# Two-player games on graphs

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# Preface

In this thesis we study different kinds of combinatorial games between two players, which are played on a board that consists of the edges of some given graph $G$. We distinguish unbiased games, in which both players claim (or orient) one edge in each round, and $b$-biased games in which the second player claims (or orients) $b$ edges in each round.

The first game in this regard is the strict oriented-cycle game, which was introduced by Bollobás and Szabó, and later studied by Ben-Eliezer, Krivelevich and Sudakov. This game is played by two players, OMaker and OBreaker, who assign orientations to the edges of the complete graph $K_n$ on $n$ vertices alternately. OMaker has the goal to create a directed cycle, while OBreaker wants to prevent such. It has been asked by Bollobás and Szabó to find the largest value $b$ for which OMaker has a winning strategy in the $b$-biased strict oriented-cycle game, i.e. when OBreaker orients $b$ edges in each round. They conjectured this value to be $n-3$, which turned out to be false, when, in an earlier work with Liebenau, we were able to show an upper bound of size $n-\Theta(\sqrt{n})$. In this thesis we improve further on this bound, and we show that even for a bias $b \geq \frac{37}{40}n$, OBreaker has a strategy to prevent cycles.

The second game that we discuss is the tournament game, which was introduced by Beck. Here, the two players, TMaker and TBreaker, alternately claim edges of a given graph $G$, where TMaker additionally assigns orientations to her edges. She wins, if her directed graph contains at least one copy of a pre-defined tournament $T$, while TBreaker wins otherwise. We consider this game when $G$ is the complete graph $K_n$ or a random graph sampled according to the random graph model $\mathcal{G}_{n,p}$, denoted by $G \sim \mathcal{G}_{n,p}$, whereas $T$ is a tournament on a constant number $k$ of vertices. For both variants we study thresholds (for the bias $b$ and the probability $p$) around which a TMaker's win suddenly turns into a TBreaker's win. We discuss relations between these thresholds and compare these with results for the $k$-clique game, which has almost the same rules as the tournament game besides that the first player does not need to care about orientations.

As third, we study the tree embedding game, which belongs to the family of Maker-Breaker games. The latter became very popular throughout the last decades, as many researchers, including Beck, Bednarska, Erdős, Hefetz, Krivelevich, Łuczak, Stojaković and Szabó, contributed to this particular field. The tree embedding game is played as follows. Maker and Breaker alternately claim edges of the complete graph on $n$ vertices, where each player claims exactly one edge per round. Maker wins if, by the end of the game, her edges contain a copy of some pre-defined spanning tree $T$; Breaker wins otherwise. For large $n$, we show that Maker has a strategy to win this game within $n+1$ rounds, in case the maximum degree of $T$ is bounded by a constant. By studying random trees, we also show that for almost every choice of $T$, Maker can win the tree embedding game within $n-1$ rounds.

Finally, we consider Walker-Breaker games, as they were introduced recently by Espig, Frieze, Krivelevich and Pegden. Walker and Breaker alternately choose edges of the complete graph $K_n$, but with the constraint that Walker has to choose her edges according to a walk. We discuss some questions of Espig et. al. In particular, we determine how large cycles Walker can create.

**Organization.** In Chapter 1, we introduce positional games in general. After summarizing all necessary concepts including some known results, we present all the theorems that are proven in this thesis. In Chapter 2 we study the strict oriented-cycle game, where we prove that OBreaker wins the $b$-biased variant of this game, when $b \geq \frac{37}{40}n$. The tournament game will be discussed in Chapter 3. Afterwards, in Chapter 4, we describe fast winning strategies for Maker in the tree embedding game. Here, we start with the description of general strategies when Maker's goal is to occupy a copy of some tree with bounded maximum degree, and afterwards we study trees which Maker can claim in optimal time. Finally, we discuss Walker-Breaker games in Section 5.

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## General notation and terminology

Many graph-theoretic notation in this thesis is rather standard and follows that of [43], but we also may use different notation and terminology as introduced in [11, 14, 15, 16].

For short notation, we write $[n] := \{1, 2, \ldots, n\}$. A set $[k]$ with $k \leq n$ is called a *down set* of $[n]$, while every subset $[n] \setminus [k]$ with $k \leq n$ is called an *upset* of $[n]$. For a set $M$ and a positive integer $k$ we let $\binom{M}{k} := \{A \subseteq M : |A| = k\}$.

A *graph* $G = (V, E)$ is a pair consisting of a *vertex set* $V$ and an *edge set* $E \subseteq \binom{V}{2}$. For an edge $e = \{u, v\} \in E$, we also write $e = uv$ for short notation. We say that $u \in V$ and $v \in V$ are *adjacent*, if $uv \in E$. A vertex $v \in V$ and an edge $e \in E$ are called *incident* if $v \in e$. Moreover, two edges $e_1, e_2 \in E$ are *adjacent* if $e_1 \cap e_2 \neq \emptyset$, otherwise they are *independent*.

Let a graph $G$ be given. Then we let $V(G)$ denote its set of vertices, and by $E(G)$ we denote its set of edges. Their sizes are denoted by $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For every vertex $v \in V(G)$ and every set $A \subseteq V(G)$, we let $N_G(v, A) := \{w \in A : vw \in E(G)\}$ denote the *neighborhood* of $v$ in $A$, and we set $N_G(v) := N_G(v, V)$ to be the *neighborhood* of $v$ in $G$. The *degree* $d_G(v)$ of a vertex is the size of its neighborhood, i.e. $d_G(v) := |N_G(v)|$. More generally, $d_G(v, A) := |N_G(v, A)|$ for every $v \in V(G)$ and $A \subseteq V(G)$. The *minimum degree* and the *maximum degrees* of $G$ are denoted by $\delta(G) := \min\{d_G(v) : v \in V(G)\}$ and $\Delta(G) := \max\{d_G(v) : v \in V(G)\}$, respectively. For two (not necessarily disjoint) sets $A, B \subseteq V(G)$, we let $E_G(A, B) := \{e = vw \in E(G) : v \in A, w \in B\}$, and we set $e_G(A, B) := |E_G(A, B)|$. We abbreviate $E_G(v, A) := E_G(\{v\}, A)$ and $e_G(v, A) := |E_G(v, A)|$. Often, when there is no risk of confusion, we omit the subscript $G$ from the notation above.

Let $G$ and $H$ be two graphs. Then $H$ is a *subgraph* of $G$, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. $H$ is *isomorphic* to $G$, denoted by $H \cong G$, if there is a bijection $\phi : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. The map $\phi$ is called an *isomorphism* between $G$ and $H$. We also say that $H$ is a *copy* of $G$, if $H \cong G$. Furthermore, assume that $V = V(H) = V(G)$, then we set $G \setminus H := (V, E(G) \setminus E(H))$. For a set $A \subseteq V(G)$, we let $G[A]$ denote the subgraph of $G$ which is *induced* by $A$; formally, $G[A] := (A, \{e \in E(G) : e \subseteq A\})$. Moreover, for a vertex $x \in V(G)$, we write $G - x := G[V(G) \setminus \{x\}]$; and for $A \subseteq V(G)$, we let $G - A := G[V(G) \setminus A]$. Similarly, for $B \subseteq E(G)$, we write $G - B := (V(G), E(G) \setminus B)$.

Let $G = (V, E)$ be a graph. Then a subset $E' \subseteq E$ is a *matching*, if no two edges in $E'$ are adjacent. A vertex $v \in V$ is *saturated* by $E'$ if there is an edge $e \in E'$ with $v \in e$. In case $E'$ is a matching in $G$ that saturates all vertices of $V$, $E'$ is called a *perfect matching* in $G$.

A graph $G = (V, E)$ is called *bipartite* if there is a partition $V = A \cup B$ of the vertex set such that $E \subseteq \{vw : \ v \in A, \ w \in B\}$. We may indicate this property by writing $G = (A \cup B, E)$ instead of $G = (V, E)$. The graph $K_n = ([n], \binom{[n]}{2})$ is called the *complete graph* on $n$ vertices. and the graph $K_{m,n} = ([m] \times \{1\} \cup [n] \times \{2\}, \{\{(a,1),(b,2)\} : \ a \in [m], \ b \in [n]\})$ is called the *complete bipartite graph* with vertex classes of size $m$ and $n$. Let $G = (V, E)$ be some graph and $A \subseteq V$. Then $A$ is a *clique of order* $k$ in $G$, or $k$-clique, if $G[A] \cong K_k$. Moreover, $A$ is called an *independent set* in $G$, if $G[A]$ has no edges.

A graph $P$ is called a *path* if there exist distinct vertices $v_i$ with $V(P) = \{v_i : \ 1 \leq i \leq k\}$ and $E(P) = \{v_i v_{i+1} : \ 1 \leq i \leq k-1\}$. For short notation, we write $P = (v_1, v_2, \ldots, v_k)$. The *length* of $P$ is its number of edges. Moreover, with $P_k$ we denote a representative from the isomorphism class of all paths with $k$ vertices. Now, let $P = (v_1, v_2, \ldots, v_k)$ be a path in some graph $G$. Then the vertices $v_1$ and $v_k$ are called the *endpoints* of $P$, and the vertices of $V(P) \setminus \{v_1, v_k\}$ are called the *interior vertices* of $P$. With $End(P) = \{v_1, v_k\}$ we denote the set of endpoints of $P$. Furthermore, $P \subseteq G$ is called a *bare path* of $G$ if $d_G(v) = 2$ for every interior vertex $v \in V(P)$; and $P \subseteq G$ is called a *Hamilton path* of $G$ if $V(P) = V(G)$.

A *walk* $W$ in a graph $G = (V, E)$ is an alternating sequence of (not necessarily distinct) vertices and edges $v_1, e_1, v_2, e_2, v_3, \ldots, v_{k-1}, e_{k-1}, v_k$, starting and ending with a vertex, such that $e_i = v_i v_{i+1} \in E$ for every $1 \leq i \leq k-1$.

A *cycle* $C$ of length $k$ is a graph with vertex set $V(C) = \{v_i : \ 1 \leq i \leq k\}$, and edge set $E(C) = \{v_i v_{i+1} : \ 1 \leq i \leq k-1\} \cup \{v_k v_1\}$ for distinct vertices $v_i$. With $C_k$ we denote a representative from the isomorphism class of all cycles of length $k$. If $G$ is a graph and $C \subseteq G$ is a cycle, then $C$ is called a *Hamilton cycle* in $G$ if $V(C) = V(G)$.

A graph $G$ is called *connected*, if between each pair of vertices there is a path in $G$ . If a subgraph $H \subseteq G$ is a maximal connected subgraph of $G$ (with respect to the number of edges), then $H$ is a *component* of $G$. $G$ is called a *forest* if it does not contain a cycle; and if $G$ is a connected forest, then $G$ is called a *tree*. If $T$ is a forest, then a vertex $v \in V(T)$ is called a *leaf* if $d_T(v) = 1$. If a tree $T$ is a subgraph of a graph $G$, then $T$ is called a *spanning tree* of $G$ if $V(T) = V(G)$. We may also consider *rooted trees*, where one vertex is designated as the *root* of $T$.

For two vertices $v$ and $w$ in a graph $G$, we define their *distance* $d_G(u, v)$ to be length of the shortest path between $u$ and $v$. In case such a path does not exist, we set $d_G(u, v) = \infty$. The *diameter* $D(G)$ of a graph $G$ is defined as $D(G) := \max_{u,v \in V(G)} d_G(u, v)$.

Let a rooted tree $T$ be given, with root $v \in V(T)$. Then the *depth* of $T$ is the shortest distance between $v$ and the leaves of $T$. Furthermore, let $u, w \in V(T)$ be distinct vertices. If $uw \in E(T)$ and $dist_T(v, w) = dist_T(v, u) + 1$, then $u$ is called a *parent* of $w$. If there exists a path between the root $v$ and the vertex $w$ which contains $u$, then $w$ is a *descendant* of $u$.

A pair $D = (V, E)$ is called a *digraph* if $E \subseteq V \times V$. Again, $V$ is called the vertex set of $D$. An ordered pair $(v, w) \in D$ is called an *arc* or a *directed edge*. Sometimes, we will identify $D$ with its edge set $E$. If $v = w$, then $(v, w) = (v, v)$ is called a *loop*. We say that $(v, w)$ is the *reverse arc* of $(w, v)$, and for an arbitrary arc $e$ we use $\overleftarrow{e}$ to denote its reverse arc. We will only focus on *simple* digraphs, i.e. those digraphs that do not contain loops and reverse arcs. Given such a digraph $D$, we let $\overleftarrow{D}$ denote the set of all reverse arcs of $D$, i.e. $\overleftarrow{D} := \{\overleftarrow{e} : e \in D\}$. Moreover, to delete or to add some edge $e$, we write $D + e := D \cup \{e\}$ and $D - e := D \setminus \{e\}$.

The definitions of adjacency, incidence, subgraphs, isomorphy etc. transfer to digraphs in the natural way. We also use the following. For two sets $A, B \subseteq V$, we let $D(A, B) := D \cap (A \times B)$ be the set of those edges in $D$ that start in $A$ and end in $B$. Moreover, we let $D(A) := D(A, A)$ denote the subgraph of $D$ *induced* by $A$. We write $D(v, B) := D(\{v\}, B)$ and $D(A, v) := D(A, \{v\})$ for every $v \in V$. To describe the sizes of certain arc sets, we let $e_D(A) := |D(A)|$, $e_D(A, B) := |D(A, B)|$, $e_D(v, B) := |D(v, B)|$ and $e_D(A, v) := |D(A, v)|$. Again, when there is no risk of confusion, we may omit the subscript $D$ from the notation above.

Moreover, a digraph $D$ is called a *tournament* if between each pair of its vertices there exists exactly one directed edge in $D$. A digraph $P = (V, E)$ is a *directed path* if there exist vertices $v_i$ with $V(P) = \{v_i : 1 \leq i \leq k\}$ and $E(P) = \{(v_i, v_{i+1}) : 1 \leq i \leq k-1\}$. For short, we again write $P = (v_1, \ldots, v_k)$. Similarly, a digraph $C = (V, E)$ is called a *directed cycle* if there exist vertices $v_i$ with $V(C) = \{v_i : 1 \leq i \leq k\}$ and $E(C) = \{(v_i, v_{i+1}) : 1 \leq i \leq k-1\} \cup \{(v_k, v_1)\}$.

Following [2], we denote with $\mathcal{G}_{n,p}$ the *Binomial random graph model*, which is the probability space of all labeled graphs on the vertex set $[n]$, where the probability for such a graph $G$ to be chosen is $p^{e(G)}(1-p)^{\binom{n}{2}-e(G)}$. Similarly, $\mathcal{G}(n, M)$ denotes the probability space of all labeled graphs with vertex set $[n]$ and exactly $M$ edges, together with the uniform distribution. If a graph $G$ is sampled according to one of these models, we write $G \sim \mathcal{G}_{n,p}$ and $G \sim \mathcal{G}(n, M)$, respectively. Let $A = A_n$ be an event depending on some integer $n \in \mathbb{N}$. Then we say that $A$ happens *asymptotically almost surely* (a.a.s.) if $Pr[A] \to 1$, when $n \to \infty$.

We abbreviate "without loss of generality" with "w.l.o.g."

Let $f, g : \mathbb{N} \to \mathbb{R} \setminus \{0\}$ be two functions. We write $f = o(g)$ or $f \ll g$, if $\frac{f(n)}{g(n)} \to 0$, as $n$ tends to infinity. Similarly, we write $f = \omega(g)$ or $f \gg g$, if $g = o(f)$. In case $\frac{f(n)}{g(n)} \to 1$, as $n$ tends to infinity, we say that $f$ and $g$ are *asymptotically equal*, and write $f \sim g$. Moreover, if there exists a constant $C > 0$ such that for every large enough $n$, $|\frac{f(n)}{g(n)}| \leq C$ holds, we write $f = O(g)$. Moreover, we write $f = \Omega(g)$ if $g = O(f)$; and $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$.

Throughout this thesis, log denotes the natural logarithm, with bases $e = 2,71828...$. If another basis $b$ is used, we write $\log_b$ instead.

Finally, in each of the following chapters we may introduce further notation, which is used only in the particular chapter.

# Chapter 1

# Introduction

The main focus of this thesis lies on the study of positional games and variations of these games. As games like these were studied a lot throughout the last decades [3, 30], and as such, a general language for such games was developed, the terminology in this thesis is rather standard.

A *positional game* is a perfect information game, played by two players, which can be represented by the use of some *hypergraph* $(X, \mathcal{F})$. (For simplicity, we let the first and the second player be female and male, respectively.) The (finite) set $X$ is called the *board* of the positional game, where we usually choose $X$ to be the edge set of some pre-defined graph $G$. Moreover, $\mathcal{F} \subseteq 2^X$ is a family of subsets of $X$, whose elements are called the *winning sets* of the positional game $(X, \mathcal{F})$. Both players alternately claim elements from $X$ which were not claimed before by either of the players, and finally, the winner of the game is determined with the help of the family $\mathcal{F}$. Of course, the last sentence appears to be rather vague, since it does not tell us precisely the way the outcome of the positional game is determined. The reason is that there exist many different types of positional games, which are distinguished by the goal that the two players have.

**Strong games.** The most natural type of positional games is probably given by what we call a *strong game* $(X, \mathcal{F})$. In these games, the first and the second player alternately claim elements of $X$, until one of the players succeeds in occupying all the elements of one of the winning sets $F \in \mathcal{F}$. We then also say that he or she *occupies/claims* this winning set. The player who claims a winning set first, is declared to be the winner of the strong game. However, in case both players manage to prevent the opponent from occupying a complete winning set until the end of the game, the game ends in a *draw*. A typical example of such a game is *Tic-Tac-Toe*, which probably every child learns to play in kindergarten or in elementary school. Here, $X$ is given by the $(3 \times 3)$-board $[3]^2$, while $\mathcal{F}$ consists of all horizontal, vertical and diagonal lines of length 3, i.e.

$$\begin{aligned}\mathcal{F} = &\Big\{\{(i,j):\ j \in [3]\} : i \in [3]\Big\}\\ &\cup\Big\{\{(i,j):\ i \in [3]\} : j \in [3]\Big\}\\ &\cup\Big\{\{(1,1),(2,2),(3,3)\},\ \{(1,3),(2,2),(3,1)\}\Big\}.\end{aligned}$$

By playing Tic-Tac-Toe for some number of times, one shall verify easily that in case both players play according to an optimal strategy, the game always ends in a draw. Indeed, by means of a basic logic argument, it turns out that for every positional game $(X, \mathcal{F})$ which is played by two optimal players the outcome of the game is determined uniquely. That is, for every such game exactly one of the following three statements holds (see Strategy Theorem [3]):

(a) The first player has a winning strategy, i.e. she has a strategy that wins against any strategy of the second player.

(b) The second player has a winning strategy, i.e. he has a strategy that wins against any strategy of the first player.

(c) Both players have a strategy to force at least a draw, i.e. they can prevent the opponent from occupying a winning set.

In fact, in case of strong games one can restrict the number of possible outcomes even further. Using the so-called *Strategy stealing argument* [3], one observes that there cannot exist a winning strategy for the second player in any strong game. The main reason for this fact is that it cannot be a disadvantage for a player to claim the first element in the game. So, the first player can force at least a draw in every strong game. Moreover, in case $\mathcal{F}$ is chosen in such a way that a draw is impossible, we immediately know that the first player needs to have a winning strategy.

Nevertheless, finding such strategies for strong games usually turns out to be tremendously difficult, since one can hardly avoid doing huge case distinctions. Already for the innocent looking game Tic-Tac-Toe, which lasts at most five rounds, one needs to come up with a small case analysis in order to prove that both players have a drawing strategy. Let us consider the strong 5-*clique game* [3] as a further example. In this game, $X = E(K_n)$ is the edge set of the complete graph on $n$ vertices, while the winning sets are precisely all the edge sets of cliques of order 5 in $K_n$. By the well-known Ramsey Theorem [43], it is clear that for large enough $n$, the game cannot end in a draw, as every coloring of the edges of $K_n$ with two colors will produce a monochromatic clique of order 5. Thus, provided $n$ is large enough, we know that the first player needs to possess a winning strategy. However, so far nobody could describe such a strategy explicitly. In particular, we do not know whether there exists an absolute constant $C > 0$ such that the first player can always win the 5-clique game on $K_n$ within at most $C$ rounds, for every large enough $n$ (see the list of open problems in [3]).

**Maker-Breaker games.** Partially due to the difficulties described above, but also because of the fact that the second player in a strong game can only hope for a draw, one may come up with positional games where the second player is already declared to be the winner in case he *blocks* every winning set. These games are referred to as *weak games* or *Maker-Breaker games*, which are defined as follows. Let the hypergraph $(X, \mathcal{F})$ be given. The players, *Maker* and *Breaker*, alternately claim elements of $X$. In case Maker manages to occupy a winning set $F \in \mathcal{F}$ (not necessarily first), she wins the game $(X, \mathcal{F})$. Otherwise, i.e. when Breaker manages to claim at least one element in each of the winning sets, Breaker is said to be the winner.

Maker-Breaker games were studied a lot throughout the previous decades and many beautiful results were proven; see e.g. [3, 4, 5, 9, 18, 23, 24, 30, 35]. One reason for this, of course, is the simplification with respect to strong games. However, another and even more important reason is the fact that throughout the last decades, many beautiful tools were developed and applied in order to solve problems about Maker-Breaker games. One of the first remarkable publications in this regard comes from Erdős and Selfridge [18], who proved a general winning criterion for Breaker, which was motivated by probabilistic tools. Their argument is said to be the first proof using the idea of derandomization. Today, the latter is an essential method in the theory of algorithms. Moreover, their argument motivated further the study of *potential functions* in the field of positional games, as it found its climax in the monograph of Beck [3].

In fact, many natural Maker-Breaker games played on the edge set of the complete graph $K_n$ turned out to be really easy wins for Maker when both players claim exactly one edge from $X = E(K_n)$ in each round. For instance, let $\mathcal{PM}_n$, $\mathcal{HAM}_n$ and $\mathcal{C}_n^k$ denote the family of edge sets of perfect matchings, Hamiltonian cycles and $k$-connected spanning subgraphs of $K_n$, respectively, where we assume $n$ to be even when we study perfect matchings. By Lehman's Theorem [3] it holds that Maker wins the *connectivity game* $(E(K_n), \mathcal{C}_n^1)$ within $n - 1$ rounds, even if she restricts herself to a sub-board of $E(K_n)$ consisting only of two edge-disjoint spanning trees of $K_n$. Hefetz, Krivelevich, Stojaković and Szabó [28] showed that Maker wins the *perfect matching game* $(E(K_n), \mathcal{PM}_n)$ within $\frac{n}{2} + 1$ rounds for every large enough even integer $n$. Moreover, as shown by Hefetz and Stich [31], there exists a strategy for Maker to win the *Hamilton cycle game* $(E(K_n), \mathcal{HAM}_n)$ within $n + 1$ rounds, and, following Ferber and Hefetz [20], she also wins $(E(K_n), \mathcal{C}_n^k)$, for $k \geq 2$, within $\lfloor \frac{kn}{2} \rfloor + 1$ rounds, provided $n$ is large enough. So, in all of these games we observe that the number of rounds, which Maker needs to play until she wins, is just one larger than the minimal size of a winning set; at the moment Maker finishes the game, almost every edge of $K_n$ is still not claimed by either of the players.

**Biased games.** Considering this stupendous power which Maker seems to have in such (and many further) natural Maker-Breaker games on $K_n$, it seems reasonable to increase Breaker's power. There are several typical options to do so. The first option leads us to *biased* Maker-Breaker games, initiated by Chvátal and Erdős [9]. Here we allow the players to claim more than one element from the board $X$ in each round. So, let $a, b \in \mathbb{N}$ be positive integers, which we call the *biases* of Maker and Breaker, respectively. Then in the $(a : b)$ *biased Maker-Breaker game* $(X, \mathcal{F})$, the players alternately claim previously unclaimed elements from the board $X$, where in each round Maker claims $a$ elements and then Breaker claims $b$ elements (but maybe for the very last round, where some player may claim all the remaining elements of $X$ if their number is smaller than the given bias). We will restrict ourselves to the case when $a = 1$. We refer to the $(1 : 1)$ game $(X, \mathcal{F})$ as the *unbiased game* $(X, \mathcal{F})$, while for $b > 1$ the $(1 : b)$ game $(X, \mathcal{F})$ will be called the *$b$-biased game* $(X, \mathcal{F})$.

As increasing $b$ should increase Breaker's chances to win, one natural question now becomes to find the smallest value $b$ for which Breaker has a strategy to win a $b$-biased game $(X, \mathcal{F})$. Indeed, it is not difficult to show that Maker-Breaker games are *bias monotone* in the following sense. If Maker possesses a strategy for winning the $b$-biased game $(X, \mathcal{F})$, then she also wins the $(b-1)$-biased game $(X, \mathcal{F})$, and if Breaker knows to win the $b$-biased game $(X, \mathcal{F})$, then he also does so for the $(b+1)$-biased game $(X, \mathcal{F})$; see e.g. [3, 30]. Thus, provided that $\mathcal{F} \neq \emptyset$ and $|F| \geq 2$ for every $F \in \mathcal{F}$, we know that there needs to exist a unique integer $b_{\mathcal{F}}$ such that Breaker wins the $b$-biased game $(E(K_n), \mathcal{F})$ if $b \geq b_{\mathcal{F}}$, and Maker wins otherwise. We refer to $b_{\mathcal{F}}$ as the *threshold bias* for the game $(E(K_n), \mathcal{F})$.

For many natural types of Maker-Breaker games played on $K_n$, the threshold biases $b_{\mathcal{F}} = b_{\mathcal{F}}(n)$ have been determined throughout the last years; see e.g. [4, 5, 24, 35]. In particular, for the games introduced above we have

$$b_{\mathcal{C}_n^1},\ b_{\mathcal{PM}_n},\ b_{\mathcal{HAM}_n} = (1 - o(1)) \frac{n}{\log(n)},$$

as proven by Gebauer and Szabó [24], and Krivelevich [35]. A very interesting fact about these results is that they underline an impressive connection to random graphs. Let us assume that the two players would play completely at random instead of playing according to optimal strategies. That is, whenever some player wants to claim an edge, he or she chooses an edge uniformly at random from the set of all unclaimed edges. Then, in the $b$-biased

Maker-Breaker game on $X = E(K_n)$, Maker would create a random graph $G \sim \mathcal{G}(n, M)$ with $M = \lceil \frac{1}{b+1}\binom{n}{2} \rceil$ edges. From the theory of random graphs [2, 33], we know that such a graph a.a.s. contains a Hamilton cycle if $M \geq (1 + o(1))\frac{n\log(n)}{2}$, while for $M \leq (1 - o(1))\frac{n\log(n)}{2}$ such a graph a.a.s. contains an isolated vertex. Thus, in case both players play randomly, we observe the following. For $b \geq (1+o(1))\frac{n}{\log(n)}$, it happens that Maker wins a.a.s. the $b$-biased Hamilton cycle game on $K_n$ (and thus the connectivity and the perfect matching game), and for $b \leq (1 - o(1))\frac{n}{\log(n)}$, Breaker wins a.a.s. the games above by isolating a vertex. In other words, the breaking point $b$, where a Maker's win suddenly turns into a Breaker's win, is asymptotically the same for the deterministic game and the random game. For almost every value of the bias $b$ the outcome of a game played by two intelligent players is the same as the typical outcome of the game played by two random players.

This connection described above usually is referred to as the *probabilistic intuition*, *random graph intuition* or *Erdős paradigm*; see e.g. [3, 24, 30]. One of the most important problems in the field of positional games is to find out for which examples of games a similar relation between optimal plays and random plays holds.

Later we will discuss different biased games, which can be regarded as variations of Maker-Breaker games; see Chapters 2, 3 and 5. We will also return to the random graph intuition in Chapter 3.

**Random graphs.** Motivated by the connection between positional games and random graphs it seems natural to study Maker-Breaker games on random graphs rather than on complete graphs. This is our second approach to increase Breaker's power, which was initiated first by Stojaković and Szabó [42]. The main idea is as follows. Before the game starts, we toss a biased coin for every edge of $K_n$ in order to decide whether it shall belong to the board $X$, where each edge is put into $X$ independently at random with probability $p$. This way we reduce the family of winning sets, and thus intensify Breaker's chances to win. The resulting board $X$ then is described by the well-known *Binomial random graph* model $\mathcal{G}_{n,p}$ [2, 32]. A natural questions then is for which edge probability $p$, it is more likely that Breaker (or Maker) wins a particular game on $X$.

In the following let us be more precise. The Binomial random graph model $\mathcal{G}_{n,p}$, with $n \in \mathbb{N}$ and $p = p(n) \in [0, 1]$, is the probability space over all labeled graphs on the vertex set $[n]$, where for any labeled graph $G = ([n], E)$ the probability of being chosen is $p^{|E|}(1-p)^{\binom{n}{2}-|E|}$. For short notation, we write $G \sim \mathcal{G}_{n,p}$, when $G$ is chosen randomly according to this random graph model. Then, a well-known fact [8] is that for every *graph property* $\mathcal{P}$ (i.e. a family of graphs which is closed under isomorphisms), which is *monotone increasing* (i.e. the property is preserved under addition of edges), there exists a *threshold probability* for a random graph

$G \sim \mathcal{G}_{n,p}$ to satisfy $G \in \mathcal{P}$. That is, there is a probability $p_{\mathcal{P}} = p_{\mathcal{P}}(n)$ such that for $G \sim \mathcal{G}_{n,p}$ we have

$$Pr(G \in \mathcal{P}) \rightarrow \begin{cases} 1 & \text{if } p \gg p_{\mathcal{P}} \\ 0 & \text{if } p \ll p_{\mathcal{P}} \end{cases}$$

as $n$ tends to infinity. In particular, as the property "Maker has a strategy to occupy a graph with property $\mathcal{F}$ in the unbiased Maker-Breaker game on $G$" is a monotone increasing graph property, we can study threshold probabilities for Maker-Breaker games as follows. Let $\mathcal{F}$ be a monotone increasing graph property, and let $G$ be some graph, then we define the game $(E(G), \mathcal{F}_G)$, where the board is the edge set $E(G)$ of $G$, and the winning sets are given with $\mathcal{F}_G := \{E(F) : F \in \mathcal{F} \text{ and } E(F) \subseteq E(G)\}$. By the discussion above, it then follows that there exists a threshold probability $p_{\mathcal{F}} = p_{\mathcal{F}}(n)$ such that for $G \sim \mathcal{G}_{n,p}$ the following holds as $n$ tends to infinity:

$$Pr(\text{Maker wins the unbiased game } (E(G), \mathcal{F}_G)) \rightarrow \begin{cases} 1 & \text{if } p \gg p_{\mathcal{F}} \\ 0 & \text{if } p \ll p_{\mathcal{F}}. \end{cases}$$

In the recent years quite a lot of research was done regarding games on random graphs; see e.g. [12, 27, 40, 41, 42]. For instance, considering the same games as before, it follows from [27, 42] that

$$p_{\mathcal{C}_n^1},\ p_{\mathcal{PM}_n},\ p_{\mathcal{HAM}_n} = \frac{\log(n)}{n}.$$

Surprisingly, for these examples we observe that the threshold probability for Maker to win the unbiased game on $G \sim \mathcal{G}_{n,p}$ is related to the inverse of the threshold bias of the biased game on $K_n$ – another evidence for the connection between deterministic games and random graphs, which is also referred to as the probabilistic intuition.

Later, we will study such relations for the so-called *tournament game*, which can be regarded as a variant of the Maker-Breaker *clique game*; see Chapter 3.

**Variations of Maker-Breaker games.** Finally, a third way for increasing Breaker's power obviously is to change the rules of a game. For instance, instead of allowing Maker to claim edges in a game $(E(K_n), \mathcal{F})$ arbitrarily, one may restrict her choices in every round according to some pre-defined rule. In this regard, we will later discuss *Walker-Breaker games*, which were introduced recently by Espig, Frieze, Pegden and Krivelevich [17].

Differently, one may also think of variants where the players have to assign orientations to those edges which they claim, thus having the goal to occupy a pre-defined digraph structure. Variants like this will also be discussed later.

In the following, let us give a short overview on the main chapters of this thesis. We will introduce all the games that will be discussed later, and we will state all the results that we are going to prove.

## 1.1 Oriented-cycle game

In Chapter 2 we study an example from the class of orientation games, which were discussed recently by Ben-Eliezer, Krivelevich and Sudakov [6], and which can be seen as a modification of biased Maker-Breaker games. The most general framework of these games is as follows. Given a graph $G$, in the *orientation game* $\mathcal{O}(G, \mathcal{P}, a, b)$, two players called OMaker and OBreaker, alternately direct previously undirected edges of the given graph $G$. OMaker, who starts the game, directs at least one edge and at most $a$ edges in every round. OBreaker then answers her move by directing at least one edge and at most $b$ edges of the graph $G$. The values $a$ and $b$ are called the *biases* of OMaker and OBreaker, respectively, analogously to Maker-Breaker games. Finally, after all edges of $G$ received an orientation, the game results in a digraph consisting of all directed edges that were chosen by either of the two players. (In case $G = K_n$, we obtain a *tournament* on $n$ vertices.) OMaker wins if this digraph fulfills the given property $\mathcal{P}$, and otherwise OBreaker is declared to be the winner.

Similar to Maker-Breaker games, these games are bias monotone in the following sense. Whenever OBreaker has a strategy to win the game $\mathcal{O}(G, \mathcal{P}, a, b)$, he also has such a strategy for $\mathcal{O}(G, \mathcal{P}, a, b+1)$. Indeed, for the game where he plays with bias $b+1$ he just needs to copy his strategy from $\mathcal{O}(G, \mathcal{P}, a, b)$. Analogously, if OMaker has a strategy to win the game $\mathcal{O}(G, \mathcal{P}, a, b)$, then she also does for the game $\mathcal{O}(G, \mathcal{P}, a+1, b)$. For that reason, the games defined above are also called *monotone orientation games*.

In contrast, one can also consider the *strict orientation game* $\mathcal{O}_s(G, \mathcal{P}, a, b)$, where both players have the constraint to orient exactly $a$ and $b$ edges in every round (besides maybe the very last round, where one player may have to orient the remaining edges whose amount is less than the given bias). These games in general are not bias monotone. For instance, consider the strict orientation game $\mathcal{O}_s(G_n, \mathcal{P}, 1, b)$ where $G_n$ is a disjoint union of $n$ paths of length 2, and $\mathcal{P}$ is the property of containing at least one directed path of length 2. Then, one easily verifies that OMaker has a winning strategy for this game if and only if $b \leq 2(n-2)$ and $b$ is even, and so we do not have the monotonicity as described earlier.

In the following, we will concentrate only on orientation games with $a = 1$ and $G = K_n$. Similar to Maker-Breaker games, we refer to the game $\mathcal{O}(K_n, \mathcal{P}, 1, b)$ as the *b-biased monotone orientation game* (with respect to property $\mathcal{P}$), and $\mathcal{O}_s(K_n, \mathcal{P}, 1, b)$ as the *b-biased strict* *orientation game* (with respect to property $\mathcal{P}$). By the previous discussion we know that for the monotone game, there exists a *threshold bias* $t_{\mathcal{P}} = t_{\mathcal{P}}(n)$ such that OMaker wins the game $\mathcal{O}(K_n, \mathcal{P}, 1, b)$ when $b \leq t_{\mathcal{P}}$ and OBreaker wins otherwise. As for the strict games such a threshold does not necessarily exist, we define the *upper threshold bias* $t^+_{\mathcal{P}} = t^+_{\mathcal{P}}(n)$ to be the largest bias $b$ for which OMaker wins $\mathcal{O}_s(K_n, \mathcal{P}, 1, b)$, and the *lower threshold bias* $t^-_{\mathcal{P}} = t^-_{\mathcal{P}}(n)$ to be the largest integer such $\mathcal{O}_s(K_n, \mathcal{P}, 1, b)$ is won by OMaker for every bias $b \leq t^-_{\mathcal{P}}$. (The definitions of $t_{\mathcal{P}}$, $t^+_{\mathcal{P}}$ and $t^-_{\mathcal{P}}$ are motivated by the study of threshold functions for strict and monotone Avoider-Enforcer games; see e.g. [10, 26, 29].)

Ben-Eliezer, Krivelevich and Sudakov [6] studied the threshold bias $t_{\mathcal{P}}(n)$ for several properties $\mathcal{P}$. They showed that $t_{\mathcal{HAM}}(n) = (1 - o(1))\frac{n}{\log(n)}$, when $\mathcal{HAM}$ is the property to contain a directed Hamilton cycle. Interestingly, this threshold is asymptotically of the same size as the threshold bias for the corresponding Maker-Breaker Hamilton cycle game, mentioned earlier. Moreover, they gave some partial results for the case when $\mathcal{P} = \mathcal{P}_H$ is the property of containing a pre-defined directed graph $H$ with a constant number of vertices.

In Chapter 2 we study the *(strict) oriented-cycle game*, which was introduced by Bollobás and Szabó [7] already before Ben-Eliezer, Krivelevich and Sudakov [6] introduced orientation games in a more general setting. In this game, played on $K_n$, we choose $\mathcal{P} = \mathcal{C}$ to be the family of all directed graphs containing a directed cycle (of any length). Thus, OMaker wins if she can force a directed cycle throughout the game, while OBreaker wins if the game ends in a transitive tournament on $n$ vertices.

For the strict oriented-cycle game $\mathcal{O}_s(K_n, \mathcal{C}, 1, b)$, Bollobás and Szabó [7] showed that $t^+_{\mathcal{C}}(n) \geq \lfloor (2 - \sqrt{3})n \rfloor$. Moreover, they observed that $t^+_{\mathcal{C}}(n) \leq n - 3$, and conjectured this bound to be tight. Later, Ben-Eliezer, Krivelevich and Sudakov [6] gave an easy argument for OMaker to win $\mathcal{O}(K_n, \mathcal{C}, 1, b)$, and $\mathcal{O}_s(K_n, \mathcal{C}, 1, b)$, when $b \leq \frac{n}{2} - 2$. Thus, we have $t_{\mathcal{C}}(n), t^+_{\mathcal{C}}(n) \geq \frac{n}{2} - 2$.

In our earlier work with Liebenau, we were able to disprove the above mentioned conjecture by showing that $t^+_{\mathcal{C}}(n) \leq n - c\sqrt{n}$ for every constant $0 < c < 1$, provided $n$ is large enough. The proof can be found in [37]. Moreover, in case of the monotone game we showed [14] that $t_{\mathcal{C}}(n) \leq \frac{5}{6}n$. The latter finally led to the question whether the upper bound of Bollobás and Szabó could still be tight asymptotically, or whether we can hope for a constant factor improvement as for the monotone oriented-cycle game. In this thesis, we settle the aforementioned question and prove that OBreaker has a winning strategy in the strict oriented-cycle game even when $b \geq \frac{37}{40}n$.

**Theorem 1.1.1 ([14])** *For large enough $n$, $t^+(n, C) < \frac{37}{40}n$.*

**Reference:** We present the proof of Theorem 1.1.1 in Chapter 2. The proof, which is a generalization of the proof of $t_{\mathcal{C}}^{+}(n) < n - c\sqrt{n}$ from [37], is joint work with Anita Liebenau [14] and submitted for publication. In this thesis, we prove an improved upper bound of $\frac{37}{40}n$ over the upper bound of $\frac{19}{20}n$ from [14] using the refined calculations found in Sections 2.1 – 2.3.

## 1.2 Tournament games

In Chapter 3 we continue with the study of games in which the players give orientations to those edges that they claim. We will take a look at the *tournament game*, which was introduced by Beck [3], and which can be seen as a variant of the so-called Maker-Breaker *clique game*.

In the *$k$-clique game* (or sometimes abbreviated just as *clique game* when the value of $k$ is not crucial), Maker's goal is to create a graph that contains a clique of order at least $k$. We denote this game by $(E(G), \mathcal{K}_k)$, i.e. $X = E(G)$ is the edge set of a given graph $G$, and $\mathcal{F} = \mathcal{K}_k$ is the family of all edge sets of $k$-cliques in $G$.

Similarly, given some *tournament* $T$ of order $k$ (i.e. a complete graph on $k$ vertices whose edges are oriented), we define the *$T$-tournament game*, denoted by $(E(G), \mathcal{K}_T)$, as follows. TMaker and TBreaker in turns claim edges of $G$ and also for each claimed edge, they choose one of the two possible orientations. TMaker wins if her digraph contains a copy of the given tournament $T$ by the end of the game; otherwise TBreaker wins.

Notice that the tournament game does not belong to the family of orientation games, as they are discussed in Chapter 2. Indeed, in the tournament game it is irrelevant for TMaker which orientations TBreaker chooses. In fact, for TBreaker we could assume that he just claims edges and does not orient them. Moreover, notice that it follows analogously to the usual Maker-Breaker games, that the tournament game is bias monotone.

However, let us first consider the unbiased games when played on $G = K_n$. Erdős and Selfridge [18] initiated the study of the largest value of $k$, $k_c = k_c(n)$, such that Maker can win the unbiased $k_c$-clique game on $K_n$. They proved that $k_c \leq (2 - o(1)) \log_2 n$. Later, with an impressive application of the method of potential functions, Beck [3] was able to show that this upper bound is tight asymptotically, i.e. $k_c = (2 - o(1)) \log_2 n$. In particular, this game also supports some random graph intuition. Indeed, if both players would play randomly in an unbiased game on $K_n$, then Maker's graph would be a random graph $G \sim \mathcal{G}(n, M)$ with $M = \lceil \frac{1}{2}\binom{n}{2} \rceil$ edges. The order of a largest clique in such a random graph is known to be $(2 - o(1)) \log_2 n$ a.a.s.; see e.g. [2]. Thus, for most values of $k$, the outcome of the unbiased $k$-clique game on $K_n$ played by two intelligent players is the same as the typical outcome of this game played by two random players.

Motivated by the study of $k_c$, Beck [3] also asked for the largest value $k$, $k_t = k_t(n)$, for which TMaker wins the unbiased $T$-tournament game on $K_n$, for every choice of a tournament $T$ on $k_t$ vertices. By analyzing the random analogue of this game, he conjectured its value to be around $(1-o(1))\log_2 n$, for which at first Gebauer [23] could verify a corresponding lower bound. However, recently it was shown [13] that $k_t = (2-o(1))\log_2 n$. So, the unbiased tournament game turned out not to support the random graph intuition. Moreover, as the two values, $k_c$ and $k_t$, are very close to each other, we could observe that it does not make a big difference for the first player whether she is building a large clique or whether she also has to care about orientations.

Looking at this observation it seems natural to ask what happens if we consider biased games or games on random graphs instead, where TMaker's goal is not to create a large tournament, but a tournament of given constant order. Bednarska and Łuczak [4] showed that, for fixed $k \geq 3$, the threshold bias in the $b$-biased $k$-clique game on $K_n$ is $b_{\mathcal{K}_k} = \Theta(n^{\frac{2}{k+1}})$. We will observe first that for the corresponding tournament game, the additional constraint of orienting edges does not make TMaker's life much harder. That is, similar to the argument of [4], we show that for every given tournament $T$ of order $k$, the threshold bias $b_{\mathcal{K}_T}$ for the biased game $(E(K_n), \mathcal{K}_T)$ is of the order $n^{\frac{2}{k+1}}$. So, the order of the threshold bias is independent of the orientations in $T$.

**Proposition 1.2.1 ([15])** *Let $T$ be an arbitrary tournament on $k \geq 3$ vertices, then the threshold bias for the $T$-tournament game on $K_n$ is $b_{\mathcal{K}_T} = \Theta(n^{\frac{2}{k+1}})$, and thus of the same order as the threshold bias for the corresponding clique game.*

Hereby, $b_{\mathcal{K}_T} = \Theta(n^{\frac{2}{k+1}})$ means that there exist constants $c_1 = c_1(k)$ and $c_2 = c_2(k)$ with $c_1 n^{\frac{2}{k+1}} \leq b_{\mathcal{K}_T} \leq c_2 n^{\frac{2}{k+1}}$, where we note that $c_2$ is given by a result from [4] and far away from the constant $c_1$.

Now, let $p_{\mathcal{K}_k}$ be the threshold probability for the property that Maker has a winning strategy in the unbiased $k$-clique game on a random graph $G \sim \mathcal{G}_{n,p}$. Stojaković and Szabó [42] showed that for $k = 3$, we have $p_{\mathcal{K}_3} = n^{-\frac{5}{9}}$. For $k \geq 4$, it was proven by Stojaković and Szabó [42], and Müller and Stojaković [40], that $p_{\mathcal{K}_k} = n^{-\frac{2}{k+1}}$. In particular, for $k \geq 4$, they obtained that $p_{\mathcal{K}_k} = \Theta(\frac{1}{b_{\mathcal{K}_k}})$. So, in this case we see that the threshold probability $p_{\mathcal{K}_k}$ and the inverse of the threshold bias $b_{\mathcal{K}_k}$ are related to each other, similar to the connectivity, perfect matching and Hamilton cycle game, as discussed earlier.

However, the triangle game is an exception in this regard, as in this case Maker can win also for probabilities below the so-called critical probability $\frac{1}{b_{\mathcal{K}_3}}$, since $n^{-\frac{5}{9}} < n^{-\frac{1}{2}}$.

We show that the tournament game behaves similarly to the clique game when played on a random graph $G \sim \mathcal{G}_{n,p}$. Denote with $p_{\mathcal{K}_T}$ the threshold probability that for $G \sim \mathcal{G}_{n,p}$, TMaker has a winning strategy in the unbiased $T$-tournament game on $G$. Then, with an argument similar to [42], we show at first that for $k \geq 4$ the probabilistic intuition is supported, i.e. the threshold probability is $n^{-\frac{2}{k+1}}$.

**Proposition 1.2.2 ([15])** *Let $T$ be an arbitrary tournament on $k \geq 4$ vertices, then the threshold probability for winning the unbiased $T$-tournament game on $G \sim \mathcal{G}_{n,p}$ is $p_{\mathcal{K}_T} = n^{-\frac{2}{k+1}}$. Thus, we obtain $p_{\mathcal{K}_T} = \Theta(\frac{1}{b_{\mathcal{K}_{T_k}}})$.*

However, we then show that the tournament on three vertices behaves differently from the larger tournaments, and that it represents an even bigger exception with respect to the probabilistic intuition compared to the corresponding Maker-Breaker triangle game. In case of the acyclic tournament $T_A$ on 3 vertices we easily verify that we have the same threshold probability as in the triangle game on $G \sim \mathcal{G}_{n,p}$; but in case of the cyclic triangle $T_C$ on 3 vertices, the threshold probability for a TMaker's win suddenly is closer to the critical probability, but still not equal.

**Theorem 1.2.3 ([15])** *The threshold probability for winning the unbiased $T_A$-tournament game on $G \sim \mathcal{G}_{n,p}$ is $p_{\mathcal{K}_{T_A}} = n^{-\frac{5}{9}}$, while for the unbiased $T_C$-tournament game this threshold probability is $p_{\mathcal{K}_{T_C}} = n^{-\frac{8}{15}}$.*

**Reference:** All results in this section are joint work in progress with Mirjana Mikalački [15]. The proofs can be found in Chapter 3, and they are based on ideas from [4, 42].

## 1.3 Tree embedding game

As mentioned already, most strong games appear to be very difficult to analyze, as huge case distinctions can hardly be avoided. However, very recently Ferber and Hefetz [19, 20] were able to describe explicit and fast winning strategies for the first player in different strong games, by using modifications of already known fast winning strategies for the corresponding Maker-Breaker games. For that reason, the study of fast winning strategies for (unbiased) Maker-Breaker games became of particular interest, see e.g. [12, 21, 38].

In this regard, in Chapter 4 we will study the following Maker-Breaker game, which was introduced by Ferber, Hefetz and Krivelevich [21]. Given $n \in \mathbb{N}$, let $T$ be a fixed labeled tree on $n$ vertices. Then the *tree embedding game* $(E(K_n), \mathcal{F}_T)$ is defined such that $X = E(K_n)$

is the board of the game, while all the edge sets of copies of $T$ in $K_n$ form the winning sets in $\mathcal{F}_T$.

Ferber, Hefetz and Krivelevich [21] studied the biased version of the tree embedding game. They were able to prove that for sufficiently small real numbers $\alpha, \varepsilon > 0$ and every sufficiently large integer $n$, Maker has a strategy to win the $b$-biased game $(E(K_n), \mathcal{F}_T)$ within $n + o(n)$ moves, when $b \leq n^{\alpha}$ and when the maximum degree $\Delta(T)$ of $T$ satisfies $\Delta(T) \leq n^{\varepsilon}$. Moreover they pointed out that it would be interesting to improve further on the number of moves, even under restriction of the bias and the maximum degree.

In the following we will provide Maker with fast winning results, in case the maximum degree of the goal tree $T$ is bounded by a constant and Breaker's bias is $b = 1$.

Obviously, no matter how $T$ is chosen before the game starts, Maker needs at least $n - 1$ rounds to win the game $(E(K_n), \mathcal{F}_T)$. In fact, as proven by Hefetz, Krivelevich, Stojaković and Szabó [28], there exist trees (here: Hamilton paths), for which this trivial lower bound is sharp. However, it also not hard to come up with examples for the goal tree $T$ for which Maker needs to play a larger number of rounds, against an optimally playing Breaker. Indeed, if we first ignore the constraint of having a bounded maximum degree, then we easily find trees that Maker cannot build at all, like stars with $n - 1$ leaves. But, even if we focus on bounded degree trees, there exist several examples that Maker cannot hope to create within $n-1$ rounds, like complete binary trees. In the first result of Chapter 4, we nevertheless show that Maker does not need to waste more than two edges.

**Theorem 1.3.1 ([11])** *Let $\Delta \in \mathbb{N}$. Then for every large enough integer $n$ (depending only on $\Delta$) the following holds. If $T$ is a tree on $n$ vertices and with maximum degree at most $\Delta$, then, in an unbiased game on $E(K_n)$, Maker has a strategy to occupy a copy of $T$ within $n+1$ moves.*

Indeed, we prove that the family of bounded degree trees can be split into two large families for which Maker can create copies of the trees of the first family with a waste of at most one edge, and she can create copies of the trees of the other family with a delay of at most two rounds. The splitting hereby depends on the existence of long bare paths, where we call a path $P$ inside a tree $T$ a *bare path* if all of its inner vertices $v$ satisfy $d_T(v) = 2$. The idea of this splitting is motivated by a publication of Krivelevich [34], where the embedding of spanning trees into a random graph $G \sim \mathcal{G}_{n,p}$ is studied. The proof of Theorem 1.3.1 now reduces to proving the following two statements.

**Theorem 1.3.2 ([11])** *Let $\Delta \in \mathbb{N}$. Then there exists an integer $m_1 = m_1(\Delta)$ such that for every large enough integer $n$ (depending only on $\Delta$) the following holds. If $T$ is a tree on $n$ vertices and with maximum degree at most $\Delta$, such that $T$ additionally contains a bare path of length at least $m_1$, then, in an unbiased game on $E(K_n)$, Maker has a strategy to occupy a copy of $T$ within $n$ moves.*

**Theorem 1.3.3 ([11])** *Let $\Delta,\ m_1 \in \mathbb{N}$. Then for every large enough integer $n$ (depending only on $\Delta$ and $m_1$) the following holds. If $T$ is a tree on $n$ vertices and with maximum degree at most $\Delta$, such that $T$ additionally does not contain a bare path of length at least $m_1$, then, in an unbiased game on $E(K_n)$, Maker has a strategy to occupy a copy of $T$ within $n+1$ moves.*

Although we cannot hope to prove that Maker can occupy every pre-defined tree of bounded maximum degree within an optimal number of rounds, the following two theorems show us that besides the Hamilton path, studied in [28], there exist many further examples that can be occupied within $n-1$ rounds. At first we give an explicit construction for a family of such trees, by this giving a strengthening of Theorem 1.4 in [28]. Secondly, applying our methods to random trees, we conclude that indeed most choices for the pre-defined tree $T$ on $n$ vertices have the property that Maker can occupy a copy of $T$ within $n-1$ rounds. Notice that for these trees $T$ it then also follows that, if we look at the corresponding strong games, where both players aim to occupy a copy of $T$ first, we have a strategy for the first player to win within $n-1$ rounds. We prove the following theorems.

