CHAPTER 2

Diagonal crossed products by duals of quasi–quantum groups

In Chapter 1 we have reviewed the notions of left and right $\mathcal{G}$–coactions and crossed products and we have introduced as new concepts the notions of two–sided $\mathcal{G}$–coactions and diagonal crossed products, where throughout $\mathcal{G}$ had been supposed to be a standard coassociative Hopf algebra. We have seen that this is the mathematical ctructure underlying the quantum group spin chains of [NS97] and also the lattice current algebras of [AFFS98]. Moreover also the Drinfel'd double $\mathcal{D}(\mathcal{G})$ appeared as a diagonal crossed product.

We now proceed to generalize the above ideas to quasi–Hopf algebras $\mathcal{G}$. In Section 2.1 we give a short review of the definitions and properties of quasi–Hopf algebras as introduced by Drinfel'd [Dri90]. In Section 2.2 we propose an obvious generalization of the notion of right $\mathcal{G}$–coactions ρ on an algebra $\mathcal{M}$ to the case of quasi–Hopf algebras $\mathcal{G}$ (and similarly for left coactions λ). As for the coproduct on $\mathcal{G}$, the basic idea here is that $(\rho \otimes \mathrm{id}) \circ \rho$ and $(\mathrm{id} \otimes \Delta) \circ \rho$ are still related by an inner automorphism, implemented by a reassociator $\phi_\rho \in \mathcal{M} \otimes \mathcal{G} \otimes \mathcal{G}$. Similarly as for Drinfel'd's reassociator $\phi \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$, ϕ_ρ is required to obey a pentagon equation to guarantee McLane's coherence condition under iterated rebracketings. We also generalize Drinfel'd's notion of a twist transformation from coproducts to coactions.

It is important to realize that ϕ_ρ has to be non–trivial, if ϕ is non–trivial. On the other hand, ϕ_ρ might be non–trivial even if $\phi = \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G}$, in which case the above mentioned pentagon equation reduces to a cocycle condition for ϕ_ρ as already considered by [DT86], [BCM86], [BM89].

In Section 2.3 we pass to two–sided $\mathcal{G}$–coactions (δ, Ψ), which could alternatively be considered as right $(\mathcal{G} \otimes \mathcal{G}^{cop})$–coactions in the above sense. Correspondingly, $\Psi \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G} \otimes \mathcal{G}$ is the reassociator for δ, which is again required to obey the appropriate pentagon equation. As in Chapter 1, associated with any two–sided $\mathcal{G}$-coaction (δ, Ψ) we have a pair (λ, ϕ_λ) and (ρ, ϕ_ρ) of left and right $\mathcal{G}$–coactions, respectively, which however in this case only *quasi–commute*. This means that there exists another reassociator $\phi_{\lambda\rho} \in \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ such that

$$\phi_{\lambda\rho}\,(\lambda \otimes \mathrm{id}_\mathcal{G})(\rho(m)) = (\mathrm{id}_\mathcal{G} \otimes \rho)(\lambda(m))\,\phi_{\lambda\rho}, \quad \forall m \in \mathcal{M}.$$

Also, $\phi_{\lambda\rho}$ obeys in a natural way two pentagon identities involving (λ, ϕ_λ) and (ρ, ϕ_ρ), respectively. We show that twist equivalence classes of two–sided coactions are in one–to–one correspondence with twist equivalence classes of quasi–commuting pairs of coactions, i.e. any two–sided coaction δ is twist–equivalent to $(\lambda \otimes \mathrm{id}) \circ \rho$ (and also to $(\mathrm{id} \otimes \rho) \circ \lambda$) where $\lambda = (\mathrm{id}_\mathcal{G} \otimes \mathrm{id}_\mathcal{M} \otimes \epsilon) \circ \delta$ and $\rho = (\epsilon \otimes \mathrm{id}_\mathcal{M} \otimes \mathrm{id}_\mathcal{G}) \circ \delta$.

In Appendix A we give a representation theoretic interpretation of the notions of left, right and two–sided coactions by showing that they give rise to functors $\mathrm{Rep}\,\mathcal{G} \times \mathrm{Rep}\,\mathcal{M} \to \mathrm{Rep}\,\mathcal{M}$, $\mathrm{Rep}\,\mathcal{M} \times \mathrm{Rep}\,\mathcal{G} \to \mathrm{Rep}\,\mathcal{M}$ and $\mathrm{Rep}\,\mathcal{G} \times \mathrm{Rep}\,\mathcal{M} \times \mathrm{Rep}\,\mathcal{G} \to \mathrm{Rep}\,\mathcal{M}$, respectively, furnished with natural associativity isomorphisms, obeying the analogue of McLane's coherence conditions for monoidal categories [Mac71].

