

Graphical calculus

The graphical calculus introduced below has been developed and used in many papers, e.g. [RT90, AC92b, Tur94], mainly in the setting of ribbon–Hopf algebras. Formally speaking, it consists of a functor from the braided monoidal category $\text{Rep } \mathcal{G}$ into a category of colored graphs. For an introduction into the category terminology see [Kas95, Tur94]. We will use the graphical notation to have a pictorial way to understand - and deduce - certain relations and identities between morphisms (intertwiners) in $\text{Rep } \mathcal{G}$, which - written out algebraically - would look very complicated.

As a central technical result we establish in Section B.2 a formula for $(S \otimes S)(R)$ and the relations between R^{-1} , $(S \otimes \text{id})(R)$ and $(\text{id} \otimes S^{-1})(R)$ for a quasitriangular $R \in \mathcal{G} \otimes \mathcal{G}$ in any (weak) quasi–Hopf algebra. Recall that in ordinary Hopf algebras the last three quantities coincide and therefore $(S \otimes S)(R) = R$. Section B.3 is devoted to the proof of the antipode properties for the antipode of the quantum double, which completes the proof of Theorem 4.4. Finally Section B.4 provides a graphical description of the construction of the diagonal crossed product.

We would like to point out that the graphical calculus allows to give nice intuitive interpretations of many of the very complicated identities arising in Chapter 2. In fact, without this graphical machinery we would have been lost in proving (or sometimes even trying to guess) these formulas. Also a purely algebraic proof of the formulas for R^{-1} and $(S \otimes S)(R)$ in Theorem B.2 would most likely be unreadable and therefore also untrustworthy. This is why we think it to be worthwhile to introduce this graphical technique in some detail and in particular to include Section B.4, rephrasing parts of Chapter 2.

B.1. Basic definitions

Let $\text{Rep } \mathcal{G}$ be the category of finite dimensional representations of a finite dimensional quasi–Hopf algebra as introduced in Section 2.1. Recall that morphisms (also called intertwiners) in $\text{Rep } \mathcal{G}$ are elements $t \in \text{Hom}_{\mathcal{G}}(U, V)$, i.e. linear maps $t : U \rightarrow V$ satisfying $t\pi_U(a) = \pi_V(a)t$, $\forall a \in \mathcal{G}$. As discussed in Section 2.1, the n –fold tensor product of \mathcal{G} –modules is again a \mathcal{G} –module (where one has to take care of the bracketing of the tensor factors). A morphism t from an n –fold to an m –fold tensor product of \mathcal{G} –modules is represented by a planar graph consisting of a “coupon” with n lower legs and m upper legs “colored” with the source and target modules respectively. The upper and lower legs are always equipped with a definite bracketing corresponding to the bracketing defining the associated tensor module. For example a morphism $t : X \boxtimes [(Y \boxtimes Z) \boxtimes U] \rightarrow (U \boxtimes V) \boxtimes X$ is represented by the following picture

$$\begin{array}{c}
 \begin{array}{ccc}
 U & V & X \\
 (& | &) \\
 \hline
 & t & \\
 \hline
 & [& (& | &) &] \\
 & X & Y & Z & U
 \end{array}
 \end{array}
 \tag{B.1}$$

Recall that we use the symbol \boxtimes to denote the tensor product in $\text{Rep } \mathcal{G}$ whereas \otimes denotes the tensor product in the category of vector spaces (with morphisms given by linear maps). The tensor product of two morphisms is represented as follows: just place a picture of the first morphism to the left of a picture of the second morphism. The composition $t \circ k$ of two morphisms is depicted by putting a picture of t on the top of a picture of k and gluing the corresponding free legs. Here one has to take care that the gluing $t \circ k$ is only admissible

if $\text{source}(t) = \text{target}(k)$, which in particular implies that the bracketing conventions of the associated tensor factors have to coincide. Note that a vertical arrow colored with (\mathbb{C}, ϵ) may be effaced from any picture without changing the morphism represented by the picture. This reflects the fact that the trivial representation (\mathbb{C}, ϵ) is the unit in $\text{Rep } \mathcal{G}$ with respect to the tensor product.

Following the conventions of [AC92b, AC92a] we now give a list of some special morphisms depicted by the following graphs:

$$\begin{array}{c} V \\ | \\ \hline \end{array} := \text{id}_V, \quad (B.2)$$

$$\begin{array}{c} V \quad *V \\ \searrow \quad \swarrow \\ \text{---} \\ \swarrow \quad \searrow \\ \mathbb{C} \end{array} := b_V, \quad \begin{array}{c} *V \quad V \\ \swarrow \quad \searrow \\ \text{---} \\ \swarrow \quad \searrow \\ *V \quad V \end{array} := a_V, \quad (B.3)$$

$$\begin{array}{c} W \quad V \\ \searrow \quad \swarrow \\ \text{---} \\ \swarrow \quad \searrow \\ V \quad W \end{array} := B_{VW}, \quad \begin{array}{c} V \quad W \\ \swarrow \quad \searrow \\ \text{---} \\ \swarrow \quad \searrow \\ W \quad V \end{array} := B_{VW}^{-1}, \quad (B.4)$$

where \mathbb{C} stands for the one dimensional representation given by the counit and where

$$\begin{aligned} b_V : \mathbb{C} &\longrightarrow V \otimes *V, \quad 1 \longmapsto \sum_i \beta \cdot v_i \otimes v^i \\ a_V : *V \otimes V &\longrightarrow \mathbb{C}, \quad \hat{v} \otimes w \longmapsto \langle \hat{v} | \alpha \cdot w \rangle \\ B_{VW} &= \tau_{VW} \circ R_{VW}, \quad B_{VW}^{-1} = R_{VW}^{-1} \circ \tau_{WV}, \end{aligned}$$

