APPENDIX A

Representation theoretic interpretation

As has already been indicated in Section 2.1, the algebraic properties of a quasi–Hopf algebra $\mathcal{G}$ may be translated into corresponding properties of its representation category $\operatorname{Rep}\mathcal{G}$. This chapter gives a more detailed discussion, where we also provide a representation theoretic interpretation of our notions of left, right and two–sided coactions. We also describe the diagonal crossed products in representation theoretical terms. (In fact this has already be done by proving Theorem 2.1, since algebra maps may be viewed as representations and vice versa.) In particular the complicated formulas given in Lemma 2.21 will be shown to be quite natural by identifying them with certain commuting diagrams. Also the δ–implementers and $\lambda\rho$–intertwiners will give rise to certain morphism, where the coherence conditions may as well be expressed by some commuting diagrams. As an application we show that the category $\operatorname{Rep}\mathcal{D}(\mathcal{G})$ of finite dimensional representations of the quantum double $\mathcal{D}(\mathcal{G})$ coincides with what has been called the double category of $\mathcal{G}$–modules by S. Majid [Maj97].

This chapter is strongly related to Appendix B, where we introduce a graphical calculus, which also relies on representation categorical considerations.

Let $\mathcal{G}$ be a quasi–Hopf algebra with invertible antipode S. Let $\operatorname{Rep}\mathcal{M}$ and $\operatorname{Rep}\mathcal{G}$ be the category of unital representations of $\mathcal{M}$ and $\mathcal{G}$, respectively, where in $\operatorname{Rep}\mathcal{G}$ we only mean to speak of finite dimensional representations. We denote the objects in $\operatorname{Rep}\mathcal{G}$ by $(U,\pi_U),(V,\pi_V),(W,\pi_W),\ldots$, where $U,V,W\ldots$ denote the underlying representation spaces and $\pi_V:\mathcal{G}\to\operatorname{End}_{\mathbb{C}}(V)$ the representation maps. Similarly, we denote the objects in $\operatorname{Rep}\mathcal{M}$ by $(\mathfrak{H},\gamma_{\mathfrak{H}}),(\mathfrak{K},\gamma_{\mathfrak{K}}),(\mathfrak{L},\gamma_{\mathfrak{L}}),\ldots$, where the Gothic symbols denote the representation spaces and where $\gamma_{\mathfrak{H}}:\mathcal{M}\to\operatorname{End}_{\mathbb{C}}(\mathfrak{H})$, etc. We will also freely use the $\mathcal{G}$–module notation by writing $a\cdot v:=\pi_V(a)v$ and $V\equiv(V,\pi_V)$ (and analogously for $\mathcal{M}$–modules $\mathfrak{H}$). The set of morphisms $\operatorname{Hom}_{\mathcal{G}}(U,V)$ (also called intertwiners) is given by the linear maps $f:U\to V$ satisfying $f\,\pi_U(a)=\pi_V(a)\,f,\ \forall a\in\mathcal{G}$.

The representation category of a quasi–Hopf algebra. It is well known (see e.g. [Dri90]) that for quasi–Hopf algebras $\mathcal{G}$ the category $\operatorname{Rep}\mathcal{G}$ becomes a rigid monoidal category, where the tensor product $(V\boxtimes W,\pi_V\boxtimes\pi_W)$ of two representations (V,π_V) and (W,π_W) is given by

$$V\boxtimes W:=V\otimes W,\quad \pi_V\boxtimes\pi_W:=(\pi_V\otimes\pi_W)\circ\Delta \tag{A.1}$$

whereas for morphisms ($\equiv\mathcal{G}$–module intertwiners) f,g one has $f\boxtimes g:=f\otimes g$. (The symbol $\otimes$ always denotes the usual tensor product in the category of vector spaces.) The associativity isomorphisms are given in terms of the reassociator ϕ by the natural family of $\mathcal{G}$–module isomorphisms

$$\phi_{UVW}:(U\boxtimes V)\boxtimes W\longrightarrow U\boxtimes(V\boxtimes W),\quad \phi_{UVW}:=(\pi_U\otimes\pi_V\otimes\pi_W)(\phi)$$

