CHAPTER 4

The quantum double $\mathcal{D}(\mathcal{G})$

As our first important application we will now propose the definition of the quantum double $\mathcal{D}(\mathcal{G})$ of a (weak) quasi–Hopf algebra $\mathcal{G}$. We will show that, similarly as for ordinary Hopf algebras, the quantum double $\mathcal{D}(\mathcal{G})$ is a quasitriangular weak quasi–Hopf algebra, where $\mathcal{D}(\mathcal{G})$ is weak if and only if $\mathcal{G}$ is weak, i.e. iff $\Delta(\mathbf{1}) \neq \mathbf{1}$. We will give explicit formulas for the coproduct, the antipode and the R–matrix. These results are formulated in Theorem 4.3 and Theorem 4.4.

In view of the identification of the quantum double $\mathcal{D}(\mathcal{G})$ of an ordinary Hopf algebra $\mathcal{G}$ with the diagonal crossed product $\mathcal{G} \bowtie \hat{\mathcal{G}}$ in (1.26) we propose the following

DEFINITION 4.1. Let $(\mathcal{G}, \Delta, \epsilon, \phi)$ be a weak quasi–Hopf algebra. The diagonal crossed product $\mathcal{D}(\mathcal{G}) := \hat{\mathcal{G}} \bowtie {}_\Delta\mathcal{G}_\Delta \cong {}_\Delta\mathcal{G}_\Delta \bowtie \hat{\mathcal{G}}$ associated with the quasi–commuting pair ($\lambda = \rho = \Delta$, $\phi_\lambda = \phi_\rho = \phi_{\lambda\rho} = \phi$) of $\mathcal{G}$–coactions on $\mathcal{M} \equiv \mathcal{G}$ is called the *quantum double of* $\mathcal{G}$.

Following the notations of [Nil97], the universal $\lambda\rho$–intertwiner of the quantum double will be denoted by $\mathbf{D} \equiv \mathbf{\Gamma}_{\mathcal{D}(\mathcal{G})} \in \mathcal{G} \otimes \mathcal{D}(\mathcal{G})$. Hence it obeys the relations $(\epsilon \otimes \mathrm{id})(\mathbf{D}) = \mathbf{1}_{\mathcal{D}(\mathcal{G})}$ and

$$\mathbf{D}\,\Delta(\mathbf{1}) = \Delta^{op}(\mathbf{1})\,\mathbf{D} = \mathbf{D} \tag{4.1}$$

$$\mathbf{D}\,\Delta(a) = \Delta^{op}(a)\,\mathbf{D}, \quad \forall a \in \mathcal{G} \tag{4.2}$$

$$\phi^{312}\,\mathbf{D}^{13}\,(\phi^{-1})^{132}\,\mathbf{D}^{23}\,\phi = (\Delta \otimes \mathrm{id})(\mathbf{D}) \tag{4.3}$$

where we have suppressed the embedding $\mathcal{G} \hookrightarrow \mathcal{D}(\mathcal{G})$. Property (4.2) motivates to call $\mathbf{D}$ the *universal flip operator* for Δ. Clearly, the relation (4.1) may be ommitted if $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$. Note that according to Theorem 2.1, the quantum double $\mathcal{D}(\mathcal{G})$ may be realized as an algebraic structure on the vector space $\hat{\mathcal{G}} \otimes \mathcal{G}$ (or, in the weak case, a certain subspace thereof, see Theorem 3.1).

We remark that a definition of a quantum double $\mathcal{D}(\mathcal{G})$ for quasi–Hopf algebras $\mathcal{G}$ has also recently been proposed by S. Majid [Maj97] using a Tannaka–Krein type reconstruction procedure [Maj92]. Unfortunately it is hard to identify this algebra in terms of generators and relations in concrete models. It will be shown in Appendix A that our construction in fact provides a concrete realization of the abstract definition of [Maj97].

The first section of this chapter is devoted to the proof that $\mathcal{D}(\mathcal{G})$ is a (weak) quasi–bialgebra. Analogously as in Proposition 1.8 this will also guarantee that every diagonal crossed product $\mathcal{M}_1 = {}_\lambda\mathcal{M}_\rho \bowtie \hat{\mathcal{G}}$ naturally admits a quasi–commuting pair $(\lambda_D, \rho_D, \phi_{\lambda_D}, \phi_{\rho_D}, \phi_{\lambda_D\rho_D})$ of coactions of $\mathcal{D}(\mathcal{G})$ on $\mathcal{M}_1$. The last observation will be of great importance, since it implies that the quantum chains constructed as iterated diagonal crossed products in Chapter 5 admit localized $\mathcal{D}(\mathcal{G})$–coactions.

In Section 4.2 we show that $\mathcal{D}(\mathcal{G})$ possess an antipode and a quasitriangular R–matrix. Hence $\mathcal{D}(\mathcal{G})$ becomes a (weak) quasitriangular quasi–Hopf algebra, generalizing the well–known results for ordinary Hopf algebras to the weak quasi–Hopf setting. We will see that the proof of the antipode properties is fairly nontrivial. A part of this proof is postponed to Chapter B, where we use graphical methods.

As an application we discuss in Section 4.3 the twisted double $\mathcal{D}^\omega(G)$ of [DPR90] and generalize the results of [Nil97] on the relation with the monodromy algebras of [AGS95, AGS96, AS96] in Section 4.4. The results of Section 4.4 will also become important in Chapter 5, when we discuss current algebras on the lattice.