see also (A.2). Here $\{v_i\}$ is a basis of V with dual basis $\{v^i\}$ and τ_{VW} denotes the permutation of tensor factors in $V \otimes W$ and we have also used the shortcut notation $R_{VW} \equiv (\pi_V \otimes \pi_W)(R)$, etc. The properties of \mathcal{G} being a quasitriangular quasi-Hopf algebra ensure, that the above defined maps are in fact intertwiners (morphisms of \mathcal{G} -modules). Note that within higher tensor products the graphs (B.3) and (B.4) are only admissible if their legs are “bracketed together”. In order to change the bracket convention one has to use rebracketing morphisms. These are given as products of the basic elements

$$\begin{array}{c} | \quad (| \quad |) \\ \text{---} \\ (| \quad |) \quad | \\ U \quad V \quad W \end{array} = \phi_{UVW}, \quad \begin{array}{c} (| \quad |) \quad | \\ \text{---} \\ | \quad (| \quad |) \\ U \quad V \quad W \end{array} = \phi_{UVW}^{-1}, \quad (B.5)$$

where each of the three individual legs in (B.5) may again represent a tensor product of \mathcal{G} -modules. In this way we adopt the convention that any empty coupon with the same number of upper and lower legs - where the coloring only differs by the bracket convention - always represents the associated unique rebracketing morphism in $\text{Rep } \mathcal{G}$ given in terms of suitable products of ϕ 's. We have already remarked that the uniqueness of this rebracketing morphism (i.e. the independence of the chosen sequence of intermediate rebracketings) is guaranteed by McLanes coherence theorem and the “pentagon axiom” (2.9). This is why it is often not even necessary to spell out one of the possible formulas for such an intertwiner. Explicitly, the

pentagon identity (2.9) may for example be expressed as

$$(B.6)$$

which is the graphical notation for

$$(\text{id} \otimes \Delta \otimes \text{id})(\phi) (\phi \otimes \mathbf{1}) (\Delta \otimes \text{id} \otimes \text{id})(\phi^{-1}) = (\mathbf{1} \otimes \phi^{-1}) (\text{id} \otimes \text{id} \otimes \Delta)(\phi).$$

As done in the above pictures we will frequently not specify the modules sitting at the source and target legs. Also note that by Eqs.(2.11) and (2.12) the rebracketing of the (invisible) “white” leg corresponding to the trivial \mathcal{G} -module \mathbb{C} is always given by the trivial identification.

If \mathcal{G} is finite dimensional, it may itself be viewed as a \mathcal{G} -module under left multiplication and algebraic identities may directly be translated into identities of the corresponding graphs and vice versa. So e.g. axiom (2.16) is equivalent to the rigidity identities (compare with (A.3))

$$(B.7)$$

and Eqs. (2.11) and (2.12) together with (2.15) imply

$$(B.8)$$

and

$$(B.9)$$

as well as the upside-down and left-right mirror images of (B.8) and (B.9) and the graphs obtained by rotating by 180° in the drawing plane. In general, with every graphical rule, where the graph is build from elementary graphs of the above list, the rotated as well as the upside-down and left-right mirror images are also valid and are proven analogously. This induces a $\mathbb{Z}_2 \times \mathbb{Z}_2$ - symmetry action on all graphical identities given below, which in fact is already apparent in the axioms of a quasi-triangular quasi-Hopf algebra given in Section 2.1 by taking \mathcal{G}^{op} , \mathcal{G}_{cop} or \mathcal{G}_{cop}^{op} instead of \mathcal{G} .

One may also draw the graphical identities corresponding to the properties (2.18) and (2.19) of the R -matrix. Using also the antipode property (2.15) and the morphisms (B.3) one obtains

the following two identities

(B.10)

(B.11)

where we have used the trivial identification $*(V^*) = V$. Finally we point out the important “*pull through*” rule saying that morphisms built from representation matrices of special elements in \mathcal{G} (like the braiding (B.4) or the reassociator (B.5)) always “commute” with all other intertwiners in the appropriate sense i.e. by changing colors and orderings accordingly. For example one has

(B.12)

In the language of categories this means that the braiding-s and the reassociators provide *natural transformations* [Mac71], the so-called commutativity and associativity constraints of the tensor product. Moreover we may summarize the above stated properties by the following remark, which with some formal effort could as well be formulated as a Theorem.

REMARK B.1. Two diagrams, built according to the above rules from coupons as in (B.1) and from the special diagrams (B.2)-(B.4) by composing and tensoring (i.e. gluing and juxtaposition), represent the same morphism if the following conditions are satisfied:

- (i) The source legs of both diagrams (i.e. the number of legs, the coloring and the bracketing) coincide as well as the target legs.
- (ii) Take the diagrams as graphs in three dimensional space, where the picture of the braiding suggest which line is viewed as lying behind the other. Also forget about all empty coupons (i.e. the reassociators) and replace them by the corresponding number of straight vertical lines. Then the obtained graphs can be continuously deformed into each other, with source and target legs kept fixed.

More precisely the description of the category $\text{Rep } \mathcal{G}$ by the above graphical rules defines a functor into a suitably defined category of tangles in 3-dim. space, which respects all structural properties of $\text{Rep } \mathcal{G}$. This expresses the fascinating relationship between algebraic properties

of quasitriangular quasi-Hopf algebras and the topology of tangles (links, knots), see [RT90, AC92b] for more details.