In Section 2.4 we use our formalism to construct, for any two–sided $\mathcal{G}$–coaction (δ, Ψ) on $\mathcal{M}$, the left and right diagonal crossed products $\mathcal{M}_\delta \bowtie \hat{\mathcal{G}}$ and $\hat{\mathcal{G}} \bowtie \mathcal{M}_\delta$ as associative algebra extensions of $\mathcal{M}$ (they are in fact equivalent as will be shown in Section 2.5). Up to equivalence, these extensions only depend on the twist–equivalence class of δ's, and therefore on the twist–equivalence class of quasi–commuting pairs (λ, ρ). The basic strategy for defining the multiplication rules in these diagonal crossed products is to generalize the generating matrix formalism of Section 1.3 to the quasi–coassociative setting. In this way one is naturally lead to define $\lambda\rho$–intertwiners $\mathbf{T}$ as in Definition 1.10, where now the coherence condition (1.32) has to be replaced by appropriately injecting the reassociators $\phi_\lambda, \phi_{\lambda\rho}$ and ϕ_ρ into the l.h.s., similarly as in Drinfel'd's definition of a quasitriangular R–matrix for quasi–Hopf algebras. With these substitutions our main result is given by the following generalization of Theorem 1.13

THEOREM 2.1. *Let $\mathcal{G}$ be a finite dimensional quasi–Hopf algebra and let $(\lambda, \phi_\lambda, \rho, \phi_\rho, \phi_{\lambda\rho})$ be a quasi–commuting pair of (left and right) $\mathcal{G}$–coactions on an associative algebra $\mathcal{M}$.*

1. *Then their exists a unital associative algebra extension $\mathcal{M}_1 \supset \mathcal{M}$ together with a linear map $\Gamma : \hat{\mathcal{G}} \to \mathcal{M}_1$ satisfying the following universal property:*
   $\mathcal{M}_1$ is algebraically generated by $\mathcal{M}$ and $\Gamma(\hat{\mathcal{G}})$ and for any algebra map $\gamma : \mathcal{M} \to \mathcal{A}$ into some target algebra $\mathcal{A}$ the relation
   $$\gamma_T(\Gamma(\varphi)) = (\varphi \otimes \mathrm{id})(\mathbf{T}) \tag{2.1}$$
   provides a one–to–one correspondence between algebra maps $\gamma_T : \mathcal{M}_1 \to \mathcal{A}$ extending γ and normal elements $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ satisfying
   $$\mathbf{T}\,\lambda_{\mathcal{A}}(m) = \rho^{op}_{\mathcal{A}}(m)\,\mathbf{T}, \quad \forall m \in \mathcal{M} \tag{2.2}$$
   $$(\phi^{312}_\rho)_{\mathcal{A}}\,\mathbf{T}^{13}\,(\phi^{132}_{\rho\lambda})^{-1}_{\mathcal{A}}\,\mathbf{T}^{23}\,(\phi_\lambda)_{\mathcal{A}} = (\Delta \otimes \mathrm{id}_{\mathcal{A}})(\mathbf{T}), \tag{2.3}$$
   where $\lambda_{\mathcal{A}}(m) := (\mathrm{id} \otimes \gamma)(\lambda(m))$, $(\phi_\lambda)_{\mathcal{A}} := (\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \gamma)(\phi_\lambda)$, etc.
2. *If $\mathcal{M} \subset \tilde{\mathcal{M}}_1$ and $\tilde{\Gamma} : \hat{\mathcal{G}} \to \tilde{\mathcal{M}}_1$ satisfy the same universality property as in part 1.), then there exists a unique algebra isomorphism $f : \mathcal{M}_1 \to \tilde{\mathcal{M}}_1$ restricting to the identity on $\mathcal{M}$, such that $\tilde{\Gamma} = f \circ \Gamma$*
3. *There exist elements $p_\lambda \in \mathcal{G} \otimes \mathcal{M}$ and $q_\rho \in \mathcal{M} \otimes \mathcal{G}$ such that the linear maps*
   $$\mu_L : \hat{\mathcal{G}} \otimes \mathcal{M} \ni (\varphi \otimes m) \mapsto (\mathrm{id} \otimes \varphi_{(1)})(q_\rho)\,\Gamma(\varphi_{(2)})\,m \in \mathcal{M}_1 \tag{2.4}$$
   $$\mu_R : \mathcal{M} \otimes \hat{\mathcal{G}} \ni (m \otimes \varphi) \mapsto m\,\Gamma(\varphi_{(1)})\,(\varphi_{(2)} \otimes \mathrm{id})(p_\lambda) \in \mathcal{M}_1 \tag{2.5}$$
   provide isomorphisms of vector spaces.

