CHAPTER 3

Generalization to weak quasi–quantum groups

In this chapter we will generalize the definitions and constructions of Chapter 2 to *weak quasi–Hopf algebras* as introduced in [MS92]. This contains the physical relevant examples such as truncated quantum groups at roots of unity. As will be shown, it is nearly straightforward to extend all results obtained so far to the case of weak quasi–Hopf algebras. The new feature of *weak* quasi–Hopf algebras is to allow the coproduct to be non unital, i.e. $\Delta(\mathbf{1}) \neq \mathbf{1} \otimes \mathbf{1}$. This results in a *truncation* of tensor products of representations, i.e. the representation $(\pi_V \otimes \pi_W) \circ \Delta$ operates only on the subspace $V \boxtimes W := (\pi_V \otimes \pi_W)(\Delta(\mathbf{1}_{\mathcal{G}}))(V \otimes W)$. Also the invertibility requirement on certain universal elements - such as the reassociator or the R–matrix - is weakened by only postulating the existence of so–called *quasi–inverses*. Correspondingly coactions and two–sided coactions of a weak quasi-Hopf algebra are no longer supposed to be unital and the associated reassociators are only required to possess quasi–inverses. The diagonal crossed products $\mathcal{M}_1 \equiv \hat{\mathcal{G}} \bowtie \mathcal{M}$ may then again be defined by the same relations as before, with the additional requirement that the universal $\lambda\rho$–intertwiner $\mathbf{\Gamma} \in \mathcal{G} \otimes \mathcal{M}_1$ has to satisfy $\mathbf{\Gamma}\, \lambda(\mathbf{1}_{\mathcal{M}}) \equiv \rho^{op}(\mathbf{1}_{\mathcal{M}})\, \mathbf{\Gamma} = \mathbf{\Gamma}$. This will also imply that now as a linear space $\mathcal{M}_1$ is only isomorphic to a certain subspace $\hat{\mathcal{G}} \bowtie {}_\lambda\mathcal{M}_\rho \subset \hat{\mathcal{G}} \otimes \mathcal{M}$ (or ${}_\lambda\mathcal{M}_\rho \bowtie \hat{\mathcal{G}} \subset \mathcal{M} \otimes \hat{\mathcal{G}}$). More specifically Theorem 2.1 now reads

THEOREM 3.1. *Let $\mathcal{G}$ be a finite dimensional weak quasi–Hopf algebra and let $(\lambda, \phi_\lambda, \rho, \phi_\rho, \phi_{\lambda\rho})$ be a quasi–commuting pair of (left and right) $\mathcal{G}$–coactions on an associative algebra $\mathcal{M}$. Then part 1. and part 2. of Theorem 2.1 stay valid with the additional requirement that the normal elements $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ satisfy not only (2.2)/(2.3) but also*

$$\mathbf{T}\, \lambda_{\mathcal{A}}(\mathbf{1}_{\mathcal{M}}) \equiv \rho^{op}_{\mathcal{A}}(\mathbf{1}_{\mathcal{M}})\, \mathbf{T} = \mathbf{T} \tag{3.1}$$

Part 3. is modified as follows

3'. *There exist elements $p_\lambda \in \mathcal{G} \otimes \mathcal{M}$ and $q_\rho \in \mathcal{M} \otimes \mathcal{G}$ such that the linear maps*

$$\mu_L : \hat{\mathcal{G}} \otimes \mathcal{M} \ni (\varphi \otimes m) \mapsto (\mathrm{id} \otimes \varphi_{(1)})(q_\rho)\, \Gamma(\varphi_{(2)})\, m \ \in \mathcal{M}_1$$
$$\mu_R : \mathcal{M} \otimes \hat{\mathcal{G}} \ni (m \otimes \varphi) \mapsto m\, \Gamma(\varphi_{(1)})\, (\varphi_{(2)} \otimes \mathrm{id})(p_\lambda) \ \in \mathcal{M}_1$$

are surjective.