B.2. The antipode image of the R-matrix

In this section we exploit the full power of our graphical machinery by proving various important identities involving the action of the antipode on a quasitriangular R -matrix. We recall that for ordinary Hopf algebras (i.e. where ϕ, α and β are trivial) one has $(S \otimes \text{id})(R) = R^{-1} = (\text{id} \otimes S^{-1})(R)$ and $(S \otimes S)(R) = R$. To generalize these identities to the quasi-Hopf setting recall the definitions of the four elements $p_\lambda, q_\lambda, p_\rho, q_\rho$ given in (2.77)-(2.78). For the special case $\lambda = \rho = \Delta$ they are elements of $\mathcal{G} \otimes \mathcal{G}$ and, using the notation (2.13), are given by

$$\begin{aligned} p_\lambda &:= Y^i S^{-1}(X^i \beta) \otimes Z^i, & p_\rho &:= \bar{X}^i \otimes \bar{Y}^i \beta S(\bar{Z}^i) \\ q_\lambda &:= S(\bar{X}^i) \alpha \bar{Y}^i \otimes \bar{Z}^i, & q_\rho &:= X^i \otimes S^{-1}(\alpha Z^i) Y^i \end{aligned} \quad (\text{B.13})$$

A graphical interpretation of these elements will be given in (B.18) below. With these definitions we now have

THEOREM B.2. *Let $(\mathcal{G}, \Delta, \phi, S, \alpha, \beta)$ be a finite dimensional quasi-Hopf algebra, let $\gamma \in \mathcal{G} \otimes \mathcal{G}$ be as in (2.24) and let $R \in \mathcal{G} \otimes \mathcal{G}$ be quasitriangular. Then*

$$\begin{aligned} R^{-1} &= [X^j \beta S(\bar{X}^i Y^j) \otimes \mathbf{1}] \cdot [(S \otimes \text{id})(q_\rho^{op} R)] \cdot [(\bar{Z}^i \otimes \bar{Y}^i) \Delta^{op}(Z^j)] \\ &= [\Delta(\bar{Z}^j) (Y^i \otimes Z^i)] \cdot [(\text{id} \otimes S^{-1})(R p_\rho)] \cdot [\mathbf{1} \otimes S^{-1}(\alpha \bar{Y}^j X^i) \bar{X}^j] \end{aligned} \quad (\text{B.14})$$

$$(S \otimes S)(R) \gamma = \gamma^{21} R \quad (\text{B.15})$$

The formula (B.14) has already been given in [AC92a]. A direct implication of equation (B.15) is the following formula, which has been stated without proof in [AC92b].

COROLLARY B.3. *Under the conditions of Theorem B.2 let $f \in \mathcal{G} \otimes \mathcal{G}$ be the twist defined in (2.26), then*

$$f^{op} R f^{-1} = (S \otimes S)(R) \quad (\text{B.16})$$

PROOF OF COROLLARY B.3. Using the formula (2.26) for f and (B.15) one computes

$$\begin{aligned} f^{op} R &= (S \otimes S)(\Delta(\bar{X}^i)) \gamma^{op} \Delta^{op}(\bar{Y}^i \beta \bar{Z}^i) R \\ &= (S \otimes S)(\Delta(\bar{X}^i)) (S \otimes S)(R) \gamma \Delta(\bar{Y}^i \beta \bar{Z}^i) \\ &= (S \otimes S)(R) (S \otimes S)(\Delta^{op}(\bar{X}^i)) \gamma \Delta(\bar{Y}^i \beta \bar{Z}^i) = (S \otimes S)(R) f. \end{aligned}$$

□

To prepare the proof of Theorem B.2 we state some further graphical identities. First we have

LEMMA B.4.

$$(\text{Diagrammatic expression}) = (\text{Diagrammatic expression}) \quad (\text{B.17})$$

and three mirror images.

PROOF. This is straightforward and left to the reader. (Use first (B.6), then (B.8) and (B.9) (or the suitable mirror image) and finally (B.7)). □

To give an algebraic formulation of the four identities of Lemma B.4 let us introduce the notation

$$\begin{array}{cc}
P_{VW}^\lambda := \begin{array}{c} V^* \quad V \quad W \\ | \quad (| \quad |) \\ \hline (| \quad |) \\ | \end{array} & P_{WV}^\rho := \begin{array}{c} W \quad V \quad *V \\ (| \quad |) \quad | \\ \hline | \quad (| \quad |) \\ | \end{array} \\
Q_{VW}^\lambda := \begin{array}{c} (| \quad |) \\ \hline | \quad (| \quad |) \\ *V \quad V \quad W \end{array} & Q_{WV}^\rho := \begin{array}{c} | \quad (| \quad |) \\ \hline (| \quad |) \quad | \\ W \quad V \quad V^* \end{array}
\end{array} \tag{B.18}$$

Algebraically, these morphisms are given by (using the module notation)

$$\begin{aligned}
P_{VW}^\lambda : w &\mapsto v^i \otimes p_\lambda \cdot (v_i \otimes w), & P_{WV}^\rho : w &\mapsto p_\rho \cdot (w \otimes v_i) \otimes v^i, \\
Q_{VW}^\lambda : \hat{v} \otimes v \otimes w &\mapsto (\hat{v} \otimes \text{id})(q_\lambda \cdot (v \otimes w)), & Q_{WV}^\rho : w \otimes v \otimes \hat{v} &\mapsto (\text{id} \otimes \hat{v})(q_\rho \cdot (w \otimes v))
\end{aligned} \tag{B.19}$$

Note that the identities (2.103a/2.103b) and (2.104a/2.104b) precisely reflect the fact that these maps are morphisms in $\text{Rep } \mathcal{G}$. In this way Lemma B.4 is also contained in Lemma 2.21, since it is equivalent to the four identities (2.103c/2.103d), (2.104c/2.104d). We remark that the morphisms (B.19) have already been defined in Chapter A, (A.8). The reader is invited to translate also the commuting diagram (A.10) into a graphical identity.

