CHAPTER 5

Quantum group spin chains and lattice current algebras

In this chapter we finally arrive at the construction of Hopf spin chains and lattice current algebras based on a weak quasi–Hopf algebra $\mathcal{G}$. Again we emphasize that this covers the important case of quantum groups at roots of unity. In Section 5.1 we generalize the notion of *two–sided crossed products*, defined as special examples of diagonal crossed products in Section 1.4.1 to the quasi–coassociative setting. We then define Hopf spin chains as iterated two–sided crossed products in Section 5.2 and arrive at the definition of lattice current algebras in Section 5.3 by imposing periodic boundary conditions again using our diagonal crossed product construction. Finally we investigate the representation theory of these models in Section 5.4.

5.1. Two–sided crossed products

As in the associative case, a simple recipe to produce two–sided $\mathcal{G}$–comodule algebras $(\mathcal{M}, \delta)$ is by tensoring a right $\mathcal{G}$–comodule algebra $(\mathcal{A}, \rho)$ and a left $\mathcal{G}$–comodule algebra $(\mathcal{B}, \lambda)$, i.e. by setting $\mathcal{M} = \mathcal{A} \otimes \mathcal{B}$ and

$$\delta(A \otimes B) := B_{(-1)} \otimes (A_{(0)} \otimes B_{(0)}) \otimes A_{(1)}$$

as in (1.41). Obviously $\delta = (\lambda \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \rho) \circ \lambda$, where we use the same symbols (λ, ϕ_λ) and (ρ, ϕ_ρ) for the trivially extended left and right coactions, i.e. $\lambda \equiv \lambda \otimes \mathrm{id}_{\mathcal{B}}$, etc. . Hence $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho} = \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\mathcal{G}})$ is a *strictly commuting* pair of $\mathcal{G}$-coactions on $\mathcal{M} = \mathcal{A} \otimes \mathcal{B}$. In the terminology of (2.52a)-(2.52d) we have $\delta = \delta_r = \delta_l$, whereas $\Psi = \Psi_r = \Psi_l$ is given by

$$\Psi = [\mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}} \otimes \phi_\rho^{-1}]\,[\phi_\lambda \otimes \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}}]. \tag{5.1}$$

According to Theorem 2.1 the diagonal crossed product $\mathcal{M}_1 = (\mathcal{A} \otimes \mathcal{B})_{\delta_r} \bowtie \hat{\mathcal{G}} =: (\mathcal{A}_\rho \otimes_\lambda \mathcal{B}) \bowtie \hat{\mathcal{G}}$ is generated by $\{A, B, \Gamma(\varphi) \mid A \in \mathcal{A},\ B \in \mathcal{B},\ \varphi \in \hat{\mathcal{G}}\}$ satisfying the defining relations

$$A\,B = B\,A \tag{5.2a}$$

$$[\mathbf{1}_{\mathcal{G}} \otimes B]\,\mathbf{\Gamma} = \mathbf{\Gamma}\,\lambda(B) \tag{5.2b}$$

$$\rho^{op}(A)\,\mathbf{\Gamma} = \mathbf{\Gamma}\,[\mathbf{1}_{\mathcal{G}} \otimes A] \tag{5.2c}$$

$$(\Delta \otimes \mathrm{id})(\mathbf{\Gamma}) = \phi_\rho^{312}\,\mathbf{\Gamma}^{13}\,\mathbf{\Gamma}^{23}\,\phi_\lambda, \tag{5.2d}$$

where $\mathbf{\Gamma} = e_\mu \otimes \Gamma(e^\mu)$ is the universal $\lambda\rho$–intertwiner.

The next Proposition is an analogue of Proposition 1.14 saying that the diagonal crossed product $(\mathcal{A}_\rho \otimes_\lambda \mathcal{B}) \bowtie \hat{\mathcal{G}}$ may be realized as a *two–sided crossed product* $\mathcal{A}_\rho \rtimes \hat{\mathcal{G}} \ltimes_\lambda \mathcal{B}$. Note that the isomorphism μ in (5.3) below is different from the isomorphisms μ_R and μ_L constructed in Theorem 2.1.

PROPOSITION 5.1. *Let $\mathcal{G}$ be a quasi–Hopf algebra and $\triangleright : \hat{\mathcal{G}} \otimes \mathcal{A} \to \mathcal{A}$ and $\triangleleft : \mathcal{B} \otimes \hat{\mathcal{G}} \to \mathcal{B}$ be the left and the right action corresponding to a right $\mathcal{G}$–coaction (ρ, ϕ_ρ) on $\mathcal{A}$ and a left $\mathcal{G}$–coaction (λ, ϕ_λ) on $\mathcal{B}$, respectively. Extend λ and ρ trivially to $\mathcal{A} \otimes \mathcal{B}$ and let $\mathcal{M}_1 := (\mathcal{A}_\rho \otimes_\lambda \mathcal{B}) \bowtie \hat{\mathcal{G}} \equiv (\mathcal{A} \otimes \mathcal{B})_{\delta_r} \bowtie \hat{\mathcal{G}}$, $\delta_r := (\mathrm{id} \otimes \rho) \circ \lambda$, be the diagonal crossed product with universal $\lambda\rho$–intertwiner $\mathbf{\Gamma} \in \mathcal{G} \otimes \mathcal{M}_1$.*

(i) *There is a linear bijection $\mu : \mathcal{A} \otimes \hat{\mathcal{G}} \otimes \mathcal{B} \to \mathcal{M}_1$ given by*

$$\mu(A \otimes \varphi \otimes B) = A\,\Gamma(\varphi)\,B, \tag{5.3}$$

where we have suppressed the embeddings $\mathcal{A} \hookrightarrow \mathcal{M}_1$ and $\mathcal{B} \hookrightarrow \mathcal{M}_1$.

(ii) *Denote the induced algebra structure on $\mathcal{A} \otimes \hat{\mathcal{G}} \otimes \mathcal{B}$ by $\mathcal{A}_\rho \rtimes \hat{\mathcal{G}} \ltimes_\lambda \mathcal{B} \equiv \mu^{-1}(\mathcal{M}_1)$. Then we get the following multiplication structure with unit $\mathbf{1}_{\mathcal{A}} \rtimes \hat{\mathbf{1}} \ltimes \mathbf{1}_{\mathcal{B}}$ on $\mathcal{A}_\rho \rtimes \hat{\mathcal{G}} \ltimes_\lambda \mathcal{B}$*

$$\begin{aligned}&\big(A \rtimes \varphi \ltimes B\big)\,\big(A' \rtimes \psi \ltimes B'\big)\\ &\quad = A(\varphi_{(1)} \triangleright A')\bar{\phi}_\rho^1 \rtimes [\bar{\phi}_\lambda^1 \rightharpoonup \varphi_{(2)} \leftharpoonup \bar{\phi}_\rho^2]\,[\bar{\phi}_\lambda^2 \rightharpoonup \psi_{(1)} \leftharpoonup \bar{\phi}_\rho^3] \ltimes \bar{\phi}_\lambda^3(B \triangleleft \psi_{(2)})B'\end{aligned} \tag{5.4}$$

PROOF. Let $p_\lambda \in \mathcal{G} \otimes \mathcal{B} \equiv \mathcal{G} \otimes (\mathbf{1}_\mathcal{A} \otimes \mathcal{B})$ be given by (2.77) and define

$$f : A \otimes \varphi \otimes B \mapsto A \otimes \varphi_{(1)} \otimes (\varphi_{(2)} \otimes \mathrm{id}_\mathcal{B})\big(\lambda(B) p_\lambda\big). \tag{5.5}$$

Using (5.2b) we obtain

$$\begin{aligned}(\mu \circ f)\big(A \otimes \varphi \otimes B\big) &= A\,(\varphi \otimes \mathrm{id}_{\mathcal{M}_1})\big(\mathbf{\Gamma}\,\lambda(B)\,p_\lambda\big) \\ &= A\,B\,(\varphi \otimes \mathrm{id}_{\mathcal{M}_1})(\mathbf{\Gamma}\,p_\lambda) \\ &= \mu_R(A \otimes B \otimes \varphi)\end{aligned}$$

where $\mu_R : (\mathcal{A} \otimes \mathcal{B}) \otimes \hat{\mathcal{G}} \to \mathcal{M}_1$ is the linear bijection constructed in part 3. of Theorem 2.1, see also (2.87). Hence μ is surjective. Conversely, let $\mathbf{R} := \mathbf{\Gamma}\,p_\lambda \in \mathcal{G} \otimes \mathcal{M}_1$ then by Proposition 2.19 $\mathbf{R}$ is a right δ_r–implementer and

$$\mathbf{\Gamma} = [\mathbf{1}_\mathcal{G} \otimes q^2_\lambda]\,\mathbf{R}\,[S^{-1}(q^1_\lambda) \otimes \mathbf{1}_{\mathcal{M}_1}],$$

where $q_\lambda \in \mathcal{G} \otimes \mathcal{B}$ is given by (2.78) and where we have used that ρ is trivial on $\mathcal{B}$. Hence we get for all $B \in \mathcal{B}$, using again (5.2b)

$$\mathbf{\Gamma}\,[\mathbf{1}_\mathcal{G} \otimes B] = [\mathbf{1}_\mathcal{G} \otimes q^2_\lambda B_{(0)}]\,\mathbf{R}\,[S^{-1}(q^1_\lambda B_{(-1)}) \otimes \mathbf{1}_{\mathcal{M}_1}] \tag{5.6}$$

where $B_{(-1)} \otimes B_{(0)} = \lambda(B)$. Setting

$$\bar{f}(A \otimes B \otimes \varphi) := \hat{S}^{-1}(\varphi_{(2)}) \otimes \mathrm{id}_\mathcal{B}\big)\big(q_\lambda \lambda(B)\big) \otimes \varphi_{(1)},$$

(5.6) implies

$$\mu(A \otimes \varphi \otimes B) \equiv A\,\Gamma(\varphi)\,B = \mu_R \circ \bar{f}(A \otimes B \otimes \varphi)$$

But since $\bar{f}$ is invertible (one directly verifies that $\bar{f} = f^{-1}$, with f given in (5.5)), the injectivity of μ_R implies the injectivity of μ.

This proves part (i). Part (ii) follows since one straightforward checks that the multiplication rule (5.4) is equivalent to the defining relations (5.2). □

Next we show, that analogously as in (1.51) the two–sided crossed product construction given in Proposition 5.1 may be iterated if one of the two algebras $\mathcal{A}$ and $\mathcal{B}$ admits a quasi–commuting pair of coactions.

PROPOSITION 5.2. *Let* $(\mathcal{A}, \rho_\mathcal{A}, \phi_{\rho_\mathcal{A}})$, $(\mathcal{C}, \lambda_\mathcal{C}, \phi_{\lambda_\mathcal{C}})$ *and* $(\mathcal{B}, \rho_\mathcal{B}, \lambda_\mathcal{B}, \phi_{\lambda_\mathcal{B}}, \phi_{\rho_\mathcal{B}}, \phi_{\lambda_\mathcal{B}, \rho_\mathcal{B}})$ *be a right, a left and a two–sided comodule algebra, respectively, and denote the universal* $\lambda\rho$*–intertwiners*

$$\begin{aligned}\mathbf{\Gamma}_{\mathcal{AB}} &:= e_\mu \otimes (\mathbf{1}_\mathcal{A} \rtimes e^\mu \ltimes \mathbf{1}_\mathcal{B}) \in \mathcal{A}_\rho \rtimes \hat{\mathcal{G}} \ltimes_\lambda \mathcal{B} \\ \mathbf{\Gamma}_{\mathcal{BC}} &:= e_\mu \otimes (\mathbf{1}_\mathcal{B} \rtimes e^\mu \ltimes \mathbf{1}_\mathcal{C}) \in \mathcal{B}_\rho \rtimes \hat{\mathcal{G}} \ltimes_\lambda \mathcal{C},\end{aligned} \tag{5.7}$$

where $\mathcal{A}_\rho \equiv \mathcal{A}_{\rho_\mathcal{A}}$, *etc. , then*

(i) $\mathcal{A}_\rho \rtimes \hat{\mathcal{G}} \ltimes_\lambda \mathcal{B}$ *admits a right* $\mathcal{G}$*–coaction* $(\tilde{\rho}, \phi_{\tilde{\rho}})$ *given by* $\phi_{\tilde{\rho}} := \phi_{\rho_\mathcal{B}}$ *(trivially embedded),* $\tilde{\rho}\restriction_{(\mathcal{A}\otimes\mathcal{B})} := \mathrm{id}_\mathcal{A} \otimes \rho_\mathcal{B}$ *and*

$$(\mathrm{id}_\mathcal{G} \otimes \tilde{\rho})(\mathbf{\Gamma}_{\mathcal{AB}}) := (\mathbf{\Gamma}_{\mathcal{AB}} \otimes \mathbf{1}_\mathcal{G})\,\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}} \tag{5.8}$$

(ii) $\mathcal{B}_\rho \rtimes \hat{\mathcal{G}} \ltimes_\lambda \mathcal{C}$ *admits a left* $\mathcal{G}$*–coaction* $(\tilde{\lambda}, \phi_{\tilde{\lambda}})$ *given by* $\phi_{\tilde{\lambda}} := \phi_{\lambda_\mathcal{B}}$ *(trivially embedded),* $\tilde{\lambda}\restriction_{(\mathcal{B}\otimes\mathcal{C})} := \lambda_\mathcal{B} \otimes \mathrm{id}_\mathcal{C}$ *and*

$$(\mathrm{id}_\mathcal{G} \otimes \tilde{\lambda})(\mathbf{\Gamma}_{\mathcal{BC}}) := (\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}})^{231}\,\mathbf{\Gamma}^{13}_{\mathcal{BC}} \tag{5.9}$$

(iii) *The trivial identification*

$$(\mathcal{A}_\rho \rtimes \hat{\mathcal{G}} \ltimes_\lambda \mathcal{B})_{\tilde{\rho}} \rtimes \hat{\mathcal{G}} \ltimes_\lambda \mathcal{C} \equiv \mathcal{A}_\rho \rtimes \hat{\mathcal{G}} \ltimes_{\tilde{\lambda}} (\mathcal{B}_\rho \rtimes \hat{\mathcal{G}} \ltimes_\lambda \mathcal{C}) \tag{5.10}$$

is an algebra isomorphism.

