces.

Corollary III.8.4. *If E is a US Köthe function space over a complete, σ-finite measure space and X is a uacs Banach space, then $E(X)$ is also uacs.*

Proof. Since uacs is a self-dual property (see Corollary I.4.2) X^* is also uacs and since E is US we have that $E^* = E'$ is UR (cf. [41, Theorem 9.10]). So by Theorem III.8.2 $E'(X^*)$ is uacs. But as a uacs space X^* is reflexive and hence it has the Radon-Nikodým property. It follows from Theorem III.1.2 that $E(X)^*$ is isometrically isomorphic to $E'(X^*)$, so $E(X)^*$ and hence also $E(X)$ is uacs. □

III.9 The properties mluacs and msluacs in Köthe-Bochner spaces

In this section we will prove a result concerning the midpoint properties mluacs and msluacs in Köthe-Bochner spaces, namely the following.

Theorem III.9.1. *Let E be an MLUR Köthe function space over a complete, σ-finite measure space $(S, \mathcal{A}, \mu)$ and let X be a Banach space. If X is mluacs, then so is $E(X)$. If X is msluacs and in addition E^* has the Kadets-Klee* property and B_{E^*} is weak*-sequentially compact, then $E(X)$ is also msluacs.*

Proof. Let us first recall that ℓ^∞ has no equivalent MLUR norm (not even an equivalent WMLUR norm, cf. for example [93, Theorem 2.1.5]) and so by [93, Propositions 3.1.4 and 3.1.5] (and since every Köthe function space is σ-order complete) E must be order continuous.
Now let us assume that X is msluacs and E^* has the Kadets-Klee* property and weak*-sequentially compact unit ball. To show that $E(X)$ is msluacs we will proceed in an analogous way to the proof of Proposition II.7.1, which in turn uses techniques from the proof of [38, Proposition 4].
So let us take two sequences $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ in $S_{E(X)}$ and $f \in S_{E(X)}$ such that $\|f_n + g_n - 2f\|_{E(X)} \to 0$. Also, take a sequence $(l_n)_{n\in\mathbb{N}}$ of norm-one funcionals on $E(X)$ such that $l_n(f_n) \to 1$. As usual, l_n will be represented by $[F_n] \in E'(X^*, w^*)$ and we conclude

$$\lim_{n\to\infty} \int_S \|F_n(t)\|\|f_n(t)\| \,\mathrm{d}\mu(t) = 1 \tag{III.9.1}$$

and, after passing to an appropriate subsequence,

$$\lim_{n\to\infty} (\|F_n(t)\|\|f_n(t)\| - F_n(t)(f_n(t))) = 0 \quad \text{a. e.} \tag{III.9.2}$$

We also have

$$\begin{aligned}&\|2\|f(\cdot)\| - \|f_n(\cdot) + g_n(\cdot)\|\|_E = \||2\|f(\cdot)\| - \|f_n(\cdot) + g_n(\cdot)\||\|_E \\ &\leq \|\|2f(\cdot) - f_n(\cdot) - g_n(\cdot)\|\|_E = \|2f - f_n - g_n\|_{E(X)},\end{aligned}$$

hence

$$\|2\|f(\cdot)\| - \|f_n(\cdot) + g_n(\cdot)\|\|_E \to 0. \tag{III.9.3}$$

As before we can also show

$$\|\|f_n(\cdot)\| + \|g_n(\cdot)\|\|_E \to 2. \tag{III.9.4}$$

Also, because of $\|f_n + g_n - 2f\|_{E(X)} \to 0$ and Lemma III.1.1, we may pass to a further subsequence such that

$$\lim_{n\to\infty} \|f_n(t) + g_n(t) - 2f(t)\| = 0 \quad \text{a. e.} \tag{III.9.5}$$

Let us define for every $n \in \mathbb{N}$

$$\begin{aligned}a_n(t) &:= 2\|f(t)\| - \frac{1}{2}(\|f_n(t)\| + \|g_n(t)\|),\\ b_n(t) &:= \|f(t)\| - \frac{1}{2}\|f_n(t) + g_n(t)\|.\end{aligned}$$

Note that

$$\|f(t)\| \leq b_n(t) + \frac{1}{2}(\|f_n(t)\| + \|g_n(t)\|).$$

So if $a_n(t) \geq 0$, then

$$|a_n(t)| = 2\|f(t)\| - \frac{1}{2}(\|f_n(t)\| + \|g_n(t)\|) \leq 2|b_n(t)| + \frac{1}{2}(\|f_n(t)\| + \|g_n(t)\|).$$

If $a_n(t) < 0$, then

$$|a_n(t)| = \frac{1}{2}(\|f_n(t)\| + \|g_n(t)\|) - 2\|f(t)\| \leq 2|b_n(t)| + \frac{1}{2}(\|f_n(t)\| + \|g_n(t)\|).$$

So we always have

$$|a_n(t)| \leq 2|b_n(t)| + \frac{1}{2}(\|f_n(t)\| + \|g_n(t)\|).$$

It follows that

$$\begin{aligned}&\frac{1}{2}\|\|f_n(\cdot)\| + \|g_n(\cdot)\|\|_E + 2\|b_n\|_E \geq \left\|2|b_n| + \frac{1}{2}(\|f_n(\cdot)\| + \|g_n(\cdot)\|)\right\|_E \\ &\geq \|a_n\|_E \geq 2 - \frac{1}{2}\|\|f_n(\cdot)\| + \|g_n(\cdot)\|\|_E\end{aligned}$$

and we can conclude with (III.9.3) and (III.9.4) that $\|a_n\|_E \to 1$.
Using this together with (III.9.3), $\|f_n(\cdot)\| + \|g_n(\cdot)\| + 2a_n = 4\|f(\cdot)\|$ and the fact that E is MLUR we get that

$$\lim_{n\to\infty} \|2\|f(\cdot)\| - \|f_n(\cdot)\| - \|g_n(\cdot)\|\|_E = 0. \tag{III.9.6}$$

Again, since E is MLUR this implies

$$\lim_{n\to\infty} \|\|f_n(\cdot)\| - \|g_n(\cdot)\|\|_E = 0. \tag{III.9.7}$$

Because of (III.9.6), (III.9.7) and Lemma III.1.1 we can pass to a further subsequence such that

$$\lim_{n\to\infty} \|f_n(t)\| = \|f(t)\| = \lim_{n\to\infty} \|g_n(t)\| \quad \text{a. e.} \tag{III.9.8}$$

Since B_{E^*} is weak*-sequentially compact we may also assume that the sequence $(\|F_n(\cdot)\|)_{n\in\mathbb{N}}$ weak*-converges to some $g \in B_{E'}$.
(III.9.6) and (III.9.7) imply $\|\|f_n(\cdot)\| - \|f(\cdot)\|\|_E \to 0$. Together with (III.9.1) this gives us

$$\lim_{n\to\infty} \int_S \|F_n(t)\|\|f(t)\| \,\mathrm{d}\mu(t) = 1, \tag{III.9.9}$$

hence we also have

$$\int_S \|g(t)\|\|f(t)\| \,\mathrm{d}\mu(t) = 1,$$

thus $\|g\|_{E'} = 1$. Since E^* has the Kadets-Klee* property it follows that $\|\|F_n(\cdot)\| - g\|_{E'} \to 0$. So by Lemma III.1.1 we can once more pass to a subsequence and assume that

$$\lim_{n\to\infty} \|F_n(t)\| = g(t) \quad \text{a. e.} \tag{III.9.10}$$

and moreover that there exists $\psi \in E'$ with $\psi \geq 0$ and

$$||F_n(t)\| - g(t)| \leq \psi(t) \quad \text{a. e. } \forall n \in \mathbb{N}. \tag{III.9.11}$$

Now if we combine (III.9.2), (III.9.5), (III.9.8) and (III.9.10) we obtain

$$\lim_{n \to \infty} (\|F_n(t)\| \|f(t)\| - F_n(t)(f(t))) = 0 \quad \text{a. e.},$$

since X is msluacs.
Using this together with (III.9.11) and Lebesgue's Theorem, as we have done in previous proofs, we can conclude

$$\lim_{n \to \infty} \int_S (\|F_n(t)\| \|f(t)\| - F_n(t)(f(t))) \, \mathrm{d}\mu(t) = 0. \tag{III.9.12}$$

Combining (III.9.12) and (III.9.9) gives us $l_n(f) \to 1$ and we are done.
The statement about mluacs spaces can be proved similarly. □

This Theorem admits the following corollary for Lebesgue-Bochner spaces.

Corollary III.9.2. *If X is mluacs/msluacs, then $L^p(\mu, X)$ is also mluacs/msluacs for every $p \in (1, \infty)$.*

III.10 Summary of the results on Köthe-Bochner spaces

Let us conclude this chapter by summarising the obtained results on Köthe-Bochner function spaces in the following table. We always assume that E is a Köthe function space over a complete, σ-finite measure space $(S, \mathcal{A}, \mu)$. The abbreviation oc stands for "order continuous", KK resp. KK* stands for "Kadets-Klee" resp. "Kadets-Klee* property", ref stands for "reflexive" and w*-sc stands for "weak*-sequentially compact".

Table III.1: *Summary of the results on Köthe-Bochner spaces*

E	X	$E(X)$
acs + oc	acs	acs
WLUR	luacs	luacs
luacs$^+$ + oc + E' oc	luacs	luacs
LUR	luacs$^+$	luacs$^+$
luacs$^+$ + ref + KK	luacs$^+$	luacs$^+$
wuacs + ref + KK	wuacs	wuacs
LUR	sluacs	sluacs
sluacs$^+$ + ref + KK + E^* KK	sluacs	sluacs
sluacs$^+$ + ref + KK + E^* KK	sluacs$^+$	sluacs$^+$
LUR + E^* KK* + B_{E^*} w*-sc	sluacs$^+$	sluacs$^+$
(u^+)	uacs	uacs
US	uacs	uacs
MLUR	mluacs	mluacs
MLUR + E^* KK* + B_{E^*} w*-sc	msluacs	msluacs

Unfortunately, the author does not know any result concerning the directional property uacsed in Köthe-Bochner spaces (not even in Lebesgue-Bochner spaces).

Finally, let us remark that in some of the results that are listed above, the assumptions could be formally weakened using the results from Section I.10. For example, by Proposition I.10.6, every reflexive space with the Kadets-Klee property which is WLUR in the sense of Lovaglia is automatically wuacs and sluacs$^+$.

IV Spaces with the Opial property and related notions

We will now leave the topic of acs spaces and consider some other geometric properties of Banach spaces. This chapter is concerned with the so called Opial property and its variants, the nonstrict and the uniform Opial property, as well as two more related notions, the WORTH property and the García-Falset coefficient. All these properties are connected to the important fixed point property for nonexpansive mappings. We will study (infinite) absolute sums of such spaces (in some cases only classical ℓ^p-sums, in some cases more general types of sums). Furthermore, we will prove some Opial-type convergence results in Lebesgue-Bochner spaces (such spaces cannot have the usual Opial property, but some analogous results for weak convergence pointwise almost everywhere instead of weak convergence can be proved, see Section IV.6 for details). The results presented in the first six sections of this chapter first appeared in the author's preprint [61], which has recently been submitted (in a slightly revised form) to Commentationes Mathematicae for possible publication.

In the last two sections we will also consider Opial properties in infinite sums associated to Cesàro sequence spaces and some Opial-type results in Cesàro spaces of vector-valued functions. These results were not published before (not even in preprint form).

IV.1 Definitions and background

We begin by recalling the notion of fixed point property. A real Banach space X is said to have the fixed point property (resp. weak fixed point property) if for every closed and bounded (resp. weakly compact) convex subset $C \subseteq X$, every nonexpansive mapping $F : C \to C$ has a fixed point (where F is called nonexpansive if $\|F(x) - F(y)\| \leq \|x - y\|$ for all $x, y \in C$, in other words, if F is 1-Lipschitz continuous).

In connection with the fixed point property, the geometric notion of normal structure is of great importance. A bounded, closed, convex subset $C \subseteq X$ is said to have normal structure if for each subset $B \subseteq C$ which contains at least two elements there exists a point $x \in B$ such that

$$\sup_{y \in B} \|x - y\| < \operatorname{diam} B,$$

where $\operatorname{diam} B$ denotes the diameter of B. The space X itself is said to have normal structure if every bounded, closed, convex subset of X has normal structure. It is well known that if C is weakly compact and has normal structure, then every nonexpansive mapping $F : C \to C$ has a fixed point (see for example [53, Theorem 2.1]), thus spaces with normal structure have the weak fixed point property.

As we have already mentioned in Chapter I, uacs spaces have normal structure (cf. [46, Theorem 3.2] or [123, Theorem 3.1]), and since they are also reflexive they have the fixed point property. By [47, Theorem 3] even the condition $u_X(\varepsilon) > 0$ for some $\varepsilon \in (0, \frac{1}{2})$ is enough to ensure that X has normal structure, where u_X denotes the modulus of u-convexity which coincides with our uacs modulus δ^X_{uacs} (see Definition I.1.5 and the following remarks). In [118, Proposition 3.3] an even stronger result was obtained: if $u_X(1) > 0$, then X and X^* have normal structure.

An example of a Banach space which fails the weak fixed point property is $L^1[0, 1]$ (see [4]).

The space X is said to have the Opial property provided that

$$\limsup_{n\to\infty} \|x_n\| < \limsup_{n\to\infty} \|x_n - x\|$$

holds for every weakly null sequence $(x_n)_{n\in\mathbb{N}}$ in X and every $x \in X \setminus \{0\}$ (one could as well use lim inf instead of lim sup or assume from the beginning that both limits exist).

This property was first considered by Opial in [109] (starting from the Hilbert spaces as canonical example) to provide a result on iterative approximations of fixed points of nonexpansive mappings. It is shown in [109] that the spaces ℓ^p for $1 \leq p < \infty$ enjoy the Opial property, whereas $L^p[0, 1]$ for $1 < p < \infty, p \neq 2$ fails to have it. Note further that every Banach space with the Schur property (i. e. weak and norm convergence of sequences coincide) trivially has the Opial property. Also, X is said to have the nonstrict Opial property if it fulfils the definition of the Opial property with "$\leq$" instead of "$<$" ([122], in [48] it is called weak Opial property).[1] It is known that every weakly compact convex set in a Banach space with the Opial property has normal structure (see for instance [114, Theorem 5.4]) and thus the Opial property implies the weak fixed point property.

[1] Note that one always has $\limsup \|x_n\| \leq \limsup \|x_n - x\| + \|x\| \leq 2 \limsup \|x_n - x\|$ if $(x_n)_{n\in\mathbb{N}}$ converges weakly to zero, since the norm is weakly lower semicontinuous. In general, the constant 2 is the best possible. Consider, for example, in the space c of all convergent sequences (with sup-norm) the weak null sequence $(2e_n)_{n\in\mathbb{N}}$ and $x = (1, 1, 1, \dots)$.

Prus introduced the notion of uniform Opial property in [113]: a Banach space X has the uniform Opial property if for every $c > 0$ there is some $r > 0$ such that

$$1 + r \leq \liminf_{n\to\infty} \|x_n - x\|$$

holds for every $x \in X$ with $\|x\| \geq c$ and every weakly null sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\liminf\|x_n\| \geq 1$. In [113] it was proved that a Banach space is reflexive and has the uniform Opial property if and only if it has the so called property (L) (see [113] for the definition), and that X has the fixed point property whenever X^* enjoys said property (L).

A modulus corresponding to the uniform Opial property was defined in [91]:

$$r_X(c) := \inf\Big\{\liminf_{n\to\infty}\|x_n - x\| - 1\Big\} \quad \forall c > 0,$$

where the infimum is taken over all $x \in X$ with $\|x\| \geq c$ and all weakly null sequences $(x_n)_{n\in\mathbb{N}}$ in X with $\liminf\|x_n\| \geq 1$ (if X has the Schur property, we agree to set $r_X(c) := 1$ for all $c > 0$). Then X has the uniform Opial property if and only $r_X(c) > 0$ for every $c > 0$.

In this work we will mostly use the following equivalent formulation of the uniform Opial property ([83, Definition 3.1]): X has the uniform Opial property if and only if for every $\varepsilon > 0$ and every $R > 0$ there is some $\eta > 0$ such that

$$\eta + \liminf_{n\to\infty}\|x_n\| \leq \liminf_{n\to\infty}\|x_n - x\|$$

holds for all $x \in X$ with $\|x\| \geq \varepsilon$ and every weakly null sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\limsup\|x_n\| \leq R$.

We can also associate a modulus to this formulation in the following way:

$$\eta_X(\varepsilon, R) := \inf\Big\{\liminf_{n\to\infty}\|x_n - x\| - \liminf_{n\to\infty}\|x_n\|\Big\} \quad \forall \varepsilon, R > 0,$$

where the infimum is taken over all $x \in X$ with $\|x\| \geq \varepsilon$ and all weakly null sequences $(x_n)_{n\in\mathbb{N}}$ in X with $\limsup\|x_n\| \leq R$. So X has the uniform Opial property if and only if $\eta_X(\varepsilon, R) > 0$ for all $\varepsilon, R > 0$. Actually, it is enough that for every $\varepsilon > 0$ there exists some $R > 2$ with $\eta_X(\varepsilon, R) > 0$. More precisely, we have the following connection between the two moduli r_X and η_X.

Lemma IV.1.1. *Let X be a Banach space which does not have the Schur property.*

(i) *For every $c > 0$ and every $R > 2$ we have*

$$\min\Big\{\eta_X(c, R), \frac{R}{2} - 1\Big\} \leq r_X(c).$$

(ii) *For all* $\varepsilon, R > 0$ *with* $r_X(\varepsilon/R) > 0$ *we have*

$$\frac{\varepsilon r_X(\varepsilon/R)}{2 + r_X(\varepsilon/R)} = \max_{\beta\in[0,\varepsilon/2]} \min\Big\{\beta r_X(\frac{\varepsilon}{R}), \varepsilon - 2\beta\Big\} \leq \eta_X(\varepsilon, R).$$

Proof. (i) Let $c > 0$ and $R > 2$. Put $\tau := \min\{\eta_X(c, R), \frac{R}{2} - 1\}$. Let $(x_n)_{n\in\mathbb{N}}$ be any weakly null sequence in X with $\liminf\|x_n\| \geq 1$ and let $x \in X$ with $\|x\| \geq c$. By passing to a subsequence, we may assume that $\lim_{n\to\infty}\|x_n - x\|$ and $s := \lim_{n\to\infty}\|x_n\|$ exist. If $s \leq R$ then $1 + \tau \leq s + \eta_X(c, R) \leq \lim_{n\to\infty}\|x_n - x\|$.
If $s > R$ and $\|x\| > R/2$, then $\lim_{n\to\infty}\|x_n - x\| \geq \|x\| > R/2 \geq 1 + \tau$ by the weak lower semicontinuity of the norm. Finally, if $s > R$ and $\|x\| \leq R/2$, then $\lim_{n\to\infty}\|x_n - x\| \geq s - \|x\| > R/2 \geq 1 + \tau$.
(ii) The first equality is easily verified. Now chose any $\beta \in (0, \varepsilon/2)$ and put $\nu := \min\{\beta r_X(\frac{\varepsilon}{R}), \varepsilon - 2\beta\}$. Let $(x_n)_{n\in\mathbb{N}}$ be a weakly null sequence in X with $\limsup\|x_n\| \leq R$ and let $x \in X$ with $\|x\| \geq \varepsilon$. Again we may assume that $\lim_{n\to\infty}\|x_n - x\|$ and $s := \lim_{n\to\infty}\|x_n\|$ exist. By the definition of r_X we get $s(1 + r_X(\varepsilon/R)) \leq \lim_{n\to\infty}\|x_n - x\|$, which implies $s + \nu \leq \lim_{n\to\infty}\|x_n - x\|$ if $s > \beta$. But if $s \leq \beta$ then $\lim_{n\to\infty}\|x_n - x\| \geq \|x\| - s \geq \varepsilon - \beta \geq \nu + \beta \geq \nu + s$ and the proof is finished. $\square$

In [48] J. García-Falset introduced the following coefficient of a Banach space X:

$$R(X) := \sup\Big\{\liminf_{n\to\infty}\|x_n + x\| : x \in B_X, (x_n)_{n\in\mathbb{N}} \in \mathrm{WN}(B_X)\Big\},$$

where we denote by $\mathrm{WN}(B_X)$ the set of all weakly null sequences in B_X. Obviously, $1 \leq R(X) \leq 2$ and $R(X) = 1$ if X has the Schur property (in particular if X is finite-dimensional or $X = \ell^1$). One has $R(c_0) = 1$ and $R(\ell^p) = 2^{1/p}$ for $1 < p < \infty$ (see [48, Corollary 3.2]). In [49, Theorem 3] it was proved that the condition $R(X) < 2$ implies that X has the weak fixed point property. The reflexive spaces with $R(X) < 2$ are precisely the so called weakly nearly uniformly smooth spaces ([48, Corollary 4.4]), which were introduced in [85] and include in particular all uniformly smooth spaces. By [102, Theorem 5] $R(X) < 2$ if $\delta^X_{\mathrm{uacs}}(\varepsilon) = u_X(\varepsilon) > 0$ for some $0 < \varepsilon < 1$.

In [121] Sims introduced the notion of WORTH (weak orthogonality) property: X is said to have the WORTH property provided that for all weakly null sequences $(x_n)_{n\in\mathbb{N}}$ in X and every $x \in X$ one has $\|x_n + x\| - \|x_n - x\| \to 0$.

Again spaces with the Schur property obviously enjoy the WORTH property. Hilbert spaces are easily seen to have the WORTH property as well. Also, the class of spaces with the WORTH property includes all so called weakly orthogonal Banach lattices (a notion introduced earlier by Borwein and Sims in [14]), which in turn includes in particular all spaces $\ell^p(I)$ for $1 \leq p < \infty$ and $c_0(I)$. However, the spaces $L^p[0,1]$ with $1 \leq p \leq \infty, p \neq 2$

do not have the WORTH property (see the remark at the end of [122]). In [121] it was proved that the WORTH property implies the nonstrict Opial property, and in [122] it was shown that a space with the WORTH property which is ε-inquadrate in every direction for some $0 < \varepsilon < 2$ (see [122] for the definition) has the weak fixed point property (even more, every weakly compact convex subset of such a space has normal structure). By [48, Proposition 3.6], a uniformly non-square[2] Banach space X with the WORTH property satisfies $R(X) < 2$.

The degree $w(X)$ of WORTHness of X was also introduced in [122] as the supremum of all $r \geq 0$ such that

$$r \liminf_{n\to\infty} \|x_n + x\| \leq \liminf_{n\to\infty} \|x_n - x\|$$

holds for all $x \in X$ and all weakly null sequences $(x_n)_{n\in\mathbb{N}}$ in X. Then $1/3 \leq w(X) \leq 1$ and X has the WORTH property if and only if $w(X) = 1$.

We are going to study the WORTH property and the García-Falset coefficient for infinite absolute sums, and the different Opial properties specifically for infinite ℓ^p-sums of Banach spaces (for normal structure in (finite and infinite) direct sums of Banach spaces see [36] and references therein, for more information about the fixed point property and normal structure in general, see [53]). Some Opial-type results for Lebesgue-Bochner spaces will also be obtained in Section IV.6. In Section IV.7 we will have a look at Opial properties in infinite sums associated to the so called Cesàro sequence spaces. Finally, Section IV.8 is devoted to Opial-type results in Cesàro spaces of vector-valued functions.

IV.2 WORTH property of absolute sums

This section concerns the WORTH property in absolute sums of Banach spaces. We will use the same notation and terminology as in Chapter II.