Next we define intertwiners $g_{VW} : (*W \boxtimes *V) \boxtimes (V \boxtimes W) \rightarrow \mathbb{C}$ and $d_{VW} : \mathbb{C} \rightarrow (V \boxtimes W) \boxtimes (*W \boxtimes *V)$ by

$$\begin{array}{cc}
g_{VW} := \begin{array}{c} \text{---} \text{---} \text{---} \\ | \quad (| \quad |) \\ \hline (| \quad |) \quad | \\ *W \quad *V \quad V \quad W \end{array} & d_{VW} := \begin{array}{c} (| \quad |) \quad (| \quad |) \\ \hline | \quad (| \quad |) \\ \text{---} \text{---} \text{---} \\ V \quad W \quad *W \quad *V \end{array}
\end{array} \tag{B.20}$$

One directly verifies that these morphisms represent the elements $\gamma, \delta \in \mathcal{G} \otimes \mathcal{G}$ defined in (2.24), (2.25) as follows

$$\begin{aligned}
g_{VW}(\hat{w} \otimes \hat{v} \otimes v \otimes w) &= \langle \hat{v} \otimes \hat{w} | \gamma \cdot (v \otimes w) \rangle \\
d_{VW}(1) &= \sum_i (\delta \cdot (v_i \otimes w_i)) \otimes (w^i \otimes v^i).
\end{aligned} \tag{B.21}$$

Here $\hat{v} \in *V$, $\hat{w} \in *W$, $v \in V$, $w \in W$ and $\{v_i \otimes w_i\}$ is a basis of $V \otimes W$ with dual basis $\{v^i \otimes w^i\} \in *V \otimes *W$.

We remark that in terms of these intertwiners the first identity in (2.29) may now be depicted as (a similar picture may be drawn for the second identity)

$$\begin{array}{c} (V \otimes W)^* \quad V \otimes W \\ | \quad | \\ \text{---} \text{---} \text{---} \\ | \quad | \\ \hline (| \quad |) \quad (| \quad |) \\ \hline | \quad | \\ (V \otimes W)^* \quad V \otimes W \end{array} = \begin{array}{c} (V \otimes W)^* \quad V \otimes W \\ | \quad | \\ \boxed{f} \quad \boxed{\text{id}_{V \otimes W}} \\ (| \quad |) \quad (| \quad |) \\ \hline \boxed{d_{V^*W^*}} \\ | \quad | \\ (V \otimes W)^* \quad V \otimes W \end{array} \tag{B.22}$$

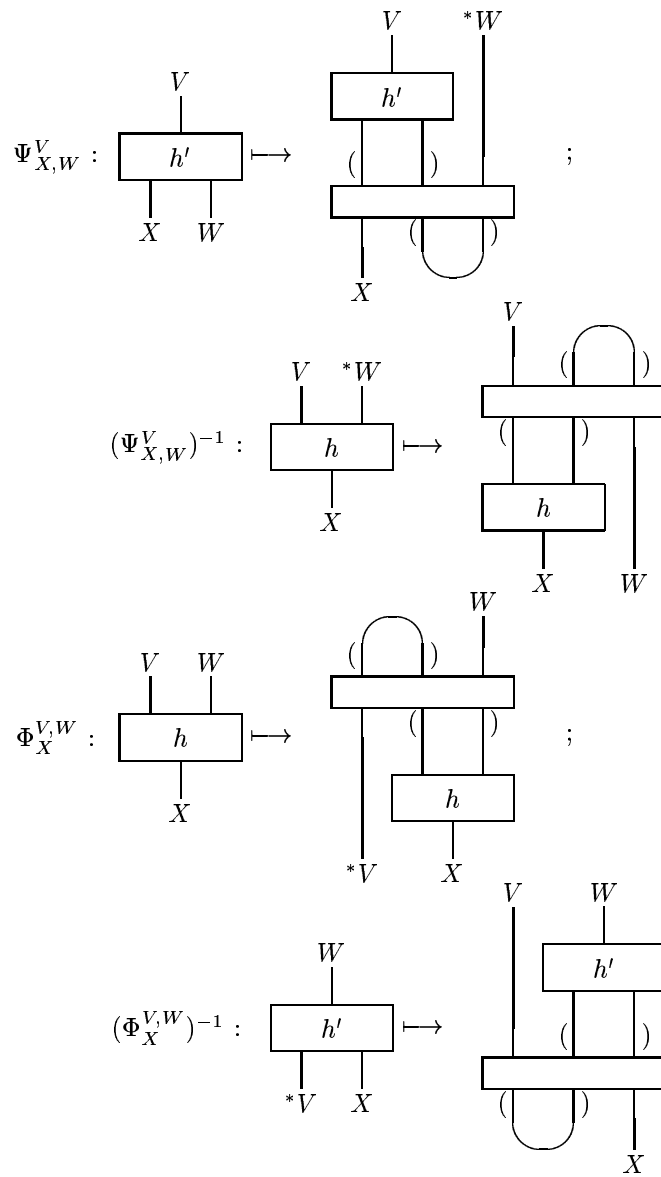
Finally we note the following linear isomorphisms of intertwiner spaces holding in fact in any rigid monoidal category.