The unit object (with respect to $\boxtimes$) in $\operatorname{Rep}\mathcal{G}$ is given by $(\mathbb{C},\epsilon)$. Throughout, if $\mathbb{C}$ is viewed as a $\mathcal{G}$–module it is always meant to be equipped with the module structure given by the one dimensional representation ϵ. The left and right dual of any representation (V,π_V) are defined by ${}^*V=V^*=\hat{V}:=\operatorname{Hom}_{\mathbb{C}}(V,\mathbb{C})$ and

$$\pi_{{}^*V}:=\pi_V^t\circ S,\quad \pi_{V^*}:=\pi_V^t\circ S^{-1},$$

where ^{t} denotes the transposed map. The (left) rigidity structure is given by the family of morphisms ($\mathcal{G}$–module intertwiners)

$$\begin{aligned} a_V&:{}^*V\boxtimes V\longrightarrow\mathbb{C}, && \hat{v}\otimes v\longmapsto\langle\hat{v}\mid\alpha\cdot v\rangle\\ b_V&:\mathbb{C}\longrightarrow V\boxtimes{}^*V, && 1\longmapsto\beta\cdot v_i\otimes v^i,\end{aligned} \tag{A.2}$$

where $v_i \in V$ and $v^i \in \hat{V}$ are a choice of dual bases and where $\alpha, \beta \in \mathcal{G}$ are the elements defined in (2.15), (2.16). Drinfel'd's antipode axioms for $\mathcal{G}$ precisely reflect the fact that a_V and b_V are morphisms in $\operatorname{Rep}\mathcal{G}$ fulfilling the *rigidity identities*

$$\begin{aligned}(\mathrm{id}_V \boxtimes a_V) \circ \phi_{V(^*V)V} \circ (b_V \boxtimes \mathrm{id}_V) &= \mathrm{id}_V \\ (a_V \boxtimes \mathrm{id}_{^*V}) \circ \phi^{-1}_{(^*V)V(^*V)} \circ (\mathrm{id}_{^*V} \boxtimes b_V) &= \mathrm{id}_{^*V}.\end{aligned} \tag{A.3}$$

Also note that one has $^*(V^*) = (^*V)^* = V$ with trivial identification.

Next, we recall that in any left–rigid monoidal category one has natural isomorphisms $^*(U \boxtimes V) \cong {^*V} \boxtimes {^*U}$. As already mentioned in our case these isomorphisms are given by

$$f_{UV} : U \boxtimes V \longrightarrow (^*V \boxtimes {^*U})^*, \quad u \otimes v \mapsto (\pi_V \otimes \pi_U)(f)\,(v \otimes u) \tag{A.4}$$

where we trivially identify the vector spaces $V \otimes W \equiv (\hat{V} \otimes \hat{W})^\wedge$ and where the twist $f \in \mathcal{G} \otimes \mathcal{G}$ is given by (2.26). The fact that f_{UV} is indeed a morphism in $\operatorname{Rep}\mathcal{G}$ follows from (2.28). Similarly, by (2.31), we have a natural family of isomorphisms

$$h_{UV} : U \boxtimes V \longrightarrow {^*(V^* \boxtimes U^*)}, \quad u \otimes v \mapsto (\pi_V \otimes \pi_U)(h)\,(v \otimes u)$$

Coactions. Now a left $\mathcal{G}$–coaction (λ, ϕ_λ) on $\mathcal{M}$ naturally induces a *left action* of $\operatorname{Rep}\mathcal{G}$ on $\operatorname{Rep}\mathcal{M}$. By this we mean a functor

$$\odot : \operatorname{Rep}\mathcal{G} \times \operatorname{Rep}\mathcal{M} \longrightarrow \operatorname{Rep}\mathcal{M},$$

where for $(V, \pi) \in \operatorname{Rep}\mathcal{G}$ and $(\mathfrak{H}, \gamma) \in \operatorname{Rep}\mathcal{M}$ we define $(V \odot \mathfrak{H}, \pi \odot \gamma) \in \operatorname{Rep}\mathcal{M}$ by

$$V \odot \mathfrak{H} = V \otimes \mathfrak{H}, \quad \pi \odot \gamma := (\pi \otimes \gamma) \circ \lambda,$$

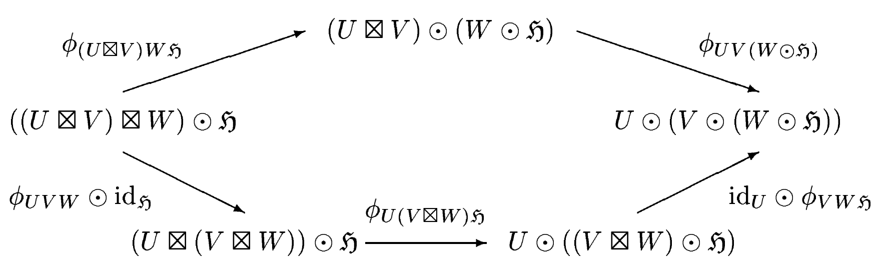
whereas for morphisms we put $f \odot g := f \otimes g$. The counit axiom for λ implies $\epsilon \odot \gamma_\mathfrak{H} = \gamma_\mathfrak{H}$ for all $(\mathfrak{H}, \gamma_\mathfrak{H}) \in \operatorname{Rep}\mathcal{M}$ and the axioms for ϕ_λ imply the quasi–associativity relations

$$(\pi_V \boxtimes \pi_W) \odot \gamma_\mathfrak{H} \cong \pi_V \boxtimes (\pi_W \odot \gamma_\mathfrak{H})$$

where the isomorphism is given by

$$\phi_{VW\mathfrak{H}} := (\pi_V \otimes \pi_W \otimes \gamma_\mathfrak{H})(\phi_\lambda).$$