By [81, Theorem 4.7], $w(X \oplus_E Y) = \min\{w(X), w(Y)\}$ holds for all Banach spaces X and Y and every absolute, normalised norm $\|\cdot\|_E$ on $\mathbb{R}^2$ (actually, the aforementioned equivalent notion of ψ-direct sums is used in [81] (see section 2 in [81])). In particular, $X \oplus_E Y$ has the WORTH property if and only if X and Y have the WORTH property (for this, see also [81, Theorem 4.2]). It is possible to generalise [81, Theorem 4.7] to sums of arbitrarily many Banach spaces.

Proposition IV.2.1. *Let I be any index set and E a subspace of $\mathbb{R}^I$ endowed with an absolute, normalised norm $\|\cdot\|_E$ such that* $\operatorname{span}\{e_i : i \in I\}$ *is dense*

[2] Recall that X is said to be uniformly non-square if there is some $\delta > 0$ such that whenever $x, y \in B_X$ one has $\|x + y\| < 2(1 - \delta)$ or $\|x - y\| < 2(1 - \delta)$, see Section I.1.

in E. Let $(X_i)_{i\in I}$ be any family of Banach spaces. Then

$$w\left(\left[\bigoplus_{i\in I} X_i\right]_E\right) = \inf\{w(X_i) : i \in I\}.$$

In particular, $\left[\bigoplus_{i\in I} X_i\right]_E$ has the WORTH property if and only if X_i has the WORTH property for every $i \in I$.

Proof. Let us write $X = \left[\bigoplus_{i\in I} X_i\right]_E$ and $s = \inf\{w(X_i) : i \in I\}$. We clearly have $w(X) \leq s$. Now let $x_n = (x_{n,i})_{i\in I} \in X$ for every $n \in \mathbb{N}$ such that $(x_n)_{n\in\mathbb{N}}$ converges weakly to zero and let $x = (x_i)_{i\in I} \in X$. Without loss of generality, we may assume that the limits $a := \lim_{n\to\infty}\|x_n + x\|_E$ and $b := \lim_{n\to\infty}\|x_n - x\|_E$ exist.
Since $\operatorname{span}\{e_i : i \in I\}$ is dense in E we have $(\|x_i\|)_{i\in I} = \sum_{i\in I}\|x_i\|e_i$. So if $\varepsilon > 0$ is given, we find a finite set $J \subseteq I$ such that

$$\left\|\sum_{i\in I\setminus J}\|x_i\|e_i\right\|_E = \left\|(\|x_i\|)_{i\in I} - \sum_{i\in J}\|x_i\|e_i\right\|_E \leq \varepsilon. \tag{IV.2.1}$$

By passing to an appropriate subsequence we may assume that the limits $a_i := \lim_{n\to\infty}\|x_{n,i} + x_i\|$ and $b_i := \lim_{n\to\infty}\|x_{n,i} - x_i\|$ exist for each $i \in J$. Since $(x_{n,i})_{n\in\mathbb{N}}$ is weakly convergent to zero in X_i for every $i \in I$ it follows that $sa_i \leq b_i \leq s^{-1}a_i$ and consequently

$$|a_i - b_i| \leq \frac{1-s}{s}b_i \quad \forall i \in J. \tag{IV.2.2}$$

For every $n \in \mathbb{N}$ we have, because of (IV.2.1),

$$\begin{aligned}
&|\|x_n + x\|_E - \|x_n - x\|_E| \leq \|(\|x_{n,i} + x_i\| - \|x_{n,i} - x_i\|)_{i\in I}\|_E \\
&\leq \left\|\sum_{i\in J}(\|x_{n,i} + x_i\| - \|x_{n,i} - x_i\|)e_i\right\|_E + 2\left\|\sum_{i\in I\setminus J}\|x_i\|e_i\right\|_E \\
&\leq \left\|\sum_{i\in J}(\|x_{n,i} + x_i\| - \|x_{n,i} - x_i\|)e_i\right\|_E + 2\varepsilon.
\end{aligned}$$

So for $n \to \infty$ we obtain

$$|a - b| \leq \left\|\sum_{i\in J}(a_i - b_i)e_i\right\|_E + 2\varepsilon.$$

Taking (IV.2.2) into account we arrive at

$$|a - b| \leq \frac{1-s}{s}\left\|\sum_{i\in J} b_i e_i\right\|_E + 2\varepsilon.$$

But $\|\sum_{i\in J}\|x_{n,i}-x_i\|e_i\|_E \leq \|x_n - x\|_E$ for each n, thus $\|\sum_{i\in J} b_i e_i\|_E \leq b$ and hence

$$|a-b| \leq \frac{1-s}{s}b + 2\varepsilon.$$

Letting $\varepsilon \to 0$ leaves us with $|a-b| \leq (1-s)b/s$, which implies $sa \leq b$ and we are done. □

IV.3 García-Falset coefficient of absolute sums

Now we turn to the García-Falset coefficient of absolute sums. In [32, Theorem 7] it was proved that $R((X_1 \oplus X_2 \oplus \cdots \oplus X_n)_E) < 2$ if $R(X_i) < 2$ for $i = 1, \ldots, n$ and $\|\cdot\|_E$ is any strictly convex, absolute, normalised norm on $\mathbb{R}^n$. For absolute sums of two Banach spaces a stronger result was obtained in [81, Theorem 3.6] (in the equivalent formulation of ψ-direct sums): $R(X \oplus_E Y) < 2$ provided that $R(X), R(Y) < 2$ and $\|\cdot\|_E$ is any absolute, normalised norm on $\mathbb{R}^2$ with $\|\cdot\|_E \neq \|\cdot\|_1$. A complete characterisation of ψ-direct sums of finitely many spaces having García-Falset coefficient less than 2 is given in [82, Theorem 6.2].

For infinite sums we have the following partial result (for $J \subseteq I$ we denote by $\big[\bigoplus_{i\in J} X_i\big]_E$ the sum of the family whose i-th member is X_i for $i \in J$ and $\{0\}$ for $i \in I \setminus J$).

Theorem IV.3.1. *If I is an infinite index set, E a subspace of $\mathbb{R}^I$ with absolute, normalised norm such that* $\operatorname{span}\{e_i : i \in I\}$ *is dense in E and $(X_i)_{i\in I}$ is a family of Banach spaces with*

$$\alpha := \sup\left\{ R\left(\Big[\bigoplus_{i\in J} X_i\Big]_E\right) : J \subseteq I \text{ finite}\right\} < 2 \tag{IV.3.1}$$

and[3] $\delta_E((1-\alpha/2)^2) > 0$*, then* $R\big(\big[\bigoplus_{i\in I} X_i\big]_E\big) < 2$.

Proof. Let us write $X = \big[\bigoplus_{i\in I} X_i\big]_E$ for short. It is well known that δ_E is continuous on $(0,2)$ (see for example [52, Lemma 5.1]), so we can find $0 < \tau < (1-\alpha/2)^2$ with $\delta_E(\tau) > 0$. Let $\gamma := \sqrt{\tau}$ and choose $0 < \eta < \min\{\delta_E(\tau), 1/2 - \gamma\}$.

Assume that $R(X) = 2$. Then there would be a weakly null sequence $(x_n)_{n\in\mathbb{N}} = ((x_{n,i})_{i\in I})_{n\in\mathbb{N}}$ in B_X and an element $x = (x_i)_{i\in I} \in B_X$ such that $\lim_{n\to\infty}\|x_n + x\|_E > 2 - \eta$. We may assume $\|x_n + x\|_E > 2 - 2\eta$ for all $n \in \mathbb{N}$. Since $\|(\|x_{n,i}\| + \|x_i\|)_{i\in I}\|_E \geq \|x_n + x\|_E$ and $\eta < \delta_E(\tau)$ it follows that

$$\|(\|x_{n,i}\| - \|x_i\|)_{i\in I}\|_E < \tau \quad \forall n \in \mathbb{N}. \tag{IV.3.2}$$

Similarly,

$$4(1-\eta) < 2\|x_n + x\|_E \leq \|(\|x_{n,i}\| + \|x_i\| + \|x_{n,i} + x_i\|)_{i\in I}\|_E \leq 4$$

[3] δ_E denotes the modulus of convexity of E, see Section I.1.

and hence

$$\|(\|x_{n,i}\| + \|x_i\| - \|x_{n,i} + x_i\|)_{i\in I}\|_E < 2\tau \quad \forall n \in \mathbb{N}. \tag{IV.3.3}$$

We further have $\|x\|_E \geq \|x_n + x\|_E - 1 > 1 - 2\eta > 2\gamma$. Since $(\|x_i\|)_{i\in I} = \sum_{i\in I}\|x_i\|e_i$ we can find a finite set $J \subseteq I$ such that

$$\left\|\sum_{i\in J}\|x_i\|e_i\right\|_E > 2\gamma. \tag{IV.3.4}$$

Put $y := (x_i)_{i\in J}, y_n = (x_{n,i})_{i\in J} \in \left[\bigoplus_{i\in J} X_i\right]_E$ as well as $a := \sum_{i\in J}\|x_i\|e_i$ and $a_n := \sum_{i\in J}\|x_{n,i}\|e_i$. By (IV.3.2) we have

$$\|a_n - a\|_E \leq \|(\|x_{n,i}\| - \|x_i\|)_{i\in I}\|_E < \tau \quad \forall n \in \mathbb{N}, \tag{IV.3.5}$$

which implies in particular $|\|y\|_E - \|y_n\|_E| = |\|a\|_E - \|a_n\|_E| < \tau$, hence $\|y_n\|_E > \|y\|_E - \tau > 2\gamma - \tau > 0$, by (IV.3.4).
Furthermore, for every $n \in \mathbb{N}$,

$$|\|a_n + a\|_E - \|y_n + y\|_E| \leq \left\|\sum_{i\in J}(\|x_{n,i}\| + \|x_i\| - \|x_{n,i} + x_i\|)e_i\right\|_E,$$

so because of (IV.3.3) it follows that

$$|\|a_n + a\|_E - \|y_n + y\|_E| < 2\tau \quad \forall n \in \mathbb{N}.$$

Also, by (IV.3.5), we have $|\|a_n + a\|_E - 2\|y\|_E| = |\|a_n + a\|_E - 2\|a\|_E| \leq \|a_n - a\| < \tau$ for each n. Consequently,

$$|\|y_n + y\|_E - 2\|y\|_E| < 3\tau \quad \forall n \in \mathbb{N}.$$

Since $\|y_n/\|y_n\|_E - y_n/\|y\|_E\|_E = |1 - \|y_n\|_E/\|y\|_E| < \tau/\|y\|_E$ we get

$$\left|2 - \left\|\frac{y}{\|y\|_E} + \frac{y_n}{\|y_n\|_E}\right\|_E\right| < \frac{4\tau}{\|y\|_E} < \frac{2\tau}{\gamma} \quad \forall n \in \mathbb{N}, \tag{IV.3.6}$$

where the last inequality holds because of (IV.3.4). Note that $(x_{n,i})_{n\in\mathbb{N}}$ converges weakly to zero in X_i for each $i \in I$ and thus, by the representation of the dual of $\left[\bigoplus_{i\in J} X_i\right]_E$ as $\left[\bigoplus_{i\in J} X_i^*\right]_{E'}$ and finiteness of J, the sequence $(y_n/\|y_n\|_E)_{n\in\mathbb{N}}$ is also a weakly null sequence (as noted above, $(\|y_n\|)_{n\in\mathbb{N}}$ is bounded away from zero).
So from (IV.3.6) and the definition of α it follows that $\alpha \geq 2(1 - \tau/\gamma)$. But $\gamma = \sqrt{\tau}$ and $\tau < (1 - \alpha/2)^2$, thus $2(1 - \tau/\gamma) > \alpha$ and with this contradiction the proof is finished. □

The above theorem reduces the case of infinite sums to the one of finite sums. The condition $\alpha < 2$ is clearly necessary for $R(X) < 2$. Unfortunately, the author does not know whether the simpler condition $\beta := \sup_{i\in I} R(X_i) < 2$ would be already enough to ensure that $\alpha < 2$. The proofs of [81, Theorem 3.6] and [82, Theorem 6.2] do not give quantitative bounds for the García-Falset coefficient of finite sums. The proof of [32, Theorem 7] shows that for $\beta < 2$ one has for every finite subset $J \subseteq I$ with $|J| = N$ that $R(\big[\bigoplus_{i\in J} X_i\big]_E) \le 2 - \delta$, where first $\varepsilon > 0$ is chosen such that $\beta(1 + N\varepsilon) < 2$ and then $0 < \delta < \min\{2\delta_E(\varepsilon), 2 - \beta(1 + N\varepsilon)\}$, so it still might be that $R(\big[\bigoplus_{i\in J} X_i\big]_E)$ tends to 2 for $N \to \infty$.

Next we will discuss some applications of Theorem IV.3.1. First, since the Schur property is inherited by finite sums, we get the following corollary.

Corollary IV.3.2. *If $(X_i)_{i\in I}$ is a family of Banach spaces with the Schur property (in particular, a family of finite-dimensional Banach spaces) and $\operatorname{span}\{e_i : i \in I\}$ is dense in E with $\delta_E(1/4) > 0$, then $R\big(\big[\bigoplus_{i\in I} X_i\big]_E\big) < 2$. In particular, $R\Big(\big[\bigoplus_{i\in I} X_i\big]_p\Big) < 2$ for all $1 < p < \infty$.*

For another application of Theorem IV.3.1 consider the following example.

Example IV.3.3. If $N \ge 2$ and $I_1, \dots, I_N$ are non-empty sets at least one of which is infinite, then

$$R\left(\Big[\bigoplus_{k=1}^{N} c_0(I_k)\Big]_p\right) = 2^{1/p} \tag{IV.3.7}$$

for every $1 \le p < \infty$. Consequently, by Theorem IV.3.1, if $(I_k)_{k\in I}$ is any family of non-empty sets we have that

$$R\left(\Big[\bigoplus_{k\in I} c_0(I_k)\Big]_p\right) < 2 \quad \text{for } 1 < p < \infty.$$

Proof. To prove (IV.3.7) put $X := \big[\bigoplus_{k=1}^{N} c_0(I_k)\big]_p$ and suppose without loss of generality that I_1 is infinite. Fix a sequence $(i_n)_{n\in\mathbb{N}}$ of distinct elements of I_1 and any $j \in I_2$ and put $x_n := (e_{i_n}, 0, \dots, 0) \in S_X$ as well as $x := (0, e_j, 0, \dots, 0) \in S_X$. Then $x_n \to 0$ weakly in X and $\|x_n + x\|_p = 2^{1/p}$ for each n, thus $2^{1/p} \le R(X)$.
To prove the reverse inequality let $x_n = (x_{n,1}, \dots, x_{n,N}) \in B_X$ for each $n \in \mathbb{N}$ such that $x_n \to 0$ weakly and let $x = (x_1, \dots, x_N) \in B_X$. Without loss of generality we can suppose that $\lim_{n\to\infty}\|x_n + x\|_p$ and also $a_k := \lim_{n\to\infty}\|x_{n,k}\|_\infty$ exist for each $k \in \{1, \dots, N\}$.
Take an arbitrary $\varepsilon > 0$. Then for each $k \in \{1, \dots, N\}$ the set $J_k := \{i \in I_k : |x_k(i)| > \varepsilon\}$ is finite. Since $x_n \to 0$ weakly we have $x_{n,k}(i) \to 0$ for all $k \in \{1, \dots, N\}$ and all $i \in I_k$. It follows that there exists $n_0 \in \mathbb{N}$ such that $|x_{n,k}(i)| \le \varepsilon$ for all $k \in \{1, \dots, N\}$, all $i \in J_k$ and all $n \ge n_0$.

But then $|x_{n,k}(i)+x_k(i)| \leq |x_{n,k}(i)|+|x_k(i)| \leq \max\{|x_{n,k}(i)|,|x_k(i)|\}+\varepsilon$ for all $k \in \{1,\dots,N\}$, all $i \in I_k$ and all $n \geq n_0$. From this we can conclude

$$\|x_n+x\|_p^p = \sum_{k=1}^N \|x_{n,k}+x_k\|_\infty^p \leq \sum_{k=1}^N (\max\{\|x_{n,k}\|_\infty, \|x_k\|_\infty\}+\varepsilon)^p \quad \forall n \geq n_0.$$

For $n \to \infty$ it follows that

$$\lim_{n\to\infty} \|x_n+x\|_p^p \leq \sum_{k=1}^N (\max\{a_k, \|x_k\|_\infty\}+\varepsilon)^p.$$

Letting $\varepsilon \to 0$ we obtain

$$\begin{aligned}&\lim_{n\to\infty} \|x_n+x\|_p^p \leq \sum_{k=1}^N \max\left\{a_k^p, \|x_k\|_\infty^p\right\} \leq \sum_{k=1}^N (a_k^p + \|x_k\|_\infty^p) \\ &= \lim_{n\to\infty} \|x_n\|_p^p + \|x\|_p^p \leq 2.\end{aligned}$$

Hence $\lim_{n\to\infty} \|x_n+x\|_p \leq 2^{1/p}$ and we are done. $\square$

As mentioned in Section IV.1, one has $R(X) < 2$ whenever $\delta_{\mathrm{uacs}}^X(\varepsilon) > 0$ for some $\varepsilon \in (0,1)$, by [102, Theorem 5]. Putting several results together it is now possible to obtain the following corollary.

Corollary IV.3.4. *Let $(X_i)_{i\in I}$ be a family of Banach spaces and $1 < p < \infty$. Suppose that there exist four pairwise disjoint (possibly empty) subsets $I_1, I_2, I_3, I_4 \subseteq I$ such that*

(i) *X_i has the Schur property for each $i \in I_1$,*

(ii) *$\inf_{i\in I_2} \delta_{\mathrm{uacs}}^{X_i}(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$,*

(iii) *for each $i \in I_3$ there is a set J_i with $X_i = c_0(J_i)$.*

(iv) *I_4 is finite and $R(X_i) < 2$ for all $i \in I_4$.*

Then $R(\left[\bigoplus_{i\in I} X_i\right]_p) < 2$.

Proof. Let us put $X := \left[\bigoplus_{i\in I} X_i\right]_p$ and $X_k := \left[\bigoplus_{i\in I_k} X_i\right]_p$ for $k = 1,2,3,4$ (or $X_k = \{0\}$ if $I_k = \emptyset$). By Corollary IV.3.2 we have $R(X_1) < 2$ and by Example IV.3.3 we have $R(X_3) < 2$. Also, by Theorem II.6.3 X_2 is again uacs, so $R(X_2) < 2$. From the aforementioned result [32, Theorem 7] it follows that $R(X_4) < 2$ and since $X \cong X_1 \oplus_p X_2 \oplus_p X_3 \oplus_p X_4$, [32, Theorem 7] implies that $R(X) < 2$. $\square$

The case of c_0-sums is not covered by the above results. However, it is easy to prove the following Proposition directly.

Proposition IV.3.5. *Let $(X_i)_{i\in I}$ be any family of Banach spaces and $X := \left[\bigoplus_{i\in I} X_i\right]_{c_0(I)}$. Then*

$$R(X) = \sup_{i\in I} R(X_i).$$

Proof. We clearly have $\alpha := \sup_{i\in I} R(X_i) \leq R(X)$. To prove the reverse inequality, fix any weakly null sequence $(x_n)_{n\in\mathbb{N}} = ((x_{n,i})_{i\in I})_{n\in\mathbb{N}}$ in B_X and any $x = (x_i)_{i\in I} \in B_X$. Without loss of generality, we may assume that $\lim_{n\to\infty}\|x_n + x\|_\infty$ exists.
Let $\varepsilon > 0$ be arbitrary. Then $J := \{i \in I : \|x_i\| \geq \varepsilon\}$ is finite, so by passing to an appropriate subsequence once more we may also assume that $\lim_{n\to\infty}\|x_{n,i} + x_i\|$ exists for all $i \in J$.
Since $x_{n,i} \to 0$ weakly for all $i \in I$ it follows that $\lim_{n\to\infty}\|x_{n,i} + x_i\| \leq R(X_i) \leq \alpha$ for all $i \in J$, so $\|x_{n,i}+x_i\| \leq \alpha+\varepsilon$ for all $i \in J$ and all sufficiently large n. But for $i \in I \setminus J$ we have $\|x_{n,i} + x_i\| \leq \|x_{n,i}\| + \|x_i\| \leq 1 + \varepsilon \leq \alpha + \varepsilon$. Consequently, $\|x_n + x\|_\infty \leq \alpha + \varepsilon$ for all sufficiently large n, hence $\lim_{n\to\infty}\|x_n + x\|_\infty \leq \alpha + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we are done. $\square$

Concerning ℓ^1-sums it was already proved in [81, Theorem 3.13] that $R(X \oplus_1 Y) < 2$ if and only if both X and Y have the Schur property (in [82, Proposition 6.7] this is generalised to finite ℓ^1-sums together with various other equivalent conditions). The proof of the "only if" part from [81, Theorem 3.13] directly generalises to sums of infinitely many spaces and since it was proved in [133] that the ℓ^1-sum of any family of Banach spaces has the Schur property if and only if each summand has the Schur property, we obtain the following characterisation.

Proposition IV.3.6. *Let I be any index set with at least two elements. Let $(X_i)_{i\in I}$ be a family of Banach spaces and $X := \left[\bigoplus_{i\in I} X_i\right]_1$. The following assertions are equivalent:*

(i) $R(X) < 2$,

(ii) *X_i has the Schur property for each $i \in I$,*

(iii) *X has the Schur property,*

(iv) $R(X) = 1$.

IV.4 Opial properties of finite absolute sums

In this section we will briefly consider Opial properties of finite sums. This is surely well-known, but we will include the results and some of their proofs here as the author was not able to find them explicitly in the literature.

First we recall that an absolute, normalised norm $\|\cdot\|_E$ on $\mathbb{R}^m$ is said to be strictly monotone if for all $a = (a_1, \ldots, a_m), b = (b_1, \ldots, b_m) \in \mathbb{R}^m$ we have

$$\|a\|_E = \|b\|_E \text{ and } |a_i| \leq |b_i| \ \forall i = 1, \ldots, m \ \Rightarrow \ |a_i| = |b_i| \ \forall i = 1, \ldots, m.$$

Obviously, the p-norm is strictly monotone for every $p \in [1, \infty)$, but the maximum norm is not. Also, it is easy to see that strictly convex, absolute, normalised norms are strictly monotone.

Proposition IV.4.1. *Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^m$ and $X_1, \ldots, X_m$ Banach spaces with the nonstrict Opial property. Then $\left[\bigoplus_{i=1}^m X_i\right]_E$ has the nonstrict Opial property. If moreover $\|\cdot\|_E$ is strictly monotone and each X_i has the Opial property, then $\left[\bigoplus_{i=1}^m X_i\right]_E$ also has the Opial property.*

The proof is straightforward and will be omitted.

As is well known, every strictly monotone, absolute, normalised norm on $\mathbb{R}^m$ is actually uniformly monotone in the following sense (the proof consists in an easy compactness argument).

Lemma IV.4.2. *Let $\|\cdot\|_E$ be a strictly monotone, absolute, normalised norm on $\mathbb{R}^m$. Let $\varepsilon, R > 0$. The there exists $\delta > 0$ such that for all $a = (a_1, \ldots, a_m), b = (b_1, \ldots, b_m) \in \mathbb{R}^m$ with $\|b\|_E \leq R$ and $|a_i| \leq |b_i|$ for $i = 1 \ldots, m$ we have*

$$\|b\|_E - \|a\|_E < \delta \ \Rightarrow \ |b_i| - |a_i| < \varepsilon \ \forall i = 1, \ldots, m.$$

Utilizing this fact, one can see the following.