3". *Let $P_L : \hat{\mathcal{G}} \otimes \mathcal{M} \to \hat{\mathcal{G}} \otimes \mathcal{M}$ and $P_R : \mathcal{M} \otimes \hat{\mathcal{G}} \to \mathcal{M} \otimes \hat{\mathcal{G}}$ be the linear maps given by*

$$P_L(\varphi \otimes m) \equiv \varphi \bowtie m := \varphi_{(2)} \otimes (\varphi_{(3)} \otimes \mathrm{id} \otimes \hat{S}^{-1}(\varphi_{(1)}))\big(\delta_l(\mathbf{1}_{\mathcal{M}})\big) m$$
$$P_R(m \otimes \varphi) \equiv m \bowtie \varphi := m(\hat{S}^{-1}(\varphi_{(3)}) \otimes \mathrm{id}_{\mathcal{M}} \otimes \varphi_{(1)})\big(\delta_r(\mathbf{1}_{\mathcal{M}})\big) \otimes \varphi_{(2)}$$

where $\delta_l = (\lambda \otimes \mathrm{id}) \circ \rho$ and $\delta_r = (\mathrm{id} \otimes \rho) \circ \lambda$. Then P_L and P_R are projections with the same kernels as μ_L and μ_R, respectively.

Part 3.' and 3". of Theorem 3.1 imply that we may put $\hat{\mathcal{G}} \bowtie {}_\lambda\mathcal{M}_\rho := P_L(\hat{\mathcal{G}} \otimes \mathcal{M})$ and ${}_\lambda\mathcal{M}_\rho \bowtie \hat{\mathcal{G}} := P_R(\mathcal{M} \otimes \hat{\mathcal{G}})$ to conclude that analogously as in (2.6) and (2.7) the restrictions

$$\mu_L : \hat{\mathcal{G}} \bowtie {}_\lambda\mathcal{M}_\rho \longrightarrow \mathcal{M}_1$$
$$\mu_R : {}_\lambda\mathcal{M}_\rho \bowtie \hat{\mathcal{G}} \longrightarrow \mathcal{M}_1$$

are linear isomorphisms inducing a concrete realization of the abstract algebra $\mathcal{M}_1$ on the subspaces $\hat{\mathcal{G}} \bowtie {}_\lambda\mathcal{M}_\rho \subset \hat{\mathcal{G}} \otimes \mathcal{M}$ and ${}_\lambda\mathcal{M}_\rho \bowtie \hat{\mathcal{G}} \subset \mathcal{M} \otimes \hat{\mathcal{G}}$, respectively. As before, we call these concrete realizations the *left and right diagonal crossed products*, respectively, associated with the quasi–commuting pair of $\mathcal{G}$–coactions $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ on $\mathcal{M}$. To actually prove Theorem 3.1 we follow the same strategy as in Chapter 2, i.e. we first construct these diagonal crossed products explicitly and then show that they solve the universal properties defining $\mathcal{M}_1$.

3.1. Weak quasi–quantum groups

Quasi–inverses. We start with a little digression on the notion of *quasi–inverses*. Let $\mathcal{A}$ be an associative algebra and let $x, p, q \in \mathcal{A}$ satisfy

$$px = x = xq \tag{3.2}$$

$$p^2 = p, \qquad q^2 = q \tag{3.3}$$

Then we say that $y \in \mathcal{A}$ is a *quasi–inverse* of x with respect to (p, q), if

$$yx = q, \qquad xy = p, \qquad yxy = y \tag{3.4}$$

Clearly, given (p, q), a quasi–inverse of x is uniquely determined, provided it exists. This is why we also write $y = x^{-1}$, if the idempotents (p, q) are understood. Also note that we have $qy = y = yp$ and $xyx = x$ and therefore x is the quasi–inverse of y with respect to (q, p). All this generalizes in the obvious way to $\mathcal{A}$–module morphisms $x \in \mathrm{Hom}_{\mathcal{A}}(V, W)$, $p \in \mathrm{End}_{\mathcal{A}}(W)$ and $q \in \mathrm{End}_{\mathcal{A}}(V)$, in which case the quasi–inverse would be an element $x^{-1} \in \mathrm{Hom}_{\mathcal{A}}(W, V)$. Note that in place of (3.2) we could equivalently add to (3.4) the requirement