PROOF. (i) To show that (5.8) provides a well defined algebra map $\tilde{\rho} : \mathcal{A}_\rho \rtimes \hat{\mathcal{G}} \ltimes_\lambda \mathcal{B} \to (\mathcal{A}_\rho \rtimes \hat{\mathcal{G}} \ltimes_\lambda \mathcal{B}) \otimes \mathcal{G}$ extending $\mathrm{id}_\mathcal{A} \otimes \rho_\mathcal{B}$ we have to check that the relations (5.2) are respected. To this end we put $\mathbf{T}_{\mathcal{AB}} := (\mathbf{\Gamma}_{\mathcal{AB}} \otimes \mathbf{1}_\mathcal{G})\,\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}}$ and compute for $B \in \mathcal{B}$

$$\begin{aligned}[\mathbf{1}_\mathcal{G} \otimes \tilde{\rho}(B)]\,\mathbf{T}_{\mathcal{AB}} &= (\mathbf{\Gamma}_{\mathcal{AB}} \otimes \mathbf{1}_\mathcal{G})\,(\lambda_\mathcal{B} \otimes \mathrm{id})(\rho_\mathcal{B}(B))\,\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}} \\ &= \mathbf{T}_{\mathcal{AB}}\,(\mathrm{id}_G \otimes \tilde{\rho})(\lambda_\mathcal{B}(B)),\end{aligned}$$

which is the relation (5.2b). Trivially one also has (since $\phi_{\lambda_\mathcal{B}\rho_\mathcal{B}} \in \mathcal{G}\otimes(\mathbf{1}_\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{G}$)

$$\mathbf{T}_{\mathcal{AB}}\,[\mathbf{1}_\mathcal{G}\otimes A\otimes\mathbf{1}_\mathcal{G}] = [\rho^{op}(A)\otimes\mathbf{1}_\mathcal{G}]\,\mathbf{T}_{\mathcal{AB}}$$

verifying (5.2c). Finally, the coherence condition (5.2d) is respected, since in $\mathcal{G}\otimes\mathcal{G}\otimes(\mathcal{A}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{B})\otimes\mathcal{G}$ we have

$$\begin{aligned}
(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\mathbf{T}_{\mathcal{AB}}) &= (\Delta\otimes\mathrm{id})(\mathbf{\Gamma}_{\mathcal{AB}})^{123}\,(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}})\\
&= (\Delta\otimes\mathrm{id})(\mathbf{\Gamma}_{\mathcal{AB}})^{123}\,[\phi^{-1}_{\lambda_\mathcal{B}}\otimes\mathbf{1}_\mathcal{G}]\,(\mathrm{id}\otimes\lambda_\mathcal{B}\otimes\mathrm{id})(\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}})\,[\mathbf{1}_\mathcal{G}\otimes\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}}]\,(\mathrm{id}\otimes\mathrm{id}\otimes\rho_\mathcal{B})(\phi_{\lambda_\mathcal{B}})\\
&= \phi^{312}_{\rho_\mathcal{A}}\,\mathbf{\Gamma}^{13}_{\mathcal{AB}}\,\mathbf{\Gamma}^{23}_{\mathcal{AB}}\,(\mathrm{id}\otimes\lambda_\mathcal{B}\otimes\mathrm{id})(\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}})\,[\mathbf{1}_\mathcal{G}\otimes\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}}]\,(\mathrm{id}\otimes\mathrm{id}\otimes\rho_\mathcal{B})(\phi_{\lambda_\mathcal{B}})\\
&= \phi^{312}_{\rho_\mathcal{A}}\,\mathbf{\Gamma}^{13}_{\mathcal{AB}}\,(\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}})^{134}\,\mathbf{\Gamma}^{23}_{\mathcal{AB}}(\phi^{-1}_{\lambda\rho})^{234}\,(\mathrm{id}\otimes\mathrm{id}\otimes\rho_\mathcal{B})(\phi_{\lambda_\mathcal{B}})\\
&= (\mathrm{id}\otimes\mathrm{id}\otimes\tilde\rho)(\phi^{312}_{\rho_\mathcal{A}})\,\mathbf{T}^{13}_{\mathcal{AB}}\,\mathbf{T}^{23}_{\mathcal{AB}}\,(\mathrm{id}\otimes\mathrm{id}\otimes\tilde\rho)(\phi_{\lambda_\mathcal{B}})
\end{aligned}$$

Here we have used the pentagon identity (2.51b) in the second line, the coherence property (5.2d) of $\mathbf{\Gamma}_{\mathcal{AB}}$ in the third line and finally the intertwining property (5.2b). Thus $\tilde\rho$ provides a well defined algebra map, which is also unit preserving since $(\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(\mathbf{T}_{\mathcal{AB}}) = (\mathbf{1}_\mathcal{A}\rtimes\hat{\mathbf{1}}\ltimes\mathbf{1}_\mathcal{B})\otimes\mathbf{1}_\mathcal{G}$. Similarly one shows by a straight forward calculation that the pair $(\tilde\rho,\phi_{\tilde\rho})$ satisfies (2.36a). Since $\phi_{\tilde\rho} = \phi_{\rho_\mathcal{B}}$, the pentagon equation (2.36b) and the counit equations (2.36c) and (2.36d) are clearly satisfied. This proves part (i). Part (ii) follows analogously. To prove part (iii) we have to check that we may consistently identify

$$\mathcal{G}\otimes\Big[(\mathcal{A}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{B})\rtimes\hat{\mathcal{G}}\ltimes\mathcal{C}\Big]\ni\mathbf{\Gamma}_{\mathcal{AB}}\overset{!}{\equiv}\mathbf{\Gamma}_{\mathcal{A}(\mathcal{B}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{C})}\in\mathcal{G}\otimes\Big[\mathcal{A}\rtimes\hat{\mathcal{G}}\ltimes(\mathcal{B}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{C})\Big] \tag{5.11}$$

$$\mathcal{G}\otimes\Big[(\mathcal{A}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{B})\rtimes\hat{\mathcal{G}}\ltimes\mathcal{C}\Big]\ni\mathbf{\Gamma}_{(\mathcal{A}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{B})\mathcal{C}}\overset{!}{\equiv}\mathbf{\Gamma}_{\mathcal{BC}}\in\mathcal{G}\otimes\Big[\mathcal{A}\rtimes\hat{\mathcal{G}}\ltimes(\mathcal{B}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{C})\Big] \tag{5.12}$$

For this, the nontrivial commutation relations to be looked at are according to (5.2b),(5.2c)

$$\begin{aligned}
\mathbf{\Gamma}_{(\mathcal{A}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{B})\mathcal{C}}\,[\mathbf{1}_\mathcal{G}\otimes X] &= \tilde\rho^{op}(X)\,\mathbf{\Gamma}_{(\mathcal{A}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{B})\mathcal{C}}, \quad X\in\mathcal{A}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{B}\\
[\mathbf{1}_\mathcal{G}\otimes Y]\,\mathbf{\Gamma}_{\mathcal{A}(\mathcal{B}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{C})} &= \mathbf{\Gamma}_{\mathcal{A}(\mathcal{B}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{C})}\,\tilde\lambda(Y), \quad Y\in\mathcal{B}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{C}
\end{aligned}$$

The remaining cases being trivial, it is enough to consider $X\in\Gamma_{\mathcal{AB}}(\hat{\mathcal{G}})$ and $Y\in\Gamma_{\mathcal{BC}}(\hat{\mathcal{G}})$ for which we get

$$\begin{aligned}
\mathbf{\Gamma}^{23}_{(\mathcal{A}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{B})\mathcal{C}}\,\mathbf{\Gamma}^{13}_{\mathcal{AB}} &= (\mathrm{id}_\mathcal{G}\otimes\tilde\rho)(\mathbf{\Gamma}_{\mathcal{AB}})^{132}\,\mathbf{\Gamma}^{23}_{\mathcal{A}(\mathcal{B}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{C})} = \mathbf{\Gamma}^{13}_{\mathcal{AB}}\,(\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}})^{132}\,\mathbf{\Gamma}^{23}_{(\mathcal{A}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{B})\mathcal{C}}\\
\mathbf{\Gamma}^{23}_{\mathcal{BC}}\,\mathbf{\Gamma}^{13}_{\mathcal{A}(\mathcal{B}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{C})} &= \mathbf{\Gamma}^{13}_{\mathcal{A}(\mathcal{B}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{C})}\,(\mathrm{id}_\mathcal{G}\otimes\tilde\lambda)(\mathbf{\Gamma}_{\mathcal{BC}})^{213} = \mathbf{\Gamma}^{13}_{\mathcal{A}(\mathcal{B}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{C})}\,(\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}})^{132}\,\mathbf{\Gamma}^{23}_{\mathcal{BC}}
\end{aligned}$$

by the definitions (5.8) and (5.9). This shows that the identifications (5.11) and (5.12) are indeed consistent and therefore proves part (iii) □

Note that due to part (iii) of the above proposition the notations $\mathbf{\Gamma}_{\mathcal{AB}}$ and $\mathbf{\Gamma}_{\mathcal{BC}}$ as in (5.7) are still well defined in iterated two–sided crossed products and the commutation relations of "neighboring" $\lambda\rho$–intertwiners are given by

$$\mathbf{\Gamma}^{13}_{\mathcal{AB}}\,\big(\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}}\big)^{132}\,\mathbf{\Gamma}^{23}_{\mathcal{BC}} = \mathbf{\Gamma}^{23}_{\mathcal{BC}}\,\mathbf{\Gamma}^{13}_{\mathcal{AB}} \tag{5.13}$$

Adaption to the weak case. The definition of *two–sided crossed products* as in Proposition 5.1 has to be slightly modified if $\mathcal{G}$ is a weak quasi–Hopf algebra, i.e. if $\Delta(\mathbf{1})\neq\mathbf{1}\otimes\mathbf{1}$. The unital algebra $\mathcal{A}_\rho\rtimes\hat{\mathcal{G}}\ltimes_\lambda\mathcal{B}$ is now defined on the subspace of $\mathcal{A}\otimes\hat{\mathcal{G}}\otimes\mathcal{B}$ given by

$$\mathcal{A}\rtimes\hat{\mathcal{G}}\ltimes\mathcal{B} := \mathrm{span}\big\{A\rtimes\varphi\ltimes B\equiv A\mathbf{1}^{(0)}_\mathcal{A}\otimes\mathbf{1}^{(-1)}_\mathcal{B}\triangleright\varphi\triangleleft\mathbf{1}^{(1)}_\mathcal{A}\otimes\mathbf{1}^{(0)}_\mathcal{B}B \mid A\in\mathcal{A},B\in\mathcal{B},\varphi\in\hat{\mathcal{G}}\big\}, \tag{5.14}$$

where $\mathbf{1}^{(0)}_\mathcal{A}\otimes\mathbf{1}^{(1)}_\mathcal{A}\equiv\rho(\mathbf{1}_\mathcal{A})$ and $\mathbf{1}^{(-1)}_\mathcal{B}\otimes\mathbf{1}^{(0)}_\mathcal{B}\equiv\lambda(\mathbf{1}_\mathcal{B})$. Again we have a linear bijection

$$\mu:\mathcal{A}\ltimes\hat{\mathcal{G}}\rtimes\mathcal{B}\to\mathcal{M}_1 = (\mathcal{A}_\rho\otimes_\lambda\mathcal{B})\bowtie\hat{\mathcal{G}},\quad A\rtimes\varphi\ltimes B\mapsto A\,\Gamma(\varphi)\,B$$

inducing the multiplication rule described by Eq. (5.4).

Also Proposition 5.2 stays valid. In particular we note that (5.14) still allows the identification

$$(A\rtimes\varphi\ltimes B)\rtimes\psi\ltimes C = A\rtimes\varphi\ltimes(B\rtimes\psi\ltimes C)$$

5.2. Quantum group spin chains

In this section we describe how the Hopf algebraic quantum chains considered in [NS97] generalize to (weak) quasi–Hopf algebras $\mathcal{G}$. To this end we use the two–sided crossed product theory of Section 5.1 to generalize the constructions (1.49),(1.50) and (1.51).

Due to Proposition 5.2 the definition of Hopf spin chains as reviewed in Section 1.4.2 immediately generalizes to the quasi–coassociative case. As in Section 1.4.2 we interpret even integers as sites and odd integers as links of a one dimensional lattice and we set $\mathcal{A}_{2i} \cong \mathcal{G}$, $\mathcal{A}_{2i+1} \cong \hat{\mathcal{G}}$, the latter just being a linear space. A local net of associative algebras $\mathcal{A}_{n,m}$ is then constructed inductively for all $n, m \in 2\mathbb{Z}, n \leq m$, by first putting

$$\mathcal{A}_{2i,2i+2} := \mathcal{A}_{2i} \rtimes \mathcal{A}_{2i+1} \ltimes \mathcal{A}_{2i+2} \cong \mathcal{G} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G}$$

where $\mathcal{G}$ is equipped with its canonical two–sided comodule structure ($\lambda = \rho = \Delta$, $\phi_\lambda = \phi_\rho = \phi_{\lambda\rho} = \phi$). Due to Proposition 5.2 this procedure may be iterated as in (1.49), by setting

$$\mathcal{A}_{2i,2j+2} := \mathcal{A}_{2i,2j} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G} \equiv \mathcal{G} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{A}_{2i-2,2j+2}$$

where the last equality follows from (5.10) by iteration. More generally one has as in (1.50) for all $i \leq \mu \leq j-1$

$$\mathcal{A}_{2i,2j} = \mathcal{A}_{2i,2\mu} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{A}_{2\mu+2,2j} \tag{5.15}$$

We will now give a description of the finite Hopf spin chains (i.e. the local algebras $\mathcal{A}_{2i,2j}$) in terms of generating matrices. Defining the generating "link operators" $\mathbf{L}_{2i+1} := \mathbf{\Gamma}_{\mathcal{A}_{2i}\mathcal{A}_{2i+2}}$ as in (5.7), we get

LEMMA 5.3. *Let $\mathcal{G}$ be a weak quasi–Hopf algebra. The finite open Hopf spin chain $\mathcal{A}_{2i,2j}$ is the unique unital algebra generated by $\mathcal{G}^{\otimes^{j-i}} \equiv \mathcal{A}_{2i} \otimes \mathcal{A}_{2i+1} \otimes \cdots \otimes \mathcal{A}_{2j}$ and the entries of generating matrices $\mathbf{L}_{2\nu+1} \in \mathcal{G} \otimes \mathcal{A}_{2i,2j}$, $i \leq \nu \leq j$, obeying the relations*

$$\mathbf{L}^{13}_{2k+1}\,\mathbf{L}^{23}_{2l+1} = \mathbf{L}^{23}_{2l+1}\,\mathbf{L}^{13}_{2k+1}, \quad \forall k \neq l, l-1, l+1 \tag{5.16a}$$

$$(\phi^{312})_{2k}\,\mathbf{L}^{13}_{2k+1}\,\mathbf{L}^{23}_{2k+1}\,\phi_{2k+2} = (\Delta \otimes \mathrm{id})(\mathbf{L}_{2k+1}) \tag{5.16b}$$

$$\mathbf{L}^{13}_{2k-1}\,\left((\phi^{-1})^{132}\right)_{2k}\mathbf{L}^{23}_{2k+1} = \mathbf{L}^{23}_{2k+1}\,\mathbf{L}^{13}_{2k-1} \tag{5.16c}$$

$$[\mathbf{1} \otimes A_{2k}(a)]\,\mathbf{L}_{2l+1} = \mathbf{L}_{2l+1}\,[\mathbf{1} \otimes A_{2k}(a)], \quad \forall k \neq l, l+1 \tag{5.16d}$$

$$[\mathbf{1} \otimes A_{2k}(a)]\,\mathbf{L}_{2k-1} = \mathbf{L}_{2k-1}\,[a_{(1)} \otimes A_{2k}(a_{(2)})] \tag{5.16e}$$

$$\mathbf{L}_{2k+1}\,[\mathbf{1} \otimes A_{2k}(a)] = [a_{(2)} \otimes A_{2k}(a_{(1)})]\,\mathbf{L}_{2k+1}, \quad \forall a \in \mathcal{G} \tag{5.16f}$$

where A_{2k} denotes the identification $\mathcal{G} \equiv \mathcal{A}_{2k}$ and where $\phi_{2k} := (\mathrm{id} \otimes \mathrm{id} \otimes A_{2k})(\phi)$.