Proposition IV.4.3. *Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^m$ which is strictly monotone and $X_1, \ldots, X_m$ Banach spaces with the uniform Opial property. Then $X := \left[\bigoplus_{i=1}^m X_i\right]_E$ also has the uniform Opial property.*

Proof. Let $\varepsilon, R > 0$ and put $\eta := \min\{\eta_{X_i}(\varepsilon/m, R) : i = 1, \ldots, m\}$. Choose a $0 < \delta \leq 1$ according to Lemma IV.4.2 corresponding to the values η and $3R + 1$.
Now consider a weakly null sequence $(x_n)_{n\in\mathbb{N}} = ((x_{n,1}, \ldots, x_{n,m}))_{n\in\mathbb{N}}$ in X with $\limsup\|x_n\|_E \leq R$ and an element $y = (y_1, \ldots, y_m) \in X$ with $\|y\|_E \geq \varepsilon$. Since $\|y\|_E \leq \sum_{i=1}^m \|y_i\|$ there is some $i_0 \in \{1, \ldots, m\}$ with $\|y_{i_0}\| \geq \varepsilon/m$. There is no loss of generality in assuming that all the limits in the following calculations exist. From the definition of η we get

$$\lim_{n\to\infty} \|x_{n,i_0}\| + \eta \leq \lim_{n\to\infty} \|x_{n,i_0} - y_{i_0}\|.$$

Since each X_i has in particular the nonstrict Opial property, we also have

$$\lim_{n\to\infty} \|x_{n,i}\| \leq \lim_{n\to\infty} \|x_{n,i} - y_i\| \ \ \forall i \in \{1, \ldots, m\} \setminus \{i_0\}.$$

If $\|y\|_E \leq 2R+1$, then $\lim_{n\to\infty}\|x_n - y\|_E \leq \lim_{n\to\infty}\|x_n\|_E + 2R+1 \leq 3R+1$ and the choice of δ implies $\lim_{n\to\infty}\|x_n\|_E + \delta \leq \lim_{n\to\infty}\|x_n - y\|_E$.
If on the other hand $\|y\|_E > 2R+1$, then $\lim_{n\to\infty}\|x_n - y\|_E \geq \|y\|_E - \lim_{n\to\infty}\|x_n\|_E \geq R+1 \geq \lim_{n\to\infty}\|x_n\|_E + \delta$. So X has the uniform Opial property. □

IV.5 Opial properties of some infinite sums

Now we turn to infinite sums, but restricted to the special cases of ℓ^p- and c_0-sums. First we will show that the Opial and the nonstrict Opial property are preserved under infinite ℓ^p-sums.

Proposition IV.5.1. *If $1 \leq p < \infty$, I is any index set and $(X_i)_{i\in I}$ a family of Banach spaces with the Opial property (nonstrict Opial property), then $X := \left[\bigoplus_{i\in I} X_i\right]_p$ also has the Opial property (nonstrict Opial property).*

Proof. We will only prove the strict case, the nonstrict case is treated analogously. Let $x_n = (x_{n,i})_{i\in I} \in X$ for every $n \in \mathbb{N}$ such that $(x_n)_{n\in\mathbb{N}}$ converges weakly to zero and let $x = (x_i)_{i\in I} \in X \setminus \{0\}$. Fix $i_0 \in I$ with $x_{i_0} \neq 0$. We may assume that $\lim_{n\to\infty}\|x_n\|_p$ and $\lim_{n\to\infty}\|x_n - x\|_p$ as well as $a := \lim_{n\to\infty}\|x_{n,i_0}\|$ and $b := \lim_{n\to\infty}\|x_{n,i_0} - x_{i_0}\|$ exist. Note also that $(x_{n,i})_{n\in\mathbb{N}}$ is a weakly null sequence in X_i for each $i \in I$. So since X_{i_0} has the Opial property it follows that $\delta := b^p - a^p > 0$. Put $K := \sup_{n\in\mathbb{N}}\|x_n\|_p$ and let $0 < \varepsilon \leq 1$. We can find a finite set $J \subseteq I$ with $i_0 \in J$ such that

$$\left\|(\|x_i\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p \leq \varepsilon, \tag{IV.5.1}$$

where $\chi_{I\setminus J}$ denotes the characteristic function of $I \setminus J$. By passing to a further subsequence, we can assume that $\lim_{n\to\infty}\|x_{n,i}\|$ and $\lim_{n\to\infty}\|x_{n,i} - x_i\|$ exist for all $i \in J$. Then, using the Opial property of each of the summands X_i, the definition of δ and (IV.5.1), we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\|x_n\|_p^p = \sum_{i\in J\setminus\{i_0\}} \lim_{n\to\infty}\|x_{n,i}\|^p + a^p + \lim_{n\to\infty}\left\|(\|x_{n,i}\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p^p \\
&\leq \lim_{n\to\infty} \sum_{i\in J\setminus\{i_0\}} \|x_{n,i} - x_i\|^p + b^p - \delta + \lim_{n\to\infty}\left\|(\|x_{n,i}\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p^p \\
&\leq \lim_{n\to\infty} \sum_{i\in J}\|x_{n,i} - x_i\|^p - \delta + \lim_{n\to\infty}\left(\left\|(\|x_{n,i} - x_i\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p + \varepsilon\right)^p.
\end{aligned}$$

But, since $|s^p - t^p| \leq pA^{p-1}|s-t|$ for all $0 \leq s,t \leq A$, we also have

$$\begin{aligned}
&\lim_{n\to\infty}\left(\left\|(\|x_{n,i} - x_i\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p + \varepsilon\right)^p \leq \lim_{n\to\infty}\left\|(\|x_{n,i} - x_i\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p^p \\
&+ \lim_{n\to\infty}\left|\left(\left\|(\|x_{n,i} - x_i\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p + \varepsilon\right)^p - \left\|(\|x_{n,i} - x_i\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p^p\right| \\
&\leq \lim_{n\to\infty}\left\|(\|x_{n,i} - x_i\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p^p + p(K + \|x\|_p + 1)^{p-1}\varepsilon.
\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\|x_n\|_p^p \le \lim_{n\to\infty}\|x_n - x\|_p^p - \delta + p(K + \|x\|_p + 1)^{p-1}\varepsilon.$$

Since $\varepsilon \in (0, 1]$ was arbitrary and δ independent of ε, we conclude

$$\lim_{n\to\infty}\|x_n\|_p^p \le \lim_{n\to\infty}\|x_n - x\|_p^p - \delta < \lim_{n\to\infty}\|x_n - x\|_p^p$$

and the proof is finished. □

c_0 is a typical example of a Banach space which has the nonstrict Opial property but not the usual (strict) Opial property. Next we will see that c_0-sums preserve the nonstrict Opial property.

Proposition IV.5.2. *Let I be any index set and $(X_i)_{i\in I}$ a family of Banach spaces with the nonstrict Opial property. Then $X := \left[\bigoplus_{i\in I} X_i\right]_{c_0(I)}$ has the nonstrict Opial property.*

Proof. Let $x_n = (x_{n,i})_{i\in I} \in X$ for every $n \in \mathbb{N}$ such that $(x_n)_{n\in\mathbb{N}}$ converges weakly to zero and let $x = (x_i)_{i\in I} \in X$. Take $\varepsilon > 0$ to be arbitrary and find a finite subset $J \subseteq I$ such that $\|x_i\| \le \varepsilon$ for every $i \in I \setminus J$. Again there is no loss of generality in assuming that all the limits involved in the following calculations exist. Since each X_i has the nonstrict Opial property, we have

$$\lim_{n\to\infty}\|x_{n,i}\| \le \lim_{n\to\infty}\|x_{n,i} - x_i\| \quad \forall i \in J.$$

Therefore we obtain

$$\begin{aligned}
\lim_{n\to\infty}\|x_n\|_\infty &= \max\left\{\max_{i\in J}\lim_{n\to\infty}\|x_{n,i}\|, \lim_{n\to\infty}\|(\|x_{n,i}\|\chi_{I\setminus J}(i))_{i\in I}\|_\infty\right\}\\
&\le \max\left\{\max_{i\in J}\lim_{n\to\infty}\|x_{n,i} - x_i\|, \lim_{n\to\infty}\|(\|x_{n,i}\|\chi_{I\setminus J}(i))_{i\in I}\|_\infty\right\}\\
&\le \lim_{n\to\infty}\max\left\{\max_{i\in J}\|x_{n,i} - x_i\|, \|(\|x_{n,i} - x_i\|\chi_{I\setminus J}(i))_{i\in I}\|_\infty\right\} + \varepsilon\\
&= \lim_{n\to\infty}\|x_n - x\|_\infty + \varepsilon
\end{aligned}$$

and since $\varepsilon > 0$ was arbitrary we are done. □

Concerning the uniform Opial property, we have the following result for infinite ℓ^p-sums, resembling in structure Theorem IV.3.1.

Theorem IV.5.3. *Let $1 \le p < \infty$ and let I be an infinite index set. For a family $(X_i)_{i\in I}$ of Banach spaces put $X_J := \left[\bigoplus_{i\in J} X_i\right]_p$ for every finite $J \subseteq I$. Suppose that*

$$\omega(\varepsilon, R) := \inf\{\eta_{X_J}(\varepsilon, R) : J \subseteq I \text{ finite}\} > 0 \quad \forall \varepsilon, R > 0.$$

Then $X := \left[\bigoplus_{i\in I} X_i\right]_p$ has the uniform Opial property.

Proof. Let $0 < \varepsilon \leq 1$ and $R > 0$. We put $\nu := \min\{3R+1, \omega(\varepsilon/2, R)\}$ and $\tau := \min\big\{1, 3R+1-((3R+1)^p - \nu^p)^{1/p}\big\}$. Now let us consider a weakly null sequence $(x_n)_{n\in\mathbb{N}} = ((x_{n,i})_{i\in I})_{n\in\mathbb{N}}$ in X with $\limsup\|x_n\|_p \leq R$ and let $x = (x_i)_{i\in I} \in X$ with $\|x\|_p \geq \varepsilon$. As before, we may assume that $\lim_{n\to\infty}\|x_n\|_p$ and $\lim_{n\to\infty}\|x_n - x\|_p$ exist. Let $K := \sup_{n\in\mathbb{N}}\|x_n\|_p$. For $0 < \alpha \leq \varepsilon/2$ we can find a finite subset $J \subseteq I$ such that

$$\big\|(\|x_i\|\chi_{I\setminus J}(i))_{i\in I}\big\|_p \leq \alpha.$$

It follows that

$$\Big\|\sum_{i\in J}\|x_i\|e_i\Big\|_p \geq \|x\|_p - \alpha \geq \varepsilon/2. \tag{IV.5.2}$$

We may also assume that $\lim_{n\to\infty}\|x_{n,i}\|$ and $\lim_{n\to\infty}\|x_{n,i} - x_i\|$ exist for all $i \in J$. Analogously to the proof of Proposition IV.5.1 we can show that

$$\begin{aligned}\lim_{n\to\infty}\|x_n\|_p^p &\leq \lim_{n\to\infty}\sum_{i\in J}\|x_{n,i}\|^p + \lim_{n\to\infty}\big\|(\|x_{n,i} - x_i\|\chi_{I\setminus J}(i))_{i\in I}\big\|_p^p \\ &+ p(K + \|x\|_p + 1)^{p-1}\alpha. \end{aligned}\tag{IV.5.3}$$

If we put $y_n := (x_{n,i})_{i\in J}$ for each $n \in \mathbb{N}$ and $y := (x_i)_{i\in J}$, then $(y_n)_{n\in\mathbb{N}}$ is a weakly null sequence in X_J with $\lim_{n\to\infty}\|y_n\|_p \leq \lim_{n\to\infty}\|x_n\|_p \leq R$ and $y \in X_J$ with $\|y\|_p \geq \varepsilon/2$ (because of (IV.5.2)), thus

$$\lim_{n\to\infty}\|y_n\|_p + \eta_{X_J}(\varepsilon/2, R) \leq \lim_{n\to\infty}\|y_n - y\|_p. \tag{IV.5.4}$$

Since $(a-b)^p \leq a^p - b^p$ for all $a \geq b \geq 0$ we can deduce from (IV.5.3) and (IV.5.4) that

$$\begin{aligned}\lim_{n\to\infty}\|x_n\|_p^p &\leq \lim_{n\to\infty}\|y_n - y\|_p^p - \eta_{X_J}(\varepsilon/2, R)^p \\ &+ \lim_{n\to\infty}\big\|(\|x_{n,i} - x_i\|\chi_{I\setminus J}(i))_{i\in I}\big\|_p^p + p(K + \|x\|_p + 1)^{p-1}\alpha \\ &\leq \lim_{n\to\infty}\|x_n - x\|_p^p - \nu^p + p(K + \|x\|_p + 1)^{p-1}\alpha.\end{aligned}$$

Letting $\alpha \to 0$ we obtain

$$\lim_{n\to\infty}\|x_n\|_p^p \leq \lim_{n\to\infty}\|x_n - x\|_p^p - \nu^p. \tag{IV.5.5}$$

If $\|x\|_p \geq 2R+1$, then $\lim_{n\to\infty}\|x_n - x\|_p \geq 2R + 1 - \lim_{n\to\infty}\|x_n\|_p \geq \lim_{n\to\infty}\|x_n\|_p + 1 \geq \lim_{n\to\infty}\|x_n\|_p + \tau$.

Now consider the case $\|x\|_p < 2R+1$. Define $f(s) := s - (s^p - \nu^p)^{1/p}$ for all $s \geq \nu$. It is easily checked that f is decreasing. Since $\lim_{n\to\infty}\|x_n - x\|_p \leq \lim_{n\to\infty}\|x_n\|_p + \|x\|_p \leq 3R+1$ it follows from (IV.5.5) that

$$\begin{aligned}\lim_{n\to\infty}\|x_n\|_p &\leq \lim_{n\to\infty}\|x_n - x\|_p - f(\lim_{n\to\infty}\|x_n - x\|_p) \\ &\leq \lim_{n\to\infty}\|x_n - x\|_p - f(3R+1) \leq \lim_{n\to\infty}\|x_n - x\|_p - \tau\end{aligned}$$

(where the last inequality holds by the definition of τ) and the proof is complete. $\square$

As a corollary we obtain again the already known result that the ℓ^p-sum of any family of Banach spaces with the Schur property has the uniform Opial property (see [114, Example 4.23 (2.)] or [119, Theorem 7]).

Corollary IV.5.4. *Let $1 \leq p < \infty$ and let $(X_i)_{i \in I}$ be a family of Banach spaces with the Schur property. Then $\left[\bigoplus_{i \in I} X_i\right]_p$ has the uniform Opial property.*

Let us also remark that in [97, Theorem 3.4] it is claimed that certain spaces $\ell_\Phi^A(X)$ of vector-valued sequences have the uniform Opial property, where the class of sequence spaces ℓ_Φ^A includes in particular the spaces ℓ^p for $1 \leq p < \infty$, but no assumptions on the Banach space X are made. This cannot be true since, for example, the fact that $\ell^p(X) = \left[\bigoplus_{n \in \mathbb{N}} X\right]_p$ has the uniform Opial property clearly implies that X itself has the uniform Opial property. Examining the proof of [97, Theorem 3.4], one finds that it is implicitly used that X has the Schur property.

The author does not know whether the conditon $\inf_{i \in I} \eta_{X_i} > 0$ is already enough to ensure that $\left[\bigoplus_{i \in I} X_i\right]_p$ has the uniform Opial property (the proof of Proposition IV.4.3 does not give a uniform lower bound for the moduli of the finite sums).

IV.6 Opial-type properties in Lebesgue-Bochner spaces

In this section we will consider Lebesgue-Bochner spaces $L^p(\mu, X)$ over a complete, finite measure space $(S, \mathcal{A}, \mu)$.

As was mentioned in Section IV.1, even the spaces $L^p[0,1]$, $1 < p < \infty, p \neq 2$, of scalar-valued functions do not have the Opial property. However, some results which are in a certain sense analogous to the Opial property are available. For example it was shown in [17] that any bounded sequence $(f_n)_{n \in \mathbb{N}}$ in $L^p(\mu)$ $(0 < p < \infty)$ which converges pointwise almost everywhere to a function $f \in L^p(\mu)$ satisfies

$$\lim_{n \to \infty} \left(\|f_n\|_p^p - \|f_n - f\|_p^p \right) = \|f\|_p^p$$

and hence

$$\liminf_{n \to \infty} \|g_n - g\|_p^p = \liminf_{n \to \infty} \|g_n\|_p^p + \|g\|_p^p$$

for any bounded sequence $(g_n)_{n \in \mathbb{N}}$ in $L^p(\mu)$ which converges pointwise almost everywhere to zero and every $g \in L^p(\mu)$.

In [11, Chapter 2, Lemma 3.3] it was shown that any sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1(\mu, X)$ (where $(S, \mathcal{A}, \mu)$ is a probability space and X an arbitrary Banach space) and any $f \in L^1(\mu, X)$ such that

$$\lim_{n\to\infty} \mu(\{t \in S : \|f_n(t) - f(t)\| \geq \varepsilon\}) = 0 \quad \forall \varepsilon > 0$$

satisfy the equality

$$\liminf_{n\to\infty} \|f_n - f\|_1 + \|f - g\|_1 = \liminf_{n\to\infty} \|f_n - g\|_1$$

for every $g \in L^1(\mu, X)$.

We are now going to consider pointwise weak convergence almost everywhere in Lebesgue-Bochner spaces and prove some results analogous to the Opial property in this setting. We begin with the following general Theorem.

Theorem IV.6.1. *Let $(S, \mathcal{A}, \mu)$ be a complete, finite measure space, $1 \leq p < \infty$ and X a Banach space with the nonstrict Opial property. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^p(\mu, X)$ such that $(f_n(t))_{n\in\mathbb{N}}$ converges weakly to zero for almost every $t \in S$. Suppose further that there is a function $g \in L^p(\mu)$ such that $\|f_n(t)\| \to g(t)$ for almost every $t \in S$. Then*

$$\begin{aligned}&\int_A \liminf_{n\to\infty} \|f_n(t) - f(t)\|^p \, \mathrm{d}\mu(t) - \int_A g(t)^p \, \mathrm{d}\mu(t) \\ &\leq \limsup_{n\to\infty} \|f_n - f\|_p^p - \limsup_{n\to\infty} \|f_n\|_p^p\end{aligned}$$

holds for every $f \in L^p(\mu, X)$ and every $A \in \mathcal{A}$. In particular,

$$\limsup_{n\to\infty} \|f_n\|_p \leq \limsup_{n\to\infty} \|f_n - f\|_p \quad \forall f \in L^p(\mu, X).$$

Proof. Without loss of generality we can assume that $\lim_{n\to\infty} \|f_n(t)\| = g(t)$ and $f_n(t) \to 0$ weakly for every $t \in S$ and also that $\lim_{n\to\infty} \|f_n\|_p$ and $\lim_{n\to\infty} \|f_n - f\|_p$ exist.
Note that it follows from Fatou's Lemma and the boundedness of $(f_n)_{n\in\mathbb{N}}$ that $\liminf_{n\to\infty} \|f_n - f\|^p$ is integrable over S.
Since X has the nonstrict Opial property we have $g(t) \leq \liminf \|f_n(t) - f(t)\|$ for every $t \in S$. Therefore it suffices to prove the statement for $A = S$.
Now let $0 < \varepsilon < 1$. By the equi-integrability of finite subsets of $L^1(\mu)$ there exists $\delta > 0$ such that

$$B \in \mathcal{A},\ \mu(B) \leq \delta \ \Rightarrow \ \int_B h(t) \, \mathrm{d}\mu(t) \leq \varepsilon \tag{IV.6.1}$$

$$\text{for each } h \in \Big\{\|f(\cdot)\|^p, g^p, \liminf_{n\to\infty} \|f_n(\cdot) - f(\cdot)\|^p\Big\}.$$

By Egorov's theorem (cf. [56, Theorem A, p.88]) there exists $C \in \mathcal{A}$ with $\mu(S \setminus C) \leq \delta$ such that $\lim_{n\to\infty} \|f_n(t)\|^p = g(t)^p$ uniformly in $t \in C$, which implies

$$\lim_{n\to\infty} \int_C \|f_n(t)\|^p \, \mathrm{d}\mu(t) = \int_C g(t)^p \, \mathrm{d}\mu(t). \tag{IV.6.2}$$

We can find a subsequence such that

$$\lim_{k\to\infty}\int_C \|f_{n_k}(t)-f(t)\|^p \,\mathrm{d}\mu(t)$$

exists. Now we can calculate, using (IV.6.2),

$$\begin{aligned}
&\lim_{n\to\infty}\|f_n\|_p^p = \int_C g(t)^p \,\mathrm{d}\mu(t) + \lim_{n\to\infty}\int_{S\setminus C}\|f_n(t)\|^p \,\mathrm{d}\mu(t)\\
&= \int_C \liminf_{n\to\infty}\|f_n(t)-f(t)\|^p \,\mathrm{d}\mu(t)\\
&+ \int_C \Big(g(t)^p - \liminf_{n\to\infty}\|f_n(t)-f(t)\|^p\Big)\,\mathrm{d}\mu(t) + \lim_{n\to\infty}\int_{S\setminus C}\|f_n(t)\|^p \,\mathrm{d}\mu(t).
\end{aligned} \tag{IV.6.3}$$

But

$$\left|\int_C g(t)^p \,\mathrm{d}\mu(t) - \int_S g(t)^p \,\mathrm{d}\mu(t)\right| = \int_{S\setminus C} g(t)^p \,\mathrm{d}\mu(t) \le \varepsilon, \tag{IV.6.4}$$

because of $\mu(S\setminus C)\le\delta$ and (IV.6.1). Analogously,

$$\int_{S\setminus C}\liminf_{n\to\infty}\|f_n(t)-f(t)\|^p \,\mathrm{d}\mu(t) \le \varepsilon. \tag{IV.6.5}$$

Putting (IV.6.3), (IV.6.4) and (IV.6.5) together we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\|f_n\|_p^p \le \int_C \liminf_{n\to\infty}\|f_n(t)-f(t)\|^p \,\mathrm{d}\mu(t) + \lim_{n\to\infty}\int_{S\setminus C}\|f_n(t)\|^p \,\mathrm{d}\mu(t)\\
&+ \int_S \Big(g(t)^p - \liminf_{n\to\infty}\|f_n(t)-f(t)\|^p\Big)\,\mathrm{d}\mu(t) + 2\varepsilon\\
&\le \lim_{k\to\infty}\int_C \|f_{n_k}(t)-f(t)\|^p \,\mathrm{d}\mu(t) + \lim_{k\to\infty}\int_{S\setminus C}\|f_{n_k}(t)\|^p \,\mathrm{d}\mu(t)\\
&+ \int_S \Big(g(t)^p - \liminf_{n\to\infty}\|f_n(t)-f(t)\|^p\Big)\,\mathrm{d}\mu(t) + 2\varepsilon,
\end{aligned} \tag{IV.6.6}$$

where we have used Fatou's lemma in the second step.
Since $\mu(S\setminus C)\le\delta$ we have $\int_{S\setminus C}\|f(t)\|^p \,\mathrm{d}\mu(t)\le\varepsilon$, by (IV.6.1). Hence

$$\lim_{k\to\infty}\int_{S\setminus C}\|f_{n_k}(t)\|^p \,\mathrm{d}\mu(t) \le \lim_{k\to\infty}\left(\left(\int_{S\setminus C}\|f_{n_k}(t)-f(t)\|^p \,\mathrm{d}\mu(t)\right)^{1/p} + \varepsilon^{1/p}\right)^p.$$

Since $|s^p - t^p| \le pA^{p-1}|s-t|$ for all $0\le s,t\le A$, we obtain as in the proof of Proposition IV.5.1

$$\lim_{k\to\infty}\int_{S\setminus C}\|f_{n_k}(t)\|^p \,\mathrm{d}\mu(t) \le \lim_{k\to\infty}\int_{S\setminus C}\|f_{n_k}(t)-f(t)\|^p \,\mathrm{d}\mu(t) + pL^{p-1}\varepsilon^{1/p}, \tag{IV.6.7}$$

where $L := \|f\|_p + 1 + \sup_{n\in\mathbb{N}}\|f_n\|_p$.
From (IV.6.6) and (IV.6.7) it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\|f_n\|_p^p \le \lim_{k\to\infty}\int_S \|f_{n_k}(t) - f(t)\|^p \,\mathrm{d}\mu(t) + pL^{p-1}\varepsilon^{1/p} \\
&+ \int_S \Big(g(t)^p - \liminf_{n\to\infty}\|f_n(t) - f(t)\|^p\Big)\,\mathrm{d}\mu(t) + 2\varepsilon \\
&= \lim_{n\to\infty}\|f_n - f\|_p^p + pL^{p-1}\varepsilon^{1/p} + 2\varepsilon \\
&+ \int_S \Big(g(t)^p - \liminf_{n\to\infty}\|f_n(t) - f(t)\|^p\Big)\,\mathrm{d}\mu(t).
\end{aligned}$$

Letting $\varepsilon \to 0$ now leads to the desired inequality. □

If X has the Opial property, we have the following corollary.