$$xyx = x \tag{3.5}$$

In our setting of weak quasi–Hopf algebras the idempotents p, q always appear as images of $\mathbf{1} \in \mathcal{G}$ of non–unital algebra maps defined on $\mathcal{G}$, like $\Delta(\mathbf{1})$, $\Delta^{op}(\mathbf{1})$, $(\Delta \otimes \mathrm{id})(\Delta(\mathbf{1})), \ldots$ etc., whereas the element x will be an intertwiner between two such maps, like a reassociator ϕ, and R–matrix R, etc. Hence, throughout we will adopt the convention that if $\alpha : \mathcal{G} \to \mathcal{A}$ and $\beta : \mathcal{G} \to \mathcal{A}$ are two algebra maps and $x \in \mathcal{A}$ satisfies

$$x\,\alpha(g) = \beta(g)\,x, \quad \forall g \in \mathcal{G}$$

then the quasi–inverse $y = x^{-1} \in \mathcal{A}$ is defined to be the unique (if existing) element satisfying

$$yx = \alpha(\mathbf{1}), \quad xy = \beta(\mathbf{1})$$
$$xyx = x, \quad yxy = y$$

Clearly this implies conversely

$$\alpha(g)\,y = y\,\beta(g), \quad \forall g \in \mathcal{G}$$

and therefore $x = y^{-1}$. We also note the obvious identities

$$\beta(g) = x\,\alpha(g)\,x^{-1}, \quad \alpha(g) = x^{-1}\,\beta(g)\,x$$
$$x\,\alpha(\mathbf{1}) = \beta(\mathbf{1})\,x = x, \quad \alpha(\mathbf{1})\,x^{-1} = x^{-1}\,\beta(\mathbf{1}) = x^{-1}$$

Weak quasi–Hopf algebras. After this digression we now define, following [MS92] a *weak quasi–bialgebra* $(\mathcal{G}, \mathbf{1}, \Delta, \epsilon, \phi)$ to be an associative algebra $\mathcal{G}$ with unit $\mathbf{1}$, a non–unital algebra map $\Delta : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$, an algebra map $\epsilon : \mathcal{G} \to \mathbb{C}$ and an element $\phi \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$ satisfying (2.8)-(2.10), whereas (2.11) is replaced by

$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\phi) = \Delta(\mathbf{1}) \tag{3.6}$$

and where in place of invertibility ϕ is supposed to have a quasi–inverse $\bar{\phi} \equiv \phi^{-1}$ with respect to the intertwining property (2.8). Hence we have $\phi\bar{\phi}\phi = \phi$, $\bar{\phi}\phi\bar{\phi} = \bar{\phi}$ as well as

$$\phi\,\bar{\phi} = (\mathrm{id} \otimes \Delta)(\Delta(\mathbf{1})), \quad \bar{\phi}\,\phi = (\Delta \otimes \mathrm{id})(\Delta(\mathbf{1})) \tag{3.7}$$

implying the further identities

$$(\mathrm{id} \otimes \Delta)(\Delta(a)) = \phi\,(\Delta \otimes \mathrm{id})(\Delta(a))\,\bar{\phi}, \quad \forall a \in \mathcal{G} \tag{3.8}$$

$$\phi = \phi\,(\Delta \otimes \mathrm{id})(\Delta(\mathbf{1})), \quad \bar{\phi} = \bar{\phi}\,(\mathrm{id} \otimes \Delta)(\Delta(\mathbf{1})) \tag{3.9}$$

$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\bar{\phi}) = \Delta(\mathbf{1}) \tag{3.10}$$

A weak quasi–bialgebra is called *weak quasi–Hopf algebra*, if there exists a unital algebra antimorphism S and elements $\alpha, \beta \in \mathcal{G}$ satisfying (2.15) and (2.16). We will also always suppose that S is invertible. The remarks about the quasi–Hopf algebras $\mathcal{G}_{op}$, $\mathcal{G}^{cop}$ and $\mathcal{G}_{op}^{cop}$ remain valid as in Section 2.1.