PROOF. Follows immediately from (5.2) and (5.13). □

Writing $A_{2i+1}(\varphi) = (\varphi \otimes \mathrm{id})(\mathbf{L}_{2i+1})$, (5.16c) is equivalent to

$$A_{2i-1}(\bar{\phi}^1 \rightharpoonup \psi)\,A_{2i}(\bar{\phi}^2)\,A_{2i+1}(\varphi \leftharpoonup \bar{\phi}^3) = A_{2i+1}(\varphi)\,A_{2i-1}(\psi)$$

Thus link operators on neighboring links do not commute any more in contrast to the coassociative setting! But the algebras $\mathcal{A}_{2i-2,2i}$ and $\mathcal{A}_{2i+2,2i+4}$ still commute, which means that the above construction still yields a local net of algebras now indexed by intervals in $2\mathbb{Z}$.

Next we remark that Theorem 4.3 applied to the special case of two–sided crossed products provides us with localized left and right coactions of the quantum double $\mathcal{D}(\mathcal{G})$ on the above quantum chain generalizing (1.53). Indeed, using (5.15) and (5.8),(4.7), we obtain (right) $\mathcal{D}(\mathcal{G})$–coactions

$$\rho^{2i}_D : \mathcal{A}_{2k,2j} \to \mathcal{A}_{2k,2j} \otimes \mathcal{D}(\mathcal{G}), \quad k < i < j, \tag{5.17}$$

acting trivially on $\mathcal{A}_{2k,2i-2} \cup \mathcal{A}_{2i+2,2j}$, and given elsewhere by

$$\rho^{2i}_D\big(A_{2i}(a)\big) := A_{2i}(a_{(1)}) \otimes \big(a_{(2)} \bowtie_D \mathbf{1}\big), \quad a \in \mathcal{G}$$

$$(\mathrm{id}_\mathcal{G} \otimes \rho^{2i}_D)(\mathbf{L}_{2i-1}) := \big[\mathbf{L}_{2i-1} \otimes (\mathbf{1} \bowtie_D \hat{\mathbf{1}})\big]\,(\mathrm{id}_\mathcal{G} \otimes \mathcal{A}_{2i} \otimes i_D)(\phi)$$

$$(\mathrm{id}_\mathcal{G} \otimes \rho^{2i}_D)(\mathbf{L}_{2i+1}) := (\phi^{-1})^{231}\,\mathbf{D}^{13}\,\phi^{213}\,\mathbf{L}^{12}_{2i+1}\,\phi^{-1},$$

where i_D denotes the embedding $\mathcal{G} \equiv \mathcal{G} \bowtie_D \hat{\mathbf{1}} \subset \mathcal{D}(\mathcal{G})$ and where we have suppressed the embedding $\mathrm{id}_\mathcal{G} \otimes A_{2i} \otimes i_D$ of the three reassociators in the last line. As before, $\mathbf{D}$ denotes the universal Δ–flip operator in $\mathcal{G}\otimes\mathcal{D}(\mathcal{G})$, see chapter 4. One similarly obtains localized left $\mathcal{D}(\mathcal{G})$–coactions $\lambda_{\mathcal{D}}^{2i}$ using (5.9),(4.5). This generalizes the $\mathcal{D}(\mathcal{G})$–cosymmetry discovered by [NS97] to weak quasi–Hopf algebras.

5.3. Lattice current algebras

Also the construction of the periodic chain by closing a finite open chain may again be described by a diagonal crossed product. As in Section 1.4.3 we define the periodic chain $\mathcal{K}_n$ with n sites as

$$\mathcal{K}_n := {}_{\lambda}(\mathcal{A}_{2,2n})_{\rho} \bowtie \hat{\mathcal{G}}, \tag{5.18}$$

where λ and ρ are nontrivial only on $\mathcal{A}_2 \rtimes \mathcal{A}_3 \ltimes \mathcal{A}_4$ and $\mathcal{A}_{2n-2} \rtimes \mathcal{A}_{2n-1} \ltimes \mathcal{A}_{2n}$, respectively, where they are defined as in Proposition 5.2, i.e. λ extends Δ viewed as a left coaction on $\mathcal{A}_2 \cong \mathcal{G}$ to $\mathcal{A}_2 \rtimes \mathcal{A}_3 \ltimes \mathcal{A}_4$ and ρ extends the right coaction Δ on $\mathcal{A}_{2n}$ to $\mathcal{A}_{2n-2} \rtimes \mathcal{A}_{2n-1} \ltimes \mathcal{A}_{2n}$.

We now show how the equivalence of the Hopf spin chains of [NS97] and the lattice current algebras of [AFFS98] as shown in [Nil97] generalizes to the (weak) quasi–Hopf setting. Suppose $\mathcal{G}$ to be quasitriangular with R–matrix R. As in (1.56) we define the generating current operators by

$$\mathbf{J}_{2i+1} := R_{2i}\,\mathbf{L}_{2i+1}, \quad R_{2i} := (\mathrm{id}\otimes A_{2i})(R^{op}), \tag{5.19}$$

Clearly this relation is invertible (in the weak case use that $\Delta^{op}(\mathbf{1})\,\mathbf{L} = \mathbf{L}$). We then get

PROPOSITION 5.4. *Denote* $\hat{R}_{2i} := (\mathrm{id}\otimes\mathrm{id}\otimes A_{2i})(\phi^{213}\,R^{12}\,\phi^{-1})$ *and* $\phi_{2i} := (\mathrm{id}\otimes\mathrm{id}\otimes A_{2i})(\phi)$. *The relations* (5.16) *are equivalent to the following set of relations*

$$\mathbf{J}^{13}_{2k+1}\,\mathbf{J}^{23}_{2l+1} = \mathbf{J}^{23}_{2l+1}\,\mathbf{J}^{13}_{2k+1}, \quad \forall k \neq l, l-1, l+1 \tag{5.20a}$$
$$\mathbf{J}^{13}_{2k+1}\,\mathbf{J}^{23}_{2k+1} = \hat{R}_{2k}\,\phi_{2k}\,(\Delta\otimes\mathrm{id})(\mathbf{J})\,\phi^{-1}_{2k+2} \tag{5.20b}$$
$$\mathbf{J}^{13}_{2k-1}\,\hat{R}_{2k}\,\mathbf{J}^{23}_{2k+1} = \mathbf{J}^{23}_{2k+1}\,\mathbf{J}^{13}_{2k-1} \tag{5.20c}$$
$$[\mathbf{1}\otimes A_{2k}(a)]\,\mathbf{J}_{2l+1} = \mathbf{J}_{2l+1}\,[\mathbf{1}\otimes A_{2k}(a)], \quad \forall k \neq l, l+1 \tag{5.20d}$$
$$[\mathbf{1}\otimes A_{2k}(a)]\,\mathbf{J}_{2k-1} = \mathbf{J}_{2k-1}\,[a_{(1)}\otimes A_{2k}(a_{(2)})], \quad \forall a \in \mathcal{G} \tag{5.20e}$$
$$[a_{(1)}\otimes A_{2k}(a_{(2)})]\,\mathbf{J}_{2k+1} = \mathbf{J}_{2k+1}\,[\mathbf{1}\otimes A_{2k}(a)] \tag{5.20f}$$

These relations generalize the defining relations of lattice current algebras as given in [AFFS98] to (weak) quasi–Hopf algebras. Thus we propose the following alternative

DEFINITION 5.5. Let $(\mathcal{G}, R)$ be a finite dimensional weak quasi–Hopf algebra with quasitriangular R–matrix $R \in \mathcal{G}\otimes\mathcal{G}$. We define the *lattice current algebra* $\mathcal{K}_n$ with n sites, $n \geq 2$, to be the unique unital algebra extension of $\mathcal{G}^{\otimes^n}$ generated by $\mathcal{G}^{\otimes^n} \equiv \mathcal{A}_2\otimes\mathcal{A}_4\otimes\cdots\otimes\mathcal{A}_{2n}$ and the entries of n generating lattice currents $\mathbf{J}_{2k+1} \in \mathcal{G}\otimes\mathcal{K}_n$, $k = 1, 2, \ldots, n \equiv 0$ satisfying the relations (5.20).

The lattice current algebra $\mathcal{K}_1$ consisting of one site link pair is defined to be the unique unital algebra extension of $\mathcal{G}$ generated by elements of $\mathcal{G}$ and the entries of a *monodromy matrix* $\mathbf{M}$ obeying the relations given in (4.28)-(4.30).

Note that due to Corollary 4.9 the above definition of $\mathcal{K}_1$ is also consistent with (5.18).

PROOF OF PROPOSITION 5.4. Let us show that the relations (5.20) follow from (5.16), the converse implication may be shown similarly. Relations (5.20a), and (5.20d-5.20f) are obvious. To see (5.20b) we compute

$$\begin{aligned}\mathbf{J}^{13}_{2k+1}\,\mathbf{J}^{23}_{2k+1} &= R^{13}_{2k}\,\mathbf{L}^{13}_{2k+1}\,R^{23}_{2k}\,\mathbf{L}^{23}_{2k+1} \\ &= (\mathrm{id}\otimes\mathrm{id}\otimes A_{2k})\Big(R^{31}(\Delta\otimes\mathrm{id})(R)^{312}\Big)\mathbf{L}^{13}_{2k+1}\,\mathbf{L}^{23}_{2k+1},\end{aligned} \tag{5.21}$$

where we have used the intertwiner property (5.16f) of $\mathbf{L}_{2k+1}$. But quasitriangularity of R implies

$$\begin{aligned}R^{31}\big[(\Delta\otimes\mathrm{id})(R)\big]^{312} &= \big[(R\otimes\mathbf{1})\,(\Delta\otimes\mathrm{id})(R)\big]^{312} \\ &= \big[\phi^{321}\,(\mathbf{1}\otimes R)\,(\mathrm{id}\otimes\Delta)(R)\,\phi\big]^{312} \\ &= \phi^{213}\,R^{12}\,(\Delta\otimes\mathrm{id})(R^{op})\,\phi^{312}\end{aligned}$$

Plugging this into (5.21) and using (5.16b) yields

$$
\begin{aligned}
\mathbf{J}^{13}_{2k+1}\,\mathbf{J}^{23}_{2k+1} &= (\mathrm{id}\otimes\mathrm{id}\otimes A_{2k})\Big(\phi^{213}\,R^{12}\,(\Delta\otimes\mathrm{id})(R^{op})\,\phi^{312}\Big)\,\mathbf{L}^{13}_{2k+1}\,\mathbf{L}^{23}_{2k+1} \\
&= (\mathrm{id}\otimes\mathrm{id}\otimes A_{2k})\Big(\phi^{213}\,R^{12}\,(\Delta\otimes\mathrm{id})(R^{op})\Big)\,(\Delta\otimes\mathrm{id})(\mathbf{L}_{2k+1})\,\phi^{-1}_{2k+2} \\
&= \hat{R}_{2k}\,\phi_{2k}\,(\Delta\otimes\mathrm{id})(\mathbf{J}_{2k+1})\,\phi^{-1}_{2k+2}
\end{aligned}
$$

Thus we have proven (5.20b).
To show (5.20c) we compute

$$
\begin{aligned}
\mathbf{J}^{23}_{2k+1}\,\mathbf{J}^{13}_{2k-1} &= R_{2k}\,\mathbf{L}^{23}_{2k+1}\,R_{2k-2}\,\mathbf{L}^{13}_{2k-1} \\
&= R_{2k-2}\,R_{2k}\,\mathbf{L}^{23}_{2k+1}\,\mathbf{L}^{13}_{2k-1} \\
&= R_{2k-2}\,R_{2k}\,\mathbf{L}^{13}_{2k-1}\,\big((\phi^{-1})^{132}\big)_{2k}\,\mathbf{L}^{23}_{2k+1} \\
&= R_{2k-2}\,\mathbf{L}^{13}_{2k-1}\,\Big(\big[(\Delta\otimes\mathrm{id})(R)\,\phi^{-1}\big]^{132}\Big)_{2k}\,\mathbf{L}^{23}_{2k+1} \\
&= \mathbf{J}^{13}_{2k-1}\,\Big(\big[(\Delta\otimes\mathrm{id})(R)\,\phi^{-1}\,(\mathbf{1}\otimes R^{-1})\big]^{132}\Big)_{2k}\,\mathbf{J}^{23}_{2k+1} \\
&= \mathbf{J}^{13}_{2k-1}\,\hat{R}_{2k}\,\mathbf{J}^{23}_{2k+1},
\end{aligned}
$$

where in the last equality we have used the identity

$$
(\Delta\otimes\mathrm{id})(R)\,\phi^{-1}\,(\mathbf{1}\otimes R^{-1}) = \phi^{312}\,R^{13}\,(\phi^{-1})^{132}
$$

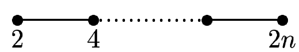
following from (2.18). □

The relations (5.20) have also appeared in [AGS95, AGS96, AS96] as lattice Chern–Simons algebras (restricted to the boundary of a disk) in the weak quasi–Hopf algebra setting, where the copies of $\mathcal{G}$ sitting at the sites are interpreted as gauge transformations.

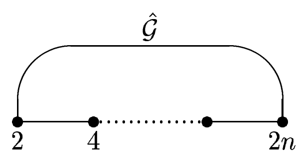
5.4. Representation theory

In this section we investigate the representation theory of finite open Hopf spin chains $\mathcal{O}_n$ and periodic Hopf spin chains $\mathcal{K}_n$, where $n\in\mathbb{N}$ denotes the number of sites, i.e.

$\mathcal{O}_n := \mathcal{A}_{2,2n} \doteq$



$\mathcal{K}_n := \mathcal{A}_{2,2n} \bowtie \hat{\mathcal{G}} \doteq$



as defined in Section 5.2 and 5.3. Recall that we have $\mathcal{O}_1 = \mathcal{G}$ and $\mathcal{K}_1 = \mathcal{D}(\mathcal{G})$. Our main result is given in

THEOREM 5.6. *Let $\mathcal{G}$ be a finite dimensional semisimple weak quasi–Hopf algebra. Moreover assume that there exists an element $\Sigma\in\hat{\mathcal{G}}\otimes\hat{\mathcal{G}}$ satisfying* (5.25) *and for part (ii) that there exists an invertible $g\in\mathcal{G}$ obeying* (5.38),(5.39). *Then*

(i) *For all $n\in\mathbb{N}$ the algebra $\mathcal{O}_n$ is semisimple and the center of $\mathcal{O}_n$ is isomorphic to the center of $\mathcal{G}$, i.e. the inequivalent irreducible representations of the finite Hopf spin chain $\mathcal{O}_n$ may be labeled by the set of inequivalent irreducible representations of the underlying weak quasi–Hopf algebra $\mathcal{G}$.*
(ii) *For all $n\in\mathbb{N}$ the algebra $\mathcal{K}_n$ is semisimple and the center of $\mathcal{K}_n$ is isomorphic to the center of the quantum double $\mathcal{D}(\mathcal{G})$, i.e. the inequivalent irreducible representations of the periodic Hopf spin chain $\mathcal{K}_n$ may be labeled by the inequivalent irreducible representations of the quantum double $\mathcal{D}(\mathcal{G})$.*

Part (i) is proven below in Corollary 5.12 and part (ii) in Corollary 5.14.

We remark that Theorem 5.6 especially applies to semisimple quotients of the quantum groups $U_q(\mathfrak{g})$ at roots of unity ($q^N = 1$). Also part (ii) covers the important cases of lattice current algebras "at roots of unity", since they are special examples of periodic Hopf spin chains as has been shown in Section 5.3.