Corollary IV.6.2. *Let $(S, \mathcal{A}, \mu)$ be a complete, finite measure space, $1 \le p < \infty$ and X a Banach space with the Opial property. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^p(\mu, X)$ such that $(f_n(t))_{n\in\mathbb{N}}$ converges weakly to zero for almost every $t \in S$. Suppose further that there is a function $g \in L^p(\mu)$ such that $\|f_n(t)\| \to g(t)$ for almost every $t \in S$. Then*

$$\limsup_{n\to\infty}\|f_n\|_p < \limsup_{n\to\infty}\|f_n - f\|_p \quad \forall f \in L^p(\mu, X)\setminus\{0\}.$$

Proof. Just put $A := \{t \in S : f(t) \ne 0\}$ in Theorem IV.6.1. Then $\mu(A) > 0$ and since X has the Opial property we have $\liminf\|f_n(t) - f(t)\| < g(t)$ for every $t \in A$, so the result follows from Theorem IV.6.1. □

In the case that X even has the uniform Opial property, we have the following two results.

Theorem IV.6.3. *Let $(S, \mathcal{A}, \mu)$ be a complete, finite measure space, $1 \le p < \infty$ and X a Banach space with the uniform Opial property. Let $M, R > 0$ and $f \in L^p(\mu, X)\setminus\{0\}$. Then there exists $\eta > 0$ such that the following holds: whenever $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^p(\mu, X)$ with $\sup_{n\in\mathbb{N}}\|f_n\|_p \le R$ such that $(f_n(t))_{n\in\mathbb{N}}$ converges weakly to zero and $\lim_{n\to\infty}\|f_n(t)\| \le M$ for almost every $t \in S$, then*

$$\limsup_{n\to\infty}\|f_n\|_p + \eta \le \limsup_{n\to\infty}\|f_n - f\|_p.$$

Proof. We define $\tau := \|f\|_p(2\mu(S))^{-1/p}$ and $A := \{t \in S : \|f(t)\| \ge \tau\}$. If $\mu(A) = 0$, then we would obtain $\|f\|_p^p \le \mu(S\setminus A)\tau^p \le \|f\|_p^p/2$, contradicting the fact that $f \in L^p(\mu, X)\setminus\{0\}$. Thus $\mu(A) > 0$.
Next we put $w := \eta_X(\tau, M)$, $\delta := \min\{(3R+1)^p, \mu(A)w^p\}$, $\omega := R + 1 - ((R+1)^p - \delta)^{1/p}$ and finally $\eta := \min\{\omega, 1\}$.

Now let $(f_n)_{n\in\mathbb{N}}$ be as above. Without loss of generality we may assume that $g(t) := \lim_{n\to\infty}\|f_n(t)\| \le M$ and $f_n(t) \to 0$ weakly for every $t \in S$. The definition of η_X implies

$$\liminf_{n\to\infty}\|f_n(t) - f(t)\| - g(t) \ge \eta_X(\tau, M) = w \quad \forall t \in A.$$

Since $(a-b)^p \le a^p - b^p$ for all $a \ge b \ge 0$, it follows that

$$\liminf_{n\to\infty}\|f_n(t) - f(t)\|^p - g(t)^p \ge w^p \quad \forall t \in A.$$

Combinig this with Theorem IV.6.1 leads to

$$\limsup_{n\to\infty}\|f_n - f\|_p^p - \limsup_{n\to\infty}\|f_n\|_p^p \ge \mu(A)w^p \ge \delta.$$

As in the proof of Theorem IV.5.3, by distinguishing the two cases $\|f\|_p \ge 2R+1$ and $\|f\|_p < 2R+1$, we can deduce from this that

$$\limsup_{n\to\infty}\|f_n\|_p + \eta \le \limsup_{n\to\infty}\|f_n - f\|_p.$$

□

Theorem IV.6.4. *Let $(S, \mathcal{A}, \mu)$ be a complete, finite measure space, $1 \le p < \infty$ and X a Banach space with the uniform Opial property. Let $p < r \le \infty$ and $\varepsilon, M, R, K > 0$. Then there exists $\eta > 0$ such that the following holds: whenever $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^p(\mu, X)$ with $\sup_{n\in\mathbb{N}}\|f_n\|_p \le R$ such that $(f_n(t))_{n\in\mathbb{N}}$ converges weakly to zero and $\lim_{n\to\infty}\|f_n(t)\| \le M$ for almost every $t \in S$ and $f \in L^r(\mu, X) \subseteq L^p(\mu, X)$ such that $\|f\|_r \le K$ and $\|f\|_p \ge \varepsilon$, then*

$$\limsup_{n\to\infty}\|f_n\|_p + \eta \le \limsup_{n\to\infty}\|f_n - f\|_p.$$

Proof. We put $s := r/p \in (1, \infty]$. Let $s' \in [1, \infty)$ such that $1/s' + 1/s = 1$. Choose $0 < \tau < \varepsilon\mu(S)^{-1/p}$ and put $Q := (\varepsilon^p - \mu(S)\tau^p)^{s'}K^{-ps'}$. Let $w := \eta_X(\tau, M)$ and $\delta := \min\{Qw^p, (3R+1)^p\}$. ω and η are also defined as in the previous proof.
Now let $(f_n)_{n\in\mathbb{N}}$ and f be as above. For $A := \{t \in S : \|f(t)\| \ge \tau\}$ we have

$$\begin{aligned}
\varepsilon^p \le \|f\|_p^p &= \int_A \|f(t)\|^p \,\mathrm{d}\mu(t) + \int_{S\setminus A} \|f(t)\|^p \,\mathrm{d}\mu(t) \\
&\le \int_A \|f(t)\|^p \,\mathrm{d}\mu(t) + \mu(S\setminus A)\tau^p \le \mu(A)^{1/s'}\|f\|_r^p + \mu(S)\tau^p \\
&\le \mu(A)^{1/s'}K^p + \mu(S)\tau^p,
\end{aligned}$$

where we have used Hölder's inequality in the second line. It follows that $\mu(A) \ge Q$. As in the previous proof we can deduce that

$$\limsup_{n\to\infty}\|f_n - f\|_p^p - \limsup_{n\to\infty}\|f_n\|_p^p \ge \mu(A)w^p \ge Qw^p \ge \delta$$

and from there we get to

$$\limsup_{n\to\infty}\|f_n\|_p + \eta \leq \limsup_{n\to\infty}\|f_n - f\|_p$$

as in the proof of Theorem IV.5.3. □

IV.7 Opial properties in Cesàro sums

In this section we will have a brief look at Cesàro sequence spaces and Opial properties in the associated sums. For $1 \leq p < \infty$, the Cesàro sequence space ces_p is defined as the space of all sequences $a = (a_n)_{n\in\mathbb{N}}$ of real numbers such that

$$\|a\|_{\mathrm{ces}_p} := \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|a_i|\right)^p\right)^{1/p} < \infty.$$

$\|\cdot\|_{\mathrm{ces}_p}$ defines a norm on ces_p. Leibowitz [88] and Jagers [68] proved that $\mathrm{ces}_1 = \{0\}$ and ces_p is separable and reflexive for $1 < p < \infty$. In [10] it was proved that for any $p \in (1, \infty)$, the space ces_p is not isomorphic to ℓ^q for any $q \in [1, \infty]$.

Note that the spaces ces_p are not in the class of sequence spaces with absolute, normalised norms that we have considered in Chapter II. They satisfy the monotonicity condition, i. e. if $b \in \mathrm{ces}_p$ and $a \in \mathbb{R}^{\mathbb{N}}$ with $|a_n| \leq |b_n|$ for every $n \in \mathbb{N}$, then $a \in \mathrm{ces}_p$ and $\|a\|_{\mathrm{ces}_p} \leq \|b\|_{\mathrm{ces}_p}$, and for $1 < p < \infty$ one also has $e_m \in \mathrm{ces}_p$ for each m, but the norm $\|e_m\|_{\mathrm{ces}_p} = (\sum_{n=m}^{\infty} 1/n^p)^{1/p}$ depends on m.

Nonetheless, given a sequence $(X_n)_{n\in\mathbb{N}}$ of Banach spaces and $p \in (1, \infty)$, we may define the p-Cesàro sum $\left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{\mathrm{ces}_p}$ of $(X_n)_{n\in\mathbb{N}}$ as the space of all sequences $x = (x_n)_{n\in\mathbb{N}}$ with $x_n \in X_n$ for each n such that $(\|x_n\|)_{n\in\mathbb{N}} \in \mathrm{ces}_p$, equipped with the norm $\|x\|_{\mathrm{ces}_p} := \|(\|x_n\|)_{n\in\mathbb{N}}\|_{\mathrm{ces}_p}$.

In [24] it was proved that ces_p has the uniform Opial property for every $p \in (1, \infty)$. In [119, Theorem 1] Saejung proved that ces_p can be regarded as a subspace of the sum $X_p := \left[\bigoplus_{n\in\mathbb{N}} \ell^1(n)\right]_p$ (where $\ell^1(n)$ denotes the n-dimensional space with ℓ^1-norm) via the isometric embedding $T : \mathrm{ces}_p \to X_p$ defined by

$$(Ta)(n) := \frac{1}{n}(a_1, \dots, a_n) \quad \forall n \in \mathbb{N}, \forall a \in \mathrm{ces}_p.$$

As mentioned before, in [119, Theorem 7] it is proved that the ℓ^p-sum of any sequence of finite-dimensional spaces has the uniform Opial property (see also [114, Example 4.23 (2.)] and Corollary IV.5.4 above). Thus Saejung obtains a new proof that ces_p has the uniform Opial property ([119, Corollary 9]).

Saejung's embedding idea directly generalises to ces_p-sums. For a given sequence $(X_n)_{n\in\mathbb{N}}$ of Banach spaces we consider the mapping S from the

Cesàro sum $\big[\bigoplus_{n\in\mathbb{N}} X_n\big]_{\mathrm{ces}_p}$ to $\big[\bigoplus_{n\in\mathbb{N}}(X_1 \oplus_1 \cdots \oplus_1 X_n)\big]_p$ defined by

$$(Sx)(n) := \frac{1}{n}(x_1, \dots, x_n) \quad \forall n \in \mathbb{N}, \forall x \in \Big[\bigoplus_{n\in\mathbb{N}} X_n\Big]_{\mathrm{ces}_p}.$$

Then S is an isometric embedding.

Now if each X_n has the Opial property (nonstrict Opial property), then $X_1 \oplus_1 \cdots \oplus_1 X_n$ also has the Opial property (nonstrict Opial property) for every $n \in \mathbb{N}$ (Proposition IV.4.1).

Hence by Proposition IV.5.1 $\big[\bigoplus_{n\in\mathbb{N}}(X_1 \oplus_1 \cdots \oplus_1 X_n)\big]_p$ also has the Opial property (nonstrict Opial property), and since $\big[\bigoplus_{n\in\mathbb{N}} X_n\big]_{\mathrm{ces}_p}$ can be viewed as a subspace of $\big[\bigoplus_{n\in\mathbb{N}}(X_1 \oplus_1 \cdots \oplus_1 X_n)\big]_p$ we have proved the following.

Proposition IV.7.1. *Let $p \in (1,\infty)$. If $(X_n)_{n\in\mathbb{N}}$ is a sequence of Banach spaces such that each X_n has the Opial property (nonstrict Opial property), then $\big[\bigoplus_{n\in\mathbb{N}} X_n\big]_{ces_p}$ also has the Opial property (nonstrict Opial property).*

By an analogous argument using Theorem IV.5.3 one obtains the following result concerning the uniform Opial property.

Proposition IV.7.2. *Let $p \in (1,\infty)$ and $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. Put $Y_m := \big[\bigoplus_{n=1}^{m}(X_1 \oplus_1 \cdots \oplus_1 X_n)\big]_p$ for each $m \in \mathbb{N}$. If*

$$\inf_{m\in\mathbb{N}} \eta_{Y_m}(\varepsilon, R) > 0 \quad \forall \varepsilon, R > 0,$$

then $\big[\bigoplus_{n\in\mathbb{N}} X_n\big]_{ces_p}$ has the uniform Opial property.

For more information on Cesàro sequence spaces, see for example the introduction of [7] and references therein.

IV.8 Opial-type properties in Cesàro spaces of vector-valued functions

Now we come to Cesàro function spaces on the interval $[0,1]$. For $1 \le p < \infty$ the Cesàro function space Ces_p is defined as the space of all measurable functions $f : [0,1] \to \mathbb{R}$ such that

$$\int_0^1 \left(\frac{1}{t}\int_0^t |f(s)|\,\mathrm{d}s\right)^p \mathrm{d}t < \infty,$$

where, as usual, two functions are identified if they agree a. e. The norm in Ces_p is given by

$$\|f\|_{\mathrm{Ces}_p} := \left(\int_0^1 \left(\frac{1}{t}\int_0^t |f(s)|\,\mathrm{d}s\right)^p \mathrm{d}t\right)^{1/p}.$$

We will first list some known results on these spaces:

(1) Ces_1 is the weighted Lebesgue space $L^1_w[0,1]$, where $w(t) := \log(1/t)$.

(2) $\mathrm{Ces}_p|_{[0,a]}$ is a subspace of $L^p[0,a]$ for every $p \in [1,\infty)$ and every $a \in (0,1)$, but not for $a = 1$.

(3) Ces_p is separable and nonreflexive for every $p \in [1,\infty)$.

(4) For $1 < p < \infty$ one has $L^p[0,1] \subseteq \mathrm{Ces}_p$ and $\|f\|_{\mathrm{Ces}_p} \leq q\|f\|_p$ for all $f \in L^p[0,1]$, where q is the conjugated exponent to p.

These and further results are collected in [7, Theorem 1]. Also, by [7, Theorem 7], for $p \in (1,\infty)$ the space Ces_p is not isomorphic to $L^q[0,1]$ for any $q \in [1,\infty]$. In [6] it was proved that Ces_p does not have the fixed point property and the dual space Ces_p^* does not even have the weak fixed point property.

For further information on Cesàro function spaces see [6, 7] and references therein.

Since $\mathrm{Ces}_p \not\subseteq L^1[0,1]$, the Cesàro function spaces do not belong to the class of Köthe function spaces that we have considered in Chapter III, but they satisfy the important monotonicity property: if $f \in \mathrm{Ces}_p$ and $g : [0,1] \to \mathbb{R}$ is measurable with $|g(t)| \leq |f(t)|$ a. e., then $g \in \mathrm{Ces}_p$ and $\|g\|_{\mathrm{Ces}_p} \leq \|f\|_{\mathrm{Ces}_p}$.

Therefore, given a Banach space X, we can define the space $\mathrm{Ces}_p(X)$ of all (equivalence classes of) Bochner-measurable functions $f : [0,1] \to X$ such that $\|f(\cdot)\| \in \mathrm{Ces}_p$, equipped with the norm $\|f\|_{\mathrm{Ces}_p(X)} := \|\|f(\cdot)\|\|_{\mathrm{Ces}_p}$.

We will prove a result for sequences of functions in $\mathrm{Ces}_p(X)$ which are pointwise almost everywhere convergent to zero with respect to the weak topology of X, where X is assumed to have the nonstrict Opial property. The result is similar Theorem IV.6.1 for Lebesgue-Bochner spaces. The proof also makes use of similar techniques.

Theorem IV.8.1. *Let $1 \leq p < \infty$ and let X be a Banach space with the nonstrict Opial property. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $Ces_p(X)$ such that $(f_n(t))_{n\in\mathbb{N}}$ converges weakly to zero for almost every $t \in [0,1]$. Suppose further that there exists a $g \in Ces_p$ such that $\|f_n(t)\| \to g(t)$ a. e. Let $f \in Ces_p(X)$ and $\varphi(t) := \liminf_{n\to\infty}\|f_n(t) - f(t)\|$ for $t \in [0,1]$. Then*

$$\begin{aligned}
&2^{p-1}\int_0^1 \frac{1}{t^p}\left(\left(\int_0^t \varphi(s)\,ds\right)^p - \left(\int_0^t g(s)\,ds\right)^p\right)dt \\
&\leq 2^{p-1}\limsup_{n\to\infty}\|f_n - f\|^p_{Ces_p(X)} - \limsup_{n\to\infty}\|f_n\|^p_{Ces_p(X)}.
\end{aligned}$$

In particular,

$$\limsup_{n\to\infty}\|f_n\|_{Ces_p(X)} \leq 2^{1-1/p}\limsup_{n\to\infty}\|f_n - f\|_{Ces_p(X)} \quad \forall f \in Ces_p(X).$$

Proof. Using the identification of Ces_1 with $L^1_w[0,1]$ from [7, Theorem 1] (where $w(t) = \log(1/t)$) the assertion for $p = 1$ easily follows from Theorem IV.6.1. We will therefore assume $p > 1$.

So let $f \in \mathrm{Ces}_p(X)$. Without loss of generality, we may assume that $\lim_{n\to\infty}\|f_n\|_{\mathrm{Ces}_p(X)}$ and $\lim_{n\to\infty}\|f_n - f\|_{\mathrm{Ces}_p(X)}$ exist and also that $\|f_n(t)\| \to g(t)$ and $f_n(t) \to 0$ weakly for every $t \in [0,1]$.
Since $M := \sup_{n\in\mathbb{N}}\|f_n\|_{\mathrm{Ces}_p(X)} < \infty$ it follows from Fatou's Lemma that $\varphi \in \mathrm{Ces}_p$.
Let us put

$$a := \|\varphi\|^p_{\mathrm{Ces}_p} - \|g\|^p_{\mathrm{Ces}_p} = \int_0^1 \frac{1}{t^p}\left(\left(\int_0^t \varphi(s)\,\mathrm{d}s\right)^p - \left(\int_0^t g(s)\,\mathrm{d}s\right)^p\right)\mathrm{d}t.$$

Since X has the nonstrict Opial property we have $\varphi(t) \geq g(t)$ for every $t \in [0,1]$. Hence $a \geq 0$.
Let $0 < \varepsilon < 1$. Denote by λ the Lebesgue measure on $[0,1]$.
The equi-integrability of finite subsets of L^1 enables us to find a $0 < \tau < \varepsilon$ such that for every measurable set $A \subseteq [0,1]$ one has

$$\lambda(A) \leq \tau \;\Rightarrow\; \int_A \left(\frac{1}{t}\int_0^t F(s)\,\mathrm{d}s\right)^p \mathrm{d}t \leq \varepsilon \quad \forall F \in \{\|f(\cdot)\|, \varphi, g\}. \tag{IV.8.1}$$

Next we choose $0 < \theta < \tau$ such that

$$\frac{p\theta}{1-p}\left(\left(1-\frac{\tau}{3}\right)^{1-p} - \left(\frac{\tau}{3}\right)^{1-p}\right)\left(\int_0^{1-\frac{\tau}{3}} F(s)\,\mathrm{d}s\right)^{p-1} \leq \varepsilon \tag{IV.8.2}$$

for $F \in \{\varphi, g\}$ and then, again by equi-integrability, we find $\delta > 0$ such that for every measurable subset $D \subseteq [0, 1-\frac{\tau}{3}]$ one has

$$\lambda(D) \leq \delta \;\Rightarrow\; \int_D F(s)\,\mathrm{d}s \leq \theta \quad \forall F \in \{\|f(\cdot)\|, \varphi, g\} \tag{IV.8.3}$$

(remember that $\mathrm{Ces}_p|_{[0,b]} \subseteq L^1[0,b]$ for every $b \in (0,1)$).
Now we apply Egorov's theorem (cf. [56, Theorem A, p.88]) to find a measurable set $C \subseteq [0,1]$ with $\lambda([0,1]\setminus C) \leq \delta$ such that $\|f_n(t)\| \to g(t)$ uniformly in $t \in C$. It follows that

$$\lim_{n\to\infty}\int_{[0,t]\cap C}\|f_n(s)\|\,\mathrm{d}s = \int_{[0,t]\cap C} g(s)\,\mathrm{d}s \quad \forall t \in [0,1).$$

Thus we can apply Egorov's theorem once more to deduce that there exists a measurable set $F \subseteq [0,1)$ with $\lambda([0,1]\setminus F) \leq \tau/3$ such that

$$\lim_{n\to\infty}\left(\frac{1}{t}\int_{[0,t]\cap C}\|f_n(s)\|\,\mathrm{d}s\right)^p = \left(\frac{1}{t}\int_{[0,t]\cap C} g(s)\,\mathrm{d}s\right)^p \text{ uniformly in } t \in F.$$

Put $B := F \cap [\frac{\tau}{3}, 1-\frac{\tau}{3}]$. Then $\lambda([0,1]\setminus B) \leq \lambda([0,1]\setminus F) + 2\tau/3 \leq \tau$ and

$$\lim_{n\to\infty}\int_B\left(\frac{1}{t}\int_{[0,t]\cap C}\|f_n(s)\|\,\mathrm{d}s\right)^p \mathrm{d}t = \int_B\left(\frac{1}{t}\int_{[0,t]\cap C} g(s)\,\mathrm{d}s\right)^p \mathrm{d}t. \tag{IV.8.4}$$

Since $M = \sup_{n\in\mathbb{N}}\|f_n\|_{\mathrm{Ces}_p(X)} < \infty$, we can find a subsequence $(n_k)_{k\in\mathbb{N}}$ of indices such that all the limits involved in the following calculations exist.

We have

$$
\begin{aligned}
&\lim_{n\to\infty}\|f_n\|^p_{\mathrm{Ces}_p(X)}\\
&= \lim_{k\to\infty}\left(\int_B\left(\frac{1}{t}\int_0^t\|f_{n_k}(s)\|\,\mathrm{d}s\right)^p\mathrm{d}t + \int_{[0,1]\setminus B}\left(\frac{1}{t}\int_0^t\|f_{n_k}(s)\|\,\mathrm{d}s\right)^p\mathrm{d}t\right)\\
&\le 2^{p-1}\lim_{k\to\infty}\left(\int_B\left(\frac{1}{t}\int_{[0,t]\cap C}\|f_{n_k}(s)\|\,\mathrm{d}s\right)^p\mathrm{d}t + \int_B\left(\frac{1}{t}\int_{[0,t]\setminus C}\|f_{n_k}(s)\|\,\mathrm{d}s\right)^p\mathrm{d}t\right)\\
&+ \lim_{k\to\infty}\int_{[0,1]\setminus B}\left(\frac{1}{t}\int_0^t\|f_{n_k}(s)\|\,\mathrm{d}s\right)^p\mathrm{d}t,
\end{aligned}
$$

where we have used the inequality $(a+b)^p \le 2^{p-1}(a^p+b^p)$ for $a,b\ge 0$, which is due to the convexity of the function $t\mapsto t^p$.