**Theorem 1.3.4 ([11])** *Let $\Delta \in \mathbb{N}$. Then there exists an integer $m_2 = m_2(\Delta)$ such that for every large enough integer $n$ (depending only on $\Delta$) the following holds. If $T$ is a tree on $n$ vertices and with maximum degree at most $\Delta$, such that $T$ additionally contains a bare path of length at least $m_2$ which ends in a leaf of $T$, then, in an unbiased game on $E(K_n)$, Maker has a strategy to occupy a copy of $T$ within $n-1$ moves.*

**Theorem 1.3.5 ([11])** *Let $T$ be a random tree, chosen uniformly at random among the family of all labeled trees on $n$ vertices, then the following holds a.a.s. In an unbiased game on $E(K_n)$, Maker has a strategy to occupy a copy of $T$ within $n-1$ rounds.*

**Reference:** All the theorems above are joint work with Asaf Ferber, Roman Glebov, Dan Hefetz and Anita Liebenau [11], and they are submitted for publication. In Chapter 4, we present a version of the proofs from [11] and add two further proofs which were omitted in [11].

## 1.4 Walker-Breaker games

Finally, in Chapter 5 we study *Walker-Breaker games* $(E(G), \mathcal{F}_G)$, which were introduced recently by Espig, Frieze, Krivelevich and Pegden [17]. Their significant difference to the usual Maker-Breaker games is that Walker (in the role of Maker) has the constraint to choose edges of a walk. To be precise, the rules of these games are as follows: Playing on some graph $G$, Walker and Breaker alternately choose edges of $G$. At any possible moment throughout the game, Walker has a position at exactly one vertex $v \in V(G)$, and for her next move she has to choose an edge from $G$ which is incident with $v$ and was not chosen by Breaker before. Hereby, we allow Walker to choose an edge which she already chose in an earlier round. If this is not the case, i.e. she chooses this edge for the first time, then she additionally claims this edge. Moreover, by choosing $vw \in E(G)$, Walker makes $w$ to become her *new position*, where she needs to choose an incident edge in the following round. In contrast, Breaker has no such restrictions. He plays as it is usual for Maker-Breaker games. That is, in every move he chooses and claims any edge that was not chosen by Walker so far. Walker finally wins, if she occupies a winning set $F \in \mathcal{F}_G$ completely; and Breaker is the winner otherwise.

As for Maker-Breaker games, we will consider unbiased games, where both players choose one edge per round, and $b$-biased games, where Breaker chooses and claims $b$ edges in each round. Moreover, we will restrict to the case $G = K_n$.

Notice first that in these games, Breaker can easily isolate one vertex from Maker's graph. Indeed, after Maker's first move, Breaker just needs to fix a vertex $v$ which is not incident with Maker's first edge, and then he always claims the edge between $v$ and the current position of Walker. In particular, Walker has no chance to occupy a spanning structure like a spanning tree or a Hamilton cycle, in contrast to Maker-Breaker games.

Thus, it becomes natural to ask questions about how much Walker can achieve, i.e. how large structures Walker is able to create. In this regard, Espig, Frieze, Krivelevich and Pegden [17] studied the question how many vertices Walker is able to visit, for different variants of Walker-Breaker games. For instance, if $b$ is a constant, they proved that Walker has a strategy to visit $n - 2b + 1$ vertices, while Breaker can prevent Walker from achieving any better result. The main idea of Maker's strategy hereby is to create a tree in depth first manner, and thus it highly uses the fact that Maker is allowed to repeat edges. The authors of [17] also considered the variant in which Walker is not allowed to return to previously visited vertices, that is, when Walker has the constraint to choose her edges according to a path. For the unbiased game in this variant they showed [17] that the largest number of vertices that Walker can visit is $n - 2$. Moreover, in case Walker proceeds randomly, they proved that she a.a.s. visits $n - \Theta(\log n)$ vertices, against an optimally playing opponent.

Besides the question of how many vertices Walker can visit, there are many other natural structures one may ask for. For instance, Espig, Frieze, Krivelevich and Pegden [17] suggested to study the problem of claiming as many edges as possible. They also asked for the largest cycle that Walker can occupy, and they asked which subgraphs Walker can create. In the following we want to give some answers to these questions, where our main focus lies on the discussion of creating long cycles. We prove the following statement.

**Theorem 1.4.1 ([16])** *Let $n \in \mathbb{N}$, and let $b \leq \frac{n}{\log^2(n)}$ be a positive integer. Then, in the $b$-biased Walker-Breaker game on $K_n$, Walker has a strategy to occupy a cycle of length $n-\Theta(b)$, while Breaker can prevent any longer cycles.*

Moreover, in case $b = 1$, we determine the size of the largest cycle precisely.

**Theorem 1.4.2 ([16])** *Let $n$ be a large enough integer. Then, in the unbiased Walker-Breaker game on $K_n$ where Breaker starts, Walker has a strategy to create a cycle of length $n - 2$, while Breaker can prevent any longer cycles.*

The proof of Theorem 1.4.2 will be done by giving Walker a strategy which provides a precise description and properties for the graph that Walker maintains throughout the game, until she finishes a cycle of length $n - 2$. In contrast, for Theorem 1.4.1, we will make use of Walker's ability to create a large graph of constant diameter, and then we will show that on the vertex set of this graph she can create a larger graph which looks almost random and thus contains a long cycle as claimed.

We use these methods further to conclude the following two theorems.

**Theorem 1.4.3 ([16])** *Let $b \in \mathbb{N}$ be a constant. Then, in the $b$-biased Walker-Breaker game on $K_n$, Walker has a strategy to claim $\frac{1}{b+1}\binom{n}{2} - \Theta(n)$ edges, while Breaker can prevent her from doing better.*

**Theorem 1.4.4 ([16])** *Let $G$ be a graph containing a cycle. Then there exists a constant $c_W$ such that the following holds for every large enough $n$. For $b \leq c_W n^{\frac{1}{m_2(G)}}$, Walker has a strategy to occupy a copy of $G$ in the $b$-biased Walker-Breaker game on $K_n$.*

The last theorem in particular tells us that for Walker it is not much harder to occupy a copy of some pre-defined graph $G$ (containing a cycle) than it is for Maker in the $b$-biased game on $K_n$. Indeed, as it was proven by Bednarska and Łuczak [4], we know that the threshold bias for the $b$-biased Maker-Breaker game on $K_n$, where Maker aims to occupy a copy of the graph $G$, is of size $\Theta(n^{\frac{1}{m_2(G)}})$.

**Reference:** In Chapter 5 we present proofs for Theorem 1.4.1 – Theorem 1.4.4, which are joint work in progress with Tuan Tran [16]. For the creation of an almost random graph, as explained above, we will also prove a version of a recent result of Ferber, Krivelevich and Naves [22].

# Chapter 2

# Strict oriented-cycle game

In this chapter, we study the strict oriented-cycle game $\mathcal{O}_s(K_n, \mathcal{C}, 1, b)$. Recall that this game is played by two players, OMaker and OBreaker, who alternately direct previously undirected edges of the complete graph $K_n$. OMaker, starting the game, directs exactly one edge in each round, while OBreaker directs exactly $b$ edges in every round (besides maybe the last round). OMaker's goal is to force a cycle for the final tournament which will be created by the directed edges of both players; and OBreaker aims to prevent her from doing so. We will show that for $b \geq \frac{37}{40}n$, OBreaker has a strategy to guarantee a transitive tournament for the end of the game, thus proving Theorem 1.1.1. In Section 2.1 we will describe that strategy and include all the lemmas that imply that OBreaker can really follow the strategy. In Section 2.2 and Section 2.3 we then give the proofs of all these lemmas.

**Notation and terminology.** Additionally to the notation and terminology introduced at the beginning of this thesis, we will make use of the following, which was used in [37] as well. Let $D = (V, E)$ be some digraph. We say that a $k$-tuple $(v_1, \ldots, v_k)$ of distinct vertices $v_i \in V$ *is fully transitive in* $D$, if all arcs are oriented from left to right, meaning that $(v_i, v_j) \in D$ whenever $i < j$. For two disjoint sets $A, B \subseteq V$ we call the pair $(A, B)$ a *uniformly directed biclique* ($UDB$) in $D$, if we have $D(A, B) = A \times B$. Moreover, a player is said to *direct an edge* $(v, w)$ or to *choose the arc* $(v, w)$ in the oriented-cycle game if he or she chooses the orientation of the edge $vw$ from $v$ to $w$. Assume a game is in progress, we usually let $D = (V, E)$ denote the digraph consisting of all the arcs that are chosen so far by either of the players. We set $\mathcal{A}(D)$ to be the set of *available* arcs, i.e. those arcs which can still be chosen by either of the players. That is, $(v, w) \in \mathcal{A}(D)$ holds if the edge $vw$ has no orientation, which is equivalent to $(w, v) \in \mathcal{A}(D)$. Moreover, we say that a player *closes* a directed path $P = (v_1, \ldots, v_k)$ in $D$ to a directed cycle, if he or she chooses the arc $(v_k, v_1)$ to belong to $D$.

## 2.1 The main strategy

Whenever necessary we will assume that $n$ is a large enough integer. Bollobás and Szabó [7] observed that for $b > n-3$ OBreaker easily wins the oriented-cycle game. So, we can assume that $\frac{37}{40}n \leq b \leq n$. In this section, we describe a strategy for OBreaker for the strict oriented-cycle game when playing with bias $b$, and in the subsequent sections we verify that this strategy indeed is a winning strategy when $\frac{37}{40}n \leq b \leq n$.

The main idea of OBreaker's strategy is to maintain a certain structure for the digraph $D$, which consists of all directed edges that have been chosen by either of the players. Globally, the goal is to maintain a $UDB$ $(A, B)$ for which at some point during the game we have $V = A \cup B$ and $|A|, |B| \leq b$, so that from that point on we can focus on each of the two parts $A$ and $B$ separately. OBreaker creates this structure in a few number of rounds to avoid "dangerous" situations. Locally, OBreaker maintains structures that are acyclic even after one further arc is added by OMaker. We will distinguish two types of local structures. Hereby, in a first stage, we ensure that OBreaker increases certain buffer sets inside the parts $A$ and $B$ of the $UDB$. In the second stage, these buffer sets then have the property that in each round at least one of these sets does not need to be touched by OBreaker's new arcs. In fact, this is what helps us to keep OBreaker's bias as small as $\frac{37}{40}n$.

The idea of maintaining a $UDB$ $(A, B)$ plus certain local structures was already introduced in [14, 37], where the bounds $t_{\mathcal{C}}(n) \leq \frac{5}{6}n$ and $t_{\mathcal{C}}^{+}(n) \leq n - \Theta(\sqrt{n})$ have been proven. The improvement towards $t_{\mathcal{C}}^{+}(n) \leq \frac{37}{40}n$ is obtained by a change of the local structures and by splitting the game into two stages. In the first stage our local structure is rather simple, but it allows us to generate the buffer sets which are mentioned above. Here lies the main difference with respect to the proof in [37]. Indeed, in the strategy for that proof, buffer sets are generated only for one round, while in the new proof we increase these sets for a number of rounds which is linear in $n$. Then, in the second stage, OBreaker maintains a structure which is a refinement of the local structure that was used in [37]. As OMaker could have chosen some arcs in Stage I, which OBreaker in Stage II needs to keep control on, this new structure is slightly more involved, which makes the argument more technical as in [37]. Still, we partially make use of the same notation as introduced in [37].

OBreaker's strategy is as follows.

**Stage I** lasts exactly $\lfloor \frac{3n}{125} \rfloor$ rounds.

We call the structure of $D$ that OBreaker maintains during Stage I *riskless*.

**Definition 2.1.1** *A digraph $D$ is called* riskless of rank $r$ *if there is a UDB $(A, B)$ in $D$ with partitions $A = A_S \cup A_0$ and $B = B_S \cup B_0$ such that the following properties hold:*

$(R1)$ Sizes: $||A| - |B|| \leq 1$ *and* $|A_S| = |B_S| = r$.

$(R2)$ Structure of $A_S$ and $B_S$: *The vertices of $A_S$ and $B_S$ can be enumerated in such a way that $A_S = \{v_1, \ldots v_r\}$ and $B_S = \{w_1, \ldots w_r\}$, and such that the following properties hold.*

- $(R2.1)$ *$(v_1, \ldots, v_r)$ and $(w_1, \ldots, w_r)$ are fully transitive in $D$.*
- $(R2.2)$ *For all $z \in A_0 \cup V \setminus (A \cup B)$: $\{i : (v_i, z) \in D\}$ is a down set of $[r]$.*
- $(R2.3)$ *For all $z \in B_0 \cup V \setminus (A \cup B)$: $\{i : (z, w_i) \in D\}$ is an upset of $[r]$.*

$(R3)$ Stars attached to $A \cup B$: *For every $1 \leq i \leq r$:*

- $(R3.1)$ *$e_D(v_i, A_0) \leq r + 1 - i$ and $e_D(v_i, V \setminus (A \cup B)) \leq \max(|A|, |B|)$.*
- $(R3.2)$ *$e_D(B_0, w_i) \leq i$ and $e_D(V \setminus (A \cup B), w_i) \leq \max(|A|, |B|)$.*

$(R4)$ Edge set: $D = D(A, B) \cup D(A_S, V \setminus B) \cup D(V \setminus A, B_S)$.

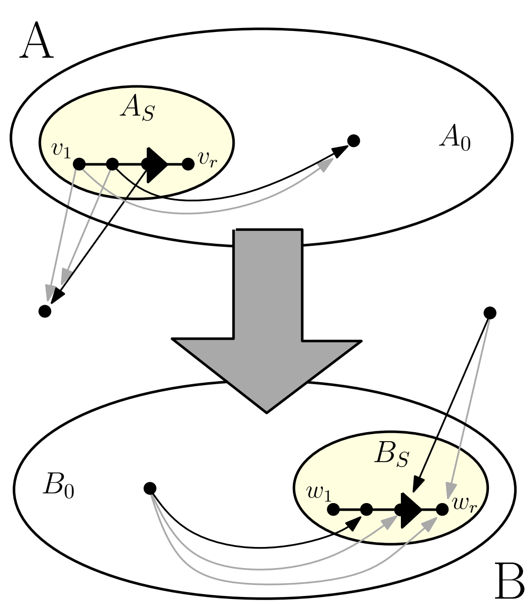


Figure 2.1: *The structure of a riskless digraph. The big grey arrow represents the edges of the UDB $(A, B)$.*

Sometimes, we call the $UDB$ $(A, B)$, from the definition above, the *underlying* $UDB$ of $D$. An illustration of the properties can be found in Figure 2.1.

One important ingredient throughout the proof is that the definition of riskless digraphs is robust under switching of the orientations of all edges. That is, the following statement holds.

**Observation 2.1.2** *If a digraph $D$ is riskless of rank $r$ with $UDB$ $(A, B)$, then $\overleftarrow{D}$ is riskless of rank $r$ with $UDB$ $(B, A)$.*

**Proof** Let $D$ be a digraph satisfying the properties of a riskless digraph, with $A = A_S \cup A_0$ and $B = B_S \cup B_0$, as given in Definition 2.1.1, and rank $r = |A_S| = |B_S|$. Then $\overleftarrow{D}$ satisfies the same properties with $UDB$ $(A', B') = (B, A)$ and $B'_S = \{w'_1, \ldots, w'_r\}$ and $A'_S = \{v'_1, \ldots, v'_r\}$ with $w'_i = v_{r-i+1}$ and $v'_i = w_{r-i+1}$ for every $i \in [r]$. □

Note that the empty graph, which is present before the first round of the game (for technical reasons we say "after round 0"), is riskless of rank 0. For some $0 \le r \le \frac{3n}{125} - 1$, after round $r$, assume $D$ is riskless of rank $r$ with $UDB$ $(A, B)$. We additionally have $|D| = r(b+1)$, as in each round exactly $b+1$ edges receive an orientation.

Let $e = (v, w)$ be the arc that OMaker directs in round $r+1$. First we consider the case that $e \in \mathcal{A}(D(V \setminus B))$. In a first step, OBreaker now chooses *at most* $b$ arcs given by Lemma 2.1.3, with the goal to restore the properties of a riskless digraph, while increasing the rank only by one. In a second step, right after all properties are restored, OBreaker then adds further edges to $D$ without destroying its structural properties until exactly $b$ arcs are chosen.

**Lemma 2.1.3** *Let $n$ be a large enough positive integer and let $\frac{37}{40}n \le b \le n$. For a non-negative integer $r \le \frac{3n}{125} - 1$, let $D$ be a digraph which is riskless of rank $r$ with underlying $UDB$ $(A, B)$ as given in Definition 2.1.1. Assume that $|D| = r(b+1)$. Let $e \in \mathcal{A}(D(V \setminus B))$ be an available arc in $V \setminus B$. Then there exist at most $b$ available arcs $f_1, \ldots, f_t \in \mathcal{A}(D+e)$ such that $D' := D \cup \{e, f_1, \ldots, f_t\}$ is a riskless digraph of rank $r+1$. Moreover, $e_{D'}(V \setminus (A' \cup B'), w'_1) = 0$, where $(A', B')$ is the underlying $UDB$ of $D'$, with $B' = B'_S \cup B'_0$ and $B'_S = \{w'_1, \ldots, w'_{r+1}\}$ as in Definition 2.1.1.*

The property $e_{D'}(V \setminus (A' \cup B'), w'_1) = 0$ is useful to accommodate OBreaker's remaining arcs that he still needs to direct in round $r+1$. Let $t$ be the number of edges that OBreaker directs in round $r+1$, when he chooses his arcs according to Lemma 2.1.3. In order to complete his move for the current round, OBreaker then adds $b-t$ further arcs using the following lemma, which ensures that the properties of a riskless digraph are maintained.

**Lemma 2.1.4** *Let $n$ be a large enough positive integer and let $\frac{37}{40}n \leq b \leq n$. For a non-negative integer $r \leq \frac{3n}{125} - 1$, let $D$ be a digraph which is riskless of rank $r+1$ with underlying UDB $(A, B)$ as given in Definition 2.1.1. Assume that $r(b+1) \leq |D| \leq (r+1)(b+1) < \binom{n}{2}$. Let $w_1$ be the top vertex in the tournament inside $B$, as given in Property $(R2)$, and assume that $e_D(V \setminus (A \cup B), w_1) = 0$ holds. Then OBreaker can direct a set of $(r+1)(b+1) - |D| \leq b$ available arcs $\mathcal{F} \subseteq \mathcal{A}(D)$ such that $D' := D \cup \mathcal{F}$ is a riskless digraph of rank $r+1$.*

Secondly, consider the case that $e = (v, w) \notin \mathcal{A}(D(V \backslash B))$. Then $e \in \mathcal{A}(D(V \backslash A))$, since $(A, B)$ forms a $UDB$ in $D$. Assume the previous two lemmas to be correct. By Observation 2.1.2, $\overleftarrow{D}$ is also riskless of rank $r$ with $UDB$ $(B, A)$. So, applying Lemma 2.1.3 and Lemma 2.1.4 we thus can find a set $\mathcal{F}$ of exactly $b$ available arcs such that $\overleftarrow{D} \cup \{\overleftarrow{e}\} \cup \mathcal{F}$ is riskless of rank $r+1$. But then, again by Observation 2.1.2, the digraph $D \cup \{e\} \cup \overleftarrow{\mathcal{F}}$ is riskless of rank $r+1$. OBreaker's strategy thus is to choose the $b$ arcs from $\overleftarrow{\mathcal{F}}$ in this case.

**Stage II** starts in round $\left\lfloor \frac{3n}{125} \right\rfloor + 1$.

The structure that OBreaker now aims to maintain is similar to the one given for Stage I. The most important difference is that from now on we partition the sets of the $UDB$ further to distinguish the vertices according to their chance to become part of a directed cycle. Motivated by the proof in [37], we maintain partitions $A = A_D \cup A_{AD} \cup A_S \cup A_0$ and $B = B_D \cup B_{AD} \cup B_S \cup B_0$. Its subsets $A_0 \cup A_D$ and $B_0 \cup B_D$ form the aforementioned buffer sets, for which we ensure that in each or the remaining rounds, OBreaker needs to touch at most one of these two sets.

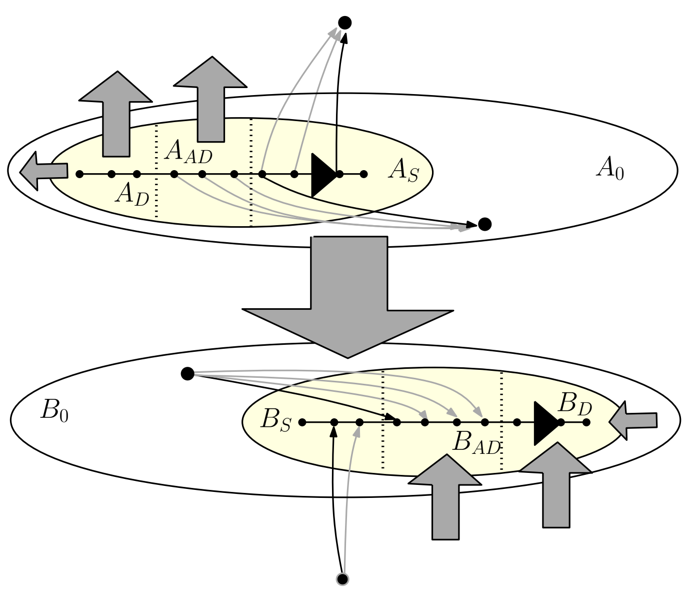


Figure 2.2: *The structure of a protected digraph. Big arrows indicate UDB's between two sets.*

**Definition 2.1.5** *A digraph $D$ on $n$ vertices is called* protected *if there is a $UDB$ $(A, B)$ with partitions $A = A_D \cup A_{AD} \cup A_S \cup A_0$ and $B = B_D \cup B_{AD} \cup B_S \cup B_0$ such that the following properties hold:*

$(P1)$ Sizes: $|A|, |B| \geq \frac{99n}{1000} + 1$, *and* $|A_D \cup A_0|$, $|B_D \cup B_0| \geq \frac{99n}{1000}$.

$(P2)$ Structure of $A_D$ and $B_D$: $(A_D, V \setminus A_D)$ *and* $(V \setminus B_D, B_D)$ *are* $UDB'$*s in* $D$; $D(A_D)$ *and* $D(B_D)$ *are transitive tournaments.*

$(P3)$ Structure of $A_{AD}$ and $B_{AD}$: $(A_{AD}, V \setminus A)$ *and* $(V \setminus B, B_{AD})$ *are* $UDB'$*s in* $D$.

$(P4)$ Structure of $A_{AD} \cup A_S$ and $B_{AD} \cup B_S$: *There exist integers* $k_1, \ell_1 \geq 0$ *and* $0 \leq k_2, \ell_2 \leq \frac{3n}{125}$ *such that* $|A_{AD}| = k_1$, $|A_S| = k_2$, $|B_{AD}| = \ell_1$, *and* $|B_S| = \ell_2$. *Moreover, the vertices can be enumerated in such a way that*

$$A_{AD} = \{v_1, \ldots, v_{k_1}\},\ A_S = \{v_{k_1+1}, \ldots, v_{k_1+k_2}\} \text{ and}$$
$$B_S = \{w_1, \ldots, w_{\ell_2}\},\ B_{AD} = \{w_{\ell_2+1}, \ldots, w_{\ell_2+\ell_1}\} \text{ and such that}$$

*the following properties hold.*

$(P4.1)$ $(v_1, \ldots, v_{k_1+k_2})$ *and* $(w_1, \ldots, w_{\ell_1+\ell_2})$ *are fully transitive in* $D$.

$(P4.2)$ *For all* $z \in A_0 \cup V \setminus (A \cup B)$: $\{i : (v_i, z) \in D\}$ *is a down set of* $[k_1 + k_2]$

$(P4.3)$ *For all* $z \in B_0 \cup V \setminus (A \cup B)$: $\{i : (z, w_i) \in D\}$ *is an upset of* $[\ell_1 + \ell_2]$.

$(P5)$ Stars attached to $A \cup B$:

$(P5.1)$ *For all* $1 \leq i \leq k_2$: $e(v_{k_1+i}, A_0) \leq \frac{3n}{125} + 1 - i$.

$(P5.2)$ *For all* $1 \leq i \leq \ell_2$: $e(B_0, w_i) \leq \frac{3n}{125} - \ell_2 + i$.

$(P6)$ Edge set: $D = D(A, B) \cup D(A \setminus A_0, V \setminus B) \cup D(V \setminus A, B \setminus B_0)$.

As before, the $UDB$ $(A, B)$ from the definition above is called the underlying $UDB$ of $D$. An illustration of the properties of a protected digraph can be found in Figure 2.2.

We also have the following observation, similar to the discussion of riskless digraphs.

**Observation 2.1.6** *If a digraph $D$ is protected with $UDB$ $(A, B)$, then $\overleftarrow{D}$ is protected with $UDB$ $(B, A)$.*

As OBreaker wants to maintain the structural properties of protected digraphs after each of his moves in the second stage, we first need to show that immediately after OBreaker's last move in Stage I, the digraph $D$ is protected. Note that, on the assumption of Lemma 2.1.3 and Lemma 2.1.4, $D$ is riskless of rank $\lfloor \frac{3n}{125} \rfloor$.

**Lemma 2.1.7** *Let $n$ be large enough, and let $\frac{37}{40}n \le b \le n$. Let $D$ be a digraph on $n$ vertices which is riskless of rank $r = \lfloor \frac{3n}{125} \rfloor$, and assume that $|D| = r(b+1)$. Then $D$ is protected.*

Now, we proceed similarly to Stage I. For some $r \ge \lfloor \frac{3n}{125} \rfloor$, after round $r$, assume that $D$ is protected with underlying $UDB$ $(A, B)$. Let $e = (v, w)$ be the arc OMaker directs in round $r + 1$. Again, we first consider the case that $e \in \mathcal{A}(D(V \setminus B))$. Then, as a first step, OBreaker chooses at most $b$ arcs given by the following lemma to restore the properties of a protected digraph.

**Lemma 2.1.8** *Let $D$ be a digraph which is protected, with underlying $UDB$ $(A, B)$ according to Definition 2.1.5, and let $e = (v, w) \in \mathcal{A}(D(V \setminus B))$ be an available arc in $V \setminus B$. Then there exist at most $b$ available arcs $f_1, \dots, f_t \in \mathcal{A}(D + e)$ such that $D' := D \cup \{e, f_1, \dots, f_t\}$ is protected.*

Let $t$ be the number of edges that OBreaker directs in round $r + 1$, when he chooses his arcs according to Lemma 2.1.8. He then adds $b - t$ further arcs using the following lemma iteratively, or in case fewer than $b - t$ arcs can be chosen (which may happen in the very last round of the game), he applies the following lemma to direct all remaining edges.

**Lemma 2.1.9** *Let $D$ be a protected digraph on $n$ vertices with $|D| < \binom{n}{2}$. Then, there exists an available arc $f \in \mathcal{A}(D)$ such that $D + f$ is protected.*

Secondly, assume that $e = (v, w) \notin \mathcal{A}(D(V \setminus B))$. We can argue as in Stage I. It follows then that $e \in \mathcal{A}(D(V \setminus A))$, since $(A, B)$ forms a $UDB$. By Observation 2.1.6, $\overleftarrow{D}$ is also protected with $UDB$ $(B, A)$. So, applying Lemma 2.1.8 and Lemma 2.1.9 we thus can find a set $\mathcal{F}$ of exactly $b$ available arcs (or at most $b$ in the last round of the game) such that $\overleftarrow{D} \cup \{\overleftarrow{e}\} \cup \mathcal{F}$ is protected. Then, by Observation 2.1.6, $D \cup \{e\} \cup \overleftarrow{\mathcal{F}}$ is protected, and so OBreaker chooses all the arcs of $\overleftarrow{\mathcal{F}}$ in this case.

The strategy for OBreaker is given implicitly in Lemma 2.1.3, 2.1.4, 2.1.8 and 2.1.9, where Lemma 2.1.9 is proven analogously to Proposition 3.3.1 in [37]. These lemmas, together with Lemma 2.1.7, also contain the proof that OBreaker can follow his strategy until every edge has an orientation. We prove Lemma 2.1.3 and Lemma 2.1.4 in the next section; Lemma 2.1.7, Lemma 2.1.8 and Lemma 2.1.9 are proved in Section 2.3.

For now, let us finish this section with the proof that, under the assumption that all above mentioned lemmas are true and that OBreaker follows the proposed strategy, OBreaker maintains a transitive tournament at the end of the game. In Stage I, Lemma 2.1.3 and

Lemma 2.1.4 guarantee that the digraph $D$ is riskless of rank $r$ after OBreaker's move in every round $r \leq \lfloor \frac{3n}{125} \rfloor$. Furthermore, after round $r = \lfloor \frac{3n}{125} \rfloor$, $D$ is protected by Lemma 2.1.7. Then, in Stage II, Lemma 2.1.8 and Lemma 2.1.9 guarantee that the digraph $D$ is protected after each move of OBreaker, and, in particular, at the end of the game. Therefore, to show that OMaker can never close a cycle, it is enough to prove the following.

**Lemma 2.1.10** *If $D$ is a protected digraph, then $D$ is acyclic.*

**Proof** Let $D$ be a protected digraph with $UDB$ $(A, B)$, and assume that there is a directed cycle $C$ in $D$. By property $(P6)$, for each $(v, w) \in D$, we have $v \in A$ or $w \in B$. Therefore, the edges of $C$ either only contain vertices from $A$ or only contain vertices from $B$. By Observation 2.1.6, we may assume w.l.o.g. that $C \subseteq D(A)$. Again by Property $(P6)$, $C$ must use only vertices from $A \setminus A_0$. However, by Property $(P2)$ and Property $(P4.1)$, $A \setminus A_0$ induces a transitive tournament in $D$, and thus does not contain a directed cycle, a contradiction. □

Thus, in order to prove Theorem 1.1.1 it remains to prove all the previously mentioned lemmas.

## 2.2 OBreaker's strategy – Stage I

In the following we prove Lemma 2.1.3 and Lemma 2.1.4. Before doing so, we prove the following proposition which we later refer to several times.

**Proposition 2.2.1** *Let $n$ be large enough, and let $b \leq n$ and $r \leq \frac{3n}{125}$. Let $D$ be a riskless digraph of rank $r$, with underlying UDB $(A, B)$, such that $|D| \leq r(b+1)$. Then*

*(i)* $|A|, |B| < \frac{n}{6}$ *and* $|V \setminus (A \cup B)| > \frac{2n}{3}$.

*(ii)* *For* $X_A := \{z \in V \setminus (A \cup B) : e_D(A, z) = 0\}$ *and* $Y_B := \{z \in V \setminus (A \cup B) : e_D(z, B) = 0\}$, *we have* $|X_A|, |Y_B| > \frac{n}{2}$.

**Proof** Note first that since $(A, B)$ is a $UDB$, it holds that $|A| \cdot |B| \leq |D| \leq r(b+1)$. By assumption on $r$ and $b$ and since $||A| - |B|| \leq 1$, by Property $(R1)$, it follows that $\max(|A|, |B|) \leq \sqrt{r(b+1)} + 1 < \frac{n}{6}$, and that $|A| + |B| < \frac{n}{3}$. Therefore, $|V \setminus (A \cup B)| > \frac{2n}{3}$. Let $\overline{X_A} := \{z \in V \setminus (A \cup B) : e_D(A, z) > 0\}$. Then, by Property $(R4)$ and $(R2.2)$,

$$\begin{aligned}\overline{X_A} &= \{z \in V \setminus (A \cup B) : (x, z) \in D \text{ for some } x \in A\} \\ &= \{z \in V \setminus (A \cup B) : (x, z) \in D \text{ for some } x \in A_S\} \\ &= \{z \in V \setminus (A \cup B) : (v_1, z) \in D\}.\end{aligned}$$

So, by Property $(R3.1)$ and (i), $|\overline{X_A}| \le \max(|A|,|B|) < \frac{n}{6}$ and hence,

$$|X_A| = |V \setminus (A \cup B)| - |\overline{X_A}| > \frac{n}{2}.$$

Analogously, $|Y_B| > \frac{n}{2}$. □

With this proposition in hand, we now prove Lemma 2.1.3. It ensures that OBreaker has a strategy to reestablish the properties of a riskless digraph throughout Stage I.

**Proof of Lemma 2.1.3** Let $e = (v, w) \in \mathcal{A}(D(V \setminus B))$ be given by the lemma. At first, let us fix distinct vertices $x_A, y_B \in V \setminus (A \cup B \cup \{v, w\})$ with $e_D(A, x_A) = 0$ and $e_D(y_B, B) = 0$. Note that Proposition 2.2.1 guarantees their existence since $r \le \frac{3n}{125}$ and $b \le n$. Define $u_S \in A_0 \cup V \setminus (A \cup B)$ and $u_A \in V \setminus (A \cup B)$ by

$$u_S := \begin{cases} x_A & \text{if } v \in A_S \\ v & \text{if } v \notin A_S \end{cases} \qquad \text{and} \qquad u_A := \begin{cases} x_A & \text{if } v \in A \\ v & \text{if } v \notin A. \end{cases}$$

Our goal is to add $u_S$ to the set $A_S$ of star centers, $u_A$ to $A$ and $y_B$ to $B$. Note that the two vertices $u_S$ and $u_A$ are equal unless $v \in A_0$. Moreover, let

$$\ell := \begin{cases} \min(i : \ (v_i, v) \notin D) & \text{if minimum exists} \\ r + 1 & \text{otherwise} \end{cases}$$

and observe that $v_\ell = v$ if $v \in A_S$, by Property $(R2.1)$. Set

$$v'_i := \begin{cases} v_i & \text{if } 1 \le i \le \ell - 1 \\ u_S & \text{if } i = \ell \\ v_{i-1} & \text{if } \ell + 1 \le i \le r + 1 \end{cases} \qquad \text{and} \qquad w'_i := \begin{cases} y_B & \text{if } i = 1 \\ w_{i-1} & \text{if } 2 \le i \le r + 1. \end{cases}$$

These vertices are used to form the new centers of the stars in Property $(R3)$. Now, choose $\{f_1, \ldots, f_t\}$ to be $\{f_1, \ldots, f_t\} = (\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_7) \cap \mathcal{A}(D)$, where

$$\begin{aligned}
\mathcal{F}_1 &:= \big\{(u_A, y) : y \in B\big\}, \\
\mathcal{F}_2 &:= \big\{(x, w'_1) : x \in A \cup \{u_A\}\big\}, \\
\mathcal{F}_3 &:= \big\{(w'_1, w'_i) : 2 \le i \le r + 1\big\}, \\
\mathcal{F}_4 &:= \big\{(v'_i, w) : 1 \le i \le \ell\big\}, \\
\mathcal{F}_5 &:= \big\{(v'_i, v'_\ell) : 1 \le i \le \ell - 1\big\}, \\
\mathcal{F}_6 &:= \big\{(v'_\ell, v'_i) : \ell + 1 \le i \le r + 1\big\}, \\
\mathcal{F}_7 &:= \big\{(v'_\ell, z) : z \in V \setminus (A \cup B) \cup A_0 \text{ and } (v'_{\ell+1}, z) \in D\big\},
\end{aligned}$$

where we use the convention that $\mathcal{F}_7 = \emptyset$ if $\ell = r+1$ (and thus $v'_{\ell+1}$ does not exist).

To show that this choice of arcs is suitable for the lemma, we first show that $f_i \neq \overleftarrow{e}$ for all $1 \le i \le t$, that $\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_7 \subseteq D \cup \mathcal{A}(D)$, and that $t \le b$. Note that $\mathcal{F}_i \subseteq D \cup \mathcal{A}(D)$ implies that $\mathcal{F}_i \subseteq D'$, where $D' = D \cup \{e, f_1, \ldots, f_t\}$. We use this information to show that $D'$ is riskless of rank $r+1$ with some $UDB$ $(A', B')$. Finally, we show that $e_{D'}(V \setminus (A' \cup B'), w'_1) = 0$, where $w'_1$ is the top vertex in the tournament in $B'$, given by Property ($R2.1$).

For the first part, $\overleftarrow{e} = (w,v) \notin \mathcal{F}_1$ since by assumption $v \notin B$, $\overleftarrow{e} \notin \mathcal{F}_2 \cup \mathcal{F}_3$ by choice of $w'_1 = y_B$, and $\overleftarrow{e} \notin \mathcal{F}_4$ since $v \neq w$. Assume now that $(w,v) = (v'_i, v'_\ell) \in \mathcal{F}_5$ for some $1 \le i \le \ell - 1$. Then, $(w,v) = (v_i, v) \in D$ by definition of $\ell$, a contradiction to $e = (v,w) \in \mathcal{A}(D(V \setminus B))$. So, $\overleftarrow{e} \notin \mathcal{F}_5$. For $\mathcal{F}_6$, assume that $(w,v) = (v'_\ell, v'_i) \in \mathcal{F}_6$ for some $\ell+1 \le i \le r+1$. Then $v \neq v'_\ell$, so $v'_\ell = u_S = x_A \neq w$ by definition of $x_A$, a contradiction. So, $\overleftarrow{e} \notin \mathcal{F}_6$. Finally, if $(w,v) \in \mathcal{F}_7$ then $v \in V \setminus (A \cup B) \cup A_0$, so $w = v'_\ell = u_S = v$ by definition of $u_S$, a contradiction since $(v,w) \in \mathcal{A}(D)$ and thus, it is not a loop. Hence, $\overleftarrow{e} \notin \mathcal{F}_7$, as well.

To see that $\mathcal{F}_1 \subseteq D \cup \mathcal{A}(D)$ note that $(y, u_A) \notin D$ for all $y \in B$, since $u_A \in V \setminus (A \cup B)$ and by Property ($R4$). Similarly, $\mathcal{F}_2 \cup \mathcal{F}_3 \subseteq D \cup \mathcal{A}(D)$ since $w'_1 = y_B \in V \setminus (A \cup B)$ and by Property ($R4$). Assume now that $(w, v'_i) \in D$ for some $1 \le i \le \ell$. Then $w = v_j \in A_S$ for some $1 \le j \le r$, by Property ($R4$). Moreover, $v'_i \neq v$, since $e = (v,w) \in \mathcal{A}(D(V \setminus B))$, and $v'_i \neq x_A$ by choice of $x_A$ and Property ($R4$). This yields $v'_i \neq u_S$ and therefore $i \neq \ell$. But then $v'_i = v_i$, and $j < i \le \ell - 1$, by Property ($R2.1$). By definition of $\ell$ we conclude $(w,v) = (v_j, v) \in D$, in contradiction to $e \in \mathcal{A}(D(V \setminus B))$. Thus, $\mathcal{F}_4 \subseteq D \cup \mathcal{A}(D)$. For $\mathcal{F}_5$, we note that $(v'_\ell, v'_i) = (u_S, v_i) \notin D$ for all $1 \le i < \ell$, by Property ($R4$) and since $u_S \notin A_S$ and $v_i \notin B$. Now, assume that $(v'_i, v'_\ell) = (v_{i-1}, u_S) \in D$ for some $\ell+1 \le i \le r+1$. Then $u_S \neq x_A$ by the choice of $x_A$. It follows $(v_{i-1}, v) = (v_{i-1}, u_S) \in D$. But then, by definition of $\ell$ and ($R2.2$), we obtain $i - 1 < \ell$, a contradiction. Thus $\mathcal{F}_6 \subseteq D \cup \mathcal{A}(D)$. Finally, $\mathcal{F}_7 \subseteq D \cup \mathcal{A}(D)$ since for all $z \in V \setminus (A \cup B) \cup A_0$ we have that $(z, v'_\ell) \notin D$, by Property ($R4$) and since $v'_\ell = u_S \notin B$.

To bound $t$ by the bias $b$ we note that

$$\begin{aligned}
t \le \sum_{i=1}^{7} |\mathcal{F}_i| &\le |B| + (|A|+1) + r + (2\ell - 1) + (r - \ell + 1) \\
&\qquad + e_D(v'_{\ell+1}, V \setminus (A \cup B) \cup A_0) \\
&= |A| + |B| + 2r + \ell + 1 + e_D(v_\ell, V \setminus (A \cup B)) + e_D(v_\ell, A_0) \\
&\le |A| + |B| + 2r + \ell + 1 + \max(|A|, |B|) + (r + 1 - \ell) \\
&\le 3\max(|A|, |B|) + 3r + 2
\end{aligned}$$

where the third inequality follows from Property ($R3.1$).

Now, by Proposition 2.2.1 (i) and since $r \leq \frac{3n}{125} - 1$, it follows that

$$t \leq \frac{3n}{6} + 3 \cdot \frac{3n}{125} \leq b.$$

We now show that $D' = D \cup \{e, f_1, \ldots, f_t\}$ is a riskless digraph of rank $r+1$.

For this, consider the sets

$$\begin{aligned} A'_S &:= \{v'_1, \ldots, v'_{r+1}\} = A_S \cup \{u_S\}, \\ A'_0 &:= (A_0 \cup \{u_A\}) \setminus \{u_S\}, \\ A' &:= A \cup \{u_A\} \end{aligned}$$

and $B'_S := \{w'_1, \ldots, w'_{r+1}\} = B_S \cup \{y_B\}$, $B'_0 := B_0$ and $B' = B \cup \{y_B\}$. We claim that $(A', B')$ is a $UDB$ in $D'$ with partitions $A' = A'_0 \cup A'_S$ and $B' = B'_0 \cup B'_S$ such that $(R1)$–$(R4)$ are satisfied for $r+1$ and such that $e_{D'}(V \setminus (A' \cup B'), w'_1) = 0$.

Since $(A, B)$ is a $UDB$ in $D$ and $D \cup \mathcal{F}_1 \cup \mathcal{F}_2 \subseteq D'$, $(A', B')$ forms a $UDB$ in $D'$. For Property $(R1)$, note that $|A'| = |A| + 1$, $|B'| = |B| + 1$ and $|A'_S| = |B'_S| = r + 1$.

For Property $(R2.1)$, note first that $A_S = A'_S \setminus \{v'_\ell\}$ induces a transitive tournament in $D \subseteq D'$. Furthermore, for all $i < \ell$ we have that $(v'_i, v'_\ell) \in \mathcal{F}_5 \subseteq D'$, and for all $i > \ell$ we have that $(v'_\ell, v'_i) \in \mathcal{F}_6 \subseteq D'$. Hence, $(v'_1, \ldots, v'_{r+1})$ is fully transitive in $D'$. Furthermore, $B_S = B'_S \setminus \{w'_1\}$ induces a transitive tournament in $D \subseteq D'$, and for all $2 \leq i \leq r+1$, we have that $(w'_1, w'_i) \in \mathcal{F}_3 \subseteq D'$. Hence, $(w'_1, \ldots, w'_{r+1})$ is fully transitive in in $D'$.

For $(R2.2)$, let $z \in A'_0 \cup V \setminus (A' \cup B')$. Then note that only arcs from $D \cup \{e\} \cup \mathcal{F}_4 \cup \mathcal{F}_7$ contribute to the set $\{i : (v'_i, z) \in D'\}$. Moreover, $D(v'_\ell, A'_0 \cup V \setminus (A' \cup B')) = \emptyset$ by Property $(R4)$ and since $v'_\ell \in A_0 \cup V \setminus (A \cup B)$. Note also that $\{i : (v_i, z) \in D\}$ is a down set of $[r]$ and the relative order of the $v'_i$ for $i \neq \ell$ does not change. If $z \neq w$, then the arcs from $\mathcal{F}_7$ reestablish the down-set property for $D'$. If $z = w$, then the arcs from $\mathcal{F}_4 \cup \{e\}$ reestablish the down-set property for $D'$.

For $(R2.3)$, let $z \in B'_0 \cup V \setminus (A' \cup B')$. Then note that only arcs from $D$ contribute to the set $\{i : (z, w'_i) \in D'\}$. Moreover, $\{i : (z, w_i) \in D\}$ is an upset of $[r]$ by assumption. Further, note that $w'_{i+1} = w_i$ for every $1 \leq i \leq r$, and that $(z, w'_1) = (z, y_B) \notin D$, by Property $(R4)$ and since $y_B \in V \setminus (A \cup B)$. Thus, $\{i : (z, w'_i) \in D'\}$ is an upset of $[r+1]$.

For $(R3.1)$, let first $1 \leq i \leq \ell - 1$. Then only the arcs in $D(v_i, A_0) \cup \mathcal{F}_4$ contribute to $D'(v'_i, A'_0)$; and only the arcs in $D(v_i, V \setminus (A \cup B)) \cup \mathcal{F}_4$ contribute to $D'(v'_i, V \setminus (A' \cup B'))$. Therefore, we obtain that $e_{D'}(v'_i, A'_0) \leq e_D(v_i, A_0) + 1 \leq (r+1) + 1 - i$ holds as well as $e_{D'}(v'_i, V \setminus (A' \cup B')) \leq e_D(v_i, V \setminus (A \cup B)) + 1 \leq \max(|A|, |B|) + 1 = \max(|A'|, |B'|)$.
Now, let $i = \ell$. Observe that $e_D(v'_\ell, A_0 \cup V \setminus (A \cup B)) = 0$ by $(R4)$ and since $v'_\ell \in A_0 \cup V \setminus (A \cup B)$.

So, only the arcs in $\mathcal{F}_7 \cup \{e\}$ contribute to $D'(v'_\ell, A'_0)$ and $D'(v'_\ell, V \setminus (A' \cup B'))$. Therefore, we obtain that $e_{D'}(v'_\ell, A'_0) \leq e_D(v_\ell, A_0) + 1 \leq (r+1) + 1 - \ell$ holds as well as $e_{D'}(v'_\ell, V \setminus (A' \cup B')) \leq e_D(v_\ell, V \setminus (A \cup B)) + 1 \leq \max(|A|, |B|) + 1 = \max(|A'|, |B'|)$.
Finally, let $\ell + 1 \leq i \leq r + 1$. Then we know that $D'(v'_i, A'_0) = D(v_{i-1}, A_0)$ holds and $D'(v'_i, V \setminus (A' \cup B')) \subseteq D(v_{i-1}, V \setminus (A \cup B))$. Hence, $e_{D'}(v'_i, A'_0) \leq (r+1) + 1 - i$ and $e_{D'}(v'_i, V \setminus (A' \cup B')) \leq \max(|A'|, |B'|)$.

For $(R3.2)$, first let $2 \leq i \leq r+1$. Then only the arcs in $D(B_0, w_{i-1})$ contribute to $D'(B'_0, w'_i)$; and only the arcs in $D(V \setminus (A \cup B), w'_i)$ contribute to $D'(V \setminus (A' \cup B'), w'_i)$. Therefore we have $e_{D'}(B'_0, w'_i) \leq i - 1$ and $e_{D'}(V \setminus (A' \cup B'), w'_i) \leq \max(|A|, |B|) \leq \max(|A'|, |B'|)$.
Now, for $i = 1$ we have $w'_1 = y_B \in V \setminus (A \cup B)$. Similarly, only the arcs in $D(B_0, y_B)$ contribute to $D'(B'_0, w'_1)$; and only the arcs in $D(V \setminus (A \cup B), y_B)$ contribute to $D'(V \setminus (A' \cup B'), w'_1)$. Then by Property $(R4)$ for the digraph $D$ and by the choice of $w'_1 = y_B$, we conclude $e_{D'}(B'_0, w'_1) = e_D(B_0, y_B) = 0$ and analogously $e_{D'}(V \setminus (A' \cup B'), w'_1) = 0$.

For $(R4)$, note that it is enough to prove that $D' \subseteq D'(A', B') \cup D'(A'_S, V) \cup D'(V, B'_S)$. This indeed holds, since

$$D(A, B) \cup \mathcal{F}_1 \cup \mathcal{F}_2 \subseteq D'(A', B'),$$
$$D(A_S, V \setminus B) \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{F}_6 \cup \mathcal{F}_7 \subseteq D'(A'_S, V), \text{ and}$$
$$D(V \setminus A, B_S) \cup \mathcal{F}_3 \subseteq D'(V, B'_S).$$

This finishes the proof of Lemma 2.1.3. □

Next, we prove Lemma 2.1.4 which ensures that OBreaker can add arcs to a riskless digraph without destroying its structural properties.

**Proof of Lemma 2.1.4** We first provide OBreaker with a strategy to direct $(r+1)(b+1) - |D|$ available arcs, then we show that OBreaker can follow that strategy, and that the resulting digraph $D'$ is riskless of rank $r + 1$.

Initially, set $t = (r+1)(b+1) - |D|$ and let $\mathcal{F} := \emptyset$ be the set of arcs that OBreaker will add to the digraph $D$. Both $t$ and $\mathcal{F}$ are dynamic in the following proof. That is, whenever OBreaker chooses a new arc, he adds this arc to $\mathcal{F}$, and he decreases the number $t$ of edges which he still needs to orient. We proceed iteratively: As long as $t \geq \max(|A|, |B|)$, OBreaker enlarges $A_0$ and $B_0$ (and thus $A$ and $B$) alternately. As soon as $t < \max(|A|, |B|)$, he fills up the stars with centers $w_i$, $1 \leq i \leq r$, starting with $w_{r+1}$.

**Step 1: $t \geq \max(|A|, |B|)$.** If $|B| = |A| - 1$, then let $y_B \in V \setminus (A \cup B)$ be an arbitrary vertex such that $e_{D \cup \mathcal{F}}(y_B, B) = 0$. For all $x \in A$, if $(x, y_B) \notin D \cup \mathcal{F}$, OBreaker directs $(x, y_B)$, decreases $t$ by one and updates $\mathcal{F} := \mathcal{F} \cup \{(x, y_B)\}$. Afterwards, he sets $B := B \cup \{y_B\}$, $B_0 := B_0 \cup \{y_B\}$ and repeats Step 1.

If $|B| \geq |A|$, then let $x_A \in V \setminus (A \cup B)$ be an arbitrary vertex such that $e_{D\cup\mathcal{F}}(A, x_A) = 0$. For all $y \in B$, if $(x_A, y) \notin D \cup \mathcal{F}$, OBreaker directs $(x_A, y)$, updates $\mathcal{F} := \mathcal{F} \cup \{(x_A, y)\}$ and decreases $t$ by one. Afterwards, he sets $A := A \cup \{x_A\}$, $A_0 := A_0 \cup \{x_A\}$ and repeats Step 1.

**Step 2:** $t < \max(|A|, |B|)$**.** If $t = 0$, there is nothing to do. Otherwise, OBreaker proceeds as follows.

If $e_{D\cup\mathcal{F}}(V \setminus (A \cup B), w_{r+1}) < \max(|A|, |B|)$, then let $z \in V \setminus (A \cup B)$ such that $e_{D\cup\mathcal{F}}(z, B) = 0$. Then OBreaker directs $(z, w_{r+1})$, updates $\mathcal{F} := \mathcal{F} \cup \{(z, w_{r+1})\}$, decreases $t$ by one and repeats Step 2.

Otherwise, $e_{D\cup\mathcal{F}}(V \setminus (A \cup B), w_{r+1}) = \max(|A|, |B|)$. Then let $\ell$ be the maximal index $i \in [r]$ such that $e_{D\cup\mathcal{F}}(V \setminus (A \cup B), w_i) < \max(|A|, |B|)$. Let $z \in V \setminus (A \cup B)$ be an arbitrary vertex with $(z, w_\ell) \notin D \cup \mathcal{F}$ and $(z, w_{\ell+1}) \in D \cup \mathcal{F}$. OBreaker then directs $(z, w_\ell)$, updates $\mathcal{F} := \mathcal{F} \cup \{(z, w_\ell)\}$, decreases $t$ by one and repeats Step 2.

We first show that OBreaker can follow the strategy. First note, by Property $(R4)$ of a riskless digraph, that for all $z \in V \setminus (A \cup B)$, for all $x \in A$ we have that $(z, x) \notin D$, and for all $y \in B$ we have that $(y, z) \notin D$. Hence, under the assumption that $x_A$ and $y_B$ in Step 1 exist, OBreaker can follow the proposed strategy in Step 1.

Now, since $D$ is a riskless digraph of rank $r + 1 \leq \frac{3n}{125}$ with $|D| \leq (r + 1)(b + 1)$, we have that

$$|X_A| = |\{z \in V \setminus (A \cup B) : e_D(A, z) = 0\}| > \frac{n}{2} \text{ and}$$
$$|Y_B| = |\{z \in V \setminus (A \cup B) : e_D(z, B) = 0\}| > \frac{n}{2},$$

before the first update in Step 1. Moreover, $D(X_A) = \emptyset$ and $D(Y_B) = \emptyset$ by Property $(R4)$. Now, in each iteration of Step 1, $A$ or $B$ increases by one vertex from $V \setminus (A \cup B)$ (alternately). Since by $(R4)$ there are no arcs inside $V \setminus (A \cup B)$, and since in Step 1, OBreaker directs all edges between the new vertices in $A$ and those in $B$, the size of each of these two sets can increase by at most $\sqrt{b} \leq \sqrt{n} \leq \frac{n}{100}$, for large enough $n$. Since $X_A$ and $Y_B$ consist of at least $\frac{n}{2}$ elements each before entering Step 1, the existence of $x_A$ and $y_B$ in each iteration of Step 1 follows.