Putting $\mathbf{\Gamma} := e_\mu \otimes \Gamma(e^\mu) \in \mathcal{G} \otimes \mathcal{M}_1$, Theorem 2.1 implies that $\mathbf{\Gamma}$ itself satisfies the defining relations (2.2) and (2.3). As before, we call $\mathbf{\Gamma}$ the *universal $\lambda\rho$–intertwiner* in $\mathcal{M}_1$. We remark that it is more or less straightforward to check that the relations (2.2) and (2.3) satisfy all associativity constraints, such that the existence of $\mathcal{M}_1$ and its uniqueness up to isomorphy may not be too much of a surprise to the experts. In this way part 1. and 2. of Theorem 2.1 could also be proven without requiring an antipode on $\mathcal{G}$. The main non–trivial content of Theorem 2.1 is stated in part 3., saying that $\mathcal{M}_1$ may still be modeled on the underlying spaces $\hat{\mathcal{G}} \otimes \mathcal{M}$ or $\mathcal{M} \otimes \hat{\mathcal{G}}$, respectively [1]. However, as a warning against likely misunderstandings we emphasize that in general (i.e. for $\phi_{\lambda\rho} \neq \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\mathcal{G}}$) neither of the maps

$$\mathcal{M} \otimes \hat{\mathcal{G}} \ni (m \otimes \varphi) \mapsto m\,\Gamma(\varphi) \in \mathcal{M}_1$$
$$\hat{\mathcal{G}} \otimes \mathcal{M} \ni (\varphi \otimes m) \mapsto \Gamma(\varphi)\,m \in \mathcal{M}_1$$

need to be injective (nor surjective)[2]. Also, in general neither of the linear subspaces $\Gamma(\hat{\mathcal{G}})$, $\mu_L(\hat{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{M}})$ or $\mu_R(\mathbf{1}_{\mathcal{M}} \otimes \hat{\mathcal{G}})$ will be a subalgebra of $\mathcal{M}_1$. Still, the invertibility of the maps $\mu_{L/R}$ guarantees that there exist well defined associative algebra structures induced on $\hat{\mathcal{G}} \otimes \mathcal{M}$ and $M \otimes \hat{\mathcal{G}}$ via $\mu^{-1}_{L/R}$ from $\mathcal{M}_1$. As in Chapter 1 we denote these by

$$\hat{\mathcal{G}} \bowtie {}_\lambda\mathcal{M}_\rho \equiv \mu_L^{-1}(\mathcal{M}_1) \tag{2.6}$$
$${}_\lambda\mathcal{M}_\rho \bowtie \hat{\mathcal{G}} \equiv \mu_R^{-1}(\mathcal{M}_1). \tag{2.7}$$

They are the analogues of the left and right diagonal crossed products, respectively, constructed in Proposition 1.6 and Corollary 1.9.

To actually prove Theorem 2.1 we go the opposite way, i.e. for any two–sided coaction (δ, Ψ) we will first explicitly construct left and right diagonal crossed products $\hat{\mathcal{G}} \bowtie \mathcal{M}_\delta$ and $\mathcal{M}_\delta \bowtie \hat{\mathcal{G}}$ as equivalent algebra extensions of $\mathcal{M}$ in Section 2.4. As in Chapter 1 these are defined on the underlying spaces $\hat{\mathcal{G}} \otimes \mathcal{M}$ and $\mathcal{M} \otimes \hat{\mathcal{G}}$, respectively. In Section 2.5.1 we describe these constructions in terms of so–called left and right *diagonal δ–implementers* $\mathbf{L}$ and $\mathbf{R}$ obeying the relations of Lemma 1.11(iii) and (ii), respectively, together with certain coherence conditions reflecting the multiplication rules in $\hat{\mathcal{G}} \bowtie \mathcal{M}$ and $\mathcal{M} \bowtie \hat{\mathcal{G}}$. In Section 2.5.2 we generalize

[1] To define the elements p_λ and q_ρ one needs an invertible antipode, see (2.77), (2.80)

[2] In fact, we don't even know whether the map $\Gamma : \hat{\mathcal{G}} \to \mathcal{M}_1$ necessarily has to be injective.