LEMMA B.5. *Let X, V, W be finite dimensional \mathcal{G} -modules. Then there exist linear bijections*

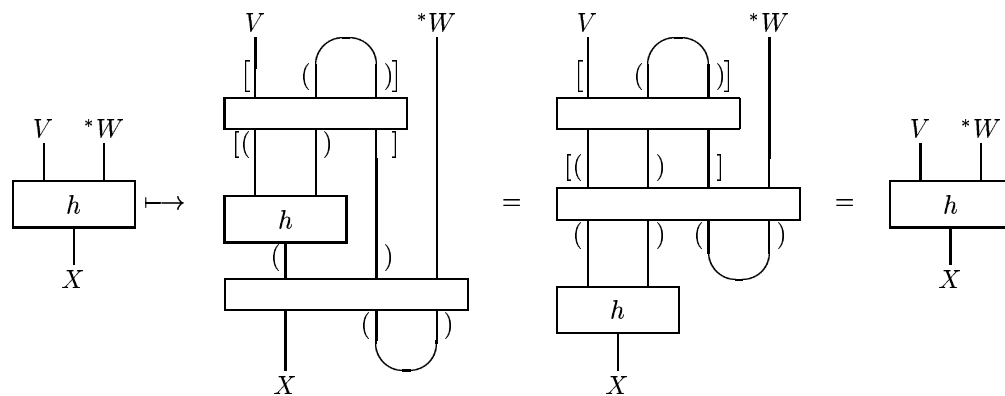
$$\Psi_{X,W}^V : \text{Hom}_{\mathcal{G}}(X \otimes W, V) \longrightarrow \text{Hom}_{\mathcal{G}}(X, V \otimes *W) \tag{B.23}$$

$$\Phi_X^{V,W} : \text{Hom}_{\mathcal{G}}(X, V \otimes W) \longrightarrow \text{Hom}_{\mathcal{G}}(*V \otimes X, W) \tag{B.24}$$

given by



PROOF. We prove $\Psi_{X,W}^V \circ (\Psi_{X,W}^V)^{-1} = \text{id}$ by determine its action on $h \in \text{Hom}_{\mathcal{G}}(X, V \otimes *W)$ as follows:



where in the first equality we have used a “pull through” rule for h , and in the second equality a left–right mirror image of (B.17). Analogously one shows that $\Psi^{-1} \circ \Psi = \text{id}$ and $\Phi \circ \Phi^{-1} = \text{id} = \Phi^{-1} \circ \Phi$. \square

This implies the following

COROLLARY B.6. Let V, W, X, Y be finite dimensional \mathcal{G} -modules with intertwiners

$$h : {}^*X \otimes {}^*Y \longrightarrow {}^*W \otimes {}^*V, \quad h' : V \otimes W \longrightarrow Y \otimes X$$

then the following two identities are equivalent

(B.25)

(B.26)

PROOF. This is proved by bending up the *X leg to the left and the W leg to the right in both sides of (B.25) using the isomorphisms stated in Lemma B.5 after a suitable rebracketing of the source legs. Then one uses remark B.1 to identify the obtained pictures with (B.26). Instead of using Remark B.1 one may also give an explicit chain of rebracketings, see [HNb]. We leave the details to the reader. \square

We are now in the position to prove Eqs. (B.14) and (B.15) stated in Theorem B.2 by rewriting them as graphical identities as follows:

LEMMA B.7. For all finite dimensional \mathcal{G} -modules the inverse braiding B_{UV}^{-1} obeys

(B.27)

and the (left) conjugate braiding $B_{{}^*U{}^*V}$ obeys

(B.28)

Taking \mathcal{G} itself as a \mathcal{G} -module yields (B.27) \Leftrightarrow (B.14) and (B.28) \Leftrightarrow (B.15) and therefore the above identities prove Theorem B.2.

PROOF. To prove the first equation in (B.27) we recall the identity (B.10). “Bending up” the left leg by applying the isomorphism $(\Phi_{U \otimes V}^{U,V})^{-1}$ of Lemma B.5 to both sides of (B.10) and using Lemma B.4 for the r.h.s. we end up with

$$(B.29)$$

Hence we get the first equality in (B.27)¹. Analogously one starts with (B.11) and takes the mirror image of the above proof, which yields the second equality in (B.27).

Eq. (B.28) follows from Corollary B.6 by putting $h' = B_{UV}$ and $h = B_{U^*V}$. Note that one could alternatively apply the Remark B.1.

The identifications (B.27) \Leftrightarrow (B.14) and (B.28) \Leftrightarrow (B.15) are straight forward and are left to the reader. This concludes the proof of Lemma B.7 and therefore also of Theorem B.2. \square

Adaption to the weak case. We have already discussed in Chapter A that the representation category $\text{Rep } \mathcal{G}$ of a weak quasi Hopf algebra is still monoidal and rigid, where due to the fact that $\Delta(\mathbf{1}) \neq \mathbf{1} \otimes \mathbf{1}$, the tensor product is truncated, see (A.12). Thus the graphical calculus described so far carries over to the present case. As before the collection of colored upper (or lower) legs represent the (truncated) tensor product of \mathcal{G} -modules associated with the individual legs. One just has to take care when translating the pictures into algebraic identities. For example the graph

$$(B.30)$$

is a pictorial representation of $\Delta(\mathbf{1})$ and not of $\mathbf{1} \otimes \mathbf{1}$! Thus the graph (B.17) is equivalent to the algebraic identity $[S(p_\lambda^1) \otimes \mathbf{1}] q_\lambda \Delta(p_\lambda^2) = \Delta(\mathbf{1})$ in place of $[S(p_\lambda^1) \otimes \mathbf{1}] q_\lambda \Delta(p_\lambda^2) = \Delta(\mathbf{1})$, etc. In this way all graphical identities stated so far stay valid as well as Theorem B.2, where now R^{-1} is meant to be the quasi-inverse of R .

B.3. The antipode in the quantum double $\mathcal{D}(\mathcal{G})$

We will now use the graphical calculus to conclude the proof of the antipode properties for the antipode S_D of the quantum double as defined in Theorem 4.4.