For Step 2, the existence of $z \in V \setminus (A \cup B)$ such that $e_{D\cup\mathcal{F}}(z, B) = 0$ is guaranteed by the following: Consider a vertex $z$ in $Y_B$ before Step 1. Then $e_D(z, B) = 0$ by definition, and $e_D(z, V \setminus (A \cup B)) = 0$ by $(R4)$. Now, if $z$ is not added to $A$ or $B$ during Step 1, then $e_D(z, B) = 0$ holds still after the update of $B$. Since in Step 1, $\mathcal{F}$ contains only arcs between $A$ and $B$, it follows, under the assumption that $z \in V \setminus (A \cup B)$ after the update, that $e_{D\cup\mathcal{F}}(z, B) = 0$ before entering Step 2. Since $|Y_B| > \frac{n}{2}$ before entering Step 1, and since in Step 1 at most $2\sqrt{n}$ vertices are moved from $Y_B$ to $A \cup B$, if follows that at the beginning of Step 2, there are more than $\frac{n}{4}$ vertices $z$ in $V \setminus (A \cup B)$ such that $e_{D\cup\mathcal{F}}(z, B) = 0$, for large $n$. Note that in Step 2 at most $\max(|A|, |B|) - 1 \le \frac{n}{6} + \sqrt{n} < \frac{n}{4}$ of those vertices $z \in V \setminus (A \cup B)$ with $e_{D\cup\mathcal{F}}(z, B) = 0$ are used. The existence of $\ell$ is always guaranteed since $e_D(V \setminus (A \cup B), w_1) = 0$ by assumption, and since Step 2 is executed at most $\max(|A|, |B|) - 1$ times. Note that by choice of $z \in V \setminus (A \cup B)$, OBreaker can always direct $(z, w_{r+1})$ or $(z, w_\ell)$ as required.

Finally, we prove that $D' := D \cup \mathcal{F}$ is a riskless digraph of rank $r + 1$, where $\mathcal{F}$ is the set of all arcs that OBreaker directs in Step 1 and Step 2. In Step 1, the sets $A$ and $B$ are enlarged (alternately) by one in each iteration. Since for each new element $x_A$ (or $y_B$ respectively) all arcs $(x_A, y)$ for $y \in B$ (or $(x, y_B)$ for $x \in A$ respectively) are directed by OBreaker (unless they are in $D \cup \mathcal{F}$ already), the pair $(A, B)$ is a $UDB$ in $D \cup \mathcal{F}$. Since $A$ and $B$ are increased alternately (except for the first executions of Step 1 in case $|B| = |A| + 1$), it follows that $||A| - |B|| \le 1$. Since $A_S$ and $B_S$ are unchanged, Property $(R1)$ follows.

Since $A_S$ and $B_S$ are untouched, there is nothing to prove for $(R2.1)$. For $(R2.2)$, note that, after the last update of Step 2, for all $z \in A_0 \cup V \setminus (A \cup B)$, the set $\{i : (v_i, z) \in D \cup \mathcal{F}\}$ is the same as $\{i : (v_i, z) \in D\}$. Now, for all $z \in B_0 \cup V \setminus (A \cup B)$, the arc $(z, w_i)$ is directed by OBreaker for some $1 \le i < r + 1$ only if $(z, w_{i+1}) \in D \cup \mathcal{F}$. So $(R2.3)$ follows as well. For $(R3.1)$, note that in Step 1, all vertices $z$ that are added to $A_0$ fulfill $e_D(A, z) = 0$. Hence for all $1 \le i \le r + 1$, $e(v_i, A_0)$ does not increase when proceeding from $D$ to $D \cup \mathcal{F}$. Also, $e(v_i, V \setminus (A \cup B))$ does not increase, since all arcs of the form $(v_i, z)$, that are directed by OBreaker, fulfill $z \in B$ after the update. In Step 2, only edges of the form $(z, w_i)$ for $z \in V \setminus (A \cup B)$ are directed, hence $(R3.1)$ follows. For $(R3.2)$, similar to $(R3.1)$, the quantity $e(B_0, w_i)$ does not increase in Step 1, for all $1 \le i \le r + 1$, since all vertices $z \in V \setminus (A \cup B)$ added to $B_0$ fulfill $e_D(z, B) = 0$. In Step 2, no vertices are added to $B_0$, so the quantity $e(B_0, w_i)$ stays unchanged for all $1 \le i \le r + 1$. Now for $1 \le i \le r + 1$, the quantity $e(V \setminus (A \cup B), w_i)$ only increases in Step 2, and only if $e(V \setminus (A \cup B), w_i) < \max(|A|, |B|)$ by the strategy description. Therefore, $(R3.2)$ follows. Finally, Property $(R4)$ follows since OBreaker updates $A$ and $B$ accordingly in Step 1, and since in Step 2, he only directs arcs of the form $(x, w_i)$ for $x \in V \setminus (A \cup B)$ and $w_i \in B_S$. □

## 2.3 OBreaker's strategy – Stage II

**Proof of Lemma 2.1.7** By assumption, $D$ is riskless of rank $r = \left\lfloor \frac{3n}{125} \right\rfloor$. Let $A = A_S \cup A_0$ and $B = B_S \cup B_0$ be given according to Definition 2.1.1. We claim that $D$ is protected with $UDB$ $(A, B)$ with partitions $A = A_D \cup A_{AD} \cup A_S \cup A_0$ and $B = B_D \cup B_{AD} \cup B_S \cup B_0$, where

$A_D = A_{AD} = B_D = B_{AD} = \emptyset$, and $k_1 = \ell_1 = 0$, $k_2 = \ell_2 = r$.

For Property $(P1)$, let $a_0 := |A_0|$ and note that by Property $(R1)$, $|a_0 - |B_0|| \leq 1$. By assumption on $|D|$ and by Property $(R4)$,

$$r(b+1) = |D| = e_D(A,B) + e_D(A_S, V \setminus B) + e_D(V \setminus A, B_S). \tag{2.1}$$

Now, $e_D(A,B) = (r+a_0)(r+|B_0|) \leq (r+a_0+1)^2$, whereas

$$\begin{aligned} e_D(A_S, V \setminus B) &= e_D(A_S, A_S) + e_D(A_S, A_0) + e_D(A_S, V \setminus (A \cup B)) \\ &\leq \binom{r}{2} + \frac{r(r+1)}{2} + r(r+a_0+1) \\ &= r^2 + r(r+a_0+1) \end{aligned}$$

where the inequality follows from Property $(R1)$ and $(R3.1)$. Similarly, by Property $(R1)$ and $(R3.2)$, $e_D(V \setminus A, B_S) \leq r^2 + r(r+a_0+1)$. Thus, (2.1) yields

$$r(b+1) \leq (r+a_0+1)^2 + 4r^2 + 2r(a_0+1).$$

Standard calculations give

$$\begin{aligned} & a_0^2 + a_0(4r+2) + 3r + 1 + 5r^2 - rb \geq 0 \\ \Rightarrow\ & a_0 \geq -2r - 1 + \sqrt{-r^2 + r + rb} \\ \Rightarrow\ & a_0 \geq \frac{99n}{1000} + 2 \end{aligned}$$

where in the last step we use that $r = \left\lfloor \frac{3n}{125} \right\rfloor$, $b \geq \frac{37n}{40}$ and $n$ is large enough. By this we then obtain $|B_0| \geq a_0 - 1 \geq \frac{99n}{1000} + 1$.

There is nothing to prove for Property $(P2)$ and $(P3)$ since $A_D = A_{AD} = B_D = B_{AD} = \emptyset$. For Property $(P4)$, note that $A_{AD} = B_{AD} = \emptyset$ and the enumerations $A_S = \{v_1, \ldots, v_r\}$ and $B_S = \{w_1, \ldots, w_r\}$ given by Property $(R2)$ fulfill $(P4.1)$-$(P4.3)$, with $k_1 = \ell_1 = 0$ and $k_2 = \ell_2 = r$. Property $(P5.1)$ and $(P5.2)$ follow from $(R3.1)$ and $(R3.2)$ respectively. Finally, $(P6)$ follows from $(R4)$. □

**Proof of Lemma 2.1.8** Let $A = A_D \cup A_{AD} \cup A_S \cup A_0$ and $B = B_D \cup B_{AD} \cup B_S \cup B_0$ be given by Definition 2.1.5, and let $A_{AD} = \{v_1, \ldots, v_{k_1}\}$, $A_S = \{v_{k_1+1}, \ldots, v_{k_1+k_2}\}$ and $B_S = \{w_1, \ldots, w_{\ell_2}\}$, $B_{AD} = \{w_{\ell_2+1}, \ldots, w_{\ell_1+\ell_2}\}$ as given by Property $(P4)$. Moreover, let $e = (v,w) \in \mathcal{A}(D(V \setminus B))$ be given by the lemma. For notation reasons we divide into two cases.

**Case 1:** $v \in A_{AD} \cup A_S$**.** Then $v = v_\ell$ for some $1 \leq \ell \leq k_1 + k_2$, and $w \notin A_D$. Note that the only properties that may not be fulfilled anymore in $D + e$ are $(P4.2)$ and $(P5.1)$. Let

$\{f_1,\dots,f_t\} = (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3) \cap \mathcal{A}(D)$, where

$$\begin{aligned}\mathcal{F}_1 &:= \big\{(v_i,w) : 1 \le i \le \ell - 1\big\},\\ \mathcal{F}_2 &:= \big\{(v_1,z) : z \in A_0\big\},\\ \mathcal{F}_3 &:= \big\{(v_{k_1+1},z) : z \in V \setminus (A \cup B)\big\},\end{aligned}$$

where we use the convention that $\mathcal{F}_3 = \emptyset$ if $A_S = \emptyset$. To show that this choice of arcs is suitable for the lemma, we now follow the structure of the proof of Lemma 2.1.3. That is, we prove that $f_i \neq \overleftarrow{e}$ for all $1 \le i \le t$, that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \subseteq D \cup \mathcal{A}(D)$, that $t \le b$, and finally we deduce that $D' = D \cup \{e, f_1, \dots, f_t\}$ is protected.

For the first part, $\overleftarrow{e} = (w,v) \notin \mathcal{F}_1$ since $v \neq w$, and $\overleftarrow{e} = (w,v) \notin \mathcal{F}_2 \cup \mathcal{F}_3$ since $v \in A_{AD} \cup A_S$ by assumption. To see that $\mathcal{F}_1 \subseteq D \cup \mathcal{A}(D)$, assume that $(w,v_i) \in D$ for some $1 \le i \le \ell - 1$. Then $w = v_j \in A_{AD} \cup A_S$ for some $j < i < \ell$, by Property $(P6)$, $(P4.1)$ and since $w \notin A_D$. By Property $(P4.1)$ again, we conclude $(w,v) = (v_j, v_\ell) \in D$, a contradiction to $e \in \mathcal{A}(D(V \setminus B))$. Thus, $\mathcal{F}_1 \subseteq D \cup \mathcal{A}(D)$. Now, $\mathcal{F}_2 \cup \mathcal{F}_3 \subseteq D \cup \mathcal{A}(D)$ since every arc of the form $(z, v_i)$ in $D$ satisfies $z \in A \setminus A_0$, by Property $(P6)$. To see that $t \le b$ note that

$$t \le |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| \le \ell + |A_0| + |V \setminus (A \cup B)| \le |V \setminus B| \le b,$$

since by Property $(P1)$, $|B| \ge \frac{99n}{1000} + 1 \ge n - b$.

To check that $D'$ is protected, consider the partition $A = A'_D \cup A'_{AD} \cup A'_S \cup A_0$ where we set

$$\begin{aligned}A'_D &:= A_D \cup \{v_1\},\\ A'_{AD} &:= \{v_2,\dots,v_{k_1+1}\} \text{ (or } A'_{AD} := \{v_2,\dots,v_{k_1}\} \text{ if } k_2 = 0),\\ A'_S &:= \{v_{k_1+2},\dots,v_{k_1+k_2}\}.\end{aligned}$$

Clearly, $(A,B)$ is still a $UDB$ in $D'$ with $|A|,|B| \ge \frac{99n}{1000} + 1$, and Property $(P1)$ holds since $|A'_D \cup A_0| = |A_D \cup A_0| + 1$.

For Property $(P2)$, $B_D$ did not change; and $D'(A'_D)$ is a transitive tournament, since $D(A_D) \subseteq D'(A'_D)$ is such, and since $(z,v_1) \in D \subseteq D'$ for every $z \in A_D$ by the $UDB$-property for $A_D$. To see that $(A'_D, V \setminus A'_D)$ forms a $UDB$ we need to observe that $(v_1,z) \in D'$ for every $z \in V \setminus A'_D$. If $v_1 \in A_{AD}$, then this follows by Property $(P3)$ and $(P4.1)$ for $D$, and since $\mathcal{F}_2 \subseteq D'$. If $v_1 \in A_S$ (and thus $k_1 = |A_{AD}| = 0$), then this follows since $(A,B)$ is a $UDB$ in $D$, by Property $(P4.1)$ for $D$, and since $\mathcal{F}_2 \cup \mathcal{F}_3 \subseteq D'$.

To see that Property $(P3)$ holds in $D'$, observe first that $A'_{AD} \setminus \{v_{k_1+1}\} \subseteq A_{AD}$. Moreover, $(v_{k_1+1}, z) \in D'$ for every $z \in V \setminus A$ since $(A,B)$ is a $UDB$ in $D$ and since $\mathcal{F}_3 \subseteq D'$. For Property $(P4)$, it obviously holds that $|A'_S| \le k_2 \le \frac{3n}{125}$ and that $|B_S| \le \frac{3n}{125}$. Property $(P4.1)$ and $(P4.3)$ follow trivially, Property $(P4.2)$ follows from Property $(P4.2)$ for $D$ and since

$\mathcal{F}_1 \subseteq D'$.

For $(P5.1)$, observe that, since we made an index shift (from $A_S$ to $A'_S$), we have to prove that $e_{D'}(v_{(k_1+1)+i}, A_0) \leq \frac{3n}{125} + 1 - i$ for every $1 \leq i < k_2$. First, let $1 \leq i < k_2$ be such that $(k_1+1)+i \leq \ell$. Then only arcs from $D(v_{k_1+1+i}, A_0) \cup \mathcal{F}_1 \cup \{e\}$ contribute to $D'(v_{k_1+1+i}, A_0)$. Therefore, $e_{D'}(v_{k_1+1+i}, A_0) \leq e_D(v_{k_1+1+i}, A_0)+1 \leq (\frac{3n}{125}+1-(1+i))+1$. Now let $k_1+1+i > \ell$. Then $D'(v_{k_1+1+i}, A_0) = D(v_{k_1+1+i}, A_0)$, and therefore, $e_{D'}(v_{k_1+1+i}, A_0) \leq \frac{3n}{125} + 1 - i$.

There is nothing to prove for Property $(P5.2)$. Finally, Property $(P6)$ follows as we have that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \subseteq D'(A \setminus A_0, V \setminus B)$ and therefore,

$$\begin{aligned} D' &= D \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \\ &= D(A,B) \cup D(A \setminus A_0, V \setminus B) \cup D(V \setminus A, B \setminus B_0) \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \\ &= D'(A,B) \cup D'(A \setminus A_0, V \setminus B) \cup D'(V \setminus A, B \setminus B_0). \end{aligned}$$

**Case 2: $v \notin A_{AD} \cup A_S$.** By Property $(P2)$ and since $e = (v,w) \in \mathcal{A}(D(V \setminus B))$, we may assume that $v \in A_0 \cup V \setminus (A \cup B)$. Moreover, $w \notin A_D$. We now want to incorporate $v$ into the tournament $A_{AD} \cup A_S$. Set

$$\ell := \begin{cases} \min(i : \ (v_i, v) \notin D) & \text{if minimum exists} \\ k_1 + k_2 + 1 & \text{otherwise,} \end{cases}$$

and

$$v'_i := \begin{cases} v_i & 1 \leq i \leq \ell - 1 \\ v & i = \ell \\ v_{i-1} & \ell + 1 \leq i \leq k_1 + k_2 + 1. \end{cases}$$

Consider the following families of arcs.

$$\begin{aligned} \mathcal{F}_1 &:= \big\{(v'_i, w) : 1 \leq i \leq \ell - 1\big\}, \\ \mathcal{F}_2 &:= \big\{(v'_\ell, v'_i) : \ell + 1 \leq i \leq k_1 + k_2 + 1\big\}, \\ \mathcal{F}_3 &:= \big\{(v'_\ell, z) : z \in V \setminus (A \cup B) \cup A_0 \text{ and } (v'_{\ell+1}, z) \in D\big\}, \\ \mathcal{F}_4 &:= \begin{cases} \big\{(v'_1, z) : z \in A_0 \text{ and } (v'_{\ell+1}, z) \notin D\big\} & \text{if } v \in A_0 \\ \big\{(v'_\ell, y) : y \in B\big\} & \text{if } v \in V \setminus (A \cup B), \end{cases} \\ \mathcal{F}_5 &:= \big\{(v'_{k_1+1}, z) : z \in V \setminus (A \cup B),\ z \neq v \text{ and } (v'_{\ell+1}, z) \notin D\big\}, \end{aligned}$$

where we use the convention that if $\ell = k_1 + k_2 + 1$ (and thus $v'_{\ell+1}$ does not exist) then we take $\mathcal{F}_3 = \emptyset$, $\mathcal{F}_5 := \big\{(v'_{k_1+1}, z) : z \in V \setminus (A \cup B),\ z \neq v\big\}$, and $\mathcal{F}_4 := \big\{(v'_1, z) : z \in A_0\big\}$ when $v \in A_0$.

We choose $\{f_1,\ldots,f_t\}$ to be $\{f_1,\ldots,f_t\} = (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots \cup \mathcal{F}_5) \cap \mathcal{A}(D)$. We proceed as before and show that $f_i \neq \overleftarrow{e}$ for all $1 \leq i \leq t$, that $\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_5 \subseteq D \cup \mathcal{A}(D)$, that $t \leq b$, and finally we deduce that $D' = D \cup \{e, f_1, \ldots, f_t\}$ is protected.

For the first part, $\overleftarrow{e} = (w,v) \notin \mathcal{F}_1$ since $v \neq w$. Similarly, $\overleftarrow{e} = (w,v) \notin \mathcal{F}_2 \cup \mathcal{F}_3$ since $v'_\ell = v \neq w$. For the same reason, $\overleftarrow{e} \notin \mathcal{F}_4$ in the case when $v \in V \setminus (A \cup B)$. In the case when $v \in A_0$, assume that $(w,v) = (v'_1, z)$ for some $z \in A_0$. Then $(v'_1, v) = (w,v) \in \mathcal{A}(D)$, by assumption on $e$, and $v'_1 = v_1$. That is, $(v'_1, v) \notin D$, which implies $\ell = 1$ by definition of $\ell$. But then $w = v'_1 = v'_\ell = v$, by definition of $v'_i$, a contradiction. Also, $\overleftarrow{e} \notin \mathcal{F}_5$ by definition of $\mathcal{F}_5$. So, $f_i \neq \overleftarrow{e}$ for all $1 \leq i \leq t$.

To see that $\mathcal{F}_1 \subseteq D \cup \mathcal{A}(D)$, assume that $(w, v'_i) = (w, v_i) \in D$ for some $1 \leq i \leq \ell - 1$. Then $w = v_j \in A_{AD} \cup A_S$ for some $j < i < \ell$, by Property $(P6)$, $(P4.1)$ and since $w \notin A_D$. By definition of $\ell$ we conclude $(w,v) = (v_j, v) \in D$, a contradiction to $e \in \mathcal{A}(D(V \setminus B))$. Thus, $\mathcal{F}_1 \subseteq D \cup \mathcal{A}(D)$. To see that $\mathcal{F}_2 \subseteq D \cup \mathcal{A}(D)$, assume that $(v'_i, v'_\ell) = (v_{i-1}, v) \in D$ for some $i > \ell$. Then $(v_\ell, v) \in D$, by Property $(P4.2)$, in contradiction to the definition of $\ell$. Thus, $\mathcal{F}_2 \subseteq D \cup \mathcal{A}(D)$. Also, $\mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \subseteq D \cup \mathcal{A}(D)$ since every arc of the form $(w', v'_i)$ in $D$ satisfies $w' \in A \setminus A_0$, by Property $(P6)$ and since $v'_i \in A \cup V \setminus (A \cup B)$.

To bound the number $t$ by the bias $b$ note that in case $v \in A_0$,

$$\begin{aligned} t &\leq |\mathcal{F}_1 \cup \mathcal{F}_2| + |\mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5| \\ &\leq k_1 + k_2 + |A_0 \cup V \setminus (A \cup B)| \\ &= |A_{AD} \cup A_S| + |A_0 \cup V \setminus (A \cup B)| \leq |V \setminus B| < b, \end{aligned}$$

since by Property $(P1)$ we have $|B| \geq \frac{99n}{1000} + 1 > n - b$. When $v \in V \setminus (A \cup B)$, then we estimate

$$\begin{aligned} t &\leq |\mathcal{F}_1 \cup \mathcal{F}_2| + |\mathcal{F}_3 \cup \mathcal{F}_5| + |\mathcal{F}_4| \\ &\leq |A_{AD} \cup A_S| + (|V \setminus (A \cup B)| + e_D(v'_{\ell+1}, A_0)) + |B| \\ &= |V \setminus (A_0 \cup A_D)| + e_D(v'_{\ell+1}, A_0). \end{aligned}$$

Since $v \in V \setminus (A \cup B)$, we have that $\ell \geq k_1 + 1$, by Property $(P3)$ and definition of $\ell$. Since $v'_{\ell+1} = v_\ell$, it follows that

$$t \leq |V \setminus (A_0 \cup A_D)| + e_D(v_\ell, A_0) \leq \frac{901n}{1000} + \frac{3n}{125} \leq b,$$

by Property $(P1)$, $(P5.1)$ and choice of $b$.

Finally, we show that $D' = D \cup \{e, f_1, \ldots, f_t\}$ is a protected digraph. Set

$$A'_D := \begin{cases} A_D \cup \{v'_1\} & \text{if } v \in A_0, \\ A_D & \text{if } v \in V \setminus (A \cup B), \end{cases}$$

$$A'_{AD} := \begin{cases} \{v'_2, \ldots, v'_{k_1+1}\} & \text{if } v \in A_0, \\ \{v'_1, \ldots, v'_{k_1+1}\} & \text{if } v \in V \setminus (A \cup B), \end{cases}$$

$$A'_S := \{v'_{k_1+2}, \ldots, v'_{k_1+k_2+1}\},$$

$$A'_0 := A_0 \setminus \{v\},$$

$$A' := A \cup \{v\}.$$

Moreover, let $B' = B$ with the same partition as for $B$. Then $(A', B')$ is a $UDB$ in $D'$, since $(A, B)$ is a $UDB$ in $D \subseteq D'$ and since in case $v = v'_\ell \in V \setminus (A \cup B)$ we have $\mathcal{F}_4 \subseteq D'$. For Property $(P1)$, $|B_D \cup B_0| \geq \frac{99n}{1000}$ and $|A'|, |B'| \geq \frac{99n}{1000} + 1$ obviously hold. Now, observe that $|A'_D| = |A_D| + 1$ and $|A'_0| = |A_0| - 1$ in case $v \in A_0$, while $A_D = A'_D$ and $A_0 = A'_0$ in case $v \in V \setminus (A \cup B)$. Thus $|A'_D \cup A'_0| = |A_D \cup A_0| \geq \frac{99n}{1000}$.

For Property $(P2)$, there is nothing to prove when $v \in V \setminus (A \cup B)$, since then $A'_D = A_D$. So, let $v \in A_0$. Again $B_D$ does not change. $D'(A'_D)$ is a transitive tournament, since $D(A_D) \subseteq D'(A'_D)$ is such, and since $(z, v'_1) \in D \subseteq D'$ for every $z \in A_D$ by the $UDB$-property for $A_D$. To see that $(A'_D, V \setminus A'_D)$ forms a $UDB$ we need to observe that $(v'_1, z) \in D'$ for every $z \in V \setminus A'_D$. By definition of $v'_i$ we have $v'_1 = v_1$ or $v_1 = v$, and so we distinguish two cases.

Assume first that $v'_1 = v_1 \in A_{AD} \cup A_S$. Then $(v'_1, z) \in D \subseteq D'$ for every $z \in B$ since $(A, B)$ is a $UDB$ in $D$; and for every $z \in (A_{AD} \cup A_S) \setminus \{v'_1\}$ by Property $(P4.1)$. For every $z \in A_0$, if $(v'_{\ell+1}, z) \notin D$ (or $\ell = k_1 + k_2 + 1$ where $v'_{\ell+1}$ does not exist), then $(v'_1, z) \in \mathcal{F}_4 \subseteq D'$; and if $(v'_{\ell+1}, z) \in D$ then $(v'_1, z) \in D \subseteq D'$ by Property $(P4.2)$. For $z \in V \setminus (A \cup B)$, if $v'_1 \in A_{AD}$ then $(v'_1, z) \in D \subseteq D'$ by Property $(P3)$ for $D$; if $v'_1 \in A_S$ (and thus $k_1 = |A_{AD}| = 0$) and if $(v'_{\ell+1}, z) \notin D$ (or $\ell = k_1 + k_2 + 1$ where $v'_{\ell+1}$ does not exist), then $(v'_1, z) \in \mathcal{F}_5 \subseteq D'$; and if $v'_1 \in A_S$ and $(v'_{\ell+1}, z) \in D$, then $(v'_1, z) \in D \subseteq D'$ by Property $(P4.2)$.

Now, assume that $v'_1 = v \in A_0$ and thus $\ell = 1$. Then again $(v'_1, z) \in D \subseteq D'$ for every $z \in B$ since $(A, B)$ is a $UDB$ in $D$. If $v'_2 = v'_{\ell+1} \in A_{AD}$, then $(v'_2, z) \in D$ for every $z \in V \setminus (A \cup B)$ by Property $(P3)$. Therefore, for every $z \in (A_0 \cup V \setminus (A \cup B)) \setminus \{v\}$ we have that $(v'_1, z) \in \mathcal{F}_3 \cup \mathcal{F}_4 \subseteq D'$. For every $z \in A_{AD} \cup A_S$, we have that $(v'_1, z) \in \mathcal{F}_2 \subseteq D'$. If $v'_2 \in A_S$ (or $v'_2$ does not exist) and therefore $k_1 = 0$, then similarly $(v'_1, z) \in \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup D \subseteq D'$ for all $z \in V \setminus A'_D$.

For Property $(P3)$, observe that the statement for $B_{AD}$ does not change. To see that the pair $(A'_{AD}, V \setminus A')$ forms a $UDB$ we need to observe that $(v'_{k_1+1}, z) \in D'$ for every $z \in V \setminus A'$.

If $z \in B'$, then this is clear, since $(A', B')$ forms a $UDB$ as we showed already above. Let now $z \in V \setminus (A' \cup B') = V \setminus (A \cup B \cup \{v\})$. If $\ell \leq k_1$, then $v'_{k_1+1} = v_{k_1} \in A_{AD}$. Therefore, $(v'_{k_1+1}, z) \in D \subseteq D'$ by Property $(P3)$ for $D$. If $\ell \geq k_1 + 1$, then $(v'_{k_1+1}, z) \in \mathcal{F}_3 \cup \mathcal{F}_5 \cup D \subseteq D'$ (where we use Property $(P4.2)$ which says that if $(v'_{\ell+1}, z) = (v_\ell, z) \in D$ then $(v_{k_1+1}, z) \in D$).

For Property $(P4)$, note that $|A'_S| = |A_S| = k_2 \leq \frac{3n}{125}$. For $(P4.1)$ observe that the statement for $\{w_1, \ldots, w_{\ell_1+\ell_2}\}$ does not change. To see that $(v'_2, \ldots, v'_{k_1+k_2+1})$ or $(v'_1, \ldots, v'_{k_1+k_2+1})$ is fully transitive in $D'$, note first that the vertex set without $v'_\ell$ is fully transitive in $D \subseteq D'$. We have $(v'_i, v'_\ell) = (v_i, v) \in D \subseteq D'$ for every $i \leq \ell - 1$, by definition of $\ell$. Moreover, $(v'_\ell, v'_i) \in D'$ for every $i \geq \ell + 1$, since $\mathcal{F}_2 \subseteq D'$.

For $(P4.2)$, let $z \in A'_0 \cup V \setminus (A' \cup B')$. We show that $\{i : (v'_i, z) \in D\}$ is a down set of $[k_1 + k_2 + 1]$. Note that this then implies $(P4.2)$ for $D'$, even when $v \in A_0$ where we have that $A'_{AD} \cup A'_S = \{v'_2, \ldots, v'_{k_1+k_2+1}\}$. Since $z \in A'_0 \cup V \setminus (A' \cup B')$, only arcs from $D \cup \{e\} \cup \mathcal{F}_1 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5$ contribute to the set $\{i : (v'_i, z) \in D'\}$. Note that $\{i : (v_i, z) \in D\}$ is a down set of $[k_1 + k_2]$ and the relative order of the $v'_i$ for $i \neq \ell$ does not change. So, if $z \neq w$, then the arcs from $\mathcal{F}_3$ reestablish the down-set property for $D'$. If $z = w$, then the arcs from $\mathcal{F}_1 \cup \{e\}$ reestablish the down-set property for $D'$. Now, $\mathcal{F}_4$ contributes at most the element $\{1\}$ to $\{i : (v'_i, z) \in D'\}$ which is of no harm. The family $\mathcal{F}_5$ may contribute the element $\{k_1 + 1\}$ to $\{i : (v'_i, z) \in D'\}$ for some $z \in V \setminus (A \cup B)$. So, in case this happens, we need to show that $[k_1] \subseteq \{i : (v'_i, z) \in D'\}$. If $k_1 < \ell$, we then know that $v'_i = v_i \in A_{AD}$ for every $i \leq k_1$ and thus $(v'_i, z) \in D \subseteq D'$, by Property $(P3)$ for $D$. Otherwise, we have $k_1 \geq \ell$. Then, for every $i \in [k_1] \setminus \{\ell\}$, we have $v'_i \in \{v_i, v_{i-1}\} \subseteq A_{AD}$ and analogously $(v'_i, z) \in D \subseteq D'$. Moreover, as $v'_{\ell+1} = v_\ell \in A_{AD}$ and thus $(v'_{\ell+1}, z) \in D$ by $(P3)$, we obtain $(v'_\ell, z) \in \mathcal{F}_3 \subseteq D'$.

There is nothing to prove for Property $(P4.3)$, since $B$ and $\{w_1, \ldots, w_{\ell_1+\ell_2}\}$ are unchanged.

For $(P5.1)$, observe that, since we make an index shift (from $A_S$ to $A'_S$), we have to prove that $e_{D'}(v'_{(k_1+1)+i}, A'_0) \leq \frac{3n}{125} + 1 - i$ for every $1 \leq i \leq k_2$. First, let $1 \leq i \leq k_2$ be such that $(k_1 + 1) + i < \ell$. Then only arcs from $D(v_{k_1+1+i}, A_0) \cup \mathcal{F}_1$ contribute to $D'(v'_{k_1+1+i}, A'_0)$. Therefore, $e_{D'}(v'_{k_1+1+i}, A'_0) \leq e_D(v_{k_1+1+i}, A_0) + 1 \leq (\frac{3n}{125} + 1 - (1 + i)) + 1$.
Now, let $(k_1 + 1) + i = \ell$. Then $e_D(v'_\ell, A_0) = e_D(v, A_0) = 0$ since $v \in A_0 \cup V \setminus (A \cup B)$ and by Property $(P6)$ for $D$. So, only $\mathcal{F}_3 \cup \{e\}$ contributes to $D'(v'_\ell, A'_0)$. Therefore, we obtain $e_{D'}(v'_\ell, A'_0) \leq e_D(v'_{\ell+1}, A_0) + 1 = e_D(v_\ell, A_0) + 1 = e_D(v_{k_1+1+i}, A_0) + 1 \leq \frac{3n}{125} + 1 - i$. Finally, let $(k_1 + 1) + i > \ell$. Then $v'_{k_1+1+i} = v_{k_1+i}$ and only arcs from $D(v_{k_1+i}, A_0)$ contribute to $D'(v'_{k_1+1+i}, A'_0)$. This again proves $e_{D'}(v'_{k_1+1+i}, A'_0) \leq \frac{3n}{125} + 1 - i$.

There is nothing to prove for Property $(P5.2)$.

For $(P6)$ note that it is enough to prove that

$$D' \subseteq D'(A', B') \cup D'(A' \setminus A'_0, V) \cup D'(V, B' \setminus B'_0).$$

This indeed holds, since

$$\begin{aligned} D(A, B) &\subseteq D'(A', B'), \\ D(A \setminus A_0, V \setminus B) \cup \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_5 &\subseteq D'(A' \setminus A'_0, V), \\ D(V \setminus A, B \setminus B_0) &\subseteq D'(V, B' \setminus B'_0). \end{aligned}$$

This finishes the proof of Lemma 2.1.8. □

**Proof of Lemma 2.1.9** Let $D$ be given according to the assumption of the lemma, and let $(A, B)$ be the underlying $UDB$, with the partitions $A = A_D \cup A_{AD} \cup A_S \cup A_0$ and $B = B_D \cup B_{AD} \cup B_S \cup B_0$. Consider first the case that $A = A_D$ and $B = B_D$. Then $A$ and $B$ induce transitive tournaments in $D$, such that $(A, V \setminus A)$ and $(V \setminus B, B)$ are $UDB$'s in $D$. But then, since $|D| < \binom{n}{2}$, there needs to exist an available arc $(a, x) \in \mathcal{A}(D)$, such that $a, x \in V \setminus (A \cup B)$. We then can choose $f = (a, x)$, and update $A_S := \{a\}$ and $A := A \cup \{a\}$. It is easily checked that after this update $D + f$ satisfies the properties $(P1)$-$(P6)$ with $UDB$ $(A, B)$.

So, let us assume that $A \neq A_D$ or $B \neq B_D$. Then the idea of the proof is as follows. We show that we can find an available edge $f$ such that $D + f$ is protected, or that we can move one vertex from $A$ to the "left" (from $A_{AD}$ to $A_D$, or from $A_S$ to $A_{AD}$, or from $A_0$ to $A_S$) or that we can similarly move one vertex from $B$ (from $B_{AD}$ to $B_D$, or from $B_S$ to $B_{AD}$, or from $B_0$ to $B_S$) such that, after an update of the partitions of $A$ and $B$, the digraph $D$ remains protected with underlying $UDB$ $(A, B)$.

Thus, applying this argument iteratively, we eventually find an edge $f$ as required, in which case we are done; or this process ends, when all vertices of $A$ are moved to $A_D$, and all vertices from $B$ are moved to $B_D$, in which case we are done, as discussed above.

By Observation 2.1.6 we may assume that $A \neq A_D$. Let $v_1$ be the first vertex of the tournament $(v_1, \ldots, v_{k_1+k_2})$ in $A$. If $v_1 \in A_{AD}$, then proceed as follows. In case there is an available arc $(v_1, x) \in \mathcal{A}(D)$ with $x \in A_0$, then choose $f = (v_1, x)$ and observe that all properties of a protected digraph are maintained for $D + f$. Otherwise, we have $(v_1, x) \in D$ for all $x \in A_0$, in which case we can move $v_1$ from $A_{AD}$ to $A_D$ without destroying the properties $(P1)$-$(P6)$. If $v_1 \in A_S$, then proceed similarly. If there is a vertex $x \in V \setminus (A \cup B)$ such that $(v_1, x) \in \mathcal{A}(D)$, then choose $f = (v_1, x)$. Otherwise, move $v_1$ from $A_S$ to $A_{AD}$.
Finally, if $v_1$ does not exist, then we have $A_{AD} \cup A_S = \emptyset$, and by assumption, $A_0 \neq \emptyset$. If $V \setminus (A \cup B) \neq \emptyset$, then choose $f = (a, x)$ with $a \in A_0$ and $x \in V \setminus (A \cup B)$, and observe

that $D+f$ remains protected with underlying $UDB$ $(A,B)$ after moving $a$ from $A_0$ to $A_S$. The properties $(P2)$-$(P6)$ are easily checked; for $(P1)$, note that, by assumption, we have $|A_D \cup A_0| + 1 = |A| \geq \frac{99n}{1000} + 1$, immediately after $a$ is moved to the set $A_S$. If $V \setminus (A \cup B)$ is empty, but $|A_0| \geq 2$, then choose $f = (a,x)$ with $a, x \in A_0$ and proceed analogously to the previous case. Similarly, if $V \setminus (A \cup B)$ is empty, but $|A_0| = 1$, then move the unique vertex $a \in A_0$ to $A_S$. □

## 2.4 Concluding remarks and open problems

We finish this chapter with some remarks and open problems.

**Threshold biases.** It would be interesting to know the sizes of threshold biases for the oriented-cycle games $\mathcal{O}(K_n, \mathcal{C}, 1, b)$ and $\mathcal{O}_s(K_n, \mathcal{C}, 1, b)$ exactly. However, as a first step it seems to be challenging to find out whether there is an $\varepsilon > 0$ such that, for every large enough $n$, OMaker has a winning strategy for $\mathcal{O}(K_n, \mathcal{C}, 1, (1+\varepsilon)\frac{n}{2})$ and $\mathcal{O}_s(K_n, \mathcal{C}, 1, (1+\varepsilon)\frac{n}{2})$. If this were true, then we would know that the threshold bias for the oriented-cycle game is not asymptotically equal to the threshold bias of the corresponding Maker-Breaker game on $K_n$, where Maker aims to occupy a cycle in $K_n$. The latter was studied by Bednarska and Pikhurko [5], where the threshold bias was proven to be roughly $\frac{n}{2}$.

**Playing on random graphs.** Instead of playing on the complete graph $K_n$, one could also study the oriented-cycle game when played on a sparse random graph $G \sim \mathcal{G}_{n,p}$. That is, we consider the games $\mathcal{O}(G, \mathcal{C}, 1, b)$ and $\mathcal{O}_s(G, \mathcal{C}, 1, b)$. The following was asked by Łuczak during the Berlin-Poznań Seminar in 2013.

**Problem 2.4.1** *Let $p = p(n) \in [0,1]$. What is the largest bias $b = b(n,p)$ such that for $G \sim \mathcal{G}_{n,p}$, a.a.s. OMaker wins the game $\mathcal{O}(G, \mathcal{C}, 1, b)$ and $\mathcal{O}_s(G, \mathcal{C}, 1, b)$, respectively?*

**Creating a Hamilton cycle fast.** For the usual unbiased Maker-Breaker Hamilton cycle game on $K_n$ it is known that Maker has a strategy which gives a Hamilton cycle within $n+1$ rounds [31]. We wonder how fast OMaker can ensure a directed Hamilton cycle in the unbiased orientation game on $K_n$.

# Chapter 3

# Tournament games

In this chapter, we study the $T$-tournament game $(E(G), \mathcal{K}_T)$. Recall that this games is played by two players, TMaker and TBreaker, who alternately direct previously undirected edges of the given graph $G$. TMaker starts the game, and she aims to create a copy of $T$ only with her edges. If she succeeds, she wins the game; otherwise TBreaker does. In Section 3.2 we first prove Proposition 1.2.1, i.e. we show that the threshold bias for the biased $T$-tournament game on $K_n$ is $b_{\mathcal{K}_T} = \Theta(n^{\frac{2}{k+1}})$, if $T$ has $k \geq 3$ vertices. Moreover, we prove Proposition 1.2.2, which says that the threshold probability for winning the unbiased $T$-tournament game on a random graph $G \sim \mathcal{G}_{n,p}$ is $p_{\mathcal{K}_T} = n^{-\frac{2}{k+1}}$, if $T$ has $k \geq 4$ vertices. Afterwards, in Section 3.3, we prove Theorem 1.2.3. That is, for the cyclic triangle $T_C$ we show that $p_{\mathcal{K}_{T_C}} = n^{-\frac{8}{15}}$, while for the acyclic triangle $T_A$ we have $p_{\mathcal{K}_{T_A}} = n^{-\frac{5}{9}}$.

**Notation and terminology.** For a graph $G$, with $v(G) \geq 1$, we set $d(G) = \frac{e(G)}{v(G)}$ as the *density* of $G$, while its *maximum density* is $m(G) = \max_{H \subseteq G;\ v(H) \geq 1} d(H)$. Similarly, the *2-density* of a graph $G$, with at least 3 vertices, is defined as $d_2(G) = \frac{e(G)-1}{v(G)-2}$ and its *maximum 2-density* is defined as $m_2(G) = \max_{H \subseteq G;\ v(H) \geq 3} d_2(H)$.

Let $n, k \in \mathbb{N}$ be positive integers. Then with $T_{n,k}$ we denote the *Turán graph* [43] with $n$ vertices and $k$ vertex classes. That is, its vertex set $V(T_{n,k}) = [n]$ comes with a partition $V(T_{n,k}) = V_1 \cup \ldots \cup V_k$ such that $\Big| |V_i| - |V_j| \Big| \leq 1$ for all $1 \leq i < j \leq k$, and such that its edge set is $E(T_{n,k}) = \{vw \mid v \in V_i,\ w \in V_j,\ 1 \leq i < j \leq k\}$. Moreover, let $G$ be a graph on at most $k$ vertices, then we say that a subgraph $H \subseteq T_{n,k}$ is a *good copy* of $G$ in $T_{n,k}$, if $G \cong H$ and $|V(H) \cap V_i| \leq 1$ for every $i \in [k]$. Let $p \in [0, 1]$ and moreover let $M \in [e(T_{n,r})]$. Then with $\mathcal{G}(T_{n,k}, p)$ we denote the random graph model obtained from $T_{n,k}$ by deleting each edge of $T_{n,k}$ independently with probability $1 - p$. That is, $\mathcal{G}(T_{n,k}, p)$ is the probability space of all subgraphs $G$ of $T_{n,k}$, where the probability for a subgraph to be chosen is $p^{e(G)}(1-p)^{e(T_{n,k})-e(G)}$. Similarly, with $\mathcal{G}(T_{n,k}, M)$ we denote the probability space of all subgraphs $G$ of $T_{n,k}$ with $M$ edges, together with the uniform distribution.

Finally, $W_k = (V, E)$ is called a *k-wheel*, if it is obtained from the cycle $C_k$ by adding one further vertex $z$ which is made adjacent to every vertex of $C_k$. The special vertex $z$ is called the *center* of $C_k$.

## 3.1 Preliminaries

Let $Bin(n, p)$ denote the binomial distribution, i.e. the distribution of the number of successes among $n$ independent experiments, where in each experiment we have success with probability $p$. Moreover, let us write $X \sim Bin(n, p)$ if $X$ is a random variable with distribution $Bin(n, p)$. The following estimate is usually referred to as a Chernoff inequality [32].

**Lemma 3.1.1 (Theorem 2.1 in [32])** *Let $X \sim Bin(n, p)$ and $\lambda = \mathbb{E}(X) = np$. Then for $t \geq 0$, it holds that $Pr(X \geq \mathbb{E}(X) + t) \leq \exp\left(-\frac{t^2}{2\lambda} + \frac{t^3}{6\lambda^2}\right)$.*

As indicated above, we will consider the random graph models $\mathcal{G}(T_{n,k}, p)$ and $\mathcal{G}(T_{n,k}, M)$. For this, we will make use of some general results about random sets.

Following [32], let $\Gamma$ be a set of size $N \in \mathbb{N}$. For $p \in [0, 1]$, we let $\Gamma_p$ denote the probability space of all subsets $A \subseteq \Gamma$, where the probability of choosing $A$ is $p^{|A|}(1-p)^{|\Gamma \setminus A|}$. So, $\mathcal{G}_{n,p}$ is a special case of this model, with $\Gamma = E(K_n)$. Moreover, for $M \in [N]$, we let $\Gamma_M$ denote the probability space of all subsets $A \subseteq \Gamma$ of size $M$, together with the uniform distribution. So, every set of size $M$ is chosen with probability $\binom{N}{M}^{-1}$. The random graph model $\mathcal{G}(n, M)$ is a special case of this model, again with $\Gamma = E(K_n)$. In case we choose a random set $A$ according to the model $\Gamma_p$, we shortly write $A \sim \Gamma_p$. Similarly, we write $A \sim \Gamma_M$, when $A$ is chosen according to the uniform model $\Gamma_M$.

One important fact about the two models above is that in many cases they are closely related to each other when $p \sim \frac{M}{N}$; see Section 1.4 in [32]. In particular, we will make use of the following two statements, which help us to transfer results from one model to the other.

**Lemma 3.1.2 (Pittel's Inequality, Equation (1.6) in [32])** *Let $\Gamma$ be a set of size $N$, let $M \in [N]$, and $p = \frac{M}{N} \in [0, 1]$. Let $\mathcal{P}$ be a family of subsets of $\Gamma$. Moreover, let $H_p \sim \Gamma_p$ and $H_M \sim \Gamma_M$, then*

$$Pr(H_M \notin \mathcal{P}) \leq 3\sqrt{M} \cdot Pr(H_p \notin \mathcal{P}).$$

**Lemma 3.1.3 (Corollary 1.16 (iii) in [32])** *Let $\Gamma$ be a set of size $N$ and let $M \in [N]$. Let $\delta > 0$ be such that $0 \leq (1+\delta)\frac{M}{N} \leq 1$, and let $p = (1+\delta)\frac{M}{N}$. Let $\mathcal{P}$ be a family of subsets of $\Gamma$. Moreover, let $H_p \sim \Gamma_p$ and $H_M \sim \Gamma_M$, then*

$$Pr(H_M \in \mathcal{P}) \to 1 \text{ implies } Pr(H_p \in \mathcal{P}) \to 1.$$

Later we want to know whether a certain random graph contains a copy of a fixed graph with high probability. In this regard, we make use of the following two theorems.

**Theorem 3.1.4 (Theorem 2.18 (ii) in [32])** *Let $\Gamma$ be a set, $p \in [0,1]$ and let $H \sim \Gamma_p$. Let $\mathcal{S}$ be a family of subsets of $\Gamma$. Moreover, for every $A \in \mathcal{S}$ let $I_A$ be the indicator variable which is 1 if $A \subseteq H$, and 0 otherwise. Finally, let $X = \sum_{A \in \mathcal{S}} I_A$ be the random variable counting the number of elements of $\mathcal{S}$ that are contained in $H$. Then*

$$Pr(X = 0) \leq \exp\Big(-\frac{\mathbb{E}(X)^2}{\sum_{A \in \mathcal{S}} \sum_{\substack{B \in \mathcal{S} \\ A \cap B \neq \emptyset}} \mathbb{E}(I_A I_B)}\Big).$$

**Theorem 3.1.5 (Theorem 3.4 in [32])** *Let $H$ be a graph, and let $X_H$ denote random variable counting the number of copies of $H$ in a random graph $G \sim \mathcal{G}_{n,p}$. Then, as $n$ tends to infinity, we have*

$$Pr(X_H > 0) \to \begin{cases} 0 & \text{if } p \ll n^{-\frac{1}{m(H)}} \\ 1 & \text{if } p \gg n^{-\frac{1}{m(H)}}. \end{cases}$$

## 3.2 Most tournaments behave like cliques

The main idea for the proof of the propositions is as follows: Let $G$ be the graph on which the game is to be played. Let $T$ be the goal tournament with vertices $v_1, \ldots, v_k$. Then, before the game starts TMaker splits the vertex set of $G$ into $k$ parts $V_1, \ldots, V_k$ with $\Big||V_i| - |V_j|\Big| \leq 1$ for all $1 \leq i < j \leq k$, and she identifies each class $V_i$ with the vertex $v_i$ according to the following rule: Whenever TMaker claims an edge between some classes $V_i$ and $V_j$, she always chooses the direction of this edge according to the direction of the edge $v_i v_j$ in $T$. Because of this identification, it then remains to show that Maker has a strategy for the usual Maker-Breaker game on $G$ to occupy a copy of $K_k$ with exactly one vertex in each $V_i$.

In order to show that Maker has such a strategy for this game, we will make use of results from [32], and follow the proof ideas from [4, 42]. As most parts are proven analogously to results in the aforementioned publications, we rather keep our argument short and, whenever possible, we refer back to the known results. At first, analogously to Theorem 3.9 in [32], we bound the probability that a random graph $G \sim \mathcal{G}(T_{n,k}, p)$ does not contain a good copy of $K_k$.

**Claim 3.2.1** *Let $k \geq 3$ be a positive integer. Then there is a constant $c_1 = c_1(k) > 0$ such that for every large enough $n$ the following is true: If $n^{-\frac{2}{k+1}} \leq p \leq 4n^{-\frac{2}{k+1}}$ and if $X$ denotes the random variable counting the number of good copies of $K_k$ in a random graph $G \sim \mathcal{G}(T_{n,k}, p)$, then $Pr(X = 0) \leq \exp(-c_1 n^2 p)$.*

**Proof** Let $G \sim \mathcal{G}(T_{n,k}, p)$. Let $\mathcal{S}$ be the family of good copies of $K_k$ in $T_{n,k}$. For each such copy $C_i \in \mathcal{S}$ let $I_{C_i}$ be the indicator variable which is 1 if and only if $C_i \subseteq G$. By Theorem 3.1.4,

$$Pr(X = 0) \leq \exp\Big(-\frac{(\mathbb{E}(X))^2}{\sum_{C_1}\sum_{C_2:\ E(C_1)\cap E(C_2)\neq\emptyset} \mathbb{E}(I_{C_1} I_{C_2})}\Big).$$

The denominator in the above expression can be bounded from above by

$$\begin{aligned}\sum_{t=2}^{k}\sum_{C_1\in\mathcal{S}}\sum_{\substack{C_2\in\mathcal{S}:\\ C_1\cap C_2\cong K_t}} p^{2\binom{k}{2}-\binom{t}{2}} &\leq \sum_{t=2}^{k} n^{2k-t} p^{2\binom{k}{2}-\binom{t}{2}} \\ &= \Theta(\mathbb{E}(X)^2)\cdot\sum_{t=2}^{k} n^{-t}p^{-\binom{t}{2}} \\ &= \Theta(\mathbb{E}(X)^2\cdot n^{-2}p^{-1})\sum_{t=2}^{k}\left(n^{-1}p^{-\frac{t+1}{2}}\right)^{t-2} \\ &= \Theta(\mathbb{E}(X)^2\cdot n^{-2}p^{-1}),\end{aligned}$$

where in the last equality we use that $p = \Theta(n^{-\frac{2}{k+1}})$. Thus, the claim follows. □

**Corollary 3.2.2** *Let $k \geq 3$ be a positive integer. Then there is a constant $c_1' = c_1'(k) > 0$ such that for every large enough $n$ the following is true: If $M = \lfloor n^{2-\frac{2}{k+1}} \rfloor$ and if $X'$ denotes the random variable counting the number of good copies of $K_k$ in a random graph $G \sim \mathcal{G}(T_{n,k}, M)$, then $Pr(X' = 0) \leq \exp(-c_1' M)$.*

**Proof** Set $p = \frac{M}{e(T_{n,k})}$ and observe that $n^{-\frac{2}{k+1}} \leq p \leq 4n^{-\frac{2}{k+1}}$. The statement now follows by Claim 3.2.1 and Lemma 3.1.2. □

**Corollary 3.2.3** *Let $k \geq 3$ be a positive integer. Then there is a constant $\delta = \delta(k) > 0$ such that for every large enough $n$ and $M = 2\lfloor n^{2-\frac{2}{k+1}} \rfloor$, a random graph $G \sim \mathcal{G}(T_{n,k}, M)$ satisfies the following property a.a.s.: Every subgraph of $G$ with at least $\lfloor (1-\delta)M \rfloor$ edges contains a good copy of $K_k$.*

**Proof** We proceed analogously to [4]. Let $\delta > 0$ such that $\delta - \delta\log(\delta) < c_1'/3$, with $c_1'$ from Corollary 3.2.2, and count the number of pairs $(H, H')$ where $H$ is a subgraph of $T_{n,k}$ with $M$ edges and where $H' \subseteq H$ is a subgraph with $\lfloor(1-\delta)M\rfloor$ edges that does not contain a good copy of $K_k$. Then using Corollary 3.2.2 (and simplifying the notation slightly by ignoring floor signs) we obtain that the number of such pairs is at most

$$\begin{aligned}\exp(-\frac{c_1'M}{2})\binom{e(T_{n,r})}{(1-\delta)M}\binom{e(T_{n,r})-(1-\delta)M}{\delta M} &\leq \exp(-\frac{c_1'M}{2})\binom{M}{\delta M}\binom{e(T_{n,r})}{M}\\ &\leq \exp\Big(-\frac{c_1'M}{2}+\delta M(1-\log(\delta))\Big)\binom{e(T_{n,r})}{M}\\ &= o(1)\binom{e(T_{n,r})}{M}.\end{aligned}$$ □

Using this last corollary, we can start proving the existence of Maker strategies. The following claim is an analogue statement to Theorem 19 in [42], and thus its proof is analogous to [42].