Section 5.4.1 is devoted to the investigation of semisimple weak quasi–Hopf algebras. We will indicate, how elements of the well-known integral theory for Hopf algebras may be generalized to (weak) quasi–Hopf algebras [Nil], which allows to formulate the condition under which a diagonal crossed product $\mathcal{M} \bowtie \hat{\mathcal{G}}$ is semisimple. We also collect some properties of semisimple algebras and their commutants. Sections 5.4.2 and 5.4.3 are concerned with the proofs of part (i) and part (ii) of Theorem 5.6, respectively. Some of the more technical proofs are collected in the Appendix 5.5 of this chapter.

5.4.1. Semisimplicity. In the following the weak quasi–Hopf algebra $\mathcal{G}$ is always assumed to be finite dimensional (f.d.). Also recall that in this case $\mathcal{G}$ being semisimple (s.s.) means that $\mathcal{G}$ is isomorphic to a multi-matrix algebra, i.e.

$$\mathcal{G} \cong \oplus_I \mathrm{End}(V_I) \cong \oplus_I \pi_I(\mathcal{G}),$$

where the V_I are finite dimensional vector spaces and the index I runs through the (finite) set of inequivalent irreducible representations of $\mathcal{G}$. Thus $\mathcal{G}$ is "the direct sum of its irreducible representations". The center of $\mathcal{G}$ is spanned by the minimal central idempotents given by the identities (matrix units) $\mathrm{id}_{V_I} \in \mathrm{End}(V_I)$. Moreover, semisimplicity of $\mathcal{G}$ implies the existence of a unique normalized left integral in $\mathcal{G}$ which is also a right integral, i.e. an element $e \in \mathcal{G}$ satisfying

$$e^2 = e, \quad \epsilon(e) = 1 \tag{5.22}$$

$$ae = ea = \epsilon(a)\,\mathbf{1}, \quad \forall a \in \mathcal{G} \tag{5.23}$$

$$S(e) = e \tag{5.24}$$

The idempotent e is the unique central projection onto the one dimensional representation given by the counit.

For ordinary Hopf algebras $\mathcal{G}$ it is known that semisimplicity of $\mathcal{G}$ already implies semisimplicity of the dual Hopf algebra $\hat{\mathcal{G}}$ [LR88]. Passing to quasi–Hopf algebras, the dual $\hat{\mathcal{G}}$ is not an algebra any more and we need some appropriate substitute for semisimplicity of $\hat{\mathcal{G}}$, which ensures that the diagonal crossed product of a semisimple algebra $\mathcal{M}$ and $\hat{\mathcal{G}}$ is again semisimple. In turns out that all we need is an element $\Sigma \in \hat{\mathcal{G}} \otimes \hat{\mathcal{G}}$ satisfying

$$\Sigma \circ \Delta = \epsilon \tag{5.25a}$$

$$(\mathrm{id} \otimes \Sigma)\Big(\phi\,(\Delta(a) \otimes b)\,\phi^{-1}\Big) = (\Sigma \otimes \mathrm{id})\Big(\phi^{-1}\,(a \otimes \Delta(a))\,\phi\Big) \tag{5.25b}$$

$$\Sigma \leftharpoonup \Delta(a) = \epsilon(a)\,\Sigma = \Delta(a) \rightharpoonup \Sigma, \quad \forall a \in \mathcal{G} \tag{5.25c}$$

Let us shortly comment on these properties. First we remark that in the Hopf case (where $\phi = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$) the existence of Σ obeying (5.25a), (5.25b) is equivalent to cosemisimplicity of $\mathcal{G}$, see e.g. [Abe80]. Also, being f.d., in this case $\mathcal{G}$ is cosemisimple if and only if $\hat{\mathcal{G}}$ is semisimple. In the Hopf case, the element Σ obeying (5.25) may then be constructed as follows. First one realizes that the definition

$$\Sigma(a \otimes b) := \sigma(aS(b)) \tag{5.26}$$

provides a one-to-one correspondence between normalized left integrals $\sigma \in \hat{\mathcal{G}}$ and elements $\Sigma \in \hat{\mathcal{G}} \otimes \hat{\mathcal{G}}$ satisfying (5.25a) and (5.25b), where (5.25a) implies the normalization $\sigma(\mathbf{1}) = 1$ and (5.25b) the left integral property $\varphi\sigma = \varphi(1)\sigma$. This is in agreement with the wellknown fact that a Hopf algebra is semisimple if and only if it possess a normalized left integral [LS69]. Next note that the second equality in (5.25c) follows immidiately from (5.26), whereas the first equality is satisfied if one demands the 'q–trace' property

$$\sigma(ab) = \sigma\big(bS^2(a)\big) \tag{5.27}$$

Thus for Hopf algebras to find an element Σ satisfying (5.25) it is sufficient to find a normalized left integral, which is a 'q-trace'. Generalizing the notion of left integrals in $\hat{\mathcal{G}}$, a similar statement holds for weak quasi–Hopf algebras [Nil], which then allows to prove the following

PROPOSITION 5.7. [Nil] *The semisimple quotients $U_q^{tr}(\mathfrak{g})$ of quantum groups $U_q(\mathfrak{g})$ at roots of unity ($q^N = 1$) possess an element $\Sigma \in \hat{\mathcal{G}} \otimes \hat{\mathcal{G}}$ satisfying* (5.25).

We will sketch the proof of Proposition 5.7 in Section 5.5. Let us now state the important

THEOREM 5.8. [Nil] *Let $\mathcal{M}$ be a f.d. and s.s. algebra which admits a two-sided $\mathcal{G}$–coaction (δ, Ψ) of a f.d. weak quasi–Hopf algebra $\mathcal{G}$. Moreover assume that there exists $\Sigma \in \hat{\mathcal{G}} \otimes \hat{\mathcal{G}}$ obeying* (5.25). *Then the diagonal crossed product $\mathcal{M}_\delta \bowtie \hat{\mathcal{G}}$ is semisimple.*

Theorem 5.8 is a Maschke type Theorem and is also proven by an "averaging" procedure involving the element Σ. We postpone the proof to Section 5.5. We emphasize that semisimplicity of diagonal crossed products $\mathcal{M}_\delta \bowtie \hat{\mathcal{G}}$ does not depend on the chosen two–sided coaction δ.

We conclude this subsection with the following Proposition, concerning commutants and bicommutants of multi-matrix (= finite dimensional semisimple) algebras:

PROPOSITION 5.9. *Let V be a finite dimensional $\mathbb{C}$–vector space and $\mathcal{M} \subset \mathrm{End}(V)$ a semisimple unital subalgebra. Denote by $\mathcal{M}'$ the relative commutant of $\mathcal{M}$ in $\mathrm{End}(V)$, i.e. $\mathcal{M}' := \{X \in \mathrm{End}(V) : Xm = mX \quad \forall m \in \mathcal{M}\}$, then*

(i) *$\mathcal{M}'$ is semisimple and has the same minimal central idempotents as $\mathcal{M}$*
(ii) $(\mathcal{M}')' = \mathcal{M}$
(iii) *Let $P \in \mathcal{M} \cup \mathcal{M}'$ be a nonzero idempotent, then $(P\,\mathcal{M}\,P)' = P\,\mathcal{M}'\,P$*

Recall that the center of semisimple algebras is generated by the minimal central idempotents so that (i) implies that the center of $\mathcal{M}$ and of $\mathcal{M}'$ coincide. Part (ii) is known as the *bicommutant theorem* and is usually formulated in the context of von Neumann algebras. The algebra $P\mathcal{M}P$ in (iii) is called the *reduction* of $\mathcal{M}$ by P. For a proof of Proposition 5.9 see e.g. [GHJ89], Prop. 2.2.3 and Prop. 2.2.5.

Choosing in Proposition 5.9 $V \equiv \mathcal{M}$ and denoting by $L(\mathcal{M})$ and $R(\mathcal{M})$ the subalgebras of $\mathrm{End}(\mathcal{M})$ given by left and right multiplication, respectively, one verifies immediately that

$$L(\mathcal{M}) \cong \mathcal{M}, \qquad R(\mathcal{M}) \cong \mathcal{M}^{op} \tag{5.28}$$
$$L(\mathcal{M})' = R(\mathcal{M}), \qquad R(\mathcal{M})' = L(\mathcal{M}) \tag{5.29}$$

Now let $P := R(p)$, where $p = p^2 \in \mathcal{M}$. Since $P \in L(\mathcal{M})'$, the left-action of $\mathcal{M}$ may be restricted to the subspace $P\mathcal{M} \equiv \mathcal{M}p$. Moreover $\mathcal{M}p$ still allows for a right-action of the subalgebra $p\mathcal{M}p$. Denoting these actions by L_p and R_p, respectively, Proposition 5.9 implies the

COROLLARY 5.10. *Let $L_p(\mathcal{M}) \subset \mathrm{End}(\mathcal{M}p)$ and $R_p(p\mathcal{M}p) \subset \mathrm{End}(\mathcal{M}p)$ be defined as above, then*

$$L_p(\mathcal{M})' = R_p(p\mathcal{M}p) \tag{5.30}$$
$$R_p(p\mathcal{M}p)' = L_p(\mathcal{M}) \tag{5.31}$$

PROOF. Using the identifications

$$\begin{aligned}\mathrm{End}(\mathcal{M}p) &\equiv P\,\mathrm{End}(\mathcal{M})\,P \\ L_p(\mathcal{M}) &\equiv L(\mathcal{M})\,P = P\,L(\mathcal{M})\,P \\ R_p(p\mathcal{M}p) &\equiv P\,R(p\mathcal{M}p)P = R(p\mathcal{M}p),\end{aligned}$$

Proposition 5.9(iii) implies

$$\begin{aligned}L_p(\mathcal{M})' &\equiv (P\,L(\mathcal{M})\,P)' = P\,R(\mathcal{M})\,P = R(p\mathcal{M}p) \equiv R_p(p\mathcal{M}p) \\ R_p(p\mathcal{M}p)' &\equiv (P\,R(\mathcal{M})P)' = P\,L(\mathcal{M})\,P \equiv L_p(\mathcal{M})\end{aligned}$$

□

5.4.2. Representation theory of finite open chains. This subsection is devoted to the proof of part (i) of Theorem 5.6. Let us first recall how one proceeds in the case of ordinary Hopf algebras $\mathcal{G}$. Here the claim follows directly from a duality theorem for iterated crossed products [BM85], saying that

$$\mathcal{A} \rtimes \hat{\mathcal{G}} \rtimes \mathcal{G} \cong \mathcal{A} \otimes \mathrm{End}(\hat{\mathcal{G}}). \tag{5.32}$$

Iterating (5.32) one immediately arrives at

$$\mathcal{O}_n \cong \mathcal{G} \otimes \mathrm{End}(\hat{\mathcal{G}}^{n-1}), \tag{5.33}$$

which implies center($\mathcal{O}_n$) = center($\mathcal{G}$). At least for the truncated case $\Delta(\mathbf{1}) \neq \mathbf{1} \otimes \mathbf{1}$ we may no longer expect (5.32) to hold, since due to (5.14), see also part (3") of Theorem 3.1, the dimension of $\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G}$ is strictly smaller than the dimension of $\mathcal{A} \otimes \mathrm{End}(\hat{\mathcal{G}})$. As a weaker substitute for (5.32) we will prove the following

THEOREM 5.11. *Let $(\mathcal{A}, \rho, \phi_\rho)$ be a finite dimensional semisimple $\mathcal{G}$–comodule algebra, where $\mathcal{G}$ is a finite dimensional semisimple weak quasi–Hopf algebra, and let also the two–sided crossed product $\mathcal{A}_\rho \rtimes \hat{\mathcal{G}} \ltimes_\Delta \mathcal{G}$ be semisimple, then*

$$\mathit{center}(\mathcal{A}) \cong \mathit{center}(\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G}) \tag{5.34}$$

Theorem 5.11 may be viewed as a generalized duality theorem. It implies the

COROLLARY 5.12. *Let $\mathcal{G}$ be a f.d. semisimple weak quasi–Hopf algebra and assume that there exists an element $\Sigma \in \hat{\mathcal{G}} \otimes \hat{\mathcal{G}}$ satisfying* (5.25). *Then*

$$\mathit{center}(\mathcal{O}_n) \cong \mathit{center}(\mathcal{G})$$

PROOF OF COROLLARY 5.12. Follows by induction from Theorem 5.11 and Theorem 5.8 □

PROOF OF THEOREM 5.11. We will prove Theorem 5.11 by explicitly constructing a faithful representation of $\mathcal{M} := \mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G}$ and showing that the commutant of $\mathcal{M}$ in this representation is isomorph to $\mathcal{A}^{op}$. Employing Proposition 5.9 this proves (5.34). We arrive at such a representation by using Corollary 5.10. We also use the notations introduced there. First note that the integral $e \in \mathcal{G}$ provides an idempotent

$$p := \mathbf{1}_\mathcal{A} \rtimes \hat{\mathbf{1}} \ltimes e \in \mathcal{M}$$

Using (5.22/5.23) and the multiplication rule (5.4) in $\mathcal{M}$ together with (5.14) one easily verifies

$$\begin{aligned} \big(A \rtimes \varphi \ltimes a\big)\, p &= A\mathbf{1}_\mathcal{A}^{(0)} \otimes \varphi \leftharpoonup \mathbf{1}_\mathcal{A}^{(1)} \otimes e\epsilon(a) \\ p\,\big(A \rtimes \varphi \ltimes a\big)\, p &= (A \otimes \hat{\mathbf{1}} \otimes e)\, \epsilon(a)\varphi(\mathbf{1}) \end{aligned}$$

implying that

$$\mathcal{M}p \equiv \mathcal{A}\mathbf{1}_\mathcal{A}^{(0)} \otimes \hat{\mathcal{G}} \leftharpoonup \mathbf{1}_\mathcal{A}^{(1)} \tag{5.35}$$

$$p\mathcal{M}p \cong \mathcal{A}, \tag{5.36}$$

where the first line is an isomorphy of vector spaces and the second line an isomorphy of algebras.