Finally, the pentagon axiom (2.35b) provides us with the analogue of McLane's coherence conditions, i.e. the following commuting diagram



(A.5)

With the obvious substitutions analogue statements hold for right coactions (ρ, ϕ_ρ), where now these induce a right action $\odot : \operatorname{Rep}\mathcal{M} \times \operatorname{Rep}\mathcal{G} \to \operatorname{Rep}\mathcal{M}$ given for $(\gamma, \mathfrak{H}) \in \operatorname{Rep}\mathcal{M}$ and $(\pi, V) \in \operatorname{Rep}\mathcal{G}$ by

$$\mathfrak{H} \odot V := \mathfrak{H} \otimes V, \quad \gamma \odot \pi := (\gamma \otimes \pi) \circ \rho.$$

Finally, a quasi–commuting pair $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ provides us with both, a left and a right action of $\operatorname{Rep}\mathcal{G}$ on $\operatorname{Rep}\mathcal{M}$, together with a further family of associativity equivalences $(\pi_U \odot \gamma_\mathfrak{H}) \odot \pi_V \cong \pi_U \odot (\gamma_\mathfrak{H} \odot \pi_V)$, where now the isomorphisms are given by

$$\phi_{U\mathfrak{H}V} := (\pi_U \otimes \gamma_\mathfrak{H} \otimes \pi_V)(\phi_{\lambda\rho}).$$

Again, the conditions (2.51b) and (2.51c) imply further pentagon diagrams of the type (A.5) with objects of the type $UV\mathfrak{H}W$ or $U\mathfrak{H}VW$, respectively, in appropriate bracket positions.

Two–sided coactions. In the obvious way the above may also be generalized to arbitrary two–sided coactions (δ, Ψ), in which case we would obtain a functor

$$\begin{aligned} &\operatorname{Rep}\mathcal{G} \times \operatorname{Rep}\mathcal{M} \times \operatorname{Rep}\mathcal{G} \longrightarrow \operatorname{Rep}\mathcal{M} \\ &U > \mathfrak{H} < V := U \otimes \mathfrak{H} \otimes V, \quad \pi_U > \gamma_{\mathfrak{H}} < \pi_V := (\pi_U \otimes \gamma_{\mathfrak{H}} \otimes \pi_V) \circ \delta \end{aligned} \tag{A.6}$$

with associativity isomorphisms $(\pi_U \boxtimes \pi_V) > \gamma_{\mathfrak{H}} < (\pi_W \boxtimes \pi_Z) \cong \pi_U > (\pi_V > \gamma_{\mathfrak{H}} < \pi_W) < \pi_Z$ given by

$$\Psi_{UV\mathfrak{H}WZ} := (\pi_U \otimes \pi_V \otimes \gamma_{\mathfrak{H}} \otimes \pi_W \otimes \pi_Z)(\Psi) \tag{A.7}$$

and obeying analogue "two–sided" pentagon diagrams. Note that the operations $>$ and $<$ are *not* defined individually, i.e. only the two–sided operation $> \cdot <$ makes sense. According to Proposition 2.8 the relation between two–sided and one–sided $\operatorname{Rep}\mathcal{G}$–actions is given by

$$\begin{gathered} \pi_V \odot \gamma_{\mathfrak{H}} := \pi_V > \gamma_{\mathfrak{H}} < \epsilon, \quad \gamma_{\mathfrak{H}} \odot \pi_V := \epsilon > \gamma_{\mathfrak{H}} < \pi_V \\ (\pi_V \odot \gamma_{\mathfrak{H}}) \odot \pi_U \cong \pi_V > \gamma_{\mathfrak{H}} < \pi_U \cong \pi_V \odot (\gamma_{\mathfrak{H}} \odot \pi_U), \end{gathered}$$

where here the intertwiners are given by $(\pi_V \otimes \gamma_{\mathfrak{H}} \otimes \pi_U)(U_{l/r})$, respectively, see Proposition 2.8.