Lemma 1.11 by showing that coherent (left or right) diagonal δ–implementers are always in one–to–one correspondence with (although not identical to) *coherent $\lambda\rho$–intertwiners* $\mathbf{T}$, i.e. generating matrices satisfying the relations (2.2) and (2.3) of Theorem 2.1. This will finally lead to a proof of Theorem 2.1 by showing that for $\delta_l := (\lambda \otimes \mathrm{id}) \circ \rho$ and $\delta_r := (\mathrm{id} \otimes \rho) \circ \lambda$ any of the four choices $\hat{\mathcal{G}} \bowtie \mathcal{M}_{\delta_l}$, $\hat{\mathcal{G}} \bowtie \mathcal{M}_{\delta_r}$, $\mathcal{M}_{\delta_l} \bowtie \hat{\mathcal{G}}$, or $\mathcal{M}_{\delta_r} \bowtie \hat{\mathcal{G}}$ explicitly solve all properties claimed in Theorem 2.1. Moreover, in terms of the notations (2.6), (2.7) we will have

$$\hat{\mathcal{G}} \bowtie {}_\lambda\mathcal{M}_\rho = \hat{\mathcal{G}} \bowtie \mathcal{M}_{\delta_l}$$
$${}_\lambda\mathcal{M}_\rho \bowtie \hat{\mathcal{G}} = \mathcal{M}_{\delta_r} \bowtie \hat{\mathcal{G}}$$

with trivial identification.

To keep the main part of this chapter more readable, we have postponed some proofs and technical Lemmata to Section 2.6.

2.1. Quasi–quantum groups

In this section we review the basic definitions and properties of quasitriangular quasi–Hopf algebras (quasitriangular quantum groups) as introduced by Drinfeld [Dri90], where the interested reader will find a more detailed discussion. As before algebra morphisms are always supposed to be unital.

A *quasi-bialgebra* $(\mathcal{G}, \Delta, \epsilon, \phi)$ is an associative algebra $\mathcal{G}$ with unit together with algebra morphisms $\Delta : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ (the coproduct) and $\epsilon : \mathcal{G} \to \mathbb{C}$ (the counit), and an invertible element $\phi \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$, such that

$$(\mathrm{id} \otimes \Delta)(\Delta(a))\phi = \phi(\Delta \otimes \mathrm{id})(\Delta(a)), \quad a \in \mathcal{G} \tag{2.8}$$
$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\phi)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi) = (\mathbf{1} \otimes \phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi)(\phi \otimes \mathbf{1}), \tag{2.9}$$
$$(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta, \tag{2.10}$$
$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\phi) = \mathbf{1} \otimes \mathbf{1} \tag{2.11}$$

A coproduct with the above properties is called *quasi-coassociative* and the element ϕ will be called the *reassociator*. The identities (2.8)-(2.11) also imply

$$(\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(\phi) = (\mathrm{id} \otimes \mathrm{id} \otimes \epsilon)(\phi) = \mathbf{1} \otimes \mathbf{1}. \tag{2.12}$$

As for Hopf algebras we will use the Sweedler notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$, but since Δ is only quasi–coassociative we adopt the further convention

$$(\Delta \otimes \mathrm{id}) \circ \Delta(a) = a_{(1,1)} \otimes a_{(1,2)} \otimes a_{(2)} \quad \text{and} \quad (\mathrm{id} \otimes \Delta) \circ \Delta(a) = a_{(1)} \otimes a_{(2,1)} \otimes a_{(2,2)}, \text{ etc..}$$

Furthermore, here and throughout we use the notation

$$\phi = X^j \otimes Y^j \otimes Z^j; \quad \phi^{-1} = \bar{X}^j \otimes \bar{Y}^j \otimes \bar{Z}^j, \tag{2.13}$$

where we have suppressed the summation symbol. To give an example, Eq. (2.8) written with this notation looks like

$$a_{(1)}X^j \otimes a_{(2,1)}Y^j \otimes a_{(2,2)}Z^j = X^i a_{(1,1)} \otimes Y^i a_{(1,2)} \otimes Z^i a_{(2)}$$