4.1. $\mathcal{D}(\mathcal{G})$ as a quasi–bialgebra and $\mathcal{D}(\mathcal{G})$–coactions

We begin with constructing $\lambda_D : \mathcal{M}_1 \to \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1$ and $\rho_D : \mathcal{M}_1 \to \mathcal{M}_1 \otimes \mathcal{D}(\mathcal{G})$ as algebra maps extending the left and right coactions $\lambda : \mathcal{M}_1 \supset \mathcal{M} \to \mathcal{G} \otimes \mathcal{M} \subset \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1$ and

$\rho : \mathcal{M}_1 \supset \mathcal{M} \to \mathcal{M} \otimes \mathcal{G} \subset \mathcal{M}_1 \otimes \mathcal{D}(\mathcal{G})$, respectively (see Proposition 1.8). The detailed proof of the next Lemma will also give some flavour of the calculations with generating marices. (Try to give a proof without using generating matrices !).

LEMMA 4.2. *Let $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ be a quasi-commuting pair of $\mathcal{G}$–coactions on $\mathcal{M}$ and let $\mathcal{M}_1 \equiv {}_\lambda\mathcal{M}_\rho \bowtie \hat{\mathcal{G}}$ be the associated diagonal crossed product with universal $\lambda\rho$–intertwiner $\mathbf{\Gamma} \in \mathcal{G} \otimes \mathcal{M}_1$. Then there exist uniquely determined algebra maps $\lambda_D : \mathcal{M}_1 \to \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1$ and $\rho_D : \mathcal{M}_1 \to \mathcal{M}_1 \otimes \mathcal{D}(\mathcal{G})$ satisfying (we suppress all embeddings $\mathcal{M} \hookrightarrow \mathcal{M}_1$ and $\mathcal{G} \hookrightarrow \mathcal{D}(\mathcal{G})$)*

$$\lambda_D(m) = \lambda(m), \quad \forall m \in \mathcal{M} \subset \mathcal{M}_1 \tag{4.4}$$

$$(\mathrm{id} \otimes \lambda_D)(\mathbf{\Gamma}) = (\phi_{\lambda\rho}^{-1})^{231}\, \mathbf{\Gamma}^{13}\, \phi_\lambda^{213}\, \mathbf{D}^{12}\, \phi_\lambda^{-1} \;\in \mathcal{G} \otimes \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1 \tag{4.5}$$

$$\rho_D(m) = \rho(m), \quad \forall m \in \mathcal{M} \subset \mathcal{M}_1 \tag{4.6}$$

$$(\mathrm{id} \otimes \rho_D)(\mathbf{\Gamma}) = (\phi_\rho^{-1})^{231}\, \mathbf{D}^{13}\, \phi_\rho^{213}\, \mathbf{\Gamma}^{12}\, \phi_{\lambda\rho}^{-1} \;\in \mathcal{G} \otimes \mathcal{M}_1 \otimes \mathcal{D}(\mathcal{G}) \tag{4.7}$$

Moreover the algebra maps λ_D, ρ_D are unital if $\mathcal{G}$ is not weak, i.e. if $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$.

Note that for the case that $\mathcal{G}$ is an ordinary Hopf algebra and all reassociators are trivial, we recover the definition of λ_D, ρ_D given in Proposition 1.8.

PROOF. Let us first suppose that $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$. Viewing the left $\mathcal{G}$–coaction $\lambda : \mathcal{M} \longrightarrow \mathcal{G} \otimes \mathcal{M}$ as a map $\lambda : \mathcal{M} \longrightarrow \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1$, Theorem 2.1 states that λ_D is a unital algebra map extending λ if and only if $\mathbf{T}_D := (\mathrm{id} \otimes \lambda_D)(\mathbf{\Gamma}) \in \mathcal{G} \otimes (\mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1)$ is a normal coherent $\lambda\rho$–intertwiner. Now normality of $\mathbf{T}_D$ follows from the normality of $\mathbf{\Gamma}$. To prove that $\mathbf{T}_D$ is a $\lambda\rho$–intertwiner we compute for all $m \in \mathcal{M}$

$$\begin{aligned}
\mathbf{T}_D\,(\mathrm{id}_\mathcal{G} \otimes \lambda_D)(\lambda(m)) &= (\phi_{\lambda\rho}^{-1})^{231}\, \mathbf{\Gamma}^{13}\, \phi_\lambda^{213}\, \mathbf{D}^{12}\, \phi_\lambda^{-1}\, (\mathrm{id}_\mathcal{G} \otimes \lambda_D)(\lambda(m)) \\
&= \big[(\lambda_D \otimes \mathrm{id}_\mathcal{G})(\rho(m))\big]^{231}\, (\phi_{\lambda\rho}^{-1})^{231}\, \mathbf{\Gamma}^{13}\, \phi_\lambda^{213}\, \mathbf{D}^{12}\, \phi_\lambda^{-1} \\
&= (\mathrm{id}_\mathcal{G} \otimes \lambda_D)(\rho^{op}(m))\, \mathbf{T}_D
\end{aligned}$$

where both sides are viewed as elements in $\mathcal{G} \otimes \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1$. Here we have used the intertwining properties of $\mathbf{\Gamma}$ and $\mathbf{D}$ and of the three reassociators.

To show that $\mathbf{T}_D$ also satisfies the coherence condition, i.e. Eq. (2.3), we compute in $\mathcal{G} \otimes \mathcal{G} \otimes \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1$ - again suppressing all embeddings