Since the map $p\mathcal{M}p \to R_p(p\mathcal{M}p)$ is clearly faithful, (5.36) implies $R_p(p\mathcal{M}p) \cong \mathcal{A}^{op}$. Hence we get from (5.31)

$$L_p(\mathcal{M}) = R_p(pMp)' \equiv R_p(\mathcal{A})' \cong (\mathcal{A}^{op})' \tag{5.37}$$

In view of Proposition 5.9(i) this proves the isomorphy (5.34), provided L_p is faithful, i.e. provided $L_p(\mathcal{M}) \cong \mathcal{M}$, which will be shown below in Proposition 5.16 in the Appendix 5.5. □

5.4.3. Representation theory of periodic chains. Similarly as for the open Hopf spin chains, we arrive at a proof of part (ii) of Theorem 5.6 by showing that the periodic chain with n sites may be viewed as the relative commutant of the periodic chain with $n+1$ sites. More generally we have

THEOREM 5.13. *Let $\mathcal{A}$ be a semisimple f.d. algebra with a quasi–commuting pair of coactions $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ of a weak quasi–Hopf algebra $\mathcal{G}$. We also require the existence of an invertible element $g \in \mathcal{G}$ satisfying*

$$g\, a = S^2(a)\, g \tag{5.38}$$

$$\Delta(g) = f^{-1}\,(g \otimes g)\, h, \tag{5.39}$$

where $f, g \in \mathcal{G} \otimes \mathcal{G}$ are the twists defined in (2.26) and (2.31). Furthermore denote

$$\mathcal{M}_0 := \hat{\mathcal{G}} \bowtie \mathcal{A}_{\delta_0}, \quad \delta_0 = (\mathrm{id} \otimes \lambda) \circ \rho$$

$$\mathcal{M}_1 := \mathcal{A}_\rho \rtimes \hat{\mathcal{G}} \ltimes {}_\Delta\mathcal{G} \cong \hat{\mathcal{G}} \bowtie (\mathcal{A}_\rho \otimes {}_\Delta\mathcal{G})$$

$$\mathcal{M}_2 := \hat{\mathcal{G}} \bowtie (\mathcal{M}_1)_{\delta_2}, \quad \delta_2 = (\tilde{\lambda} \otimes \mathrm{id}) \circ \tilde{\Delta}$$

where $\tilde{\lambda}$ and $\tilde{\Delta}$ are the extensions of λ and Δ, respectively, as defined in Proposition 5.2. Let us also assume $\mathcal{M}_0$ and $\mathcal{M}_2$ to be semisimple. Then

$$\textit{center}(\mathcal{M}_0) \cong \textit{center}(\mathcal{M}_2)$$

Note that Theorem 5.13 especially applies to the case where $\mathcal{M}_0 = \mathcal{K}_{n-1}$, $\mathcal{M}_1 = \mathcal{O}_n$ and $\mathcal{M}_2 = \mathcal{K}_n \equiv \mathcal{O}_n \bowtie \hat{\mathcal{G}}$ yielding the following

COROLLARY 5.14. *Let $\mathcal{G}$ be a semisimple f.d. weak quasi–Hopf algebra. Also assume the existence of $g \in \mathcal{G}$ being invertible and satisfying (5.38),(5.39) and of $\Sigma \in \hat{\mathcal{G}} \otimes \hat{\mathcal{G}}$ satisfying (5.25). Then the periodic Hopf spin chains $\mathcal{K}_n$ are semisimple and their center is isomorph to the center of $\mathcal{K}_1 \cong \mathcal{D}(\mathcal{G})$.*

PROOF OF COROLLARY 5.14. Follows by induction from Theorem 5.13 and Theorem 5.8. □

PROOF OF THEOREM 5.13. As for the proof of Theorem 5.11 we will construct a faithful representation of $\mathcal{M}_2$ such that the relative commutant of $\mathcal{M}_2$ is isomorphic to $\mathcal{M}_0^{op}$, which then proves the claim due to Proposition 5.9. Again we will use the canonical integral $e \in \mathcal{G}$ to define the idempotent

$$\tilde{p} := \hat{\mathbf{1}} \bowtie \big(\hat{\mathbf{1}} \bowtie (\mathbf{1}_\mathcal{A} \otimes e)\big) \; \in \mathcal{M}_2$$

Using the notation of Corollary 5.10 we get a representation $L_{\tilde{p}}$ of $\mathcal{M}_2$ on $\mathcal{M}_2\tilde{p}$ by left multiplication, obeying

$$L_{\tilde{p}}(\mathcal{M}_2)' = R_{\tilde{p}}(\tilde{p}\mathcal{M}_2\tilde{p}) \tag{5.40}$$

Clearly $R_{\tilde{p}}$ is injective and we are left to show that $L_{\tilde{p}}$ is injective and that $\tilde{p}\mathcal{M}_2\tilde{p} \cong \mathcal{M}_0$. To see the injectivity of $L_{\tilde{p}}$ note that

$$L_{\tilde{p}}(\psi \bowtie m)\big[(\hat{\mathbf{1}} \bowtie n)\tilde{p}\big] = \psi \bowtie (mnp) = \psi \bowtie \big(L_p(m)[np]\big),$$

where L_p is the representation of $\mathcal{M}_1$ discussed in Proposition 5.16. Hence the injectivity of $L_{\tilde{p}}$ follows from the injectivity of L_p.

To prove the isomorphy $\tilde{p}\mathcal{M}_2\tilde{p} \cong \mathcal{M}_0$ we proceed as follows. First we compute

$$\tilde{p}\big[\psi \bowtie \big(\varphi \bowtie (A \otimes a)\big)\big]\tilde{p} = \psi \leftharpoondown S^{-1}(e_{(2)}) \bowtie \big(e_{(1)} \rightharpoonup \varphi \bowtie (A \otimes e)\big)$$

Employing Corollary 5.21 given below in the Appendix 5.5 yields

$$\begin{aligned}\tilde{p}\mathcal{M}_2\tilde{p} &= \mathrm{lin}\Big\{\psi_{(2)} \leftharpoondown \beta g \bowtie \big(\psi_{(1)} \bowtie (A \otimes e)\big), \quad \psi \in \hat{\mathcal{G}},\, A \in \mathcal{A}\Big\} \\ &= \mathrm{lin}\Big\{\mathbf{1}^\mathcal{A}_{(-1)} \rightharpoonup \psi_{(2)} \leftharpoondown \beta g \otimes \big(\psi_{(1)} \leftharpoondown S^{-1}(\mathbf{1}^\mathcal{A}_{(0,1)})\big) \otimes \mathbf{1}^\mathcal{A}_{(0)}\, A \otimes e\Big\},\end{aligned}$$

where, as also stated in Corollary 5.21, the assignment $\psi \mapsto (\psi_{(2)} \leftharpoondown \beta g \otimes \psi_{(1)})$ is injective. Thus the linear map

$$f : \mathcal{M}_0 \to \tilde{p}\mathcal{M}_2\tilde{p}, \quad \psi \bowtie A \mapsto \psi_{(2)} \leftharpoondown \beta g \bowtie \big(\psi_{(1)} \bowtie (A \otimes e)\big) \tag{5.41}$$

is bijective and we are left to show that f is also an algebra map, which will be done in Lemma 5.22 below in Appendix 5.5.

In view of Proposition 5.9 this concludes the proof of Theorem 5.13, since by (5.40)

$$\mathcal{M}_0^{op} \cong R_{\tilde{p}}(\tilde{p}\mathcal{M}_2\tilde{p}) = L_{\tilde{p}}(\mathcal{M}_2)' \cong \mathcal{M}_2'.$$

□

5.5. Proofs

Proofs of Section 5.4.1. To prove Proposition 5.7 we use the following Proposition stated in [Nil].

PROPOSITION 5.15. *Let $\mathcal{G}$ be a f.d. s.s. weak quasi–Hopf algebra $\mathcal{G}$. Define $\sigma_0 \in \hat{\mathcal{G}}$ by*

$$\sigma_0 = \sum_\mu \big(S^2(e_\mu)S(\beta)\alpha\big) \rightharpoonup e^\mu, \tag{5.42}$$

where $e_\mu \in \mathcal{G}$ is a basis in $\mathcal{G}$ with dual basis $e^\mu \in \hat{\mathcal{G}}$. Then $\Sigma_0 \in \hat{\mathcal{G}} \otimes \hat{\mathcal{G}}$ defined by

$$\Sigma_0(a \otimes b) := \sigma_0\big(a\beta S(\alpha b)\big), \quad a, b \in \mathcal{G} \tag{5.43}$$

satisfies (5.25b) *and* (5.25c).

Note that in general σ_0 may be zero. But if $\mathcal{G}$ possess a nonzero left integral e (i.e. in particular if $\mathcal{G}$ is semisimple), then $\sigma_0 \neq 0$ since one immidiately computes from (5.42) that $S(e) \rightharpoonup \sigma_0 = \epsilon$. We are now in the position to give the

PROOF OF PROPOSITION 5.7. Since $\mathcal{G} = U_q^{tr}(\mathfrak{g})$ is semisimple we may apply Proposition 5.15 to obtain a nonzero element $\Sigma_0 \in \hat{\mathcal{G}} \otimes \hat{\mathcal{G}}$ expressed in terms of $\sigma_0 \in \hat{\mathcal{G}}$ as in (5.43) and obeying (5.25b), (5.25c). Thus we are left to show that Σ_0 may be normalized to fulfill also (5.25a). But since

$$\Sigma_0\big(\Delta(a)\big) = \sigma_0\big(a_{(1)}\beta S(\alpha a_{(2)})\big) = \epsilon(a)\,\sigma_0\big(\beta S(\alpha)\big)$$

this amounts to showing that $\sigma_0\big(\beta S(\alpha)\big) \neq 0$. We may then define $\Sigma := \big[\sigma_0\big(\beta S(\alpha)\big)\big]^{-1}\Sigma_0$ which now satisfies (5.25a)-(5.25c). We therefore compute

$$\begin{aligned}
\sigma_0\big(\alpha S(\beta)\big) &= \langle S^2(e_\mu)S(\beta)\alpha \rightharpoonup e^\mu \mid \beta S(\alpha)\rangle \\
&= \langle e^\mu \mid \beta S(\alpha)S^2(e_\mu)S(\beta)\alpha\rangle \\
&= \langle e^\mu \mid \beta S(\alpha)g\, e_\mu\, g^{-1}S(\beta)\alpha\rangle \\
&= Tr\big(L(\beta S(\alpha)g) \circ R(g^{-1}S(\beta)\alpha)\big) \\
&= \sum_I Tr_I\big(L(\beta S(\alpha)g) \circ R(g^{-1}S(\beta)\alpha)\big)
\end{aligned}$$

where Tr denotes the trace in $\mathrm{End}(\mathcal{G})$, Tr_I the trace in $\mathrm{End}\big(\mathrm{End}(V_I)\big)$, where $\mathcal{G} = \oplus_I \mathrm{End}(V_I)$, and $L(a)$ and $R(a)$ the left and right multiplication, respectively, with $a \in \mathcal{G}$. Denoting the trace in $\mathrm{End}(V_I)$ by tr_I we now use that

$$Tr_I\big(L(a) \circ R(b)\big) = tr_I\big(\pi_I(a)\big)\, tr_I\big(\pi_I(b)\big),$$

which may be checked by identifying $\mathrm{End}(V) \cong V^* \otimes V$ and noting that $L(a)$ operates on V and $R(b)$ on V^* only. Hence we get

$$\sigma_0\big(\alpha S(\beta)\big) = \sum_I tr_I\big(\pi_I(\beta S(\alpha)g)\big)\, tr_I\big(\pi_I(g^{-1}S(\beta)\alpha)\big) = \sum_I d_I\, d_{\bar{I}} > 0,$$

where $d_I > 0$ are the well–known quantum dimensions of $U_q(\mathfrak{g})$. Here we have used that

$$\begin{aligned}
d_{\bar{I}} &= tr_{\bar{I}}\big(\pi_{\bar{I}}(\beta S(\alpha)g)\big) \\
&= tr_I\big(\pi_I(S(g)S^2(\alpha)S(\beta))\big) \\
&= tr_I\big(\pi_I(g^{-1}S^2(\alpha)S(\beta))\big) \\
&= tr_I\big(\pi_I(\alpha g^{-1}S(\beta))\big) \\
&= tr_I\big(\pi_I(g^{-1}S(\beta)\alpha)\big)
\end{aligned}$$

□

PROOF OF THEOREM 5.8. Recall that a finite dimensional algebra is s.s. if and only if all its finite dimensional left modules are reducible (Wedderburn's structure Theorem, see e.g. [Abe80]). Thus it suffices to show that for every two $\mathcal{M} \bowtie \hat{\mathcal{G}}$ modules V, W, where $W \subset V$ is a submodule, there exists a $\mathcal{M} \bowtie \hat{\mathcal{G}}$-linear surjective map $\bar{p} : V \to W$. Denoting the canonical embedding $i : W \hookrightarrow V$ it therefore suffices to prove the following three identities

$$\bar{p} \circ i = \mathrm{id}_W \tag{5.44}$$

$$\bar{p} \circ m = m \circ \bar{p}, \quad \forall m \in \mathcal{M} \tag{5.45}$$

$$(\mathrm{id}_{\mathcal{G}} \otimes \bar{p}) \circ \mathbf{R}_V = \mathbf{R}_W \circ (\mathrm{id}_{\mathcal{G}} \otimes \bar{p}), \tag{5.46}$$

where at the l.h.s. and r.h.s. of (5.45) we have used the shortcut notation $m \equiv \pi_V(m) \in \mathrm{End}(V)$ and $m \equiv \pi_W(m) \in \mathrm{End}(W)$, respectively. This notation will also be used frequently below. We proceed as follows. Viewing V and W as $\mathcal{M}$–modules, the semisimplicity of $\mathcal{M}$ implies the existence of an $\mathcal{M}$–linear map $p : V \to W$ onto W, i.e. satisfying (5.44) and (5.45). We now define the map $\bar{p} : V \to W$ in terms of p by setting

$$\bar{p} := (\Sigma \otimes \mathrm{id}_W) \circ (\bar{\Omega}^4_R \otimes \bar{\Omega}^5_R \otimes \bar{\Omega}^3_R) \circ \mathbf{R}^{13}_W \circ (\mathrm{id} \otimes \mathrm{id} \otimes p) \circ \mathbf{R}^{23}_V \circ (\bar{\Omega}^2_R \otimes \bar{\Omega}^1_R \otimes \mathrm{id}_V), \tag{5.47}$$

where $\mathbf{R}_W = (\mathrm{id} \otimes \pi_W)(\mathbf{R}) \in \mathcal{G} \otimes \mathrm{End}(W)$ and $\mathbf{R}_V = (\mathrm{id} \otimes \pi_V)(\mathbf{R})$ are the canonical (normal and coherent) right δ–implementers.

To show (5.44) note that $\mathbf{R}_V \circ (\mathbf{1} \otimes i) = (\mathbf{1} \otimes i) \circ \mathbf{R}_W$, since by assumption the embedding i is $\mathcal{M} \bowtie \hat{\mathcal{G}}$–linear. Using $p \circ i = \mathrm{id}_W$ this implies

$$\mathbf{R}^{13}_W \circ (\mathrm{id} \otimes \mathrm{id} \otimes p) \circ \mathbf{R}^{23}_V \circ (\mathrm{id} \otimes \mathrm{id} \otimes i) = \mathbf{R}^{13}_W \mathbf{R}^{23}_W$$

and therefore

$$\begin{aligned} \bar{p} \circ i &= (\Sigma \otimes \mathrm{id}_W) \circ (\bar{\Omega}^4_R \otimes \bar{\Omega}^5_R \otimes \bar{\Omega}^3_R) \circ \mathbf{R}^{13}_W \circ \mathbf{R}^{23}_W \circ (\bar{\Omega}^2_R \otimes \bar{\Omega}^1_R \otimes \mathrm{id}_W) \\ &= (\Sigma \otimes \mathrm{id}_W) \circ (\Delta \otimes \mathrm{id})(\mathbf{R}_W) \\ &= (\epsilon \otimes \mathrm{id}_W)(\mathbf{R}_W) \\ &= \mathrm{id}_W, \end{aligned}$$

where we have used the coherence property (2.67) of $\mathbf{R}$, (5.25a) and then the normality of $\mathbf{R}$. Thus we have proven (5.44).