From (IV.8.4) it now follows that

$$
\begin{aligned}
&\lim_{n\to\infty}\|f_n\|^p_{\mathrm{Ces}_p(X)}\\
&\le 2^{p-1}\left(\int_B\left(\frac{1}{t}\int_{[0,t]\cap C}g(s)\,\mathrm{d}s\right)^p\mathrm{d}t + \lim_{k\to\infty}\int_B\left(\frac{1}{t}\int_{[0,t]\setminus C}\|f_{n_k}(s)\|\,\mathrm{d}s\right)^p\mathrm{d}t\right)\\
&+ \lim_{k\to\infty}\int_{[0,1]\setminus B}\left(\frac{1}{t}\int_0^t\|f_{n_k}(s)\|\,\mathrm{d}s\right)^p\mathrm{d}t. \qquad \text{(IV.8.5)}
\end{aligned}
$$

Because of $\lambda([0,1]\setminus B)\le\tau$ and (IV.8.1) we have

$$
\int_{[0,1]\setminus B}\left(\frac{1}{t}\int_0^t\|f(s)\|\,\mathrm{d}s\right)^p\mathrm{d}t \le \varepsilon.
$$

Thus by the triangle inequality for L^p we get

$$
\begin{aligned}
&\lim_{k\to\infty}\left(\int_{[0,1]\setminus B}\left(\frac{1}{t}\int_0^t\|f_{n_k}(s)\|\,\mathrm{d}s\right)^p\mathrm{d}t\right)^{1/p}\\
&\le \lim_{k\to\infty}\left(\int_{[0,1]\setminus B}\left(\frac{1}{t}\int_0^t\|f_{n_k}(s)-f(s)\|\,\mathrm{d}s\right)^p\mathrm{d}t\right)^{1/p} + \varepsilon^{1/p}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\lim_{k\to\infty}\int_{[0,1]\setminus B}\left(\frac{1}{t}\int_0^t \|f_{n_k}(s)\|\,\mathrm{d}s\right)^p \mathrm{d}t\\
&\leq \lim_{k\to\infty}\int_{[0,1]\setminus B}\left(\frac{1}{t}\int_0^t \|f_{n_k}(s)-f(s)\|\,\mathrm{d}s\right)^p \mathrm{d}t\\
&+\lim_{k\to\infty}\left|\left(\left(\int_{[0,1]\setminus B}\left(\frac{1}{t}\int_0^t \|f_{n_k}(s)-f(s)\|\,\mathrm{d}s\right)^p \mathrm{d}t\right)^{1/p}+\varepsilon^{1/p}\right)^p\right.\\
&\left.-\int_{[0,1]\setminus B}\left(\frac{1}{t}\int_0^t \|f_{n_k}(s)-f(s)\|\,\mathrm{d}s\right)^p \mathrm{d}t\right|.
\end{aligned}
$$

Put $L := M+\|f\|_{\mathrm{Ces}_p(X)}+1$. Since $|a^p-b^p|\leq pL^{p-1}|a-b|$ for all $a,b\in[0,L]$ we obtain

$$
\begin{aligned}
&\lim_{k\to\infty}\int_{[0,1]\setminus B}\left(\frac{1}{t}\int_0^t \|f_{n_k}(s)\|\,\mathrm{d}s\right)^p \mathrm{d}t\\
&\leq \lim_{k\to\infty}\int_{[0,1]\setminus B}\left(\frac{1}{t}\int_0^t \|f_{n_k}(s)-f(s)\|\,\mathrm{d}s\right)^p \mathrm{d}t+pL^{p-1}\varepsilon^{1/p}. \qquad \text{(IV.8.6)}
\end{aligned}
$$

Next we define $h(s) := (1-p)^{-1}(s^p - s/3^{1-p})$ for $s\geq 0$.
Recall that $B\subseteq[\frac{\tau}{3},1-\frac{\tau}{3}]$, $\lambda([0,1]\setminus C)\leq\delta$ and $\theta<\tau$. Thus it follows from (IV.8.3) that

$$
\int_B\left(\frac{1}{t}\int_{[0,t]\setminus C}\|f(s)\|\,\mathrm{d}s\right)^p \mathrm{d}t\leq\tau^p\int_B\frac{1}{t^p}\,\mathrm{d}t\leq\tau^p\int_{\frac{\tau}{3}}^1\frac{1}{t^p}\,\mathrm{d}t=h(\tau).
$$

Hence

$$
\begin{aligned}
&\lim_{k\to\infty}\left(\int_B\left(\frac{1}{t}\int_{[0,t]\setminus C}\|f_{n_k}(s)\|\,\mathrm{d}s\right)^p \mathrm{d}t\right)^{1/p}\\
&\leq \lim_{k\to\infty}\left(\int_B\left(\frac{1}{t}\int_{[0,t]\setminus C}\|f_{n_k}(s)-f(s)\|\,\mathrm{d}s\right)^p \mathrm{d}t\right)^{1/p}+h(\tau)^{1/p}.
\end{aligned}
$$

Using the same trick as before we now obtain

$$
\begin{aligned}
&\lim_{k\to\infty}\int_B\left(\frac{1}{t}\int_{[0,t]\setminus C}\|f_{n_k}(s)\|\,\mathrm{d}s\right)^p \mathrm{d}t\\
&\leq \lim_{k\to\infty}\int_B\left(\frac{1}{t}\int_{[0,t]\setminus C}\|f_{n_k}(s)-f(s)\|\,\mathrm{d}s\right)^p \mathrm{d}t+ph(\tau)^{1/p}A^{p-1}, \qquad \text{(IV.8.7)}
\end{aligned}
$$

where $A := M+\|f\|_{\mathrm{Ces}_p}+K^{1/p}$ and $K := \sup_{s\in[0,1]}h(s)$.

From (IV.8.5) and Fatou's Lemma it follows that

$$
\begin{aligned}
&\lim_{n\to\infty} \|f_n\|^p_{\mathrm{Ces}_p(X)} \\
&\leq 2^{p-1} \lim_{k\to\infty} \int_B \left(\frac{1}{t} \int_{[0,t]\cap C} \|f_{n_k}(s) - f(s)\| \,\mathrm{d}s \right)^p \mathrm{d}t \\
&+ 2^{p-1} \lim_{k\to\infty} \int_B \left(\frac{1}{t} \int_{[0,t]\setminus C} \|f_{n_k}(s)\| \,\mathrm{d}s \right)^p \mathrm{d}t + 2^{p-1} \int_B \left(\frac{1}{t} \int_{[0,t]\cap C} g(s) \,\mathrm{d}s \right)^p \mathrm{d}t \\
&- 2^{p-1} \int_B \left(\frac{1}{t} \int_{[0,t]\cap C} \liminf_{k\to\infty} \|f_{n_k}(s) - f(s)\| \,\mathrm{d}s \right)^p \mathrm{d}t \\
&+ \lim_{k\to\infty} \int_{[0,1]\setminus B} \left(\frac{1}{t} \int_0^t \|f_{n_k}(s)\| \,\mathrm{d}s \right)^p \mathrm{d}t.
\end{aligned}
$$

Combining this with (IV.8.6) and (IV.8.7) we obtain (by using $x^p + y^p \leq (x+y)^p$ for $x, y \geq 0$)

$$
\begin{aligned}
&\lim_{n\to\infty} \|f_n\|^p_{\mathrm{Ces}_p(X)} \\
&\leq 2^{p-1} \lim_{k\to\infty} \|f_{n_k} - f\|^p_{\mathrm{Ces}_p(X)} + 2^{p-1} p h(\tau)^{1/p} A^{p-1} + p L^{p-1} \varepsilon^{1/p} \\
&+ 2^{p-1} \int_B \left(\frac{1}{t} \int_{[0,t]\cap C} g(s) \,\mathrm{d}s \right)^p \mathrm{d}t \\
&- 2^{p-1} \int_B \left(\frac{1}{t} \int_{[0,t]\cap C} \liminf_{k\to\infty} \|f_{n_k}(s) - f(s)\| \,\mathrm{d}s \right)^p \mathrm{d}t,
\end{aligned}
$$

thus

$$
\begin{aligned}
&\lim_{n\to\infty} \|f_n\|^p_{\mathrm{Ces}_p(X)} \\
&\leq 2^{p-1} \lim_{n\to\infty} \|f_n - f\|^p_{\mathrm{Ces}_p(X)} + 2^{p-1} p h(\tau)^{1/p} A^{p-1} + p L^{p-1} \varepsilon^{1/p} \\
&+ 2^{p-1} \int_B \left(\left(\frac{1}{t} \int_{[0,t]\cap C} g(s) \,\mathrm{d}s \right)^p - \left(\frac{1}{t} \int_{[0,t]\cap C} \varphi(s) \,\mathrm{d}s \right)^p \right) \mathrm{d}t. \qquad \text{(IV.8.8)}
\end{aligned}
$$

Since $\lambda([0,1] \setminus C) \leq \delta$ it follows from (IV.8.3) that for $F \in \{g, \varphi\}$ and $t \in (0, 1 - \frac{\tau}{3}]$ we have

$$
\left| \frac{1}{t} \int_0^t F(s) \,\mathrm{d}s - \frac{1}{t} \int_{[0,t]\cap C} F(s) \,\mathrm{d}s \right| \leq \frac{\theta}{t}
$$

and hence

$$
\left| \left(\frac{1}{t} \int_0^t F(s) \,\mathrm{d}s \right)^p - \left(\frac{1}{t} \int_{[0,t]\cap C} F(s) \,\mathrm{d}s \right)^p \right| \leq p \frac{\theta}{t} \left(\frac{1}{t} \int_0^t F(s) \,\mathrm{d}s \right)^{p-1}.
$$

Since $B \subseteq [\frac{\tau}{3}, 1-\frac{\tau}{3}]$ it follows that

$$\begin{aligned}
&\left| \int_B \left(\frac{1}{t} \int_0^t F(s)\,\mathrm{d}s \right)^p \mathrm{d}t - \int_B \left(\frac{1}{t} \int_{[0,t]\cap C} F(s)\,\mathrm{d}s \right)^p \mathrm{d}t \right| \\
&\leq \int_B p \frac{\theta}{t^p} \left(\int_0^t F(s)\,\mathrm{d}s \right)^{p-1} \mathrm{d}t \leq p\theta \left(\int_0^{1-\frac{\tau}{3}} F(s)\,\mathrm{d}s \right)^{p-1} \int_{\frac{\tau}{3}}^{1-\frac{\tau}{3}} \frac{1}{t^p}\,\mathrm{d}t \\
&= p\theta \left(\int_0^{1-\frac{\tau}{3}} F(s)\,\mathrm{d}s \right)^{p-1} \frac{1}{1-p} \left(\left(1-\frac{\tau}{3}\right)^{1-p} - \left(\frac{\tau}{3}\right)^{1-p} \right).
\end{aligned}$$

Thus it follows from (IV.8.2) that for $F \in \{g, \varphi\}$ one has

$$\left| \int_B \left(\frac{1}{t} \int_0^t F(s)\,\mathrm{d}s \right)^p \mathrm{d}t - \int_B \left(\frac{1}{t} \int_{[0,t]\cap C} F(s)\,\mathrm{d}s \right)^p \mathrm{d}t \right| \leq \varepsilon. \qquad \text{(IV.8.9)}$$

Since $\lambda([0,1] \setminus B) \leq \tau$ we also have

$$\left| \int_B \left(\frac{1}{t} \int_0^t F(s)\,\mathrm{d}s \right)^p \mathrm{d}t - \int_0^1 \left(\frac{1}{t} \int_0^t F(s)\,\mathrm{d}s \right)^p \mathrm{d}t \right| \leq \varepsilon \qquad \text{(IV.8.10)}$$

for $F \in \{g, \varphi\}$, by (IV.8.1).
From (IV.8.9) and (IV.8.10) we obtain

$$\left| \int_B \left(\frac{1}{t} \int_{[0,t]\cap C} F(s)\,\mathrm{d}s \right)^p \mathrm{d}t - \int_0^1 \left(\frac{1}{t} \int_0^t F(s)\,\mathrm{d}s \right)^p \mathrm{d}t \right| \leq 2\varepsilon$$

for $F \in \{g, \varphi\}$.
Together with (IV.8.8) this implies

$$\begin{aligned}
&\lim_{n\to\infty} \|f_n\|^p_{\mathrm{Ces}_p(X)} \\
&\leq 2^{p-1} \lim_{n\to\infty} \|f_n - f\|^p_{\mathrm{Ces}_p(X)} + 2^{p-1} p h(\tau)^{1/p} A^{p-1} + pL^{p-1}\varepsilon^{1/p} \\
&+ 2^{p-1} \int_0^1 \left(\left(\frac{1}{t} \int_0^t g(s)\,\mathrm{d}s \right)^p - \left(\frac{1}{t} \int_0^t \varphi(s)\,\mathrm{d}s \right)^p \right) \mathrm{d}t + 2^{p-1} 4\varepsilon.
\end{aligned}$$

Hence by definition of a we have

$$\begin{aligned}
\lim_{n\to\infty} \|f_n\|^p_{\mathrm{Ces}_p(X)} &\leq 2^{p-1} \lim_{n\to\infty} \|f_n - f\|^p_{\mathrm{Ces}_p(X)} \\
&+ 2^{p-1} p h(\tau)^{1/p} A^{p-1} + pL^{p-1}\varepsilon^{1/p} - 2^{p-1} a + 2^{p+1}\varepsilon.
\end{aligned}$$

Since $h(\tau) \to 0$ for $\tau \to 0$ and $\tau < \varepsilon$, we obtain for $\varepsilon \to 0$

$$\lim_{n\to\infty} \|f_n\|^p_{\mathrm{Ces}_p(X)} \leq 2^{p-1} \lim_{n\to\infty} \|f_n - f\|^p_{\mathrm{Ces}_p(X)} - 2^{p-1} a$$

and the proof is finished. $\square$

We have the following Corollary in the case that X even has the Opial property.

Corollary IV.8.2. *Let $1 \leq p < \infty$ and let X be a Banach space with the Opial property. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $Ces_p(X)$ such that $(f_n(t))_{n\in\mathbb{N}}$ converges weakly to zero for almost every $t \in [0,1]$. Suppose further that there exists a $g \in Ces_p$ such that $\|f_n(t)\| \to g(t)$ a. e. Then*

$$\limsup_{n\to\infty}\|f_n\|_{Ces_p(X)} < 2^{1-1/p}\limsup_{n\to\infty}\|f_n - f\|_{Ces_p(X)} \quad \forall f \in Ces_p(X) \setminus \{0\}.$$

Proof. Let a be defined as in the previous proof. Since X has the Opial property we have $\varphi(t) \geq g(t)$ for every $t \in [0,1]$ and even "$>$" if $f(t) \neq 0$, which by assumption happens on a set of positive measure. Thus $a > 0$ and hence the desired inequality follows from Theorem IV.8.1. □

Concerning the uniform Opial property, we also have the following analogue of Theorem IV.6.3 for Cesàro function spaces (the proof is similar as well, but we will write out the details here for the readers' convenience).

Theorem IV.8.3. *Let $1 \leq p < \infty$ and let X be a Banach space with the uniform Opial property. Let $M, R > 0$ and $f \in Ces_p(X) \setminus \{0\}$. Then there exists $\eta > 0$ such that the following holds: whenever $(f_n)_{n\in\mathbb{N}}$ is a sequence in $Ces_p(X)$ with $\sup_{n\in\mathbb{N}}\|f_n\|_{Ces_p(X)} \leq R$ such that $(f_n(t))_{n\in\mathbb{N}}$ converges weakly to zero and $\lim_{n\to\infty}\|f_n(t)\| \leq M$ for almost every $t \in [0,1]$, then*

$$\limsup_{n\to\infty}\|f_n\|_{Ces_p(X)} + \eta \leq 2^{1-1/p}\limsup_{n\to\infty}\|f_n - f\|_{Ces_p(X)}.$$

Proof. Fix $0 < \tau < \|f\|_{\mathrm{Ces}_p(X)}$ and put $A := \{s \in [0,1] : \|f(s)\| \geq \tau\}$. If $\lambda(A) = 0$, then we would obtain $\|f\|^p_{\mathrm{Ces}_p(X)} \leq \int_0^1 t^p\tau^p 1/t^p \,\mathrm{d}t = \tau^p$. Thus we must have $\lambda(A) > 0$. Let $w := \eta_X(\tau, M)$.
Define $A_t := A \cap [0,t]$ for $t \in [0,1]$. Then $\lambda(A_t) \to \lambda(A)$ for $t \to 1$ and hence we can find $t_0 \in (0,1)$ such that $\lambda(A_t) \geq \lambda(A)/2$ for $t \in [t_0, 1]$.
Put $\theta := \int_{t_0}^1 1/t^p \,\mathrm{d}t$ and $\nu := \min\{(w^p\lambda(A)^p\theta/2)^{1/p}, 2^{1-1/p}(3R+1)\}$.
Next we define $\omega := 2^{1-1/p}(3R+1) - (2^{p-1}(3R+1)^p - \nu^p)^{1/p}$ and finally $\eta := \min\{\omega, 1\}$.
Now let $(f_n)_{n\in\mathbb{N}}$ be as above. Without loss of generality we may assume that $g(t) := \lim_{n\to\infty}\|f_n(t)\| \leq M$ and $f_n(t) \to 0$ weakly for every $t \in [0,1]$. Let $\varphi(t) := \liminf\|f_n(t) - f(t)\|$ for all $t \in [0,1]$. Then we have $\varphi \geq g$ and the definition of η_X implies that even $\varphi(s) - g(s) \geq \eta_X(\tau, M) = w$ for all $s \in A$.
Using the relation $(a-b)^p \leq a^p - b^p$ for $a \geq b \geq 0$ we obtain

$$\begin{aligned}&\left(\int_0^t \varphi(s)\,\mathrm{d}s\right)^p - \left(\int_0^t g(s)\,\mathrm{d}s\right)^p \geq \left(\int_0^t (\varphi(s) - g(s))\,\mathrm{d}s\right)^p \\ &\geq \left(\int_{A_t} (\varphi(s) - g(s))\,\mathrm{d}s\right)^p \geq w^p\lambda(A_t)^p\end{aligned}$$

for every $t \in [0,1]$.
Theorem IV.8.1 now implies that

$$\begin{aligned}
&2^{p-1} \limsup_{n\to\infty} \|f_n - f\|^p_{\mathrm{Ces}_p(X)} - \limsup_{n\to\infty} \|f_n\|^p_{\mathrm{Ces}_p(X)} \\
&\geq 2^{p-1} \int_0^1 \frac{w^p}{t^p} \lambda(A_t)^p \,\mathrm{d}t \geq 2^{p-1} w^p \int_{t_0}^1 \frac{\lambda(A_t)^p}{t^p} \,\mathrm{d}t \geq w^p \frac{\lambda(A)^p}{2} \theta \geq \nu^p,
\end{aligned} \tag{IV.8.11}$$

by the choice of t_0 and the definition of θ and ν.
Next we define $h(s) := 2^{1-1/p}s - (2^{p-1}s^p - \nu^p)^{1/p}$ for $s \geq 2^{1/p-1}\nu$. It is easy to see that h is decreasing on $[2^{1/p-1}\nu, \infty)$.
Now we proceed analogously to the proof of Theorem IV.5.3 to see that for $\|f\|_{\mathrm{Ces}_p(X)} \geq 2R+1$ we have $2^{1-1/p} \limsup \|f_n - f\|_{\mathrm{Ces}_p(X)} \geq \limsup \|f_n - f\|_{\mathrm{Ces}_p(X)} \geq \limsup \|f_n\|_{\mathrm{Ces}_p(X)} + \eta$, while in the case $\|f\|_{\mathrm{Ces}_p(X)} < 2R+1$ we have $\limsup \|f_n - f\|_{\mathrm{Ces}_p(X)} \leq 3R+1$ and hence (by (IV.8.11))

$$\begin{aligned}
&\limsup_{n\to\infty} \|f_n\|_{\mathrm{Ces}_p(X)} \\
&\leq 2^{1-1/p} \limsup_{n\to\infty} \|f_n - f\|_{\mathrm{Ces}_p(X)} - h\Big(\limsup \|f_n - f\|_{\mathrm{Ces}_p(X)}\Big) \\
&\leq 2^{1-1/p} \limsup_{n\to\infty} \|f_n - f\|_{\mathrm{Ces}_p(X)} - h(3R+1) \\
&\leq 2^{1-1/p} \limsup_{n\to\infty} \|f_n - f\|_{\mathrm{Ces}_p(X)} - \eta,
\end{aligned}$$

where the last inequality holds by the definition of η. □

Finally, we have the following analogue of Theorem IV.6.4.

Theorem IV.8.4. *Let $1 < p < \infty$ and let X be a Banach space with the uniform Opial property. Let $p < r \leq \infty$ and $\varepsilon, M, K, R > 0$. Then there exists $\eta > 0$ such that the following holds: whenever $(f_n)_{n\in\mathbb{N}}$ is a sequence in $Ces_p(X)$ with $\sup_{n\in\mathbb{N}} \|f_n\|_{Ces_p(X)} \leq R$ such that $(f_n(t))_{n\in\mathbb{N}}$ converges weakly to zero and $\lim_{n\to\infty} \|f_n(t)\| \leq M$ for almost every $t \in [0,1]$ and $f \in L^r([0,1],X) \subseteq L^p([0,1],X) \subseteq Ces_p(X)$ is such that $\|f\|_r \leq K$ and $\|f\|_{Ces_p(X)} \geq \varepsilon$, then*

$$\limsup_{n\to\infty} \|f_n\|_{Ces_p(X)} + \eta \leq 2^{1-1/p} \limsup_{n\to\infty} \|f_n - f\|_{Ces_p(X)}.$$

Proof. Let $s := r/p \in (1, \infty]$ and let s' and q be the conjugated exponents to s and p. Choose $0 < \tau < 1$ such that $q^p\tau^p < \varepsilon^p$ and put $Q := \min\Big\{(\varepsilon^p/q^p - \tau^p)^{s'} K^{-ps'}, 1\Big\}$, $w := \eta_X(\tau, M)$ and $t_0 := 1 - Q/2$.
We also put $\theta := \int_{t_0}^1 1/t^p \,\mathrm{d}t$ and $\nu := \min\big\{(w^p Q^p \theta/2)^{1/p}, 2^{1-1/p}(3R+1)\big\}$, as well as $\omega := 2^{1-1/p}(3R+1) - (2^{p-1}(3R+1)^p - \nu^p)^{1/p}$ and finally $\eta := \min\{\omega, 1\}$.

Now let $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{Ces}_p(X)$ and $f \in L^r([0,1],X)$ be as above. We assume without loss of generality that $g(t) := \lim_{n\to\infty}\|f_n(t)\| \leq M$ and $f_n(t) \to 0$ weakly for every $t \in [0,1]$.
Let $A := \{s \in [0,1] : \|f(s)\| \geq \tau\}$. Since $\varepsilon \leq \|f\|_{\mathrm{Ces}_p(X)} \leq q\|f\|_p$ (see (4) on page 141) we can proceed analogously to the proof of Theorem IV.6.4 to show that $\lambda(A) \geq Q$.
Let $A_t := A \cap [0,t]$ for $t \in [0,1]$. We have $\lambda(A) - \lambda(A_{t_0}) = \lambda(A \cap (t_0,1]) \leq 1 - t_0 = Q/2$ and hence $\lambda(A_t) \geq \lambda(A_{t_0}) \geq Q/2$ for $t \in [t_0,1]$.
As in the previous proof we can now use Theorem IV.8.1 to conclude

$$2^{p-1}\limsup_{n\to\infty}\|f_n - f\|^p_{\mathrm{Ces}_p(X)} - \limsup_{n\to\infty}\|f_n\|^p_{\mathrm{Ces}_p(X)} \geq \nu^p$$

and from this obtain, also as in the previous proof, that

$$\limsup_{n\to\infty}\|f_n\|_{\mathrm{Ces}_p(X)} \leq 2^{1-1/p}\limsup_{n\to\infty}\|f_n - f\|_{\mathrm{Ces}_p(X)} - \eta.$$

$\square$

V Banach spaces with the ball generated property

This chapter concerns yet another geometric notion for Banach spaces, the so called ball generated property (BGP). It was proved by S. Basu in [8] that this property is stable under (infinite) c_0- and ℓ^p-sums for $1 < p < \infty$. We will show here that for any absolute, normalised norm $\|\cdot\|_E$ on $\mathbb{R}^2$ satisfying a certain smoothness condition the sum $X \oplus_E Y$ of two Banach spaces X and Y has the BGP whenever X and Y have the BGP. In the proof we will use a characterisation of the smoothness of absolute, normalised norms on $\mathbb{R}^2$ via the boundary curve of their unit ball (this characterisation is quite probably well known, but it is included here with its proof as the author was not able to find a reference).