**Claim 3.2.4** *Let $k \geq 3$ and $n$ be positive integers. Then there is a constant $c_2 = c_2(k) > 0$ such that for every $M \geq c_2^{-1}n^{2-\frac{2}{k+1}}$, every $1 \leq b \leq c_2Mn^{-2+\frac{2}{k+1}}$, for a random graph $G \sim \mathcal{G}(T_{n,k}, M)$ the following a.a.s. holds: Maker has a strategy to occupy a good copy of $K_k$ in the $b$-biased Maker-Breaker game on $G$.*

**Proof** Choose $\delta = \delta(G)$ according to Corollary 3.2.3 and let $c_2 = \delta/10$. Maker's strategy is as follows: in each of her moves she chooses an edge from $G$ uniformly at random among all edges from $G$ that have not been claimed so far by herself. If she chooses an edge that is not claimed by Breaker so far, she claims this edge. Otherwise, Maker declares her move as a failure and skips it. Similar to [42], we consider the first $M' := 2\lfloor n^{2-\frac{2}{k+1}}\rfloor \leq \frac{\delta}{2}\cdot\frac{1}{b+1}M$ rounds of the game. As only a $\frac{\delta}{2}$-fraction of all edges are claimed in these rounds, the probability for a failure is at most $\frac{\delta}{2}$ in each round. So, the number of failures can be "upper bounded" by a binomial random variable $X \sim Bin(M', \frac{\delta}{2})$, which by Chernoff's inequality (Theorem 3.1.1) satisfies $Pr(X \geq 2\mathbb{E}(X)) \leq \exp(-\frac{\mathbb{E}(X)}{3}) = o(1)$. That is, the number of failures will be at most $\delta M'$ a.a.s. Thus, Maker a.a.s. creates a graph $H \setminus R$ with $H \sim \mathcal{G}(T_{n,k}, M')$ and $e(R) \leq \delta M'$, against any strategy of Breaker, which by Corollary 3.2.3 a.a.s. contains a good copy of $K_k$. Thus, a.a.s. Breaker cannot have a strategy to prevent good copies of $K_k$, and as either Maker or Breaker needs to have a winning strategy, the claim follows. □

**Corollary 3.2.5** *Let $k \geq 3$ and $n$ be positive integers Then there is a constant $c_3 = c_3(k) > 0$ such that for every $p \geq c_3n^{-\frac{2}{k+1}}$ and $G \sim \mathcal{G}(T_{n,k}, p)$ the following a.a.s. holds: Maker has a strategy to occupy a good copy of $K_k$ in the unbiased Maker-Breaker game on $G$.*

**Proof** The statement follows immediately from Corollary 3.2.4 and Lemma 3.1.3, where we choose $\mathcal{P}$ to be the family of all graphs $G \subseteq T_{n,k}$ for which Maker has a strategy to occupy a good copy of $K_k$ in the unbiased Maker-Breaker game on $E(G)$. □

Finally, we can prove the two propositions.

**Proof of Proposition 1.2.1.** Let $T$ be the tournament, with $k \geq 3$ vertices, of which TMaker aims to create a copy on $K_n$. By Theorem 1 in [4], we know that there is a constant $c > 0$ such that for large enough $n$ and for every $b \geq cn^{\frac{2}{k+1}}$, Breaker has a strategy to prevent cliques of order $k$. Using this strategy, TBreaker wins the $T$-tournament game on $K_n$. Now, let $c_2 = c_2(k)$ be given according to Claim 3.2.4, and let $M = e(T_{n,k})$, $b = 0.25c_2n^{\frac{2}{k+1}}$. Then Claim 3.2.4 implies that Maker has a strategy to occupy a good copy of $K_k$ in the $b$-biased Maker-Breaker game on $T_{n,k}$. But, as we argued earlier, this also gives TMaker a strategy for the $b$-biased $T$-tournament game on $K_n$. □

**Proof of Proposition 1.2.2.** Let $T$ be the tournament, with $k \geq 4$ vertices, of which TMaker aims to create a copy in an unbiased game on $G \sim \mathcal{G}_{n,p}$. By Theorem 1.1 in [40], we know that there is a constant $c > 0$ such that for $p \leq cn^{-\frac{2}{k+1}}$, Breaker a.a.s. has a strategy to block cliques of order $k$ in the unbiased Maker-Breaker game on $G$, which again gives a winning strategy for TBreaker in the $T$-tournament game on $G$. Now, let $p \geq c_3n^{-\frac{2}{k+1}}$, with $c_3 = c_3(k)$ from Corollary 3.2.5. Before sampling the random graph $G \sim \mathcal{G}_{n,p}$ fix a partition $V_1 \cup \ldots \cup V_k = [n]$ as before. Then, after sampling $G \sim \mathcal{G}_{n,p}$, we know that the subgraph induced by those edges which intersect two different parts $V_i$ and $V_j$ is sampled like a random graph $F \sim \mathcal{G}(T_{n,k}, p)$. According to Corollary 3.2.5, Maker a.a.s. has a strategy to occupy a good copy of $K_k$ in $F \subseteq G$, and thus TMaker a.a.s. has a strategy to create a copy $T$ in the unbiased tournament game on $G$. □

## 3.3 The triangle case

In the following we prove **Theorem 1.2.3**.

For the acyclic triangle $T_A$, the result can be obtained from [42] as follows: For $p \ll n^{-\frac{5}{9}}$ Breaker a.a.s. has a strategy to prevent triangles in the unbiased Maker-Breaker game on $G \sim \mathcal{G}_{n,p}$. Applying such a strategy in the $T_A$-tournament game as TBreaker obviously blocks acyclic triangles. For $p \gg n^{-\frac{5}{9}}$ a.a.s. Maker has a strategy to gain an undirected triangle in the unbiased Maker-Breaker game on $G \sim \mathcal{G}_{n,p}$. In the $T_A$-game, TMaker now can proceed as follows. She fixes an arbitrary ordering $\{v_1, \ldots, v_n\}$ of $V(G)$ before the game starts. Then she applies the mentioned strategy of Maker for gaining an undirected triangle, where she always chooses orientations from vertices of smaller index to vertices of larger index. This way, every triangle claimed by her will be an acyclic triangle, and thus she wins.

Thus, from now on, we can restrict the problem to the discussion of the cyclic triangle $T_C$. To show that $n^{-\frac{8}{15}}$ is the threshold probability for the existence of a winning strategy for TMaker in the $T_C$-tournament game on $G \sim \mathcal{G}_{n,p}$, we will study TMaker's and TBreaker's strategy separately.

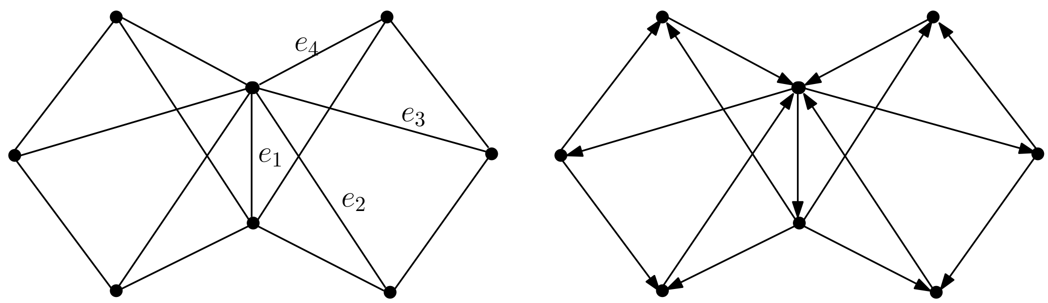


Figure 3.1: Graph $H$ without and with orientation.

We start with **TMaker's strategy**. Let $p \gg n^{-\frac{8}{15}}$. Then, by Theorem 3.1.5, a.a.s. $G \sim \mathcal{G}_{n,p}$ contains the graph $H$, presented in the left half of Figure 3.1, as $m(H) = \frac{15}{8}$. As indicated in the right half of the same figure, its edges can be oriented in such a way that each triangle has a cyclic orientation, and thus, it is enough to prove that Maker has a strategy to claim an undirected triangle in the unbiased Maker-Breaker game on $H$. Her strategy is as follows. At first she claims the edge $e_1$, as indicated in the figure. By symmetry, we can assume that afterwards Breaker claims an edge which is on the "left side" of $e_1$. Then in the next moves, as long as she cannot close a triangle, Maker claims the edges $e_2$, $e_3$ and $e_4$, always forcing Breaker to block an edge which could close a triangle, and Maker will surely be able to complete a triangle in the next round.

Now, let $p \ll n^{-\frac{8}{15}}$. We are going to show that a.a.s. there exists a **TBreaker's strategy** which blocks copies of $T_C$, when playing on $G \sim \mathcal{G}_{n,p}$. We start with some preparations. Among others, we will consider *triangle collections*, as studied in [42].

**Definition 3.3.1** *Let $G = (V, E)$ be some graph without isolated vertices. Further, let $T_G = (V_T, E_T)$ be the graph where $V_T = \{H \subseteq G : H \cong K_3\}$ is the set of all triangles in $G$, and $E_T = \{H_1H_2 : E(H_1) \cap E(H_2) \neq \emptyset\}$ is the (binary) relation on $V_T$ of having a common edge. Then:*

- *$G$ is called* very basic *if $T_G$ is a subgraph of a copy of $K_3^+$ (triangle plus a pending edge), or a subgraph of a copy of $P_k$ with $k \in \mathbb{N}$.*

- *$G$ is called* basic *if there are distinct edges $e_1, e_2 \in E(G)$ such that $G - e_i$ is very basic for both $i \in \{1, 2\}$.*

- *$G$ is a triangle collection if every edge of $G$ is contained in some triangle and $T_G$ is connected.*

*If $G$ is a triangle collection we further call it a bunch (of triangles) if we can find triangles $F_1, \ldots F_r \in V_T$ covering all edges of $G$ with the property that $|V(F_i) \setminus \cup_{j<i} V(F_j)| = 1$ and $|E(F_i) \setminus \cup_{j<i} E(F_j)| \geq 2$ for every $i \in [r]$.*

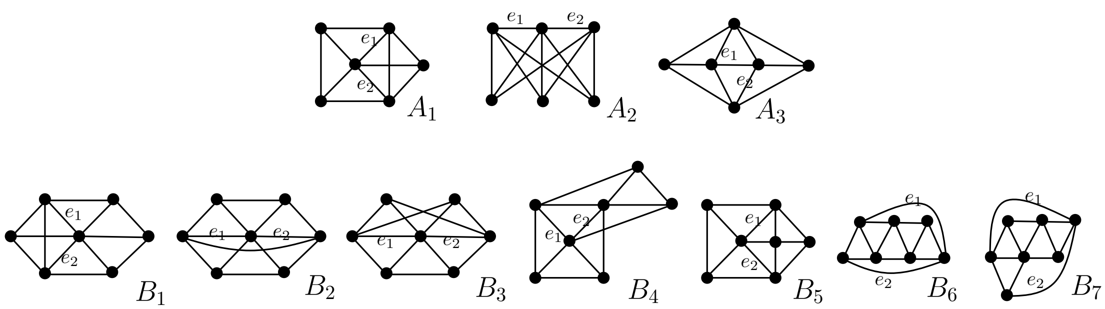


Figure 3.2: Basic triangle collections.

Note that every collection on a given number $n$ of vertices, contains a bunch on the same number of vertices with at least $2n - 3$ edges. Figure 3.2 shows some collections that are easily checked to be basic. For each of the graphs, the edges $e_1$ and $e_2$ indicated in the figure satisfy the condition from the definition of basic graphs. Moreover, the following observation is easily verified.

**Observation 3.3.2** *Let $G = (V, E)$. Maker (TMaker) has a strategy to create a triangle (a copy of $T_C$) on $G$ if and only if $G$ contains a collection $C$ such that she has a strategy to create a triangle (a copy of $T_C$) on $C$.*

In the following we show now that Breaker can prevent Maker from occupying a triangle when playing on basic graphs. This also ensures a winning strategy for TBreaker in the corresponding $T_C$-tournament game. We start with the following proposition.

**Proposition 3.3.3** *Let $G = (V, E)$ be very basic, then Breaker can block every triangle in the unbiased Maker-Breaker game on $E(G)$, even if Maker is allowed to claim two edges in the very first round.*

**Proof** W.l.o.g. we can assume that $T_G \cong P_k$ for some $k$, or $T_G \cong K_3^+$, with $T_G$ as given in Definition 3.3.1. We further can assume that Maker in the first round claims two edges $f_1, f_2 \in E(G)$ that participate in triangles of $G$. If $T_G \cong P_k$ then observe that there is an ordering $F_1, \ldots, F_k$ of the elements in $T_G$, such that $f_1 \in E(F_1)$, and $|V(F_i) \setminus \cup_{j<i} V(F_j)| = 1$, and $|E(F_i) \setminus \cup_{j<i} E(F_j)| = 2$ for every $2 \leq i \leq k$. To see this one just has to start the sequence with a triangle $F_1$ containing $f_1$, and to extend the sequence along the path-like structure of $T_G$. Finally, let $A_1 := E(F_1) \setminus \{f_1\}$ and $A_i := E(F_i) \setminus \cup_{j<i} E(F_j)$ for every $i \in [k] \setminus \{1\}$. These sets are pairwise disjoint, have cardinality 2 and satisfy $A_i \subseteq E(F_i)$ for each $i \in [k]$. That is, Breaker can block triangles by an easy pairing strategy. (In particular, for his first move, Breaker claims the unique edge $f$ for which there is an $i \in [k]$ with $A_i = \{f_2, f\}$.) If $T_G \cong K_3^+$, then it can be shown that $G$ contains exactly four triangles and that one can find an ordering $F_1, \ldots F_k$ (with $k = 4$) with the properties from the previous case. So, Breaker wins similarly. □

**Corollary 3.3.4** *Let $G = (V, E)$ be basic, then Breaker can block every triangle in the unbiased Maker-Breaker game on $E(G)$.*

**Proof** Let $e_1, e_2$ be the edges given by the definition of a basic graph. Breaker's strategy is to claim $e_1$ or $e_2$ in the first round. Afterwards, the game reduces to the graph $G - e_i$ for some $i \in [2]$, where Maker claims 2 edges, before Breaker claims his first edge. Now, since $G - e_i$ is very basic for both $i \in \{1, 2\}$, Breaker then succeeds by the previous proposition. □

We further observe the following two statements which can be checked by easy case distinctions.

**Observation 3.3.5** *TBreaker has a strategy to prevent cyclic triangles in an unbiased game on $E(K_4)$, even if TMaker is allowed to claim and orient two edges in her first turn.*

**Observation 3.3.6** *TBreaker has a strategy to prevent cyclic triangles in an unbiased game on $E(W_4)$, even if TMaker is allowed to claim and orient two edges in her first turn, as long as not both edges are incident with the center vertex of $W_4$.*

Now, using the previous statements we will show that for $p \ll n^{-\frac{8}{15}}$ a.a.s. every collection $C$ in $G \sim G_{n,p}$ is such that TBreaker has a strategy to prevent cyclic triangles in an unbiased game on $C$. It follows then by Observation 3.3.2 that a.a.s. Breaker wins on $G$. To do so, we start with the following propositions, motivated by [42], which helps to restrict the set of collections we need to consider.

**Proposition 3.3.7** *Let $p \ll n^{-\frac{8}{15}}$, then a.a.s. every triangle collection $C$ in $G \sim \mathcal{G}_{n,p}$. satisfies $m(C) < \frac{15}{8}$.*

**Proof** Each collection $C$ on at least 25 vertices contains a bunch $B$ on exactly 25 vertices with

$$d(B) = \frac{e(B)}{v(B)} \geq \frac{2v(B) - 3}{v(B)} > \frac{15}{8}.$$

Since there are only finitely many such bunches and each of them a.a.s. does not appear in $G$ according to Theorem 3.1.5, together with the union bound we obtain that a.a.s. each collection in $G$ lives on at most 25 vertices. Since there are only finitely many collections with at most 25 vertices, we also know by the same reason that a.a.s. each collection in $G$ on at most 25 vertices needs to have maximum density smaller than $\frac{15}{8}$. □

**Proposition 3.3.8** *Let $C$ be a triangle collection with $m(C) < \frac{15}{8}$ such that TMaker has a strategy to create a cyclic triangle in an unbiased game on $C$, but there is no such strategy for any collection $C' \subset C$. Then the following properties hold:*

*(a)* $5 \leq v(C) \leq 7$,

*(b)* $e(C) = 2v(C) - 1$,

*(c)* $\delta(C) \geq 3$,

*(d) $C$ is not basic.*

**Proof** Property (d) obviously holds, using Corollary 3.3.4. Moreover, (c) follows immediately. Indeed, if there were a vertex $v$ with $d_C(v) \leq 2$, then TBreaker could prevent cycles on $C - v$ by the minimality condition on $C$, and cycles containing $v$ by simply pairing the edges incident with $v$ (if there exist two such edges), a contradiction. Furthermore, $v(C) \geq 5$ is needed, according to Observation 3.3.5. Now, let $B$ be a bunch contained in $C$ with $v(C)$ vertices, then $e(C) > e(B)$, since $\delta(B) = 2 < \delta(C)$. As such a bunch contains at least $2v(B) - 3$ edges, it follows that $e(C) \geq e(B) + 1 \geq 2v(C) - 2$. Furthermore $e(C) \leq 2v(C) - 1$, since otherwise $m(C) \geq 2$. If $e(C) = 2v(C) - 1$, then together with $m(C) < \frac{15}{8}$, we deduce that $v(C) \leq 7$. Otherwise, we have $e(C) = 2v(C) - 2$ and $e(C) = e(B) + 1$. Analogously to the proof of Theorem 23 in [42] it then follows that $C$ can only be a wheel; for completeness let us include the argument here: Let $E(C) \setminus E(B) = \{v_1 v_2\}$. By the definition of a bunch, we can find triangles $F_1, \dots F_r$ in $B$ covering all edges of $B$ with the property that $|V(F_i) \setminus \cup_{j<i} V(F_j)| = 1$ and $|E(F_i) \setminus \cup_{j<i} E(F_j)| \geq 2$ for every $i \in [r]$. As $e(B) = e(C) - 1 = 2v(B) - 3$ it then follows that $r = v(C) - 2$ and $|E(F_i) \setminus \cup_{j<i} E(F_j)| = 2$ for every $i \in [r] \setminus \{1\}$, as otherwise

$e(B) > 3 + 2(r-1) = 2v(C) - 3$, a contradiction. Thus, for every $i \in [r] \setminus \{1\}$, $F_i$ needs to share exactly one edge with $\cup_{j<i} F_j$. From this, we can conclude that $B$ needs to contain at least two vertices of degree 2. However, as $\delta(C) \geq 3$ and $E(C) \setminus E(B) = \{v_1v_2\}$, we know that $v_1$ and $v_2$ must be the only vertices in $B$ of degree 2. Now, by the definition of a triangle collection, $v_1v_2$ needs to be part of a triangle in $C$. Thus, there needs to be a vertex $v_3$ such that $v_1v_3, v_3v_2 \in E(B)$. But this is only possible if $v_3$ belongs to every triangle $F_i$, $i \in [r]$, and thus, $C$ needs to be a wheel. Now, to finish the proof, observe that TBreaker can always prevent triangles in an unbiased game on a wheel by a simple pairing strategy, a contradiction to our assumption. □

So, the goal will be to show that there exists no collection $C$ which satisfies all the conditions given in Proposition 3.3.8.

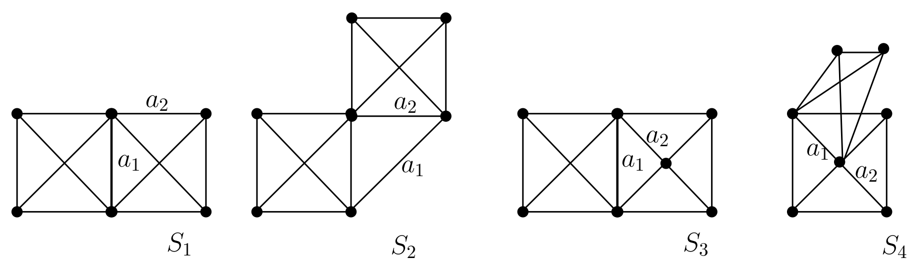


Figure 3.3: Special collections.

**Lemma 3.3.9** *If a collection $C$ satisfies (a) - (d) from Proposition 3.3.8, then either $C$ is isomorphic to $K_5^-$ ($K_5$ minus one edge) or $C$ is isomorphic to one of the graphs $S_i$, $1 \leq i \leq 4$, given in Figure 3.3.*

**Proof** If $v(C) = 5$, then $e(C) = 9$, by Property (b), and the statement follows obviously. So, let $v(C) \neq 5$. We will show now that a collection satisfying (a) - (c) either is isomorphic to one of the collections $S_i$, or it is isomorphic to one of the basic collections $A_i$ or $B_i$ from Figure 3.2, thus contradicting Property (d).

Let us start with $v(C) = 6$. Assume first that $C$ contains a subgraph $H \cong K_4$ and let $\{x, y\} = V(C) \setminus V(H)$. With $e(C) = 11$ and $\delta(C) \geq 3$ we conclude $xy \in E(C)$, and by the definition of a collection it follows that $x$ and $y$ have a common neighbor $v_1 \in V(H)$. Because of (c), we further have $xv_2 \in E(C)$ for some $v_2 \in V(H) \setminus \{v_1\}$. Now, if $yv_2 \in E(C)$, then $C \cong S_1$, otherwise by (c) we have $yv_3 \in E(C)$ for some $v_3 \in V(H) \setminus \{v_1, v_2\}$ and so $C \cong A_1$. Assume then that $C$ does not contain a clique of order 4. We still find a subgraph $H' \subseteq C$ with four vertices $V(H') = \{v_1, v_2, v_3, v_4\}$ and five edges, say $v_1v_3 \notin E(H)$. Since $C$ is a triangle collection, there needs to be some $x \in V(C) \setminus V(H')$ that is part of the same triangle as an edge $e$ from $H'$. Let $y$ be the unique vertex in $V(C) \setminus (V(H') \cup \{x\})$.

Assume first that $e = v_2v_4$. We know then that $\{x, v_1, v_3\}$ is an independent set in $C$, since otherwise we would have a 4-clique in $C$. By (b) and (c), it thus follows that $N(y) = \{x, v_1, v_3, v_i\}$ for some $i \in \{2, 4\}$, which gives $C \cong A_2$.

Assume then that $e \neq v_2v_4$ and w.l.o.g. $e = v_3v_4$ by symmetry of $H'$. If $v_1x \in E(C)$, it then follows that $d(y) = 3$, since (b) and (c) need to hold; moreover, $C[V(C) \setminus \{y\}] \cong W_4$ where $v_4$ represents the center of the wheel. In case $v_4y \in E(C)$, we can only have $C \cong A_2$, as $C$ does not contain a 4-clique; and in case $v_4y \notin E(C)$, we can assume that $N(y) = \{v_1, v_2, v_3\}$ (because of the symmetry of the 4-wheel), which yields $C \cong A_3$. If otherwise $v_1x \notin E(C)$, then, since there is no 4-clique in $C$, we immediately obtain $d(y) = 4$ and $v_1, x \in N(y)$, as $e(C) = 11$ and $\delta(C) \geq 3$. Moreover, $v_4 \notin N(y)$, since we otherwise would obtain a 4-clique, independently of the choice of the fourth neighbor of $y$. Thus, we conclude $N(y) = \{v_1, v_2, v_3, x\}$ and $C \cong A_3$.

Now, let $v(C) = 7$. We distinguish three cases.

**Case 1.** Assume that $C$ contains a subgraph $H \cong K_4$. Let $\{x, y, z\} = V(C) \setminus V(H) =: V'$. With $e(C) = 13$ and $\delta(C) \geq 3$ it follows that $\{x, y, z\}$ is not an independent set, w.l.o.g. $xy \in E(C)$. By the definition of a collection it further follows that $x$ and $y$ have a common neighbor – the vertex $z$ or some vertex $v \in V(H)$.

Assume first that $z \in N(x) \cap N(y)$. By $\delta(C) \geq 3$ each vertex in $V'$ needs to have at least one neighbor in $V(H)$. If there were a matching of size 3 between $V'$ and $V(H)$, then by (b), one of the matching edges could not be part of a triangle, a contradiction. If all the three vertices have a common neighbor in $V(H)$, then one easily deduces $C \cong S_2$. Otherwise, by symmetry we can assume that there is a vertex $v_1 \in V(H)$ such that $v_1x, v_1y \in E(C)$ and $v_1z \notin E(C)$, and moreover, $v_2z \in E(C)$ for some $v_2 \in V(H) \setminus \{v_1\}$. Now, let $\{v_3, v_4\} = V(H) \setminus \{v_1, v_2\}$. To ensure that $v_2z$ belongs to some triangle in $C$, we finally need to have exactly one of the edges from $\{v_3z, v_4z, v_2x, v_2y\}$ to be an edge in $C$. The first two edges however do not result in a triangle collection, while for the other two edges we get $C \cong S_3$.

Assume then that $z \notin N(x) \cap N(y)$, but $v \in N(x) \cap N(y)$ for some $v \in V(H)$. Because of (b) and (c), either $xz \in E(C)$ or $yz \in E(C)$, w.l.o.g. say $xz \in E(C)$ and $yz \notin E(C)$. As $\delta(C) \geq 3$, we then immediately get $yw \in E(C)$ for some $w \in V(H) \setminus \{v\}$. Moreover, we then need two other edges incident with $z$ besides $xz$, of which one is $zv$ to ensure that $xz$ belongs to a triangle. If the second edge is $zw$, then $C \cong S_4$; otherwise $C \cong B_1$.

**Case 2.** Assume that $C$ does not contain a clique of order 4, but there is some $H \subseteq C$ with $H \cong W_4$. Let $\{x, y\} = V(C) \setminus V(H) =: V'$ and let $z$ be the unique vertex with $d_H(z) = 4$. By (b) and (c), it follows that $xy \in E(C)$, and since $C$ is a collection, there is a common neighbor of $x$ and $y$ in $V(H)$.

Assume first that $z \in N(x) \cap N(y)$. As $\delta(C) \geq 3$, both vertices $x$ and $y$ have another neighbor in $V(H) \setminus \{z\}$, however there cannot be a second common neighbor, since there is no 4-clique in $C$. One easily checks that $C \cong B_2$ or $C \cong B_3$ follows.

Assume then that $z \notin N(x) \cap N(y)$, but $v \in N(x) \cap N(y)$ for some $v \in V(H) \setminus \{z\}$. If $xz \in E(C)$ (or $yz \in E(C)$), we then need $yw \in E(C)$ (or $xw \in E(C)$) for some $w \in N_H(v) \setminus \{z\}$ to ensure that $e(C) = 13$ and $\delta(C) \geq 3$ holds while $C$ is a triangle collection. This gives $C \cong B_4$. Otherwise, we have $z \notin N(x) \cup N(y)$. In this case, let $w'$ to be the unique vertex of $H$ not belonging to $N(v) \cup \{v\}$. Then we also have $w' \notin N(x) \cup N(y)$. Indeed, if we had $yw' \in E(C)$ say, then as $yw'$ needs to be part of some triangle and as $d(x) \geq 3$ and $e(C) = 13$, we would need $xw' \in E(C)$, in which case it is easily checked that $C$ is not a triangle collection. So, we can assume that $xv_1 \in E(C)$ for some $v_1 \in V(H) \setminus \{v, w', z\}$, and $yv_1 \notin E(C)$, because $C$ does not have a 4-clique. Finally, since $\delta(C) \geq 3$, we need $v_2 y \in E(C)$ for the unique vertex $v_2 \in V(H) \setminus \{v, w', z, v_1\}$, i.e. $C \cong B_5$.

**Case 3.** Finally assume that $C$ neither contains a 4-clique nor a 4-wheel. It is easy to check that $C_0 \subseteq C$ (with notation of vertices as given in Figure 3.4), and by the assumption of this case we further have $v_1v_3, v_1v_4, v_3v_5 \notin E(C)$. Since $C$ is a triangle collection, we find a vertex $x \in V' := V(C) \setminus V(C_0)$ which belongs to a triangle that also contains an edge $e \in E(C_0)$. Let $\{y\} = V' \setminus \{x\}$. By symmetry of $C_0$ we may assume that $e \in \{v_2v_5, v_4v_5, v_1v_5, v_1v_2\}$.

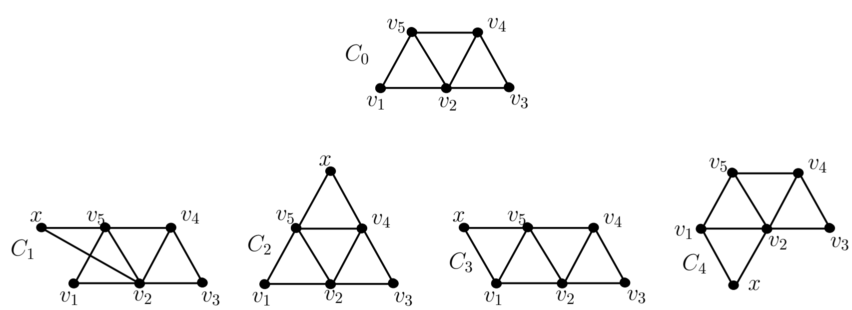


Figure 3.4: Subgraphs.

Assume first that $e = v_2v_5$ were possible, i.e. $C_1 \subseteq C$. Then by assumption of Case 3, every edge in $E(C) \setminus E(C_1)$ would need to be incident with $y$. Because of (b) and (c) we then had that $d(y) = 4$ and $v_1y, v_3y, xy \in E(C)$. Since these three edges would need to belong to triangles, we further would need $yv_2 \in E(C)$, which would create a 4-wheel on $V(C) \setminus \{v_3, v_4\}$ with center $v_2$, in contradiction to the assumption.

So, as next assume that $e = v_4v_5$ were possible, i.e. $C_2 \subseteq C$. Then analogously every edge in $E(C) \setminus E(C_2)$ would need to be incident with $y$, and $d(y) = 4$ and $\{v_1, v_3, x\} \subseteq N(y)$, because of (b) and (c). But then, independently of what the fourth neighbor of $y$ is, one of the edges $v_1y, v_3y, xy$ could not belong to a triangle, again a contradiction.

As third, assume that $e = v_1v_5$, i.e. $C_3 \subseteq C$. By the assumption of Case 3, every edge in $E(C) \setminus (E(C_3) \cup \{xv_3\})$ needs to be incident with $y$. If $xv_3 \notin E(C)$, then we have $d(y) = 4$ and $xy, v_3y \in E(C)$, because of $e(C) = 13$ and $\delta(C) \geq 3$. Depending on how the other two edges incident with $y$ are chosen, we either obtain a contradiction by creating a 4-clique or a 4-wheel, or we see that $C \cong B_6$. So, let $xv_3 \in E(C)$. Then $d(y) = 3$, by (b) and (c), and to have $xv_3$ in a triangle, we need $yx, yv_3 \in E(C)$. It follows that $C \cong B_6$, if $yv_1 \in E(C)$ or $yv_4 \in E(C)$, or $C \cong B_7$, if $yv_2 \in E(C)$ or $yv_5 \in E(C)$.

As last, assume that $e = v_1v_2$, i.e. $C_4 \subseteq C$. If $xv_3 \in E(C)$ were possible, then we had $d(y) = 3$ because of $e(C) = 13$ and $\delta(C) \geq 3$. But then, depending on the three edges incident with $y$, we would get a 4-clique or a 4-wheel in $C$, or we would find an edge which is not contained in a triangle, a contradiction. So, we can assume that $xv_3 \notin E(C)$. Then, by (b), (c) and the assumption of Case 3, we deduce that $d(y) = 4$ and $yx, yv_3 \in E(C)$. If $yv_2 \in E(C)$ were also an edge of $C$, then for any choice of the fourth edge incident with $y$, we would create a 4-clique or a 4-wheel in $C$. That is, we can assume that $yv_2 \notin E(C)$. But then we need $v_1y, v_4y \in E(C)$ to ensure that $yx$ and $yv_3$ belong to triangles, which yields $C \cong B_7$. □

**Lemma 3.3.10** *For any collection given by Lemma 3.3.9, TBreaker has a strategy to prevent cyclic triangles.*

**Proof** If $C \cong S_i$ for some $i$, note that $C$ is covered by two (not necessarily disjoint) graphs $C(1)$, $C(2)$, plus at most one additional edge if $C \cong S_2$, where each of the $C(i)$ is isomorphic to $K_4$ or $W_4$. Choose edges $a_1$ and $a_2$ as indicated in Figure 3.3. In his first move, TBreaker claims the edge $a_1$ if TMaker did not orient it before; otherwise he claims the edge $a_2$. Afterwards, TBreaker plays on $C(1)$ and $C(2)$ separately, meaning: each time TMaker orients an edge of $C(i)$, TBreaker claims an edge of $C(i)$ if there remains one. Now, using Proposition 3.3.3 and Observation 3.3.5, TBreaker can do this in a way such that he prevents cyclic triangles on each $C(i)$, and therefore in $C$.

Finally, we need to look at the case when $C \cong K_5^-$. By an easy case analysis, it can be proven that TBreaker has a strategy to prevent cyclic triangles on $C$. We give a sketch in the following. Let $V(C) = X \cup Y$ with $X = \{v_1, v_2, v_3\}$ and $Y = \{v_4, v_5\}$, and let $E(C) = \binom{X}{2} \cup \{xy : x \in X, y \in Y\}$.

**Case 1.** TMaker orients an edge in $E(X, Y)$ in her first turn.

W.l.o.g. let $e = v_1v_4 \in E(X, Y)$ be the edge to which TMaker gives an orientation in her first move. Then TBreaker's strategy is to delete the edge $v_1v_2$. Note that $C - \{v_1v_2\}$ is isomorphic to the 4-wheel $W_4$, here with center $v_3$, and TMaker's first arc is not incident with $v_3$. Thus, TBreaker can win by Observation 3.3.6.

**Case 2.** TMaker orients an edge inside $E(X)$ in her first turn.

W.l.o.g. let TMaker's first oriented edge be $(v_1, v_2)$. Then TBreaker's first move will be to delete the edge $v_2v_4$. Afterwards, TBreaker's second move will depend on TMakers second move, as follows:

If TMaker orients $(v_1, v_3)$ or $(v_3, v_2)$ for her second move, then TBreaker claims $v_2v_5$ and afterwards he wins by an easy pairing strategy, with the pairs $\{v_1v_4, v_3v_4\}$ and $\{v_1v_5, v_3v_5\}$.

If TMaker for her second move chooses one of the arcs $(v_1, v_4)$, $(v_4, v_1)$, $(v_3, v_4)$, $(v_4, v_3)$, $(v_1, v_5)$, $(v_5, v_2)$, $(v_2, v_3)$ and $(v_3, v_5)$, then TBreaker for his second move claims the edge $v_1v_3$. As he claims $v_2v_4$ and $v_1v_3$ then, the only triplets on which TMaker could create a triangle are $\{v_1, v_2, v_5\}$ and $\{v_2, v_3, v_5\}$. In either of the cases it is easy to check that from now on TBreaker can prevent cyclic triangles.

If TMaker for her second move chooses $(v_2, v_5)$ or $(v_5, v_3)$, then TBreaker claims $v_1v_5$ for his second move. Afterwards there remain three triplets on which TMaker still could create a triangle, namely $\{v_1, v_3, v_4\}$, $\{v_1, v_2, v_3\}$ and $\{v_2, v_3, v_5\}$. To block a triangle on $\{v_1, v_3, v_4\}$, TBreaker can consider a pairing $\{v_1v_4, v_3v_4\}$. For the other two triplets it is easy to check then that TBreaker can prevent cyclic triangles, since the orientation which $v_2v_3$ needs, to create a cyclic triangle, is different for these two remaining triplets.

If TMaker for her second move chooses $(v_3, v_1)$, then TBreaker needs to claim $v_2v_3$. Afterwards there remain three triplets on which TMaker still could create a triangle, namely $\{v_1, v_3, v_4\}$, $\{v_1, v_2, v_5\}$ and $\{v_1, v_3, v_5\}$. To block a triangle on $\{v_1, v_3, v_4\}$, TBreaker can consider a pairing $\{v_1v_4, v_3v_4\}$. For the other two triplets it again is easy to check that TBreaker can prevent cyclic triangles, since the orientation which $v_1v_5$ needs, to create a cyclic triangle, is different for these two triplets.

Finally, if TMaker for her second move chooses $(v_5, v_1)$, then TBreaker needs to claim $v_2v_5$. Afterwards there remain three triplets on which TMaker still could create a triangle, namely $\{v_1, v_3, v_4\}$, $\{v_1, v_2, v_3\}$ and $\{v_1, v_3, v_5\}$. To block a triangle on $\{v_1, v_3, v_4\}$, TBreaker can consider a pairing $\{v_1v_4, v_3v_4\}$. For the other two triplets it again is easy to check that TBreaker can prevent cyclic triangles, since the orientation which $v_1v_3$ needs, to create a cyclic triangle, is different for these two triplets. □

To summarize, we have shown now that for $p \ll n^{-\frac{8}{15}}$, a.a.s. TBreaker can prevent cyclic triangles in the tournament game on $G \sim \mathcal{G}_{n,p}$. Indeed, by Proposition 3.3.8, Lemma 3.3.9

and Lemma 3.3.10, we know that there exists no collection $C$ with $m(C) < \frac{15}{8}$ on which TMaker has a strategy to create a copy of $T_C$. By Proposition 3.3.7 we however know that for $p \ll n^{-\frac{8}{15}}$ a random graph $G \sim \mathcal{G}_{n,p}$ a.a.s. only contains such collections, and using Observation 3.3.2 we thus conclude that a.a.s. TMaker does not have a winning strategy when playing on $G \sim \mathcal{G}_{n,p}$, which at the same time guarantees a winning strategy for TBreaker. □

# Chapter 4

# Tree embedding game

In this chapter, we study the unbiased tree embedding game $(E(K_n), \mathcal{F}_T)$. Recall that this game is played by two players, Maker and Breaker, who alternately claim previously unclaimed edges of the complete graph $K_n$. Maker, who starts the game, claims one edge in each round and aims to occupy a copy of a pre-defined (labeled) spanning tree $T$ of $K_n$. Breaker, also claiming one edge in each round, wants to prevent Maker from claiming a copy of $T$. Hereby, besides for the study of random trees, we will focus on the case that $T$ has a bounded maximum degree.

**Notation and terminology.** Let $G$ be a graph and let $T$ be a forest. Then for every $S \subseteq V(T)$, a function $\phi : S \to V(G)$ is an *embedding* of $T[S]$ into $G$, if $\phi$ is injective and if for every edge $xy \in E(T[S])$, we have $\phi(x)\phi(y) \in E(G)$. Let $\phi : S \to V(G)$ be such an embedding, then the vertices of $S$ are said to be *embedded* (into $G$), while those of $V(T) \setminus S$ are not embedded. If $v' \in S$ is an embedded vertex, then we call $v'$ *closed* with respect to the forest $T$ and the embedding $\phi$, if $N_T(v') \subseteq S$, i.e. all neighbors of $v'$ are embedded. Its image $v = \phi(v')$ will be called *closed* as well. Otherwise, when $N_T(v') \setminus S \neq \emptyset$, then $v'$ is called *open* with respect to the forest $T$ and the embedding $\phi$, and we also say that $v = \phi(v')$ is *open*. Moreover, if a vertex $v \in V(G)$ does not belong to the (image of the) embedding, i.e. $v \in V(G) \setminus \phi(S)$, then we say that $v$ is *available*. In all the proofs of this chapter, Maker always creates only one embedding, where she increases the set $S$ of embedded vertices in each round. Therefore, we leave away the phrasing "with respect to the forest $T$ and the embedding $\phi$", if it is clear from the context.

Assume now that some Maker-Breaker game, played on the edge set of some graph $G$, is in progress. Following standard notation for Maker-Breaker games [30], with $M$ we denote the graph consisting of all current Maker's edges, while $B$ denotes Breaker's graph, and

$F := G \setminus (M \cup B)$ is the graph containing all *free* edges. Assume further that Maker's goal is to occupy a copy of a forest $T$, and that at any given moment during the game, there is an embedding $\phi : S \to V(M)$ of $T[S]$ into $M$, with $S \subseteq V(T)$. Then with $\mathcal{O}_T = \mathcal{O}_T(\phi)$ we denote the set of vertices of $V(T)$ that are open with respect to $T$ and $\phi$, i.e. $\mathcal{O}_T = \{s \in S : N_T(s) \setminus S \neq \emptyset\}$.

Finally, we say that Maker *wastes* an edge (or a move), if this edge does not belong to her winning set at the end of the game, meaning that even without this edge Maker would be the winner.

## 4.1 Trees with a long bare path

The goal of this section is to prove Theorem 1.3.2. That is, we show that Maker can create a copy of a tree, which contains a long bare path, while wasting at most one edge. Let $T$ be such a tree and let $P$ be a long bare path in $T$. The strategy which we use in order to conclude the statement can be summarized as follows. In a first stage, Maker embeds $T \setminus P$ more or less greedily and without wasting any edges, while using a *potential function* argument to care about the distribution of Breaker's edges. Then, in a second stage, we use a modification of the strategy from Hefetz and Stich [31] for the Hamilton cycle game, to show that Maker can extend her copy of $T \setminus P$ to the desired copy of $T$, with a waste of at most one edge.

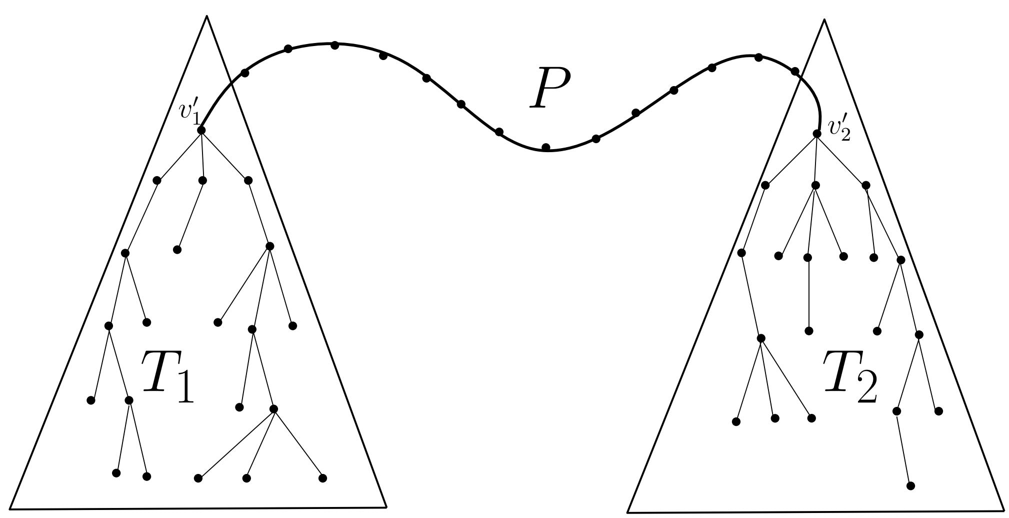


Figure 4.1: Splitting of $T$ into a long bare path $P$ and $T \setminus P = T_1 \cup T_2$.

The mentioned stages are given implicitly by the following statements.

**Theorem 4.1.1** *Let $n, m, \Delta \in \mathbb{N}$ with $\Delta \geq 3$ and $n > m \geq (\Delta+1)^2$. Let $T_1 \cup T_2$ be a forest consisting of two vertex disjoint trees $T_i = (V_i, E_i)$ with $\Delta(T_i) \leq \Delta$ and $|V_1 \cup V_2| = n-m$. Let $v_i' \in V_i$, with $i \in [2]$, be arbitrary vertices. Then, in an unbiased game on $E(K_n)$, Maker has a strategy to occupy a copy of $T_1 \cup T_2$ within $n-m-2$ moves, such that immediately after Maker's $(n-m-2)^{nd}$ move, the following holds. There exists an embedding $\phi : V(T_1 \cup T_2) \to V(M)$ of $T_1 \cup T_2$ into $M$ such that, with $A = V(K_n) \setminus \phi(V_1 \cup V_2)$, we have*

$$e_B\Big(A \cup \phi(\{v_1', v_2'\})\Big) \leq \binom{\Delta+1}{2}.$$

**Lemma 4.1.2** *Let $k \in \mathbb{N}$. Then there exists an integer $m' = m'(k)$ such that the following holds. Given any graph $G = (V, E)$ on $m \geq m'$ vertices and with $|E| \geq \binom{m}{2} - k$, and given any distinct vertices $v_1, v_2 \in V$, then in an unbiased game on $E(G)$, Maker has a strategy to occupy a Hamilton path $P$ of $G$, within $m$ moves, such that $End(P) = \{v_1, v_2\}$.*

Before we give the proofs of these two statements in the following subsections, let us see at first how **Theorem 1.3.2** can be concluded. Let $\Delta \in \mathbb{N}$ as given in the statement of Theorem 1.3.2. We choose $k = (\Delta+1)^4$ and $m_1 = m'(k)$, where $m'(k)$ satisfies Lemma 4.1.2. Observe then that $m'(k) > (\Delta+1)^2$ follows immediately. Now, let $T$ be the a tree satisfying the assumptions from Theorem 1.3.2. In particular, let $P \subseteq T$ be a bare path of length $m_1$, and let $End(P) = \{v_1', v_2'\}$. We need to describe a strategy for Maker which guarantees that she claims a copy of $T$ within $n$ rounds. For this, let $T_1$ and $T_2$ be the two non-trivial components of $T \setminus P$ that contain the vertices $v_1'$ and $v_2'$, respectively. Maker at first occupies a copy of $T_1 \cup T_2$ according to Theorem 4.1.1, which is possible as $e(P) > (\Delta+1)^2$ and thus $v(T_1 \cup T_2) \leq n - (\Delta+1)^2$. When she is done with this, her graph is isomorphic to $T_1 \cup T_2$ (plus isolated/available vertices). Moreover, according to Theorem 4.1.1, Breaker then claims at most $\binom{\Delta+1}{2}$ edges among the set $U \subseteq V(K_n)$, which consists of all available vertices plus those vertices $v_1$ and $v_2$ in $K_n$ that are the images of $v_1'$ and $v_2'$ with respect to Maker's embedding. Now, as a second step, Maker claims a copy of $P$, on the vertex set $U$ and with endpoints $v_1$ and $v_2$, while wasting at most one move. As $|U| = m_1 + 1 > m'(k)$ and $k > \binom{\Delta+1}{2}$, this is possible by Lemma 4.1.2. This way Maker occupies a copy of the goal tree $T$ within at most $n$ rounds, as she wastes at most one move (when she embeds $P$). □

### 4.1.1 A careful greedy embedding I

In this subsection we prove **Theorem 4.1.1**.

Let $n, m, \Delta \in \mathbb{N}$ and $T_1 \cup T_2$ be given according to the assumptions of Theorem 4.1.1, and let $v_1' \in V_1$ and $v_2' \in V_2$ be designated vertices. In the following we describe a strategy for Maker when playing on $K_n$ in order to create a copy of $T_1 \cup T_2$ as desired. Afterwards, we prove that she can indeed follow that strategy.

In her strategy, Maker will create a copy of $T_1 \cup T_2$ more or less greedily (by maintaining an embedding $\phi : S \to V(M)$ of $(T_1 \cup T_2)[S]$ into $M$, with $S \subseteq V(T_1 \cup T_2)$). That is, she starts from the vertices $v_1'$ and $v_2'$ (respectively from the images of $v_1'$ and $v_2'$ with respect to Maker's embedding $\phi$), and then she embeds the trees $T_i$ step by step, towards the leaves. For this, at each moment throughout the game, we let $S$ denote the set of vertices of $T_1 \cup T_2$ that are already embedded. Initially, we set $S = \{v_1', v_2'\}$. Moreover, with $\phi$ we will denote Maker's embedding of $(T_1 \cup T_2)[S]$ into $M$, as indicated above. Initially, we set $\phi(v_i') = v_i$ for every $i \in [2]$, where $v_1, v_2 \in V(K_n)$ are distinct and arbitrarily chosen vertices of $K_n$. With $A = V(K_n) \setminus \phi(S)$ we then denote the set of available vertices, i.e. those vertices which were not chosen for the embedding so far, and moreover, we set $U = A \cup \{v_1, v_2\}$. All these sets and the embedding will be updated after each of Maker's moves. Hereby, one may keep in mind that, whenever Maker removes a vertex from $A$, this vertex is also removed from $U$.

The main idea for Maker's strategy is to increase the set $S$ of embedded vertices, until all vertices of $T_1 \cup T_2$ are embedded, in such a way that $e_B(U) \le \binom{\Delta+1}{2}$ holds after Maker's (last) move. To do so, Maker considers a potential function to keep control on the distribution of Breaker's edges. She sets $p(v) := \max\{0, d_B(v, U) - d_M(v)\}$ for every $v \in V(K_n)$ as the *potential* assigned to $v$. Moreover, recalling that $\mathcal{O}_{T_i}$ denotes the set of open vertices with respect to $\phi$ and $T_i$ at any given moment throughout the game, she defines a cumulative potential $\psi$ after every step of the game as follows:

$$\psi := e_B(U) + \sum_{v \in \phi(\mathcal{O}_{T_1}) \cup \phi(\mathcal{O}_{T_2})} p(v).$$

In the very beginning of the game, Maker then closes the vertices $v_1$ and $v_2$. That is, she proceeds as follows. Let $d_1 = d_{T_1}(v_1')$ and $d_2 = d_{T_2}(v_2')$ be the degrees of the designated vertices in $T_1$ and $T_2$, respectively. Then in the first $d_1$ rounds, Maker claims $d_1$ free edges $v_1 a_i$ with $i \in [d_1]$ and distinct vertices $a_i \in A$. She then makes the following obvious update. She removes the vertices $a_i$ from $A$, she adds all vertices from $N_{T_1}(v_1)$ to $S$ and she updates $\phi$ in such a way that $\phi(N_{T_1}(v_1)) = \{a_i : \ i \in [d_1]\}$. (Moreover, she updates $\mathcal{O}_{T_1}$ accordingly. In particular, she removes $v_1'$ from this set.) Afterwards, in the following $d_2$ rounds, she proceeds similarly for $v_2$, by claiming $d_2$ edges $v_2 b_i$ with $i \in [d_2]$ and $b_i \in A$, removing the vertices $b_i$ from $A$, adding the vertices of $N_{T_2}(v_2)$ to $S$ and updating $\phi$ such that $\phi(N_{T_2}(v_2)) = \{b_i : \ i \in [d_2]\}$.

Once this part is done, for every further round $t$, Maker continues her embedding of $T_1 \cup T_2$, under consideration of the potential $\psi$. So, let $t > d_1 + d_2$. Maker from now on plays her $t^{\text{th}}$

move by considering the following cases.

**Case 1.** Assume first that $\psi \leq \binom{\Delta+1}{2}$. Then, Maker claims a free edge $uw$ such that $u \in \phi(\mathcal{O}_{T_1}) \cup \phi(\mathcal{O}_{T_2})$ is an open vertex (with respect to her current embedding $\phi$), and where $w \in A$ is an available vertex. Accordingly, Maker deletes $w$ from $A$, she chooses an arbitrary vertex $w' \in N_{T_1 \cup T_2}(\phi^{-1}(u)) \setminus S$, for that she sets $\phi(w') = w$ and adds $w'$ to $S$.

**Case 2.** Assume then that $\psi > \binom{\Delta+1}{2}$ and $\psi > e_B(U)$. Then, Maker claims a free edge $uw$ as in Case 1 with the additional constraint that must $d_B(u, U) > d_M(u)$ hold. Accordingly, she finishes her move with an update as described in Case 1.

**Case 3.** Otherwise, as $\psi \geq e_B(U)$ always holds, we have $\psi = e_B(U) > \binom{\Delta+1}{2}$. Then, Maker again claims a free edge $uw$ as in Case 1, this time with the constraint that $d_B(w, U) > 0$ must hold. Accordingly, she finishes her move with an update as described in Case 1.