LEMMA B.8. *In the setting of Theorem 4.4 the following holds.*

- (i) *The universal flip operator $\mathbf{D} \in \mathcal{G} \otimes \mathcal{D}(\mathcal{G})$ is (quasi-)invertible where the (quasi-)inverse is given by*

$$\mathbf{D}^{-1} = [X^j \beta S(\bar{X}^i Y^j) \otimes \mathbf{1}] \cdot [(S \otimes \text{id})(q_\rho^{op} \mathbf{D})] \cdot [(\bar{Y}^i \otimes \bar{Z}^i) \Delta_{op}(Z^j)], \quad (B.31)$$

with $q_\rho \in \mathcal{G} \otimes \mathcal{G}$ being defined in (B.13).

- (ii) *Let μ_D denote the multiplication map $\mu_D : \mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{G}) \longrightarrow \mathcal{D}(\mathcal{G})$, then*

$$(\text{id} \otimes \mu_D) \circ (\text{id} \otimes S_D \otimes \text{id}) \left((\text{id} \otimes \Delta_D)(\mathbf{D}) \cdot (\mathbf{1}_G \otimes \mathbf{1}_G \otimes \alpha_D) \right) = \mathbf{1}_G \otimes \alpha_D \quad (B.32)$$

$$(\text{id} \otimes \mu_D) \circ (\text{id} \otimes \text{id} \otimes S_D) \left((\text{id} \otimes \Delta_D)(\mathbf{D}) \cdot (\mathbf{1}_G \otimes \beta_D \otimes \mathbf{1}_G) \right) = \mathbf{1}_G \otimes \beta_D \quad (B.33)$$

¹By finite dimensionality it suffices to prove the left inverse property.

PROOF. We warn the reader that the following discussion will be quite subtle. We will use the graphical methods adopted in this Section B.1/B.2. To this end let us view \mathcal{G} and $\mathcal{D} \equiv \mathcal{D}(\mathcal{G})$ as left \mathcal{G} -modules. Then, due to (4.2), $\tilde{B}_{\mathcal{G}\mathcal{D}} := \tau_{\mathcal{G}\mathcal{D}} \circ \mathbf{D}$ defines an intertwiner $\tilde{B}_{\mathcal{G}\mathcal{D}} : \mathcal{G} \otimes \mathcal{D} \rightarrow \mathcal{D} \otimes \mathcal{G}$ which will be depicted as

$$\tilde{B}_{\mathcal{G}\mathcal{D}} =: \begin{array}{c} \mathcal{D} \quad \mathcal{G} \\ \diagdown \quad \diagup \\ \mathcal{G} \quad \mathcal{D} \end{array}$$

For the left modules ${}^*\mathcal{G}$ and ${}^*\mathcal{D}$ the corresponding intertwiners $\tilde{B}_{{}^*\mathcal{G}\mathcal{D}}$, $\tilde{B}_{\mathcal{G}{}^*\mathcal{D}}$, $\tilde{B}_{{}^*\mathcal{G}{}^*\mathcal{D}}$ are defined with the help of the map S and/or S_D . Graphically they are represented by the same picture, except that the colors of the legs are replaced by ${}^*\mathcal{G}$ and(or) ${}^*\mathcal{D}$, respectively. The reason for distinguishing the \mathcal{D} -line from the \mathcal{G} -line lies in the fact that unlike in (B.4) $\tilde{B}_{\mathcal{G}\mathcal{D}}$ is **not** given in terms of a quasitriangular R -matrix in $\mathcal{G} \otimes \mathcal{G}$, which is why we write $\tilde{B}_{\mathcal{G}\mathcal{D}}$ in place of $B_{\mathcal{G}\mathcal{D}}$. Correspondingly, the identities derived in Section B.2 are **not** automatically valid for $\tilde{B}_{\mathcal{G}\mathcal{D}}$. We now show, which of them still hold. First, since S is an antipode for Δ , Eq. (4.3) together with $(\epsilon \otimes \text{id})(\mathbf{D}) = \mathbf{1}_{\mathcal{D}}$ implies the equality (compare with (B.10))

(B.34)

and a step by step repetition of the prove of (B.29) yields

(B.35)

This means that algebraically we get the analogue of the first identity in (B.14) which yields (B.31). Thus we have proven part (i).

To prove (ii) let us translate the two claims (B.32), (B.33) into the graphical language as

$$(B.32) \Leftrightarrow \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \end{array} \quad (B.36)$$

The diagram on the left of (B.36) consists of two parts. The top part shows a box with a loop on top and a vertical line on the right labeled \mathcal{G} . Below the box, a vertical line descends and then crosses a dotted line from the right to the left. The bottom part shows a box with a vertical line on the left labeled \mathcal{G} and a vertical line on the right labeled \mathcal{D} . A dotted line crosses from the left to the right between the two boxes.

The diagram on the right of (B.36) shows a box with a loop on top. Below the box, three vertical lines descend, labeled from left to right as \mathcal{G} , ${}^*\mathcal{D}$, and \mathcal{D} .

$$(B.33) \Leftrightarrow \begin{array}{c} \text{Diagram 4} \\ \text{Diagram 5} \end{array} = \begin{array}{c} \text{Diagram 6} \end{array} \quad (B.37)$$

The diagram on the left of (B.37) consists of two parts. The top part shows a box with a vertical line on the left labeled \mathcal{D} , a vertical line on the right labeled \mathcal{G} , and a dotted line crossing from right to left. Below the box, a vertical line descends and then crosses a dotted line from the left to the right. The bottom part shows a box with a vertical line on the left labeled \mathcal{G} and a loop on the right.

The diagram on the right of (B.37) shows a box with a loop on top. Below the box, three vertical lines descend, labeled from left to right as \mathcal{D} , ${}^*\mathcal{D}$, and \mathcal{G} .