As before we call $\mathcal{G}$ *quasitriangular*, if there exists an elment $R \in \mathcal{G} \otimes \mathcal{G}$ satisfying (2.17-2.19), where instead of invertibility R is supposed to be quasi–invertible with respect to the intertwiner property (2.8), i.e. there exists $\bar{R} \in \mathcal{G} \otimes \mathcal{G}$ satisfying $\bar{R}R = \Delta(\mathbf{1})$, $R\bar{R} = \Delta^{op}(\mathbf{1})$.

A quasi–invertible element $F \in \mathcal{G} \otimes \mathcal{G}$ satisfying $(\epsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \epsilon)(F) = \mathbf{1}$ induces a *twist transformation* from $(\mathcal{G}, \Delta, \epsilon, \phi)$ to $(\mathcal{G}, \Delta_F, \epsilon, \phi_F)$ as in (2.22) and (2.23), where $\mathcal{G}_F := (\mathcal{G}, \mathbf{1}, \Delta_F, \epsilon, \phi_F)$ is again a weak quasi–bialgebra. The two bialgebras $\mathcal{G}_F$ and $\mathcal{G}$ are called *twist–equivalent.*

Finally the properties of the twists f, h defined as in (2.26) and (2.31) are still valid. In particular (2.27) defines the quasi–inverse of f with respect to the intertwining property (2.28).

Let us shortly indicate the implications on the representation theory of $\mathcal{G}$. For more details see Appendix A. Due to the coproduct being non–unital the definition of the tensor product of representations has to be slightly modified. First note that $\Delta(\mathbf{1}) \in \Delta(\mathcal{G})$ is central and idempotent. Thus, given two representations $(V, \pi_V), (W, \pi_W)$, the operator $(\pi_V \otimes \pi_W)(\Delta(\mathbf{1}))$ is a projector, whose image is precisely the $\mathcal{G}$–invariant subspace of $V \otimes W$ on which the tensor product representation operates non trivial. Thus one is led to define the tensor product $\boxtimes$ of two representations of $\mathcal{G}$ by setting

$$V \boxtimes W := (\pi_V \otimes \pi_W)(\Delta(\mathbf{1}))\, V \otimes W, \quad \pi_V \boxtimes \pi_W := (\pi_V \otimes \pi_W) \circ \Delta|_{V \boxtimes W} \tag{3.11}$$

With this definition all considerations about the representation theory of $\mathcal{G}$ carry over to weak quasi–Hopf algebras. In particular $\mathrm{Rep}\,\mathcal{G}$ becomes a rigid monoidal category and a quasitriangular R–matrix defines a braiding in $\mathrm{Rep}\,\mathcal{G}$.

Weak coactions. The notion of coactions may easily be generalized as well. By a left $\mathcal{G}$–coaction (λ, ϕ_λ) of a weak quasi–bialgebra $\mathcal{G}$ on a unital algebra $\mathcal{M}$ we mean a (not necessarily unital) algebra map $\lambda : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M}$ and a quasi–invertible element $\phi_\lambda \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{M}$ satisfying (2.35a)-(2.35c) as in Definition 2.3 and

$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\phi_\lambda) = (\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\lambda) = \lambda(\mathbf{1}_\mathcal{M}) \tag{3.12}$$

The definition of right coactions is generalized analogously. Lemma 2.4 about twist equivalencies of coactions stays valid, where one has to make the adjustments that a twist $U \in \mathcal{M} \otimes \mathcal{G}$ only is supposed to be quasi–invertible.