The material presented in this chapter is based on the author's recent preprint [62].

V.1 The ball generated property

Recall that for $x \in X$ and $r > 0$ we denote by $B_r(x)$ the closed ball with center x and radius r.

A real Banach space X is to have the ball generated property (BGP) if every closed, bounded, convex subset $C \subseteq X$ is ball generated, i. e. it can be written as an intersection of finite unions of closed balls, formally: there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $\bigcap \mathcal{A} = C$, where

$$\mathcal{B} := \left\{ \bigcup_{i=1}^{n} B_{r_i}(x_i) : n \in \mathbb{N},\, r_1, \dots, r_n > 0,\, x_1, \dots, x_n \in X \right\}.$$

The ball topology b_X is defined to be the coarsest topology on X with respect to which every ball $B_r(x)$ is closed. A basis for b_X is given by $\{X \setminus B : B \in \mathcal{B}\} \cup \{X\}$, where $\mathcal{B}$ is as above. Obviously, X has the BGP if and only if every closed, bounded, convex subset of X is also closed with respect to b_X.

Ball generated sets and the ball topology were introduced by Godefroy and Kalton in [51] but the notions implicitly appeared before in [23]. By [51, Theorem 8.1], every weakly compact subset of a Banach space is ball generated. In particular, every reflexive space has the BGP. c_0 is an example of a nonreflexive space with the BGP (see for instance the more general result [8, Theorem 4] on c_0-sums). A standard example of a Banach space which fails to have the BGP is ℓ^1 (see the remark at the end of [23]). More generally, it is known that a dual space X^* has the BGP if and only if X is reflexive (see [20, Corollary 9]).

We now list some easy remarks on the ball topology (see [51, p.197]; some of them may be used later without further notice):

(i) For every $y \in X$, the map $x \mapsto x + y$ is continuous with respect to b_X.

(ii) For every $\lambda > 0$, the map $x \mapsto \lambda x$ is continuous with respect to b_X.

(iii) b_X is not a Hausdorff topology, but it is a T_1-topology (i. e. singletons are closed).

It follows from [51, Theorem 8.3] that X has the BGP if and only if the ball topology and the weak topology coincide on B_X.

In the paper [8] by S. Basu many stability results for the BGP are established, in particular, for any family $(X_i)_{i\in I}$ of Banach spaces and any $p \in (1, \infty)$, the ℓ^p-sum $\left[\bigoplus_{i\in I} X_i\right]_p$ has the BGP if and only if each X_i has the BGP ([8, Theorem 7]). An analogous result holds for c_0-sums ([8, Theorem 4]).

Here we will study the BGP for direct sums of two spaces only, but with respect to more general absolute, normalised norms $\|\cdot\|_E$ on $\mathbb{R}^2$. We recall once more from Lemma II.1.1 that such norms satisfy

$$\|(a, b)\|_\infty \leq \|(a, b)\|_E \leq \|(a, b)\|_1 \quad \forall (a, b) \in \mathbb{R}^2 \tag{V.1.1}$$

and

$$|a| \leq |c|, \ |b| \leq |d| \ \Rightarrow \ \|(a, b)\|_E \leq \|(c, d)\|_E. \tag{V.1.2}$$

Moreover, one also has

$$|a| < |c|, \ |b| < |d| \ \Rightarrow \ \|(a, b)\|_E < \|(c, d)\|_E \tag{V.1.3}$$

(see [13, p. 36, Lemma 1 and 2]).

We are going to prove that $X \oplus_E Y$ has the BGP if X and Y have the BGP and the norm $\|\cdot\|_E$ is Gâteaux-differentiable at $(1, 0)$ and $(0, 1)$. To do so, we will use a description of absolute, normalised norms by the boundary curve of their unit ball, which will be discussed in the next section. For further information on the ball topology, the BGP and related notions, the reader is referred to [8, 20, 21, 51, 54, 92] and references therein.

V.2 Boundary curves of unit balls of absolute norms

The following Proposition is quite probably well known (moreover, its assertion is intuitively clear) but since the author was not able to find a reference, a formal proof is included here for the readers' convenience.

Proposition V.2.1. *Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^2$. Then for every $x \in (-1,1)$ there exists exactly one $y \in (0,1]$ such that $\|(x,y)\|_E = 1$.*

Proof. Let $x \in (-1,1)$. Since the function $t \mapsto \|(x,t)\|_E$ is continuous with $\lim_{t\to\infty}\|(x,t)\|_E = \infty$ and $\|(x,0)\|_E = |x| < 1$, it follows that there exists $y > 0$ such that $\|(x,y)\|_E = 1$. We also have $y \le \|(x,y)\|_E = 1$.
Now we prove the uniqueness assertion. By symmetry it suffices to consider the case $x \ge 0$. Suppose there exist $0 < y_1 < y_2 \le 1$ such that $\|(x,y_1)\|_E = \|(x,y_2)\|_E = 1$. Let $0 < \lambda < 1 - y_1/y_2$. It follows that $z := (x,y_2) + \lambda((1,0) - (x,y_2)) = (x + \lambda(1-x), y_2(1-\lambda))$ still lies in B_E.
But $x + \lambda(1-x) > x$ and $y_2(1-\lambda) > y_1$, thus by (V.1.3) we must have $\|z\|_E > \|(x,y_1)\|_E = 1$, which is a contradiction. $\square$

We denote by f_E the function from $(-1,1)$ to $(0,1]$ which assigns to each $x \in (-1,1)$ the corresponding value y given by Proposition V.2.1. Thus $\|(x, f_E(x))\|_E = 1$ for every $x \in (-1,1)$. The function f_E will be called the upper boundary curve of the unit ball B_E.

The following properties of f_E are easily verified: f_E is a concave (and hence continuous), even function on $(-1,1)$ with $f_E(0) = 1$. Further, f_E is increasing on $(-1,0]$ and decreasing on $[0,1)$. In particular, the limits $\lim_{x\nearrow 1} f_E(x)$ and $\lim_{x\searrow -1} f_E(x)$ exist. Thus we may extend f_E to a continuous function from $[-1,1]$ to $[0,1]$, which will be again denoted by f_E.

Conversely, if any function $f : (-1,1) \to (0,1]$ with the above properties is given, then it is not difficult to show that the Minkowski functional of the closed, absolutely convex hull of the graph of f defines an absolute, normalised norm on $\mathbb{R}^2$ whose unit ball's upper boundary curve is f.

It is possible to characterise properties of the norm $\|\cdot\|_E$ by corresponding properties of the function f_E. As examples we state below characterisations of strict convexity and strict monotonicity[1]. Once again, this is probably well known and so the (anyway easy) proofs are omitted.

Proposition V.2.2. *Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^2$. The space $E := (\mathbb{R}^2, \|\cdot\|_E)$ is strictly convex if and only if f_E is strictly concave[2] on $(-1,1)$ and $f_E(1) = 0$.*

[1] Recall that the norm $\|\cdot\|_E$ is said to be strictly monotone if the following holds: whenever $a,b,c,d \in \mathbb{R}$ with $|a| \le |c|$ and $|b| \le |d|$ and one of these inequalities is strict, then $\|(a,b)\|_E < \|(c,d)\|_E$ (see Section IV.4).

[2] This means $f_E(\lambda x + (1-\lambda)y) > \lambda f_E(x) + (1-\lambda) f_E(y)$ for all $\lambda \in (0,1)$ and all $x, y \in (-1,1)$ with $x \ne y$.

The norm $\|\cdot\|_E$ is strictly monotone if and only if f_E is strictly decreasing on $[0,1)$ and $f_E(1)=0$.

Next we would like to study the smoothness of $\|\cdot\|_E$ in terms of differentiability of f_E. This, too, is quite probably known, but the author could not find a reference. Since these results are important for our main result on sums of spaces with the BGP, we will provide them here with complete proofs.

First recall that, since f_E is concave on $(-1,1)$, it possesses left and right derivatives f'_{E-} and f'_{E+} on $(-1,1)$ which are decreasing and satisfy $f'_{E+}\leq f'_{E-}$. Moreover, for every $x_0\in(-1,1)$ and $a\in\mathbb{R}$ we have

$$f_E(x)\leq f_E(x_0)+a(x-x_0)\ \ \forall x\in(-1,1)\ \Leftrightarrow\ f'_{E+}(x_0)\leq a\leq f'_{E-}(x_0). \tag{V.2.1}$$

Also, f_E is differentiable at $x_0\in(-1,1)$ if and only if f'_{E+} is continuous at x_0 if and only if f'_{E-} is continuous at x_0. All this follows immediately from the corresponding well known facts for convex functions, see for example [116, p.113ff.].

For $x\in[-1,1]$, we will denote by $S_E(x)$ the set of support functionals at $(x,f_E(x))$, i. e. $S_E(x):=\{g\in E^*:\|g\|_{E^*}=1=g(x,f(x))\}$.

Proposition V.2.3. *Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^2$ and $x_0\in(-1,1)$. For all $a\in[f'_{E+}(x_0),f'_{E-}(x_0)]$ we have $f_E(x_0)\geq ax_0+1$ and $S_E(x_0)$ consists exactly of the functionals g of the form*

$$g(x,y)=\frac{ax-y}{ax_0-f_E(x_0)}\quad\forall x,y\in\mathbb{R} \tag{V.2.2}$$

for some $a\in[f'_{E+}(x_0),f'_{E-}(x_0)]$.

Proof. Let $a\in[f'_{E+}(x_0),f'_{E-}(x_0)]$. By (V.2.1) we have $f_E(x_0)-ax_0\geq f_E(0)=1$.
If g is defined by (V.2.2) then it follows from (V.2.1) that $g(x,f_E(x))\leq 1$ for all $x\in(-1,1)$. From this it is easy to deduce that $g(x,y)\leq 1$ for all points (x,y) of norm 1, thus $\|g\|_{E^*}\leq 1$. Moreover, $g(x_0,f_E(x_0))=1$, so $g\in S_E(x_0)$.
Conversely, suppose that g is a functional belonging to $S_E(x_0)$. It is of the form $g(x,y)=Ax+By$ for constants A and B. We then have

$$Ax\pm Bf_E(x)\leq 1\ \ \forall x\in(-1,1)\ \text{ and }\ Ax_0+Bf_E(x_0)=1. \tag{V.2.3}$$

We first prove that $B>0$. If $B\leq 0$, then (V.2.3) implies $Ax_0\geq 1$. In the case $x_0>0$ we would obtain, by (V.2.3), $1\geq Ax-Bf_E(x)\geq Ax\geq x/x_0$ for all $x\in(0,1)$, which is a contradiction. A similar argument works for $x_0<0$. So we must have $B>0$ and hence it follows from (V.2.3) that

$$f_E(x)\leq\frac{1}{B}-\frac{A}{B}x\ \ \forall x\in(-1,1).$$

Since $Ax_0 + Bf_E(x_0) = 1$ we conclude

$$f_E(x) \leq f_E(x_0) - \frac{A}{B}(x - x_0) \quad \forall x \in (-1, 1).$$

Now (V.2.1) implies that $a := -A/B$ lies in $[f'_{E+}(x_0), f'_{E-}(x_0)]$.
From $Ax_0 + Bf_E(x_0) = 1$ we obtain $B = 1/(f_E(x_0) - ax_0)$ and hence $A = a/(ax_0 - f_E(x_0))$. Thus g is of the form (V.2.2). $\square$

The following is an immediate Corollary of Proposition V.2.3.

Corollary V.2.4. *Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^2$ and $x_0 \in (-1, 1)$. The norm $\|\cdot\|_E$ is Gâteaux-differentiable at $(x_0, f_E(x_0))$ if and only if f_E is differentiable at x_0. In this case, the Gâteaux-derivative of $\|\cdot\|_E$ is given by*

$$(x, y) \mapsto \frac{f'_E(x_0)x - y}{f'_E(x_0)x_0 - f_E(x_0)}.$$

It remains to characterise the support functionals at the end points $(-1, f_E(-1))$ and $(1, f_E(1))$. This requires to distinguish a number of cases. We will state the result below for completeness, but skip the proof (once again, it should be already known).

Proposition V.2.5. *Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^2$. Let $a := \inf_{x\in[0,1)} f'_{E-}(x) \in [-\infty, 0]$.*

For $A, B \in \mathbb{R}$ denote by $g_{A,B}$ the functional given by $g_{A,B}(x, y) = Ax+By$. The following holds:

(i) *If $f_E(1) > 0$, then $\|\cdot\|_E$ is Gâteaux-differentiable at each point $(1, b)$ with $b \in (-f_E(1), f_E(1))$ and the Gâteaux-derivative at each such point is $g_{1,0}$.*

(ii) *$f_E(1) = 1$ if and only if $a = 0$ if and only if $\|\cdot\|_E = \|\cdot\|_\infty$. In that case $S_E(1) = \{g_{A,B} : A, B \geq 0$ and $A + B = 1\}$.*

(iii) *If $a = -\infty$, then $\|\cdot\|_E$ is Gâteaux-differentiable at $(1, f_E(1))$ with $S_E(1) = \{g_{1,0}\}$.*

(iv) *If $f_E(1) > 0$ and $-\infty < a < 0$, then $g_{A,B} \in S_E(1)$ if and only if $(A, B) = (\frac{c}{c-f_E(1)}, \frac{-1}{c-f_E(1)})$ for some $c \in (-\infty, a]$ or $(A, B) = (1, 0)$.*

(v) *If $f_E(1) = 0$ and $-\infty < a < 0$, then $g_{A,B} \in S_E(1)$ if and only if $(A, B) = (1, \pm\frac{1}{c})$ for some $c \in (-\infty, a]$ or $(A, B) = (1, 0)$.*

By symmetry arguments, an analogous characterisation holds for the left endpoint $(-1, f_E(-1))$. Let us also remark that characterisations of support functionals of absolute, normalised norms (on $\mathbb{C}^2$ even), which are similar to Propositions V.2.3 and V.2.5, can be found for example in [13, p.38, Lemma

4]. These characterisations do not use the function f_E, but rather the function ψ given by $\psi(t) = \|(1-t, t)\|_E$ for $t \in [0,1]$.[3]

V.3 Sums of spaces with the BGP

Now we come to the announced result on sums of two spaces with the BGP. We start with the following analogue of [8, Lemmas 2 and 5].

Lemma V.3.1. *Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^2$ with the following property:*

$$\forall \varepsilon > 0 \ \exists s_0 > \varepsilon \ \forall s \geq s_0 \ \|(1, s-\varepsilon)\|_E < s. \tag{V.3.1}$$

Let X, Y be Banach spaces and $Z := X \oplus_E Y$. Let $((x_i, y_i))_{i\in I}$ be a net in B_Z which is convergent to 0 in the ball topology b_Z. Then $(y_i)_{i\in I}$ converges to 0 in the topology b_Y.
Likewise, if $\|\cdot\|_E$ satisfies

$$\forall \varepsilon > 0 \ \exists s_0 > \varepsilon \ \forall s \geq s_0 \ \|(s-\varepsilon, 1)\|_E < s, \tag{V.3.2}$$

one can conclude that $(x_i)_{i\in I}$ converges to 0 with respect to b_X.

Proof. The proof is also analogous to that of [8, Lemma 5]. We suppose that $y_i \not\to 0$ with respect to b_Y. Then, by passing to a subnet if necessary, we may assume that there are $y \in Y$ and $r > 0$ such that $y_i \in B_r(y)$ for all $i \in I$ and $0 \in Y \setminus B_r(y)$, i. e. $\|y\| > r$.
Put $\varepsilon := \|y\| - r$. By (V.3.1) we can find $s > \max\{\varepsilon, \|y\|\}$ such that $t := \|(1, s-\varepsilon)\|_E < s$.
Now if $u \in B_X$ and $v \in B_{s-\varepsilon}(sy/\|y\|)$, then by the monotonicity of $\|\cdot\|_E$,

$$\|(u,v) - (0, sy/\|y\|)\|_E = \|(\|u\|, \|v - sy/\|y\|\|)\|_E \leq \|(1, s-\varepsilon)\|_E = t,$$

in other words: $B_X \times B_{s-\varepsilon}(sy/\|y\|) \subseteq B_t((0, sy/\|y\|))$.
But for $w \in B_r(y)$ we have

$$\|w - sy/\|y\|\| \leq \|w - y\| + \|y - sy/\|y\|\| \leq r + s - \|y\| = s - \varepsilon,$$

thus $B_r(y) \subseteq B_{s-\varepsilon}(sy/\|y\|)$.
Altogether it follows that $(x_i, y_i) \in B_t((0, sy/\|y\|))$ for every $i \in I$. But $0 \notin B_t((0, sy/\|y\|))$, since $t < s$. So the complement of $B_t((0, sy/\|y\|))$ is a b_Z-neighbourhood of 0 not containing any of the points (x_i, y_i). With this contradiction the proof is finished. □

[3] This description of absolute, normalised norms via the associated function ψ is also the basis for the ψ-direct sums, the alternative formulation of absolute sums of two spaces that we have mentioned before.

As mentioned in Section V.1, X has the BGP if and only if the ball topology and the weak topology of X coincide on B_X ([51, Theorem 8.3]). Thus we can, as in [8], derive the following stability result.

Corollary V.3.2. *Let* $\|\cdot\|_E$ *be an absolute, normalised norm on* $\mathbb{R}^2$ *satisfying both* (V.3.1) *and* (V.3.2). *Let* X *and* Y *be Banach spaces with the BGP. Then* $X \oplus_E Y$ *also has the BGP.*

Proof. It follows from Lemma V.3.1 that for every bounded net $((x_i, y_i))_{i\in I}$ in $X \oplus_E Y$ which is convergent to some point (x, y) in the ball topology we also have $x_i \to x$ and $y_i \to y$ in the respective ball topologies of X and Y. Since X and Y have the BGP, it follows that these nets also converge in the weak topology of X resp. Y, which in turn implies $(x_i, y_i) \to (x, y)$ in the weak topology of $X \oplus_E Y$. Thus $X \oplus_E Y$ has the BGP. □

It remains to determine which absolute norms satisfy the conditions (V.3.1) and (V.3.2). As it turns out, (V.3.1) resp. (V.3.2) is equivalent to the Gâteaux-differentiablility of $\|\cdot\|_E$ at $(0, 1)$ resp. $(1, 0)$. To prove this we will use the description of the norm by its upper boundary curve f_E from the previous section and the following version of the mean value theorem for one-sided derivatives (see for instance [120, p.204] or [136, p.358] for an even more general statement).

Theorem V.3.3. *Let* I *be an interval and* $f : I \to \mathbb{R}$ *a continuous function. Let* J *be another interval. Suppose that the right derivative* $f'_+(x)$ *exists and lies in* J *for all but at most countably many interior points from* I. *Then*

$$\frac{f(b) - f(a)}{b - a} \in J \quad \forall a, b \in I \text{ with } a \neq b.$$

An analogous statement holds for the left derivative.

Proposition V.3.4. *Let* $\|\cdot\|_E$ *be an absolute, normalised norm on* $\mathbb{R}^2$. $\|\cdot\|_E$ *is Gâteaux-differentiable at* $(0, 1)$ *resp.* $(1, 0)$ *if and only if* (V.3.1) *resp.* (V.3.2) *holds.*

Proof. We only prove the statement for $(0, 1)$, the other case follows from this one by considering instead of $\|\cdot\|_E$ the norm given by $\|(x, y)\|_F := \|(y, x)\|_E$. Assume first that $\|\cdot\|_E$ is Gâteaux-differentiable at $(0, 1)$. By Corollary V.2.4 the function f_E is differentiable at 0 and the Gâteaux-derivative of $\|\cdot\|_E$ at $(0, 1)$ is given by

$$(x, y) \mapsto -f'_E(0)x + y.$$

But this Gâteaux-derivative must be the projection onto the second coordinate, thus $f'_E(0) = 0$.
For each real number $s > 0$ we define $f_s(x) := sf_E(x/s)$ for $x \in (-s, s)$. The functions f_s are continuous and differentiable from the right with $f'_{s+}(x) = f'_{E+}(x/s)$.

Let $\varepsilon > 0$. Since f'_{E+} is continuous at 0 (cf. the remarks preceding Proposition V.2.3) we can find $\delta \in (0,1)$ such that $\left|f'_{E+}(x)\right| < \varepsilon$ for every $x \in (-\delta, \delta)$. Let $s_0 > \max\{\varepsilon, 1/\delta\}$ and $s \geq s_0$. Then $\left|f'_{s+}(x)\right| < \varepsilon$ for all $x \in (0,1)$ and thus by Theorem V.3.3 $|f_s(1) - f_s(0)| < \varepsilon$, hence $f_s(1) > s - \varepsilon$.
This implies $\|(1, s-\varepsilon)\|_E < s$, for otherwise we would have $s = \|(1, f_s(1))\|_E \geq \|(1, s-\varepsilon)\|_E \geq s$, so $\|(1, s-\varepsilon)\|_E = s$, which would mean $f_E(1/s) = 1 - \varepsilon/s$ and thus we would obtain the contradiction $f_s(1) = s - \varepsilon$. This completes one direction of the proof.
To prove the converse we assume that (V.3.1) holds but $\|\cdot\|_E$ is not Gâteaux-differentiable at $(0,1)$. Then by Corollary V.2.4, the function f_E is not differentiable at 0. Since f_E is increasing on $(-1, 0]$ we have $a := f'_{E-}(0) \geq 0$ and because f_E is even we have $f'_{E+}(0) = -a$. Hence $a > 0$ and by (V.2.1) $f_E(x) \leq f_E(0) + f'_{E+}(0)x = 1 - ax$ for all $x \in (-1,1)$.
If we define f_s as above it follows that

$$f_s(x) \leq s - ax \quad \forall x \in (-s, s), \forall s > 0. \tag{V.3.3}$$

By (V.3.1) we can choose $s > \max\{1, a\}$ such that $\|(1, s-a)\|_E < s$. Then by (V.3.3) $f_s(1) \leq s - a$ and hence $s = \|(1, f_s(1))\|_E \leq \|(1, s-a)\|_E < s$. This contradiction finishes the proof. □

Putting Corollary V.3.2 and Proposition V.3.4 together we obtain the final result.

Corollary V.3.5. *Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^2$ which is Gâteaux-differentiable at $(0,1)$ and $(1,0)$. Let X and Y be Banach spaces with the BGP. Then $X \oplus_E Y$ also has the BGP.*

This result contains in particular the case of p-sums for $1 < p \leq \infty$ that—as we mentioned in Section V.1—was already treated in [8] (even for infinite sums). As was also mentioned in [8], the BGP cannot be stable under infinite ℓ^1-sums (since ℓ^1 itself does not have the BGP), but it is open whether $X \oplus_1 Y$ has the BGP whenever X and Y have it.