Note that as opposed to (B.34) the identities (B.36) and (B.37) are not automatically satisfied, since $S_{\mathcal{D}}$ is not yet proved to be an antipode for $\Delta_{\mathcal{D}}$. To prove (B.36) and (B.37) we now proceed backwards along the proof of Lemma B.7, i.e. we use Lemma B.5 to show that either of these two identities is equivalent to

$$\begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} = \begin{array}{c} \text{Diagram 9} \end{array} := (\tilde{B}_{\mathcal{G} \circ \mathcal{D}})^{-1} \quad (B.38)$$

The diagram on the left of (B.38) consists of two parts. The top part shows a box with a loop on top, a vertical line on the right labeled \mathcal{G} , and a vertical line on the right labeled ${}^*\mathcal{D}$. Below the box, a vertical line descends and then crosses a dotted line from the left to the right. The bottom part shows a box with a vertical line on the left labeled ${}^*\mathcal{D}$ and a vertical line on the right labeled \mathcal{G} . A dotted line crosses from the left to the right between the two boxes.

The diagram on the right of (B.38) shows a dotted line crossing from the top-left to the bottom-right, with a vertical line on the left labeled ${}^*\mathcal{D}$ and a vertical line on the right labeled \mathcal{G} .

More precisely (B.36) is equivalent to (B.38) just as (B.11) is equivalent to the second equation in (B.27), and “rotating” this proof by 180° in the drawing plane we also get (B.37) \Leftrightarrow (B.38). Thus we are left with proving (B.38). To this end we remark, that (B.34) equally holds if we replace \mathcal{D} by ${}^*\mathcal{D}$, and therefore (B.35) also holds with \mathcal{D} replaced by ${}^*\mathcal{D}$. Hence (B.38) follows

from (B.35) provided we can show

$$(B.39)$$

By Lemma B.6 this is further equivalent to

$$(B.40)$$

Using (B.20) and (B.21), Eq. (B.40) is algebraically equivalent to

$$\gamma^{op} \mathbf{D} = (S \otimes S_D)(\mathbf{D}) \gamma,$$

which finally holds by (4.14),(4.2) and (2.29). This concludes the proof of Lemma B.8 (ii) and therefore of Theorem 4.4. \square

B.4. Graphical description of the diagonal crossed product

In this Section we review the construction of diagonal crossed products using the graphical language. Since we have already identified the δ -implementers and $\lambda\rho$ -intertwiners in terms of certain morphisms in a representation category in Chapter A, we may also present them in our graphical calculus. We will only treat the special case of the quantum double $\mathcal{D}(\mathcal{G})$, i.e. where $\mathcal{M} = \mathcal{G}$ and $\lambda = \rho = \Delta$, but our graphical calculus could easily be generalized to treat the general case. We will also convince the reader that one could prove most of the results of Chapter 2, especially Proposition 2.19, graphically.

Let us proceed to the details. For simplicity let us assume that $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, the adaption to the weak should by now be obvious (the pictures do not change, whereas the corresponding algebraic identities have to be equipped with the correct projections, see (B.30)). First recall that any unital algebra map $\gamma : \mathcal{G} \rightarrow \mathcal{A}$ defines a \mathcal{G} -module structure on \mathcal{A} via $b \cdot A \equiv \pi_{\mathcal{A}}(b)A := \gamma(b)A$. Moreover, given two \mathcal{G} -modules V, W , then $(V \otimes \mathcal{A}) \otimes W$ becomes a \mathcal{G} -module by setting $\pi_{(V \otimes \mathcal{A}) \otimes W}(a) := (\pi_V \boxtimes \pi_{\mathcal{A}}) \boxtimes \pi_W = (\pi_V \otimes \pi_{\mathcal{A}} \otimes \pi_W)(\delta(a))$, where $\delta := \delta_l = (\Delta \otimes \text{id}) \circ \Delta$. Consider also \mathcal{G} as a \mathcal{G} -module by left multiplication and let $\mathbf{L} \in \mathcal{G} \otimes \mathcal{A}$ be a coherent left δ -implementer, i.e.

$$[\mathbf{1}_{\mathcal{G}} \otimes \gamma(a)] \mathbf{L} = [S^{-1}(a_{(1)}) \otimes \mathbf{1}_{\mathcal{A}}] \mathbf{L} [a_{(-1)} \otimes \gamma(a_{(0)})], \quad \forall a \in \mathcal{G} \quad (B.41)$$

$$\mathbf{L}^{13} \mathbf{L}^{23} = [\Omega^5 \otimes \Omega^4 \otimes \mathbf{1}_{\mathcal{A}}][(\Delta \otimes \text{id})(\mathbf{L})][\Omega^1 \otimes \Omega^2 \otimes \gamma(\Omega^3)], \quad (B.42)$$

where $\delta(a) \equiv a_{(-1)} \otimes a_{(0)} \otimes a_{(1)}$ and

$$\begin{aligned} \Omega &:= (\text{id}^{\otimes 3} \otimes S^{-1} \otimes S^{-1})(f^{45} \cdot \Psi^{-1}) \\ \Psi &:= [(\text{id} \otimes \Delta \otimes \text{id})(\phi) \otimes \mathbf{1}][\phi \otimes \mathbf{1} \otimes \mathbf{1}][(\delta \otimes \text{id} \otimes \text{id})(\phi^{-1})], \end{aligned} \quad (B.43)$$

see (2.41),(2.55). We now define the map

$$l_{\mathcal{G}}^{\mathcal{A}} : (\mathcal{G} \otimes \mathcal{A}) \otimes \mathcal{G}^* \longrightarrow \mathcal{A}, \quad (b \otimes A \otimes \varphi) \mapsto (\varphi \otimes \text{id})(\mathbf{L} \cdot (b \otimes A)),$$

as in (A.13). Then Eq. (B.41) is equivalent to $t_{\mathcal{G}}^A$ being an intertwiner of \mathcal{G} -modules, i.e. to $t_{\mathcal{G}}^A \in \text{Hom}_{\mathcal{G}}((\mathcal{G} \otimes \mathcal{A}) \otimes \mathcal{G}^*, \mathcal{G})$. We depict this intertwiner as

$$t_{\mathcal{G}}^A := \begin{array}{c} \mathcal{A} \\ | \\ \boxed{\mathbf{L}} \\ \begin{array}{ccc} \downarrow & | & \downarrow \\ \mathcal{G} & \mathcal{A} & \mathcal{G}^* \end{array} \end{array} .$$