Let us briefly recall some of the main consequences of these definitions for the representation theory of $\mathcal{G}$. Let $\mathrm{Rep}\,\mathcal{G}$ be the category of finite dimensional representations of $\mathcal{G}$, i.e. of pairs (π_V, V), where V is a finite dimensional vector space and $\pi_V : \mathcal{G} \to \mathrm{End}_{\mathbb{C}}(V)$ is a unital algebra morphism. We will also use the equivalent notion of a $\mathcal{G}$–module V with multiplication $g \cdot v \equiv \pi_V(g)v$. Given two pairs $(\pi_V, V), (\pi_U, U)$, the coproduct allows for the definition of a tensor product $(\pi_V \boxtimes \pi_U, V \boxtimes U)$ by setting $V \boxtimes U := V \otimes U$ and $\pi_V \boxtimes \pi_U := (\pi_V \otimes \pi_U) \circ \Delta$. The counit defines a one dimensional representation. Equation (2.10) says, that this representation is a left and right unit with respect to the tensor product $\boxtimes$, and (2.8) says that given three representations (π_U, π_V, π_W), then $(\pi_U \boxtimes \pi_V) \boxtimes \pi_W \cong \pi_U \boxtimes (\pi_V \boxtimes \pi_W)$ with intertwiner $\phi_{UVW} = (\pi_U \otimes \pi_V \otimes \pi_W)(\phi)$.

The meaning of (2.9) is the commutativity of the pentagon

$$
\begin{array}{ccccc}
((U \boxtimes V) \boxtimes W) \boxtimes X & \longrightarrow & (U \boxtimes V) \boxtimes (W \boxtimes X) & \longrightarrow & U \boxtimes (V \boxtimes (W \boxtimes X)) \\
\searrow & & & & \nearrow \\
 & (U \boxtimes (V \boxtimes W)) \boxtimes X & \longrightarrow & U \boxtimes ((V \boxtimes W) \boxtimes X) & ,
\end{array}
\tag{2.14}
$$

where the arrows stand for the corresponding rebracketing intertwiners. For example the first one is given by $\big((\pi_U \boxtimes \pi_V) \otimes \pi_W \otimes \pi_X\big)(\phi) \equiv (\pi_U \otimes \pi_V \otimes \pi_W \otimes \pi_X)\big((\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi)\big)$. The diagram (2.14) explains the name pentagon identity for equation (2.9). The importance of axiom (2.9) lies in the fact, that in any tensor product representation the intertwiner connecting two different bracket conventions is given by a suitable product of ϕ's, as in (2.14). The pentagon identity then guarantees, that this intertwiner is independent of the chosen sequence of intermediate rebracketings. This is known as Mac Lanes coherence theorem [Mac71].

A quasi–bialgebra $\mathcal{G}$ is called *quasi-Hopf algebra*, if there is a linear antimorphism $S : \mathcal{G} \to \mathcal{G}$ and elements $\alpha, \beta \in \mathcal{G}$ satisfying (for all $a \in \mathcal{G}$)

$$S(a_{(1)})\alpha a_{(2)} = \alpha\epsilon(a), \qquad a_{(1)}\beta S(a_{(2)}) = \beta\epsilon(a) \tag{2.15}$$

$$X^j \beta S(Y^j)\alpha Z^j = 1 = S(\bar{X}^j)\alpha \bar{Y}^j \beta S(\bar{Z}^j), \tag{2.16}$$

where we have used the notation (2.13). The map S is called an *antipode*. We will also always suppose that S is invertible. Note that as opposed to ordinary Hopf algebras, an antipode is not uniquely determined, provided it exists. The antipode allows to define the (left) dual representation $({}^*\pi, {}^*V)$ of (π, V), where ${}^*V$ is the dual space of V, by ${}^*\pi(a) = \pi(S(a))^t$, the superscript t denoting the transposed map. Analogously one defines a right dual representation (π^*, V^*), where $V^* \equiv {}^*V$ and $\pi^*(a) = \pi(S^{-1}(a))^t$.