VI Generalised lush spaces and the Mazur-Ulam property

The final chapter of this thesis is devoted to the class of generalised lush (GL) spaces that was introduced in [66] in connection with the so called Mazur-Ulam property (MUP) and generalises (at least for separable spaces) the concept of lushness that was introduced in [15].

We will obtain some stability results for GL-spaces, for example, the property GL is stable under ultraproducts and M-ideals (and even some more general types of ideals) in GL-spaces are again GL-spaces. Also, we will show that a space has the MUP if its bidual is a GL-space and that a GL-space with 1-unconditional space does not have any LUR points.

The results in this chapter first appeared in the author's preprint [59], which has been submitted to Studia Mathematica for possible publication.[1]

VI.1 Generalised lushness and the MUP

Let us first introduce some notation. For a subset A of X, we denote by $\overline{A}$ its norm-closure and by $\operatorname{co} A$ resp. $\operatorname{aco} A$ its convex resp. absolutely convex hull. By $\operatorname{dist}(x, A)$ we denote the distance from a point $x \in X$ to the set A. For any functional $x^* \in S_{X^*}$ and any $\varepsilon > 0$ let $S(x^*, \varepsilon) := \{x \in B_X : x^*(x) > 1 - \varepsilon\}$ be the slice of B_X induced by x^* and ε.

Now let us begin by recalling the classical Mazur-Ulam theorem (see [103]), which states that every bijective isometry T between two real normed spaces X and Y must be affine, i. e. $T(\lambda x + (1-\lambda)y) = \lambda T(x) + (1-\lambda)T(y)$ for all $x, y \in X$ and every $\lambda \in [0, 1]$ (equivalently, $T - T(0)$ is linear). A simplified proof of this theorem was given in [135]. See also the recent paper [108] for an even further simplified argument.

In 1972, Mankiewicz [98] proved the following generalisation of the Mazur-Ulam theorem: if $A \subseteq X$ and $B \subseteq Y$ are convex with non-empty interior or open and connected, then every bijective isometry $T : A \to B$ can be extended to a bijective affine isometry $\tilde{T} : X \to Y$. This result implies in

[1]Proposition VI.2.3 is not included in the first preprint version that has appeared under http://arxiv.org/abs/1309.4358, but it is included in the revised version that has been submitted to Studia Math.

particular that every bijective isometry from B_X onto B_Y is the restriction of a linear isometry from X onto Y. Tingley asked in [134] whether the same is true if one replaces the unit balls of X and Y by their respective unit spheres. As a first step towards solving this problem, Tingley proved in [134] that for finite-dimensional spaces X and Y, every bijective isometry $T : S_X \to S_Y$ satisfies $T(-x) = -T(x)$ for all $x \in S_X$.

Though Tingley's problem remains open to the present day even in two dimensions, affirmative answers have been obtained for many special classes of spaces. In particular, the answer is "yes" if Y is an (a priori) arbitrary Banach space and X is any of the classical Banach spaces $\ell^p(I)$, $c_0(I)$, for $1 \le p \le \infty$ and I any index set, or $L^p(\mu)$, for $1 \le p \le \infty$ and μ a σ-finite measure (see [35, 44, 117, 131, 132] and further references therein). The answer is also known to be positive for Y arbitrary and $X = C(K)$ if K is a compact metric space (see [43]).

The notion of Mazur-Ulam property was introduced in [22]: a real Banach space X is said to have the Mazur-Ulam property (MUP) if for every Banach space Y every bijective isometry between S_X and S_Y can be extended to a linear isometry between X and Y.

Next let us recall that a Banach space X is called a CL-space resp. an almost CL-space if for every maximal convex subset F of S_X one has $B_X = \operatorname{aco} F$ resp. $B_X = \overline{\operatorname{aco}} F$. CL-spaces were introduced by Fullerton in [45], almost CL-spaces were introduced by Lima (see [89, 90]). Lima also proved that real $C(K)$ and $L^1(\mu)$ spaces (where K is any compact Hausdorff space, μ any finite measure) are CL-spaces. The complex spaces $C(K)$ are also CL while $L^1(\mu)$ is in the complex case in general only almost CL (see [99]).

In [22] Cheng and Dong proposed a proof that every CL-space whose unit sphere has a smooth point and every polyhedral space[2] has the MUP. Unfortunately, this proof is not completely correct, as is mentioned in the introduction of [75]. Kadets and Martín proved in [75] that every *finite-dimensional* polyhedral space has the MUP. In [95] Liu and Tan showed that every almost CL-space whose unit sphere admits a smooth point has the MUP.

Now we recall the definition of lushness, which was introduced in [15] (in connection with a problem concerning the numerical index of a Banach space). The space X is said to be lush provided that for any two points $x, y \in S_X$ and every $\varepsilon > 0$ there exists a functional $x^* \in S_{X^*}$ such that $x \in S(x^*, \varepsilon)$ and

$$\operatorname{dist}(y, \operatorname{aco} S(x^*, \varepsilon)) < \varepsilon.$$

For example, every almost CL-space is lush but the converse is not true in general (see [15, Example 3.4]).

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[2] A Banach space is called polyhedral if the unit ball of each of its finite-dimensional subspaces is a polyhedron, i. e. the convex hull of finitely many points.

For completeness, we will briefly recall the concept of numerical index and its connection to lush spaces. First of all, for an operator $T \in L(X)$ the numerical radius of T is defined by

$$v(T) := \sup\{|x^*(Tx)| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}.$$

The numerical index of X is then defined as

$$n(X) := \sup\{k \geq 0 : k\|T\| \leq v(T)\ \forall T \in L(X)\}.$$

Obviously, $n(X) = 1$ if and only if $\|T\| = v(T)$ for every $T \in L(X)$. It is known (cf. [100, Lemma 2.3]) that $v(T) = \|T\|$ if and only if T satisfies the alternative Daugavet equation

$$\max_{\omega \in \mathbb{T}} \|\mathrm{id} + \omega T\| = 1 + \|T\|$$

(where $\mathbb{T}$ denotes the set of all scalars of modulus one) that we have mentioned already in Section I.10.

It has been a longstanding open problem in the theory of numerical indices whether there exists a Banach space X with $n(X) = 1$ but $n(X^*) < 1$. To solve this question, the notion of lushness was introduced in [15]. It is proved in [15] that every lush space has numerical index one and this result is then used to obtain a space X with $n(X) = 1$ but $n(X^*) < 1$.

For more information on lushness and numerical index, see for example [15, 16, 73, 74]. Specifically, for results on the numerical index of absolute sums and Köthe-Bochner spaces, see [101].

In [66] Huang, Liu and Tan proposed the following definition of generalised lush spaces: X is called a generalised lush (GL) space if for every $x \in S_X$ and every $\varepsilon > 0$ there is some $x^* \in S_{X^*}$ such that $x \in S(x^*, \varepsilon)$ and

$$\mathrm{dist}(y, S(x^*, \varepsilon)) + \mathrm{dist}(y, -S(x^*, \varepsilon)) < 2 + \varepsilon \quad \forall y \in S_X.$$

It is proved in [66] that every almost CL-space and every separable lush space is a GL-space (see [66, Example 2.4] resp. [66, Example 2.5]). Also, in [66, Example 2.7] the space $\mathbb{R}^2$ equipped with the hexagonal norm $\|(x, y)\| = \max\{|y|, |x| + 1/2|y|\}$ is given as an example of a GL-space which is not lush.

The following two Propositions are proved in [66].

Proposition VI.1.1 ([66, Proposition 3.2])**.** *If X is a GL-space, Y any Banach space and $T : S_X \to S_Y$ is a (not necessarily onto) isometry, then*

$$\|T(x) - \lambda T(y)\| \geq \|x - \lambda y\| \quad \forall x, y \in S_X, \forall \lambda \geq 0. \tag{VI.1.1}$$

Proposition VI.1.2 ([66, Proposition 3.4])**.** *If X and Y are Banach spaces and $T : S_X \to S_Y$ is an onto isometry which satisfies* (VI.1.1)*, then T can be extended to a linear isometry from X onto Y.*

It follows that every GL-space (in particular, every almost CL-space and every separable lush space) has the MUP ([66, Theorem 3.3]).

The authors of [66] further call a Banach space X a local GL-space if for every separable subspace Y of X there is a subspace Z of X which is GL and contains Y. Since lushness is separably determined (see [16, Theorem 4.2]) every lush space is a local GL-space ([66, Example 3.7]). From their Propositions 3.2 and 3.4 the authors of [66] conclude that even every local GL-space has the MUP ([66, Theorem 3.8]), thus *every* lush space (separable or not) has the MUP ([66, Corollary 3.9]).

Many stability properties for GL-spaces have already been established in [66], for example, if X is GL then so is the space $C(K, X)$ of all continuous functions from K into X, where K is any compact Hausdorff space (see [66, Theorem 2.10]). Also, the property GL is preserved under c_0-, ℓ^1- and ℓ^∞-sums (see [66, Theorem 2.11]). In the following we will establish some further stability results.

VI.2 Ultraproducts of GL-spaces

In this section we consider ultraproducts of GL-spaces. First we recall the definition of ultraproducts of Banach spaces (see for example [64]). Given a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, for every bounded sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers there exists (by a compactness argument) a number $a \in \mathbb{R}$ such that for every $\varepsilon > 0$ one has $\{n \in \mathbb{N} : |a_n - a| < \varepsilon\} \in \mathcal{U}$. Of course a is uniquely determined. It is called the limit of $(a_n)_{n\in\mathbb{N}}$ along $\mathcal{U}$ and denoted by $\lim_{n,\mathcal{U}} a_n$.

Now for a given sequence $(X_n)_{n\in\mathbb{N}}$ of Banach spaces we put

$$\mathcal{N}_{\mathcal{U}} := \left\{ (x_n)_{n\in\mathbb{N}} \in \Big[\bigoplus_{n\in\mathbb{N}} X_n\Big]_{\ell^\infty} : \lim_{n,\mathcal{U}} \|x_n\| = 0 \right\} \text{ and}$$

$$\prod_{n,\mathcal{U}} X_n := \Big[\bigoplus_{n\in\mathbb{N}} X_n\Big]_{\ell^\infty} / \mathcal{N}_{\mathcal{U}}.$$

Equipped with the (well-defined) norm $\|[(x_n)_{n\in\mathbb{N}}]\|_{\mathcal{U}} := \lim_{n,\mathcal{U}} \|x_n\|$ this quotient becomes a Banach space. It is called the ultraproduct of $(X_n)_{n\in\mathbb{N}}$ (with respect to $\mathcal{U}$). By the way, it is easy to see that the subspace $\mathcal{N}_{\mathcal{U}}$ is closed in $\big[\bigoplus_{n\in\mathbb{N}} X_n\big]_{\ell^\infty}$ with respect to the usual sup-norm and that $\|\cdot\|_{\mathcal{U}}$ coincides with the usual quotient-norm. For more information on ultraproducts the reader is referred to [64].

In [16, Corollary 4.4] it is shown that the ultraproduct of a sequence of lush spaces is again lush, in fact it even satisfies a stronger property, called ultra-lushness in [16]. We can easily prove an analogous result for GL-spaces. First we need a little remark.

Remark VI.2.1. If X is a GL-space, $x \in S_X$ and $\varepsilon > 0$ then there is some $x^* \in S_{X^*}$ such that $x \in S(x^*, \varepsilon)$ and

$$\operatorname{dist}(y, S(x^*, \varepsilon)) + \operatorname{dist}(y, -S(x^*, \varepsilon)) \leq (2+\varepsilon)\|y\| + 2|1 - \|y\|| \quad \forall y \in X.$$

Proof. The proof is analogous to the proof of [66, Lemma 2.9]. Let $x \in S_X$ and $\varepsilon > 0$. By the definition of GL-spaces there exists $x^* \in S_{X^*}$ such that $x \in S(x^*, \varepsilon)$ and

$$\operatorname{dist}(z, S(x^*, \varepsilon)) + \operatorname{dist}(z, -S(x^*, \varepsilon)) < 2 + \varepsilon \quad \forall z \in S_X.$$

Now let $y \in X \setminus \{0\}$. There exist $u \in S(x^*, \varepsilon)$ and $v \in -S(x^*, \varepsilon)$ such that

$$\|\|y\|u - y\| + \|\|y\|v - y\| < (2+\varepsilon)\|y\|.$$

It follows that

$$\begin{aligned}&\|u - y\| + \|v - y\| \leq (2+\varepsilon)\|y\| + \|u - \|y\|u\| + \|v - \|y\|v\| \\ &\leq (2+\varepsilon)\|y\| + |1 - \|y\||\|u\| + |1 - \|y\||\|v\| \leq (2+\varepsilon)\|y\| + 2|1 - \|y\||.\end{aligned}$$

□

Proposition VI.2.2. *Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and $(X_n)_{n\in\mathbb{N}}$ a sequence of GL-spaces. Let $Z = \prod_{n,\mathcal{U}} X_n$. Then the following holds: for every $z \in S_Z$ there is a functional $z^* \in S_{Z^*}$ with $z^*(z) = 1$ such that for every $y \in S_Z$ there are $z_1, z_2 \in S_Z$ with $z^*(z_1) = 1 = -z^*(z_2)$ and $\|y - z_1\| + \|y - z_2\| = 2$. In particular, Z is also a GL-space.*

Proof. Let $z = [(x_n)_{n\in\mathbb{N}}] \in S_Z$. Without loss of generality we may assume $x_n \neq 0$ for all $n \in \mathbb{N}$. By the previous remark we can find, for every $n \in \mathbb{N}$, a functional $x_n^* \in S_{X_n^*}$ such that $x_n/\|x_n\| \in S(x_n^*, 2^{-n})$ and for every $v \in X_n$

$$\operatorname{dist}(v, S(x_n^*, 2^{-n})) + \operatorname{dist}(v, -S(x_n^*, 2^{-n})) \leq (2 + 2^{-n})\|v\| + 2|1 - \|v\||. \tag{VI.2.1}$$

Define $z^* : Z \to \mathbb{R}$ by $z^*([(v_n)]) := \lim_{n,\mathcal{U}} x_n^*(v_n)$. Then z^* is a well-defined element of S_{Z^*} with $z^*(z) = 1$ (because of $x_n^*(x_n) > (1 - 2^{-n})\|x_n\|$ for all n). Now given any $y = [(y_n)] \in S_Z$ we can find, by (VI.2.1), sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ in $\left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{\ell^\infty}$ such that $-v_n, u_n \in S(x_n^*, 2^{-n})$ and

$$\|u_n - y_n\| + \|v_n - y_n\| < (2 + 2^{-n})\|y_n\| + 2|1 - \|y_n\|| + 2^{-n}.$$

Also, because of $-v_n, u_n \in S(x_n^*, 2^{-n})$, the sum on the left-hand side of the above equation is at least $2(1 - 2^{-n})$. Altogether it follows that $z_1 := [(u_n)]$ and $z_2 := [(v_n)]$ satisfy our requirements. □

Let us also include here the following result concerning the closedness of the class of GL-spaces with respect to the Banach-Mazur distance (although it is not directly related to ultraproducts).

First recall that for two isomorphic Banach spaces X and Y, their Banach-Mazur distance is defined by

$$\mathrm{d}(X,Y) := \inf\big\{\|T\|\|T^{-1}\| : T \text{ is an isomorphism between } X \text{ and } Y\big\}.$$

Then the following is valid.

Proposition VI.2.3. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of GL-spaces and X a Banach space which is isomorphic to each X_n such that $\mathrm{d}(X_n, X) \to 1$. Then X is also a GL-space.*

Proof. By passing to a subsequence we may assume that $\mathrm{d}(X_n, X) < 1+1/n$ for each $n \in \mathbb{N}$. Hence there are isomorphisms $T_n : X_n \to X$ with $\|T_n\| = 1$ and $\|T_n^{-1}\| \leq 1+1/n$.
Now let $\varepsilon > 0$ and $x \in S_X$. Put $x_n := T_n^{-1}x$ for each n. Then $1 \leq \|x_n\| \leq 1+1/n$. Let $y_n := x_n/\|x_n\|$. Since X_n is a GL-space, we can find, for each $n \in \mathbb{N}$, a functional $x_n^* \in S_{X_n^*}$ such that $y_n \in S(x_n^*, \varepsilon/2)$ and

$$\mathrm{dist}(z, S(x_n^*, \varepsilon/2)) + \mathrm{dist}(z, -S(x_n^*, \varepsilon/2)) < 2 + \frac{\varepsilon}{2} \quad \forall z \in S_{X_n}. \tag{VI.2.2}$$

We define $y_n^* := (T_n^*)^{-1}x_n^* \in X^*$ for each n. Then $\|y_n^*\| \leq 1+1/n$ and

$$y_n^*(x) = x_n^*(x_n) = \|x_n\|x_n^*(y_n) > 1-\varepsilon/2,$$

since $\|x_n\| \geq 1$ and $y_n \in S(x_n^*, \varepsilon/2)$.
Take $N \in \mathbb{N}$ such that $(1-\varepsilon/2)(1+1/N)^{-1} \geq 1-\varepsilon$ and $1/N \leq \varepsilon/4$. Let $y^* := y_N^*/\|y_N^*\|$. It follows that $y^*(x) > (1-\varepsilon/2)(1+1/N)^{-1} \geq 1-\varepsilon$, i.e. $x \in S(y^*, \varepsilon)$.
Now if $y \in S_X$, we define $z := T_N^{-1}y$. Then $1 \leq \|z\| \leq 1+1/N$. Let $z_0 := z/\|z\| \in S_{X_N}$. By (VI.2.2) we can find $u_1 \in S(x_N^*, \varepsilon/2)$ and $u_2 \in -S(x_N^*, \varepsilon/2)$ such that $\|z_0 - u_1\| + \|z_0 - u_2\| < 2+\varepsilon/2$. Let $v_i := T_N u_i \in B_X$ for $i = 1, 2$. It is easily checked that $v_1 \in S(y^*, \varepsilon)$ and $v_2 \in -S(y^*, \varepsilon)$. We further have

$$\begin{aligned}
&\|v_1 - y\| + \|v_2 - y\| = \|T_N u_1 - \|z\|T_N z_0\| + \|T_N u_2 - \|z\|T_N z_0\| \\
&\leq \|T_N(u_1 - z_0)\| + \|T_N(u_2 - z_0)\| + 2(\|z\|-1)\|T_N z_0\| \\
&\leq \|u_1 - z_0\| + \|u_2 - z_0\| + 2(\|z\|-1) < 2+\varepsilon/2+2/N \leq 2+\varepsilon.
\end{aligned}$$

This shows that X is indeed a GL-space. □

VI.3 E-ideals in GL-spaces

Now we are going to consider the inheritance of the property GL to a certain class of E-ideals, including in particular the M-ideals. We start with the necessary definitions.

First, a linear projection $P : X \to X$ is called an M-projection (see [63, Chapter I, Definition 1.1]) if

$$\|x\| = \max\{\|Px\|, \|x - Px\|\} \ \ \forall x \in X.$$

P is called an L-projection if

$$\|x\| = \|Px\| + \|x - Px\| \ \ \forall x \in X.$$

A closed subspace Y of X is said to be an M-summand (L-summand) in X if it is the range of some M-projection (L-projection) on X. Equivalently, Y is an M-summand (L-summand) in X if and only if there is some closed subspace Z in X such that $X = Y \oplus_\infty Z$ ($X = Y \oplus_1 Z$). Also, Y is called an M-ideal in X if $Y^\perp$ is an L-summand in X^* (where $Y^\perp := \{x^* \in X^* : x^*|_Y = 0\}$ is the annihilator of Y).

Every M-summand is also an M-ideal, but not conversely. For example, if K is a compact Hausdorff space and $A \subseteq K$ is closed, then the subspace $Y := \{f \in C(K) : f|_A = 0\}$ is always an M-ideal in $C(K)$ but it is an M-summand if and only if A is also open in K (see [63, Chapter I, Example 1.4(a)]).

As is pointed out in [63], the notion of an "L-ideal" (i. e. a subspace whose annihilator is an M-summand in the dual) is not introduced because every "L-ideal" is already an L-summand (see [63, Chapter I, Theorem 1.9]).

Just to give a few more examples let us mention that $L^1(\mu)$ is an L-summand in its bidual for every σ-finite measure μ (cf. [63, Chapter IV, Example 1.1(a)]) and, as can be found in [63, Chapter III, Example 1.4(f)], for a Hilbert space H the space $K(H)$ of compact operators on H is an M-ideal in $K(H)^{**} = L(H)$ (the space of all operators on H). For more information on M-ideals and L-summands the reader is referred to [63].

Of course it is also possible to consider more general types of summands and ideals (see the overview in [63, p.45f] and the papers [104–106, 111], we will just recall the basic definitions here). As before, we denote by $\|\cdot\|_E$ an absolute, normalised norm on $\mathbb{R}^2$. For a Banach space X, a linear projection $P : X \to X$ is called an E-projection if

$$\|x\| = \|(\|Px\|, \|x - Px\|)\|_E \ \ \forall x \in X$$

and of course, a closed subspace Y of X is said to be an E-summand in X if it is the range of an E-projection (equivalently, $X = Y \oplus_E Z$ for some closed subspace Z). Finally, Y is called an E-ideal if $Y^\perp$ is an $E^\sharp$-summand in X^*, where $\|\cdot\|_{E^\sharp}$ is the reversed dual norm of $\|\cdot\|_E$, i. e.

$$\|(a, b)\|_{E^\sharp} = \sup\big\{|av + bu| : (u, v) \in \mathbb{R}^2 \text{ with } \|(u, v)\|_E \leq 1\big\} \ \ \forall (a, b) \in \mathbb{R}^2.$$

Then the L- resp. M-summands (M-ideals) are just the $\|\cdot\|_1$- resp. $\|\cdot\|_\infty$-summands ($\|\cdot\|_\infty$-ideals). Every E-summand is also an E-ideal (see [111, Lemma 8]). It is known that E-summands and E-ideals coincide (in every Banach space) if and only if the point $(0,1)$ is an extreme point of the unit ball of $(\mathbb{R}^2, \|\cdot\|_E)$ (see [111, Corollary 10 and Remark 12] and the results in section 2 of [104]).

It was proved in [112] that every L-summand and every M-ideal in a lush space is again lush. In [66, Theorem 2.11] it is shown that the c_0-sum (and likewise the ℓ^∞-sum) of a family of Banach spaces is GL if *and only if* each summand is GL. So M-summands in GL-spaces are again GL. It is possible to extend this result to a class of E-ideals which includes in particular all M-ideals. The main tool of the proof is, as in [112], the principle of local reflexivity (see [3, Theorem 11.2.4]).