Obviously, if Maker can always follow the proposed strategy, then she creates an embedding $\phi : V(T_1 \cup T_2) \to V(M)$ of $T_1 \cup T_2$ into $M$, within $e(T_1 \cup T_2) = n - m - 2$ rounds. Moreover, in case $\psi \leq \binom{\Delta+1}{2}$ holds immediately after her $(n-m-2)^{\text{nd}}$ move, the statement of Theorem 4.1.1 follows then, as $\psi \geq e_B(U)$. So, it remains to prove that Maker can indeed follow the proposed strategy, until $T_1 \cup T_2$ is fully embedded, and that the mentioned inequality is maintained. This will be done through the following two claims.

**Claim 4.1.3** *Let $t \geq d_1 + d_2$. Then, as long as Maker can follow the proposed strategy, $\psi \leq \binom{\Delta+1}{2}$ holds immediately after her $t^{th}$ move.*

**Proof** Assume that Maker can follow the proposed strategy. Then, one easily verifies that Maker never increases the value of $\psi$ with her moves, since she neither adds Breaker edges to the board nor increases the set $U$. According to her strategy, she closes the vertices $v_1$ and $v_2$ within the first $d_1 + d_2$ rounds, and thus, none of Breaker's edges can contribute more than 1 to the value of $\psi$ after Maker's $(d_1 + d_2)^{\text{th}}$ move. In particular, Breaker can increase the cumulative potential $\psi$ by at most 1 in each move after that move of Maker.

Now, proceeding by induction on the number of rounds $t$, let us prove Claim 4.1.3. At first, let $t = d_1 + d_2$. Then, by the observation above, we have $\psi \leq e(B) \leq 2\Delta \leq \binom{\Delta+1}{2}$ immediately after Maker's $(d_1 + d_2)^{\text{th}}$ move.

Now, for doing induction, assume that $\psi \leq \binom{\Delta+1}{2}$ was true immediately after Maker's $t^{\text{th}}$ move, for some $t \geq d_1 + d_2$. Then, we aim to show that $\psi \leq \binom{\Delta+1}{2}$ is maintained after Maker's $(t+1)^{\text{st}}$ move. As Breaker can increase the value of $\psi$ by at most 1, we know that $\psi$ cannot be larger than $\binom{\Delta+1}{2} + 1$ immediately before Maker considers to do her $(t+1)^{\text{st}}$ move. In case we have $\psi \leq \binom{\Delta+1}{2}$, we are done already, since Maker will not increase this potential by the argument above. So, it remains to check the case when $\psi = \binom{\Delta+1}{2} + 1$ holds immediately before Maker's move. Then, following the strategy, she plays according to Case 2 or Case 3. If she plays according to Case 2, then the following happens. The potential of the open vertex $u$, which by the choice of $u$ is at least 1 before Maker's move, is decreased by 1 (or vanishes from the cumulative potential, when $u$ is closed after Maker's move). Moreover, no other positive terms are added to $\psi$, as we add at most $d_B(w, U)$ to the sum of potentials of open vertices, while this value is subtracted from $e_B(U)$. Similarly, if Maker plays according to Case 3, the following happens. The value of $e_B(U)$ decreases by $d_B(w, U)$, while the sum of potentials of open vertices can increase by at most $d_B(w, U) - 1$, as $d_M(w) = 1$ then. Notice that the latter may happen if $w$ is an open vertex (with respect to the new embedding $\phi$) after Maker's move. Thus, we obtain that Maker decreases the value of $\psi$, when she follows Case 2 or Case 3 of her strategy, and therefore she maintains $\psi \leq \binom{\Delta+1}{2}$. □

**Claim 4.1.4** *Maker can always follow the strategy until $T_1 \cup T_2$ is fully embedded, i.e. until she finishes an embedding $\phi : V(T_1 \cup T_2) \to V(M)$ of $T_1 \cup T_2$ into $M$.*

**Proof** We observe first that $|A| \geq (\Delta+1)^2 \geq 4\Delta$ is always true for the first $n-m-2$ rounds, as $|V_1 \cup V_2| \leq n - (\Delta+1)^2$ by the assumption on $T_1 \cup T_2$. Thus, it is easy to check that Maker can close the vertices $v_1$ and $v_2$ as proposed by the strategy. Indeed, to close these vertices, Maker only needs to claim at most $2\Delta$ edges between the vertices $v_i$ and the set $A$, while Breaker in the meantime can only block $2\Delta$ such edges. Thus, we can concentrate on the rounds after Maker's $(d_1 + d_2)^{\text{st}}$ move. For induction assume that Maker could so far follow the first $t$ rounds, where $t \geq d_1 + d_2$. We need to show that she can follow the strategy in round $t+1$ as well. By Claim 4.1.3 we then know that immediately after her last move $\psi \leq \binom{\Delta+1}{2}$ was fulfilled. Moreover, we already saw that in his following move Breaker can increase the value of $\psi$ by at most 1, and so $\psi \leq \binom{\Delta+1}{2} + 1$ holds before Maker considers to do her $(t+1)^{\text{st}}$ move. If she then plays according to Case 1, then $\psi \leq \binom{\Delta+1}{2}$ is fulfilled before this move. It follows then that for every open vertex $u \in \phi(\mathcal{O}_{T_1}) \cup \phi(\mathcal{O}_{T_2})$ the Breaker-degree into $A$ is bounded from above by $d_B(u, U) \leq p(u) + d_M(u) \leq \psi + \Delta < (\Delta+1)^2 \leq |A|$. Thus, Maker can find an edge $uw$ as suggested. If she plays according to Case 2 or Case 3 instead, then we have $\psi = \binom{\Delta+1}{2} + 1$ immediately before her move. In Case 2, by its assumption, there needs to be an open vertex $u$ with $\phi(u) = d_B(u, U) - d_M(u) > 0$. Moreover, analogously to the previous argument, we have $d_B(u, A) < |A|$ and therefore, Maker can find an edge $uw$ as suggested in Case 2. In Case 3, as $e_B(U) = \binom{\Delta+1}{2} + 1$, then we find at least $\Delta$ vertices $w \in A$ with $d_B(w, U) > 0$; and for every open vertex $u$, we have $d_B(u, U) < \Delta$, as $p(u) = 0$. Thus, Maker can find an edge $uw$ as suggested in Case 3. □

From the previous claims, Theorem 4.1.1 now follows. □

### 4.1.2 A Hamilton path game

In the following we want to prove Lemma 4.1.2. First of all, let us recall that in [31], the following theorem was proved.

**Theorem 4.1.5 (Theorem 1.1 in [31])** *Let $n \in \mathbb{N}$ be large enough. Then, in the unbiased Maker-Breaker game on $K_n$, Maker has a strategy to occupy a Hamilton cycle within $n+1$ rounds.*

Maker's strategy in [31] can be sketched as follows. Playing on $K_n$, Maker starts the game by building at least 2 and at most 15 paths of constant length.

In case she can close one of her paths to a cycle $S$, she does so. She then connects all the remaining paths and isolated vertices to a Hamilton path $P$ on $V(K_n) \setminus V(S)$. Finally, she occupies a Hamilton cycle of $K_n$, by attaching the endpoints of $P$ to two consecutive vertices of $S$.

Otherwise, if she cannot close one of her paths to a cycle, then this means that Maker always blocks the corresponding edges. But then Maker can extend her paths (while Breaker always blocks cycles) in such a way that all Breaker's edges block cycles or lie between inner points of Maker's paths. Maker then proceeds until every vertex belongs to one of her paths, and then she connects those paths to a Hamilton cycle, wasting at most one edge.

For Lemma 4.1.2, we need to create a Hamilton path between two fixed vertices $v_1$ and $v_2$, such that we waste at most one move. This problem is very similar to the problem discussed above. Indeed, Maker can imagine that $v_1v_2$ is an edge which already belongs to her graph, and then, her goal becomes to create a Hamilton cycle containing this particular edge. However, we also have to care about the fact that, before Maker starts to create the Hamilton path/cycle, a certain number of edges over the vertex set $[m]$ may not belong to the board $E(G)$. The exact details of Maker's strategy are given in the following proof. As we need to modify the strategy of [31] slightly, we shorten our argument whenever possible, by referring back to the proof of Hefetz and Stich [31].

**Proof of Lemma 4.1.2** Whenever necessary, let us assume that $m$ is a large enough integer. Let $G \subseteq K_m$ be given according to the lemma. Instead of playing on $G$ we can assume to play on $K_m$ with the constraint that, before the game starts, all the at most $k$ edges of $E(K_m) \setminus (E(G) \cup \{v_1v_2\})$ already belong to Breaker's graph and that the edge $v_1v_2$ belongs to Maker's graph. So, in the following we prove that, under these constraints, Maker has a strategy to occupy a Hamilton cycle on $K_m$, which contains the edge $v_1v_2$, within $m$ rounds.

We start our proof by describing a strategy for Maker. Afterwards, we prove that she can follow that strategy, and by doing so, she finishes the desired Hamilton cycle within $m$ rounds. Maker's strategy consists of the following stages.

**Stage I.** Maker claims disjoint paths $P_1, ..., P_{15}$ with $v_1v_2 \in E(P_1)$, $e(P_i) = 20$ for $i \in [15]$, and denotes her first path by $P_1 = p_0...p_{20}$. This takes her 299 rounds. Afterwards, she proceeds to Stage II, where she may extend her paths further.

**Stage II.** Let $i > 299$. If Maker is able to close one of her paths to a cycle in her $i^{\text{th}}$ move, by claiming the edge between its endpoints, she does so and proceeds to Stage IV. Otherwise, immediately before her $i^{\text{th}}$ move, she considers the following vertex sets:

$$U := \{v \in V(K_m) : d_F(v) = m - 1\},$$
$$T := \{v \in V(K_m) : d_B(v) > 0 \text{ and } v \notin V(P_j) \text{ for every } j \in [15]\},$$
$$C := \bigcup_{j=1}^{15} \{v \in End(P_j) : d_B(v, V(P_\ell)) > 0 \text{ for some } \ell \neq j\}.$$

That is, $U$ is the set of untouched vertices, $T$ is the set of vertices that are touched by Breaker but not by Maker, and $C$ is the set of endpoints of Maker's paths which are incident to a Breaker's edge that also intersects another path of Maker. Maker now plays as follows.

If $C \cup T = \emptyset$, i.e. all Breaker's edges connect two vertices of the same path $P_i$ or two inner vertices of different paths $P_i$ and $P_j$, Maker proceeds to Stage III. Otherwise she enlarges one path (in most cases this will be $P_1$) by one vertex as follows.

- If $i = 300$, Maker claims an edge between $p_0$ and some vertex $v^* \in U$. She then updates $P_1$ accordingly.
- If $i = 301$, she claims an edge between $p_{20}$ and some vertex in $U$. She then updates $P_1$ accordingly.
- If $i > 301$ is odd and $T \neq \emptyset$, she claims an edge between $U$ and the youngest endpoint of $P_1$. She then updates $P_1$ accordingly.
- If $i > 301$ is even and $T \neq \emptyset$, she claims an edge between $T$ and the youngest endpoint of $P_1$. She then updates $P_1$ accordingly.
- If $i > 301$, $T = \emptyset$ and $C \neq \emptyset$, then she claims an edge in $E(C, U)$. She then updates her paths accordingly.

Maker then repeats Stage II (which ends at the latest after $1202 + 4k$ rounds, as we will see later).

**Stage III.** Maker practically proceeds with the strategy of Hefetz and Stich [31], starting with Phase 3. If possible in the following rounds, Maker closes one of her paths $P_1, \ldots P_{15}$ to a cycle $S$. She then completes a Hamilton path on $V \setminus V(S)$ without wasting a move, i.e. $M[V \setminus V(S)]$ is a Hamilton path of $K_m - V(S)$ then; and finally she creates the required Hamilton cycle within two further rounds. Otherwise, if she cannot close a cycle in the following rounds, i.e. Breaker always blocks cycles, then she extends $P_1$ using isolated vertices, until there are only 9 isolated vertices left. Then, she creates a Hamilton path on these 9 isolated vertices. Finally, she connects all her paths to a Hamilton cycle, such that $v_1v_2$ is contained, and such that she is done after the $m^{\text{th}}$ round. (The exact details of how Maker can achieve this goal are given in the discussion of the strategy.)

**Stage IV.** Maker's graph consists of a cycle $S$, and a collection of disjoint paths. Next, Maker creates a Hamilton path of $K_m - V(S)$ without wasting any move, i.e. such that her graph is a union of the cycle $S$ and a Hamilton path $\tilde{P}$ on $V \setminus V(S)$. Afterwards, within two further rounds, she creates a Hamilton cycle that contains $v_1v_2$.

Obviously, if Maker can follow the strategy, then she occupies a Hamilton cycle/path as required. Thus, it remains to show that Maker can indeed follow the strategy. To do so, we make use of the following technical lemma, due to Hefetz and Stich [31], which helps to create a Hamilton path on a given subset of vertices, in case certain conditions on the distribution of Breaker's edges hold.

**Lemma 4.1.6 (Lemma 3.3(i) in [31])** *Let $S \subseteq V(K_m)$ be an arbitrary set. Assume that before a Maker's move, Maker's graph on $V(K_m) \setminus S$ is a linear forest $F$ (i.e. a vertex-disjoint union of paths) plus isolated vertices. Let $f$ be the number of paths in $F$, let $End_F$ denote the set of endpoints of paths in $F$, let $I = V(K_m) \setminus \big(S \cup V(F)\big)$ and let $e_b$ be the number of Breaker's edges contained in $I$. Moreover, let $B'$ be the graph of Breaker's edges among $S \cup I \cup End_F$, minus those edges that are contained in $S$ or for which both endpoints belong to the same path in $F$. Assume that $|I| \geq 9$, $e_b \leq 2$, $f \geq 3$ and $e(B') \leq \frac{|I|}{2}$. Then, Maker has a strategy to occupy a Hamilton path $P$ of $K_m - S$ within $f + |I| - 1$ moves, such that immediately after $P$ is finished, $e_B\big(S, End(P)\big) \leq 3$ holds.*

Now, in order to show that Maker can follow the strategy, let us go through the cases separately.

**Stage I.** It is obvious that Maker can follow this part of the strategy, provided that $m$ is large enough. For her paths she needs to claim 299 edges ($v_1v_2$ is already claimed), and since Breaker in the meantime claims at most $299 + k$ edges, while $m$ is large, we can always find vertices $v$ with $d_F(v) = 0$ that Maker can use to construct her 15 paths.

**Stage II.** Notice that, since Stage I lasts 299 rounds, $|C \cup T| \leq 2e(B) \leq 600 + 2k$ at the beginning of Stage II. If Maker can close a cycle, there is nothing to prove. Otherwise, during Stage II, Breaker always blocks cycles and thus never increases the size of $C \cup T$. Indeed, assuming that Maker cannot close a cycle, Maker in the first two rounds of Stage II enlarges her path $P_1$ by attaching untouched vertices to both its endpoints. Each time, after such an attachment, Breaker needs to block a cycle, as otherwise Maker closes such a cycle and proceeds immediately to Stage III. Moreover, by the choice of these vertices, the new endpoints (including $v^*$) are chosen to be independent in Breaker's graph from all vertices outside Maker's paths. Thus, as long as Maker can follow Stage II, but Breaker always blocks cycles, we know that each time Maker adds a new vertex to $P_1$ or claims an edge in $E(C, U)$, there again appears a cycle that Breaker needs to block.

However, as long as Maker follows Stage II and $C \cup T \neq \emptyset$, she ensures that the size of $C \cup T$ decreases by at least one after every second round (besides for the first two rounds). Thus, if Maker can follow the strategy, it takes her at most $2 + 2(600 + 2k)$ moves, until $C \cup T = \emptyset$ and Stage II ends. As $m$ is assumed to be large, this ensures that $U$ is always large throughout Stage II, and thus Maker can play according to the proposed strategy for Stage II. As long as $T \neq \emptyset$, Maker can obviously attach vertices from $U$ and $T$ alternately to her path $P_1$. When $T = \emptyset$, then Maker can obviously claim an edge between $C$ and $U$.

**Stage III.** When Maker enters Stage III the first time, her graph consists of 15 paths $P_1, \ldots, P_{15}$ covering at most a constant (depending on $k$) number of edges. Moreover, immediately after her last move in Stage II, each Breaker's edge connects two vertices of the same path $P_j$ or two inner vertices of different Maker's paths (as $C \cup T = \emptyset$). In fact, this equals the situation at the beginning of Phase 3 in the strategy of [31] (see page 3-4 in [31]). In the next rounds, Maker extends $P_1$ by untouched vertices, unless she can close one of her paths to a cycle or the number of isolated vertices equals 9. It is evident that Maker can follow this strategy, since untouched vertices stay untouched when Breaker always blocks a cycle. We also note that this behaviour of Breaker leaves the edges between endpoints of distinct Maker paths untouched. In particular, as long as Breaker blocks cycles, we maintain $C \cup T = \emptyset$.

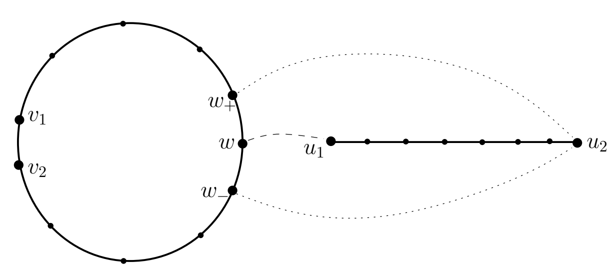


Figure 4.2: Illustration for Maker's strategy.

At first assume that Maker can close one of her paths to a cycle $S$, while the number of isolated vertices is at least 9. Then after Breaker's next move the situation is as follows. Maker's graph consists of the cycle $S$ with $v(S) \geq 20$, a collection of 14 vertex disjoint paths on $V \setminus V(S)$ and at least 9 isolated vertices. Moreover, if we define $B'$ as in Lemma 4.1.6 (where we replace $S$ with $V(S)$), then $e(B') \leq 2$ holds, as Breaker could only claim edges belonging to $B'$ in the previous two rounds. In particular, Breaker has at most 2 edges that are contained in the set $I$ of isolated vertices in Maker's graph. Following Lemma 4.1.6, Maker can build a Hamilton path $\tilde{P}$ on $K_m - S$ without wasting any move, such that immediately after $\tilde{P}$ is built, $E(M) = E(S) \cup E(\tilde{P})$ and $e_B(S, End(\tilde{P})) \leq 3$. Afterwards, let $u_1$ and $u_2$ be the endpoints of $\tilde{P}$. Assume first that $v_1v_2 \in E(S)$. Then, in her next move, Maker claims an edge between $u_1$ and an arbitrary vertex $w \in V(S) \setminus \{v_1, v_2\}$ such that both its neighbors $w_+$, $w_-$ on $S$ are neither elements of $\{v_1, v_2\}$ nor adjacent to $u_2$ in Breaker's graph. (For an illustration, see Figure 4.2.) This is easily doable, as $e_B(S, End(\tilde{P})) \leq 4$ and $v(S) \geq 20$. Afterwards, Maker finishes the desired Hamilton cycle by claiming one of the edges $u_2w_+$ and $u_2w_-$. Note that Maker occupies only one edge, which is not used for the final Hamilton cycle. Therefore, Maker plays exactly $m$ rounds in this case. Assume then that $v_1v_2 \notin E(S)$. Then the argument is exactly the same, just that Maker does not need to care about the edge $v_1v_2$ when she determines the vertex $w \in V(S)$.

In the second case, Breaker always blocks a cycle and Maker extends $P_1$ until there are only 9 isolated vertices left. Then, once the number of isolated vertices equals 9, the situation is as follows: Maker's graph consists of 15 vertex disjoint paths covering all but 9 vertices, and all of Breaker's edges still satisfy the property that each of them connects two vertices of the same path $P_j$ or two inner vertices of different Maker's paths. So, we have the same conditions as at the beginning of Phase 4 in the proof of Hefetz and Stich (see page 4 in [31]). If Maker now follows their strategy starting from Phase 4, then she first connects the 9 isolated vertices to a path and then she connects all of her paths step by step, decreasing their number by one in each round, until either Breaker lets her close a cycle or the number of paths equals 3. (Notice that in Stage 5 of the strategy of [31], Maker stops connecting the paths when their number is 2 instead of 3. However, we stop one round earlier to simplify our argument.)

In case she can close a cycle, she then plays in such a way that her graph consists of one cycle $S$ and one Hamilton path $\tilde{P}$ on $V \setminus V(S)$ such that $e_B(S, End(\tilde{P})) \leq 3$. (This can be achieved as explained in Phase X of [31]; the argument is very similar to the previous case.) Analogously to the previous discussion, she then finishes her Hamilton cycle. Otherwise, Maker stops when her graph consists of three paths covering $V(K_m)$ such that, immediately after her move, Breaker has no edge between endpoints of different paths and such that the edge between the endpoints of one of these paths, denoted by $P$, is still free. (This can be achieved as explained in Phase 5 of [31], see page 4 and page 13.) Note that these paths are all of length at least 20. Let $A = a_0...a_{\ell_A}$, $B = b_0...b_{\ell_B}$ and $C = c_0...c_{\ell_C}$ be the three paths, w.l.o.g. $v_1v_2 \in E(A)$. If Breaker does not block the edge between the endpoints of $P$, Maker can close it in her following move, and then she wins as in the previous discussions. That is, she connects the remaining two paths and then she attaches the new path to the cycle as before. Otherwise, Maker can claim the edge $b_{\ell_B}c_0$. If Breaker does not continue with $b_0c_{\ell_C}$, Maker can again close a cycle, having one cycle and one path then, so that she can finish her Hamilton cycle as before. Thus, assume that Breaker claims $b_0c_{\ell_C}$. Maker then claims $b_0c_0$. Afterwards, by an easy case analysis, Maker can claim a Hamilton cycle, which contains $v_1v_2$, within 2 further moves, just claiming edges of $E(\{a_0, a_{\ell_A}\}, \{b_0, b_{\ell_B}, c_{\ell_C}\})$. For example, if Breaker's next edge is $a_0b_0$, then Maker afterwards claims $a_0c_{\ell_C}$. Then the edges $a_{\ell_A}b_0$ and $a_{\ell_A}b_{\ell_B}$ both complete a Hamilton cycle as required, and Maker can surely claim one of these edges next.

**Stage IV.** When Maker enters Stage IV the situation is as follows. Her cycle $S$ has length at least 20; on $V \setminus V(S)$ her graph consists of a disjoint union of 14 paths, while in her graph $m(1-o(1))$ vertices are still isolated, as Stage I and II together took at most a constant $c = c(k)$ number of rounds. Let $I_j := \{v \in V(K_m) : d_M(v) = 0 \text{ before Maker's } j^{\text{th}} \text{ move}\}$, and let $B_j := \{e = vw \in E(B) : \{v,w\} \subseteq I_j \text{ before Maker's } j^{\text{th}} \text{ move}\}$. In particular, when Maker enters Stage IV in round $a_4$, for some integer $a_4$, we have that $|I_{a_4}| = m(1-o(1))$ and that $|B_{a_4}| \leq c + k$. Now, as long as $|B_j| > 1$, Maker for her $j^{\text{th}}$ move claims an edge that is adjacent to at least two edges of $B_j$ and independent of her previously claimed edges. This way, after a constant number of rounds, $|B_{j_0}| \leq 2$ holds before a Maker's move in some round $j_0$, while Maker's graph on $V \setminus V(S)$ is a disjoint union of at least 14 paths. Moreover, as only a constant number of rounds was played so far, and $m$ is assumed to be large, we further obtain $e(B) \leq \frac{|I_{j_0}|}{2}$ before Maker's $j_0^{\text{th}}$ move. By Lemma 4.1.6, it again follows that Maker can claim a Hamilton path $\tilde{P}$ on $K_m - S$ without wasting any move, such that $e_B(End(\tilde{P}), S) \leq 3$ immediately after Maker finished $\tilde{P}$. She then can proceed as before and finish her Hamilton cycle as required. □

## 4.2 Trees with many leaves

The goal of this section is to prove Theorem 1.3.3. That is, we show that Maker can create a constant degree tree on $n$ vertices within $n + 1$ rounds, even if this tree does not contain a long bare path. Let $T$ be such a tree. Then, we observe that $T$ needs to have a large number of leaves and, in particular, we can find a large matching in $T$ for which each edge saturates a leaf of $T$. Now, the main idea is similar to the previous section. In a first stage, Maker embeds $T$ minus this large matching (or at least most of its edges) more or less greedily, while caring about the distribution of Breaker's edges. Again, she does not waste any move in this phase of the game. Then, in a second stage, Maker finishes her desired copy of $T$ by playing a perfect matching game. For this she wastes at most two edges.

We start with a lemma which ensures the existence of the large matching described above.

**Lemma 4.2.1** *Let $\Delta, m, n \in \mathbb{N}$, then the following holds. If $T$ is a tree on $n$ vertices and with $\Delta(T) \leq \Delta$, such that every bare path in $T$ has length at most $m$, then $|N_T(L)| \geq \frac{n}{2\Delta(m+1)}$, where $L$ is the set of leaves in $T$.*

The statement above can be deduced from Lemma 2.1 in [34], which gives an upper bound on the number of bare paths in a tree on $n$ vertices with given number of leaves. However, for completeness, let us give a direct proof of Lemma 4.2.1 here, whose argument is analogous to the one in [34].

**Proof** With $d_1, d_2, d_{\geq 3}$ denote the number of vertices in $T$ with degree 1, 2 or at least 3, respectively, and note that $|L| = d_1$. By the Handshake-Lemma (Proposition 1.3.3 in [43]) we have $2(d_1+d_2+d_{\geq 3}-1) = 2(n-1) = \sum_{v \in V(T)} d_T(v) \geq d_1+2d_2+3d_{\geq 3}$ and therefore, $d_{\geq 3} < d_1$. The number of edge-disjoint bare paths, which start and end in vertices of degree different from 2, is $d_1+d_{\geq 3}-1 < 2d_1$, and so there are less than $2d_1 \cdot m$ vertices of degree 2 in $T$, as each bare path in $T$ has less than $m$ inner vertices. It follows that $n = d_1 + d_2 + d_{\geq 3} < 2d_1(m+1)$, i.e. $|L| \geq \frac{n}{2(m+1)}$. As $|N_T(L)| \geq \frac{|L|}{\Delta}$, the lemma follows. $\square$

Using the above lemma, Theorem 1.3.3 will be concluded from the following two statements, which represent the two stages described before.

**Theorem 4.2.2** *Let $\varepsilon > 0$ be a real number, and let $\Delta \geq 3$ be an integer. Then there is a constant $K$ (depending only on $\varepsilon$ and $\Delta$) such that for every large enough $n$ the following is true. Let $T$ be a tree on $n$ vertices with maximum degree $\Delta(T) \leq \Delta$ and $|N_T(L)| \geq \varepsilon n$, where $L$ is the set of leaves in $T$. Then, in an unbiased game on $E(K_n)$, Maker has a strategy to occupy a copy of some subtree $T_1 \subseteq T$ within $|V(T_1)|-1$ moves, such that immediately after Maker's $(|V(T_1)|-1)^{st}$ move, the following holds. There exists an embedding $\phi : V(T_1) \to V(M)$ of $T_1$ into $M$ such that*

$$\Delta(B[A \cup \phi(\mathcal{O}_T)]) \leq K$$

*where $A = V(K_n) \setminus \phi(V(T_1))$, and $\mathcal{O}_T$ is the set of open vertices of $T$ with respect to $\phi$. Moreover $T \setminus T_1$ is matching between $\mathcal{O}_T$ and $L \setminus V(T_1)$ of size at least $\frac{\varepsilon n}{2}$.*

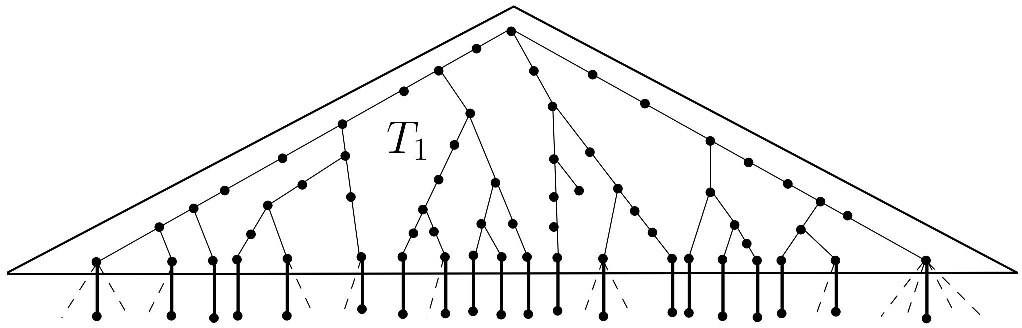


Figure 4.3: Splitting of $T$ into a subtree $T_1$, including the dashed edges, and a large matching.

**Lemma 4.2.3** *Let $r \in \mathbb{N}$. Then for every large enough integer $n$ the following holds. Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with $|V_1| = |V_2| = n$ and $\delta(G) \geq n - r$. Then, playing an unbiased Maker-Breaker game on $E(G)$, Maker has a strategy to occupy a perfect matching of $G$ within $n + 2$ rounds.*

Before we give the proofs of these two statements in the following subsections, let us see at first how **Theorem 1.3.3** can be concluded: Let $\Delta \in \mathbb{N}, m_1 \in \mathbb{N}$ as given in the statement of Theorem 1.3.3, and let $T$ be a tree on $n$ vertices as given by the same theorem. Whenever necessary, we assume that $n$ is large enough. We choose $\varepsilon := (2\Delta(m_1 + 1))^{-1}$, and we choose $r := K$ according to Theorem 4.2.2. Observe first that, by Lemma 4.2.1, we have $|N_T(L)| \geq \varepsilon n$. Now, we aim to describe a strategy for Maker which guarantees that she claims a copy of $T$ within $n+1$ rounds. For a first step, she occupies a subtree $T_1 \subseteq T$ within $|V(T_1)| - 1$ rounds, as given by Theorem 4.2.2. In particular, we then have an embedding $\phi : V(T_1) \to V(M)$ as described in Theorem 4.2.2, with $\Delta(B[A \cup \phi(\mathcal{O}_T)]) \leq r$, and such that $T \setminus T_1$ is a matching between $\mathcal{O}_T$ and $L \setminus V(T_1)$ of size $n' := |\mathcal{O}_T| = |A| \geq \frac{\varepsilon n}{2}$.

Then, in a second step, Maker completes her embedding of $T_1$ to an embedding of $T$, by claiming a perfect matching between $\phi(\mathcal{O}_T)$ and $A$, within at most $n'+2$ rounds. Notice that, for large enough $n$, she can do so by Lemma 4.2.3, as $d_F(v, \mathcal{O}_T) \geq n' - r$ and $d_F(w, A) \geq n' - r$ for every $v \in A$ and $w \in \mathcal{O}_T$, immediately after $T_1$ is embedded. This way Maker occupies a copy of the goal tree $T$ within at most $n + 1$ rounds, as she wastes at most two moves throughout the game. □

### 4.2.1 A careful greedy embedding II

In the following we prove **Theorem 4.2.2.**

Let $\varepsilon > 0$ and $\Delta \geq 3$ as given in the theorem. We choose $k$ such that $\Delta^{k-1} \geq \frac{80}{\varepsilon}$, and whenever necessary, assume that $n$ is large enough. Moreover, we set $K := \Delta^{k+1}$. Let $T$ be a tree as described in the theorem, with $\Delta(T) \leq \Delta$ and $|N_T(L)| \geq \varepsilon n$. We then find a matching of size $\varepsilon n$ in $T$ between $N_T(L)$ and some subset $L_0 \subseteq L$ of size $\varepsilon n$. From now on, set $T_0 := T - L_0$. Maker will occupy a copy of some tree $T_1$ with $T_0 \subseteq T_1 \subseteq T$ and $|V(T_1)| \leq n - \frac{\varepsilon n}{2}$, where $T_1$ will be determined during the game.

In the following we give a strategy for Maker to occupy such a tree, and afterwards we show that she can follow that strategy, and that all conditions of our theorem will be satisfied.

For the embedding, Maker proceeds as follows. Similar to the proof of Theorem 4.1.1, Maker starts from some vertex $v_0'$ in $T_0$, and then she embeds the tree $T_0$ (plus maybe some further edges of $T$) step by step, towards the leaves (by maintaining an embedding $\phi : S \rightarrow V(M)$ of a subtree $T[S]$ into $M$, with $S \subseteq V(T)$). From time to time it may happen that she needs to embed some vertex from $V(T) \setminus V(T_0)$ to be able to keep control on the distribution of Breaker's edges. These vertices belong to $V(T_1) \setminus V(T_0)$ then. Again, we denote with $S$ the set of vertices that are already embedded. So, initially $S = \{v_0'\}$. Moreover, with $\phi$ we will denote Maker's embedding of $T[S]$ into $M$, as indicated above. Initially, we set $\phi(v_0') = v_0$ for an arbitrary vertex $v_0 \in V(K_n)$. Moreover, $A = V(K_n) \setminus \phi(S)$ will denote the set of available vertices, i.e. those vertices which were not chosen for the embedding so far.

The main idea for Maker's strategy is to increase the set $S$ of embedded vertices, until (at least) all the vertices of $T_0$ are embedded, such that Breaker's degrees among the open and available vertices in $K_n$ is bounded by $K$. This time, instead of defining a cumulative potential, we distinguish between dangerous vertices and non-dangerous vertices, in order to have some control on the distribution of Breaker's edges. We say that a vertex $v$ is *dangerous* if it is open or available (with respect to the current embedding $\phi$ and the tree $T$), and if additionally its degree satisfies $d_B(v) \geq K$. With *Dang* we denote the set of all dangerous vertices, at any given moment throughout the game. So, initially we have $Dang = \emptyset$, and Maker updates *Dang* after each move of either of the players. She proceeds as follows.

In case Maker already embedded $T_0$ completely and there exists no dangerous vertex, she stops playing. (Notice that in case Maker would embed the whole tree $T$ at some point this condition would be satisfied automatically.) Otherwise she considers the following cases.

**Case 1.** If there exists a dangerous vertex, then let $v \in Dang$ be an arbitrary such vertex. According to the definition, $v$ needs to be available or open. So, we have two different subcases.

**Case 1.(i)** If $v$ is open, then Maker proceeds as follows. In the next rounds, Maker closes $v$. Let $v_1', \ldots, v_d'$ be all the neighbors of $v' := \phi^{-1}(v)$ in $T$ that Maker did not embed so far.

Then, in her next $d$ moves, Maker claims $d$ edges $vv_i$, $1 \leq i \leq d$, where $v_1, \ldots, v_d$ are distinct available vertices in $A$. Accordingly, she removes those vertices from $A$, she removes $v$ from $D$, she sets $\phi(v_i') = v_i$ and adds $v_i'$ to $S$, for every $i \in [d]$.

**Case 1.(ii)** If $v$ is available, then Maker includes this vertex into her embedding as follows.

(a) If there is an open $u \in \phi(\mathcal{O}_T)$ with $uv$ being free, then Maker immediately attaches $v$ to the current tree and identifies $v$ with a non-embedded neighbor of $\phi^{-1}(u)$. Formally, let $u' = \phi^{-1}(u)$. Maker claims the edge $uv$, she chooses an arbitrary vertex $v' \in N_T(u') \setminus S$ which is not embedded so far, she adds this vertex to $S$ and sets $\phi(v') = v$.

(b) Otherwise, assume that there exist two open vertices $u_0, w_0 \in \phi(\mathcal{O}_T)$ at which we can still attach paths of length two, so that the resulting graph is still a copy of some subtree of $T$. Formally, let $u_0' = \phi^{-1}(u_0)$ and $w_0' = \phi^{-1}(w_0)$, and assume there exist vertices $u_1', u_2', w_1', w_2' \in V(T) \setminus S$ that were not embedded so far, such that $u_i'u_{i+1}', w_i'w_{i+1}' \in E(T)$ for both $i \in \{0,1\}$. Maker then chooses an available vertex $z$ with $zv, zu_0, zw_0 \notin E(B)$, and claims one of the paths $(u_0, z, v)$ and $(w_0, z, v)$ within two rounds, by claiming the edge $zv$ first and afterwards one of the edges $zu_0$ and $zw_0$. By symmetry, assume that she claims $zu_0$, then she adds $u_1'$ and $u_2'$ to $S$, she removes $z$ and $v$ from $A$, and she set $\phi(u_1') = z$ and $\phi(u_2') = v$.

(c) Otherwise, assume that there exists an open vertex $u_0 \in \phi(\mathcal{O}_T)$ at which we can still attach a path of length three, so that the resulting graph is still a copy of some subtree of $T$. Formally, let $u_0' = \phi^{-1}(u_0)$ and let there exist three vertices $u_1', u_2', u_3' \in V(T) \setminus S$ which were not embedded so far, such that $u_i'u_{i+1}' \in E(T)$ for every $i \in \{0,1,2\}$. Within three rounds, Maker then claims a path of length three, containing $u_0$ and $v$, and identifies this path with $(u_0', u_1', u_2', u_3')$. To be precise, she first claims a free edge $vw$ where $w \in A$ is available. Then, similar to the previous case, she chooses an available vertex $z$ such that $zu_0, zv, zw \notin E(B)$, and claims one of the paths $(u_0, z, v)$ and $(u_0, z, w)$ in the following two rounds, by first claiming $u_0z$ and afterwards either $zv$ or $zw$. W.l.o.g. assume that she claim $zv$. Then she adds $u_i'$ to $S$ for every $i \in [3]$, she removes $v,\ z,\ w$ from $A$, and sets $\phi(u_1') = z$, $\phi(u_2') = v$ and $\phi(u_3') = w$.

**Case 2.** If there is no dangerous vertex immediately before Maker's move, but $T_0$ is still not embedded completely, then Maker continues her embedding of $T_0$. For this, she claims an arbitrary edge $uv \notin E(B)$ with $u \in \phi(\mathcal{O}_{T_0})$ and $v \in A$. Let $u' = \phi^{-1}(u)$ and let $v' \in N_{T_0}(u') \setminus S$ be an arbitrary vertex that is not embedded so far. Maker then removes $v$ from $A$, adds $v'$ to $S$, and she sets $\phi(v') = v$.

Obviously, if Maker can follow the proposed strategy, then she embeds some tree $T_1$, which contains $T_0$, within $|V(T_1)| - 1$ rounds. We thus need to prove that she can indeed follow the strategy. Moreover, we need to verify that, once Maker stops playing, all the conditions which are required by Theorem 4.2.2 are satisfied. We start with some useful claims first.

**Claim 4.2.4** *Until the moment when Maker stops playing according to her strategy, at most* $\frac{2n}{K}$ *vertices become dangerous.*

**Proof** This claim is obvious since Maker plays at most $n$ rounds, while a dangerous vertex needs to have a degree of at least $K$ in Breaker's graph. □

**Claim 4.2.5** *Until the moment when Maker stops playing according to the strategy of Stage I, we have at least* $0.9\varepsilon n$ *available vertices.*

**Proof** Notice that it is enough to show that at most $0.1\varepsilon n$ vertices from $L_0$ will be embedded until Maker stops playing. This is given by the following reason. Following Maker's strategy, a vertex $v' \in L_0$ is embedded, i.e. added to $S$, only in Case 1. In Case 1(i), it could happen that we embed $v'$ if for its parent $w'$ (with respect to $T$) we have that its image $w = \phi(w')$ becomes dangerous and thus Maker closes $w$. However, when we close $w$, then $v'$ is the only vertex from $L_0$ that Maker embeds. In Case 1(ii), it could happen that Maker embeds $v'$ if we embed a dangerous vertex by attaching a path of length at most three to some open vertex, as described in (a) – (c), such that the endpoint of this path corresponds to $v'$. Again, $v'$ then is the only vertex from $L_0$ which becomes embedded.

Thus, it follows that the number of vertices from $L_0$, that Maker embeds throughout Stage I, can be bounded from above by the number of vertices that become dangerous. By Claim 4.2.4 and the choice of $K$, we therefore have at most $\frac{2n}{K} < 0.1\varepsilon n$ such vertices. □

**Claim 4.2.6** *Until the moment when Maker stops playing according to the strategy of Stage I, we have* $d_B(v) \leq \frac{0.5\varepsilon n}{\Delta}$ *for every vertex* $v \in A \cup \phi(\mathcal{O}_T)$.

**Proof** Let $v \in A \cup \phi(\mathcal{O}_T)$. We can assume that $v \in Dang$ at some point during the game, as otherwise $d_B(v) \leq \frac{0.5\varepsilon n}{\Delta}$ is immediate, for large enough $n$. Now, as long as $v \in Dang$, Maker plays according to Case 1. However, in this case, Maker always cares about dangerous vertices, by attaching them to her current tree (Case 1(ii)) and closing them (Case 1(i)). As for each dangerous vertex, we play at most 3 rounds according to Case 1(ii), and at most $\Delta$ rounds according to Case 1(i), it follows by Claim 4.2.4 that Maker plays at most $(\Delta + 3) \cdot \frac{2n}{K}$

moves according to Case 1. Thus, by the choice of $K$ and for large enough $n$, we know that $d_B(v) \leq K + (\Delta + 3) \cdot \frac{2n}{K} < \frac{0.5\varepsilon n}{\Delta}$ as long as $v$ is open or available. □

The previous two claims will be useful to show that Maker can follow the proposed strategy, as for every open or available vertex we know that its Breaker-degree is much smaller than the number of available vertices. However, another important fact which we need to show first, is that our case distinction in Case 1(ii) covers all possible cases. To do so, we prove the following claim.

**Claim 4.2.7** *Assume that Maker plays according to Case 1(ii), then one of the assumptions from (a), (b) and (c) holds.*

**Proof** For contradiction, assume that the statement is wrong. Let $v$ be the available vertex that Maker chooses, playing according to Case 1(ii). Since the condition of (a) is assumed not to hold, we obtain $|N_T(L) \cap \mathcal{O}_T| \leq d_B(v) \leq \frac{0.5\varepsilon n}{\Delta}$, where the last inequalities follows from Claim 4.2.6. Moreover, each vertex in $N_T(L) \cap \mathcal{O}_T$ can have at most $\Delta$ descendants that are not embedded so far, as $\Delta(T) \leq \Delta$. Since the condition of (c) is also assumed not to hold, it follows that if $x \in \mathcal{O}_T \setminus N_T(L)$, then $x$ has at most $\Delta + \Delta^2$ descendants that were not embedded so far. However, as the condition of (b) is assumed not to hold, there can be at most one such vertex $x$. But then, $|A| \leq \frac{0.5\varepsilon n}{\Delta} \cdot \Delta + 1 \cdot (\Delta^2 + \Delta) < 0.9\varepsilon n$ for large enough $n$, which is in contradiction with Claim 4.2.5. □

Now, with all the above claims in hand, we will conclude that Maker always can follow the proposed strategy. We go through the cases separately. If Maker plays according to **Case 1**, then she focuses on an arbitrary vertex $v \in Dang$. Assume first that Case 1(i) happens, i.e. $v$ is open. Then Maker can follow the strategy as $d_B(v) < |A| - 2\Delta$, by Claim 4.2.5 and Claim 4.2.6. So, let us assume then that Case 1(ii) happens, i.e. $v$ is available, and one of conditions of the subcases (a) – (c) is satisfied, by Claim 4.2.7. In case Maker considers to play according to (a), then there is nothing to prove, since the existence of the edge that Maker needs to claim, is given by the condition of (a). In case she considers to play according to (b), then just observe that $d_B(v) + d_B(u_0) + d_B(w_0) < |A|$, by Claim 4.2.5 and Claim 4.2.6. This guarantees that Maker can find a vertex $z$ as required in (b), and having this vertex fixed it is obvious that Maker can follow the proposed strategy. Moreover, in case Maker considers to play according to (c), then first of all we have $d_B(v) < |A|$ when Maker wants to claim an edge $vw$ as described, by Claim 4.2.5 and Claim 4.2.6. By the same claims, we afterwards know that $d_B(u_0) + d_B(v) + d_B(w) < |A|$, which guarantees that Maker can find a vertex $z$ as required. Thus, Maker can analogously follow the strategy. Finally, when Maker considers to play according to **Case 2**, then $d_B(u) < K < |A|$ for every open vertex $u \in \phi(\mathcal{O}_{T_0})$, by

Claim 4.2.5, and thus, Maker can claim an edge as described.

Therefore, we know that Maker can always follow her strategy until she stops, i.e. until (at least) $T_0$ is embedded and $Dang = \emptyset$. Let $T_1 \subseteq T$ be the tree that Maker has embedded when she stops. As $T_0$ is fully embedded, we have $T_0 \subseteq T_1$. Moreover, by definition of $T_0$, it follows then that $T \setminus T_1 \subseteq T \setminus T_0$ is a matching between $\mathcal{O}_T$ and $L \setminus V(T_1)$. By Claim 4.2.5, we further obtain $e(T \setminus T_1) \geq 0.9\varepsilon n > 0.5\varepsilon n$; and as $Dang = \emptyset$, we have $\Delta(B[A \cup \phi(\mathcal{O}_T)]) \leq K$. □

### 4.2.2 A perfect matching game

In the following we aim to prove Lemma 4.2.3. For this, we observe at first that, with a slight modification of the proof of Theorem 1.2 in [28], the following can be proven.

**Theorem 4.2.8** *For every large enough integer $n$ the following holds. In the unbiased Maker-Breaker game on $E(K_{n,n})$, Maker (as the second player) has a strategy to occupy a perfect matching of $K_{n,n}$ within $n+1$ moves.*

We will use this result in order to prove the following statement which is easily seen to be equivalent to Lemma 4.2.3.

**Lemma 4.2.9** *Let $r_1, r_2 \in \mathbb{N} \cup \{0\}$. Then for every large enough integer $n$ the following holds. If $G = (V_1 \cup V_2, E)$ is a bipartite graph with $|V_i| = n$ and $d(v_i, V_{3-i}) \geq n - r_i$ for every $v_i \in V_i$ and every $i \in [2]$, then, in an unbiased Maker-Breaker game on $E(G)$, Maker has a strategy to occupy a perfect matching of $G$ within $n+2$ moves.*

**Proof** For the proof let us assume that we play on the graph $K_{n,n}$ and that the edges of $K_{n,n} \setminus G$ already belong to Breaker's graph $B$ before the game starts. We will prove the claim by induction on $r_1 + r_2$. Whenever necessary, we assume that $n$ is large enough. The main idea for the induction step is to start with a large matching touching those vertices which have a large degree in Breaker's graph. We do so until we know that the maximum degree among the unsaturated vertices of one of the partite sets is smaller than it was at the beginning, while in the other partite set the maximum degree did not increase. Then we use the induction hypothesis for $r_1 + r_2 - 1$.

Throughout the game, we let $U = \{u \in V_1 \cup V_2 : \ d_M(u) = 0\}$ denote the set of those vertices of $K_{n,n}$ which are untouched, i.e. not saturated by Maker's partial matching, and we set $U_i = U \cap V_i$ for $i \in [2]$. Moreover, as motivated by the short description above, we consider the maximum degrees $\Delta_i = \max\{d_{B[U]}(v) : v \in U_i\}$, for $i \in [2]$. Thus, at the beginning we have $\Delta_i \leq r_i$ by the assumption of the lemma, and the assumption that $B = K_{n,n} \setminus G$ at the beginning of the game.

At first, let us assume that $r_1 + r_2 \leq 2$. Then there are two subcases to consider. If $r_i = 0$ for some $i \in [2]$, then $r_{3-i} = 0$ also holds, and we are done immediately by Theorem 4.2.8. Otherwise, if $r_i > 0$ for both $i \in [2]$, then Breaker's graph $B = K_{n,n} \setminus G$ needs to be a matching. It is easy to see then that, immediately after Breaker's first move, we can cover $V_1 \cup V_2$ by two complete bipartite subgraphs $G_1$ and $G_2$ of $K_{n,n}$, with $G_1 \cong K_{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil}$ and $G_2 \cong K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$, such that Breaker may claim (exactly) one edge in $G_1$, but none in $G_2$. Then, playing on these subgraphs separately according to the strategy given for Theorem 4.2.8, Maker can claim a perfect matching of $G_1$ within $\lceil \frac{n}{2} \rceil$ rounds, and of $G_2$ within $\lfloor \frac{n}{2} \rfloor$ rounds. Hereby, she claims her first edge in $G_1$ (as it may happen that Breaker already claims one edge from this graph); and afterwards she always claims an edge in the same subgraph $G_i$ in which Breaker claimed an edge before (besides the case when she already occupies a perfect matching of $G_i$, in which case she switches to $E(G_{3-i})$).

So, let us assume now that $r_1 + r_2 \geq 3$ and $\Delta_i \leq r_i$ at the beginning of the game. Moreover, for doing induction, let us assume that in case $\Delta_1 + \Delta_2 \leq r_1 + r_2 - 1$ the statement of our lemma is true. By symmetry we may also assume that $r_2 \geq r_1$. In the following we give a strategy for Maker and then we show that she can always follow this particular strategy. Moreover, applying the induction hypothesis for $r_1 + r_2 - 1$, as indicated earlier, we will be able to conclude that Maker creates a perfect matching of $K_{n,n}$ within the required number of rounds.

The strategy consists of two stages. In case $\Delta_1 < r_1$ before the game starts, and thus $\Delta_1 + \Delta_2 \leq r_1 + r_2 - 1$, Maker immediately proceeds with Stage II.

**Stage I.** Maker starts by building a partial matching while caring about the maximum degrees $\Delta_1$ and $\Delta_2$. This stage is split into two phases, depending on whether Maker achieves to decrease the size of $\Delta_1$.

**Phase 1.** In each round in Phase 1, Maker looks for a vertex $u_1 \in U_1$ and a vertex $u_2 \in U_2$, such that $d_{B[U]}(u_1) = \Delta_1$ and $d_{B[U]}(u_2) = \max\{d_{B[U]}(v) : v \in U_2, u_1 v \notin B\} \geq 2$. If such vertices do not exist, she immediately proceeds with Phase 2. But if such vertices exist, she then claims the edge $u_1 u_2$, and she updates $U_1$ and $U_2$ by removing $u_1$ and $u_2$, respectively. In case $\Delta_1 < r_1$ holds after the update, Maker proceeds with Stage II. Otherwise, she repeats Phase 1.

**Phase 2.** Within at most two further moves, in which Maker claims independent edges from $K_{n,n}[U]$, Maker ensures that immediately afterwards, $\Delta_i \leq r_i$ for both $i \in [2]$, and $\Delta_1 + \Delta_2 < r_1 + r_2$. The exact details of how she chooses her edges in order to guarantee the mentioned inequalities will be given later in the proof.

**Stage II.** On the remaining vertex set $U$, Maker occupies a perfect matching within $\frac{|U|}{2} + 2$ rounds.

Obviously, if Maker can follow this strategy, then the lemma is proven. So, as usual, it remains to show that she can follow the strategy. To do so, we prove some useful claims first.

**Claim 4.2.10** *Assume that Maker can follow the strategy. Then, throughout Phase 1, $\Delta_i \leq r_i$ is maintained to hold after each of Maker's moves, for both $i \in [2]$.*

**Proof** The proof of the above claim follows by induction on the number $t$ of rounds. Indeed, at the beginning of the game the required inequalities hold by the assumption of the lemma. So, let us assume now that $\Delta_1 \leq r_1$ and $\Delta_2 \leq r_2$ is given immediately after Maker's $t^{\text{th}}$ move, for some $t \in \mathbb{N} \cup \{0\}$, where we allow to set $t = 0$ to represent the moment before the game starts. We then aim to show that, by following the proposed strategy, Maker ensures that the same inequalities hold immediately after her $(t+1)^{\text{st}}$ move. For this, let $x_1x_2$ denote the edge which Breaker claims in the meantime, where $x_1 \in V_1$ and $x_2 \in V_2$. Then, we may assume that $x_i \in U_i$ for both $i \in [2]$, since otherwise Breaker changes neither $\Delta_1$ nor $\Delta_2$, and we would be done already, as Maker cannot increase these maximum degrees. Now, since Breaker's edge only increases the degrees of $x_1$ and $x_2$, we observe at first, that immediately after he claims his edge, we still have $\Delta_i \leq r_i + 1$ for both $i \in [2]$, and $d_{B[U]}(v) \leq r_i$ for every $v \in U_i \setminus \{x_i\}$. Assume then that Maker follows the strategy of Phase 1 and that she claims an edge $u_1u_2$, as explained in the strategy description. In case there is some $i \in [2]$ with $u_i = x_i$, we remove $x_i$ from the set $U_i$, and we also decrease $d_{B[U]}(x_{3-i}, U_i)$ by removing $x_i$. Therefore, we maintain $\Delta_i \leq r_i$ for both $i \in [2]$. Otherwise, we have $u_i \neq x_i$ for both $i \in [2]$. As Maker chooses $u_1$ with $d_{B[U]}(u_1)$ being maximal, we conclude $d_{B[U]}(x_1) \leq d_{B[U]}(u_1) \leq r_1$, i.e. $\Delta_1 \leq r_1$ before Maker's $(t+1)^{\text{st}}$ move. If we additionally had $d_{B[U]}(x_2) \leq r_2$, then we would be done already. So, we can assume further that before Maker's $(t+1)^{\text{st}}$ move we have $d_{B[U]}(x_2) = r_2 + 1$ and therefore $x_2$ is the unique vertex in $U_2$ attaining the maximum degree $\Delta_2$. Then, since Maker claimed $u_1u_2$ according to the strategy rather than the edge $u_1x_2$, we must have $u_1x_2 \in B$, according to the description of Phase 1. Thus, Maker decreases the value of $d_{B[U]}(x_2)$ by removing $u_1$ from $U_1$, and thus, $\Delta_2 \leq r_2$ is maintained. □

Now, Maker can obviously follow the strategy of Phase 1. We further observe that the number of rounds that this phase lasts can be bounded by $\frac{r_1 n + 1}{r_1 + 1}$, which will be necessary to ensure that, when Maker enters Stage II, the number of unsaturated vertices is still large enough.