A quasi-Hopf algebra $\mathcal{G}$ is called *quasitriangular*, if there exists an invertible $R \in \mathcal{G} \otimes \mathcal{G}$, such that

$$\Delta^{op}(a)R = R\Delta(a), \quad a \in \mathcal{G} \tag{2.17}$$

$$(\Delta \otimes \mathrm{id})(R) = \phi^{312}\, R^{13}\, (\phi^{-1})^{132}\, R^{23}\, \phi \tag{2.18}$$

$$(\mathrm{id} \otimes \Delta)(R) = (\phi^{-1})^{231}\, R^{13}\, \phi^{213}\, R^{12}\, \phi^{-1}, \tag{2.19}$$

where we use the following notation: If $\psi = \sum_i \psi_i^1 \otimes \ldots \psi_i^n \in \mathcal{G}^{\otimes^m}$, then, for $m \le n$, $\psi^{n_1 n_2 \ldots n_m} \in \mathcal{G}^{\otimes^n}$ denotes the element of $\mathcal{G}^{\otimes^n}$ having ψ_i^k in the $n_k{}^{\text{th}}$ slot and $\mathbf{1}$ in the remaining ones. The element R is called the R-matrix. The above relations imply the quasi-Yang-Baxter equation

$$R^{12}\, \phi^{312}\, R^{13}\, (\phi^{-1})^{132}\, R^{23}\, \phi = \phi^{321}\, R^{23}\, (\phi^{-1})^{231}\, R^{13}\, \phi^{213}\, R^{12} \tag{2.20}$$

and the property

$$(\epsilon \otimes \mathrm{id})(R) = (\mathrm{id} \otimes \epsilon)(R) = \mathbf{1}. \tag{2.21}$$

Property (2.17) implies, that for any pair π_U, π_V the two representations $(\pi_U \boxtimes \pi_V, U \boxtimes V)$ and $(\pi_V \boxtimes \pi_U, V \boxtimes U)$ are equivalent with intertwiner $B_{UV} := \tau^{12} \circ (\pi_U \otimes \pi_V)(R)$, where τ^{12} denotes the permutation of tensor factors in $U \otimes V$. Eqs. (2.18),(2.19) imply the commutativity of two hexagon diagrams obtained by taking $\pi_U \otimes \pi_V \otimes \pi_W$ on both sides.

$\mathcal{G}$ being a quasitriangular quasi–Hopf algebra implies that $\mathrm{Rep}\,\mathcal{G}$ is a rigid monoidal category with braiding, where the associativity and commutativity constraints for the tensor product functor $\boxtimes : \mathrm{Rep}\,\mathcal{G} \times \mathrm{Rep}\,\mathcal{G} \to \mathrm{Rep}\,\mathcal{G}$ are given by the natural families ϕ_{UVW} and $\tau^{12} \circ R_{UV}$ and the (left) duality is defined with the help of the antipode S and the elements α, β, see (B.2–B.4) below.

Together with a quasi–Hopf algebra $\mathcal{G} \equiv (\mathcal{G}, \Delta, \epsilon, \phi, S, \alpha, \beta)$ we also have $\mathcal{G}_{op}$, $\mathcal{G}^{cop}$ and $\mathcal{G}_{op}^{cop}$ as quasi–Hopf algebras, where "*op*" means opposite multiplication and "*cop*" means opposite comultiplication. The quasi–Hopf structures are obtained by putting $\phi_{op} := \phi^{-1}$, $\phi^{cop} := (\phi^{-1})^{321}$, $\phi_{op}^{cop} := \phi^{321}$, $S_{op} = S^{cop} = (S_{op}^{cop})^{-1} := S^{-1}$, $\alpha_{op} := S^{-1}(\beta)$, $\beta_{op} := S^{-1}(\alpha)$, $\alpha^{cop} := S^{-1}(\alpha)$, $\beta^{cop} := S^{-1}(\beta)$, $\alpha_{op}^{cop} := \beta$ and $\beta_{op}^{cop} := \alpha$. Also if $R \in \mathcal{G} \otimes \mathcal{G}$ is quasitriangular in $\mathcal{G}$, then R^{-1} is quasitriangular in $\mathcal{G}_{op}$, R^{21} is quasitriangular in $\mathcal{G}^{cop}$ and $(R^{-1})^{21}$ is quasitriangular in $\mathcal{G}_{op}^{cop}$

Next we recall that the definition of a quasitriangular quasi-Hopf algebra is 'twist covariant' in the following sense: An invertible element $F \in \mathcal{G} \otimes \mathcal{G}$, which satisfies $(\epsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \epsilon)(F) = \mathbf{1}$, induces a so–called *twist transformation*

$$\Delta_F(a) := F\Delta(a)F, \tag{2.22}$$

$$\phi_F := (\mathbf{1} \otimes F)\,(\mathrm{id} \otimes \Delta)(F)\,\phi\,(\Delta \otimes \mathrm{id})(F^{-1})\,(F^{-1} \otimes \mathbf{1}) \tag{2.23}$$