Theorem VI.3.1. *If $\|\cdot\|_E$ is an absolute, normalised norm on $\mathbb{R}^2$ such that $(0,1)$ is an extreme point of the unit ball of $(\mathbb{R}^2, \|\cdot\|_{E^\sharp})$, X is a GL-space and Y is an E-ideal in X, then Y is also a GL-space.*

Proof. Let $X^* = Y^\perp \oplus_{E^\sharp} U$ for a suitable closed subspace $U \subseteq X^*$. It easily follows that U can be canonically identified with $X^*/Y^\perp$, which in turn can be canonically identified with Y^*, thus $X^* = Y^\perp \oplus_{E^\sharp} Y^*$.
Now let $y \in S_Y$ and $0 < \varepsilon < 1$ be arbitrary. Since $(0,1)$ is an extreme point of $B_{(\mathbb{R}^2, \|\cdot\|_{E^\sharp})}$ and by an easy compactness argument there is a $0 < \delta < \varepsilon$ such that

$$\|(a,b)\|_{E^\sharp} = 1 \text{ and } b \geq 1-\delta \;\Rightarrow\; |a| \leq \varepsilon. \tag{VI.3.1}$$

Because X is GL we can find $x^* \in S_{X^*}$ such that $y \in S(x^*, \delta)$ and

$$\operatorname{dist}(v, S(x^*,\delta)) + \operatorname{dist}(v, -S(x^*,\delta)) < 2+\delta \quad \forall v \in S_X. \tag{VI.3.2}$$

Write $x^* = (y^\perp, y^*)$ with $y^\perp \in Y^\perp, y^* \in Y^*$ and $1 = \|x^*\| = \|(\|y^\perp\|, \|y^*\|)\|_{E^\sharp}$. Then $y^*(y) = x^*(y) > 1-\delta > 1-\varepsilon$. Since $\|y^*\| \leq 1$ we get that $y \in S(y^*/\|y^*\|, \varepsilon)$. It also follows that $\|y^*\| > 1-\delta$ and hence by (VI.3.1) we must have $\|y^\perp\| \leq \varepsilon$.
Next we fix an arbitrary $z \in S_Y$. By (VI.3.2) we can find $x_1 \in S(x^*,\delta), x_2 \in -S(x^*,\delta)$ such that

$$\|x_1 - z\| + \|x_2 - z\| < 2+\delta. \tag{VI.3.3}$$

We have $X^{**} = Y^{**} \oplus_E (Y^\perp)^*$, so if we consider X canonically embedded in its bidual we can write $x_i = (y_i^{**}, f_i) \in Y^{**} \oplus_E (Y^\perp)^*$ for $i = 1,2$. It follows that

$$1-\delta < x^*(x_1) = f_1(y^\perp) + y_1^{**}(y^*).$$

Taking into account that $\|y^\perp\| \leq \varepsilon$ and $\|f_1\| \leq 1$ we obtain

$$y_1^{**}(y^*) > 1-\delta-\varepsilon > 1-2\varepsilon. \tag{VI.3.4}$$

Analogously one can see that

$$-y_2^{**}(y^*) > 1-2\varepsilon. \tag{VI.3.5}$$

It also follows from (VI.3.3) that

$$\|(\|y_1^{**}-z\|,\|f_1\|)\|_E + \|(\|y_2^{**}-z\|,\|f_2\|)\|_E < 2+\delta$$

and hence

$$\|y_1^{**}-z\| + \|y_2^{**}-z\| < 2+\delta < 2+\varepsilon. \tag{VI.3.6}$$

We put $F = \operatorname{span}\{y_1^{**}, y_2^{**}, z\}$ and choose $0 < \eta < 2\varepsilon$ such that

$$\frac{1-2\varepsilon}{1+\eta} > 1-3\varepsilon \quad \text{and} \quad (1+\eta)(2+\varepsilon) < 2+2\varepsilon.$$

Now the principle of local reflexivity ([3, Theorem 11.2.4]) comes into play. It yields a finite-dimensional subspace $V \subseteq Y$ and an isomorphism $T : F \to V$ such that $\|T\|, \|T^{-1}\| \le 1+\eta$, $T|_{F\cap Y} = \mathrm{id}$ and $y^*(Ty^{**}) = y^{**}(y^*)$ for all $y^{**} \in F$. Let $y_i = Ty_i^{**}$ for $i = 1, 2$. Then $y^*(y_i) = y_i^{**}(y_i)$ and $\|y_i\| \le 1+\eta$. By (VI.3.4), (VI.3.5) and the choice of η we obtain $\|y_i\| > 1-2\varepsilon$ as well as

$$\frac{y_1}{\|y_1\|} \in S\left(\frac{y^*}{\|y^*\|}, 3\varepsilon\right) \quad \text{and} \quad \frac{y_2}{\|y_2\|} \in -S\left(\frac{y^*}{\|y^*\|}, 3\varepsilon\right). \tag{VI.3.7}$$

From (VI.3.6) and the choice of η we get

$$\|y_1-z\| + \|y_2-z\| = \|Ty_1^{**}-Tz\| + \|Ty_2^{**}-Tz\| < (1+\eta)(2+\varepsilon) < 2+2\varepsilon.$$

Since $1-2\varepsilon < \|y_i\| \le 1+\eta < 1+2\varepsilon$ we have $\|y_i - y_i/\|y_i\|\| < 2\varepsilon$ and thus it follows that

$$\left\|\frac{y_1}{\|y_1\|} - z\right\| + \left\|\frac{y_2}{\|y_2\|} - z\right\| < 2+6\varepsilon,$$

which, in view of (VI.3.7), finishes the proof. □

As mentioned before, Theorem VI.3.1 shows in particular that M-ideals in GL-spaces are again GL and the corresponding result for M-ideals in lush spaces was proved in [112]. The proof of [112] readily extends to the case of more general ideals that we considered above (we skip the details).

Theorem VI.3.2. *If $\|\cdot\|_E$ is an absolute, normalised norm on $\mathbb{R}^2$ such that $(0,1)$ is an extreme point of the unit ball of $(\mathbb{R}^2, \|\cdot\|_{E^\sharp})$, X is a lush space and Y is an E-ideal in X, then Y is also lush.*

Theorem 2.11 in [66] also states that the ℓ^1-sum of any family of Banach spaces is GL if *and only if* every summand is GL. The "only if" part of this statement just means that L-summands in GL-spaces are again GL-spaces. However, the proof of this part given in [66] contains a slight mistake: the statement "$\|u_\lambda\| > 1/2-\varepsilon/2$ and $\|v_\lambda\| > 1/2-\varepsilon/2$" cannot be deduced from the two preceding lines (2.3) and (2.4) as claimed in [66]. For a counterexample just consider the sum $X := \mathbb{R}\oplus_1\mathbb{R}$ and take $x := (1,0) \in S_X$. Then the norm-one functional $x^* : X \to \mathbb{R}$ defined by $x^*(a,b) := a+b$ satisfies $x^*(x) = 1$ and

$\mathrm{dist}(y,S)+\mathrm{dist}(y,-S)=2$ for all $y\in S_X$, where $S:=\{z\in S_X : x^*(z)=1\}$ (we even have $\mathrm{aco}\, S=B_X$). Now for $y:=(-1,0), u:=(u_1,u_2):=(0,1)$ and $v:=y$ we have $-v,u\in S$ and $\|y-u\|_1+\|y-v\|_1=2$. So if the claim in the proof of [66] was true we would obtain the contradiction $|u_1|\geq 1/2$.

We will therefore include a slightly different proof for the inheritance of generalised lushness to L-summands here.

Proposition VI.3.3. *If X is a GL-space and Y is an L-summand in X, then Y is also a GL-space.*

Proof. Write $X=Y\oplus_1 Z$ for a suitable closed subspace $Z\subseteq X$. Let $y\in S_Y$ and $0<\varepsilon<1$. Take $0<\delta<\varepsilon^2$. Since X is GL there is a functional $x^*=(y^*,z^*)$ in the unit sphere of $X^*=Y^*\oplus_\infty Z^*$ such that $y\in S(x^*,\delta)$ and

$$\mathrm{dist}(v,S(x^*,\delta))+\mathrm{dist}(v,-S(x^*,\delta))<2+\delta \quad \forall v\in S_X. \tag{VI.3.8}$$

Since $x^*(y)=y^*(y)$ it follows that $y\in S(y^*/\|y^*\|,\delta)\subseteq S(y^*/\|y^*\|,\varepsilon)$.
Now fix an arbitrary $u\in S_Y$. Because of (VI.3.8) we can find $x_1\in S(x^*,\delta)$ and $x_2\in -S(x^*,\delta)$ such that $\|u-x_1\|+\|u-x_2\|<2+\delta$. Write $x_i=y_i+z_i$ with $y_i\in Y, z_i\in Z$ for $i=1,2$. It then follows that

$$\|u-y_1\|+\|z_1\|+\|u-y_2\|+\|z_2\|<2+\delta. \tag{VI.3.9}$$

We distinguish two cases. First we assume that $\|y_1\|,\|y_2\|\geq\varepsilon$. Since $x_1\in S(x^*,\delta)$ we have that

$$y^*(y_1)=x^*(x_1)-z^*(z_1)>1-\delta-\|z_1\|\geq\|y_1\|-\delta$$

and hence

$$y^*\Big(\frac{y_1}{\|y_1\|}\Big)>1-\frac{\delta}{\|y_1\|}\geq 1-\frac{\delta}{\varepsilon}>1-\varepsilon,$$

thus $y_1/\|y_1\|\in S(y^*/\|y^*\|,\varepsilon)$. Analogously one can see that $y_2/\|y_2\|\in -S(y^*/\|y^*\|,\varepsilon)$. Furthermore, because of (VI.3.9) and since $\|y_i\|+\|z_i\|=\|x_i\|>1-\delta$, we have

$$\begin{aligned}&\left\|u-\frac{y_1}{\|y_1\|}\right\|+\left\|u-\frac{y_2}{\|y_2\|}\right\|\leq\|u-y_1\|+\|u-y_2\|+|1-\|y_1\||+|1-\|y_2\||\\ &\leq\|u-y_1\|+\|u-y_2\|+\|z_1\|+\|z_2\|+2\delta<2+3\delta<2+3\varepsilon.\end{aligned}$$

In the second case we have $\|y_1\|<\varepsilon$ or $\|y_2\|<\varepsilon$. If $\|y_1\|<\varepsilon$, it follows that $\|z_1\|=\|x_1\|-\|y_1\|>1-\delta-\varepsilon>1-2\varepsilon$ and hence, because of (VI.3.9),

$$\|u-y_2\|+\|z_2\|<2+\delta-(1-2\varepsilon)-\|u-y_1\|<1+3\varepsilon-(1-\|y_1\|)<4\varepsilon.$$

Then in particular $|y^*(u) - y^*(y_2)| < 4\varepsilon$ and thus (since $-x_2 \in S(x^*\delta)$) we have

$$\begin{aligned} y^*(u) &< 4\varepsilon + y^*(y_2) = 4\varepsilon + x^*(x_2) - z^*(z_2) < 4\varepsilon - (1-\delta) - z^*(z_2) \\ &< 5\varepsilon - 1 + \|z_2\| \leq 5\varepsilon - \|y_2\| \leq 5\varepsilon + \|u - y_2\| - 1 < 9\varepsilon - 1. \end{aligned}$$

Hence $-u \in S(y^*/\|y^*\|, 9\varepsilon)$. But then

$$\begin{aligned} &\operatorname{dist}(u, S(y^*/\|y^*\|, 9\varepsilon)) + \operatorname{dist}(u, -S(y^*/\|y^*\|, 9\varepsilon)) \\ &= \operatorname{dist}(u, S(y^*/\|y^*\|, 9\varepsilon)) \leq 2. \end{aligned}$$

If $\|y_2\| < \varepsilon$, an analogous argument shows that $u \in S(y^*/\|y^*\|, 9\varepsilon)$ and thus the proof is complete. □

VI.4 Inheritance from the bidual

Next we would like to prove that every Banach space X whose bidual is GL has itself the MUP (in fact we will prove a little bit more). First consider the following (at least formal) weakening of the definition of GL-spaces.

Definition VI.4.1. A real Banach space X is said to have the property $(*)$ provided that for every $\varepsilon > 0$ and all $x, y_1, y_2 \in S_X$ there exists a functional $x^* \in S_{X^*}$ such that $x \in S(x^*, \varepsilon)$ and

$$\operatorname{dist}(y_i, S(x^*, \varepsilon)) + \operatorname{dist}(y_i, -S(x^*, \varepsilon)) < 2 + \varepsilon \quad \text{for } i = 1, 2.$$

The exact same proof as in [66] shows that [66, Proposition 3.2] (Proposition VI.1.1 in our notation) holds true not only for GL-spaces but for all spaces with property $(*)$ and consequently every space with property $(*)$ has the MUP. Now we will prove that property $(*)$ inherits from X^{**} to X and thus in particular X has the MUP if X^{**} is a GL-space.

Theorem VI.4.2. *If X^{**} has property $(*)$, then so does X.*

Proof. Again the principle of local reflexivity is the key to the proof. If we fix $x, y_1, y_2 \in S_X$ and $\varepsilon > 0$ and consider X canonically embedded into its bidual, then since the latter has property $(*)$ we can find $x^{***} \in S_{X^{***}}$ such that $x \in S(x^{***}, \varepsilon)$ and $u_1^{**}, u_2^{**} \in S(x^{***}, \varepsilon), v_1^{**}, v_2^{**} \in -S(x^{***}, \varepsilon)$ with

$$\|y_i - u_i^{**}\| + \|y_i - v_i^{**}\| < 2 + \varepsilon \quad \text{for } i = 1, 2. \tag{VI.4.1}$$

If we also consider X^* canonically embedded into X^{***}, then by Goldstine's theorem B_{X^*} is weak*-dense in $B_{X^{***}}$, so we can find $\tilde{x}^* \in B_{X^*}$ such that

$$\begin{aligned} &|u_i^{**}(\tilde{x}^*) - x^{***}(u_i^{**})| \leq \varepsilon, \ |v_i^{**}(\tilde{x}^*) - x^{***}(v_i^{**})| \leq \varepsilon \quad \text{for } i = 1, 2 \\ &\text{and} \ |\tilde{x}^*(x) - x^{***}(x)| \leq \varepsilon. \end{aligned}$$

We put $x^* = \tilde{x}^*/\|\tilde{x}^*\|$. It follows that $x \in S(x^*, 2\varepsilon)$, as well as $x^* \in S(u_i^{**}, 2\varepsilon)$ and $-x^* \in S(v_i^{**}, 2\varepsilon)$ for $i = 1, 2$.
Now let $V := \operatorname{span}\{x, y_1, y_2, u_1^{**}, u_2^{**}, v_1^{**}, v_2^{**}\} \subseteq X^{**}$ and choose $0 < \delta < \varepsilon$ such that

$$\frac{1-2\varepsilon}{1+\delta} > 1 - 3\varepsilon \quad \text{and} \quad (2+\varepsilon)(1+\delta) < 2 + 2\varepsilon.$$

By the principle of local reflexivity ([3, Theorem 11.2.4]) there is a finite-dimensional subspace F of X and an isomorphism $T : V \to F$ such that $\|T\|, \|T^{-1}\| \leq 1 + \delta$, $T|_{X \cap V} = \mathrm{id}$ and $x^*(Tx^{**}) = x^{**}(x^*)$ for all $x^{**} \in V$.
Put $\tilde{u}_i := Tu_i^{**}$ and $\tilde{v}_i := Tv_i^{**}$, as well as $u_i = \tilde{u}_i/\|\tilde{u}_i\|$ and $v_i = \tilde{v}_i/\|\tilde{v}_i\|$ for $i = 1, 2$. We then have

$$\frac{1-\varepsilon}{1+\delta} \leq \|\tilde{u}_i\|, \|\tilde{v}_i\| \leq 1 + \delta \quad \text{for } i = 1, 2. \tag{VI.4.2}$$

It follows that

$$x^*(u_i) = \frac{x^*(Tu_i^{**})}{\|\tilde{u}_i\|} \geq \frac{u_i^{**}(x^*)}{1+\delta} > \frac{1-2\varepsilon}{1+\delta} > 1 - 3\varepsilon,$$

so $u_i \in S(x^*, 3\varepsilon)$ and similarly also $-v_i \in S(x^*, 3\varepsilon)$ for $i = 1, 2$. Furthermore, because of (VI.4.1), we have

$$\|y_i - \tilde{u}_i\| + \|y_i - \tilde{v}_i\| = \|Ty_i - Tu_i^{**}\| + \|Ty_i - Tv_i^{**}\| < (1+\delta)(2+\varepsilon) < 2 + 2\varepsilon.$$

From (VI.4.2) we get that $\|u_i - \tilde{u}_i\|, \|v_i - \tilde{v}_i\| \leq \varepsilon + \delta < 2\varepsilon$. Hence

$$\|y_i - u_i\| + \|y_i - v_i\| < 2 + 6\varepsilon \quad \text{for } i = 1, 2$$

and we are done. $\square$

By a similar argument one could also prove that lushness inherits from X^{**} to X. This fact has already been established in [74, Proposition 4.3], albeit with a different proof (the proof in [74] is based on an equivalent formulation of lushness ([74, Proposition 2.1]) and does not use the principle of local reflexivity).

VI.5 GL-spaces and rotundity

Intuitively, the unit sphere of a GL space cannot too "round". The goal of this section is to confirm this intuition by a precise result (at least for spaces with 1-unconditional basis). We start with an easy (and surely well known) observation on Hilbert spaces.

Remark VI.5.1. Let H be a Hilbert space and put

$$A := \{(x, x^*) \in S_H \times S_{H^*} : x \in \ker x^*\}.$$

Then

$$\operatorname{dist}(x, S(x^*, \varepsilon)) = \sqrt{2(1 - \sqrt{2\varepsilon - \varepsilon^2})} \tag{VI.5.1}$$

for all $0 < \varepsilon < 1$ and all $(x, x^*) \in A$. Consequently,

$$\lim_{\varepsilon \to 0} \operatorname{dist}(x, S(x^*, \varepsilon)) = \sqrt{2} \text{ uniformly in } (x, x^*) \in A.$$

Proof. Let $0 < \varepsilon < 1$ and $(x, x^*) \in A$ be arbitrary. Take $y \in S_H$ such that $x^* = \langle \cdot, y \rangle$. Put $U = \operatorname{span}\{y\}$ and $V = U^\perp = \ker x^*$. Let P_U and P_V denote the orthogonal projections from H onto U and V, respectively. Since $x \in V$ we have $P_V x = x$. It follows that for any $z \in S(x^*, \varepsilon)$ we have

$$\begin{aligned}
&\|x - z\|^2 = \|P_V(x - z)\|^2 + \|x - z - P_V(x - z)\|^2 \\
&= \|x - P_V z\|^2 + \|P_V z - z\|^2 = \|x - P_V\|^2 + \|P_U z\|^2 \\
&= \|x - P_V z\|^2 + |\langle z, y \rangle|^2 = \|x - P_V z\|^2 + |x^*(z)|^2 \\
&\geq (1 - \|P_V z\|)^2 + (1 - \varepsilon)^2.
\end{aligned}$$

But $\|P_V z\|^2 + \|P_U z\|^2 = \|z\|^2 \leq 1$ and hence $\|P_V z\|^2 \leq 1 - |x^*(z)|^2 \leq 1 - (1 - \varepsilon)^2$. Putting everything together we get

$$\|x - z\|^2 \geq (1 - \sqrt{1 - (1 - \varepsilon)^2})^2 + (1 - \varepsilon)^2 = 2(1 - \sqrt{2\varepsilon - \varepsilon^2}),$$

which proves the "$\geq$" part of (VI.5.1). On the other hand, if $0 < \delta < \varepsilon$ and we put $\lambda := \sqrt{1 - (1 - \delta)^2}$ and $u := \lambda x + (1 - \delta) y$, then $\|u\|^2 = \lambda^2 + (1 - \delta)^2 = 1$ and $x^*(u) = \langle u, y \rangle = 1 - \delta > 1 - \varepsilon$. So $u \in S(x^*, \varepsilon)$ and thus $\operatorname{dist}(x, S(x^*, \varepsilon))^2 \leq \|x - u\|^2 = (1 - \lambda)^2 + (1 - \delta)^2 = 2(1 - \sqrt{2\delta - \delta^2})$. Letting $\delta \to \varepsilon$ gives the desired result. $\square$

It immediately follows from Remark VI.5.1 that a Hilbert space (with dimension at least two) cannot be a GL-space. The following result generalises this fact.

Proposition VI.5.2. *Let X be an infinite-dimensional GL-space with a (not necessarily countable) 1-unconditional basis. Then S_X does not have any LUR points.*[3]

Proof. Since X has a 1-unconditional basis, we can regard it as a subspace of $\mathbb{R}^I$ (for some suitable infinite index set I) which contains $\{e_i : i \in I\}$, is endowed with an absolute, normalised norm and is contained in $c_0(I)$.
Now let $x = (x_i)_{i \in I} \in S_X$ be arbitrary. We wish to show that x is not an LUR point of S_X. By replacing x with $|x|$ if necessary we may assume without loss of generality that $x_i \geq 0$ for all $i \in I$.

[3]Recall that $x \in S_X$ is said to be an LUR point of S_X if for every sequence $(x_n)_{n \in \mathbb{N}}$ in S_X with $\|x_n + x\| \to 2$ one has $\|x_n - x\| \to 0$.

If x was an LUR point of S_X, then there would be $\eta > 0$ such that for every $y \in X$

$$|\|y\| - 1| \leq \eta \text{ and } |\|x+y\| - 2| \leq \eta \;\Rightarrow\; \|x-y\| \leq \frac{1}{2}. \tag{VI.5.2}$$

Let $\varepsilon > 0$ be arbitrary. Again since x is assumed to be an LUR point there would be $0 < \delta \leq \varepsilon$ such that

$$y \in B_X,\ \|x+y\| \geq 2(1-\delta) \;\Rightarrow\; \|x-y\| \leq \varepsilon.$$

Since X is a GL-space there is some $x^* \in S_{X^*}$ with $x \in S(x^*, \delta)$ and

$$\operatorname{dist}(y, S(x^*, \delta)) + \operatorname{dist}(y, -S(x^*, \delta)) < 2 + \delta \quad \forall y \in S_X.$$

Choose $i_0 \in I$ such that $x_{i_0} \leq \varepsilon$ and find $y_1 \in S(x^*, \delta), y_2 \in -S(x^*, \delta)$ with

$$\|e_{i_0} - y_1\| + \|e_{i_0} - y_2\| < 2 + \delta. \tag{VI.5.3}$$

We have $\|x + y_1\| \geq x^*(x + y_1) > 2(1-\delta)$ and hence the choice of δ implies $\|x - y_1\| \leq \varepsilon$. Similarly, $\|x + y_2\| \leq \varepsilon$. Combining this with (VI.5.3) gives

$$\|x + e_{i_0}\| + \|x - e_{i_0}\| < 2 + \delta + 2\varepsilon \leq 2 + 3\varepsilon. \tag{VI.5.4}$$

Note that $|e_{i_0} - x| = |x + e_{i_0} - 2x_{i_0} e_{i_0}|$, hence

$$|\|e_{i_0} - x\| - \|e_{i_0} + x\|| \leq \|x + e_{i_0} - 2x_{i_0} e_{i_0} - (x + e_{i_0})\| = 2x_{i_0} \leq 2\varepsilon.$$

In view of (VI.5.4) it follows that

$$2\|x + e_{i_0}\| < 2 + 5\varepsilon. \tag{VI.5.5}$$

Next we show that $\|x + e_{i_0}\| > 1 + \eta$. If not, then since $\|x + e_{i_0}\| \geq \|x\| = 1$ (by the monotonicity of $\|\cdot\|$) we would have $|\|x + e_{i_0}\| - 1| \leq \eta$ and also $2 = 2\|x\| \leq \|2x + e_{i_0}\| \leq 2 + \eta$, hence the choice of η would imply $1 = \|e_{i_0}\| \leq 1/2$. Thus $\|x + e_{i_0}\| > 1 + \eta$.
But then by (VI.5.5) we get $\eta < 5\varepsilon/2$ and since ε was arbitrary it follows that $\eta \leq 0$. With this contradiction the proof is finished. $\square$

As regards finite-dimensional GL-spaces, virtually the same argument as in the previous proof can be used to show the following (in fact the argument is even simpler, for in finite-dimensional spaces the compactness of the unit ball allows us to take $\delta = 0 = \eta = \varepsilon$ and 0 instead 1/2 at the end of (VI.5.2)).

Proposition VI.5.3. *Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^n$ and*

$$A := \{x = (x_1, \ldots, x_n) : \|x\|_E = 1 \text{ and } x_i = 0 \text{ for some } i \in \{1, \ldots, n\}\}.$$

If $(\mathbb{R}^n, \|\cdot\|_E)$ is a GL-space and $x \in A$, then x is not an LUR point of the unit sphere of $(\mathbb{R}^n, \|\cdot\|_E)$.

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