**Claim 4.2.11** *Phase 1 lasts at most* $\frac{r_1 n+1}{r_1+1}$ *rounds.*

**Proof** Consider the value of the sum $\sum_{v \in U_1} d_{B[U]}(v) \geq 0$, which is bounded from above by $r_1 n + 1$ immediately after Breaker's first move. Now, if Maker chooses an edge $u_1 u_2$ as described in the strategy, then this sum is decreased by $d_{B[U]}(u_1) + d_{B[U]}(u_2) \geq r_1 + 2$, as $d_{B[U]}(u_1) = \Delta_1 \geq r_1$ (otherwise Maker would have finished with Phase 1) and since removing $u_2$ from $U_2$ decreases $d_{B[U]}(v)$ for every $v \in U_1$ with $vu_2 \in E(B)$. Thus, as Breaker can increase the mentioned sum by at most 1 in one move, we see that it decreases in total by at least $r_1 + 1$ for each round that is played according to Phase 1. The claim follows. □

**Claim 4.2.12** *In Phase 2, Maker can ensure that within 2 rounds,* $\Delta_i \leq r_i$ *for both* $i \in [2]$ *and* $\Delta_1 + \Delta_2 < r_1 + r_2$ *hold.*

**Proof** Maker enters Phase 2 after she played according to Phase 1. Thus, by Claim 4.2.10, we know that immediately before her first move in Phase 2 we find at most one vertex $x_i \in U_i$ with $d_{B[U]}(x_i) > r_i$, for both $i \in [2]$, and in case such a vertex exists, we have $d_{B[U]}(x_i) = r_i + 1$. For her first move in Phase 2, Maker then claims an edge $u_1 u_2 \notin E(B)$ with $u_1 \in U_1$ and $u_2 \in U_2$ such that $d_{B[U]}(u_1) = \Delta_1$. (In case $d_{B[U]}(x_1) > r_1$, we have $u_1 = x_1$.) For large enough $n$ she can do so, as $\Delta_1 \leq r_1 + 1$ by Claim 4.2.10, and $|U_2| \geq \frac{n-1}{r_1+1}$ by Claim 4.2.11. Moreover, as Maker stopped playing according to Phase 1, we know that $u_1 v \in E(B)$ for every $v \in U_2$ which has degree $d_{B[U]}(v) \geq r_2 \geq 2$. After she claimed the edge $u_1 u_2$, we have $\Delta_1 \leq r_1$, as $u_1$ is removed from $U_1$. We also obtain $\Delta_2 \leq r_2$, as $u_1$ is removed from $U_1$, and thus $d_{B[U]}(v)$ decreases for every vertex $v \in U_2$ which satisfied $d_{B[U]}(v) \geq r_2$. Moreover, after she claimed that edge, there can be at most one vertex $x_2' \in U_2$ which has degree $d_{B[U]}(x_2') = r_2$ (namely, $x_2' = x_2$), while all the other vertices in $U_2$ have smaller degrees. If this vertex does not exist, then we are already done, and Maker then proceeds with Stage II. Otherwise, Maker plays a second move in Phase 2 as follows. Let $xy$ be Breaker's next edge, with $x \in V_1$ and $y \in V_2$. We then distinguish four cases.

**Case 1.** If $x \notin U_1$ or $y \notin U_2$, then Maker claims an edge $v_1 x_2$ such that $v_1 \in U_1$, which again is possible by Claim 4.2.11. Since $x_2$ was the unique vertex with degree $r_2$ towards $U_1$, and since Breaker did not change the values $\Delta_i$ by his choice of $xy$, we then have $\Delta_1 \leq r_1$ and $\Delta_2 < r_2$ after Maker's move.

**Case 2.** If $d_{B[U]}(y) > r_2$, i.e. $y = x_2$, then Maker again claims a free edge $v_1 x_2$ with $v_1 \in U_1$, and similarly to Case 1, $\Delta_1 \leq r_1$ and $\Delta_2 < r_2$ is guaranteed.

**Case 3.** If there is a vertex $z \in U_2$ such that $d_{B[U]}(z) \geq r_2$ and additionally $xz \notin E(B)$, then $z = x_2$, as $z \in \{y, x_2\}$ is necessary for $d_{B[U]}(z) \geq r_2$, but $xy \in E(B)$. Moreover, as $y \neq x_2$, we have $d_{B[U]}(y) \leq r_1$ before Maker's move. Maker now claims $xz = xx_2$. This way $x$ is removed from $U_1$, $z = x_2$ is removed from $U_2$, and $d_{B[U]}(y)$ decreases below $r_2$ since its neighbor $x$ in $B$ is removed from $U$. Again $\Delta_1 \leq r_1$ and $\Delta_2 < r_2$ follows.

**Case 4.** If neither of the three cases above happens, then we know that every vertex $v \in U_2$ with $d_{B[U]}(v) \geq r_2$ needs to satisfy $d_{B[U]}(v) = r_2$ (as Case 2 does not happen) and $xv \in E(B)$ (as Case 3 does not happen). Maker then takes an arbitrary free edge $xz \notin E(B)$ with $z \in U_2$, which is possible analogously to previous cases. As $x$ is removed from $U_1$, the degree of all vertices $v \in U_2$, which satisfied $d_{B[U]}(v) = r_2$, decreases. Thus, $\Delta_1 \leq r_1$ and $\Delta_2 < r_2$ follows again. □

Thus, we see that Maker can follow Stage I, and that she enters Stage II after at most $\frac{r_1 n+1}{r_1+1}+2$ rounds. At this point then, we have $\Delta_1 + \Delta_2 \leq r_1 + r_2 - 1$ immediately before Breaker's previous move, while $|U_1| = |U_2|$ is large, provided that $n$ is large enough. Thus, by the induction hypothesis, Maker has a strategy to occupy a perfect matching of $K_{n,n}[U]$ within $\frac{|U|}{2} + 2$ moves. □

## 4.3 Hamilton paths with a fixed endpoint

In order to prove Theorem 1.3.4 in the following section, we at first want to show that Maker has a strategy to create a Hamilton path with some designated vertex as an endpoint in optimal time. Our strategy is motivated by the proof of Theorem 1.4 in [28], and thus it starts by creating a perfect matching.

**Lemma 4.3.1** *Let $r \in \mathbb{N}$ be large enough, then for ever large enough integer $n$ (depending on $r$) the following holds. Let $G$ be a graph with $n$ vertices and at least $\binom{n}{2} - n + r$ edges, then, in an unbiased Maker-Breaker game on $E(G)$, Maker has a strategy to occupy a perfect matching of $G$ within $\frac{n}{2} + 1$ moves.*

**Proof** As in the proof of Lemma 4.1.2, we will assume that the game is played on the graph $K_n$ and that the edges of $K_n \setminus G$ already belong to Breaker's graph $B$ before the game starts. In the first part of the game, Maker will create a large matching. We then let $U = \{u \in V(K_n) : d_M(u) = 0\}$ denote the set of those vertices of $K_n$ which are untouched, i.e. not saturated by Maker's partial matching. Maker's goal is to decrease the number of Breaker's edges inside $U$, by choosing edges for her matching that are adjacent with many (at least 3) Breaker's edges inside $U$. To make it precise, for every free edge $e \in K_n[U]$, we define its *danger* as $dang(e) := |\{f \in E(B[U]) : e \cap f \neq \emptyset\}|$.

In the following we give a strategy for Maker. As usual, we then prove that Maker can follow this strategy, and while doing so, she creates a perfect matching within $\frac{n}{2} + 1$ moves.

**Stage I.** Throughout Stage I Maker occupies the edges of a matching of $K_n \setminus B$ more or less greedily, while caring about the danger values of the free edges. If there exists a free edge $e \in K_n[U]$ with $dang(e) \geq 3$, then Maker claims an arbitrary such edge. She then repeats Stage I. Otherwise, if such an edge does not exist, Maker proceeds to Stage II.

**Stage II.** Let $U$ be the set of vertices that are not saturated by Maker's matching when Maker enters Stage II. Then in this stage, Maker occupies a perfect matching of $K_n[U]$ within $\frac{|U|}{2} + 1$ rounds.

Obviously, if Maker can follow the proposed strategy, she occupies a perfect matching as required. So, it just remains to show that Maker can always follow the proposed strategy.

**Stage I.** For this stage, there is nothing to prove.

**Stage II.** Before we show that Maker can follow the strategy for this stage, we again start with some useful observations first.

**Claim 4.3.2** *As long as Maker follows the strategy of Stage I, the value of* $e(B[U])$ *decreases by at least two in each round.*

**Proof** If Maker follows the proposed strategy, then she claims an edge $e \in K_n[U]$ with danger value $dang(e) \geq 3$, and $e(B[U])$ decreases by this danger value. As Breaker increases $e(B[U])$ by at most 1 in each of his moves, the claim is proven. □

From this claim it now follows that Stage I can last at most $(n-r)/2$ rounds, as $e(B[U]) \leq n-r$ before the game starts, by assumption of the lemma. Moreover, one concludes inductively that $e(B[U]) \leq v(B[U]) - 2$ is maintained, as long as Maker can follow Stage I. Indeed, this inequality holds immediately after Breaker's first move, as at this moment we have $e(B[U]) \leq n - r + 1 \leq n - 2$, by the assumption of the lemma, for large enough $r$. Moreover, the inequality is maintained afterwards, as $v(B[U])$ decreases by 2 in each round of Stage I, while $e(B[U])$ decreases by at least 2 (where Maker first decreases $e(B[U])$ by at least 3, before Breaker may increase $e(B[U])$ by 1).

So, we see that, when Maker enters Stage II, $|U| \geq r$ (where $r$ is chosen to be large enough) and $e(B[U]) \leq v(B[U]) - 2$ hold. At this moment then, i.e. immediately before Maker's first move in Stage II, we additionally have $dang(e) \leq 2$ for every free edge $e \in K_n[U]$, as otherwise Maker would continue with Stage I. As $e(B[U]) \leq v(B[U]) - 2$ still holds, every vertex of $u \in U$ needs to be incident with a free edge $e_u$ in $K_n[U]$, and thus it follows that $d_{B[U]}(u) \leq 2$ for every $u \in U$, as otherwise $dang(e_u) \geq 3$, a contradiction.

From this, we can conclude that we can find a partition $U = U_1 \cup U_2$ with $|U_1| = |U_2|$ such that $e_{B[U]}(U_1, U_2) \leq 1$. Indeed, if $B[U]$ is a subset of a perfect matching of $K_n[U]$, represented by edges $x_1y_1, \ldots, x_{\frac{|U|}{2}}y_{\frac{|U|}{2}}$, we then just set $U_1 = \{x_1, \ldots, x_{\lceil \frac{|U|}{4} \rceil}, y_1, \ldots, y_{\lfloor \frac{|U|}{4} \rfloor}\}$ and $U_2 = U \setminus U_1$, giving a partition as required. Otherwise, there is a vertex $u \in U$ with $d_{B[U]}(u) = 2$, where we may assume that $uv, uw \in E(B[U])$. Then, we choose $U_1 \subseteq U \setminus \{u, v, w\}$ arbitrarily of size $\frac{|U|}{2}$, and $U_2 = U \setminus U_1$, and we observe that $e_{B[U]}(U_1, U_2) = 0$. Indeed, otherwise we would have an edge $xy \in E(B)$ with $x \in U_1$ and $y \in U_2$. But then, as $d_{B[U]}(u) = 2$, we would know that $e = ux$ is a free edge, and moreover $dang(e) \geq 3$, a contradiction.

Finally, for Stage II, Maker just claims a perfect matching between $U_1$ and $U_2$, by following the strategy given for Theorem 4.2.8. This she can do, as $r$ is assumed to be large enough, and as $e_{B[U]}(U_1, U_2) \leq 1$, immediately before she makes her first move in Stage II. She just pretends to be the second player, in case $e_{B[U]}(U_1, U_2) = 1$, and occupies a perfect matching between $U_1$ and $U_2$, within $|U_1| + 1 = \frac{|U|}{2} + 1$ rounds. □

Starting from a perfect matching as described above, our goal now is to create a Hamilton path rapidly with one designated vertex being one of its endpoints. We show the following lemma, whose statement should remind of Lemma 4.1.2.

**Lemma 4.3.3** *Let $k$ be a positive integer. Then there exists an integer $m' = m'(k)$ such that the following holds. If $G = (V, E)$ is a graph on $m \geq m'$ vertices and $e(G) \geq \binom{m}{2} - k$ edges, and if $v_1 \in V$, then in an unbiased game on $E(G)$, Maker has a strategy to occupy a Hamilton path $P$ of $G$, within $m - 1$ moves, such that $v_1 \in End(P)$.*

**Proof** As in previous proofs, we will assume that the game is played on the graph $K_m$ and that the edges of $K_m \setminus G$ already belong to Breaker's graph $B$ before the game starts. Whenever necessary, we will assume $m$ to be large enough. Moreover, we may assume that $m$ is odd. Otherwise, in case $m$ is even, Maker for her first move just claims an arbitrary free edge $v_1v_1'$ which is incident with the designated vertex $v_1$. She then reduces the board by deleting $v_1$, and considers $v_1'$ to be the new designated endpoint.

The main idea of Maker's strategy is to start with a matching and then to connect its edges step by step, until we create a Hamilton path. To shorten the notation we write $P \circ p_tq_1 \circ Q$ for the path $(p_1 \ldots p_sq_1 \ldots q_t)$, which is obtained by connecting two vertex-disjoint paths $P = (p_1 \ldots p_s)$ and $Q = (q_1 \ldots q_t)$ through the edge $p_sq_1$. As we need to care about the designated vertex $v_1$ especially, we will allow one path to consist only of one vertex. Initially, this will be the path consisting only of $v_1$; but as we may connect $v_1$ to some path, we may replace $v_1$ by the other endpoint $v_1'$ of the resulting path $P'$, similar to the argument above.

Throughout the strategy, Maker thus will maintain a collection $\mathcal{P} = \{P_0, P_1, \ldots, P_\ell\}$ of vertex-disjoint paths, where $P_0 = \{p_0\}$ consists only of one vertex, and where $v(P_i) \geq 2$ for every other path. As Maker aims to connect the paths at their endpoints, we set $End_{\mathcal{P}} = \bigcup_{i=0}^{\ell} End(P_i)$, where $End(P_0) = \{p_0\}$, for the set of all endpoints of paths in $\mathcal{P}$, and we let $X_{\mathcal{P}}$ denote the set of those edges that connect two endpoints of different paths in $\mathcal{P}$. Formally, $X_{\mathcal{P}} = \{uv |\ u \in End(P_i) \text{ and } v \in End(P_j) \text{ for some } 1 \leq i < j \leq \ell\}$. Moreover, we let $X_B$ denote the set of edges in $X_{\mathcal{P}}$ that belong to Breaker's graphs. Maker then is mainly interested in claiming edges from $X_{\mathcal{P}} \setminus X_B$; these edges will be called *good* edges. Moreover, similar to the previous proof for the perfect matching game, we define a *danger* for every good edge $e$, setting $dang(e) := |\{f \in X_B : e \cap f \neq \emptyset\}|$.

In the following we describe a strategy for Maker. Then, we show that Maker can follow this strategy, and that, by following this strategy, she occupies a Hamilton path as required.

**Stage I.** Applying the strategy from the perfect matching game, Maker occupies a collection $\mathcal{P} = \{P_0, P_1, \ldots, P_{\frac{m-3}{2}}\}$ of vertex-disjoint paths, such that $V(P_0) = \{v_1\}$, $e(P_1) = 3$ and $e(P_i) = 1$ for all $2 \leq i \leq \frac{m-3}{2}$, and $\bigcup_{i=0}^{(m-3)/2} V(P_i) = V(K_m)$. This stage lasts exactly $\frac{m+1}{2}$ rounds. Afterwards, Maker proceeds to Stage II.

**Stage II.** Let $\mathcal{P} = \{P_0, P_1, \ldots, P_\ell\}$, $\ell = \frac{m-3}{2}$, be the collection of paths that Maker occupied in Stage I, and let $P_0 = \{p_0\}$ be the unique path consisting of one vertex, where we have $p_0 = v_1$. In each of the following rounds, as long as possible, Maker connects her paths by claiming good edges which have a large danger value (where large means at least 3). So, in each round of Stage II, she looks for a good edge $e = uv \in X_{\mathcal{P}} \setminus X_B$ with $dang(e) \geq 3$. In case such an edge does not exist for the first time, Maker stops playing according to Stage II and proceeds to Stage III. Otherwise, Maker claims such an edge $uv$ arbitrarily and updates her family $\mathcal{P}$ as follows. Let $u \in End(P_i)$ and $v \in End(P_j)$ with $i < j$. If $i \neq 0$, then she removes $P_j$ from $\mathcal{P}$, and updates $P_i := P_i \circ uv \circ P_j$. Otherwise, if $i = 0$, then she removes $P_j$ from $\mathcal{P}$, and updates $P_0 = \{p_0\} = \{v'\}$, where $v' \in End(P_j) \setminus \{v\}$. In every case, she updates $X_{\mathcal{P}}$ and $X_B$ accordingly.

**Stage III.** Maker now aims to ensure that $X_B$ forms a matching. If this is the case immediately before her move, then she proceeds to Stage IV. Otherwise, there is a vertex $v \in End_{\mathcal{P}}$ which is incident with at least two edges of $X_B$. Maker then chooses such a vertex $v$ and claims a good edge $vw \in X_{\mathcal{P}} \setminus X_B$, where the choice of $vw$ is made more explicit later in the strategy discussion. Afterwards, Maker updates $\mathcal{P}$ and $X_{\mathcal{P}}$ as described in Stage II, and then she repeats Stage III. Later, when the choice of $vw$ is made more precise, we will see that

Maker needs at most two rounds in order to ensure that $X_B$ is a matching.

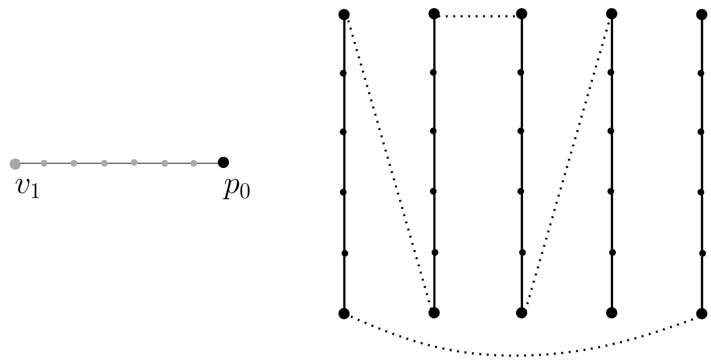


Figure 4.4: The shape of $X_B$, represented by the dotted lines, during Stage IV.

**Stage IV.** If $|\mathcal{P}| = 3$, then Maker proceeds to Stage V. Otherwise, if $|\mathcal{P}| > 3$, then Maker claims a good edge $uv \in X_{\mathcal{P}} \setminus X_B$ such that, after the same update for $\mathcal{P}$ and $X_{\mathcal{P}}$ as in Stage II, the following property holds: $X_B$ forms a matching and none of the edges in $X_B$ is incident with the special vertex $p_0$. (An illustration for this property is given in Figure 4.4. The precise choice of $uv$ will be explained later in the strategy discussion.) Then, Maker repeats Stage IV.

**Stage V.** Maker enters this stage when $|\mathcal{P}| = 3$. Within two more rounds, she connects her three paths to a Hamilton path $P$ with $v_1 \in End(P)$.

Obviously, if Maker can follow the strategy, then she creates a Hamilton path $P$ such that $v_1 \in End(P)$. As she never closes a cycle, she does so within $m - 1$ moves. Therefore, it remains to prove that Maker can always follow the proposed strategy.

**Stage I.** To follow this stage of her strategy, Maker just plays according to the strategy which is given by Lemma 4.3.1 on the graph $K_m - v_1$. This way she occupies paths $P_1, \ldots, P_{\frac{m-3}{2}}$ as required, and sets $P_0 = \{v_1\}$. (Just notice that, in case Maker creates a perfect matching of $K_m - v_1$ without wasting a move, then she just claims an arbitrary further edge.)

**Stage II.** For this stage there is nothing to prove, because if Maker wants to claim an edge $e$ in this stage, then its existence is given by the assumption of this stage.

**Stage III.** Before we can show that Maker also can follow this part of the strategy, we observe the following useful statements about Stage II, analogously to the proof of Lemma 4.3.1.

**Claim 4.3.4** *As long as Maker can follow the strategy of Stage II, $|X_B|$ decreases by at least two in each round.*

**Proof** The proof is analogous to the proof of Claim 4.3.2. □

**Claim 4.3.5** *Let $m$ be large enough. Then, $|X_B| \leq |End_{\mathcal{P}}| - 3$ is maintained throughout Stage II.*

**Proof** The above claim follows by induction on the number of rounds. At the beginning of Stage II, the collection $\mathcal{P}$ consists of $\frac{m-1}{2}$ paths, among which exactly one consists of exactly one vertex. Thus, we have $|End_{\mathcal{P}}| = m-2$, $|X_B| \leq \frac{m+1}{2} + 1 + k < \frac{2m}{3}$ and $|X_B| \leq |End_{\mathcal{P}}| - 3$ before Maker's first move in Stage II. Finally, the induction step follows be Claim 4.3.4. Indeed, from now on $|End_{\mathcal{P}}|$ decreases by 2 within a sequence of consecutive moves of Maker and Breaker, while $|X_B|$ decreases by at least 2 (where Maker starts with decreasing this value by at least 3). □

**Claim 4.3.6** *Let $m$ be large enough. Stage II lasts less than $\frac{m}{3}$ rounds. In particular, we have $|End_{\mathcal{P}}| \geq \frac{m}{3} - 2$ throughout Stage II.*

**Proof** As seen in the previous proof, we have $|X_B| < \frac{2m}{3}$, immediately before Maker's first move in Stage II. Thus, by Claim 4.3.4, the bound on the number of rounds follows. Moreover, as $|End_{\mathcal{P}}| = m - 2$ holds when Maker enters Stage II, while $|End_{\mathcal{P}}|$ decreases by 2 in each round, the bound on $|End_{\mathcal{P}}|$ follows. □

As next, let us study the structure of Breaker's graph at the beginning of Stage III.

**Claim 4.3.7** *Set $H_B = (End_{\mathcal{P}}, X_B)$ immediately before Maker's first move in Stage III. Then $\Delta(H_B) \leq 2$. Moreover, in case $\Delta(H_B) = 2$ holds, $H_B$ is a subgraph of a copy of $K_3$, or $H_B$ is a subgraph of a copy of $C_4$ whose vertex set is given by the endpoints of two distinct paths $P_i, P_j \in \mathcal{P}$.*

**Proof** We first prove that the maximum degree of $H_B$ is at most 2. Indeed, for contradiction, assume that there is a vertex $v \in End_{\mathcal{P}}$ with $d_{H_B}(v) \geq 3$. By Claim 4.3.5 we know that there needs to exist a good edge $e$, which is incident with $v$. However, this edge then satisfies $dang(e) \geq d_{H_B}(v) \geq 3$, in contradiction to the fact that Maker stopped playing according to Stage II.

Now, assume that $\Delta(H_B) = 2$, i.e. there exist vertices $u, v_1, v_2 \in End_{\mathcal{P}}$ with $uv_1, uv_2 \in X_B$. In case $u \neq p_0$, let $u'$ be the other endpoint of the path $P_i$ with $u \in End(P_i)$. In case $u = p_0$, set $u' = u$. Then, for every other endpoint $w \in End_{\mathcal{P}} \setminus \{u, v_1, v_2, u'\}$, we have $d_{H_B}(w) = 0$, since otherwise $e = uw$ would be a good edge with $dang(e) \geq 3$, in contradiction to the fact that Maker stopped playing according to Stage II. In case $u = u'$ or in case $d_{H_B}(u') = 0$ holds, we immediately obtain $X_B \subseteq \{uv_1, uv_2, v_1v_2\}$. Otherwise, $u \neq u'$ and

$\emptyset \neq N_{H_B}(u') \subseteq \{v_1, v_2\}$. W.l.o.g. let $u'v_1 \in X_B$. Then $v_1v_2 \notin X_B$, as $d_{H_B}(v_1) \leq \Delta(H_B) \leq 2$, and $v_1v_2 \notin X_{\mathcal{P}} \setminus X_B$, since otherwise $e = v_1v_2$ would be a good edge with $dang(e) \geq 3$, which would be a contradiction as before. It follows that $v_1$ and $v_2$ need to be endpoints of the same path in $\mathcal{P}$, and $X_B \subseteq \{uv_1, uv_2, u'v_1, u'v_2\}$. □

Using the above claims, we now have everything what we need to describe how Maker should claim her edges in Stage III, and to prove that Maker can follow the strategy of this stage.

In case $X_B$ forms a matching when Maker enters Stage III, then there is nothing to prove, as she proceeds to Stage IV immediately. Otherwise, Claim 4.3.7 tells us that $H_B = (End_{\mathcal{P}}, X_B)$ is a subgraph of a copy of $K_3$, or a subgraph of a copy of $C_4$ living on the endpoints of two distinct paths $P_i$ and $P_j$ in $\mathcal{P}$.

Assume first that $H_B$ is a subgraph of a copy of $K_3$, and thus $\{uv, uw\} \subseteq X_B \subseteq \{uv, uw, vw\}$ for some endpoints $u, v, w \in End_{\mathcal{P}}$. Then in her first move of Stage III, Maker claims an arbitrary good edge that is incident to $u$, which needs to exist as, by Claim 4.3.6, we have $|End_{\mathcal{P}}| \geq \frac{m}{3} - 2$ after Maker's last move in Stage II, while $d_{H_B}(u) = 2$. Let $xy$ be Breaker's next edge, and observe that $X_B \subseteq \{vw, xy\}$ holds immediately after his move. If $X_B$ now forms a matching, then Maker immediately proceeds with Stage IV, and thus there is nothing to prove. Otherwise, let us assume that $v = x$. Then, Maker as next claims an arbitrary good edge, which is incident with $v$, which is possible as $|End_{\mathcal{P}}| \geq \frac{m}{3} - 4$ still holds. No matter what Breaker does next, immediately after his move, $X_B$ will be a matching, and Maker thus proceeds with Stage IV.

Assume then that $H_B$ is a subgraph of a copy of $C_4$, whose vertices are the endpoints of two distinct paths $P_i$ and $P_j$ in $\mathcal{P}$. So, $X_B \subseteq \{uv, uv', u'v, u'v'\}$ and $End(P_i) = \{u, u'\}$, $End(P_j) = \{v, v'\}$. Then in a first move of Stage III, Maker claims an arbitrary good edge that is incident to $u$, which needs to exist analogously to the previous argument. Again, let $xy$ be Breaker's next edge and thus $X_B \subseteq \{u'v, u'v', xy\}$ immediately after he claimed that edge. If $X_B$ now forms a matching, then Maker immediately proceeds with Stage IV, and thus there is nothing to prove. Otherwise, we can assume that $X_B = \{u'v, u'v', xy\}$ and $x \notin \{u', v, v'\}$. In her second move of Stage III, Maker then claims the edge $u'x$ if $u' \neq y \in End_{\mathcal{P}}$, and otherwise she claims an arbitrary good edge which is incident with $u'$, and whose existence is guaranteed analogously to the previous argument. No matter what Breaker does next, immediately after his move $X_B$ will be a (maybe empty) matching, and Maker thus proceeds with Stage IV.

So, to summarize, in all cases Maker can follow the strategy of Stage III in such a way that after at most two rounds, $X_B$ forms a matching.

**Stage IV.** Observe first that, when Maker enters Stage IV, $|\mathcal{P}| \geq \frac{m}{6} - 3 \geq 4$ holds, as we have $|End_{\mathcal{P}}| \geq \frac{m}{3} - 6$, by Claim 4.3.6, and since Stage III lasts at most two rounds. Moreover, as Maker stopped playing according to Stage III, $X_B$ forms a matching. Again, let $H_B = (End_{\mathcal{P}}, X_B)$. If $d_{H_B}(p_0) = 0$, then for her first move in Stage IV, Maker claims an arbitrary good edge, which is not incident with $p_0$, and updates $\mathcal{P}$ accordingly (as in Stage II). Afterwards $X_B$ is still a matching which does not touch the vertex $p_0$. Otherwise, let $d_{H_B}(p_0) = 1$ with $p_0v \in X_B$ and $v \in End(P_i)$ for some $i \neq 0$. Then, Maker claims an arbitrary good edges $vw$, where $w \in End(P_j)$ for some $j \notin \{0, i\}$. This is possible, as $|\mathcal{P}| > 3$ and $d_{H_B}(v) = 1$. Afterwards, $X_B$ satisfies the required conditions of Stage IV, as $v$ is removed from $End$, and therefore $p_0v$ is removed from $X_B$.

Now, doing induction as long as $|\mathcal{P}| > 3$, assume that immediately before a Breaker's move in Stage IV, $X_B$ forms a matching which does not saturate $p_0$. Let $vv'$ be Breaker's next edge, where we may assume that $v \in End(P_i)$ for some $i \neq 0$. Then, Maker similarly claims an arbitrary good edge $vw$, where $w \in End(P_j)$ for some $j \notin \{0, 1\}$, which again is possible, as $|\mathcal{P}| > 3$ and $d_{H_B}(v) \leq 2$. Analogously, we then conclude that $X_B$ satisfies the conditions which are required for Stage IV. Thus, Maker can follow the strategy of Stage IV, until $|\mathcal{P}| = 3$.

**Stage V.** When Maker enters Stage V, her graph is a collection of three paths, one of which is the path $\{p_0\}$ consisting of one vertex. Let $P_1$ and $P_2$ be the two paths different from $\{p_0\}$, and let $\{v, v'\} = End(P_1)$ and $\{w, w'\} = End(P_2)$. Immediately after Maker's last move of Stage IV, we know that $X_B$ was a matching which did not saturate the vertex $p_0$, and so, w.l.o.g. let $X_B = \{vw, v'w'\}$ at that moment. Let $e$ be the edge that Breaker claimed next, i.e. before Maker's first move in Stage V. If $p_0 \in e$, we can assume that $e = p_0v$. Then Maker first claims $vw'$, and in her second move of Stage IV she takes $p_0v'$ or $p_0w$. Doing so, she finishes her Hamilton path as required. If $p_0 \notin e$, then we can still assume that w.l.o.g. Breaker claims a good edge, say $e = vw'$. Then Maker at first claims $v'w$, and afterwards she finishes her Hamilton path by claiming $p_0v$ or $p_0w'$. □

## 4.4 Optimal trees

In the following we prove **Theorem 1.3.4**.

Let $\Delta \in \mathbb{N}$ as given in the statement of Theorem 1.3.4. As in the proof of Theorem 1.3.2, we choose $k = (\Delta + 1)^4$ and $m_2 = m'(k) > (\Delta + 1)^2$, where $m'(k)$ satisfies Lemma 4.3.3. Now, let $T$ be a tree on $n$ vertices which satisfies the assumptions from Theorem 1.3.4. We aim to show that Maker has a strategy for occupying a copy of $T$ within $n - 1$ rounds. By assumption, there exists a bare path $P \subseteq T$ of length $m_2$ with $End(P) = \{v_1', w_1'\}$, such that $w_1'$ is a leaf of $T$. Now, let $T_1 := T - (V(P) \setminus \{v_1'\})$. Moreover, let $T_2$ be a tree consisting of exactly one vertex $v_2'$ that is not contained in $T_1$.

In a first stage Maker embeds $T_1 \cup T_2$ (with fixed vertices $v_1'$ and $v_2'$ from above) using the strategy guaranteed by Theorem 4.1.1, which is possible as $m_2 > (\Delta + 1)^2$. When she is done with this, her graph is isomorphic to $T_1 \cup T_2$ (plus isolated/available vertices). Moreover, according to Theorem 4.1.1, Breaker then claims at most $\binom{\Delta+1}{2}$ edges among the set $U \subseteq V(K_n)$, which consists of all available vertices plus those vertices $v_1$ and $v_2$ in $K_n$ that are the images of $v_1'$ and $v_2'$ with respect to Maker's embedding. Now, as a second step, Maker claims a copy of $P$, on the vertex set $U$ such that $v_1$ is one of its endpoints, while wasting no move. (Notice that $v_2$ still is isolated in Maker's graph and we do not require this vertex to be an endpoint of the copy of $P$.) As $|U| = m_2 + 1 > m'(k)$ and $k > \binom{\Delta+1}{2}$, Maker succeeds by Lemma 4.3.3. This way she occupies a copy of the goal tree $T$ within $n-1$ rounds, as she wastes no moves throughout the game. □

## 4.5 Random trees

In this chapter we finally aim to prove Theorem 1.3.5. Let $T$ be a tree chosen uniformly at random from the class of all labeled trees on $n$ vertices, denoted by $T \sim \mathcal{T}_n$, then we show that Maker a.a.s. can proceed as follows. Similarly to the proof of Theorem 1.3.3, Maker starts by embedding a tree $T_1 \subseteq T$ more or less greedily. This time, $T \setminus T_1$ consists of pairwise vertex-disjoint bare paths of certain length, each ending in a leaf of $T$. Afterwards, she then embeds those paths without wasting any edges.

Before we start proving Theorem 1.3.5, let us however collect some useful facts about random trees, the first being about their maximum degrees.

**Theorem 4.5.1 (Theorem 3 in [39])** *Let $T \sim \mathcal{T}_n$, then a.a.s. the maximum degree satisfies $\Delta(T) = (1 + o(1))\frac{\log n}{\log \log n}$.*

In [1], the authors study the appearance of certain subtrees in random trees. Let $T$ be some tree and assume that $uv$ is an edge in $T$. They define a rooted (undirected) tree $T^{(u,v)}$ by fixing $v$ as a root, in order to receive a direction of parenthood in $T$, and then removing $u$ and all its descendants from $T$. (An illustration is given in Figure 4.5.) For any given rooted tree $R$ they say that $T$ has an *R-leaf* if $R$ is isomorphic to $T^{(u,v)}$, shortly written as $R \cong T^{(u,v)}$, for some $uv \in E(T)$. Finally, they prove the following statement.

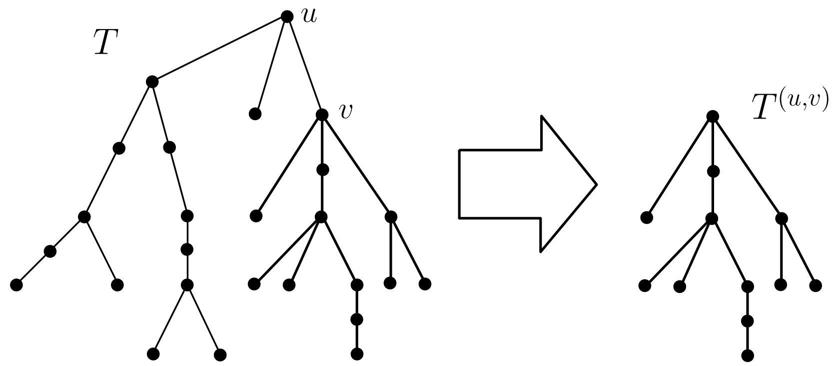


Figure 4.5: Illustration for the definition of $T^{(u,v)}$.

**Lemma 4.5.2 (Lemma 3 in [1])** *Let $R$ be a rooted tree. Then there exists a constant $c = c(R) > 0$ such that for $T \sim \mathcal{T}_n$ the following holds:*

$$Pr(\exists uv \in E(T): \; R \cong T^{(u,v)}) > 1 - \exp(-cn).$$

Thus, we know that a given rooted tree on a constant number of vertices appears a.a.s. as an $R$-leaf in a random tree $T \sim \mathcal{T}_n$. With a quick look on the proof of Lemma 3 in [1], one even verifies that any such rooted tree $R$ a.a.s. can be found a linear number of times as an $R$-leaf in $T$. Indeed, let $X$ be the number of pairs $(u, v)$ with $R \cong T^{(u,v)}$. Then, using the so-called Joyal mapping in order to study random functions on $[n]$ instead of random trees, they prove in [1] that $X$ is concentrated around its expectation, which is of size $\Theta(n)$. Thus, the following statement holds.

**Corollary 4.5.3** *Let $R$ be a rooted tree, then there is an $\varepsilon = \varepsilon(R) > 0$ such that a.a.s. there exist $\varepsilon n$ pairs $(u, v)$ with $uv \in E(T)$ and $\mathcal{T}_n^{(u,v)} \cong R$.*

From this corollary it finally follows that in $T \sim \mathcal{T}_n$ we a.a.s. can find many vertex-disjoint bare paths such that each path ends in a leaf of a $T$.

**Lemma 4.5.4** *Let $k$ be a positive integer, then there exists a real number $\varepsilon = \varepsilon(k) > 0$ such that the following holds. Let $T \sim \mathcal{T}_n$, then a.a.s. $T$ contains a family $\mathcal{P}$ of $\varepsilon n$ vertex-disjoint paths of length $k$ such that each of these paths is incident to a leaf of $T$.*

**Proof** Let $T \sim \mathcal{T}_n$ and let $R$ be a path with $k+1$ edges, rooted at one of its endpoints. Then by Corollary 4.5.3 we a.a.s. find $\varepsilon n$ pairs $(u, v)$ with $uv \in E(T)$ and $T_n^{(u,v)} \cong R$, which means that a.a.s. $T$ contains $\varepsilon n$ bare paths of length $k+1$, each ending in a leaf of $T$. Now, these

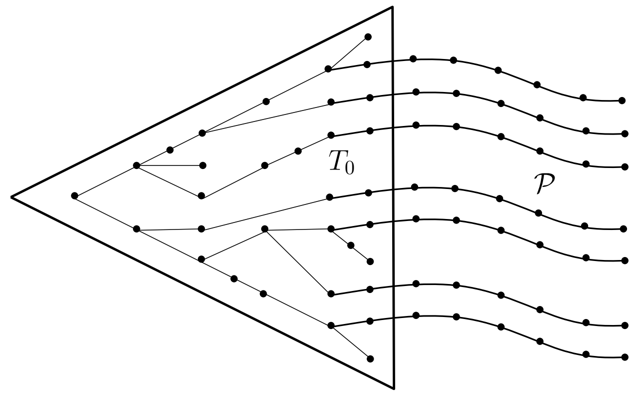


Figure 4.6: Splitting of $T$ into a subtree $T_0$ and a large family of bare paths.

paths can only intersect in their endpoints, which are non-leaf vertices of $T$. Forgetting about these vertices, we therefore find $\varepsilon n$ bare paths of length $k$ satisfying the required properties. □

In the proof of Theorem 1.3.5 we will condition on the property presented in Lemma 4.5.4. Similarly to previous proofs, Maker starts by embedding the random tree $T$ besides all the $\varepsilon n$ bare paths (or at least most of them). Once this is done, we aim to embed all the bare paths without wasting any move. To succeed, we do the following. We first split the set of available and open vertices into $\varepsilon n$ subsets of size $k+1$ which are independent in Breaker's graph, so that Maker then can care about the bare paths by playing on different boards separately. The following lemma will help us to do so.

**Lemma 4.5.5** *Let $H = (V, E)$ be a graph with a partition of its vertex set $V = U_1 \cup U_2$, and let $k \in \mathbb{N}$ be such that $|U_2| = k|U_1|$ and $\Delta(H) \leq \min\{\frac{|U_1|}{2k}, |U_1| - 1\}$. Then there exists a partition $V(H) = V_1 \cup \ldots \cup V_{|U_1|}$ such that $|U_1 \cap V_i| = 1$, $|U_2 \cap V_i| = k$ and $E(H[V_i]) = \emptyset$ for every $1 \leq i \leq |U_1|$.*

The proof will follow by a standard application of Hall's Theorem (see Theorem 4.5.6) on the existence of perfect matchings in bipartite graphs and the theorem of Hajnal and Szemerédi (see Theorem 4.5.7).

**Theorem 4.5.6 (Theorem 3.1.11 in [43])** *Let $G = (A \cup B, E)$ be a bipartite graph with vertex parts $A$ and $B$ of equal size. Then $G$ contains a perfect matching if and only if $|N_G(S)| \geq |S|$ for every $S \subseteq A$.*

**Theorem 4.5.7 (Theorem 1 in [25])** *Let $G$ be a graph on $n$ vertices and let $\Delta(G) \leq r-1$ for some positive integer $r$. Then there exists a partition of $V(G)$ into $r$ independent sets, each being of size $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$.*

**Proof of Lemma 4.5.5** Let $H$ be given according to the lemma. By the theorem of Hajnal and Szemerédi (Theorem 4.5.7) we find a partition $U_2 = W_1 \cup \ldots \cup W_{|U_1|}$ into independent sets of $H[U_2]$ of size $|W_i| = \frac{|U_2|}{|U_1|} = k$, as $\Delta(H[U_2]) \leq |U_1| - 1$. To each of the parts $W_i$, we now want to add one vertex from $U_1$ to obtain sets $V_i$ as required. To do so, let $H' = (W \cup U_1, E')$ be the bipartite graph with partite sets $W := \{W_1, \ldots, W_{|U_1|}\}$ and $U_1 := \{u_1, \ldots, u_{|U_1|}\}$, where we put an edge between $W_i$ and $u_j$ if and only if $d_H(u_j, W_i) = 0$ (which means that we could add $w_j$ to $U_i$). By assumption on $H$, we then have $\delta(H') \geq \frac{|U_1|}{2}$, and thus by Hall's Theorem (Theorem 4.5.6) it follows that $H'$ contains a perfect matching. Indeed, let $S \subseteq U_1$ be an arbitrary subset of $U_1$. If $|S| \leq \frac{|U_1|}{2}$, then we have $|N_{H'}(S)| \geq \delta(H') \geq |S|$, and if $|S| > \frac{|U_1|}{2}$, then every $W_i \in W$ has a neighbor in $S$, i.e. $|N_{H'}(S)| = |W| \geq |S|$. W.l.o.g. let $\{u_i W_i : 1 \leq i \leq |U_1|\}$ be a perfect matching in $H'$. Then the partition $V(H) = V_1 \cup \ldots \cup V_{|U_1|}$, with $V_i = W_i \cup \{u_i\}$ for every $1 \leq i \leq |U_1|$, satisfies the required properties. □

Now we have everything that we need to prove Theorem 1.3.5.

**Proof of Theorem 1.3.5** Fix $t = m'(1)$, where $m'(1)$ is given according to Lemma 4.3.3 and, whenever necessary, assume that $n$ is large enough. Let $T = (V, E) \sim \mathcal{T}_n$. We condition on the properties which a.a.s. hold according to Theorem 4.5.1 and Lemma 4.5.4. That is, we assume from now on that $\Delta(T) = (1 + o(1)) \frac{\log n}{\log \log n}$ and that $T$ contains a family $\mathcal{P}$ of $\varepsilon n$ bare paths of length $t$ that are pairwise disjoint, and such that each of them ends in a leaf of $T$. For every such path $P \in \mathcal{P}$, let $End(P) = \{v_1^P, v_2^P\}$, where $v_2^P$ is a leaf of $T$. Moreover, set $V_{\mathcal{P}} := \bigcup_{P \in \mathcal{P}} \left(V(P) \setminus \{v_1^P\}\right)$.

As mentioned earlier, Maker at first aims to embed the tree $T$ besides all the edges that belong to (most of) the paths from $\mathcal{P}$. That is, she first focuses on the subtree $T_0 := T[V \setminus V_{\mathcal{P}}]$ with $|V(T) \setminus V(T_0)| = t\varepsilon n$. For her embedding, Maker proceeds as follows. Similar to the proof of Theorem 4.2.2, Maker starts from some vertex $v_0'$ in $T_0$, and then she embeds the tree $T_0$ (plus maybe some further edges of $T$) step by step, towards the leaves (by maintaining an embedding $\phi : S \to V(M)$ of a subtree $T[S]$ into $M$, with $S \subseteq V(T)$). From time to time it may happen that she needs to embed some vertex from $V(T) \setminus V(T_0)$ to be able to keep control on the distribution of Breaker's edges. These vertices will then belong to $V(T_1) \setminus V(T_0)$. Again, we denote with $S$ the set of vertices that are already embedded. So, initially $S = \{v_0'\}$. Moreover, with $\phi$ we will denote Maker's embedding of $T[S]$ into $M$, as indicated above. Initially, we set $\phi(v_0') = v_0$ for an arbitrary vertex $v_0 \in V(K_n)$. Moreover, $A = V(K_n) \setminus \phi(S)$ will denote the set of available vertices, i.e. those vertices which were not chosen for the embedding so far.

In the following we give a strategy for Maker. We then show that she can follow that strategy and moreover, while doing so, she creates a copy of $T$ within $n - 1$ moves.

**Stage I.** Maker creates an embedding $\phi : T[S] \to M$ of a tree $T_1 = T[S]$ with $T_0 \subseteq T_1 \subseteq T$, within $|V(T_1)| - 1$ rounds, such that $T_1$ contains only a small number of edges from $\mathcal{P}$ and such that all Breaker's degrees among the remaining open and available vertices are not too large. To be precise, Maker maintains that $d_B(v) \leq \sqrt{n}\log(n)$ for every vertex $v \in A \cup \phi(\mathcal{O}_T)$, and $|S \cap V_{\mathcal{P}}| \leq \sqrt{n}\log(n)$.

**Stage II.** In Stage I, Maker may have embedded some vertices of some paths from $\mathcal{P}$. Denote the family of these paths by $\mathcal{P}^*$. In Stage II, Maker now completes the embedding of every path $P \in \mathcal{P}^*$, therefore creating a copy of some tree $T_2$ with $T_1 \subseteq T_2 \subseteq T$. The precise details of how she can do this, will be given later in the strategy discussion. Once she is done with this, she proceeds with Stage III.

**Stage III.** Maker embeds the remaining paths of $\mathcal{P}$ in order to complete her embedding of $T$. She does so without wasting any move. The precise details of how she can do this, will be given later in the strategy discussion.

Obviously, if Maker can follow this strategy, then she creates a copy of $T$ as required. Thus, it remains to show that Maker can indeed follow the strategy.

**Stage I.** The argument for Stage I is similar to the proof of Theorem 4.2.2. This time, we have $\Delta(T) = (1 + o(1))\frac{\log n}{\log\log n}$, and $T \setminus T_0$ consists of pairwise vertex-disjoint bare paths. We define a vertex $v$ to be *dangerous* if $d_B(v) \geq \sqrt{n}$ and $v$ is either open or available, and again we let *Dang* denote the set of dangerous vertices. Now, Maker plays according to the same strategy as given in the proof of Theorem 4.2.2. We then obtain the following claims, analogously to Claim 4.2.4 – Claim 4.2.7.

**Claim 4.5.8** *Until the moment when Maker stops playing according to the strategy of Stage I, at most $2\sqrt{n}$ vertices become dangerous.*

**Claim 4.5.9** *Until the moment when Maker stops playing according to the strategy of Stage I, at most $\sqrt{n}\log(n)$ vertices of $V_{\mathcal{P}}$ are embedded, provided $n$ is large enough.*

**Proof** A vertex $w' \in V_{\mathcal{P}}$ is only embedded, when Maker plays according to Case 1 of the strategy (given for Theorem 4.2.2), i.e. when *Dang* $\neq \emptyset$. Now, for each vertex $v$, which is dangerous at some point during the game, we consider each of the four subcases Case 1(i), Case 1(ii) (a) – Case 1(ii) (c) at most once, and in each of these four subcases, Maker can embed at most three vertices of $V_{\mathcal{P}}$. Thus, using Claim 4.5.8, the number of vertices in $V_{\mathcal{P}}$ that can be embedded throughout Stage I is of size $O(\sqrt{n})$, and so the claim follows, provided $n$ is large enough. □

**Claim 4.5.10** *Until the moment when Maker stops playing according to the strategy of Stage I, we have at least* $0.9\varepsilon n$ *available vertices, provided* $n$ *is large enough.*

**Proof** By Claim 4.5.9 and since $|V_{\mathcal{P}}| = |V(T) \setminus V(T_0)| = t\varepsilon n \geq \varepsilon n$, we know that at any moment throughout Stage I, $|V(T) \setminus S| \geq |V_{\mathcal{P}}| - |V_{\mathcal{P}} \cap S| \geq \varepsilon n - \sqrt{n}\log(n) > 0.9\varepsilon n$ holds. Thus, the claim follows. $\square$

**Claim 4.5.11** *Until the moment when Maker stops playing according to the strategy of Stage I, we have* $d_B(v) < \sqrt{n}\log(n)$ *for every vertex* $v \in A \cup \phi(\mathcal{O}_T)$.

**Proof** Analogously to the proof of Claim 4.2.6, we obtain

$$d_B(v) \leq \sqrt{n} + (\Delta + 3) \cdot 2\sqrt{n} < \sqrt{n}\log(n)$$

for every $v \in A \cup \phi(\mathcal{O}_T)$, for large $n$, by Claim 4.5.8 and $\Delta(T) = (1 + o(1))\frac{\log n}{\log\log n}$. $\square$

**Claim 4.5.12** *Assume that Maker plays according to Case 1(ii), then one of the assumptions from (a), (b) and (c) holds.*

**Proof** Assume that the statement is wrong. Then, analogously to the proof of Claim 4.2.7, we obtain a contradiction to Claim 4.5.10 with $|A| < \sqrt{n}\log(n) \cdot \Delta + 1 \cdot (\Delta^2 + \Delta) = o(n)$, provided $n$ is large enough. $\square$

Using all these claims, it is shown analogously to the proof of Theorem 4.2.2 that Maker can follow the strategy of Stage I. Moreover, for large enough $n$, $|S \cap V_{\mathcal{P}}| \leq \sqrt{n}\log(n)$ is guaranteed by Claim 4.5.9; and $d_B(v) \leq \sqrt{n}\log(n)$ for every vertex $v \in A \cup \phi(\mathcal{O}_T)$ holds by Claim 4.5.11.

**Stage II.** By Claim 4.5.9, $|\mathcal{P}^*| \leq \sqrt{n}\log(n)$ holds, while every path in $\mathcal{P}^*$ has length $t$. Now, Maker completes the embedding of the paths in $\mathcal{P}^*$ in the obvious way. As long as there is a path $P \in \mathcal{P}^*$ which is not fully embedded, i.e. $V(P) \setminus S \neq \emptyset$, she fixes such a path and proceeds as follows. Let $P = (p_0, \ldots, p_t)$ and assume that $i < t$ is the largest index with $p_i \in S$. Then for her move, she claims an arbitrary free edge $\phi(p_i)u$ with $u \in A$, removes $u$ from $A$, adds $p_{i+1}$ to $S$ and sets $\phi(p_{i+1}) = u$.

As Stage II will last at most $t \cdot |\mathcal{P}^*| = O(\sqrt{n}\log(n))$ rounds, this, together with Claim 4.5.11, guarantees that $d_B(u) = O(\sqrt{n}\log(n))$ for every $u \in A \cup \phi(\mathcal{O}_T)$ throughout this stage. Moreover, as $|A| \geq 0.9\varepsilon n$ holds at the beginning of Stage I, by Claim 4.5.10, we observe that Maker can always follow the proposed strategy.

