Appendix. A Trivial Bang-Bang Example

Consider the trivial problem

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} tu(t)\,dt \to \min \quad \text{s.t.} \quad -1 \le u \le 1\,.$$

Although this is a very simple example, it illustrates the behavior of interior point approaches in the presence of bang-bang-control. Obviously, the solution is

$$u^*(t) = \begin{cases} -1 & t < 0 \\ 1 & t > 0 \end{cases}.$$

The primal-dual interior point approach

$$\begin{aligned} t - \eta_1 + \eta_2 &= 0 \\ (1-u)\eta_1 &= \mu \\ (1+u)\eta_2 &= \mu \end{aligned}$$

yields

$$\begin{aligned} u(\mu) &= \frac{-\mu + \sqrt{\mu^2 + t^2}}{t} \\ \eta_1(\mu) &= \frac{1}{2}(\mu + t + \sqrt{t^2 + \mu^2}) \\ \eta_2(\mu) &= \frac{1}{2}(\mu - t + \sqrt{t^2 + \mu^2})\,. \end{aligned}$$

For fixed $\mu > 0$ we have $u = \frac{t}{2\mu} + \mathcal{O}(t^3)$ and therefore no convergence in L_∞ for $\mu \to 0$ notwithstanding the pointwise convergence to the solution u^* almost everywhere. In contrast, the L_p-norm of the error

$$|\epsilon(\mu)| = 1 - \frac{-\mu + \sqrt{\mu^2 + t^2}}{|t|} \le \begin{cases} 1 & t < \mu \\ \frac{\mu}{t} & t \ge \mu \end{cases}$$

is bounded by

$$\begin{aligned} \|\epsilon(\mu)\|_p &\le \left(\int_{-\mu}^{\mu} 1\,dt + 2\mu^p \int_{\mu}^{\frac{1}{2}} \frac{1}{t^p}\,dt \right)^{\frac{1}{p}} \le \left(2\mu + 2\mu^p \frac{1}{1-p} \left(\frac{1}{2^{1-p}} - \mu^{1-p} \right) \right)^{\frac{1}{p}} \\ &\le \left(2\mu + 2\mu \frac{1}{p-1} \right)^{\frac{1}{p}} \le \left(\frac{2\mu p}{p-1} \right)^{\frac{1}{p}} = \mathcal{O}(\mu^{\frac{1}{p}}) \end{aligned}$$

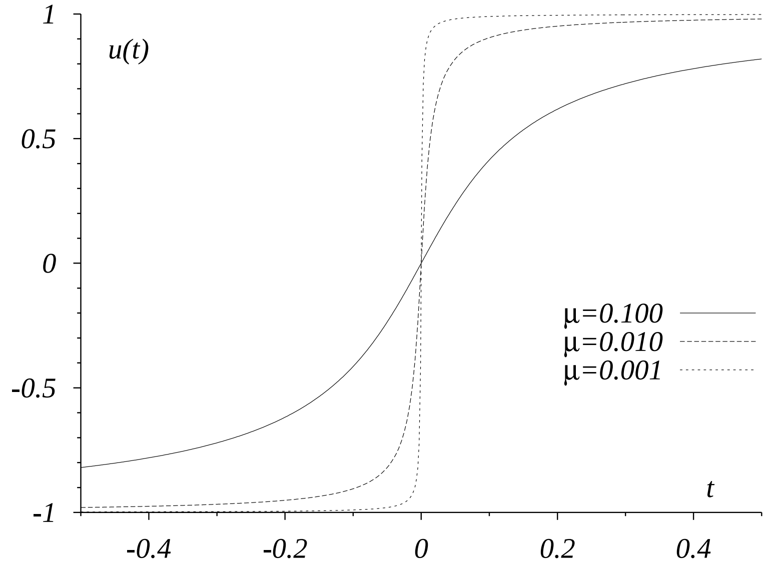

Figure 4.21: Central path solutions for $\mu = 10^{-1}, 10^{-2}, 10^{-3}$.

for $1 < p < \infty$ ($\mathcal{O}(\mu \ln \mu)$ holds for $p = 1$). Thus, the central path $u(\mu)$ converges to u^* in L^p for $p < \infty$, but its derivative is unbounded for $\mu \to 0$:

$$\|u'(\mu)\| = \frac{\mu - \sqrt{t^2 + \mu^2}}{|t|\sqrt{t^2 + \mu^2}} \leq \begin{cases} \frac{|t|}{2\mu^2} & |t| \leq \mu \\ \frac{3}{2|t|} & |t| \geq \mu \end{cases} ,$$

such that

$$\begin{aligned} \|u'(\mu)\|_p &\leq \left(\int_{-\mu}^{\mu} \left(\frac{|t|}{2\mu^2} \right)^p dt + 2 \int_{\mu}^{\frac{1}{2}} \left(\frac{3}{2t} \right)^p dt \right)^{\frac{1}{p}} \\ &= \left(2 \frac{\mu^{p+1}}{(p+1)\,(2\mu^2)^p} + 2 \frac{3^p}{2^p(1-p)} \left(\frac{1}{2^{1-p}} - \mu^{1-p} \right) \right)^{\frac{1}{p}} = \mathcal{O}(\mu^{\frac{1}{p}-1}) \end{aligned}$$

for $1 < p < \infty$. Nevertheless, the length of the path is of order

$$\int_{\mu=0}^{1} \|u'(\mu)\|_p \, d\mu = \mathcal{O}(p)\,.$$

Symbols

General Notation

K^+	dual (polar) cone of K
$\langle \cdot, \cdot \rangle$	dual pairing
int	topological interior
$\mathrm{co}\, S$	convex hull of S
$f \cdot g$	pointwise product of f and g
$\mathcal{L}(X, Y)$	continuous linear bounded operators from X to Y
$\\| \cdot \\|_{X \to Y}$	operator norm on $\mathcal{L}(X, Y)$
$\mathrm{im} A$	image (range) of A
$\lvert \cdot \rvert$	euklidean norm in $\mathbb{R}^n$

General Function Spaces

$C(\Omega)$	space of continuous functions on Ω
L_p	(real) Lebesgue space
W_p^k	(real) Sobolev space

Specific Function Spaces

X_u	space of control variables u
X_y	space of state variables y
X	$X_u \times X_y$, the space of the variables
Λ	space of the equality constraints multipliers
W_u	space of the control constraints slack variables and multipliers
W_y	space of the state constraints slack variables and multipliers
W	$W_u \times W_y$, the space of the slack variables and multipliers
V	$X \times \Lambda \times W \times W$, space of variables, slacks, and multipliers
Z	$\mathrm{im} F$, the image space of the complementarity formulation

Variables

u	control variables
y	state variables
x	(u, y) control and state variables
w^u	control constraints slack variables
w^y	state constraints slack variables
λ	equality constraints multipliers
η^u	control constraints multipliers
η^y	state constraints multipliers
η	$(\eta_u, \eta_y)^T$, control and state constraints multipliers
v	$(x, \lambda, \eta, w)^T$
μ, τ	complementarity continuation parameters, $\mu = e^{-\tau}$

Functions

J	cost functional
c	equality constraints
g	inequality constraints
L	$J - \langle \lambda, c \rangle - \langle \eta, g \rangle$, the Lagrangian
ψ, Ψ	complementarity functions