Some families of natural transformations. We now give a representation theoretic interpretation of the elements $p_\lambda, q_\lambda \in \mathcal{G} \otimes \mathcal{M}$ given in (2.77)-(2.78) by defining for $(V, \pi_V) \in \operatorname{Rep}\mathcal{G}$ and $(\mathfrak{H}, \gamma_{\mathfrak{H}}) \in \operatorname{Rep}\mathcal{M}$ the natural family of morphisms[1]

$$\begin{aligned} &P_{V\mathfrak{H}} : \mathfrak{H} \longrightarrow V^* \odot (V \odot \mathfrak{H}), \quad \mathfrak{h} \mapsto v^i \otimes p_\lambda \cdot (v_i \otimes \mathfrak{h}) \\ &Q_{V\mathfrak{H}} : {}^*V \odot (V \odot \mathfrak{H}) \longrightarrow \mathfrak{H}, \quad \hat{v} \otimes v \otimes \mathfrak{h} \mapsto (\hat{v} \otimes \mathrm{id})\Big(q_\lambda \cdot (v \otimes \mathfrak{h})\Big) \end{aligned} \tag{A.8}$$

By (2.103a) and (2.103b) these are indeed morphisms in $\operatorname{Rep}\mathcal{M}$, and (2.103c) and (2.103d) imply the "generalized left rigidity" identities

$$\begin{aligned} Q_{V^*(V\odot\mathfrak{H})} \circ (\mathrm{id}_V \odot P_{V\mathfrak{H}}) &= \mathrm{id}_{V\odot\mathfrak{H}} \\ (\mathrm{id}_V \odot Q_{V\mathfrak{H}}) \circ P_{{}^*V(V\odot\mathfrak{H})} &= \mathrm{id}_{V\odot\mathfrak{H}} \end{aligned} \tag{A.9}$$

Finally, (2.103e) implies the coherence condition given by the following commuting diagram:

$$\begin{array}{ccccc} \mathfrak{H} & \xrightarrow{P_{V\mathfrak{H}}} & V^* \odot (V \odot \mathfrak{H}) & \xrightarrow{\mathrm{id}_{V^*} \odot P_{U(V\odot\mathfrak{H})}} & V^* \odot \big[U^* \odot (U \odot (V \odot \mathfrak{H}))\big] \\ \big\downarrow{\scriptstyle P_{(U\boxtimes V)\mathfrak{H}}} & & & & \\ (U \boxtimes V)^* \odot [(U \boxtimes V) \odot \mathfrak{H}] & & & & \big\downarrow{\scriptstyle \mathrm{id}_{V^*} \odot (\mathrm{id}_{U^*} \odot \phi^{-1}_{UV\mathfrak{H}})} \\ \big\downarrow{\scriptstyle f^{-1}_{V^*U^*} \odot \mathrm{id}_{(U\boxtimes V)\odot\mathfrak{H}}} & & & & \\ (V^* \boxtimes U^*) \odot [(U \boxtimes V) \odot \mathfrak{H}] & & \xrightarrow{\phi_{V^*U^*[(U\boxtimes V)\odot\mathfrak{H}]}} & & V^* \odot \big[U^* \odot ((U \boxtimes V) \odot \mathfrak{H})\big] \end{array} \tag{A.10}$$

The reader is invited to draw the analogous diagram implied by (2.103f), now involving the morphisms $Q_{V\mathfrak{H}}$.

Similar statements hold of course for the natural family of morphisms

$$\begin{aligned} &P_{\mathfrak{H}V} : \mathfrak{H} \longrightarrow (\mathfrak{H} \odot V) \odot {}^*V, \quad \mathfrak{h} \mapsto p_\rho \cdot (\mathfrak{h} \otimes v_i) \otimes v^i \\ &Q_{\mathfrak{H}V} : (\mathfrak{H} \odot V) \odot V^* \longrightarrow \mathfrak{H}, \quad \mathfrak{h} \otimes v \otimes \hat{v} \mapsto (\mathrm{id} \otimes \hat{v})\Big(q_\rho \cdot (\mathfrak{h} \otimes v)\Big) \end{aligned} \tag{A.11}$$

[1] Again a summation is understood, where $\{v_i\}$ is a basis of V with dual basis $\{v^i\}$

Adjustments to weak quasi–Hopf algebras. Let us now shortly discuss the adjustements to be made for the case that $\mathcal{G}$ is a weak quasi–Hopf algebra. As already discussed in Section 3.2, due to the coproduct being non–unital the definition of the tensor product functor in $\mathrm{Rep}\,\mathcal{G}$ has to be slightly modified. First note that the element $\Delta(\mathbf{1})$ (as well as iterated coproducts of $\mathbf{1}$) is idempotent and commutes with all elements in $\Delta(\mathcal{G})$. Thus, given two representations $(V,\pi_V),(W,\pi_W)$, the operator $(\pi_V\otimes\pi_W)(\Delta(\mathbf{1}))$ is a projector, whose image is precisely the $\mathcal{G}$–invariant subspace of $V\otimes W$ on which the tensor product representation operates non trivial. Thus one is led to modify the tensor product $\boxtimes$ of two representations of $\mathcal{G}$ as given in (A.1) as follows

$$V\boxtimes W:=(\pi_V\otimes\pi_W)(\Delta(\mathbf{1}))\,V\otimes W,\quad \pi_V\boxtimes\pi_W:=(\pi_V\otimes\pi_W)\circ\Delta|_{V\boxtimes W}\tag{A.12}$$

One readily verifies that with these definition ϕ_{UVW} - restricted to the subspace $(U\boxtimes V)\boxtimes W$ - furnish a natural family of isomorphisms defining an associativity constraint for the tensor product functor $\boxtimes$. Moreover, $\mathrm{Rep}\,\mathcal{G}$ becomes a rigid monoidal category with rigidity structure defined as before by (A.2)-(A.3).