By now it should become clear how one has to proceed: Definition 2.5 of two–sided coactions is generalized by allowing δ to be non–unital and Ψ to be non–invertible but with quasi–inverse $\bar{\Psi} \equiv \Psi^{-1}$ and by replacing (2.39d) by

$$(\mathrm{id}_\mathcal{G} \otimes \epsilon \otimes \mathrm{id}_\mathcal{M} \otimes \epsilon \otimes \mathrm{id}_\mathcal{G})(\Psi) = (\epsilon \otimes \mathrm{id}_\mathcal{G} \otimes \mathrm{id}_\mathcal{M} \otimes \mathrm{id}_\mathcal{G} \otimes \epsilon)(\Psi) = \delta(\mathbf{1}_\mathcal{M}) \tag{3.13}$$

The definitions of quasi–commuting pairs of coactions, twist equivalencies of two–sided coactions etc. are generalized similarly. With these adjustments all results of Section 2.3 stay valid for weak quasi–Hopf algebras and are proven analogously.

The elements q_λ, p_λ and q_ρ, p_ρ are defined as in (2.77)-(2.80) and obey all the relations stated in Lemma 2.21 with the only modifications that in (2.103c) and (2.103d) the r.h.s. becomes $\lambda(\mathbf{1}_\mathcal{M})$ instead of $\mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{M}$ and similarly (2.104c/2.104d) where the r.h.s. has to be replaced by $\rho(\mathbf{1}_\mathcal{M})$, i.e.

$$\lambda(q_\lambda^2)\, p_\lambda\, [S^{-1}(q_\lambda^1) \otimes \mathbf{1}_\mathcal{M}] = \lambda(\mathbf{1}_\mathcal{M}) \tag{3.14}$$

$$[S(p_\lambda^1) \otimes \mathbf{1}_\mathcal{M}]\, q_\lambda\, \lambda(p_\lambda^2) = \lambda(\mathbf{1}_\mathcal{M}) \tag{3.15}$$

$$\rho(q_\rho^1)\, p_\rho\, [\mathbf{1}_\mathcal{M} \otimes S(q_\rho^2)] = \rho(\mathbf{1}_\mathcal{M}) \tag{3.16}$$

$$[\mathbf{1}_\mathcal{M} \otimes S^{-1}(p_\rho^2)]\, q_\rho\, \rho(p_\rho^1) = \rho(\mathbf{1}_\mathcal{M}) \tag{3.17}$$

Going through the proof of Lemma 2.21, this follows from the fact that one uses (3.12) or the corresponding identity for ϕ_ρ.

3.2. Diagonal crossed products

The definition of diagonal crossed products $\hat{\mathcal{G}} \bowtie \mathcal{M}_\delta$ and $\mathcal{M}_\delta \bowtie \hat{\mathcal{G}}$ as equivalent algebra extensions of $\mathcal{M}$, given in Definition 2.9/2.11 need some more care in the present context. We will proceed in two steps. First we define an associative algebra structure on $\hat{\mathcal{G}} \otimes \mathcal{M}$ (or $\mathcal{M} \otimes \hat{\mathcal{G}}$) exactly as in Definition 2.9 (or Definition 2.11). Unfortunately in general this algebra is not unital unless the two–sided coaction δ is unital. But the element $\hat{\mathbf{1}} \otimes \mathbf{1}_\mathcal{M}$ is still a right unit ($\mathbf{1}_\mathcal{M} \otimes \hat{\mathbf{1}}$ is still a left unit) and in particular idempotent. The second step then consists in defining the subalgebra $\hat{\mathcal{G}} \bowtie \mathcal{M}_\delta \subset \hat{\mathcal{G}} \otimes \mathcal{M}$ as the right ideal generated by $\hat{\mathbf{1}} \otimes \mathbf{1}_\mathcal{M}$, i.e.