**Stage III.** Let $U_1 = \phi(\mathcal{O}_T) = \{u_1, \dots, u_r\}$ and let $U_2 = A$ be the sets of open and available vertices, respectively, at the moment when Maker enters this stage. Then $|U_2| = t|U_1|$ and $r := |U_1| = |\mathcal{P} \setminus \mathcal{P}^*| \geq \varepsilon n - o(n) > 0.9\varepsilon n$. Moreover, by Claim 4.5.11 and since Stage II lasts at most $O(\sqrt{n}\log(n))$ rounds, $d_B(v) = O(\sqrt{n}\log n) < \frac{|U_1|}{2t}$ holds, for every vertex $v \in U_1 \cup U_2$, provided $n$ is large enough. Applying Lemma 4.5.5, we thus find a partition $A \cup \phi(\mathcal{O}_T) = W_1 \cup \ldots \cup W_r$ with $|W_i| = t+1$ such that $|W_i \cap \phi(\mathcal{O}_T)| = 1$ and $E(B[W_i]) = \emptyset$ for every $1 \leq i \leq r$. W.l.o.g assume that $u_i \in W_i$ for every $1 \leq i \leq r$. Then, by playing on the boards $K_n[W_i]$ separately, Maker now claims a Hamilton path on each $W_i$ with designated endpoint $u_i$, for every $1 \leq i \leq r$. Whenever Breaker claims an edge of $K_n[W_i]$ for some $1 \leq i \leq r$ with the property that Maker still does not occupy a Hamilton path of $K_n[W_i]$, then Maker claims an edge of $K_n[W_i]$ following the strategy given by Lemma 4.3.3. This is possible, as $|W_i| = t+1 > m'(1)$. Otherwise, if Maker already occupies a Hamilton path on $W_i$, then she applies this strategy on a different board $K_n[W_j]$, $j \neq i$, where she still does not occupy the required Hamilton path.

This way, Maker claims all Hamilton paths as required and without wasting any move. In particular, she claims a copy of $T$ within exactly $n-1$ rounds. □

## 4.6 A short discussion of the perfect matching game

In Theorem 1.3.3, we show that, for large enough $n$, Maker needs at most $n+1$ rounds in the tree embedding game, in order to occupy a copy of a given tree $T$ with maximum degree $\Delta$ which has no long bare path. So, in our strategy, Maker wastes up to two rounds. The reason for this waste is that, following Lemma 4.2.3, Maker wastes two edges when she aims to create a perfect matching of an almost complete bipartite graph.

As mentioned earlier there exist such trees, like the complete binary tree, which Maker cannot hope to create within $n-1$ rounds, if Breaker plays optimally. Still, we believe that for $\Delta \in \mathbb{N}$ and every large enough $n$, Maker can occupy every tree $T$ with $\Delta(T) \leq \Delta$ within at most $n$ rounds. So, it becomes natural to ask whether Lemma 4.2.3 could be improved in general. However, as shown in the following, Lemma 4.2.3 is best possible. Thus, in order to achieve an improvement from $n+1$ towards $n$, one may need to come up with a different proof idea.

**Lemma 4.6.1** *Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with $|V_1| = |V_2| = n$ and $d_G(v) = n-1$ for every $v \in V_1 \cup V_2$. Then, playing an unbiased Maker-Breaker game on $E(G)$, Breaker has a strategy to ensure that Maker needs at least $n+2$ rounds for occupying a perfect matching of $G$.*

**Proof** Consider $G$ as a subgraph of $K_{n,n}$. As Maker cannot claim the edges of $K_{n,n} \setminus G$, we may assume that the game is played on $K_{n,n}$ and that $B = K_{n,n} \setminus G$ before the game starts. So, before the game starts, Breaker's graph is a perfect matching of $K_{n,n}$. Moreover, we may assume that Maker is the first player, and, by the bias monotonicity of Maker-Breaker games, we may allow Breaker to claim at most one edge in each round, while Maker claims exactly one edge in each round.

Now, throughout the proof let $V = V_1 \cup V_2$, $U = \{u \in V : d_M(u) = 0\}$, and $U_i = V_i \cap U$ for both $i \in [2]$. In the following we describe a strategy for Breaker. Afterwards, we show that he can follow that strategy, and, by doing so, he prevents Maker from occupying a perfect matching of $K_{n,n}$ within $n+1$ rounds. The strategy consists of three stages.

**Stage I.** Before his move, Breaker checks whether Maker's graph is a matching of $K_{n,n}$. If this is not the case, then he proceeds with Stage II. Otherwise, if there exist vertices $u_1 \in U_1$ and $u_2 \in U_2$ such that $d_{B[U]}(u_i) = 0$ for both $i \in [2]$, then Breaker claims such an edge $u_1u_2$. If such vertices do not exist, then Breaker claims no edge and skips his move.

**Stage II.** Before his move, Breaker checks whether Maker's graph is the disjoint union of a matching and a copy $P$ of $P_3$ (path with three vertices). If this is not the case, he proceeds with Stage III. Otherwise, let $V(P) = \{x, y, z\}$ with $End(P) = \{x, z\} \subseteq V_i$ for some $i \in [2]$, and $y \in V_{3-i}$. Then, Breaker claims an edge such that immediately after his move the following holds.

(a) If $|U_{3-i}| > 1$, then there exist distinct vertices $x', z' \in U_{3-i}$ with $xx', zz' \in E(B)$.

(b) If $|U_{3-i}| = 1$, then $xv', zv' \in E(B)$ for the unique vertex $v' \in U_{3-i}$.

**Stage III.** In this stage, Breaker always checks whether $|U_1| = |U_2| = 1$ holds. If this is not the case, then he claims an arbitrary edge. Otherwise, when $|U_1| = |U_2| = 1$, he makes sure that he claims the unique edge between $U_1$ and $U_2$.

In the following we show that Breaker can always follow the strategy above. Moreover, we show that, in case Breaker follows this strategy, Maker cannot occupy a perfect matching of $K_{n,n}$ within $n+1$ rounds.

**Stage I.** For this stage, there is nothing to prove.

**Stage II.** Before we consider Stage II, let us prove some useful claims.

**Claim 4.6.2** *After each move of Breaker in Stage I, the graph $B[U]$ is a perfect matching in $K_{n,n}[U]$.*

**Proof** The proof follows by induction on the number of Breaker's moves $t$. Before the game starts, the claim is true, by our assumption. So, assume that Breaker follows Stage I in round $t$, and that before Maker's $t^{\text{th}}$ move the claim was true. Let $x_1x_2$ be the edge that Maker claims in round $t$, with $x_i \in V_i$ for $i \in [2]$. As Breaker still follows Stage I, we know that $x_i \in U_i$ for both $i \in [2]$, before Maker's $t^{\text{th}}$ move. In particular, before this move of Maker, we find vertices $x_1' \in U_2$ and $x_2' \in U_1$ such that $x_1x_1', x_2x_2' \in E(B[U])$, by the induction hypothesis. Again by this hypothesis, we know that after Maker's $t^{\text{th}}$ move, when $x_1$ and $x_2$ are removed from $U_1$ and $U_2$, respectively, Breaker's graph $B[U]$ is a matching which saturates all vertices of $U$ besides $x_1'$ and $x_2'$. According to his strategy, he then claims $x_1'x_2'$. Afterwards, the graphs $B[U]$ is a perfect matching of $K_{n,n}[U]$ again. □

**Claim 4.6.3** *If Breaker plays according to the proposed strategy, then Maker cannot avoid to create a copy of $P_3$, before claiming a perfect matching.*

**Proof** For contradiction, assume that Maker could avoid copies of $P_3$. Then, this would mean that throughout the game her graph is a matching of $K_{n,n}$, until she finally reaches a perfect matching in round $n$. In particular, Breaker would always play according to Stage I. But then, after Maker's $(n-1)^{\text{st}}$ move, we must have $|U_1| = |U_2| = 1$. According to Breaker's strategy, Breaker then ensures that after his $(n-1)^{\text{st}}$ move we have $u_1u_2 \in E(B)$, where $U_i = \{u_i\}$ for $i \in [2]$. However, this is in contradiction to the fact that Maker finishes a perfect matching in round $n$. □

So, from the last claim we conclude that Breaker will enter Stage II at some point during the game. In case Maker's graph is not a disjoint union of a matching and a copy of $P_3$ there is nothing to prove for this stage. Otherwise, Maker's graph is of this shape with some copy $P$ of $P_3$ as described in the strategy, i.e. $V(P) = \{x, y, z\}$ and $End(P) = \{x, z\} \subseteq V_i$ for some $i \in [2]$. We then have the following.

**Claim 4.6.4** *As long as Breaker plays according to Stage II, he can always claim an edge to maintain the Properties (a) and (b).*

**Proof** The claim above follows by induction on the number of rounds.

Assume that Breaker enters Stage II in round $r$. Then, after his $(r-1)^{\text{st}}$ move, we know that Maker's edges form a perfect matching of $K_{n,n}[V \setminus U]$, while $B[U]$ is a perfect matching of $K_{n,n}[U]$. In round $r$, Maker then creates $P$, and $|U_{3-i}| = |U_i| + 1 \geq 1$ holds after Maker's $r^{\text{th}}$ move. W.l.o.g. let $xy$ be the edge which she claims in round $r$. Then $yz \in E(M)$ before round $r$. Moreover, $x \in U_i$ before round $r$, as otherwise Maker's graph would contain a copy of $P_4$, and thus Breaker would proceed to Stage III. In particular, by Claim 4.6.2, there exists a vertex $x' \in U_{3-i}$ with $xx' \in E(B)$ at the end of round $r-1$. Now, if $|U_{3-i}| = 1$ immediately before Breaker's $t^{\text{th}}$ move, then Breaker claims $zx'$; and if $|U_{3-i}| > 1$, then Breaker claims $zz'$ for some $z' \in U_{3-i} \setminus \{x'\}$. This way, the Properties (a) and (b) are maintained.

Assume then that after Breaker's $t^{\text{th}}$ move, Properties (a) and (b) hold, and assume that he plays in round $t+1$ according to Stage II. Let $V(P) = \{x, y, z\}$ as before. As Breaker plays in round $t+1$ according to Stage II, we must have $|U_{3-i}| \geq 1$ immediately before his $(t+1)^{\text{st}}$ move. Thus, we had $|U_{3-i}| \geq 2$ immediately after Breaker's $t^{\text{th}}$ move, and by induction hypothesis, there must have been distinct vertices $x', z' \in U_{3-i}$ such that $xx', zz' \in E(B)$. In her $(t+1)^{\text{st}}$ move, Maker did not claim an edge incident with $x$ or $z$, as otherwise Breaker would not play according to Stage II in round $t+1$. If her edge was neither incident with $x'$ nor with $z'$, then Breaker can claim an arbitrary edge, and still (a) and (b) hold. Otherwise, assume w.l.o.g. that Maker claimed an edge incident with $z'$. Then, analogously to the induction start, Breaker can maintain the required properties. □

Thus, we conclude that Breaker can follow Stage II of her strategy.

**Stage III.** For this stage, there is nothing to prove.

So, we only need to verify that Maker needs at least $n+2$ rounds until she occupies a perfect matching.

Assume to the contrary that Maker can create a perfect matching within $n+1$ rounds. By Claim 4.6.3, Maker will create a copy $P$ of $P_3$ in some round $t \leq n$, w.l.o.g. let $V(P) = \{x, y, z\}$ and $End(P) = \{x, z\} \subseteq V_1 \setminus U_1$. We then have $|U_1| = n - t$ and $|U_2| = n - t + 1$. If Maker wants to win until round $n+1$, then, in the following $n - t + 1$ rounds, she needs to claim a matching of size $n - t$ between $U_1$ and some subset $U_2' \subseteq U_2$, and an edge between $\{x, z\}$ and the unique vertex of $U_2 \setminus U_2'$. In case she tries to claim this edge after she fully claimed the matching of size $n - t$, then she will fail, because of Property (b), and thus we get a contradiction. Otherwise, by claiming an edge between $\{x, z\}$ and $U_2$, Maker creates a copy of $P_4$, before finishing this matching of size $n - t$. But then, Breaker plays according to Stage III, and once Maker claimed all but one edge of the desired matching of size $n - t$, Breaker blocks the unique edge which Maker would need to claim to win within $n+1$ rounds, again a contradiction. □

## 4.7 Concluding remarks and open problems

**Winning as fast as possible**. We showed that, given a constant $\Delta > 0$, Maker can create a copy of any pre-defined tree on $n$ vertices of maximum degree at most $\Delta$ within $n+1$ rounds, provided $n$ is large enough. Moreover, we know that there exist trees $T$ with $\Delta(T) = 2$ which cannot be embedded within $n-1$ rounds. There is still a small gap and we wonder whether our bound of $n+1$ can be improved to $n$ in general.

**Building trees of large maximum degree**. Disregarding the goal of winning in a small number of rounds, another open question is how large the maximum degree of a tree $T$ is allowed to be such that Maker still has a strategy to create a copy of $T$.

**Question 4.7.1** *What is the largest value $\Delta = \Delta(n)$ such that for every tree $T$ on $n$ vertices of maximum degree at most $\Delta$, Maker has a strategy to occupy a copy of $T$ in the unbiased Maker-Breaker game on $E(K_n)$?*

**Embedding graph factors.** Let $n \in \mathbb{N}$ and let $H$ be a graph with a constant number $k$ of vertices. It also seems to be of interest to study such a Maker-Breaker game in which Maker's goal is to occupy a factor of $H$ on the board $E(K_n)$, i.e. a vertex disjoint union of copies of $H$ that cover all but less than $k$ vertices of $K_n$. It should be clear that, applying our methods for the unbiased Maker-Breaker game on $K_n$, Maker can occupy such a factor by wasting at most 2 edges when $H = T$ is a tree. However, it would be interesting to understand the general problem further, in particular when $H$ contains at least one cycle.

# Chapter 5

# Walker-Breaker games

In this chapter, we study Walker-Breaker games. Recall that these games are played by two players, Walker and Breaker, who alternately choose edges of a graph $G$ that were not chosen by the opponent, while Walker has the constraint to choose edges of a walk. We will discuss games in which Walker aims to occupy the edges of a large cycle or a fixed subgraph in $K_n$. We also discuss how many edges Walker can occupy.

In Section 5.1, we collect some auxiliary results, that will be useful for the proofs of the main theorems, followed by Section 5.2, where we show that Walker has a strategy to create large graphs of small diameter. Afterwards, in Section 5.3 we prove Theorem 1.4.1 and Theorem 1.4.2. In Section 5.4 we prove Theorem 1.4.4, and in Section 5.5 we prove Theorem 1.4.3. Finally, we conclude with some remarks and open problems in Section 5.6.

**Notation and terminology.** Recall that the *2-density* of a graph $G$ on at least 3 vertices is defined as $d_2(G) = \frac{e(G)-1}{v(G)-2}$ and its *maximum 2-density* is $m_2(G) = \max_{H \subseteq G; v(H) \geq 3} d_2(H)$.

Assume a Walker-Breaker game on some graph $G$ is in progress, and let $v$ be Walker's current position. According to the rules, Walker then has to choose an edge which is incident with $v$, and which was not chosen by Breaker so far. Let $vw$ be this edge. Then we say that Walker *walks* from $v$ to $w$, and $w$ becomes Walker's *new position.* Moreover, we say that Walker *visits* $w$. If Walker chose $vw$ already in an earlier round, then we say that she *repeats* or *reuses* this particular edge, and we say that she returns to $w$. Moreover, a vertex $u$ will be called *untouched* if it was not Walker's position so far. At any given moment throughout the game, $W$ will denote the graph induced by all edges that Walker chose. Similarly, $B$ denotes the graph of Breaker's edges. The remaining edges, which form the graph $F = G \setminus (W \cup B)$, are said to be *free.*

For simplicity and clarity of presentation, we do not care about optimizing the constants that appear in the following proofs. Moreover, whenever these are not crucial, we omit floor and ceiling signs.

## 5.1 Preliminaries

### 5.1.1 Resilience results

In some strategies that we present later, Walker creates a graph which looks almost like a random graph. In order to guarantee that these graphs contain large cycles or copies of fixed graphs, we make use of the following *resilience results.*

**Theorem 5.1.1 (Theorem 1.1 in [36])** *For every $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ such that for $p \geq \frac{C\log(n)}{n}$ and $G \sim \mathcal{G}_{n,p}$ the following a.a.s. holds. For every $R \subseteq G$ with $\Delta(R) \leq (\frac{1}{2} - \varepsilon)np$, the graph $G \setminus R$ contains a Hamilton cycle.*

Notice that the formulation of the above statement is slightly different from Theorem 1.1 in [36]. Here it said that a.a.s. the following holds: For every $H \subseteq G$ with $\delta(H) \geq (\frac{1}{2} + \varepsilon)np$, the graph $H$ contains a Hamilton cycle. However, by showing that the degrees in $G \sim \mathcal{G}_{n,p}$ are concentrated around $np$, the statement in Theorem 5.1.1 easily follows.

**Theorem 5.1.2 (Corollary of Theorem 15 in [41])** *Let $H$ be any graph. Then there exist constants $C, \gamma > 0$ such that for $p \geq Cn^{-\frac{1}{m_2(H)}}$ and $G \sim \mathcal{G}_{n,p}$ the following a.a.s. holds. For every $R \subseteq G$ with $\Delta(R) \leq \gamma np$, the graph $G \setminus R$ contains a copy of $H$.*

### 5.1.2 Creating an almost random graph

As already mentioned, Walker will aim to create graphs which look almost random. To do so, we will use the following slight modification of Theorem 1.5 in [22]. However, to make the terminology precise, let us first give the following definition, analogously to [22].

**Definition 5.1.3** *Let $\mathcal{P} = \mathcal{P}(n)$ be some graph property that is monotone increasing, and let $0 \leq \varepsilon, p \leq 1$. Then $\mathcal{P}$ is said to be $(p, \varepsilon)$-resilient if a random graph $G \sim \mathcal{G}_{n,p}$ a.a.s. has the following property: For every $R \subseteq G$ with $d_R(v) \leq \varepsilon d_G(v)$ for every $v \in V(G)$ it holds that $G \setminus R \in \mathcal{P}$.*

**Theorem 5.1.4 (Modification of Theorem 1.5 in [22])** *For every $\varepsilon > 0$ and every large enough $n \in \mathbb{N}$ the following holds. Let $\frac{\log(n)}{n} \ll p = p(n) \leq 1$, let $c_2 \in \mathbb{N}$ and let $\mathcal{P} = \mathcal{P}(n)$ be* *a monotone $(p, 4\varepsilon)$-resilient graph property. Assume a $(1 : \frac{\varepsilon}{30(c_2+1)p})$ Walker-Breaker game on $K_n$ is in progress, where the graph $F$ of free edges satisfies $\delta(F) \geq (1-\varepsilon)n$, and where Walker's current graph $W_0$ has the property that between every two vertices in $V(K_n)$ it contains a path of length at most $c_2$. Then, Walker has a strategy for continuing the game that creates a graph $W' \in \mathcal{P}$.*

We remark at this point that in later applications of this theorem, $W_0$ may live on a vertex set that contains $V(K_n)$. In fact, we may play on a board containing $E(K_n)$, on which Walker first creates a graph $W_0$ as described. Afterwards, she more or less reduces her game to the board $E(K_n)$, where she aims to create her desired structure, as for example a large cycle. The graph $W_0$ then has the advantage that, in case there appears a *dangerous* vertex in $V(K_n)$ having large Breaker degree, Walker is able to reach this vertex within at most $c_2$ rounds by reusing the edges of $W_0$. In fact, this is the only property of $W_0$ that we will use in the proof of the above mentioned theorem.

Now, let us turn to the proof of the above theorem. It follows very closely the proof of Theorem 1.5 in [22], with small modifications. For the convenience of the reader, we include it here. In particular, we will use the so called *MinBox game*, which was motivated by the study of the degree game [24]. The game *MinBox*$(n, D, \alpha, b)$ is a Maker-Breaker game on a family of $n$ disjoint boxes $S_1, \ldots, S_n$ with $|S_i| \geq D$ for every $i \in [n]$, where Maker claims one element and Breaker claims at most $b$ elements in each round, and where Maker wins if she manages to occupy at least $\alpha|S_i|$ elements in each box $S_i$.

Throughout such a game, by $w_M(S)$ and $w_B(S)$ we denote the number of elements that Maker and Breaker claim so far from the box $S$, respectively. As motivated by [24], we also set $dang(S) := w_B(S) - b \cdot w_M(S)$ for every box $S$. If not every element of a box $S$ is claimed so far, then $S$ is said to be *free*. Moreover, $S$ is said to be *active* if Maker still needs to claim elements of $S$, i.e. $w_M(S) < \alpha|S|$. The following statement holds.

**Theorem 5.1.5 (Theorem 2.3 in [22])** *Let $n, b, D \in \mathbb{N}$, let $0 < \alpha < 1$ be a real number, and consider the game MinBox$(n, D, \alpha, b)$. Assume that Maker plays as follows: In each turn, she chooses an arbitrary free active box with maximum danger, and then she claims one free element from this box. Then, proceeding according to this strategy,*

$$dang(S) \leq b(\log(n) + 1)$$

*is maintained for every active box $S$ throughout the game.*

We remark at this point that in [22], Breaker claims exactly $b$ elements in each round of *MinBox*$(n, D, \alpha, b)$. However, the condition that Breaker can claim at most $b$ elements does not change the theorem above, as its proof in [22] only uses the fact that in each round $\sum_{i\in[n]} w_B(S_i)$ can increase by at most $b$.

**Proof of Theorem 5.1.4** Let $F_0 = F$ be the set of free edges, which is given by the assumption of the theorem, and let $W_0$ be the current graph of Walker.

For contradiction, let us assume that Walker does not have a strategy to occupy a graph satisfying property $\mathcal{P}$. Then we know, as mentioned in the introduction of this thesis, that Breaker needs to have a strategy $S_B$ which prevents Walker from creating a graph with property $\mathcal{P}$, independent of how Walker proceeds.

In the following, we describe a randomized strategy for Walker and afterwards we show that, playing against $S_B$, this randomized strategy lets Walker create a graph from $\mathcal{P}$ with positive probability, thus achieving a contradiction. The main idea of the strategy, as motivated by [22], is as follows: throughout the game Walker generates a random graph $H \sim \mathcal{G}_{n,p}$ on the vertex set $V(K_n)$. Following her randomized strategy, she then obtains that a.a.s. $d_{W\setminus W_0}(v) \geq (1-4\varepsilon)d_H(v)$ holds for every vertex $v \in V(K_n)$. Thus, by the assumption on $\mathcal{P}$, we then know that $W' = W \setminus W_0$ a.a.s. satisfies property $\mathcal{P}$.

When generating the random graph $H$, Walker tosses a coin on each edge of $K_n$ independently at random (even if this edge belongs to $K_n \setminus F_0$), which has success with probability $p$. In case of success, Walker then declares that it is an edge of $H$, and in case this edge is still free in the game on $K_n$, she claims it.

To decide which edges to toss a coin on, Walker always identifies an *exposure vertex* $v$ (which will be marked with color red). After identification, Walker proceeds to $v$ by reusing the edges of $W_0$. Once, she reached the vertex $v$, she tosses her coin only on edges that are incident with $v$ and for which she did not toss a coin before. When she has no success or when she has success on an edge which cannot be claimed anymore (i.e. this edge belongs to $B\cup(K_n\setminus F_0)$), then she declares her move as a failure. If the first case happens, we denote this failure as a failure of type I, and following [22] we set $f_I(v)$ to be the number of *failures of type I*, for which $v$ is the exposure vertex. Otherwise, if Walker has success on an edge of $B\cup(K_n\setminus F_0)$, then it said to be a *failure of type II*, and with $f_{II}(v)$ we denote the number of edges that are incident with $v$ in $K_n$, and which were failures of type II.

To reach our goal, it suffices to prove that following Walker's random strategy we a.a.s. obtain that $f_{II}(v) < 3.9\varepsilon np$ holds for every vertex $v \in V(K_n)$ at the end of the game. Indeed, by a simple Chernoff-type argument one verifies that a.a.s. $d_H(v) \geq 0.99np$ for every $v \in V$, which then would yield $f_{II}(v) < 4\varepsilon d_H(v)$ and $d_{W\setminus W_0}(v) \geq (1-4\varepsilon)d_H(v)$.

As in the proof of Theorem 1.5 in [22], we also say that Walker *exposes* an edge $e \in E(K_n)$ whenever she tosses a coin for the edge $e$; and we also consider the set $U_v \subseteq N_G(v)$ which contains those vertices $u \neq v$ for which the edge $vu$ is still not exposed.

Now, to make sure that the failures of type II do not happen very often, we associate a box $S_v$ of size $4n$ to every vertex $v \in V(K_n)$, and we use the game $MinBox(n, 4n, \frac{p}{2}, 2b(c_2+1))$ on the family of these boxes to determine the exposure vertex. In this game, Walker imagines to play in the role of Maker. The idea behind this simulated game is to relate Breaker's degree $d_B(v)$ to the value $w_B(S_v)$, and to relate the number of Walker's exposure processes at $v$ to Maker's value $w_M(S_v)$. This way, we ensure that Walker stops doing exposure processes at $v$, once $d_B(v)$ becomes large, which helps to keep the expected number of failures of type II small.

We now come to an explicit description of Walker's (randomized) strategy. Afterwards, we show that she can follow that strategy and that, by the end of the game, a.a.s. $f_{II}(v) < 3.9\varepsilon np$ holds for every vertex $v \in V(K_n)$.

**Stage I.** Let Walker's $t^{\text{th}}$ move happen in Stage I, and let $e_1, \ldots, e_b$ be the edges that Breaker claimed in his previous move. Moreover, let $v_{t-1}$ be Walker's current position. Then Walker at first updates the simulated game $MinBox(n, 4n, \frac{p}{2}, 2b(c_2+1))$ as follows: for every vertex $u \in V$, Breaker claims $|\{i \leq b : \ u \in e_i\}|$ free elements in $S_u$. (So, in total, Breaker receives $2b$ free elements over all boxes $S_u$.) In the real game, she then looks for a vertex $v$ that is colored red. If such a vertex exists, she proceeds immediately with the case distinction below. Otherwise, if neither vertex has color red, she first does the following: she identifies a vertex $v$ for which in the simulated game $MinBox(n, 4n, \frac{p}{2}, 2b(c_2+1))$, $S_v$ is a free active box of largest danger value. If no such box exists, Walker proceeds with Stage II. Otherwise, she colors the vertex $v$ red (to identify it as her exposure vertex), Maker claims an element of $S_v$ in the simulated game $MinBox(n, 4n, \frac{p}{2}, 2b(c_2+1))$, and then Walker proceeds with the following cases:

**Case 1.** $v_{t-1} \neq v$. Let $P = (v_{t-1}, x_1, \ldots, x_r, v)$ be a shortest $v_{t-1}$-$v$-path in $W_0$. Then Walker reuses the edge $v_{t-1}x_1$ (to get closer to $v$), makes $x_1$ her new position and finishes her move.

**Case 2.** $v_{t-1} = v$, i.e. Walker's current position is the (red) exposure vertex. Then Walker starts her exposure process on the edges $vw$ with $w \in U_v$. She fixes an arbitrary ordering $\sigma : [|U_v|] \to U_v$ of the vertices of $U_v$, and she tosses her coin on the vertices of $U_v$ according to that ordering, independently at random, with $p$ being the probability of success.

**2a.** If this coin tossing brings no success, the exposure is a failure of type $I$. So, Walker increases the value of $f_I(v)$ by 1. In the simulated game $MinBox(n, 4n, \frac{p}{2}, 2b(c_2+1))$, Maker receives $2pn-1$ free elements in $S_v$ (or all remaining free elements if their number is less than $2pn-1$). In the real game, as all edges incident with $v$ are exposed, $U_v$ becomes the empty set, while $v$ is removed from every other set $U_w$. Walker removes the color from $v$ and she finishes her move by reusing an arbitrary edge of $W_0$.

**2b.** Otherwise, let Walker's first success happen at the $k^{\text{th}}$ coin tossing. We distinguish the following two subcases.

- If the edge $v\sigma(k)$ is free, then Walker claims this edge in the real game, thus setting $v_t := \sigma(k)$ for her new position. For every $i \leq k$, she removes $v$ from $U_{\sigma(i)}$ and $\sigma(i)$ from $U_v$; moreover she removes the color from $v$. In the simulated game $MinBox(n, 4n, \frac{p}{2}, 2b(c_2+1))$, Maker claims a free element from the box $S_{\sigma(k)}$.
- If the edge $v\sigma(k)$ is not free, the exposure is a failure of type II. Accordingly, Walker increases the value of $f_{II}(v)$ and $f_{II}(\sigma(k))$ by 1. She updates the sets $U_v$ and $U_{\sigma(i)}$ as in the previous case and removes the color from $v$. To finish her move, she reuses an arbitrary edge of $W_0$.

**Stage II.** In this stage, Walker tosses her coin on every unexposed edge $uv \in E(G)$. In case of success, she declares a failure of type II for both vertices $u$ and $v$.

It is easy to see that Walker can follow the proposed strategy. Indeed, the strategy always asks her to claim an edge which is known to be free or to belong to $W_0$ and which is incident with Walker's current position. We only need to check that Theorem 5.1.5 is applicable, i.e., we need to check that in the simulated game $MinBox(n, 4n, \frac{p}{2}, 2b(c_2+1))$ Breaker claims at most $2b(c_2+1)$ elements between two consecutive moves of Maker in which she claims free elements from free active boxes of maximum danger. This follows from Claim 5.1.7, and the observation that Maker claims such an element when Walker colors some vertex red, while in each round Breaker claims $2b$ elements in the simulated game.

**Claim 5.1.6** *At any point in Stage I, at most one vertex is red.*

**Proof** The claim follows from the fact that, in Stage I, Walker only colors a vertex red if there is no vertex having this color. □

**Claim 5.1.7** *After a vertex $v$ becomes red in Stage I, it takes at most $c_2+1$ rounds until the color is removed and, in the following round, a new (maybe the same) vertex is colored red.*

**Proof** Assume $v$ becomes red. Then, according to the strategy description, Walker proceeds with Stage I as long as $v$ is red. As long as her current position is different from $v$, Walker walks towards the vertex $v$ by reusing the edges of $W_0$. By the assumption on $W_0$ we know that this takes at most $c_2$ rounds. Once $v$ is Walker's position, the exposure process starts (which lasts only one round) and independent of its outcome, Walker removes the color from $v$. □

Thus, it remains to prove that, by the end of the game, a.a.s. $f_{II}(v) \leq 3.9np$ holds for every vertex $v \in V(K_n)$. To do so, we verify the following claims, which are proven analogously to Claims 3.1 – 3.4 in [22].

**Claim 5.1.8** *During Stage I, $w_B(S_v) < n$ and $w_M(S_v) < (1+2p)n$ for every $v \in V$.*

**Proof** According to the strategy description, Breaker claims an element of $S_v$ in the simulated game if and only if in the real game he claims an edge incident with $v$. Thus, $w_B(S_v) < n$ follows. Moreover, we observe the following: $w_M(S_v)$ is increased by 1 each time $v$ is colored red, and it is increased by at most 1 when Walker has success on an edge $vw$ where $w$ is the red vertex (Case 2b). Both cases together can happen at most $n-1$ times, since we can have at most $n-1$ exposure processes in which we toss a coin on an edge that is incident with $v$. Additionally, $w_M(S_v)$ increases by at most $2pn-1$ when $v$ is the exposure vertex and a failure of type I happens (Case 2a). However, this can happen at most once, since after a failure of type I (Case 2a), Walker ensures that in the simulated game $S_v$ is not free or active anymore and thus, $v$ will not become the exposure vertex again. Thus, the bound on $w_M(S_v)$ follows. □

**Claim 5.1.9** *For every vertex $v \in V(G)$, $S_v$ becomes inactive before $d_B(v) \geq \frac{\varepsilon n}{5}$.*

**Proof** Assume to the contrary that $w_B(S_v) = d_B(v) \geq \frac{\varepsilon n}{5}$ for some active box $S_v$. Then, by Theorem 5.1.5, $w_B(S_v) - 2(c_2+1)b \cdot w_M(S_v) \leq 2(c_2+1)b(\log(n)+1)$. With $b = \frac{\varepsilon}{30(c_2+1)p}$ we then conclude $w_M(S_v) \geq 3pn - (\log(n)+1) > 2pn$, where in the second inequality we used the fact that $p \gg \frac{\log(n)}{n}$. However, this contradicts with the assumption that $S_v$ is active. □

**Claim 5.1.10** *A.a.s. for every vertex $v \in V$ the following holds: As long as $U_v \neq \emptyset$ holds, we have that $S_v$ is active. In particular, a.a.s. every edge of $K_n$ will be exposed in Stage I.*

**Proof** Suppose there is a vertex, say $v$, with $U_v \neq \emptyset$ such that $S_v$ is not active. Then $f_I(v) = 0$ and $2np = \frac{p}{2}|S_v| \leq w_M(S_v)$. As discussed in the previous proof, $w_M(S_v)$ could always increase by 1 when Walker had success on an edge $vw$ where $w$ was the exposure vertex (Case 2b), or when $v$ was colored red. Notice that in the second case, Walker then exposed edges at $v$ and (besides maybe the last exposure process at $v$) she had success on some edge, as $f_I(v) = 0$. But this means that Walker had success on at least $2pn - 1$ edges incident with $v$, i.e., $d_H(v) \geq 2np - 1$. However, a simple Chernoff argument shows that for $H \sim \mathcal{G}_{n,p}$ a.a.s. for all vertices $v$ the last inequality will not happen. Thus, the first statement follows. Now, let us condition on the first statement and assume that there is an edge $uv$ of $K_n$ which is not exposed at the end of Stage I. Then $U_v \neq \emptyset$ and therefore $S_v$ is active. Moreover, by Claim 5.1.8, $S_v$ is free, as $w_M(S_v) + w_B(S_v) < |S_v|$. But this is in contradiction with the fact that Walker does not continue with Stage I. □

**Claim 5.1.11** *A.a.s. for every vertex $v \in V(G)$, we have $f_{II}(v) \leq 3.9\varepsilon np$.*

**Proof** We may condition on the statements that a.a.s. hold according to Claim 5.1.10. In particular, all failures of type II happen in Stage I. Moreover, by Claim 5.1.9 we then have $d_B(v) \leq \frac{\varepsilon n}{5}$ as long as $U_v \neq \emptyset$, for every $v \in V$. Moreover, by assumption we have $d_{K_n \setminus F_0}(v) \leq \varepsilon n$. Now, in Stage I, a failure of type II happens only if Walker has success on an edge $e$ which already belongs to Breaker's graph, i.e. $e \in E(B)$, or which is already claimed at the beginning, i.e. $e \in E(K_n \setminus F_0)$. In particular, for every $v \in V$ there is a non-negative integer $m \leq 1.2\varepsilon n$ such that $f_{II}(v)$ is dominated by $Bin(m, p)$. Applying Chernoff's inequality and union bound, while using that $p \gg \frac{\log(n)}{n}$, we obtain that a.a.s. $f_{II}(v) \leq 3.9\varepsilon np$ for every vertex $v \in V$. □

The last claim completes the proof of Theorem 5.1.4. □

## 5.2 Creating large graphs with small diameter

The proofs for most of our theorems will make use of Walker's ability to create a graph of small diameter covering almost every vertex of $V(K_n) = [n]$, within a small number of rounds. Her strategy is given by two main steps which are represented by the following two propositions.

**Proposition 5.2.1** *For every large enough integer $n$ the following holds. Let $b \leq \frac{n}{\log^2(n)}$ be a positive integer, and let $r = \frac{\log n}{\log\left(\frac{n}{200b}\right)}$. Then, in the $b$-biased Walker-Breaker game on $K_n$, Walker has a strategy to create a tree on $\frac{n}{2\lfloor r \rfloor + 2}$ vertices, of depth at most $\lfloor r \rfloor + 1$, within at most $n$ rounds.*

**Proposition 5.2.2** *For every large enough integer $n$ the following holds. Let $b \leq \frac{n}{\log^2(n)}$ be a positive integer, and let $r = \frac{\log n}{\log\left(\frac{n}{200b}\right)}$. Then, in the $b$-biased Walker-Breaker game on $K_n$, Walker has a strategy to create a graph on $n - 400b$ vertices, with diameter at most $2\lfloor r \rfloor + 6$, within at most $7n$ rounds.*

**Proof of Proposition 5.2.1** Whenever necessary, let us assume that $n$ is large enough. From the assumption it follows that

$$r = O\left(\frac{\log n}{\log\log n}\right).$$

For simplicity of notation we set $c_1 = \frac{1}{(2\lfloor r\rfloor+2)}$. Notice that $c_1 \leq \frac{1}{4}$.

The main idea is to create the tree in some kind of breadth first manner, by iteratively attaching stars of size $\frac{n}{100b}$ to the leaves of the current tree.

For a given moment throughout the game, assume that $T$ is the tree which Walker created so far and that Walker's current position $v \in V(T)$ is a leaf of $T$. Assume further that for some positive integer $s$, there are at least $(2b+1)s$ vertices $w \in V \setminus V(T)$ such that the edge $vw$ is free. Then, by *attaching a star* of size $s$ to $v$ we mean a strategy of creating a star $S$ of size $s$ with center $v$ within $2s$ rounds in the following way: As long as the star $S$ did not reach size $s$ or her current position is not $v$, Walker proceeds to an untouched vertex $w$ if her current position is $v$ (thus enlarging $S$ by one edge), or she proceeds to the vertex $v$ if her current position is a leaf of $S$. Notice that Walker can easily follow this strategy and create her star under the given assumption, since Walker needs two rounds to care about one edge of $S$, while Breaker in the meantime can claim $2b$ edges.

In the following we describe a strategy for Walker. Afterwards, as usual, we show that she can follow that strategy, and while doing so, she creates a tree as required.

Initially set $L_0 = \{v_0\}$, where $v_0$ is Walker's start vertex. We consider $v_0$ as the root of Walker's tree $T$. Initially, $V(T) = L_0 = \{v_0\}$. Now, Walker plays according to several stages, where in Stage $j$ she proceeds from a tree of depth $j-1$ to a tree of depth $j$.

**Stage 1.** In Stage 1, Walker attaches a star of size $\frac{n}{200b}$ to the vertex $v_0$. Then she proceeds with Stage 2.

**Stage $j$ ($j > 1$).** At the end of Stage $j-1$, let Walker's tree have depth $j-1$, and let $L_{j-1}$ be its set of leaves. In Stage $j$, Walker enlarges her tree by attaching stars of size $\frac{n}{100b}$ to $\frac{|L_{j-1}|}{2}$ vertices of $L_{j-1}$, as long as $v(T) \leq c_1 n$. (In case $v(T) = c_1 n$, Walker stops playing. In case she reaches the size $c_1 n$ by attaching a smaller star, she just attaches this smaller star.) To do so, Walker proceeds as follows. Assume that Walker just finished her $t^{\text{th}}$ star of Stage $j$ with $t < \frac{|L_{j-1}|}{2}$. Then afterwards she identifies a vertex $v \in L_{j-1}$ not being a center of one of her stars yet, which has smallest possible Breaker-degree at this moment. Within at most $2j$ rounds she walks towards this vertex $v$ by using the edges of her current tree $T$; and afterwards she attaches a star of size $\frac{n}{100b}$ to this vertex $v$. (We will see later that there are enough free edges to do so.)

Finally, after Walker attached $\frac{|L_{j-1}|}{2}$ stars in total (and $v(T) < c_1 n$ still holds), Walker proceeds to Stage $j+1$.

Obviously, if Walker can follow this strategy, then she creates a tree with $c_1 n$ vertices. Thus, it remains to prove that she can follow her strategy, that in total she plays at most $n$ rounds, and that the depth of $T$ is as small as required.

For this, let us use the following notation. With $T_j$ we denote Walker's tree at the end of Stage $j$, and with $L_j$ we denote, as already introduced by the strategy, the set of leaves of $T_j$. Moreover, with $R_j$ we denote the number of rounds until the end of Stage $j$ (including all previous stages). The following claim provides us with some useful inequalities for the analysis of Walker's strategy.

**Claim 5.2.3** *Let $n$ be large enough. As long as Walker can follow the strategy, the following holds.*

1. *$|L_0| = 1$ and $|L_j| \le \frac{n}{200b}|L_{j-1}|$ for every $j$. Moreover, $|L_j| = \frac{n}{200b}|L_{j-1}|$ if $v(T_j) < c_1 n$.*
2. *$R_j \le 2e(T_j) + 2j \cdot v(T_{j-1}) < 5 \cdot \left(\frac{n}{200b}\right)^j$.*

**Proof** For the first part, just observe that, if Walker can follow the strategy, then in Stage 1, she creates a star of size $\frac{n}{200b}$, ensuring that $|L_1| = \frac{n}{200b}$. Moreover, in Stage $j$ Walker attaches at most $\frac{|L_{j-1}|}{2}$ stars of size $\frac{n}{100b}$ to the vertices of $L_{j-1}$, giving $|L_j| \le \frac{n}{200b}|L_{j-1}|$. If $v(T_j) < c_1 n$, she attaches exactly $\frac{|L_{j-1}|}{2}$ stars of size $\frac{n}{100b}$ to the vertices of $L_{j-1}$, giving $|L_j| = \frac{n}{200b}|L_{j-1}|$. In particular, $|L_j| = \left(\frac{n}{200b}\right)^j$.

The second part of the claim is obtained in the following way. If $j = 1$, then the statement is obvious. If $j > 1$, then, by the first part, we conclude that $\left(\frac{n}{200b}\right)^{j-1} = |L_{j-1}| < n$, when $v(T_{j-1}) < c_1 n$. Thus, $j \le \lfloor r \rfloor + 1 = o(\log n)$. Walker's strategy consists of two different actions. On one hand, she creates stars where for each edge she makes two moves, since she walks along each edge in both possible directions. On the other hand, after finishing one star and before starting a new one, she moves along the edges of her current tree to a vertex, which she determined right after the first of the two stars was finished. In Stage $j$, such a step takes her at most $2j = o(\log n)$ moves (repeated edges), as the depth of Walker's tree is bounded from above by $j$. Thus, we can bound $R_j$ from above by $2e(T_j) + 2j \cdot v(T_{j-1})$, since $v(T_{j-1})$ is an upper bound on the number of star attachments. Now, we have $v(T_{j-1}) = \sum_{i=0}^{j-1} \left(\frac{n}{200b}\right)^i < 2 \cdot \left(\frac{n}{200b}\right)^{j-1}$ and $e(T_j) < v(T_j) < 2 \cdot \left(\frac{n}{200b}\right)^j$. Therefore, for large $n$,

$$R_j \le 2e(T_j) + 2j \cdot v(T_{j-1}) \le 4 \cdot \left(\frac{n}{200b}\right)^j + 4j \cdot \left(\frac{n}{200b}\right)^{j-1} < 5 \cdot \left(\frac{n}{200b}\right)^j,$$

where in the last inequality we used the fact that $\frac{n}{200b} = \Omega(\log n)$ and $j = o(\log n)$. □

**Claim 5.2.4** *Let $n$ be large enough. Immediately before Walker starts building a star with center vertex $v$, we have $d_B(v) \le \frac{n}{5}$.*

**Proof** When Walker starts her first star at $v_0$, there are no Breaker edges at all. Therefore, for $v = v_0$ the statement is obvious, and we can consider to look at Stage $j$, for some $j > 1$. Let $v \in L_{j-1}$ be a vertex at which Walker wants to attach a star, according to the proposed strategy. That is, $v(T) < c_1 n$ still holds, and Claim 5.2.3 can be applied. Then, before starting the star, $v$ belongs to a set of at least $\frac{|L_{j-1}|}{2}$ vertices of $L_{j-1}$ that still have degree 1 in Walker's graph. Since we played at most $R_j$ rounds so far, Breaker claims at most $b \cdot R_j$ edges, which implies that the average Breaker-degree of all these at least $\frac{|L_{j-1}|}{2}$ vertices, is bounded from above by

$$\frac{2b \cdot R_j}{|L_{j-1}|/2} < 20b \cdot \left(\frac{n}{200b}\right)^j / \left(\frac{n}{200b}\right)^{j-1} = 0.1n,$$

where the first inequality follows from Claim 5.2.3. Since Walker, by following the strategy, chooses the vertex $v$ such that its Breaker-degree is minimal, we obtain $d_B(v) \le 0.1n$ at the moment when Walker considers $v$ for attaching a star. She may walk to $v$ within in the following $2j$ rounds, but even then, $d_B(v) \le 0.1n + 2jb < 0.2n$ holds, at the moment when Walker starts to attach a star at $v$. □

With the previous claims in hand, we can finish our proof. As long as Walker can follow the strategy, all the previous statements hold. At the same time these statements ensure that Walker can always continue as long as her tree $T$ has at most $c_1 n$ vertices. Indeed, when Walker aims to attach the next star to some vertex $v$, then the number of vertices $w \in V \setminus V(T)$ with $vw$ being free is at least

$$|V \setminus V(T)| - d_B(v) \ge (1 - c_1)n - 0.2n > (2b + 1) \cdot \frac{n}{100b},$$

where the first inequality holds be Claim 5.2.4 and since $v(T) \le c_1 n$. By our argument at the beginning of this proof, we know that this is enough to guarantee that Walker can attach a star of size $\frac{n}{100b}$ to the vertex $v$. Thus, as long as $v(T) < c_1 n$, Walker can follow the strategy. When she finishes her tree of size $c_1 n$ during Stage $j$, then $\left(\frac{n}{200b}\right)^{j-1} = |L_{j-1}| < n$. Hence, her final tree has depth $j \le \lfloor r \rfloor + 1$. Moreover, the total number of rounds can be bounded as follows. For the attachment of the stars, Walker needs at most $2c_1 n$ rounds, since she walks along each edge twice, which she claims during an attachment. Between two consecutive star attachments, she identifies a new vertex to be the center of her new star. As her stars have size $\frac{n}{100b}$ (besides the star in Stage 1), she makes less than $200b$ star attachments in total; and between each two of them she reuses at most $2(\lfloor r \rfloor + 1)$ edges. So, in total she plays at most $2c_1 n + 200b \cdot 2(\lfloor r \rfloor + 1) \leq \frac{n}{2} + o(n) < n$ rounds. □

**Proof of Proposition 5.2.2** Set $c_2 = 2\lfloor r \rfloor + 2$ and $c_1 = c_2^{-1}$. We first give a simple strategy for Walker and then we prove that this strategy helps to reach her desired goal. The main idea is to create a graph of diameter at most $c_2 + 4$ by attaching stars to the tree from the previous proposition. Whenever necessary, assume $n$ to be large enough. After Walker's move in round $t$, let $U_t$ be the set of vertices not touched. Walker's strategy is as follows.

**Stage I.** Within at most $n$ rounds, Walker creates a tree $T_1$ on $c_1 n$ vertices, of depth at most $\frac{c_2}{2}$.

**Stage II.** From now on, $T_1$ is fixed as the tree which Walker occupies at the end of Stage I. Throughout Stage I, Walker maintains a tree $T_2$ of depth at most $\frac{c_2}{2} + 1$, by attaching large stars to the vertices of $T_1$. Assume Walker already attached $i-1$ such stars, and now she plays according to Stage II for the $i^{\text{th}}$ time. Let $V_i$ be the set of vertices that are not contained in her tree so far. If $|V_i| \leq 50c_2 b$, or if there is no vertex $z_i \in V(T_1)$ with $d_F(z_i, V_i) > 0$, then Walker proceeds to Stage III. Otherwise she fixes an arbitrary vertex $z_i \in V(T_1)$ with $d_F(z_i, V_i) > 0$, which minimizes $d_B(z_i, V_i)$. Then, in the following at most $c_2 + 2$ rounds, Walker walks to $z_i$ using the edges of her current tree. Afterwards, as long as possible, Walker creates a star with center $z_i$ and leaves in $V_i$. That is, as long as possible, she claims a free edge between $z_i$ and $V_i$ in every second round (by alternately walking between $z_i$ and distinct vertices from $V_i$). When this is no longer possible, she stops focusing on $z_i$, i.e. she does not attach any further edges to $z_i$, and then she repeats Stage II.

**Stage III.** From now on, $T_2$ is fixed as the tree which Walker occupies at the end of Stage II. At the beginning of Stage III, Walker walks to a vertex $z_0 \in V(T_2)$ with $d_F(z_0) \geq \frac{10n}{11}$, within at most $c_2 + 2$ rounds. (That she can do so will be proven later.) Then, throughout Stage III, Walker maintains a tree $T_3$ of depth at most $\frac{c_2}{2} + 2$, by attaching large stars to the vertices of $V(T_2)$. Assume that Walker already attached $i - 1$ such stars, and her current position is $z_{i-1}$, for some $i \geq 1$. Let $W_i$ be the set of vertices that have not been visited by Walker so far. If $|W_i| \leq 400b$, then Walker stops playing. Otherwise, she continues to attach stars to her current tree. For this, she identifies a vertex $z_i \in V(T_2)$ such that (i) $z_{i-1}z_i$ is a free edge, (ii) $d_F(z_i) \geq \frac{9n}{10}$ and (iii) $d_B(z_i) \leq 112b$. She immediately walks to $z_i$ using the edge $z_{i-1}z_i$. Afterwards, Walker creates a star of size $\frac{|W_i| - d_B(z_i, W_i)}{2b+1} - 1$ with center $z_i$ and with leaves in $W_i$. She then repeats this process. That is, if more than $400b$ vertices are still untouched, Walker attaches another star to a vertex $z_{i+1}$, as described above.

Obviously, if Walker can follow the strategy above until for some $i$, $|W_i| \leq 400b$ holds, then she creates a graph of required size and diameter. So, it remains to prove that she can indeed follow the strategy until $|W_i| \leq 400b$ happens, and that it takes her at most $7n$ rounds.

**Stage I.** This part is already given by Proposition 5.2.1.

**Stage II.** Before we show that Walker can follow this part of her strategy, let us first prove some useful claims.

**Claim 5.2.5** *Let $n$ be large enough. Assume that Walker can follow the strategy of Stage II. Then, as long as $i \leq \frac{c_1 n}{2}$, we have that $d_B(z_i, V_i) \leq 20c_2 b$ at the moment when Walker starts to attach a star at the identified vertex $z_i$.*

**Proof** As long as $i \leq \frac{c_1 n}{2}$, less than $4n$ rounds were played. Indeed, Walker's moves can be distinguished as before. Stage I lasts at most $n$ rounds. In Stage II, when Walker claims a free edge (during some star creation), she walks along this edge in both directions. Otherwise, after she identified a vertex for a new star attachment, she proceeds to this vertex by repeating at most $c_2+2$ edges of her tree $T_1$. Thus, in total we have at most $n+2n+i(c_2+2) < 4n$ rounds. It follows, when Walker identifies $z_i$, we have $e(B) \leq 4bn$ and $|V(T_1) \setminus \{z_1, \ldots, z_{i-1}\}| \geq \frac{c_1 n}{2}$. Thus, at this moment, the Breaker-degree of $z_i$ is at most

$$\frac{2e(B)}{|V(T_1) \setminus \{z_1, \ldots, z_{i-1}\}|} \leq \frac{8nb}{c_1 n/2} = 16c_2 b,$$

as Walker chooses $z_i$ with minimal degree. After Walker identified $z_i$ for her $i^{\text{th}}$ repetition of Stage II, she needs at most $c_2 + 2$ rounds to ensure that $z_i$ is her new position, as her tree has diameter at most $c_2 + 2$. So, when she starts creating the mentioned star, we have $d_B(z_i, V_i) \leq 16c_2 b + (c_2 + 2)b < 20c_2 b$. □

**Claim 5.2.6** *Let $n$ be large enough. Assume that Walker can follow the strategy of Stage II. Then, as long as $i \leq \frac{c_1 n}{2}$ and $|V_i| > 50c_2 b$, we have that $|V_{i+1}| \leq \left(1 - \frac{1}{4b+2}\right)|V_i|$.*

**Proof** When Walker starts creating the star with center $z_i$, there are at least $|V_i| - d_B(z_i, V_i)$ vertices in $V_i$ that Walker can walk to starting from $z_i$. As, during her star attachment, she claims one edge within two rounds, while Breaker claims $2b$ edges, Walker can create a star with center $z_i$ of size at least $\frac{|V_i| - d_B(z_i, V_i)}{2b+1} - 1$. By Claim 5.2.5, this yields

$$|V_{i+1}| \leq |V_i| - \left(\frac{|V_i| - d_B(z_i, V_i)}{2b+1} - 1\right) \leq \frac{2b}{2b+1}|V_i| + \frac{20c_2 b}{2b+1} + 1 \leq \left(1 - \tfrac{1}{4b+2}\right)|V_i|,$$

by the assumption on $|V_i|$. □

**Claim 5.2.7** *Let $n$ be large enough. Assume that Walker can follow the strategy of Stage II. Then, there is a positive integer $t \leq \frac{c_1 n}{2}$ such that $|V_t| \leq 50c_2 b$.*

**Proof** As long as $|V_i| > 50c_2b$, the previous claim tells us that

$$50c_2b < |V_i| \leq \left(1 - \tfrac{1}{4b+2}\right)|V_{i-1}| \leq \ldots \leq \left(1 - \tfrac{1}{4b+2}\right)^{i-1} n < e^{-(i-1)/(4b+2)}n.$$

Since $\frac{c_1 n}{b} = \Omega(\log(n)\log\log(n))$, it follows that $i \leq \frac{c_1 n}{2}$. □

Obviously, Walker can follow the strategy of Stage II. Now, by the previous claims we also see that she can do so until at most $50c_2b$ vertices do not belong to her tree. Indeed, as long as the number of untouched vertices is more than $50c_2b$, their amount decreases by at least a factor $(1 - \frac{1}{4b+2})$ with each star attachment, by Claim 5.2.6, which by the previous claim can happen less than $\frac{v(T_1)}{2}$ times. Moreover, as Walker is done with Stage II after she attached at most $\frac{c_1 n}{2}$ stars to her tree $T_1$, we conclude (as in the proof of Claim 5.2.5) that Walker plays at most $4n$ rounds until she proceeds with Stage III.