The *left action* of $\mathrm{Rep}\,\mathcal{G}$ on $\mathrm{Rep}\,\mathcal{M}$ induced by a left $\mathcal{G}$–coaction (λ,ϕ_λ) on $\mathcal{M}$ has to be modified analogously by defining

$$V\odot\mathfrak{H}:=(\pi\otimes\gamma)(\lambda(\mathbf{1}_{\mathcal{M}}))\,V\otimes\mathfrak{H},\quad \pi\odot\gamma:=(\pi\otimes\gamma)\circ\lambda|_{V\odot\mathfrak{H}}$$

The modifications of right actions and of two–sided actions of $\mathrm{Rep}\,\mathcal{G}$ on $\mathrm{Rep}\,\mathcal{M}$ (induced by (ρ,ϕ_ρ) and by (δ,Ψ), respectively) should by now be obvious and are left to the reader.

With these adjustments all categorical identities such as the definition of natural families and commuting diagrams given above stay valid. Translating these into algebraic identities one has to take some care with identities in higher tensor products of $\mathcal{G}$. The only equations which have to be modified are (2.103c),(2.103d), where the r.h.s. becomes $\lambda(\mathbf{1}_{\mathcal{M}})$ instead of $\mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{M}}$ and similarly (2.104c/2.104d), where the r.h.s. has to be replaced by $\rho(\mathbf{1}_{\mathcal{M}})$. This is rather obvious from the categorical point of view, since for example (2.103c) is directly connected with (A.9) where the r.h.s. is given by $\mathrm{id}_{V\odot\mathfrak{H}}\equiv(\pi_V\otimes\pi_{\mathfrak{H}})(\lambda(\mathbf{1}_{\mathcal{M}}))$.

δ–Implementers. We now give a representation theoretic interpretation of our notion of left and right δ–implementers. To this end let $(\mathfrak{H},\gamma_{\mathfrak{H}})$ be a fixed representation of $\mathcal{M}$. Putting $\mathcal{A}:=\mathrm{End}_{\mathbb{C}}(\mathfrak{H})$, we consider $\gamma\equiv\gamma_{\mathfrak{H}}:\mathcal{M}\to\mathcal{A}$ as an algebra map. This leads to

Definition A.1. Let (δ,Ψ) be a two–sided $\mathcal{G}$–coaction on $\mathcal{M}$. A representation $(\mathfrak{H},\gamma)$ of $\mathcal{M}$ is called *δ–coherent* if there exists a normal coherent left δ–implementer $\mathbf{L}\in\mathcal{G}\otimes\mathrm{End}_{\mathbb{C}}(\mathfrak{H})$ (equivalently right δ–implementer $\mathbf{R}\in\mathcal{G}\otimes\mathrm{End}_{\mathbb{C}}(\mathfrak{H})$).

Corollary 2.14 then says that a representation of $\mathcal{M}$ is δ–coherent if and only if it extends to a representation of the diagonal crossed products $\hat{\mathcal{G}}\bowtie\mathcal{M}_\delta\cong\mathcal{M}_\delta\bowtie\hat{\mathcal{G}}$.

We now provide a category theoretic description of δ–coherent representations $(\mathfrak{H},\gamma)$. Associated with a left δ–implementer $\mathbf{L}\in\mathcal{G}\otimes\mathcal{A}$, $\mathcal{A}:=\mathrm{End}_{\mathbb{C}}(\mathfrak{H})$, we define a family of $\mathcal{M}$–linear morphisms

$$l_V:V\gtrdot\mathfrak{H}\lessdot V^*\longrightarrow\mathfrak{H},\quad v\otimes\mathfrak{h}\otimes\hat{v}\mapsto\Big((\hat{v}\otimes\mathrm{id})\circ\mathbf{L}_V\Big)(v\otimes\mathfrak{h})\tag{A.13}$$

where $(V,\pi_V)\in\mathrm{Rep}\,\mathcal{G}$ and $\mathbf{L}_V:=(\pi_V\otimes\mathrm{id})(\mathbf{L})$ and where we have used the notation (A.6). Eq. (2.64) guarantees that l_V is in fact a morphism in $\mathrm{Rep}\,\mathcal{M}$. The normality condition for $\mathbf{L}$ implies $l_{\mathbb{C}}=\mathrm{id}_{\mathfrak{H}}$ and the coherence condition (2.66) for $\mathbf{L}$ translates into the following coherence condition for the l_V's