$\hat{\mathcal{G}} \bowtie \mathcal{M}_\delta := (\hat{\mathbf{1}} \otimes \mathbf{1}_\mathcal{M}) \cdot (\hat{\mathcal{G}} \otimes \mathcal{M})$ (the left ideal generated by $\mathbf{1}_\mathcal{M} \otimes \hat{\mathbf{1}}$, i.e. $\mathcal{M}_\delta \bowtie \hat{\mathcal{G}} := (\mathcal{M} \otimes \hat{\mathcal{G}}) \cdot (\mathbf{1}_\mathcal{M} \otimes \hat{\mathbf{1}})$). These algebras are then unital algebra extensions of $\mathcal{M} \equiv \hat{\mathbf{1}} \bowtie \mathcal{M}$ and of $\mathcal{M} \equiv \mathcal{M} \bowtie \hat{\mathbf{1}}$, respectively. As in Section 2.5 one may proceed to a description by left and right δ–implementers and equivalently by $\lambda\rho$–intertwiners, thus getting a proof of Theorem 3.1.

DEFINITION 3.2. Let (δ, Ψ) be a two–sided coaction of a weak quasi–Hopf algebra $\mathcal{G}$ on an algebra $\mathcal{M}$. We define $\hat{\mathcal{G}} \otimes_\delta \mathcal{M}$ to be the vector space $\hat{\mathcal{G}} \otimes \mathcal{M}$ with multiplication structure given as in (2.56) and the left *diagonal crossed product* $\hat{\mathcal{G}} \bowtie \mathcal{M}_\delta$ to be the subspace

$$\hat{\mathcal{G}} \bowtie \mathcal{M}_\delta := (\hat{\mathbf{1}} \otimes \mathbf{1}_\mathcal{M}) \cdot (\hat{\mathcal{G}} \otimes_\delta \mathcal{M}) \tag{3.18}$$

Analogously $\mathcal{M} \otimes_\delta \hat{\mathcal{G}}$ is defined to be the vector space $\mathcal{M} \otimes \hat{\mathcal{G}}$ with multiplication structure (2.63) and the right *diagonal crossed product* $\mathcal{M}_\delta \bowtie \hat{\mathcal{G}}$ to be the subspace

$$\mathcal{M}_\delta \bowtie \hat{\mathcal{G}} := (\mathcal{M} \otimes_\delta \hat{\mathcal{G}}) \cdot (\mathbf{1}_\mathcal{M} \otimes \hat{\mathbf{1}}) \tag{3.19}$$

The elements spanning $\hat{\mathcal{G}} \bowtie \mathcal{M}_\delta$ and $\mathcal{M}_\delta \bowtie \hat{\mathcal{G}}$ will be denoted by, respectively

$$\varphi \bowtie m := (\hat{\mathbf{1}} \otimes \mathbf{1}_\mathcal{M})(\varphi \otimes m) \equiv (\varphi \otimes \mathbf{1}_\mathcal{M})(\mathbf{1}_\mathcal{M} \otimes m) \equiv \varphi_{(2)} \otimes \big(\hat{S}^{-1}(\varphi_{(1)}) \triangleright \mathbf{1}_\mathcal{M} \triangleleft \varphi_{(3)}\big) m \tag{3.20}$$

$$m \bowtie \varphi := (m \otimes \varphi)(\mathbf{1}_\mathcal{M} \otimes \hat{\mathbf{1}}) \equiv (m \otimes \hat{\mathbf{1}})(\mathbf{1}_\mathcal{M} \otimes \varphi) \equiv m\big(\varphi_{(1)} \triangleright \mathbf{1}_\mathcal{M} \triangleleft \hat{S}^{-1}(\varphi_{(3)})\big) \otimes \varphi_{(2)} \tag{3.21}$$

Note that $\hat{\mathcal{G}} \otimes_\delta \mathcal{M} = \hat{\mathcal{G}} \bowtie \mathcal{M}_\delta$, if $\delta(\mathbf{1}_\mathcal{M}) = \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{M} \otimes \mathbf{1}_\mathcal{G}$, which means that the above definition generalizes Definition 2.9. We now state the analogue of Theorem 2.10.