$$\begin{aligned}
(\Delta \otimes \mathrm{id})(\mathbf{T}_D) &= \big[(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\phi_{\lambda\rho}^{-1})\, [\mathbf{1}_\mathcal{G} \otimes \phi_\rho]\big]^{3412} \\
&\quad \mathbf{\Gamma}^{14}\, (\phi_{\lambda\rho}^{-1})^{142}\, \mathbf{\Gamma}^{24} \big[[\mathbf{1}_\mathcal{G} \otimes \phi_\lambda]\, (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi_\lambda)[\phi \otimes \mathbf{1}_\mathcal{M}]\big]^{3124} \mathbf{D}^{13}\, (\phi^{-1})^{132}\, \mathbf{D}^{23} \\
&\quad [\phi \otimes \mathbf{1}_\mathcal{M}](\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\lambda^{-1}) \\
&= \big[(\lambda \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\rho)[\phi_{\lambda\rho}^{-1} \otimes \mathbf{1}_\mathcal{G}](\mathrm{id} \otimes \rho \otimes \mathrm{id})(\phi_{\lambda\rho}^{-1})\big]^{3412} \\
&\quad \mathbf{\Gamma}^{14}\, (\phi_{\lambda\rho}^{-1})^{142}\, \mathbf{\Gamma}^{24} \big[(\mathrm{id} \otimes \mathrm{id} \otimes \lambda)(\phi_\lambda)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\lambda)\big]^{3124} \mathbf{D}^{13}\, (\phi^{-1})^{132}\, \mathbf{D}^{23} \\
&\quad (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi_\lambda^{-1})[\mathbf{1}_\mathcal{G} \otimes \phi_\lambda^{-1}](\mathrm{id} \otimes \mathrm{id} \otimes \lambda)(\phi_\lambda) \\
&= \big[(\lambda \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\rho)[\phi_{\lambda\rho}^{-1} \otimes \mathbf{1}_\mathcal{G}]\big]^{3412}\, \mathbf{\Gamma}^{14} \\
&\quad \big[(\mathrm{id} \otimes \lambda \otimes \mathrm{id})(\phi_{\lambda\rho}^{-1})\, [\mathbf{1}_\mathcal{G} \otimes \phi_{\lambda\rho}^{-1}]\, (\mathrm{id} \otimes \mathrm{id} \otimes \rho)(\phi_\lambda)\big]^{3142}\, \mathbf{\Gamma}^{24}\, \mathbf{D}^{13} \\
&\quad \big[(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\lambda)[\phi^{-1} \otimes \mathbf{1}_\mathcal{M}]\, (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi_\lambda^{-1})\big]^{1324} \\
&\quad \mathbf{D}^{23}\, [\mathbf{1}_\mathcal{G} \otimes \phi_\lambda^{-1}](\mathrm{id} \otimes \mathrm{id} \otimes \lambda)(\phi_\lambda) \\
&= \big[(\lambda \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\rho)[\phi_{\lambda\rho}^{-1} \otimes \mathbf{1}_\mathcal{G}]\big]^{3412}\, \mathbf{\Gamma}^{14}\, \phi_\lambda^{314}\, \mathbf{D}^{13} \\
&\quad \big[(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi_{\lambda\rho}^{-1})\, (\mathrm{id} \otimes \mathrm{id} \otimes \rho)(\phi_\lambda^{-1})\big]^{1324}\, \mathbf{\Gamma}^{24} \\
&\quad \phi_\lambda^{324}\, \mathbf{D}^{23}\, [\mathbf{1}_\mathcal{G} \otimes \phi_\lambda^{-1}](\mathrm{id} \otimes \mathrm{id} \otimes \lambda)(\phi_\lambda) \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes \lambda_D)\Big(\phi_\rho^{312}\, \mathbf{\Gamma}^{13}\, (\phi_{\lambda\rho}^{-1})^{132}\, \mathbf{\Gamma}^{23}\, \phi_\lambda\Big)
\end{aligned}$$

Here we have used several pentagon identities for the reassociators involved and the intertwining and coherence properties of $\mathbf{\Gamma}$ and $\mathbf{D}$. In the first equality we used (2.3) for $\mathbf{\Gamma}$ and $\mathbf{D}$, and in the second the pentagons (2.51c) and (2.35b). For the third equality we used the intertwining properties of $\mathbf{D}$ and $\mathbf{\Gamma}$ to move two more reassociators between $\mathbf{D}^{13}$ and $\mathbf{D}^{23}$ and two more between $\mathbf{\Gamma}^{14}$ and $\mathbf{\Gamma}^{24}$. To arrive at the fourth equality we commuted $\mathbf{D}^{13}$ and $\mathbf{\Gamma}^{24}$ and used the pentagons (2.51b) and (2.35b) and then again the intertwining properties of $\mathbf{D}$ and $\mathbf{\Gamma}$ to bring two reassociators back between $\mathbf{D}^{13}$ and $\mathbf{\Gamma}^{24}$. The last equality holds by (4.4), (4.5). Thus we have shown that $\mathbf{T}_D$ is coherent and therefore the definitions (4.4), (4.5) uniquely define a unital algebra map λ_D extending λ. Similarly one shows that ρ_D defines a unital algebra map $\rho_D : \mathcal{M}_1 \longrightarrow \mathcal{M}_1 \otimes \mathcal{D}(\mathcal{G})$ extending ρ.

Now let $\Delta(\mathbf{1}) \neq \mathbf{1} \otimes \mathbf{1}$, then eventually λ is non unital implying that also λ_D may be nonunital. That λ_D is an algebra map is proved as above. □

Choosing in Lemma 4.2 also $\mathcal{M} = \mathcal{G}$ (i.e. $\mathcal{M}_1 = \mathcal{D}(\mathcal{G})$) we arrive at the following

THEOREM 4.3. *Let $(\mathcal{G}, \Delta, \epsilon, \phi)$ be a weak quasi–Hopf algebra, denote $i_D : \mathcal{G} \hookrightarrow \mathcal{D}(\mathcal{G})$ the canonical embedding and let $\mathbf{D} \in \mathcal{G} \otimes \mathcal{D}(\mathcal{G})$ be the universal flip operator.*

(i) *Then $(\mathcal{D}(\mathcal{G}), \Delta_D, \epsilon_D, \phi_D)$ is a weak quasi–bialgebra, where*

$$\phi_D := (i_D \otimes i_D \otimes i_D)(\phi) \tag{4.8}$$

$$\epsilon_D(i_D(a)) := \epsilon(a), \quad (\mathrm{id} \otimes \epsilon_D)(\mathbf{D}) := \mathbf{1}_{\mathcal{D}(\mathcal{G})} \tag{4.9}$$

$$\Delta_D(i_D(a)) := (i_D \otimes i_D)(\Delta(a)), \quad \forall a \in \mathcal{G} \tag{4.10}$$

$$(i_D \otimes \Delta_D)(\mathbf{D}) := (\phi_D^{-1})^{231}\, \mathbf{D}^{13}\, \phi_D^{213}\, \mathbf{D}^{12}\, \phi_D^{-1} \tag{4.11}$$

Moreover $\mathcal{D}(\mathcal{G})$ is weak if and only if $\mathcal{G}$ is weak.