$$l_{V\boxtimes W}\circ\Omega^L_{VW\mathfrak{H}W^*V^*}=l_V\circ(\mathrm{id}_V\otimes l_W\otimes\mathrm{id}_{V^*}),$$

where $\Omega^L_{VW\mathfrak{H}W^*V^*}:V\gtrdot(W\gtrdot\mathfrak{H}\lessdot W^*)\lessdot V^*\to(V\boxtimes W)\gtrdot\mathfrak{H}\lessdot(V\boxtimes W)^*$ is the natural $\mathcal{M}$–linear isomorphism given by $\Omega^L_{VW\mathfrak{H}W^*V^*}=(\mathrm{id}_V\otimes\mathrm{id}_W\otimes\mathrm{id}_{\mathfrak{H}}\otimes f_{W^*V^*})\circ\Psi^{-1}_{VW\mathfrak{H}W^*V^*}$ see (A.4), (A.7) and (2.55). Similarly, a right δ–implementer $\mathbf{R}\in\mathcal{G}\otimes\mathcal{A}$ gives rise to a coherent family of $\mathcal{M}$–linear morphisms

$$r_V:\mathfrak{H}\longrightarrow V^*\gtrdot\mathfrak{H}\lessdot V,\quad \mathfrak{h}\mapsto v^i\otimes\mathbf{R}^{21}_V(\mathfrak{h}\otimes v_i),\tag{A.14}$$

where $\mathbf{R}_V:=(\pi_V\otimes\mathrm{id})(\mathbf{R})$. As above, this means

$$\Omega^R_{V^*U^*\mathfrak{H}UV}\circ r_{U\boxtimes V}=(\mathrm{id}_{V^*}\otimes r_U\otimes\mathrm{id}_V)\circ r_V,$$

where $\Omega^R_{V^*U^*\mathfrak{H}UV}:=\Psi_{V^*U^*\mathfrak{H}UV}\circ(f^{-1}_{V^*U^*}\otimes\mathrm{id}_{\mathfrak{H}}\otimes\mathrm{id}_U\otimes\mathrm{id}_V)$, see (2.62) and (2.67).

$\lambda\rho$–Intertwiners. We finally give a representation categorical interpretation of the notion of $\lambda\rho$–intertwiners and of their connection with δ–implementers as stated in Proposition 2.19. As before, given a fixed representation $(\mathfrak{H}, \gamma)$ of $\mathcal{M}$ we consider $\gamma : \mathcal{M} \to \mathcal{A} \equiv \mathrm{End}_{\mathbb{C}}(\mathfrak{H})$ as an algebra map.

DEFINITION A.2. Let $(\lambda, \phi_\lambda, \rho, \phi_\rho, \phi_{\lambda\rho})$ be a quasi–commuting pair of $\mathcal{G}$–coactions on $\mathcal{M}$. A representation $(\mathfrak{H}, \gamma_{\mathfrak{H}})$ of $\mathcal{M}$ is called *$\lambda\rho$–coherent*, if there exists a normal coherent $\lambda\rho$–intertwiner $\mathbf{T} \in \mathcal{G} \otimes \mathrm{End}_{\mathbb{C}}(\mathfrak{H})$.

Proposition 2.19 then says, that $\lambda\rho$–coherence is equivalent to δ_l-coherence for $\delta_l = (\lambda \otimes \mathrm{id}) \circ \rho$ (or to δ_r–coherence for $\delta_r = (\mathrm{id} \otimes \rho) \circ \lambda$). Associated with a $\lambda\rho$–intertwiner $\mathbf{T} \in \mathcal{G} \otimes \mathrm{End}_{\mathbb{C}}(\mathfrak{H})$ we now define a family of $\mathcal{M}$–linear morphisms