It has been noticed by Drinfel'd [Dri90] that $(\mathcal{G}, \Delta_F, \epsilon, \phi_F)$ is again a quasi–bialgebra. Setting

$$\alpha_F := S(h^i)\alpha k^i, \quad \beta_F := f^i \beta S(g^i),$$

where $h^i \otimes k^i = F^{-1}$ and $f^i \otimes g^i = F$, $(\mathcal{G}, \Delta_F, \epsilon, \phi_F, S, \alpha_F, \beta_F)$ is also a quasi-Hopf algebra. Moreover, if R is quasitriangular with respect to (Δ, ϕ), then

$$R_F := F^{21} R F^{-1}$$

is quasitriangular w.r.t. (Δ_F, ϕ_F). This means that a twist preserves the class of quasitriangular quasi-Hopf algebras [Dri90].

It is well known that the antipode of a Hopf algebra is also an anti coalgebra morphism, i.e. $\Delta(a) = (S \otimes S)\big(\Delta^{op}(S^{-1}(a))\big)$. For quasi-Hopf algebras this is true only up to a twist: Following Drinfel'd we define the elements $\gamma, \delta \in \mathcal{G} \otimes \mathcal{G}$ by setting[3]

$$\gamma := (S(U^i) \otimes S(T^i)) \cdot (\alpha \otimes \alpha) \cdot (V^i \otimes W^i) \tag{2.24}$$

$$\delta := (K^j \otimes L^j) \cdot (\beta \otimes \beta) \cdot (S(N^j) \otimes S(M^j)) \tag{2.25}$$

where

$$T^i \otimes U^i \otimes V^i \otimes W^i = (\mathbf{1} \otimes \phi^{-1}) \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\phi),$$

$$K^j \otimes L^j \otimes M^j \otimes N^j = (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi) \cdot (\phi^{-1} \otimes \mathbf{1}).$$

With these definitions Drinfel'd has shown in [Dri90], that $f \in \mathcal{G} \otimes \mathcal{G}$ given by

$$f := (S \otimes S)(\Delta^{op}(\bar{X}^i)) \cdot \gamma \cdot \Delta(\bar{Y}^i \beta \bar{Z}^i). \tag{2.26}$$

defines a twist with inverse given by

$$f^{-1} = \Delta(S(\bar{X}^j)\alpha \bar{Y}^j) \cdot \delta \cdot (S \otimes S)(\Delta^{op}(\bar{Z}^i)), \tag{2.27}$$

such that for all $a \in \mathcal{G}$

$$f\Delta(a)f^{-1} = (S \otimes S)\big(\Delta^{op}(S^{-1}(a))\big). \tag{2.28}$$

The elements γ, δ and the twist f fulfill the relations

$$f\,\Delta(\alpha) = \gamma, \quad \Delta(\beta)\,f^{-1} = \delta\,. \tag{2.29}$$

Furthermore, the corresponding twisted reassociator (2.23) is given by

$$\phi_f = (S \otimes S \otimes S)(\phi^{321}). \tag{2.30}$$

Setting $h := (S^{-1} \otimes S^{-1})(f^{21})$, the above relations imply

$$h\Delta(a)h^{-1} = (S^{-1} \otimes S^{-1})\big(\Delta^{op}(S(a))\big) \tag{2.31}$$

$$\phi_h = (S^{-1} \otimes S^{-1} \otimes S^{-1})(\phi^{321}) \tag{2.32}$$

$$h\Delta(S^{-1}(\alpha)) = (S^{-1} \otimes S^{-1})(\gamma^{21}) \tag{2.33}$$

These identities will be used frequently below as well as the following

COROLLARY 2.2. *For $a \in \mathcal{G}$ let $\Delta_L(a) := h\Delta(a)$ and $\Delta_R(a) := \Delta(a)h^{-1}$ where $h \in \mathcal{G} \otimes \mathcal{G}$ is the twist in* (2.31). *Then*

$$(\mathrm{id} \otimes \Delta_L)(\Delta_L(a))\,\phi = (S^{-1} \otimes S^{-1} \otimes S^{-1})(\phi^{321})\,(\Delta_L \otimes \mathrm{id})(\Delta_L(a))$$

$$\phi\,(\Delta_R \otimes \mathrm{id})(\Delta_R(a)) = (\mathrm{id} \otimes \Delta_R)(\Delta_R(a))\,(S^{-1} \otimes S^{-1} \otimes S^{-1})(\phi^{321}), \quad \forall a \in \mathcal{G}$$