This way the ‘‘coherence condition’’ (B.42) becomes equivalent to the following graphical identity

$$\begin{array}{c} \mathcal{A} \\ | \\ \boxed{\mathbf{L}} \\ \begin{array}{ccc} \downarrow & | & \downarrow \\ \mathcal{G} & \boxed{\mathbf{L}} & \mathcal{G}^* \end{array} \end{array} = \begin{array}{c} \mathcal{A} \\ | \\ \boxed{\mathbf{L}} \\ \begin{array}{ccc} \downarrow & | & \downarrow \\ \boxed{\text{id}_{\mathcal{G} \otimes \mathcal{G}}} & | & \boxed{f} \\ \downarrow & | & \downarrow \\ \mathcal{G} & \mathcal{A} & \mathcal{G}^* \end{array} \\ \boxed{\Psi^{-1}} \\ \begin{array}{ccc} \downarrow & | & \downarrow \\ \mathcal{G} & \mathcal{A} & \mathcal{G}^* \end{array} \end{array} \quad (\text{B.44})$$

Note that the lowest box on the r.h.s. represents the rebracketing morphism Ψ^{-1} defined in (B.43). Thus we again see, how the complicated multiplication rule in $\hat{\mathcal{G}} \bowtie \mathcal{G}$ involving Ψ and f is due to the fact that the tensor product of \mathcal{G} -modules is not associative anymore if the coproduct fails to be coassociative.

To arrive at Δ -flip operators (i.e. $\lambda\rho$ -intertwiners with $\lambda = \rho = \Delta$) we use the isomorphism $\Psi_{(\mathcal{G} \otimes \mathcal{A}), \mathcal{G}^*}^A$ of Lemma B.5, to define the intertwiner $t_{\mathcal{G}}^A \in \text{Hom}_{\mathcal{G}}(\mathcal{G} \otimes \mathcal{A}, \mathcal{A} \otimes \mathcal{G})$ by

$$t_{\mathcal{G}}^A \equiv \begin{array}{c} \mathcal{A} \quad \mathcal{G} \\ \swarrow \quad \searrow \\ \mathcal{G} \quad \mathcal{A} \end{array} := \begin{array}{c} \mathcal{A} \quad \mathcal{G} \\ | \quad | \\ \boxed{\mathbf{L}} \\ \downarrow \quad | \\ \mathcal{G} \quad \mathcal{A} \\ \downarrow \quad | \\ \mathcal{G} \quad \mathcal{A} \end{array} \quad (\text{B.45})$$

Algebraically Definition (B.45) translate into $t_{\mathcal{G}}^A(b \otimes A) = \mathbf{T}^{21} \cdot (A \otimes b)$, where $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ is expressed in terms of \mathbf{L} by $\mathbf{T} := [S^{-1}(p_{\rho}^2) \otimes \mathbf{1}] \cdot \mathbf{L} \cdot (\text{id} \otimes \gamma)(\Delta(p_{\rho}^1))$, compare with (2.84). Now note that the property of $t_{\mathcal{G}}^A$ being an intertwiner of \mathcal{G} -modules is equivalent to \mathbf{T} being a Δ -flip operator. Moreover according to Lemma B.5 we also have

$$\begin{array}{c} \mathcal{A} \\ | \\ \boxed{\mathbf{L}} \\ \begin{array}{ccc} \downarrow & | & \downarrow \\ \mathcal{G} & \mathcal{A} & \mathcal{G}^* \end{array} \end{array} = \begin{array}{c} \mathcal{A} \quad \mathcal{G} \\ | \quad | \\ \boxed{\mathbf{T}} \\ \downarrow \quad | \\ \mathcal{G} \quad \mathcal{A} \\ \downarrow \quad | \\ \mathcal{G} \quad \mathcal{A} \end{array}$$

This implies that \mathbf{T} is a Δ -flip operator if and only if \mathbf{L} is a left δ -implementer. Also note that the last graphical identity is equivalent to $\mathbf{L} = q_{\rho}^{op} \mathbf{T}$. Thus we have proven the one-to-one correspondence between δ -implementers and Δ -flip operators as stated in Proposition 2.19 (recall that we have chosen $\delta := \delta_l$).

To see that the map $\mathbf{L} \leftrightarrow \mathbf{T}$ is also coherence preserving, we note that \mathbf{T} is coherent (see Definition 2.17), iff the following graphical identity holds

(B.46)

Thus the map $\mathbf{L} \leftrightarrow \mathbf{T}$ is coherence preserving if and only if the graphical identities (B.46) and (B.44) are equivalent. Let us prove $(B.44) \Rightarrow (B.46)$. Using the definition (B.45) we get for the l.h.s. of (B.46)

l.h.s =
of (B.46)

Here we have used a pull through rule to push the lower \mathbf{L} -box up and then we have combined all rebracketing morphisms in one box. Now plugging in (B.44), splitting the rebracketing

morphism at the bottom into four factors and pushing two of them up, one obtains

l.h.s. =
of (B.46)

=

Using the identity (B.22) the last picture equals the r.h.s. of (B.46). Hence we have shown (B.44) \Rightarrow (B.46). The implication (B.46) \Rightarrow (B.44) is shown similarly by bending the two upper \mathcal{G} -legs in (B.46) down again. We remark that instead of going carefully through the sequence of rebracketings as done above, we could have used Remark B.1.