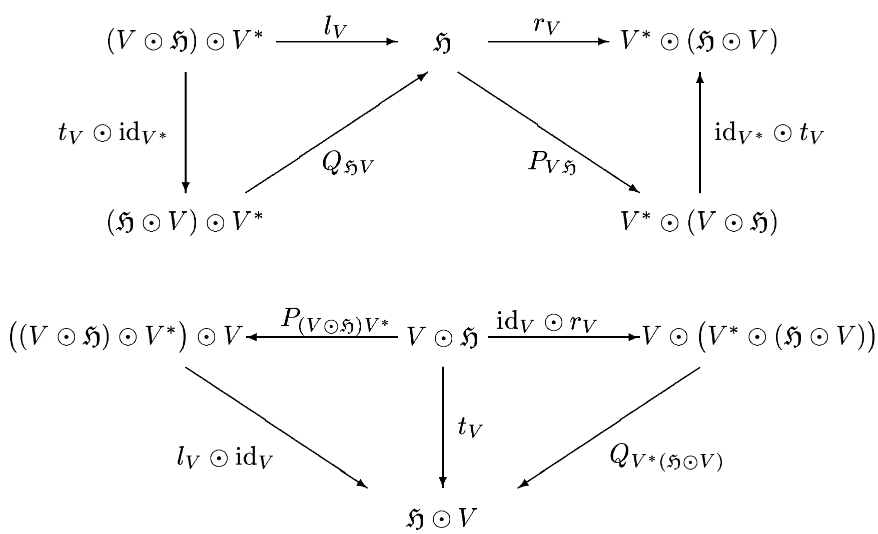
$$t_V : V \odot \mathfrak{H} \longrightarrow \mathfrak{H} \odot V, \quad v \otimes \mathfrak{h} \mapsto \mathbf{T}_V^{21}(\mathfrak{h} \otimes v), \tag{A.15}$$

where $(V, \pi_V) \in \mathrm{Rep}\,\mathcal{G}$ and $\mathbf{T}_V^{21} := (\mathrm{id} \otimes \pi_V)(\mathbf{T}^{21})$. Eq. (2.73) guarantees that t_V is a morphism in $\mathrm{Rep}\,\mathcal{M}$. The normality condition for $\mathbf{T}$ implies $t_{\mathbb{C}} = \mathrm{id}_{\mathfrak{H}}$ and the coherence condition (2.74) for $\mathbf{T}$ translates into the following coherence condition for t_V

$$t_{V \boxtimes W} = \phi_{VW\mathfrak{H}} \circ (\mathrm{id}_V \otimes t_W) \circ \phi_{V\mathfrak{H}W}^{-1} \circ (t_V \otimes \mathrm{id}_W) \circ \phi_{\mathfrak{H}VW}\,. \tag{A.16}$$

Note that (A.16) looks precisely like one of the coherence conditions for the braiding in a braided quasi–tensor category with nontrivial associativity isomorphisms. Indeed, as has been shown in Chapter 4, for the case $\mathcal{M} = \mathcal{G}$, the family of t_V's may be used to define a braiding in the representation category of the quantum double $\mathcal{D}(\mathcal{G}) \equiv \mathcal{G} \bowtie \hat{\mathcal{G}}$.

Using the morphisms $P_{V\mathfrak{H}}, P_{\mathfrak{H}V}$ and $Q_{V\mathfrak{H}}, Q_{\mathfrak{H}V}$ given in (A.8) and (A.11), the relation between $\lambda\rho$–intertwiners $\mathbf{T}$, right δ_r–implementers $\mathbf{R}$ and left δ_l–implementers $\mathbf{L}$ may now be described by the following commuting diagrams connecting the intertwiner morphisms t_V with the maps r_V (associated with $\mathbf{R}$) and l_V (associated with $\mathbf{L}$) as given in (A.13) and (A.14).



(A.17)

Note that the commutativity of the above diagrams is in fact equivalent to the statements in Proposition 2.19.

The representation category of the quantum double. We will now describe the representation category of the quantum double in terms of the representation category of the underlying quasi-Hopf algebra $\mathcal{G}$. In this way we will show that $\mathcal{D}(\mathcal{G})$ is a concrete realization of the quantum double as defined by Majid in [Maj97] with the help of a Tannaka-Krein-like reconstruction theorem. We denote the monoidal category of finite dimensional unital representations of $\mathcal{D}(\mathcal{G})$ by $\mathrm{Rep}\,\mathcal{D}(\mathcal{G})$. As an immidiate implication of Theorem 2.1 we state a necessary and sufficient condition, under which a representation of $\mathcal{G}$ extends to a representation of $\mathcal{D}(\mathcal{G})$, see also Definition A.2 for the case $\lambda = \rho = \Delta$. We only treat the case $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, the generalization to the weak case being obvious.

COROLLARY A.3.
1.) The objects of $\mathrm{Rep}\,\mathcal{D}(\mathcal{G})$ *are in one to one correspondence with pairs* $\{(\pi_V, V), \mathbf{D}_V\}$*, where* (π_V, V) *is a finite dimensional representation of* $\mathcal{G}$ *and where* $\mathbf{D}_V \in \mathcal{G} \otimes \mathrm{End}_{\mathbb{C}}(V)$ *is a normal coherent* Δ*-flip, i.e.*

(i) $(\epsilon \otimes \mathrm{id})(\mathbf{D}_V) = \mathrm{id}_V$

(ii) $\mathbf{D}_V \cdot (\mathrm{id} \otimes \pi_V)(\Delta(a)) = (\mathrm{id} \otimes \pi_V)(\Delta^{op}(a)) \cdot \mathbf{D}_V, \quad \forall a \in \mathcal{G}$

(iii) $\phi_V^{312}\, \mathbf{D}_V^{13}\, (\phi_V^{-1})^{132}\, \mathbf{D}_V^{23}\, \phi_V = (\Delta \otimes \mathrm{id})(\mathbf{D}_V)$*, where* $\phi_V := (\mathrm{id} \otimes \mathrm{id} \otimes \pi_V)(\phi)$.