THEOREM 3.3.

(i) *$\hat{\mathcal{G}} \otimes_\delta \mathcal{M}$ and $\mathcal{M} \otimes_\delta \hat{\mathcal{G}}$ are associative algebras with left unit $\hat{\mathbf{1}} \otimes \mathbf{1}_\mathcal{M}$ and right unit $\mathbf{1}_\mathcal{M} \otimes \hat{\mathbf{1}}$, respectively. Consequently, the diagonal crossed products $\hat{\mathcal{G}} \bowtie \mathcal{M}_\delta$ and $\mathcal{M}_\delta \bowtie \hat{\mathcal{G}}$ are subalgebras of $\hat{\mathcal{G}} \otimes_\delta \mathcal{M}$ and $\mathcal{M} \otimes_\delta \hat{\mathcal{G}}$, respectively, with unit given by $\hat{\mathbf{1}} \bowtie \mathbf{1}_\mathcal{M} \equiv \hat{\mathbf{1}} \otimes \mathbf{1}_\mathcal{M}$ and $\mathbf{1}_\mathcal{M} \bowtie \hat{\mathbf{1}} \equiv \mathbf{1}_\mathcal{M} \otimes \hat{\mathbf{1}}$, respectively.*

(ii) *$\mathcal{M} \equiv \hat{\mathbf{1}} \otimes \mathcal{M} = \hat{\mathbf{1}} \bowtie \mathcal{M} \subset \hat{\mathcal{G}} \bowtie \mathcal{M}_\delta$ and $\mathcal{M} \equiv \mathcal{M} \otimes \hat{\mathbf{1}} = \mathcal{M} \bowtie \hat{\mathbf{1}} \subset \mathcal{M}_\delta \bowtie \hat{\mathcal{G}}$ are unital algebra inclusions.*

PROOF. We will sketch the proof of part (i) for $\hat{\mathcal{G}} \otimes \mathcal{M}$, the case $\mathcal{M} \otimes \hat{\mathcal{G}}$ being analogous. From (2.56) one computes that

$$\begin{aligned}
(\varphi \otimes m)(\hat{\mathbf{1}} \otimes \mathbf{1}_\mathcal{M}) &= (\varphi \otimes m) \\
(\hat{\mathbf{1}} \otimes \mathbf{1}_\mathcal{M})(\varphi \otimes m) &= \varphi_{(2)} \otimes \big(\hat{S}^{-1}(\varphi_{(1)}) \triangleright \mathbf{1}_\mathcal{M} \triangleleft \varphi_{(3)}\big)\, m \\
&= \big(\mathbf{1}_{(-1)} \rightharpoonup \varphi \leftharpoonup S^{-1}(\mathbf{1}_{(1)})\big) \otimes \mathbf{1}_{(0)}\, m
\end{aligned}$$

where $\delta(\mathbf{1}_\mathcal{M}) = \mathbf{1}_{(-1)} \otimes \mathbf{1}_{(0)} \otimes \mathbf{1}_{(1)}$. This shows that $\hat{\mathbf{1}} \otimes \mathbf{1}_\mathcal{M}$ is a right unit in $\hat{\mathcal{G}} \otimes_\delta \mathcal{M}$ but in general not a left unit.