(ii) *Under the setting of Lemma 4.2 denote $i_{\mathcal{M}_1} : \mathcal{M} \hookrightarrow \mathcal{M}_1$ the embedding and define*

$$\phi_{\lambda_D} := (i_D \otimes i_D \otimes i_{\mathcal{M}_1})(\phi_\lambda) \quad \in \mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1$$
$$\phi_{\rho_D} := (i_{\mathcal{M}_1} \otimes i_D \otimes i_D)(\phi_\rho) \quad \in \mathcal{M}_1 \otimes \mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{G})$$
$$\phi_{\lambda_D \rho_D} := (i_D \otimes i_{\mathcal{M}_1} \otimes i_D)(\phi_{\lambda\rho}) \quad \in \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1 \otimes \mathcal{D}(\mathcal{G})$$

Then $(\lambda_D, \rho_D, \phi_{\lambda_D}, \phi_{\rho_D}, \phi_{\lambda_D \rho_D})$ provides a quasi–commuting pair of $\mathcal{D}(\mathcal{G})$–coactions on $\mathcal{M}_1 \equiv {}_\lambda\mathcal{M}_\rho \bowtie \hat{\mathcal{G}}$.

PROOF. Setting $\mathcal{M} := \mathcal{G}$ and $\lambda = \Delta$ in Lemma 4.2 implies that Δ_D is an algebra morphism, which is unital if and only if Δ is unital. The property of ϵ_D being a counit for Δ_D follows directly from (4.1) and the fact that $(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\phi) = \Delta(\mathbf{1})$. To show that Δ_D is quasi–coassociative one computes that

$$[\mathbf{1}_\mathcal{G} \otimes \phi_D] \cdot (\mathrm{id} \otimes \Delta_D \otimes \mathrm{id})\big((\mathrm{id} \otimes \Delta_D)(\mathbf{D})\big) = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta_D)\big((\mathrm{id} \otimes \Delta_D)(\mathbf{D})\big) \cdot [\mathbf{1}_\mathcal{G} \otimes \phi_D],$$

where one has to use (4.11), the pentagon equation for ϕ and the intertwiner property (4.2) of $\mathbf{D}$ similarly as in the proof of Lemma 4.2. Thus Δ_D is quasi–coassociative and this concludes the proof of part (i).

Part (ii) is shown by direct calculation using the intertwiner properties of $\mathbf{\Gamma}$ and $\mathbf{D}$ and several pentagon identities for the reassociators involved. The details are left to the reader. □

4.2. The quasitriangular quasi–Hopf structure

Note that viewed in $\mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{G})$ and $\mathcal{D}(\mathcal{G})^{\otimes^3}$, respectively, the relations (4.3) and (4.11) are the defining properties of a quasitriangular R–matrix, see (2.18)-(2.19). Hence $R_D := (i_D \otimes \mathrm{id})(\mathbf{D})$ is an R–matrix for $\mathcal{D}(\mathcal{G})$ provided we can also show the intertwiner property (2.17).

To arrive at a suitable definition of an antipode S_D for $\mathcal{D}(\mathcal{G})$ extending the antipode of $\mathcal{G}$, we anticipate the result of Chapter B, Cor. B.3, that a quasitriangular R–matrix of a quasi–Hopf algebra obeys $(S \otimes S)(R) = f^{op}\, R\, f^{-1}$, where f is the twist defined in (2.26). This suggests the following

THEOREM 4.4. *Let $\mathcal{D}(\mathcal{G})$ be the (weak) quasi–bialgebra defined in Theorem 4.3. Then $(\mathcal{D}(\mathcal{G}), \Delta_D, \epsilon_D, \phi_D, S_D, \alpha_D, \beta_D, R_D)$ is a quasitriangular (weak) quasi-Hopf algebra with R-Matrix R_D and antipode S_D given by*

$$R_D := (i_D \otimes \mathrm{id})(\mathbf{D}) \tag{4.12}$$

$$S_D(i_D(a)) := i_D(S(a)), \quad \forall a \in \mathcal{G} \tag{4.13}$$

$$(S \otimes S_D)(\mathbf{D}) := (\mathrm{id} \otimes i_D)(f^{op})\, \mathbf{D} (\mathrm{id} \otimes i_D)\ (f^{-1}) \tag{4.14}$$

where $f \in \mathcal{G} \otimes \mathcal{G}$ *is the twist defined in* (2.26)*. The elements* α_D, β_D *are given by*

$$\alpha_D := i_D(\alpha), \quad \beta_D := i_D(\beta). \tag{4.15}$$

Clearly, if $\mathcal{G}$ is a Hopf algebra and $\phi = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$, one recovers the well-known definitions of Δ_D, S_D and R_D in Drinfel'd's quantum double

$$\begin{aligned} \Delta_D(i_D(g)) &= (i_D \otimes i_D)(\Delta(g)) \\ \Delta_D(D(\varphi)) &= (D \otimes D)(\hat{\Delta}^{op}(\varphi)) \\ S_D(i_D(g)) &= i_D(S(g)) \\ S_D(D(\varphi)) &= D(S^{-1}(\varphi)) \\ R_D &= (\hat{\mathbf{1}} \otimes e_\mu) \otimes (e^\mu \otimes \mathbf{1}), \end{aligned}$$

where $D(\varphi) := (\varphi \otimes \mathrm{id})(\mathbf{D})$, $\varphi \in \hat{\mathcal{G}}$.