2.) Let $\{(\pi_V, V), \mathbf{D}_V\}$ *and* $\{(\pi_W, W), \mathbf{D}_W\}$ *be as above, then*

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{G})} = \{t \in \mathrm{Hom}_{\mathcal{G}}(V, W) \mid (\mathrm{id} \otimes t)(\mathbf{D}_V) = \mathbf{D}_W\}$$

PROOF. Part 1.) follows from Theorem 2.1 by choosing $\lambda = \rho = \Delta$, $\mathcal{A} = \mathrm{End}\,(V)$ and $\gamma = \pi_V$, see also (A.15). We shortly repeat the arguments. Define the extended representation π_V^D on the generators of $\mathcal{D}(\mathcal{G})$ by

$$\pi_V^D(i_D(a)) := \pi_V(a), \quad a \in \mathcal{G} \tag{A.18}$$

$$\pi_V^D(D(\varphi)) := (\varphi \otimes \mathrm{id}_{End_V})(\mathbf{D}_V), \quad \varphi \in \hat{\mathcal{G}} \tag{A.19}$$

Condition (i) implies that π_V^D is unital whereas conditions (ii),(iii) just reflect the defining relations (4.2) and (4.3) of $\mathcal{D}(\mathcal{G})$, which ensures, that π_V^D is a well defined algebra morphism. On the other hand, given a representation (π_V^D, V) of $\mathcal{D}(\mathcal{G})$, we define

$$\mathbf{D}_V := (\mathrm{id}_{\mathcal{G}} \otimes \pi_V^D)(\mathbf{D})$$

which clearly satisfies conditions (i) - (iii). This proves part 1.). Part 2.) follows trivially. □

To get the relation with Majid's formalism [Maj97] we now write $a \cdot v := \pi_V(a)v$, $a \in \mathcal{G}, v \in V$ and define $\beta_V : V \longrightarrow \mathcal{G} \otimes V; \quad v \mapsto v^{(\bar{1})} \otimes v^{(\bar{2})} := \mathbf{D}_V(\mathbf{1}_{\mathcal{G}} \otimes v)$. With this notation we get

COROLLARY A.4. *The conditions (i)-(iii) of Corollary A.3 are equivalent to the following three conditions for* β_V *(as before denoting* $\bar{X}^i \otimes \bar{Y}^i \otimes \bar{Z}^i = \phi^{-1}$*):*

(i') $(\epsilon \otimes \mathrm{id}_V) \circ \beta_V = \mathrm{id}_V$

(ii') $(a_{(2)} \cdot v)^{(\bar{1})} a_{(1)} \otimes (a_{(2)} \cdot v)^{(\bar{2})} = a_{(2)} v^{(\bar{1})} \otimes a_{(1)} \cdot v^{(\bar{2})}, \quad \forall v \in V$

(iii') $\bar{Z}^i v^{(\bar{1})} \otimes (\bar{Y}^i \cdot v^{(\bar{2})})^{(\bar{1})}\, \bar{X}^i \otimes (\bar{Y}^i \cdot v^{(\bar{2})})^{(\bar{2})} =$
$(\phi^{-1})^{321} \cdot \Big[(\bar{Z}^i \cdot v)^{(\bar{1})}{}_{(2)}\, \bar{Y}^i \otimes (\bar{Z}^i \cdot v)^{(\bar{1})}{}_{(1)}\, \bar{X}^i \otimes (\bar{Z}^i \cdot v)^{(\bar{2})} \Big], \quad \forall v \in V$

PROOF. The equivalences (i) ⇔ (i') and (ii) ⇔ (ii') are obvious. The equivalence (iii) ⇔ (iii') follows by multiplying (iii) with $(\phi_V^{-1})^{312}$ from the left and with ϕ_V^{-1} from the right and permuting the first two tensor factors. □

The conditions stated in the above Corollary agree with those formulated in [Maj97],Prop.2.2, by taking $\mathcal{G}^{cop} \equiv (\mathcal{G}, \Delta^{op}, (\phi^{-1})^{321})$ instead of $(\mathcal{G}, \Delta, \phi)$ as the underlying quasi–bialgebra. Thus we have identified the category $\mathrm{Rep}\,\mathcal{D}(\mathcal{G})$ with what is called the double category of modules over $\mathcal{G}$ in [Maj97]. This proves that our quantum double is a concrete realization of the abstract definition given by S. Majid.