The $\mathcal{M}$–linearity (5.45) follows from the fact that $\mathbf{R}$ is a δ–implementer (see (2.65)) and from property (5.25c) of Σ since one computes

$$\begin{aligned} m \circ \bar{p} &= (\Sigma \otimes m) \circ (\cdots) \\ &= \Big((\Delta(S^{-1}(m_{(-1)})) \rightharpoonup \Sigma \leftharpoonup \Delta(m_{(1)})) \otimes \mathrm{id}_W \Big) \circ (\cdots) \circ m_{(0)} \\ &= \bar{p} \circ m. \end{aligned}$$

We are left to show (5.46). For the l.h.s. we get (denoting $\Omega := \Omega_R$)

$$\begin{aligned} (\mathrm{id}_{\mathcal{G}} \otimes \bar{p}) \circ \mathbf{R}_V &= (\Sigma \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_W) \circ (\bar{\Omega}^4 \otimes \bar{\Omega}^5 \otimes \mathrm{id}_{\mathcal{G}} \otimes \bar{\Omega}^3) \circ \mathbf{R}^{14}_W \circ (\mathrm{id}^3_{\mathcal{G}} \otimes p) \circ \\ &\qquad \mathbf{R}^{24}_V \circ \mathbf{R}^{34}_V \circ (\bar{\Omega}^2 \otimes \bar{\Omega}^1 \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_V) \\ &= (\Sigma \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_W) \circ \big(\bar{\Omega}^4 \Omega^3_{(1)} \otimes \bar{\Omega}^5 \Omega^4 \otimes \Omega^5 \otimes \bar{\Omega}^3 \Omega^3_{(0)}\big) \circ \mathbf{R}^{14}_W \circ (\mathrm{id}^3_{\mathcal{G}} \otimes p) \\ &\qquad \circ \big[(\Delta \otimes \mathrm{id})(\mathbf{R}_V)\big]^{234} \circ \big(S^{-1}(\Omega^3_{(-1)})\bar{\Omega}^2 \otimes \Omega^2 \bar{\Omega}^1 \otimes \Omega^1 \otimes \mathrm{id}_V\big) \\ &= (\Sigma \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_W) \circ (\phi^{-1} \otimes \mathrm{id}_W) \circ \big(\bar{\Omega}^4 \otimes \Delta(\bar{\Omega}^5) \otimes \bar{\Omega}^3\big) \circ \mathbf{R}^{14}_W \circ (\mathrm{id}^3 \otimes p) \\ &\qquad \circ \big[(\Delta \otimes \mathrm{id})(\mathbf{R}_V)\big]^{234} \circ \big(\bar{\Omega}^2 \otimes \Delta(\bar{\Omega}^1) \otimes \mathrm{id}_V\big) \circ (\phi \otimes \mathrm{id}_V). \end{aligned} \tag{5.48}$$

Here we have used the identity

$$\begin{aligned} &\Omega^1 \otimes \Omega^2 \bar{\Omega}^1 \otimes S^{-1}(\Omega^3_{(-1)})\bar{\Omega}^2 \otimes \bar{\Omega}^3 \Omega^3_{(0)} \otimes \bar{\Omega}^4 \Omega^3_{(1)} \otimes \bar{\Omega}^5 \Omega^4 \otimes \Omega^5 \\ &\quad = \Big[(\Delta^{op}(\bar{\Omega}^1) \otimes \bar{\Omega}^2) \phi^{321} (\Omega^1 \otimes \Delta^{op}(\Omega^2)) \Big] \otimes \Omega^3 \bar{\Omega}^3 \otimes \Big[(\Delta(\Omega^4) \otimes \Omega^5) \phi^{-1} (\bar{\Omega}^4 \otimes \Delta(\bar{\Omega}^5)) \Big] \end{aligned}$$

- following from (2.62), the pentagon identity (2.39b) for Ψ and formula (2.32) - and then property (5.25c) of Σ. A similar calculation yields for the r.h.s. of (5.46)

$$\begin{aligned} \mathbf{R}_W \circ (\mathrm{id}_{\mathcal{G}} \otimes \bar{p}) &= (\mathrm{id}_{\mathcal{G}} \otimes \Sigma \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_W) \circ (\phi \otimes \mathrm{id}_W) \circ \big(\Delta(\bar{\Omega}^4) \otimes \bar{\Omega}^5 \otimes \bar{\Omega}^3\big) \circ \big[(\Delta \otimes \mathrm{id})(\mathbf{R}_W)\big]^{124} \\ &\qquad \circ (\mathrm{id}^3_{\mathcal{G}} \otimes \bar{p}) \circ \mathbf{R}^{34}_V \circ \big(\Delta(\bar{\Omega}^2) \otimes \bar{\Omega}^1 \otimes \mathrm{id}_V\big) \circ (\phi^{-1} \otimes \mathrm{id}_V) \end{aligned} \tag{5.49}$$

Comparing (5.48) and (5.49), both expressions coincide due to (5.25b). Hence we have proved (5.46) which concludes the proof of Theorem 5.8. □

Proofs of Section 5.4.2.

PROPOSITION 5.16. *Let $\mathcal{M} := \mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G}$ and $p^2 = p := \mathbf{1}_{\mathcal{A}} \rtimes \hat{\mathbf{1}} \ltimes e$, where $e \in \mathcal{G}$ is the integral in $\mathcal{G}$. Define the representation*

$$L_p : \mathcal{M} \to \mathrm{End}(\mathcal{M}p), \quad L_p(m)[np] := mnp$$

Then L_p is injective.

PROOF. Since the proof will be quite lengthy, we first treat the case, where $\mathcal{G}$ is an ordinary Hopf algebra. In this case we identify

$$\mathcal{M}p = \mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes e \equiv \mathcal{A} \otimes \hat{\mathcal{G}}$$

as a vector space. Using the multiplication rule (1.43) in $\mathcal{M}$, the integral properties (5.22), (5.23) of e and the identity $\psi_{(1)} \otimes a \leftharpoonup \psi_{(2)} = a_{(1)} \rightharpoonup \psi \otimes a_{(2)}$, one computes

$$\begin{aligned} L_p(A \rtimes \varphi \ltimes a)[B \otimes \psi] &= [A(\varphi_{(1)} \triangleright B) \otimes \varphi_{(2)}\psi_{(1)}]\, \epsilon(a \leftharpoonup \psi_{(2)}) \\ &= [A(\varphi_{(1)} \triangleright B) \otimes \varphi_{(2)}(a \rightharpoonup \psi)] \end{aligned}$$

We claim that the subspace $\mathbf{1}_{\mathcal{A}} \otimes \hat{\mathcal{G}} \subset \mathcal{M}p$ is separating for $\mathcal{M}$. To see this, we define the linear map

$$\begin{aligned} \tau : \mathcal{M} &\longrightarrow \mathrm{H}om_{\mathbb{C}}(\mathcal{G} \otimes \hat{\mathcal{G}}, \mathcal{A}) \\ \tau_m(b \otimes \psi) &:= (\mathrm{id}_{\mathcal{A}} \otimes b)\Big(L_p(m)[\mathbf{1}_{\mathcal{A}} \otimes \psi]\Big) \end{aligned}$$

and continuation by linearity, where $(\mathrm{id}_{\mathcal{A}} \otimes b)(B \otimes \chi) := B\langle b \mid \chi\rangle$. Note, that τ is not an algebra morphism. Choosing $X := \psi \leftharpoonup S^{-1}(b_{(2)}) \otimes b_{(1)}$ and $m = A \rtimes \varphi \ltimes a$ this yields

$$\begin{aligned} \tau_m(X) &= A\,\langle b_{(1)} \mid \varphi\,(a \rightharpoonup \psi \leftharpoonup S^{-1}(b_{(2)}))\rangle \\ &= A\,\langle b_{(1)} \mid \varphi\rangle\langle b_{(2)} \mid a \rightharpoonup \psi \leftharpoonup S^{-1}(b_{(3)})\rangle \\ &= A\,\langle b_{(1)} \mid \varphi\rangle\langle S^{-1}(b_{(3)})b_{(2)}\, a \mid \psi\rangle \\ &= A\,\langle b \mid \varphi\rangle\langle a \mid \psi\rangle \end{aligned}$$

Since $\mathcal{G}$ is finite dimensional, the dual pairing $\langle \cdot \mid \cdot\rangle$ is non degenerate. Thus the map τ is injective and therefore also L_p has to be injective.

Now let us treat the general case, i.e. let $\mathcal{G}$ be a weak quasi–Hopf algebra. In view of (5.14) we may identify

$$\mathcal{M}p = \mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes e \equiv \mathcal{A}\mathbf{1}_{\mathcal{A}}^{(0)} \otimes \hat{\mathcal{G}} \leftharpoonup \mathbf{1}_{\mathcal{A}}^{(1)}. \tag{5.50}$$

as vector spaces. The representation L_p now involves the reassociator ϕ_ρ. Using (5.4) one arrives at

$$\begin{aligned} L_p(A \rtimes \varphi \ltimes a)[B\mathbf{1}_{\mathcal{A}}^{(0)} \otimes \psi \leftharpoonup \mathbf{1}_{\mathcal{A}}^{(1)}] &= [A(\varphi_{(1)} \triangleright B)\bar{X}_\rho^k \otimes (\varphi_{(2)} \leftharpoonup \bar{Y}_\rho^k)\,(a \rightharpoonup \psi \leftharpoonup \bar{Z}_\rho^k)] \\ &= [AB_{(0)}\bar{X}_\rho^k \otimes (\varphi \leftharpoonup B_{(1)}\bar{Y}_\rho^k)\,(a \rightharpoonup \psi \leftharpoonup \bar{Z}_\rho^k)], \end{aligned} \tag{5.51}$$

where we denote $\phi_\rho^{-1} = \bar{X}_\rho^i \otimes \bar{Y}_\rho^i \otimes \bar{Z}_\rho^i$ and where in the second equality we have used the identity $\varphi_{(1)} \triangleright B \otimes \varphi_{(2)} = B_{(0)} \otimes \varphi \leftharpoonup B_{(1)}$. Next we modify the linear map τ by setting

$$\begin{aligned} \tau : \mathcal{M} &\longrightarrow \mathrm{H}om_{\mathbb{C}}(\mathbf{1}_{(1)}\mathcal{G} \otimes \hat{\mathcal{G}} \leftharpoonup S^{-1}(\mathbf{1}_{(2)}), \mathcal{A}) \\ \tau_m\big(\mathbf{1}_{(1)}b \otimes \psi \leftharpoonup S^{-1}(\mathbf{1}_{(2)})\big) &:= (\mathrm{id}_{\mathcal{A}} \otimes \bar{Y}_\rho^i\mathbf{1}_{(1)}b)\Big(L_p(m)[q_\rho^1 \otimes \psi \leftharpoonup S^{-1}(\bar{Z}_\rho^i\mathbf{1}_{(2)})q_\rho^2]\Big)\, \bar{X}_\rho^i, \end{aligned}$$

where $q_\rho \in \mathcal{A} \otimes \mathcal{G}$ is given in (2.80). Choosing $X = b_{(1)}p^1 \otimes \psi \leftharpoonup S^{-1}(b_{(2)}p^2) \in \mathbf{1}_{(1)}\mathcal{G} \otimes \hat{\mathcal{G}} \leftharpoonup S^{-1}(\mathbf{1}_{(2)})$, where $p := p_{\rho=\Delta} \in \mathcal{G} \otimes \mathcal{G}$ is defined in (2.79), we get for $m = A \rtimes \varphi \ltimes a \in \mathcal{M}$:

$$\begin{aligned}
\tau_m(X) &= (\mathrm{id}_\mathcal{A} \otimes \bar{Y}^i_\rho b_{(1)}p^1)\Big(L_p(m)[q^1_\rho \otimes \psi \leftharpoonup S^{-1}(\bar{Z}^i_\rho b_{(2)}p^2)q^2_\rho]\Big)\,\bar{X}^i_\rho \\
&= Aq^1_{\rho(0)}\bar{X}^k_\rho \bar{X}^i_\rho \,\langle \bar{Y}^i_\rho b_{(1)}p^1 \mid (\varphi \leftharpoonup q^1_{\rho(1)}\bar{Y}^k_\rho)(a \rightharpoonup \psi \leftharpoonup S^{-1}(\bar{Z}^i_\rho b_{(2)}p^2)q^2_\rho \bar{Z}^k_\rho)\rangle \\
&= Aq^1_{\rho(0)}\bar{X}^k_\rho \bar{X}^i_\rho \,\langle \Delta(\bar{Y}^i_\rho b_{(1)}p^1) \mid (\varphi \leftharpoonup q^1_{\rho(1)}\bar{Y}^k_\rho) \otimes (a \rightharpoonup \psi \leftharpoonup S^{-1}(\bar{Z}^i_\rho b_{(2)}p^2)q^2_\rho \bar{Z}^k_\rho)\rangle \\
&= Aq^1_{\rho(0)}\bar{X}^k_\rho \bar{X}^i_\rho \,\langle (q^1_{\rho(1)}\bar{Y}^k_\rho \otimes S^{-1}(\bar{Z}^i_\rho b_{(2)}p^2)q^2_\rho \bar{Z}^k_\rho) \cdot \Delta(\bar{Y}^i_\rho b_{(1)}p^1) \cdot (\mathbf{1} \otimes a) \mid \varphi \otimes \psi\rangle \\
&\overset{(5.52)}{=} A\mathbf{1}^{(0)}_\mathcal{A} \,\langle (\mathbf{1}^{(1)}_\mathcal{A} \otimes S^{-1}(b_{(2)}p^2)) \cdot q \cdot \Delta(b_{(1)}p^1) \cdot (\mathbf{1} \otimes a) \mid \varphi \otimes \psi\rangle \\
&= A\mathbf{1}^{(0)}_\mathcal{A} \,\langle \mathbf{1}^{(1)}_\mathcal{A} b\, \mathbf{1}_{(1)} \mid \varphi\rangle \langle \mathbf{1}_{(2)} a \mid \psi\rangle \\
&= A\mathbf{1}^{(0)}_\mathcal{A} \,\langle b \mid \mathbf{1}_{(1)} \rightharpoonup \varphi \leftharpoonup \mathbf{1}^{(1)}_\mathcal{A}\rangle \langle \mathbf{1}_{(2)} a \mid \psi\rangle \\
&= (\mathrm{id}_\mathcal{A} \otimes b \otimes \psi)\Big(A \rtimes \varphi \ltimes a\Big)
\end{aligned}$$

Here we have plugged in (5.51) in the second line, then used the identity (5.52), which will be proved below in Lemma 5.17, and finally the identities (2.104b) and (3.17) for the elements p, q. Hence, τ is injective, implying that L_p is injective. □

LEMMA 5.17. *Let $q := q_{\rho=\Delta} \in \mathcal{G} \otimes \mathcal{G}$ and $q_\rho \in \mathcal{A} \otimes \mathcal{G}$ be given by (2.80) and denote $\phi_\rho^{-1} = \bar{X}^i_\rho \otimes \bar{Y}^i_\rho \otimes \bar{Z}^i_\rho$. Then the following identity holds.*

$$\big(\mathbf{1}_\mathcal{A} \otimes \mathbf{1} \otimes S^{-1}(\bar{Z}^i_\rho)\big) \cdot (\rho \otimes \mathrm{id})(q_\rho) \cdot \phi_\rho^{-1} \cdot \big(\bar{X}^i_\rho \otimes \Delta(\bar{Y}^i_\rho)\big) = \big(\rho(\mathbf{1}_\mathcal{A}) \otimes \mathbf{1}\big) \cdot \big(\mathbf{1}_\mathcal{A} \otimes q\big) \tag{5.52}$$