PROOF. To simplify the notation we will frequently suppress the embedding i_D, if no confusion is possible, i.e. we write $\alpha \equiv i_D(\alpha) = \alpha_D$, $(\mathrm{id} \otimes \mathrm{id} \otimes i_D)(\phi) \equiv \phi$ etc. . To show quasitriangularity we first note that the element $R_D = (i_D \otimes \mathrm{id})(\mathbf{D})$ fulfills (2.18) and (2.19) so to say by definition because of (4.3) and (4.11). The (quasi-) invertibility of R_D is equivalent to the (quasi-) invertibility of the generating matrix $\mathbf{D}$ which will be proved below in Lemma B.8 of Chapter B using graphical methods. We are left to show that R_D intertwines Δ_D and Δ_D^{op}, i.e.

$$\Delta_D^{op}(i_D(a)) \cdot R_D = R_D \cdot \Delta_D(i_D(a)), \quad \forall a \in \mathcal{G} \tag{4.16}$$

$$(\mathrm{id} \otimes \Delta_{op}^D)(\mathbf{D})) \cdot R_D^{23} = R_D^{23} \cdot (\mathrm{id} \otimes \Delta_D)(\mathbf{D}). \tag{4.17}$$

Now Eq. (4.16) follows from (4.10). Hence we also get in $\mathcal{D}(\mathcal{G})^{\otimes^3}$

$$R_D^{12} \cdot (\Delta_D \otimes \mathrm{id})(R_D) = (\Delta_D^{op} \otimes \mathrm{id})(R_D) \cdot R_D^{12}, \tag{4.18}$$

which together with (2.18) implies the quasi-Yang Baxter equation

$$(\phi_D^{-1})^{321}\, R_D^{12}\, \phi_D^{312}\, R_D^{13}\, (\phi_D^{-1})^{132}\, R_D^{23} = R_D^{23}\, (\phi_D^{-1})^{231}\, R_D^{13}\, \phi_D^{213}\, R_D^{12}\, \phi_D^{-1}. \tag{4.19}$$

Using (4.11), Eq. (4.19) is further equivalent to

$$(i_D \otimes \Delta_D^{op})(\mathbf{D}) \cdot R_D^{23} = R_D^{23} \cdot (i_D \otimes \Delta_D)(\mathbf{D})$$

which also proves (4.17). Hence R_D is quasitriangular.

In order to prove that the definition of S_D in (4.13),(4.14) may be extended anti-multiplicatively to the entire algebra $\mathcal{D}(\mathcal{G})$, we have to show that this continuation is consistent with the defining relations (4.2),(4.3). This amounts to showing

$$(S \otimes S_D)(\mathbf{D}) \cdot (S \otimes S_D)(\Delta^{op}(a)) = (S \otimes S_D)(\Delta(a)) \cdot (S \otimes S_D)(\mathbf{D}), \quad \text{and} \tag{4.20}$$

$$\begin{aligned} (S \otimes S \otimes S_D)\big((\Delta \otimes \mathrm{id})(\mathbf{D})\big) &= (S \otimes S \otimes S_D)(\phi) \cdot (S \otimes S \otimes S_D)(\mathbf{D}^{23}) \\ &\cdot (S \otimes S \otimes S_D)((\phi^{-1})^{132}) \cdot (S \otimes S \otimes S_D)(\mathbf{D}^{13}) \cdot (S \otimes S \otimes S_D)(\phi^{312}). \end{aligned} \tag{4.21}$$

Since by definition $(S \otimes S_D)(\mathbf{D}) = f^{op} \mathbf{D} f^{-1}$ [1], equation (4.20) follows directly from (4.2) and the fact, that by (2.28) f has the property $f \cdot \Delta(S(a)) = (S \otimes S)(\Delta_{op}(a)) \cdot f$. For the proof of (4.21) let us recall, that $\Delta_f := f \Delta(\cdot) f^{-1}$ defines a twist equivalent quasi-coassociative coproduct on $\mathcal{G}$ with twisted reassociator ϕ_f defined in (2.23) satisfying $\phi_f = (S \otimes S \otimes S)(\phi^{321})$ (see (2.30)). Thus we get for the l.h.s. of (4.21) (with $\mathbf{D}_f := f^{op} \mathbf{D} f^{-1}$)

$$\begin{aligned} (S \otimes S \otimes S_D)\big((\Delta \otimes \mathrm{id})(\mathbf{D})\big) &= (\Delta_f^{op} \otimes \mathrm{id})\big((S \otimes S_D)(\mathbf{D})\big) \\ &= (\Delta_f^{op} \otimes \mathrm{id})(\mathbf{D}_f) \\ &= \phi_f^{321}\, \mathbf{D}_f^{23}\, (\phi_f^{-1})^{231} \mathbf{D}_f^{13} \phi_f^{213}, \end{aligned}$$

where the last equality is exactly the transformation property of a quasitriangular R-matrix under a twist [Dri90] and may be proven analogously using (4.2). By (2.30) this equals the r.h.s. of (4.21). Hence S_D defines an anti-algebra morphism on $\mathcal{D}(\mathcal{G})$.