To proof the associativity of the product one proceeds as in the proof of Theorem 2.10. Here one has to take some notational care when translating (2.94/2.95) into relations of a generating matrix $\mathbf{L}$. First, it is necessary to distinguish $\tilde{\mathbf{L}} \equiv \tilde{L}^1 \otimes \tilde{L}^2 := e_\mu \otimes (e^\mu \otimes \mathbf{1}_\mathcal{M}) \in \mathcal{G} \otimes (\hat{\mathcal{G}} \otimes \mathcal{M})$ and $\mathbf{L} := e_\mu \otimes (e^\mu \bowtie \mathbf{1}_\mathcal{M}) \in \mathcal{G} \otimes (\hat{\mathcal{G}} \bowtie \mathcal{M})$. Eq. (2.98) must then be replaced by

$$[\mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{M}]\, \tilde{\mathbf{L}} = \mathbf{L} \tag{3.22}$$

and (2.58), (2.59) are at first only valid for $\mathbf{L}$ but not for $\tilde{\mathbf{L}}$, since $\mathbf{1}_\mathcal{M} \equiv \hat{\mathbf{1}} \otimes \mathbf{1}_\mathcal{M}$ is not a left unit in $\hat{\mathcal{G}} \otimes_\delta \mathcal{M}$. This can be cured by rewriting for example (2.58) for $\tilde{\mathbf{L}}$ more carefully as

$$(\tilde{L}^1 \otimes m\tilde{L}^2) = S^{-1}(m_{(1)})\, \tilde{L}^1\, m_{(-1)} \otimes \tilde{L}^2\, m_{(0)}, \quad \forall m \in \mathcal{M}$$

in which form it would still be valid. Taking this into account and using that $\bar{\Psi}\,(\mathbf{1}_\mathcal{G} \otimes \delta(\mathbf{1}_\mathcal{M}) \otimes \mathbf{1}_\mathcal{G}) = \bar{\Psi}$, the proof proceeds as the one of Theorem 2.10 (i). The proof of the remaining parts of Theorem 2.10 is straightforwardly adjusted in the same spirit. □

From now on we will disregard the "unphysical" non–unital algebras $\mathcal{M} \otimes_\delta \hat{\mathcal{G}}$ and $\hat{\mathcal{G}} \otimes_\delta \mathcal{M}$, and stay with $\hat{\mathcal{G}} \bowtie \mathcal{M}$ and $\mathcal{M} \bowtie \hat{\mathcal{G}}$ as our objects of interest. With the appropriate notations (3.20),(3.21) all relations of Section 2.4 remain valid for these algebras. We also remark that the left multiplication by $\hat{\mathbf{1}} \otimes \mathbf{1}_\mathcal{M}$ (right multiplication by $\mathbf{1}_\mathcal{M} \otimes \hat{\mathbf{1}}$) precisely gives the projections $P_{L/R}$ mentioned in part 3". of Theorem 3.1.

Defining left and right δ–implementers as in Definition 2.13 all results in Section 2.5.1 also carry over to the present setting. A $\lambda\rho$–intertwiner is then supposed to have the additional property that

$$\mathbf{T}\,\lambda_{\mathcal{A}}(\mathbf{1}_{\mathcal{M}}) \equiv \rho^{op}_{\mathcal{A}}(\mathbf{1}_{\mathcal{M}})\,\mathbf{T} = \mathbf{T} \tag{3.23}$$

With this the one–to–one correspondence $\mathbf{T} \leftrightarrow \mathbf{R}$ and $\mathbf{T} \leftrightarrow \mathbf{L}$ of Proposition 2.19 is still valid, where (3.23) becomes equivalent to

$$\mathbf{L} = [S^{-1}(\mathbf{1}_{(1)}) \otimes \mathbf{1}_{\mathcal{A}}]\,\mathbf{L}\,[\mathbf{1}_{(-1)} \otimes \gamma(\mathbf{1}_{(0)})]$$
$$\mathbf{R} = [\mathbf{1}_{(1)} \otimes \gamma(\mathbf{1}_{(0)})]\,\mathbf{R}\,[S^{-1}(\mathbf{1}_{(-1)}) \otimes \mathbf{1}_{\mathcal{A}}]$$

respectively, which follow from (2.64) and (2.65). One may now prove Theorem 3.1 analogously as Theorem 2.1 where the modifications in part 3'. and 3". have their origin in (3.23).