PROOF. The l.h.s. of (5.52) may be written as

$$\text{l.h.s. of (5.52)} = \Lambda^1 \otimes \Lambda^2 \otimes S^{-1}(\alpha\Lambda^4)\Lambda^3$$

where

$$\Lambda = (\rho \otimes \mathrm{id}_\mathcal{G} \otimes \mathrm{id}_\mathcal{G})(\phi_\rho) \cdot (\phi_\rho^{-1} \otimes \mathbf{1}) \cdot (\mathrm{id}_\mathcal{A} \otimes \Delta \otimes \mathrm{id}_\mathcal{G})(\phi_\rho^{-1})$$

But the pentagon equation (2.36b) for ϕ_ρ implies the equality

$$\Lambda = (\mathrm{id}_\mathcal{A} \otimes \mathrm{id}_\mathcal{G} \otimes \Delta)(\phi_\rho^{-1}) \cdot (\mathbf{1}_\mathcal{A} \otimes \phi)$$

Therefore (denoting $\phi = X^k \otimes Y^k \otimes Z^k$)

$$\begin{aligned}
\text{l.h.s. of (5.52)} &= \bar{X}^i_\rho \otimes \bar{Y}^i_\rho X^k \otimes S^{-1}(\alpha \bar{Z}^i_{\rho(2)} Z^k)\bar{Z}^i_{\rho(1)} Y^k \\
&= \mathbf{1}^{(0)}_\mathcal{A} \otimes \mathbf{1}^{(1)}_\mathcal{A} X^k \otimes S^{-1}(\alpha Z^k) Y^k \\
&= \big(\rho(\mathbf{1}_\mathcal{A}) \otimes \mathbf{1}\big)\big(\mathbf{1}_\mathcal{A} \otimes q\big),
\end{aligned}$$

where for the second equality we have used the antipode property (2.15) and the identity $(\mathrm{id}_\mathcal{A} \otimes \mathrm{id}_\mathcal{G} \otimes \epsilon)(\phi_\rho^{-1}) = \rho(\mathbf{1}_\mathcal{A})$. □

Proofs of Section 5.4.3. We start with some technical considerations needed later on. First recall the canonical left and right actions of a weak quasi–Hopf algebra $\mathcal{G}$ on its dual $\hat{\mathcal{G}}$ already introduced in (2.34). They are given by

$$a \rightharpoonup \varphi := \varphi_{(1)} \,\langle \varphi_{(2)} \mid a\rangle, \quad \varphi \leftharpoonup a := \varphi_{(2)} \,\langle \varphi_{(1)} \mid a\rangle, \quad a \in \mathcal{G},\ \varphi \in \hat{\mathcal{G}}. \tag{5.53}$$

We list a few properties of these actions:

LEMMA 5.18. *The following identities hold for all $a, b \in \mathcal{G}$, $\varphi \in \hat{\mathcal{G}}$*

$$\hat{\Delta}(a \rightharpoonup \varphi \leftharpoonup b) = \varphi_{(1)} \leftharpoonup b \otimes a \rightharpoonup \varphi_{(2)} \tag{5.54}$$

$$a \rightharpoonup \varphi_{(1)} \otimes \varphi_{(2)} = \varphi_{(1)} \otimes \varphi_{(2)} \leftharpoonup a \tag{5.55}$$

$$\hat{S}\big(S(a) \rightharpoonup \varphi\big) = \hat{S}(\varphi) \leftharpoonup a, \quad \hat{S}\big(\varphi \leftharpoonup S(a)\big) = a \rightharpoonup \hat{S}(\varphi). \tag{5.56}$$

PROOF. The first two identities follow by direct computation from definition (5.53) (note that $\hat{\Delta}$ is strictly coassociative also for quasi–Hopf algebras $\mathcal{G}$). To prove the first identity in (5.56), we pair the l.h.s. with an arbitrary element $b \in \mathcal{G}$ to obtain

$$\begin{aligned}\langle \hat{S}(S(a)) \rightharpoonup \varphi \mid b\rangle &= \langle S(a) \rightharpoonup \varphi \mid S(b)\rangle \\ &= \langle \varphi \mid S(b)S(a)\rangle \\ &= \langle \hat{S}(\varphi) \mid ab\rangle \\ &= \langle \hat{S}(\varphi) \leftharpoonup a \mid b\rangle\end{aligned}$$

The second identity is proven analogously. □

Next we recall the definition of the so called *Pentagon operator* $U : \hat{\mathcal{G}} \otimes \hat{\mathcal{G}} \to \hat{\mathcal{G}} \otimes \hat{\mathcal{G}}$ [], where $\hat{\mathcal{G}}$ is the dual of a Hopf algebra $\mathcal{G}$:

$$\begin{aligned}U(\varphi \otimes \psi) &= \varphi \hat{S}(\psi_{(1)}) \otimes \psi_{(2)} \\ U^{-1}(\varphi \otimes \psi) &= \varphi \psi_{(1)} \otimes \psi_{(2)}.\end{aligned}$$

The operator U is also called Takesaki operator or multiplicative unitary, see e.g. [BS93]. Denoting the left $\mathcal{G}$–actions

$$l_a(\varphi \otimes \psi) := a_{(1)} \rightharpoonup \varphi \otimes \psi \leftharpoonup S(a_{(2)}), \quad L_a(\varphi \otimes \psi) := a \rightharpoonup \varphi \otimes \psi, \tag{5.57}$$

U has the following intertwiner property

$$U\, l_a = L_a\, U, \quad \forall a \in \mathcal{G}.$$

We now propose the appropriate generalization to weak quasi–Hopf algebras.

LEMMA 5.19. *Let $\mathcal{G}$ be a weak quasi–Hopf algebra and let $p := p_{\rho=\Delta}, q := q_{\rho=\Delta} \in \mathcal{G} \otimes \mathcal{G}$ be the elements given in* (2.79),(2.80). *Define $U, \bar{U} : \hat{\mathcal{G}} \otimes \hat{\mathcal{G}} \to \hat{\mathcal{G}} \otimes \hat{\mathcal{G}}$ by*

$$U(\varphi \otimes \psi) := (\varphi \leftharpoonup q^1)(\hat{S}(\psi_{(1)}) \leftharpoonup q^2) \otimes \psi_{(2)} \tag{5.58}$$

$$\bar{U}(\varphi \otimes \psi) := (p^1 \rightharpoonup \varphi)(p^2 \rightharpoonup \psi_{(1)}) \otimes \psi_{(2)}. \tag{5.59}$$

Then $U\bar{U}$ and $\bar{U}U$ are idempotents given by

$$U\bar{U}(\varphi \otimes \psi) = \varphi \leftharpoonup \mathbf{1}_{(1)} \otimes \psi \leftharpoonup \mathbf{1}_{(2)} \tag{5.60}$$

$$\bar{U}U(\varphi \otimes \psi) = \mathbf{1}_{(1)} \rightharpoonup \varphi \otimes \psi \leftharpoonup S(\mathbf{1}_{(2)}), \tag{5.61}$$

and U has the intertwiner property

$$U\, l_a = L_a\, U, \quad \forall a \in \mathcal{G}. \tag{5.62}$$

We remark that the above Lemma also implies, that the restricted map $U : \bar{U}U(\hat{\mathcal{G}} \otimes \hat{\mathcal{G}}) \to U\bar{U}(\hat{\mathcal{G}} \otimes \hat{\mathcal{G}})$ is bijective with inverse $\bar{U}$.

PROOF. We show (5.60). According to (2.104a) and (3.16) the elements $p, q \in \mathcal{G} \otimes \mathcal{G}$ obey

$$\begin{aligned}\Delta(a_{(1)})\, p\, [\mathbf{1} \otimes S(a_{(2)})] &= p\, [a \otimes \mathbf{1}], \quad \forall a \in \mathcal{G} \\ \Delta(q^1)\, p\, [\mathbf{1} \otimes S(q^2)] &= \Delta(\mathbf{1}),\end{aligned}$$

which together imply the identity

$$\Delta(\mathbf{1})\, [a \otimes \mathbf{1}] = \Delta(q^1\, a_{(1)})\, p\, [\mathbf{1} \otimes S(q^2\, a_{(2)})], \quad \forall a \in \mathcal{G}. \tag{5.63}$$

We now compute for $a, b \in \mathcal{G}$, $\varphi, \psi \in \hat{\mathcal{G}}$

$$\begin{aligned}\langle U\bar{U}(\varphi \otimes \psi) \mid a \otimes b\rangle &= \langle \{[(p^1 \rightharpoonup \varphi)(p^2 \rightharpoonup \psi_{(1)})] \leftharpoonup q^1\} \,(\hat{S}(\psi_{(2)}) \leftharpoonup q^2) \otimes \psi_{(3)} \mid a \otimes b\rangle \\ &= \langle [(p^1 \rightharpoonup \varphi)(p^2 \rightharpoonup \psi_{(1)})] \otimes \hat{S}(\psi_{(2)}) \otimes \psi_{(3)} \mid q\Delta(a) \otimes b\rangle \\ &= \langle \varphi \otimes \psi_{(1)} \otimes \hat{S}(\psi_{(2)}) \otimes \psi_{(3)} \mid \Delta(q^1 a_{(1)})p \otimes q^2 a_{(2)} \otimes b\rangle \\ &= \langle \varphi \otimes \psi \mid \Delta(q^1 a_{(1)})\, p\, [\mathbf{1} \otimes S(q^2 a_{(2)})b]\rangle \\ &= \langle \varphi \leftharpoonup \mathbf{1}_{(1)} \otimes \psi \leftharpoonup \mathbf{1}_{(2)} \mid a \otimes b\rangle\end{aligned}$$

where in the last equality we have used (5.63). This proves (5.60). The identity (5.61) is shown similarly using the analogue of (5.63) following from (2.104b) and (3.17).

We are left to show the intertwiner property (5.62).

$$\begin{aligned} U\, l_a(\varphi \otimes \psi) &= \big(a_{(1)} \rightharpoonup \varphi \leftharpoonup q^1\big)\big(\hat{S}(\psi_{(1)} \leftharpoonup S(a_{(2)})) \leftharpoonup q^2\big) \otimes \psi_{(2)} \\ &= \big(a_{(1)} \rightharpoonup \varphi \leftharpoonup q^1\big)\big(a_{(2)} \rightharpoonup \hat{S}(\psi_{(1)}) \leftharpoonup q^2\big) \otimes \psi_{(2)} \\ &= L_a\, U(\varphi \otimes \psi), \end{aligned}$$

where we have used (5.56) in the second equality. This concludes the proof of Lemma 5.19. □

COROLLARY 5.20. *Consider the following two subspaces of* $\hat{\mathcal{G}} \otimes \hat{\mathcal{G}}$

$$\mathcal{C}_{diag} := \{\beta \rightharpoonup \psi_{(1)} \otimes \psi_{(2)} \mid \psi \in \hat{\mathcal{G}}\} \tag{5.64}$$

$$\mathcal{C}_{inv} := \{X \in \hat{\mathcal{G}} \otimes \hat{\mathcal{G}} \mid l_a(X) = \epsilon(a)X,\ \forall a \in \mathcal{G}\}, \tag{5.65}$$

where $\beta \in \mathcal{G}$ *is defined in* (2.15) *and* l_a *is the* $\mathcal{G}$*–action given in* (5.57).
Then $\mathcal{C}_{diag} = \mathcal{C}_{inv}$*, and as a linear space* $\mathcal{C}_{diag}$ *is isomorph to* $\hat{\mathcal{G}}$*.*

PROOF. First we note that due to (5.55) $\mathcal{C}_{diag} \subset \bar{U}U(\hat{\mathcal{G}}\otimes\hat{\mathcal{G}})$. We also have $\mathcal{C}_{inv} \subset \bar{U}U(\hat{\mathcal{G}}\otimes\hat{\mathcal{G}})$ since $\epsilon(\mathbf{1}) = 1$. We now compute, again using (5.55),

$$a_{(1)}\beta \rightharpoonup \psi_{(1)} \otimes \psi_{(2)} \leftharpoonup S(a_{(2)}) = a_{(1)}\beta S(a_{(2)}) \rightharpoonup \psi_{(1)} \otimes \psi_{(2)} = \epsilon(a)\big(\beta \rightharpoonup \psi_{(1)} \otimes \psi_{(2)}\big),$$

which proves the inclusion $\mathcal{C}_{diag} \subset \mathcal{C}_{inv}$. We are left to show the inclusion $\mathcal{C}_{inv} \subset \mathcal{C}_{diag}$. First note that (using $(\epsilon \otimes \mathrm{id})(p) = \beta$)

$$\bar{U}(\hat{\mathbf{1}} \otimes \psi) = \beta \rightharpoonup \psi_{(1)} \otimes \psi_{(2)},$$

which implies $\bar{U}(\hat{\mathbf{1}} \otimes \hat{\mathcal{G}}) = \mathcal{C}_{diag}$ and, since $\hat{\mathbf{1}} \otimes \hat{\mathcal{G}} \subset U\bar{U}(\hat{\mathcal{G}} \otimes \hat{\mathcal{G}})$, also

$$U(\mathcal{C}_{diag}) = \hat{\mathbf{1}} \otimes \hat{\mathcal{G}}. \tag{5.66}$$

Thus $\mathcal{C}_{diag} \cong \hat{\mathcal{G}}$ as linear spaces. Now we invoke the intertwiner property (5.62) to conclude (supressing summation symbols $\sum_i$)

$$\varphi^i \otimes \psi^i \in \mathcal{C}_{inv} \ \Rightarrow\ L_a\, U(\varphi^i \otimes \psi^i) = \epsilon(a)\, U(\varphi^i \otimes \psi^i), \quad \forall a \in \mathcal{G}$$

which further implies $U(\varphi^i \otimes \psi^i) \in \hat{\mathbf{1}} \otimes \hat{\mathcal{G}}$, since

$$a \rightharpoonup \chi = \epsilon(a)\chi,\ \forall a \in \mathcal{G} \quad \Rightarrow \quad \chi \in \mathbb{C}\hat{\mathbf{1}}.$$

Thus due to (5.66) $U(\mathcal{C}_{inv}) \subset U(\mathcal{C}_{diag})$ implying

$$\mathcal{C}_{inv} \equiv \bar{U}U(\mathcal{C}_{inv}) \subset \bar{U}U(\mathcal{C}_{diag}) \equiv \mathcal{C}_{diag}$$

□

COROLLARY 5.21. *Consider the left action* $l^0_a(\varphi \otimes \psi) := a_{(1)} \rightharpoonup \varphi \otimes \psi \leftharpoonup S^{-1}(a_{(2)})$ *and define*

$$\begin{aligned} \mathcal{C}^0_{inv} &:= \big\{X \in \hat{\mathcal{G}} \otimes \hat{\mathcal{G}} \mid l^0_a(X) = \epsilon(a)X, \quad \forall a \in \mathcal{G}\big\} \\ \mathcal{C}^0_{diag} &:= \big\{\psi_{(1)} \otimes \psi_{(2)} \leftharpoonup \beta g\big\} \\ \mathcal{C} &:= l^0_e(\hat{\mathcal{G}} \otimes \hat{\mathcal{G}}) \equiv \big\{e_{(1)} \rightharpoonup \varphi^i \otimes \psi^i \leftharpoonup S^{-1}(e_{(2)})\big\} \end{aligned}$$

where $e \in \mathcal{G}$ *is the unique integral in* $\mathcal{G}$ *and* $g \in \mathcal{G}$ *implements the square of the antipode. Then* $\mathcal{C}^0_{inv} = \mathcal{C}^0_{diag} = \mathcal{C}$*, and as a linear space* $\mathcal{C} \cong \hat{\mathcal{G}}$*.*