[1] where we have again suppressed the embedding $\mathrm{id} \otimes i_D$ of f

We are left to show that the map S_D fulfills the antipode axioms given in (2.15) and (2.16). Axiom (2.16) is clearly fulfilled since we have $S_D \circ i_D = i_D \circ S$ and $\alpha_D = i_D(\alpha)$, $\beta_D = i_D(\beta)$, $\phi_D = (i_D \otimes i_D \otimes i_D)(\phi)$. Noting that $\Delta_D(i_D(a)) = (i_D \otimes i_D)(\Delta(a))\,, a \in \mathcal{G}$, the validity of axiom (2.15) follows from its validity in $\mathcal{G}$ and the following two identities, which will be proven in Section B.3, Lemma B.8.

$$(\mathrm{id} \otimes \mu_D) \circ (\mathrm{id} \otimes S_D \otimes \mathrm{id})\Big((\mathrm{id} \otimes \Delta_D)(\mathbf{D}) \cdot (\mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G} \otimes \alpha_D)\Big) = \mathbf{1}_\mathcal{G} \otimes \alpha_D$$

$$(\mathrm{id} \otimes \mu_D) \circ (\mathrm{id} \otimes \mathrm{id} \otimes S_D)\Big((\mathrm{id} \otimes \Delta_D)(\mathbf{D}) \cdot (\mathbf{1}_\mathcal{G} \otimes \beta_D \otimes \mathbf{1}_\mathcal{G})\Big) = \mathbf{1}_\mathcal{G} \otimes \beta_D,$$

with $\mu_D : \mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{G}) \to \mathcal{D}(\mathcal{G})$ denoting the multiplication map. □

As in the Hopf algebra case, one may take the construction of the quasitriangular R-Matrix in $\mathcal{D}(\mathcal{G})$ as the starting point and formulate Theorem 4.3(i) together with Theorem 4.4 differently:

COROLLARY 4.5. *Let $\mathcal{G}$ be a finite dimensional quasi-Hopf algebra with invertible antipode. Then there exists a unique quasi-Hopf algebra $\mathcal{D}(\mathcal{G})$ such that*

(i) $\mathcal{D}(\mathcal{G}) = \hat{\mathcal{G}} \otimes \mathcal{G}$ *as a vector space*
(ii) *the canonical embedding $i_D : \mathcal{G} \hookrightarrow \hat{\mathbf{1}} \otimes \mathcal{G} \subset \mathcal{D}(\mathcal{G})$ is a unital injective homomorphism of quasi-Hopf algebras,*
(iii) *Let $\mathbf{D} \in \mathcal{G} \otimes \mathcal{D}(\mathcal{G})$ be given by $\mathbf{D} := S^{-1}(p^2)\, e_\mu\, p^1_{(1)} \otimes \big(e^\mu \otimes p^1_{(2)}\big)$, where $p := p_{\rho=\Delta}$ is defined in (2.79), then $R_D := (i_D \otimes \mathrm{id})(\mathbf{D}) \in \mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{G})$ is quasitriangular.*

This quasi–Hopf algebra structure is given by (4.1)-(4.3) *and the definitions in Theorem 4.3 and 4.4.*

PROOF. Property (ii) implies (4.8), the first part of (4.9), (4.10) and (4.15), yielding also $f_D = (i_D \otimes i_D)(f)$. The quasitriangularity of R_D implies (4.2), (4.3), (4.11) and the second part of (4.9) and according to (B.16) $(S_D \otimes S_D)(R_D) = f_D^{op}\, R_D f_D^{-1}$. Hence the antipode is uniquely fixed to be the one defined in Theorem 4.4. □

We remark that the above Corollary is also valid for weak quasi–Hopf algebras where only part (i) has to be modified according to

(i') As a vectorspace $\mathcal{D}(\mathcal{G}) = \mathrm{lin}\{\mathbf{1}_{(-1)} \rightharpoonup \varphi \leftharpoonup S^{-1}(\mathbf{1}_{(1)}) \otimes \mathbf{1}_{(0)}\, a \mid \varphi \in \hat{\mathcal{G}}, a \in \mathcal{G}\}$, where $\mathbf{1}_{(-1)} \otimes \mathbf{1}_{(0)} \otimes \mathbf{1}_{(1)} = (\Delta \otimes \mathrm{id})(\Delta(\mathbf{1}))$.

4.3. The twisted double of a finite group

As an application we now use our definition of the quantum double to recover the "twisted" quantum double $\mathcal{D}^\omega(G)$ of [DPR90], where G is a finite group and $\omega : G \times G \times G \to U(1)$ is a normalized 3-cocycle. By definition this means $\omega(g, h, k) = 1$ whenever at least one of the three arguments is equal to the unit e of G and

$$\omega(g, x, y)\omega(gx, y, z)^{-1}\omega(g, xy, z)\omega(g, x, yz)^{-1}\omega(x, y, z) = 1, \quad \forall g, x, y, z \in \mathrm{G}.$$

The Hopf algebra $\mathcal{G} := Fun(G)$ of functions on G may then also be viewed as a quasi-Hopf algebra with its standard coproduct, counit and antipode but with reassociator given by

$$\phi := \sum_{g,h,k \in G} \omega(g, h, k) \cdot (\delta_g \otimes \delta_h \otimes \delta_k), \tag{4.22}$$

where $\delta_g(x) := \delta_{g,x}$. The identities (2.9) and (2.11) for ϕ are equivalent to ω being a normalized 3-cocycle. Also note that choosing $\alpha = \mathbf{1}_\mathcal{G}$ the antipode axioms now require $\beta = \sum_g \omega(g^{-1}, g, g^{-1})\delta_g$. In this special example our quantum double $\mathcal{D}(\mathcal{G}) \equiv \hat{\mathcal{G}} \bowtie \mathcal{G}$ allows for another identification with the linear space $\hat{\mathcal{G}} \otimes \mathcal{G}$.