**Stage III.** When Walker enters Stage III, we have $v(T_2) \geq n - 50c_2b$, while at most $4n$ rounds were played so far. Now, similarly to the discussion of Stage II, one verifies the following claims.

**Claim 5.2.8** *Let $n$ be large enough. As long as $i \leq \frac{n}{4}$ and $|W_i| \geq 400b$, Walker can always identify a vertex $z_i$, as described by her strategy in Stage III.*

**Proof** At the beginning of Stage III, less than $4n$ rounds were played. The number of vertices in $K_n$ of free-degree less than $\frac{10n}{11} - 1$ is at most

$$\frac{2(e(B)+e(W))}{n/11} \leq \frac{2(4bn+4n)}{n/11} = o(n) = o(v(T_2)).$$

Thus, there exists a vertex $z_0 \in V(T_2)$ such that $d_F(z_0) \geq \frac{10n}{11}$. Walker then spends at most $c_2 + 2$ rounds to reach $z_0$ from the position which she has at the end of Stage II. Hence, at the time when Walker visits $z_0$, we must have $d_F(z_0) \geq \frac{10n}{11} - (c_2+2)(b+1) > \frac{9n}{10}$.

Let us consider now the remaining vertices $z_i$ with $i \leq \frac{n}{4}$. As long as $i \leq \frac{n}{4}$, we see that at most $4n + (c_2+2) + i + 2n < 7n$ rounds were played. Indeed, until the end of Stage II the game lasts at most $4n$ rounds, we may need $c_2 + 2$ rounds to reach the vertex $z_0$ at the beginning of Stage III, for each new identification of a vertex $z_j$ Walker chooses the edge $z_{j-1}z_j$, and the star attachments last in total at most $2n$ rounds. Thus, right before Walker's move from $z_{i-1}$ to $z_i$, we must have $e(B) \leq 7bn$ and $e(W) \leq 7n$. Now, let

$$X = \{v \in V(T_2) \setminus \{z_1, \ldots, z_{i-1}\} : \ d_F(v) \geq \frac{9n}{10}\}.$$

The number of vertices in $K_n$ of free-degree at most $\frac{9n}{10} - 1$ is at most

$$\frac{2(e(B)+e(W))}{n/10} \leq \frac{2(7bn+7n)}{n/10} = o(n).$$

Thus, $|X| \geq |V(T_2) \setminus \{z_1, \dots, z_{i-1}\}| - o(n) > \frac{n}{2}$.

Moreover, by the choice of $z_{i-1}$ and the size of the star attached to $z_{i-1}$, we obtain

$$d_F(z_{i-1}) \geq \frac{9n}{10} - (2b+1) \cdot \frac{|W_{i-1}| - d_B(z_{i-1}, W_{i-1})}{2b+1} \geq \frac{9n}{10} - |W_{i-1}| > \frac{8n}{9},$$

as $|W_i| \leq 50c_2 b$. Hence $d_F(z_{i-1}, X) > \frac{n}{8}$. Furthermore, since $e(B) < 7bn$, at most $\frac{n}{8}$ vertices in $X$ have Breaker-degree larger than $112b$. It follows that there is a vertex $z_i \in X$ such that $z_{i-1}z_i$ is a free edge and $d_B(z_i) \leq 112b$. □

**Claim 5.2.9** *Let $n$ be large enough. Assume that Walker can follow the strategy of Stage III. Then, as long as $i \leq \frac{n}{4}$ and $|W_i| \geq 400b$, we have that $|W_{i+1}| \leq \left(1 - \frac{1}{4b+2}\right)|W_i|$.*

**Proof** Analogously to Claim 5.2.7, we obtain

$$|W_{i+1}| \leq |W_i| - \left(\frac{|W_i| - d_B(z_i, W_i)}{2b+1} - 1\right) \leq \frac{2b}{2b+1}|W_i| + 60 \leq \left(1 - \frac{1}{4b+2}\right)|W_i|,$$

by the assumption on $|W_i|$ and since $d_B(z_i, W_i) \leq 112b$ before Walker starts to attach the star at $z_i$. □

**Claim 5.2.10** *Let $n$ be large enough. Assume that Walker can follow the strategy of Stage III. Then, there is a positive integer $t \leq \frac{n}{4}$ such that $|W_t| < 400b$.*

**Proof** As long as $|W_i| \geq 400b$, the previous claim tells us that

$$400b \leq |W_i| \leq \left(1 - \frac{1}{4b+2}\right)|W_{i-1}| \leq \ldots \leq \left(1 - \frac{1}{4b+2}\right)^{i-1} \cdot 50c_2 b < e^{-(i-1)/(4b+2)} \cdot 50c_2 b.$$

As $\frac{n}{b} = \Omega(\log^2 n)$, it follows that $i \leq \frac{n}{4}$. □

By the last claims, we now conclude that Walker can follow the strategy of Stage III until her graph touches all vertices but at most $400b$. Moreover, she needs to attach at most $\frac{n}{4}$ stars during Stage III until she finishes her graph. In total, she needs at most $7n$ rounds, as already discussed in the proof of Claim 5.2.8. Moreover, her final graph obviously has diameter at most $c_2 + 4 = 2\lfloor r \rfloor + 6$. □

## 5.3 Occupying a long cycle

### 5.3.1 The unbiased game

In the following we prove **Theorem 1.4.2**.

We start with **Breaker's part**. Assuming he plays as first player, his strategy is as follows. At the beginning of the game, he fixes a vertex $w_1$ which is not the start vertex $v_0$ of Walker. As long as Walker has a component of size less than $n-2$, Breaker's strategy is to claim the edge between $w_1$ and Walker's current position. In case this edge is not free, he claims another arbitrary edge. Note that this in particular means that Breaker's first edge is $v_0w_1$, and inductively Walker has no chance to make $w_1$ to become her next position, as long as her graph is a component of size smaller than $n-2$.

If Walker does not manage to create a component of size $n-2$, then there will not be a cycle of length $n-2$, and we are done. So, we can assume that there is a point in the game where Walker's component $K$ reaches size $n-2$, with $v'$ being the last vertex added to it. Let $w_2$ be the other vertex besides $w_1$ which does not belong to $K$, and note that the only free edges incident with $w_1$ are $w_1w_2$ and $w_1v'$. From now on, Breaker always claims the edge between $w_2$ and the current position of Walker, starting with $v'w_2$. If again this edge is not free, Breaker claims another arbitrary edge.

Now the following is easy to see. Walker will never visit the vertex $w_2$. In particular, she never claims the edge $w_1w_2$, which guarantees $d_W(w_1) \leq 1$ and $d_W(w_2) = 0$ throughout the game. That is, both vertices $w_1$ and $w_2$ will not participate in a cycle of Walker, and thus Breaker prevents cycles of length larger than $n-2$.

So, from now on let us focus on **Walker's part**, and let us assume that Breaker is the first player. We start with the following useful lemma, which roughly says that in case Walker manages to create a large cycle that touches almost every vertex, while certain properties on the distribution of Breaker's edges hold, Walker has a strategy to create an even larger cycle within a small number of rounds.

**Lemma 5.3.1** *Let $n$ be large enough. Assume a Walker-Breaker game is in progress, where Walker already claims a cycle $C$ with $n-125 \leq v(C) \leq n-3$, and with her current position being a vertex $x \in V(C)$. Assume further that Breaker claims at most $2n$ edges so far, and that there is at most one vertex $y \in V(K_n) \setminus V(C)$ with $d_B(y, V(C)) \geq \frac{n}{10}$. Then Walker has a strategy to create a cycle $C'$, with $V(C) \subset V(C')$ and $V(C') \neq V(C)$, within at most 25 further rounds.*

**Proof** In the next rounds, Walker goes along the edges of $C$ in an arbitrary direction, i.e. she repeats edges that she already claimed in earlier rounds, until she reaches a vertex $v \in V(C)$ with $d_B(v) \leq \frac{n}{10}$. For large $n$, this will take her at most 21 rounds, as $e(B) \leq 2n + O(1)$. Once she reached such a vertex, Walker fixes two vertices $v_1, v_2 \in V(K_n) \setminus V(C)$ such that $d_B(v_i, C) \leq \frac{n}{10} + 21$. It follows that $d_B(v_1, C) + d_B(v_2, C) + d_B(v, C) \leq \frac{3n}{10} + 42$ and so, by the pigeonhole principle, there exist three consecutive vertices $w_1, w_2, w_3$ on the cycle $C$ such that neither of the edges between $\{w_1, w_2, w_3\}$ and $\{v, v_1, v_2\}$ is claimed by Breaker. Walker's next move then is to proceed from $v$ to $w_2$. W.l.o.g. we can assume that Breaker in the following move does not claim any of the edges $w_j v_1$ with $j \in [3]$. Otherwise, we just interchange the vertices $v_1$ and $v_2$. Walker as next proceeds from $w_2$ to $v_1$, and in the following round she closes a cycle of length $v(C) + 1$ by proceeding to one of the vertices in $\{w_1, w_3\}$. □

With the above lemma in hand, we now can describe a strategy for Walker to create a cycle of length $n - 2$, for which we show later that she can always follow it, provided $n$ is large enough. Let $v_0$ be the start vertex of Walker. After Walker's move in round $t$, let $U_t$ be the set of vertices not touched by Walker so far, and let $v_t$ denote her current position. We split Walker's strategy into the following stages.

**Stage I.** Let Walker's $(t+1)^{\text{st}}$ move be in Stage I. Assume that Walker's graph is a path $P_t = (v_0, v_1, \ldots, v_t)$.

**Ia.** If $|U_t| \leq 120$, then Walker proceeds with Stage III.

**Ib.** If $|U_t| > 120$ and $v_t v_0$ is a free edge, then Walker takes this edge, closing a cycle, and sets $v_{t+1} := v_0$, and $\ell := t$. She proceeds with Stage II then.

**Ic.** If $|U_t| > 120$ and $v_t v_0$ is not free, Walker claims an arbitrary free edge $v_t w$ with $w \in U_t$, and sets $v_{t+1} := w$. She then repeats Stage I.

**Stage II.** Let Walker's $(t+1)^{\text{st}}$ move be in Stage II. Assume that Walker's graph is a cycle $C = (v_0, v_1, \ldots, v_\ell)$ of length $\ell + 1$, attached to a (maybe empty) path $P_t = (v_{\ell+1}, v_{\ell+2}, \ldots, v_t)$ with $v_{\ell+1} = v_0$, and with $v_t$ being the current position of Walker. Moreover, with $x$ denote the number of past rounds in which Walker followed Case **II.c.1**. We set

$$V_t := \{v \in U_t : \ d_B(v, V \setminus U_t) \geq \frac{n}{11}\}.$$

Moreover, in order to keep control on the distribution of Breaker's edges after each move of Walker in Stage II, we say that Property $P[t+1, x, i]$ is maintained if the following inequalities hold.

$$\text{Property } P[t+1, x, i] : \begin{cases} e_B(U_{t+1}) \leq 3x + 4 + i \\ d_B(v_{t+1}, U_{t+1}) + e_B(U_{t+1}) = e_B(U_t) \leq 3x + 5 + i \\ e_B(\{v_1, v_\ell\}, U_{t+1}) \leq 2(3x + 5 + i). \end{cases}$$

Now, Walker considers the following subcases:

**IIa.** If $|U_t| \leq 120$, then Walker proceeds with Stage IV.

**IIb.** If $|U_t| > 120$, and $V_t = \emptyset$, Walker claims an edge $v_t w$ with $w \in U_t$, and sets $v_{t+1} := w$ and $U_{t+1} := U_t \setminus \{w\}$, in such a way that immediately after her move Property $P[t+1, x, 0]$ holds. (The precise details of how to choose this edge are given later.) Walker in the next round repeats Stage II.

**IIc.** If $|U_t| > 120$, and if $V_t \neq \emptyset$, Walker considers two subcases:

**IIc.1.** If there is a free edge $v_t w$ with $w \in V_t$, Walker then claims such an edge. She sets $v_{t+1} := w$ and $U_{t+1} := U_t \setminus \{w\}$ (and thus $w \notin V_{t+1}$), and she increases $x$ by one. Moreover, she chooses $w$ in such a way that immediately after her move, $P[t+1, x, -1]$ holds with the new value of $x$. (The precise details are given later.) Then she repeats Stage II.

**IIc.2.** Otherwise, Walker proceeds with a free edge $v_t w$ with $w \in U_t$ such that $wz$ is free for every $z \in V_t$. She sets $v_{t+1} := w$ and $U_{t+1} := U_t \setminus \{w\}$. Moreover, she ensures that immediately after her move Property $P[t+1, x, 1]$ holds. (The precise details are given later.) She then repeats Stage II.

**Stage III.** Let Walker's $(t+1)^{\text{st}}$ move be in Stage III, and let Walker's graph be a path $(v_0, v_1, \ldots, v_t)$. Since $|U_t| \leq 120$, we have $t \geq n - 121$. Walker then claims an arbitrary free edge $v_t v_i$ with $0 \leq i \leq 4$, thus creating a cycle of length at least $n - 125$. Then she proceeds with Stage V.

**Stage IV.** Let $U$ be the set of untouched vertices, and $|U| \leq 120$, when Walker enters Stage IV. Within two rounds Walker creates a cycle of length at least $n - 120$, which covers every vertex that was visited by Walker so far. Then she proceeds with Stage V.

**Stage V.** When Walker enters Stage V her graph contains a cycle of length at least $n - 125$. She finally creates a cycle of length $n - 2$ by repeatedly applying the strategy given by Lemma 5.3.1.

It is obvious that if Walker can follow the proposed strategy, she will create a cycle of length $n - 2$. It thus remains to convince ourselves that, for large enough $n$, she can indeed do so. We consider all stages and substages separately.

**Stage I.** Before discussing Stage I, let us observe the following.

**Observation 5.3.2** *Assume that Walker did not leave Stage I before the $(t+1)^{st}$ round. Then immediately after her $t^{th}$ move her graph is a path $P_t = (v_0, v_1, \ldots, v_t)$ such that all but at most 2 Breaker edges do not belong to $E(v_0, V(P_t))$.*

**Proof** Walker does not leave Stage I, as long as she always plays according to Case Ic. Since in this stage she always proceeds from her current vertex to an untouched vertex, it is obvious that her graph is a path. Moreover, after round $t$, Breaker claims $t$ edges in total. Since Walker never followed Case Ib in an earlier round, Breaker needs to claim $v_0v_i \in E(v_0, V(P_t))$ for every $i \in \{2, \ldots, t-1\}$. □

It thus follows that, whenever Walker considers to play according to Stage I, her graph is a path, i.e. the assumption of Stage I is satisfied. There is nothing to prove in case Walker considers Case Ia or Case Ib. Moreover, she can follow Case Ic easily, since, by Observation 5.3.2 and by the assumption of Case Ic, before her $(t+1)^{\text{st}}$ move, we have $e_B(v_t, U_t) \leq 3 < 120 \leq |U_t|$.

**Stage II.** When Walker considers playing to Stage II, her previous move was in Stage I or Stage II. Since she only enters Stage II after closing a cycle in Stage Ib, and since in Stage II she always proceeds to a vertex from the set of untouched vertices, starting from $v_0$, it is obvious that her graph has the shape as described at the beginning of the strategy description for Stage II.

Moreover, after a move in Stage II in round $t+1$, we have $U_t = U_{t+1} \dot\cup \{v_{t+1}\}$ and thus $d_B(v_{t+1}, U_{t+1}) + e_B(U_{t+1}) = e_B(U_t)$ is guaranteed immediately.

To show that Walker can follow Stage II always, one may proceed by induction on the number of rounds in that stage.

Assume first that the $(t+1)^{\text{st}}$ round is the first round in Stage II. Then, Walker played according to Case Ib in round $t$ and in all the rounds before, she followed Case Ic. In particular, $x = 0$. Immediately before Walker's $(t+1)^{\text{st}}$ move, by Observation 5.3.2, Breaker has at most 4 edges that do not belong to $E(v_0, V(C))$, where $C$ is Walker's cycle at the end of Stage I. In particular $V_t = \emptyset$. We have $|U_t| > 120$, as Walker entered Stage II after Stage Ib, and therefore, Walker wants to follow Stage IIb. By our observation on the distribution of Breaker's edges, Walker can do so, as she can easily find a vertex $w \in U_t$ such that $v_tw$ is free. Moreover $P[t+1, x, 0]$ holds then with $v_{t+1} := w$ and $U_{t+1} = U_t \setminus \{w\}$, as $e_B(U_{t+1}) \leq e_B(U_t) \leq 4$ and $e_B(\{v_1, v_\ell\}, U_{t+1}) \leq 4$.

Assume then that the $(t+1)^{\text{st}}$ round happens in Stage II, but after the first round of Stage II, and assume that so far Walker could follow the strategy. To show that Walker can still follow the strategy, we discuss the different cases separately. In case Walker follows Case IIa, there is nothing to prove. Before discussing the other parts of Stage II, we observe the following upper bound on $x$ and the size of $V_t$.

**Observation 5.3.3** *Let $n$ be large enough. Assume Walker considers to play according to Stage II for her $(t+1)^{st}$ move, after she followed the strategy for the first $t$ rounds. Then $x, |V_t| \leq 11$.*

**Proof** The value of $x$ increases by one each time when Walker follows Stage IIc.1, where she enlarges her graph by a vertex of Breaker-degree at least $\frac{n}{11}$. If we had $x \geq 12$, then Breaker would have more than $n$ edges claimed already, also if $|V_t| \geq 12$. However, since Walker's graph contains only one cycle, we played at most $n$ rounds, a contradiction. □

**Case IIb.** Now, let us focus on Case IIb first and assume that so far, before this $(t+1)^{\text{st}}$ move, Walker could always follow the proposed strategy. Then in round $t$, Walker played according to IIb or IIc.

Assume first that Walker played according to Case IIb in round $t$. So, we know that before Breaker's $(t+1)^{\text{st}}$ move, Property $P[t, x, 0]$ was true, where $x \leq 11$.

If Breaker in his last move did not make any of the inequalities of Property $P[t, x, 0]$ invalid, then Walker takes $w \in U_t$ arbitrarily with $v_t w$ being free. This is possible, as $d_B(v_t, U_t) \leq 3x + 5 \leq 38 < |U_t|$, and it also guarantees Property $P[t+1, x, 0]$ immediately after Walker's move. Otherwise, Breaker makes at least one inequality of Property $P[t, x, 0]$ invalid. There are three cases to consider, which we discuss in the following.

**Case 1.** If Breaker with his $(t+1)^{\text{st}}$ move achieved that $e_B(U_t) = 3x + 5$, then in this move he claimed an edge in $U_t$. So, we then obtain that $d_B(v_t, U_t) + e_B(U_t) \leq 3x + 6$ and thus $d_B(v_t, U_t) \leq 1$, and that $e_B(\{v_1, v_\ell\}, U_t) \leq 2(3x + 5)$. Now, Walker finds a vertex $w \in U_t$ with $v_t w$ being free such that $d_B(w, U_t) \geq 1$. Walker claims such an edge, setting $v_{t+1} := w$, and then $P[t+1, x, 0]$ holds, since $e_B(U_{t+1}) = e_B(U_t) - d_B(w, U_t) \leq 3x + 4$, $d_B(v_{t+1}, U_{t+1}) + e_B(U_{t+1}) \leq 3x + 5$, and $e_B(\{v_1, v_\ell\}, U_{t+1}) \leq e_B(\{v_1, v_\ell\}, U_t) \leq 2(3x + 5)$.

**Case 2.** If after Breaker's $(t+1)^{\text{st}}$ move $e_B(U_t) \leq 3x + 4$ still holds, but we have $d_B(v_t, U_t) + e_B(U_t) = 3x + 6$, then we know that Breaker claimed an edge in $U_t \cup \{v_t\}$. Moreover, $d_B(v_t, U_t) \leq 3x + 6 \leq 39 < |U_t|$ and $e_B(\{v_1, v_\ell\}, U_t) \leq 2(3x + 5)$. Walker then takes $w \in U_t$ arbitrarily with $v_t w$ being free. After Walker's move we then obtain Property $P[t+1, x, 0]$, since then $d_B(v_{t+1}, U_{t+1}) + e_B(U_{t+1}) = e_B(U_t) \leq 3x + 4$, and we also have $e_B(\{v_1, v_\ell\}, U_{t+1}) \leq e_B(\{v_1, v_\ell\}, U_t) \leq 2(3x + 5)$.

**Case 3.** If the first two inequalities of Property $P[t, x, 0]$ still hold after Breaker's $(t+1)^{\text{st}}$ move, but $e_B(\{v_1, v_\ell\}, U_t) = 2(3x + 5) + 1$, then Breaker in his move claimed an edge between $\{v_1, v_\ell\}$ and $U_t$. Then there are at least $3x + 6$ vertices $w \in U_t$ with $d_B(w, \{v_1, v_\ell\}) \geq 1$, and for at least one such vertex $w$ Walker can claim the edge $v_t w$, as $d_B(v_t, U_t) \leq 3x + 5$. As before, Property $P[t+1, x, 0]$ is guaranteed to hold, as then we obtain $e_B(U_{t+1}) \leq e_B(U_t) \leq 3x + 4$, and $e_B(\{v_1, v_\ell\}, U_{t+1}) = e_B(\{v_1, v_\ell\}, U_t) - d_B(w, \{v_1, v_\ell\}) \le 2(3x+5)$.

Finally, assume that Walker played according to Case IIc in the $t^{\text{th}}$ round, and thus, before her $t^{\text{th}}$ move, we had $V_{t-1} \neq \emptyset$. Then Walker played according to IIc.1 in round $t$, since otherwise we had $V_t \supseteq V_{t-1} \neq \emptyset$, in contradiction to considering Case IIb for round $t+1$. In particular, the value of $x$ was increased in the $t^{\text{th}}$ round, so that $x \ge 1$, and immediately after Walker's $t^{\text{th}}$ move, we had Property $P[t, x, -1]$. Independent of Breaker's $(t+1)^{\text{st}}$ move, Walker just takes some vertex $w \in U_t$ with $v_t w$ being free, which she can do since $d_B(v_t, U_t) \le (3x+4) + 1 < |U_t|$. After proceeding as proposed, $P[t+1, x, 0]$ then holds, as $e_B(U_{t+1}) \le e_B(U_t) \le (3x+3)+1 = 3x+4$, and $e_B(\{v_1, v_\ell\}, U_{t+1}) \le 2(3x+4)+1 < 2(3x+5)$.

**Stage IIc.** Now, let us focus on Case IIc, and assume first that in round $t+1$ Walker wants to play according to Case IIc.1. By the assumption of Case IIc.1, Walker can claim the edge $v_t w$ easily. Now, let $x$ be given after the update of Stage II.c.1 in round $t+1$. Then, after Walker's move in round $t$ we had Property $P[t, x-1, 1]$, independent of whether round $t$ was played in Stage IIb, IIc.1 or IIc.2. Thus, no matter how Breaker chooses his $(t+1)^{\text{st}}$ edge, and how Walker chooses $w$ above, Property $P[t+1, x, -1]$ is maintained immediately after Walker sets $v_{t+1} = w$, $U_{t+1} = U_t \setminus \{w\}$. Indeed, we obtain $e_B(U_{t+1}) \le e_B(U_t) \le (3(x-1)+5)+1 = 3x+3$, and $e_B(\{v_1, v_\ell\}, U_{t+1}) \le e_B(\{v_1, v_\ell\}, U_t) \le 2(3(x-1)+6)+1 < 2(3x+4)$.

So, it remains to consider the case when Walker plays according to Stage IIc.2 for round $t+1$. Then, after Walker's move in round $t$ we had Property $P[t, x, i]$, with $i \in \{-1, 0, 1\}$ depending on whether round $t$ was played in Stage IIc.1, IIb or IIc.2, respectively. In any case this gives Property $P[t, x, 1]$. Using $x, |V_t| \le 11$, we know that after Breaker's $(t+1)^{\text{st}}$ move we have

$$\sum_{v \in V_t} d_B(v, U_t) + d_B(v_t, U_t) \le 2e_B(U_t) + d_B(v_t, U_t)$$
$$\le 2(e_B(U_t) + d_B(v_t, U_t)) \le 2((3x+6)+1) \le 80 < |U_t \setminus V_t|.$$

That is, Walker can choose a vertex $w$ as described in the strategy. In case Walker followed Case IIb or IIc.1 in round $t$, in which case we even had Property $P[t, x, 0]$ immediately after Walker's $t^{\text{th}}$ move, it can be seen easily that immediately after her $(t+1)^{\text{st}}$ move, we obtain $P[t+1, x, 1]$. So, assume Walker followed Case IIc.2 in round $t$. Then in round $t$ Walker chose $v_t \in U_t \setminus V_t$ in such a way that $v_t z$ was free for every $z \in V_{t-1} \subseteq V_t$. However, immediately before her move in round $t+1$ no such edge was free anymore, since Walker again follows Stage IIc.2. That is, in the current round we need to have $|V_t| = 1$ while Breaker in his last move claimed the unique edge $v_t z$ with $V_t = \{z\}$. It follows that immediately after Walker's move in round $t+1$ we have $e_B(U_{t+1}) \le e_B(U_t) \le 3x+5$ and $e_B(\{v_1, v_\ell\}, U_{t+1}) \le 2(3x+6)$, which implies Property $P[t+1, x, 1]$.

**Stage III.** When Walker enters Stage III in round $t+1$, then because she followed Stage Ia in the round before. In particular, all the rounds before she played according to Case Ic, and thus, when she enters Stage III, her graph is a path $P_t = (v_0, \ldots, v_t)$ with $n - v(P_t) = |U_t| \leq 120$, while Breaker claims all the edges $v_0v_i$ with $2 \leq i \leq t-1$. In particular, there has to be a free edge $v_tv_j$ with $0 \leq j \leq 4$, and Walker thus can follow the strategy and close a large cycle, which misses at most 125 vertices.

**Stage IV.** Say that Walker enters Stage IV in round $t+1$. Then her graph is a cycle $C = (v_0, v_1, \ldots, v_\ell)$ attached to a path $P_t = (v_{\ell+1}, v_{\ell+2}, \ldots, v_t)$ with $v_{\ell+1} = v_0$, and with $v_t$ being her current position. As she played according to Stage II in the $t^{\text{th}}$ round, we know that immediately after her previous move Property $P[t, x, 1]$ was true. In her first move in Stage IV, Walker proceeds to a vertex $w \in U_t$ such that $wv_1$, $wv_\ell$ and $wv_t$ are free, which is possible as after Breaker's $(t+1)^{\text{st}}$ move we have $\sum_{j\in\{1,2,t\}} d_B(v_j, U_t) \leq 3(3x+6)+1 \leq 118 < |U_t|$. In her second move, she then either claims $wv_1$ or $wv_\ell$, thus creating a cycle on $V \setminus U_{t+1}$.

**Stage V.** When Walker enters Stage V, her graph contains a cycle $C_0$ of length at least $n-125$, while less than $n$ rounds were played so far. We further observe in the following that outside the cycle there can be at most one vertex which has a large Breaker-degree towards the cycle.

**Observation 5.3.4** *When Walker enters Stage V there is at most one vertex $w \in V \setminus V(C_0)$ such that $d_B(w, V(C_0)) \geq \frac{n}{11} + 50$.*

**Proof** There are two possible ways that Walker enters Stage V.

The first way is that she played according to Stage III before, which she entered because of Stage Ia. That is, Walker created a path until $n-120$ vertices were touched, while in the meantime Breaker always blocked cycles by claiming edges that are incident with $v_0$. It follows then that $v_0$ is the only vertex which can have a Breaker-degree of size at least $\frac{n}{11}+50$.

The second way to enter Stage V is to play according to Stage IIa, until the number of untouched vertices drops down to 120, and then to reach Stage V through Stage IV. Assume in this case that there were two vertices $w_1, w_2 \in V \setminus V(C_0)$ such that $d_B(w_i, V(C_0)) \geq \frac{n}{11}+50$ for both $i \in [2]$, when Walker enters Stage V. It follows then that in all the 20 rounds $t$ before entering Stage IV both vertices were elements of the corresponding set $V_t$, as the degree $d_B(w_i, V \setminus U_t)$ can be increased by at most by 2 in each round. (Breaker may increase this value by one by claiming an edge incident with $w_i$, and Walker may increase this value by decreasing the set $U_t$ of untouched vertices.) That is, Walker always would have played according to Stage IIc in all these rounds. When she played according to Stage IIc.1, she walked to a vertex belonging to $V_t$, which then in Stage IV became part of her cycle. Otherwise, when Walker played according to IIc.2, then she proceeded to a vertex $w$ such that $ww_1$ and $ww_2$

were free. Since not both of these edges could be claimed by Breaker in the following round, Walker followed Stage IIc.1 afterwards; but as she did not proceed to $w_1$ or $w_2$, there must have been another vertex in the current set $V_t$ considered for enlarging her path. However, as at most 11 vertices may reach a Breaker-degree of at least $\frac{n}{11}$, it can happen at most 9 times that Walker chooses a vertex from $V_t$ different from $w_1$ and $w_2$ for enlarging the path. Thus, there must have been a round in Stage IIc.1 where Walker would have chosen $w_i$ to be her next position, for some $i \in [2]$. In Stage IV this vertex $w_i$ would have become a part of Walker's cycle, in contradiction to the assumption that $w_i \notin V(C_0)$. □

Now, with this observation in hand, the proof is clear. As long as Walker does not have a cycle of length $n-2$, Walker creates larger cycles $C_1, C_2, \ldots$, with $V(C_i) \subset V(C_{i+1})$, by applying Lemma 5.3.1 iteratively. As $v(C_0) \geq n - 125$ at the beginning of Stage V, and as by Lemma 5.3.1 it takes at most 25 rounds to maintain a larger cycle, Stage V will last less than 4000 rounds, until either Walker reaches a cycle of length $n-2$, or Walker cannot follow her strategy anymore. It follows, by Observation 5.3.4 and since $n$ is large, that throughout Stage V, there is always at most one vertex outside Walker's current cycle $C_i$ with Breaker-degree at least $\frac{n}{10}$ towards this cycle. Moreover, as we played at most $n$ rounds before entering Stage IV, we also have $e(B) \leq 2n$ throughout Stage V, for large $n$. Thus, throughout Stage V the conditions of Lemma 5.3.1 are always fulfilled, and therefore Walker can follow the proposed strategy until she reaches a cycle $C_i$ with $v(C_i) \geq n-2$. □

### 5.3.2 The biased game

In the following we give a proof for **Theorem 1.4.1**.

First of all, observe that Breaker can prevent any cycle of length larger than $n-b$. Indeed, assume that Walker starts the game, then immediately after her first move, Breaker fixes $b$ untouched vertices $u_1, \ldots, u_b$. From that point on, he always claims the edges between $u_1, \ldots, u_b$ and Walker's current position (in case they are still free). It follows then that Walker never visits the vertices $u_1, \ldots, u_b$ and thus, she cannot create a cycle of length larger than $n-b$.

Now, for Walker's part, let $0 < \varepsilon < 0.1$. Let $b \leq \frac{n}{\log^2(n)}$, let $p = \frac{\log(n) \cdot \log\log\log(n)}{n}$ and $c_2 = 2\lfloor r \rfloor + 6$ where $r = \frac{\log(n)}{\log(\frac{n}{200b})}$. Moreover, observe that $r = O(\frac{\log(n)}{\log\log(n)})$. In the following we explain how Walker can guarantee to create a cycle of length $n - O(b)$, provided $n$ is large enough. Walker first builds a graph $G' = (V', E')$ on $n - 400b$ vertices of diameter at most $c_2$ within at most $7n$ rounds. This she can do according to Proposition 5.2.2. Immediately afterwards, look at the induced graph $(W \cup B)[V']$. Since the number of edges in this graph is at most $(b+1)\cdot 7n \leq 14bn$, $V'$ contains a subset $V^*$ of size $N := n-(400+\frac{60}{\varepsilon})b = n(1-o(1))$ such that the induced graph $(W \cup B)[V^*]$ has maximum degree less than $\frac{\varepsilon n}{2}$. If we write $F'$ for the complement of $(W \cup B)[V^*]$ over $V^*$, i.e. $F'$ is the graph of free edges on $V^*$, then the assumptions of Theorem 5.1.4 hold. That is, $\delta(F') \geq (1-\varepsilon)N$, as $d_{F'}(v) \geq N - \frac{\varepsilon n}{2} \geq (1-\varepsilon)N$ for every $v \in V'$, for large enough $n$. Moreover, Walker claims a graph $W_0 = G'$ such that between each two vertices of $V^*$ there is a path of length at most $c_2$. Now $p \gg \frac{\log(N)}{N}$ and $\frac{\varepsilon}{30(c_2+1)p} = \omega(\frac{n}{\log^2(n)}) > b$. Moreover, the property $\mathcal{P}$ of containing a Hamilton cycle is $(p, 4\varepsilon)$-resilient, as follows from Theorem 5.1.1 (applied with $N$ instead of $n$). Thus, applying Theorem 5.1.4, we conclude that Walker has a strategy to continue the game in such a way that she creates a Hamilton cycle on $V^*$, i.e. a cycle of length $N = n - O(b)$, provided $n$ is large enough. □

## 5.4 Creating fixed graphs of constant size

With a slight modification of the proof of Theorem 1.4.1, we now can prove **Theorem 1.4.4**.

Let $G$ be given. Let $c_2 = 2\lfloor \frac{2m_2(G)}{m_2(G)-1} \rfloor + 6$ and let $\gamma > 0$ and $C > 1$ be given according to Theorem 5.1.2. Finally, set $c_W = \frac{\gamma}{400(c_2+1)C\cdot 2^{1/m_2(G)}}$ and let $b = c_W n^{\frac{1}{m_2(G)}}$.

In a first step, according to Proposition 5.2.2, Walker in the first $7n$ rounds creates a graph $G' = (V', E') \subset K_n$ on $n - 400b$ vertices with diameter at most $c_2$. Just notice that, for large enough $n$, this proposition tells us that she can ensure a diameter of size at most $2\lfloor r \rfloor + 6$, where $r = \frac{\log(n)}{\log(\frac{n}{200b})} < \frac{2m_2(G)}{m_2(G)-1}$. Immediately after Walker occupied $G'$, consider the induced graph $(W \cup B)[V']$ which has at most $7(b+1)n \leq 14bn$ edges. Then $V'$ contains a subset $V^*$ of size $N := \frac{n}{2}$ such that the induced graph $(W \cup B)[V^*]$ has maximum degree less than $30b$. If we write $F'$ for the complement of $(W \cup B)[V^*]$ on the vertex set $V^*$, then $F'$ is a graph on $N$ vertices whose minimum degree is at least $N - 30b = (1-o(1))N$. Moreover, Walker claims a graph $W_0 = G'$ such that between each two vertices in $V^*$ there is a path of length at most $c_2$. Let $p = CN^{-\frac{1}{m_2(G)}}$ and observe that $\frac{\gamma}{240(c_2+1)p} \geq b$. Moreover, by Theorem 5.1.2 the property $\mathcal{P}$ of containing a copy of $G$ is $(p, \gamma/2)$-resilient (applied with $N$ instead of $n$). Thus, by Theorem 5.1.4, has a strategy to continue the game in such a way that she creates a copy of $G$ on $V^*$ (applied with $\varepsilon = \gamma/8$). □

## 5.5 Occupying as many edges as possible

**Proof of Theorem 1.4.3** Whenever necessary assume $n$ to be large enough. In the following we first show that Walker can occupy $\frac{1}{b+1}\binom{n}{2} - c_1 n$ edges for some $c_1 > 0$, by giving a strategy for Walker; then we show that, playing against an optimally playing Breaker, she cannot occupy $\frac{1}{b+1}\binom{n}{2} - c_2 n$ edges for some constant $c_2 > 0$.

We start with the **lower bound**. Thus, in the following we give a strategy for Walker and afterwards we show that, using this strategy, Walker occupies $\frac{1}{b+1}\binom{n}{2} - c_1 n$ edges for some $c_1 > 0$. We split the strategy into two stages.

**Stage I.** Within at most $7n$ rounds, Walker creates a graph $G_1 = (X, E)$ on $n - 400b$ vertices with diameter 8. Afterwards, Walker proceeds with Stage II.

**Stage II.** Let Walker's $t^{\text{th}}$ move be in Stage II, and let $v_t$ be Walker's current position. We consider three cases.

**Case 1.** If there is a vertex $w \in X$ such that $v_t w$ is free, then Walker claims such an edge, setting $v_{t+1} := w$. Then she repeats Stage II.

**Case 2.** If there is no such vertex $w \in X$ with $v_t w$ being free, but there is a vertex $w' \in X$ with $d_F(w') \geq 500b$, then Walker sets $X := X \setminus \{v_t\}$, and in her next (at most) 8 moves, she walks to vertex $w'$, using the edges of $G_1$. Only afterwards, she repeats Stage II.

**Case 3.** Otherwise, in all other cases, Walker does an arbitrary move.

It is obvious that Walker can follow the above strategy. Indeed, Stage I is given by Proposition 5.2.2, and for Stage II there is nothing to prove. So, we only need to show that she will occupy the required number of edges, when following the proposed strategy. Observe that for this it is enough to show that Walker repeats only $O(n)$ edges during the first $\frac{1}{b+1}\binom{n}{2}$ rounds of the game.

There are three possibilities that Walker repeats edges that she already claimed. In Stage I, she may repeat edges when creating the graph $G_1$. In Case 2 of Stage II, she repeats at most 8 edges, when she returns to some vertex $w'$ as described by the strategy. However, Case 2 happens at most $n$ times, as the size of $X$ decreases each time that Walker follows that case. So, the first two possibilities lead only to $O(n)$ rounds, in which Walker repeats edges. Finally, in Case 3 of Stage II, Walker may ignore free edges by doing an arbitrary move. However, when this case happens, we have $d_F(v) < 500b$ for every $v \in X$, and $d_F(v, V(G_1)) = 0$ for every $v \in V(G_1) \setminus X$, i.e. at most $O(n)$ edges are still free.

Now, let us continue with the **upper bound.** W.l.o.g. let Breaker be the second player. As next we describe a strategy for Breaker and afterwards we prove that, by following this strategy, Breaker ensures that Walker cannot occupy more than $\frac{1}{b+1}\binom{n}{2} - c_2 n$ edges, for some constant $c_2 > 0$. In his strategy, Breaker never repeats edges. Thus, it will be enough to ensure that throughout the first $\frac{1}{b+1}\binom{n}{2}$ rounds of the game, Walker repeats $\Omega(n)$ edges.

In order to count the number of rounds in which Walker repeats edges, we define color assignments on the edges of $E(K_n)$, according to the following rules. Initially, all edges have no color (and most of them will not receive any). Assume that there is a vertex $v$ whose free-degree drops to 1, i.e. $d_F(v) = 1$. Then, at this moment, if the unique free edge $e$ incident with $v$ still has no color, we assign the color green to the edge $e$. Moreover, a green edge will be recolored with red, in case either of the players claims it.

Breaker now plays as follows. Assume that Walker's current position is the vertex $v \in V(K_n)$. Breaker then chooses his $b$ edges iteratively, by considering the following cases for each of his edges.

**Case 1.** If $d_F(v) = 1$, then Breaker claims the unique free edge incident with $v$. (If this edge is green, we recolor it with red.)

**Case 2.** If $d_F(v) \neq 1$ and there is a free edge $e$ without color assignment and $v \notin e$, then Breaker claims an arbitrary such edge. (In case the degree of some vertex drops down to 1, Breaker updates the color assignments accordingly.)

**Case 3.** If $d_F(v) \neq 1$ and all free edges are green or touch $v$, then Breaker claims a free edge, where he prefers edges without color assignments to those which are green.

Again, it is obvious that Breaker can follow the strategy, and that, each time he considers to claim an edge, exactly one of the cases needs to happen. So, we just need to prove that Walker repeats $\Omega(n)$ edges. We prove the following claim first.

**Claim 5.5.1** *Assume that Walker walks along a green edge $uw$, starting in $u$ and ending in $w$, and assume further that there will be at least one further round in the game. Then one of the following two statements is true:*

- *In the following round, Walker repeats an edge that she already claimed earlier.*
- *All free edges are green.*

**Proof** Assume that Walker walks along the green edge $e = uw$ in round $t$. As $e$ was green, we know that $e$ was free before Walker's move. Moreover, the edge $e = uw$ became green either at the moment when $d_F(u) = 1$ for the first time or at the moment when $d_F(w) = 1$ for the first time. It cannot happen that the free-degrees of both vertices $u$ and $w$ drop to 1 at the same time, as this would mean that $e = uw$ was claimed when both free-degrees dropped down to 1, in contradiction to the fact that $e$ was free.

If $e$ became green, when the free-degree of $w$ dropped to 1, then after Walker's $t^{\text{th}}$ move, we have $d_F(w) = 0$ and thus, in her next move, she needs to reuse an edge incident with $w$. So, let us assume that $e$ became green, when the free-degree of $u$ dropped to 1. In the previous round $t-1$, Walker then walked from some other vertex $z \neq u$ towards $u$. We distinguish two cases.

Case 1. After Walker's $(t-1)^{\text{st}}$ move we have $d_F(u) > 1$. Then, immediately after that move, $e$ was not green, as it becomes green when $d_F(u)$ drops to 1. However, since by assumption $e$ is green before Walker's $t^{\text{th}}$ move, we conclude that Breaker claimed edges incident with $u$ in his $(t-1)^{\text{st}}$ move. As he does so only in Case 3 of his strategy and only if all uncolored free edges are incident with $u$, it follows that after Breaker's $(t-1)^{\text{st}}$ move all free edges are green.

Case 2. After Walker's $(t-1)^{\text{st}}$ move we have $d_F(u) \leq 1$. Then, according to Breaker's strategy (Case 1), he claims the remaining edge incident with $u$, in case there is any, thus ensuring $d_F(u) = 0$ before Walker's $t^{\text{th}}$ move. But then $uw$ cannot be green immediately before Walker's $t^{\text{th}}$ move, a contradiction. □

Now, let $N$ be the number of edges that receive a color during the game. Obviously, we have $N \geq \lfloor \frac{n}{2} \rfloor$, as every vertex is incident to at least one such edge. Let $t^*$ be the the first round, after which there is no free edge without color assignment.

With $t_X^-$ denote the number of green edges claimed by player $X$ until round $t^*$ (including this round), and let $t_X^+$ denote the number of green edges claimed by player $X$ after round $t^*$. By the pigeonhole principle at least one of the values $t_W^-, t_W^+, t_B^-, t_B^+$ is of size at least $\frac{N}{4}$.

**Case 1.** $t_W^- \geq \frac{N}{4}$. By Claim 5.5.1, Walker then repeats at least $\frac{N}{4} - 1$ edges, as claiming a green edge ensures that in the following round she repeats an edge, as long as there exist free edges without color assignment.

**Case 2.** $t_B^- \geq \frac{N}{4}$. There are two cases when Breaker claims green edges during the first $t^*$ rounds. In Case 1 of her strategy, this happens if $d_F(v) = 1$ for the current position $v$ of Walker. In Case 3 of her strategy, this happens if every free edge is green. As the latter happens the earliest in round $t^*$, Breaker must claim at least $\frac{N}{4} - b$ edges according to Case 1 during the first $t^* - 1$ rounds. However, in Case 1 Breaker always claims the unique free edge incident with Walker's current position. So, each time when Case 1 happens, Walker afterwards needs to repeat some edge, and thus she repeats at least $\frac{N}{4} - b - 1$ edges.

**Case 3.** $t_X^+ \geq \frac{N}{4}$ for some $X \in \{B, W\}$. Then, there need to be at least $\frac{N}{4b}$ rounds after round $t^*$, in which the players claim green edges or repeat edges. (Note that Breaker never repeats edges according to the strategy.) But then Walker repeats at least $(\frac{N}{4b} - 2)/2$ edges, by the following claim.

**Claim 5.5.2** *For every* $1 \leq i \leq (\frac{N}{4b} - 2)/2$ *there is an* $r \in \{t^* + 2i, t^* + 2i + 1\}$ *such that Walker repeats an edge in round* $r$.

**Proof** Fix $i$ and consider $r = t^* + 2i$. If Walker repeats an edge in round $r$, then we are done. Otherwise, she claims a free edge $e = uw$, by walking from $u$ to $w$, and by definition of $t^*$, this edge is green. We will observe first that $e$ became green, at the moment when $d_F(w) = 1$ for the first time. Indeed, assume the contrary, i.e. that $d_F(u) = 1$ and that $d_F(w) > 1$ when $e$ became green. If after Walker's $(r-1)^{\text{st}}$ move we had $d_F(u) = 0$, then $uw$ would not be green anymore, a contradiction. If after Walker's $(r-1)^{\text{st}}$ move we had $d_F(u) = 1$, then according to Case 1 of his strategy, Breaker would have claimed $uw$ in round $r-1$, a contradiction. Otherwise, i.e. we had $d_F(u) \geq 2$; then after round $r - 2 \geq t^*$, $e = uw$ was a free edge without color assignment, a contradiction to the definition of $t^*$.

Thus, we know that $e$ became green when $d_F(w)$ dropped to 1, and as $e = vw$ is still green before Walker's $r^{\text{th}}$ move, we obtain that $d_F(w) = 1$ before Walker's $r^{\text{th}}$ move. But then $d_F(w) = 0$ after Walker's $r^{\text{th}}$ move, and so in round $r + 1 = t^* + 2i + 1$ she needs to repeat an edge, when proceeding from $w$ to some other vertex. □

So, in either case Walker needs to repeat $\Omega(n)$ edges. □

## 5.6 Concluding remarks

**Creating large subgraphs.** In Theorem 1.4.4 we studied (biased) Walker-Breaker games in which Walker aims to create a copy of some fixed graph of constant size. It also seems to be interesting to study which large subgraphs Walker can create in the unbiased game on $K_n$. As Breaker can prevent Walker from visiting every vertex of $K_n$, Walker cannot hope to occupy spanning structures. As shown in Theorem 1.4.2, Walker however can occupy a cycle of length $n-2$, and applying a similar method as in the proof of Theorem 1.4.2, we are also able to show that Walker can create a path of length $n-2$ (i.e. with $n-1$ vertices) within $n$ rounds. We wonder which other graphs (e.g. trees) on $n-1$ vertices Walker can create. Moreover, as already asked by Espig et. al. [17], it seems to be challenging to find the size of the largest clique that Walker can occupy. Notice that it is not hard to see that the answer is of order $\Theta(\log(n))$.

**When no repetitions are allowed.** To make Walker's life harder, it is natural to study a variant where she is not allowed to choose edges twice. All the problems discussed in this chapter can obviously be asked in this setting as well. In particular, we wonder how many edges Walker can occupy in the (unbiased) Walker-Breaker game on $K_n$, under this restriction.

The answer to this question is of order $\Theta(n^2)$, as is given by the following observation; but we do not know the precise size.

**Observation 5.6.1** *Let $\varepsilon > 0$. Playing an unbiased Walker-Breaker game on $K_n$, assume that Walker's current position is a vertex $v$ with $d_F(v) \geq \frac{n}{2} - 1$, and assume further that at most $\left(\frac{1}{16} - \epsilon\right) n^2$ rounds were played so far. Then Walker can proceed to a vertex $w$ with $vw \in E(F)$ and such that $d_F(W) \geq \frac{n}{2}$.*

**Proof** Let $G = B \cup W$ be the union of Breaker's and Walker's graph. Then the average degree of $G[N_F(v, V(K_n))]$ is bounded from above by $2e(G)/|N_F(v, V(K_n))| < \frac{n}{2} - 1$. Thus, there exists a vertex $w$ as we claimed. □

# Zusammenfassung

In dieser Dissertation werden kombinatorische Spiele auf Graphen mit zwei Spielern studiert. Der erste Spieler beansprucht oder orientiert stets genau eine Kante des gegebenen Graphen pro Runde, während der zweite Spieler $b$ Kanten wählt.

Wir betrachten das "strict oriented-cycle game", welches von Bollobás und Szabó definiert wurde. OMaker und OBreaker orientieren hierbei abwechselnd die Kanten des vollständigen Graphen $K_n$, wobei OMaker genau dann gewinnt, wenn ein gerichteter Kreis entsteht. Entgegen einer Vermutung von Bollobás und Szabó zeigten wir kürzlich in einem Projekt mit Liebenau, dass OBreaker eine Gewinnstrategie besitzt, falls $b \geq n - \Theta(\sqrt{n})$. In dieser Arbeit verbessern wir diese Schranke und zeigen, dass er sogar gewinnt, wenn $b \geq \frac{37}{40}n$.

Als zweites untersuchen wir das "tournament game", welches von Beck motiviert wurde. Wieder orientieren zwei Spieler, TMaker und TBreaker, abwechselnd die Kanten eines gegebenen Graphen $G$, wobei TMaker genau dann gewinnt, wenn ihre Kanten eine Kopie eines gegebenen Turniers $T$ mit $k$ Ecken induzieren. Wir bestimmen Schwellenwerte für den "bias" $b$, hinsichtlich der Eigenschaft, dass der erste Spieler eine Gewinnstrategie auf $G = K_n$ besitzt. Falls $G$ ein Zufallsgraph mit $n$ Ecken und Kanten-Wahrscheinlichkeit $p$ ist, bestimmen wir zudem entsprechende Schwellenwerte für die Wahrscheinlichkeit $p$.

Mit dem "tree embedding game" untersuchen wir schließlich ein Spiel, das zu den klassischen "Maker-Breaker"-Spielen gehört, welche unter anderem von Beck, Erdős, Hefetz, Krivelevich, Stojaković und Szabó studiert wurden. In diesem Spiel nehmen beide Spieler, Maker und Breaker, abwechselnd jeweils genau eine Kante von $K_n$ ein, wobei Maker das Ziel verfolgt, mit ihren Kanten eine Kopie eines aufspannenden Baums $T$ zu erzeugen. Wir zeigen, dass sie dieses Ziel für große $n \in \mathbb{N}$ stets in $n+1$ Runden erreichen kann, falls der Maximalgrad von $T$ durch eine Konstante beschränkt ist. Falls $T$ ein zufälliger aufspannender Baum ist, zeigen wir zudem, dass sie mit hoher Wahrscheinlichkeit innerhalb von $n-1$ Runden gewinnen kann.

Schließlich betrachten wir "Walker-Breaker"-Spiele, in denen Walker und Breaker abwechselnd Kanten des vollständigen Graphen $K_n$ einnehmen, aber mit der Einschränkung, dass die Kanten von Walker einen Kantenzug bilden. Bezugnehmend auf Fragen von Espig et. al. zeigen wir unter anderem, dass der größte Kreis, den Walker erzeugen kann, die Länge $n-\Theta(b)$ hat, wobei im Fall $b = 1$ die genaue Länge $n - 2$ ist. Dabei verwenden wir einen Ansatz von Ferber et. al., durch den es Walker gelingt, einen Graphen zu erzeugen, der sich nahezu wie ein Zufallsgraph verhält.

## Eidesstattliche Erklärung

Gemäß § 7(4) der Promotionsordnung des Fachbereichs Mathematik und Informatik der Freien Universität Berlin versichere ich hiermit, dass ich alle Hilfsmittel und Literatur angegeben und auf dieser Grundlage die Arbeit selbständig verfasst habe. Außerdem versichere ich, dass diese Arbeit zu keinem früheren Promotionsverfahren eingereicht wurde.

Berlin, den

Dennis Clemens

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## Curriculum Vitae

Mein Lebenslauf wird aus Gründen des Datenschutzes in der elektronischen Fassung meiner Arbeit nicht veröffentlicht.