PROOF. The identity $\mathcal{C}^0_{inv} = \mathcal{C}^0_{diag}$ follows immediately from Corollary 5.20 since we have

$$\begin{aligned} l^0_a(\varphi^i \otimes \psi^i) = \epsilon(a)\,(\varphi^i \otimes \psi^i) &\Leftrightarrow l_a(\varphi^i \otimes \psi^i \leftharpoonup g^{-1}) = \epsilon(a)\,(\varphi^i \otimes \psi^i \leftharpoonup g^{-1}) \\ &\Leftrightarrow \varphi^i \otimes \psi^i \leftharpoonup g^{-1} \in \mathcal{C}_{diag} \\ &\Leftrightarrow \varphi^i \otimes \psi^i \in \mathcal{C}^0_{diag} \end{aligned}$$

where the first equivalence uses (5.38)and the third (5.55). This also implies that as a linear space $\mathcal{C}^0_{inv} \cong \mathcal{C}_{inv} \cong \hat{\mathcal{G}}$. Thus we are left to show the equality $\mathcal{C}^0_{inv} = \mathcal{C}$. The property $\epsilon(e) = 1$ implies the inclusion $\mathcal{C}^0_{inv} \subset \mathcal{C}$. Conversely let $X \in \mathcal{C}$, then $l^0_e(X) = X$ and therefore

$$l^0_a(X) = l^0_a \circ l^0_e(X) = l^0_{ae}(X) = \epsilon(a) l^0_e(X) = \epsilon(a)X$$

which implies the inclusion $\mathcal{C} \subset \mathcal{C}^0_{inv}$. This finishes the proof of Corollary 5.21. □

We are now in the position to prove

LEMMA 5.22. *The map $f:\mathcal{M}_0\to\tilde p\mathcal{M}_2\tilde p$ defined in* (5.41) *is an algebra map.*

PROOF. Under the conditions of Theorem 5.13 denote the left $(\delta_0,\delta_1,\delta_2)$–implementers

$$\mathbf{L}_0\in\mathcal{G}\otimes\mathcal{M}_0,\quad \mathbf{L}_1\in\mathcal{G}\otimes\mathcal{M}_1,\quad \mathbf{L}_2\in\mathcal{G}\otimes\mathcal{M}_2$$

and define

$$\begin{aligned} I(\varphi) &:= L_2(\varphi_{(2)}\leftharpoonup\beta g)\,L_1(\varphi_{(1)})\,\tilde p,\quad \varphi\in\hat{\mathcal{G}}\\ i(A) &:= \hat{\mathbf{1}}\bowtie\big(\hat{\mathbf{1}}\bowtie(A\otimes e)\big),\quad A\in\mathcal{A}. \end{aligned}\tag{5.67}$$

Thus the map f given in (5.41) may be expressed as

$$f(\varphi\bowtie A)=f\big(L_0(\varphi)\,A\big)=I(\varphi)\,i(A)$$

and according to Corollary 2.14 f is an algebra map (extending i) if and only if $\mathbf{I}:=e_\mu\otimes I(e^\mu)$ is a coherent left δ_0–implementer (with respect to i). To show that $\mathbf{I}$ is a δ_0–implementer (for $\delta_0=(\mathrm{id}\otimes\rho)\circ\lambda$) we compute

$$\begin{aligned} i(A)\,I(\varphi) &= i(A)\,L_2(\varphi_{(2)}\leftharpoonup\beta g)\,L_1(\varphi_{(1)})\,\tilde p\\ &= L_2(A_{(-1)}\rightharpoonup\varphi_{(2)}\leftharpoonup\beta g)\,i(A_{(0)})\,L_1(\varphi_{(1)})\,\tilde p\\ &= L_2(A_{(-1)}\rightharpoonup\varphi_{(2)}\leftharpoonup\beta g)\,L_1\big(\varphi_{(1)}\leftharpoonup S^{-1}(A_{(0,1)})\big)\,i(A_{(0,0)})\\ &= I\big(A_{(-1)}\rightharpoonup\varphi\leftharpoonup S^{-1}(A_{(0,1)})\big)\,i(A_{(0,0)}), \end{aligned}$$

where we have used (5.54) in the last equality. Thus $\mathbf{I}$ is a left δ_0–implementer. According to (2.66), $\mathbf{I}$ is coherent iff

$$I(\bar\Omega_L^1\rightharpoonup\varphi\leftharpoonup\bar\Omega_L^5)\,I(\bar\Omega_L^2\rightharpoonup\psi\leftharpoonup\bar\Omega_L^4)\,i(\bar\Omega_L^3)\stackrel{!}{=}I(\varphi\psi)\tag{5.68}$$

with $\bar\Omega_L\equiv\Omega_L^{-1}=(h^{-1})^{54}\,(\mathrm{id}_{\mathcal{G}}\otimes\mathrm{id}_{\mathcal{G}}\otimes\mathrm{id}_{\mathcal{A}}\otimes S^{-1}\otimes S^{-1})(\psi_r)$, where ψ_r is expressed in terms of ϕ_λ,ϕ_ρ and $\phi_{\lambda\rho}$ in (2.52d). To prove (5.68), we need the multiplication rules of $\mathbf{L}_1$ and $\mathbf{L}_2$. Using (5.1) they may be expressed as

$$L_2(\varphi\psi)=L_2\big(\phi_\lambda^1\rightharpoonup\varphi\rightharpoonup\bar h^1S^{-1}(\bar\phi^3)\big)\,L_2\big(\phi_\lambda^2\rightharpoonup\psi\leftharpoonup\bar h^2S^{-1}(\bar\phi^2)\big)\,(\phi_\lambda^3\otimes\bar\phi^1)\tag{5.69}$$

$$L_1(\varphi\psi)=L_1\big(\phi^1\rightharpoonup\varphi\rightharpoonup\bar h^1S^{-1}(\bar\phi_\rho^3)\big)\,L_1\big(\phi^2\rightharpoonup\psi\leftharpoonup\bar h^2S^{-1}(\bar\phi_\rho^2)\big)\,(\bar\phi_\rho^1\otimes\phi^3),\tag{5.70}$$

where here and throughout we suppress the embedding $\mathcal{A}\otimes\mathcal{G}\subset\mathcal{M}_2$. Moreover, with $\delta_2=(\tilde\lambda\otimes\mathrm{id})\circ\tilde\Delta$, one obtains from (5.8),(5.9) (after some calculations expressing $\mathbf{L}$ in terms of $\mathbf{\Gamma}$)

$$L_1(\psi)\,L_2(\varphi)=L_2\big(\phi_{\lambda\rho}^1\rightharpoonup\varphi\leftharpoonup S^{-1}(\bar\phi^3)\big)\,L_1\big(\bar\phi^1\rightharpoonup\psi\leftharpoonup S^{-1}(\phi_{\lambda\rho}^3)\big)\,(\phi_{\lambda\rho}^2\otimes\bar\phi^2)\tag{5.71}$$

We know compute

$$\begin{aligned} I(\varphi)\,I(\psi) &= L_2(\varphi_{(2)}\leftharpoonup\beta g)\,L_1(\varphi_{(1)})\,L_2(\psi_{(2)}\leftharpoonup\beta g)\,L_1(\psi_{(1)})\,\tilde p\\ &= L_2(\varphi_{(2)}\leftharpoonup\beta g)\,L_2\big(\phi_{\lambda\rho}^1\rightharpoonup\psi_{(2)}\leftharpoonup\beta gS^{-1}(\bar\phi^3)\big)\\ &\qquad L_1\big(\bar\phi^1\rightharpoonup\varphi_{(1)}\leftharpoonup S^{-1}(\phi_{\lambda\rho}^3)\big)\,(\phi_{\lambda\rho}^2\otimes\bar\phi^2)\,L_1(\psi_{(1)})\,\tilde p\\ &= L_2(\varphi_{(2)}\leftharpoonup\beta g)\,L_2\big(\phi_{\lambda\rho}^1\rightharpoonup\psi_{(2)}\leftharpoonup\beta gS^{-1}(\bar\phi^3)\big)\\ &\qquad L_1\big(\bar\phi^1\rightharpoonup\varphi_{(1)}\leftharpoonup S^{-1}(\phi_{\lambda\rho}^3)\big)\,L_1\big(\bar\phi^2\rightharpoonup\psi_{(1)}\leftharpoonup S^{-1}(\phi_{\lambda\rho(1)}^2)\big)\,(\phi_{\lambda\rho(0)}^2\otimes e) \end{aligned}$$

where in the first equation we have plugged in definition (5.67), in the second we have used the commutation relations (5.71) and in the last equality that $\mathbf{L}_1$ is a δ_1–implementer and then the integral property of e. This yields for the l.h.s. of (5.68)

$$\begin{aligned} &I(\bar\Omega_L^1\rightharpoonup\varphi\leftharpoonup\bar\Omega_L^5)\,I(\bar\Omega_L^2\rightharpoonup\psi\leftharpoonup\bar\Omega_L^4)\,i(\bar\Omega_L^3)\\ &\quad= I\big(\phi_\lambda^1\rightharpoonup\varphi\leftharpoonup\bar h^1S^{-1}(\bar\phi_{\lambda\rho}^3\,\bar\phi_\rho^3\,\phi_{\lambda(1,2)}^3)\big)\\ &\qquad\quad I\big(\bar\phi_{\lambda\rho}^1\,\phi_\lambda^2\rightharpoonup\psi\leftharpoonup\bar h^2S^{-1}(\bar\phi_{\lambda\rho(1)}^2\,\bar\phi_\rho^2\,\phi_{\lambda(1,1)}^3)\big)\,i(\bar\phi_{\lambda\rho(0)}^2\,\bar\phi_\rho^1\,\phi_{\lambda(0)}^3)\\ &\quad= L_2\big(\tilde\varphi_{(2)}\leftharpoonup\beta g\big)\,L_2\big(\tilde\psi_{(2)}\leftharpoonup\beta gS^{-1}(\bar\phi^3)\big)\,L_1\big(\bar\phi^1\rightharpoonup\tilde\varphi_{(1)}\big)\,L_1\big(\bar\phi^2\rightharpoonup\tilde\psi_{(1)}\big)\,(\bar\phi_\rho^1\,\phi_{\lambda(0)}^3\otimes e)\\ &\quad= L_2\big(\tilde\varphi_{(2)}\leftharpoonup\bar\phi^1\beta g\big)\,L_2\big(\tilde\psi_{(2)}\leftharpoonup\bar\phi^2\beta S(\bar\phi^3)g\big)\,L_1\big(\tilde\varphi_{(1)}\big)\,L_1\big(\tilde\psi_{(1)}\big)\,(\bar\phi_\rho^1\,\phi_{\lambda(0)}^3\otimes e), \end{aligned}\tag{5.72}$$

where we have used (5.54) and the notation

$$\tilde\varphi := \phi^1_\lambda \rightharpoonup \varphi \leftharpoonup \bar h^1 S^{-1}(\bar\phi^3_\rho\, \phi^3_{\lambda(1,2)})$$
$$\tilde\psi := \phi^2_\lambda \rightharpoonup \psi \leftharpoonup \bar h^2 S^{-1}(\bar\phi^2_\rho\, \phi^3_{\lambda(1,1)})$$

in the second equality and (5.55), (5.38) in the third equality. To compute the r.h.s. of (5.68) we first note that (5.39) and (2.29) imply the identity $\Delta(\beta g) = \delta\,(g\otimes g)\,h$. Hence

$$\begin{aligned}
I(\varphi\psi) &= L_2\big((\varphi_{(2)}\psi_{(2)}) \leftharpoonup \beta g\big)\, L_1\big(\varphi_{(1)}\psi_{(1)}\big)\,\tilde p \\
&= L_2\big((\varphi_{(2)} \leftharpoonup \delta^1 g h^1)(\psi_{(2)} \leftharpoonup \delta^2 g h^2)\big)\, L_1\big(\varphi_{(1)}\psi_{(1)}\big)\,\tilde p \\
&\overset{(5.69)}{=} L_2\big(\phi^1_\lambda \rightharpoonup \varphi_{(2)} \leftharpoonup \delta^1 g S^{-1}(\bar\phi^3)\big)\, L_2\big(\phi^2_\lambda \rightharpoonup \psi_{(2)} \leftharpoonup \delta^2 g S^{-1}(\bar\phi^2)\big) \\
&\qquad L_1\big(\bar\phi^1 \rightharpoonup (\varphi_{(1)}\psi_{(1)}) \leftharpoonup S^{-1}(\phi^3_{\lambda(1)})\big)\,\big(\phi^3_{\lambda(0)}\otimes e\big) \\
&\overset{(5.70)}{=} L_2\big(\ldots\big)\, L_2\big(\ldots\big)\, L_1\big(\bar\phi_{(1)} \rightharpoonup \varphi_{(1)} \leftharpoonup \bar h^1 S^{-1}(\bar\phi^3_\rho\, \phi^3_{\lambda(1,2)})\big) \\
&\qquad L_1\big(\bar\phi^1_{(2)} \rightharpoonup \psi_{(1)} \leftharpoonup \bar h^2 S^{-1}(\bar\phi^2_\rho\, \phi^3_{\lambda(1,1)})\big)\,\big(\bar\phi^1_\rho\, \phi^3_{\lambda(0)}\otimes e\big) \\
&\overset{(5.54)}{=} L_2\big(\tilde\varphi_{(2)} \leftharpoonup \delta^1 g S^{-1}(\bar\phi^3)\big)\, L_2\big(\tilde\psi_{(2)} \leftharpoonup \delta^2 g S^{-1}(\bar\phi^2)\big) \\
&\qquad L_1\big(\bar\phi^1_{(1)} \rightharpoonup \tilde\varphi_{(1)}\big)\, L_1\big(\bar\phi^1_{(2)} \rightharpoonup \tilde\psi_{(1)}\big)\,\big(\bar\phi^1_\rho\, \phi^3_{\lambda(0)}\otimes e\big) \\
&\overset{(5.55)}{=} L_2\big(\tilde\varphi_{(2)} \leftharpoonup \bar\phi^1_{(1)}\delta^1 S(\bar\phi^3) g\big)\, L_2\big(\tilde\psi_{(2)} \leftharpoonup \bar\phi^1_{(2)}\delta^2 S(\bar\phi^2) g\big)\, L_1\big(\tilde\varphi_{(1)}\big)\, L_1\big(\tilde\psi_{(1)}\big)\,\big(\bar\phi^1_\rho\, \phi^3_{\lambda(0)}\otimes e\big)
\end{aligned}$$

where we have used the integral property of e several times and also the fact that $\mathbf{L}_1$ is a left δ_1–implementer to move elements of $\mathcal{A}\otimes\mathcal{G}$ to the right. Comparing with (5.72) this proves (5.68) provided

$$\bar\phi^1\beta\otimes\bar\phi^2\beta S(\bar\phi^3) = \Delta(\bar\phi^1)\delta\left(S(\bar\phi^3)\otimes S(\bar\phi^2)\right)$$

which is immediately verified by going back to the definition (2.25) of δ. Hence we have proven the coherence property of $\mathbf{I}$ which concludes the proof of Lemma 5.22. □