LEMMA 4.6. *Let $\mathcal{G}$ be as above and define $\sigma : \hat{\mathcal{G}} \otimes \mathcal{G} \longrightarrow \mathcal{D}(\mathcal{G})$ by $\sigma(\varphi \otimes a) := D(\varphi)\, a$, $\varphi \in \hat{\mathcal{G}}, a \in \mathcal{G}$. Then σ is a linear bijection.*

PROOF. Since $(\mathcal{G},\Delta,\epsilon,S)$ is also an ordinary Hopf algebra, the relation (4.2) is equivalent to (suppressing the symbol i_D)

$$a\,D(\varphi)=D\big(a_{(1)}\rightharpoonup\varphi\leftharpoonup S^{-1}(a_{(3)})\big)\,a_{(2)},\quad \forall a\in\mathcal{G},\varphi\in\hat{\mathcal{G}}. \tag{4.23}$$

Using (2.4) (for the special case $\mathbf{\Gamma}=\mathbf{D}$) this implies

$$\varphi\bowtie a\equiv(\mathrm{id}\otimes\varphi_{(1)})(q_\rho)\,D(\varphi_{(2)})\,a=D\big(q^1_{\rho(1)}\rightharpoonup\varphi\leftharpoonup(q^2_\rho\,S^{-1}(q^1_{\rho(3)}))\big)\,q^1_{\rho(2)}\,a,$$

which lies in the image of σ. Hence, σ is surjective and therefore also injective. □

We note that in general the map σ need not be surjective (nor injective). Due to Lemma 4.6 we may now identify $\mathcal{D}(\mathcal{G})$ with the new algebraic structure on $\hat{\mathcal{G}}\otimes\mathcal{G}$ induced by σ^{-1}. We call this algebra $\hat{\mathcal{G}}\otimes_D\mathcal{G}$. Putting $a\equiv\hat{\mathbf{1}}\otimes_D a$, $a\in\mathcal{G}$ and $\mathbf{D}:=e_\mu\otimes(e^\mu\otimes_D\mathbf{1})\in\mathcal{G}\otimes(\hat{\mathcal{G}}\otimes_D\mathcal{G})$ it is described by the relations (4.23),(4.3) and the requirement of $\mathcal{G}\equiv\hat{\mathbf{1}}\otimes_D\mathcal{G}$ being a unital subalgebra. To compute these multiplication rules we now use that the group elements $g\in G$ provide a basis in $\hat{\mathcal{G}}$ with dual basis $\delta_g\in\mathcal{G}$. Hence a basis of $\hat{\mathcal{G}}\otimes_D\mathcal{G}$ is given by $\{h\otimes_D\delta_g\}_{h,g\in\mathcal{G}}$. In this basis the generating matrix $\mathbf{D}$ is given by

$$\mathbf{D}=\sum_{k\in G}\delta_k\otimes(k\otimes_D\mathbf{1}_\mathcal{G}),\quad \mathbf{1}_\mathcal{G}=\sum_{h\in G}\delta_h. \tag{4.24}$$

Let us know compute the multiplication laws according to the (4.2),(4.3). To begin with, we have

$$(h\otimes\mathbf{1}_\mathcal{G})(e\otimes\delta_g)=(h\otimes\delta_g)\quad\text{and}\quad(g\otimes\mathbf{1}_\mathcal{G})(h\otimes\mathbf{1}_\mathcal{G})=(gh\otimes\mathbf{1}_\mathcal{G}).$$

Taking $(x\otimes\mathrm{id})$ of both sides of (4.2), where $x\in G$, and using $\Delta(\delta_g)=\sum_{k\in G}\delta_k\otimes\delta_{k^{-1}g}$ we get

$$(x\otimes\mathbf{1}_\mathcal{G})(e\otimes\delta_{x^{-1}g})=(e\otimes\delta_{gx^{-1}})(x\otimes\mathbf{1}_\mathcal{G}),$$

or equivalently

$$(e\otimes\delta_g)(x\otimes\mathbf{1}_\mathcal{G})=(x\otimes\delta_{x^{-1}gx}). \tag{4.25}$$

Finally, pairing equation (4.3) with $x\otimes y\in\hat{\mathcal{G}}\otimes\hat{\mathcal{G}}$ in the two auxiliary spaces, the l.h.s. yields

$$\begin{aligned}&\sum_{s,r,t\in G}\omega(s,x,y)(\mathbf{1}\otimes\delta_s)(x\otimes\mathbf{1}_\mathcal{G})\cdot\omega(x,r,y)^{-1}(e\otimes\delta_r)(y\otimes\mathbf{1}_\mathcal{G})\cdot\omega(x,y,t)(\mathbf{1}\otimes\delta_t)\\ &\qquad=(x\otimes\mathbf{1}_\mathcal{G})(y\otimes\mathbf{1}_\mathcal{G})\cdot\sum_{s,r,t\in G}\frac{\omega(s,x,y)\omega(x,y,t)}{\omega(x,r,y)}(e\otimes\delta_{(xy)^{-1}sxy}\delta_{y^{-1}ry}\delta_t)\\ &\qquad\qquad=\sum_{t\in G}(x\otimes\mathbf{1}_\mathcal{G})(y\otimes\mathbf{1}_\mathcal{G})(\mathbf{1}\otimes\delta_t)\,\frac{\omega(xyt(xy)^{-1},x,y)\omega(x,y,t)}{\omega(x,yty^{-1},y)},\end{aligned}$$

where we have used (4.25) in the first equality. The right hand side of (4.3) gives $(xy\otimes\mathbf{1}_\mathcal{G})$ so that we end up with

$$(x\otimes\mathbf{1}_\mathcal{G})(y\otimes\mathbf{1}_\mathcal{G})=\sum_{t\in G}\frac{\omega(x,yty^{-1},y)}{\omega(xyt(xy)^{-1},x,y)\omega(x,y,t)}(xy\otimes\delta_t). \tag{4.26}$$

Similarly the coproduct is computed as $\Delta_D(e\otimes\delta_g)=\sum_{k\in G}(e\otimes\delta_k)\otimes(e\otimes\delta_{k^{-1}g})$ and

$$\Delta_D(x\otimes\mathbf{1}_\mathcal{G})=\sum_{r,s\in G}\frac{\omega(xrx^{-1},x,s)}{\omega(x,r,s)\omega(xrx^{-1},xsx^{-1},x)}\big((x\otimes\delta_r)\otimes(x\otimes\delta_s)\big). \tag{4.27}$$

The above construction agrees with the definition of $\mathcal{D}^\omega(G)$ given in [DPR90] up to the convention, that they have build $\mathcal{D}(\mathcal{G})$ on $\mathcal{G}\otimes\hat{\mathcal{G}}$ instead of $\hat{\mathcal{G}}\otimes\mathcal{G}$.

4.4. The monodromy algebra

Having defined the quantum double of a (weak) quasi–Hopf algebra, the definition of monodromy algebras (see e.g. [AFFS98]) associated with quasitriangular Hopf algebras may now easily be generalized to the case of quasi-Hopf algebras. These algebras have already appeared in [AGS95, AGS96]. We will give an explicit proof that the defining relations of [AGS95, AGS96] indeed define an associative algebra structure on $\hat{\mathcal{G}} \otimes \mathcal{G}$ (or, in the weak case, a certain subspace thereof), which in fact is isomorphic to our quantum double $\mathcal{D}(\mathcal{G})$. For ordinary Hopf algebras this has recently been shown in [Nil97], see Section 1.4.3.

Let $\mathcal{G}$ be a finite dimensional quasi-Hopf algebra with quasitriangular R-matrix $R \in \mathcal{G} \otimes \mathcal{G}$. Following [Nil97] we define the monodromy matrix $\mathbf{M} \in \mathcal{G} \otimes \mathcal{D}(\mathcal{G})$ to be

$$\mathbf{M} := (\mathrm{id} \otimes i_D)(R^{op})\, \mathbf{D}.$$

Defining also $\hat{R} \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{D}(\mathcal{G})$ by

$$\hat{R} := \phi^{213}\, R^{12}\, \phi^{-1},$$

we get the following Lemma:

LEMMA 4.7. *The monodromy matrix* $\mathbf{M}$ *is normal, i.e.* $(\epsilon \otimes \mathrm{id})(\mathbf{M}) = \mathbf{1}_{\mathcal{D}(\mathcal{G})}$ *and satisfies (dropping the symbol* i_D*):*

$$\Delta(\mathbf{1})\, \mathbf{M} = \mathbf{M}\, \Delta(\mathbf{1}) \tag{4.28}$$

$$\Delta(a)\, \mathbf{M} = \mathbf{M}\, \Delta(a), \quad a \in \mathcal{G} \tag{4.29}$$

$$\mathbf{M}^{13}\, \hat{R}\, \mathbf{M}^{23} = \hat{R}\, \phi\, (\Delta \otimes \mathrm{id})(\mathbf{M})\, \phi^{-1} \tag{4.30}$$

PROOF. We will freely suppress the embedding i_D. Since the R-Matrix has the property $(\mathrm{id} \otimes \epsilon)(R) = \mathbf{1}$, normality of $\mathbf{M}$ follows from the normality of $\mathbf{D}$. The identities (4.28/4.29) are implied by (4.1/4.2) and the intertwiner property of the R–Matrix. Let us now compute the l.h.s. of (4.30):

$$\begin{aligned} \mathbf{M}^{13}\, \hat{R}\, \mathbf{M}^{23} &= R^{31}\, \mathbf{D}^{13} \phi^{213}\, R^{12}\, \phi^{-1}\, R^{32}\, \mathbf{D}^{23} \\ &= R^{31}\, \mathbf{D}^{13}\, [(\Delta \otimes \mathrm{id})(R) \phi^{-1}]^{132}\, \mathbf{D}^{23} \\ &= \big[(R \otimes \mathbf{1}) \cdot (\Delta \otimes \mathrm{id})(R)\big]^{312}\, \mathbf{D}^{13}\, (\phi^{-1})^{132}\, \mathbf{D}^{23}, \end{aligned}$$

where we have used the quasitriangularity of R in the second line and property (4.2) of $\mathbf{D}$ in the third line. The r.h.s. of (4.30) yields

$$\begin{aligned} \hat{R}\, \phi\, (\Delta \otimes \mathrm{id})(\mathbf{M})\, \phi^{-1} &= \phi^{213}\, R^{12}\, (\Delta \otimes \mathrm{id})(R^{op})\, \phi^{312}\, \mathbf{D}^{13}\, (\phi^{-1})^{132} \mathbf{D}^{23} \\ &= \big[\phi^{321}\, (\mathbf{1} \otimes R)\, (\mathrm{id} \otimes \Delta)(R)\, \phi\big]^{312}\, \mathbf{D}^{13}\, (\phi^{-1})^{132} \mathbf{D}^{23}, \end{aligned}$$

where we have used the definitions of $\mathbf{M}$ and $\hat{R}$ and (4.3). Now, the quasitriangularity of R implies

$$(R \otimes \mathbf{1})\, (\Delta \otimes \mathrm{id})(R) = \phi^{321}\, (\mathbf{1} \otimes R)\, (\mathrm{id} \otimes \Delta)(R)\, \phi$$

which finally proves (4.30). □

Note that the relations (4.28) - (4.30) are the defining relations postulated in a similiar form by [AGS95, AGS96] to describe the algebra generated by the entries of a monodromy matrix around a closed loop together with the quantum group of gauge transformations sitting at the initial (≡ end) point of the loop. Thus we define similarly as in [Nil97]

DEFINITION 4.8. The **gauged monodromy algebra** $M_R(\mathcal{G}) \supset \mathcal{G}$ is the algebra extension generated by $\mathcal{G}$ and elements $M(\varphi)$, $\varphi \in \hat{\mathcal{G}}$ with defining relations given by (4.28) - (4.30), where $M(\varphi) \equiv (\varphi \otimes \mathrm{id})(\mathbf{M})$.

Lemma 4.7 then implies the immediate

COROLLARY 4.9. *Let* $(\mathcal{G}, R)$ *be a finite dimensional quasitriangular weak quasi–Hopf algebra. Then the monodromy algebra* $M_R(\mathcal{G})$ *and the quantum double* $\mathcal{D}(\mathcal{G})$ *are equivalent extensions of* $\mathcal{G}$*, where the isomorphism is given on the generators by*

$$M(\varphi) \leftrightarrow (\varphi \otimes \mathrm{id})(R^{op}\, \mathbf{D})$$