PROOF. Writing Eq. (2.32) as

$$(\mathbf{1} \otimes h)\,(\mathrm{id} \otimes \Delta)(h)\,\phi = (S^{-1} \otimes S^{-1} \otimes S^{-1})(\phi^{321})\,(h \otimes \mathbf{1})\,(\Delta \otimes \mathrm{id})(h),$$

multiplicating both sides from the right with $(\Delta \otimes \mathrm{id})(\Delta(a))$ and using (2.8) yields the first equality. The second equality is proven analogously. □

[3] suppressing summation symbols

The importance of the twist f for the representation theory of $\mathcal{G}$ lies in the fact that it provides an intertwiner $U \otimes V \to ({}^*V \boxtimes {}^*U)^*$ given by $\tau^{12} \circ (\pi_U \boxtimes \pi_V)(f)$.

From the point of view of representation theory the difference between Hopf algebras and quasi–Hopf algebras may be reformulated by stating that in the first case Rep $\mathcal{G}$ is a *strict* rigid monoidal category, whereas in the latter case Rep $\mathcal{G}$ is not strict but still rigid monoidal. A more detailed discussion of these representation theoretic considerations is given in Appendix A.

Finally we introduce $\hat{\mathcal{G}}$ as the dual space of $\mathcal{G}$ with its natural coassociative coalgebra structure $(\hat{\Delta}, \hat{\epsilon})$ given by $\langle \hat{\Delta}(\varphi) \mid a \otimes b \rangle := \langle \varphi \mid ab \rangle$ and $\hat{\epsilon}(\varphi) := \langle \varphi \mid \mathbf{1}_{\mathcal{G}} \rangle$, where $\varphi \in \hat{\mathcal{G}}$, $a, b \in \mathcal{G}$ and where $\langle \cdot \mid \cdot \rangle : \hat{\mathcal{G}} \otimes \mathcal{G} \to \mathbb{C}$ denotes the dual pairing. On $\hat{\mathcal{G}}$ we have the natural left and right $\mathcal{G}$–actions

$$a \rightharpoonup \varphi := \varphi_{(1)} \langle \varphi_{(2)} \mid a \rangle, \quad \varphi \leftharpoonup a := \varphi_{(2)} \langle \varphi_{(1)} \mid a \rangle, \tag{2.34}$$

where $a \in \mathcal{G}$, $\varphi \in \hat{\mathcal{G}}$. By transposing the coproduct on $\mathcal{G}$ we also get a multiplication $\hat{\mathcal{G}} \otimes \hat{\mathcal{G}} \to \hat{\mathcal{G}}$, which however is no longer associative

$$\langle \varphi\psi \mid a \rangle := \langle \varphi \otimes \psi \mid \Delta(a) \rangle, \quad \langle \hat{\mathbf{1}} \mid a \rangle := \epsilon(a).$$

Yet, we have the identities $\hat{\mathbf{1}}\varphi = \varphi\hat{\mathbf{1}} = \varphi$, $\hat{\Delta}(\varphi\psi) = \hat{\Delta}(\varphi)\hat{\Delta}(\psi)$, $a \rightharpoonup (\varphi\psi) = (a_{(1)} \rightharpoonup \varphi)(a_{(2)} \rightharpoonup \psi)$ and $(\varphi\psi) \leftharpoonup a = (\varphi \leftharpoonup a_{(1)})(\psi \leftharpoonup a_{(2)})$ for all $\varphi, \psi \in \hat{\mathcal{G}}$ and $a \in \mathcal{G}$. We also introduce $\hat{S} : \hat{\mathcal{G}} \to \hat{\mathcal{G}}$ as the coalgebra anti-morphism dual to S, i.e. $\langle \hat{S}(\varphi) \mid a \rangle := \langle \varphi \mid S(a) \rangle$.

2.2. Coactions of quasi–quantum groups

The generalization of the definition of coactions as given in (1.1 - 1.4) to the quasi–Hopf case is straightforward:

DEFINITION 2.3. A *left coaction* of a quasi-bialgebra $(\mathcal{G}, \mathbf{1}_{\mathcal{G}}, \Delta, \epsilon, \phi)$ on a unital algebra $\mathcal{M}$ is an algebra morphism $\lambda : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M}$ together with an invertible element $\phi_\lambda \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{M}$ satisfying

$$(\mathrm{id} \otimes \lambda)(\lambda(m))\, \phi_\lambda = \phi_\lambda\, (\Delta \otimes \mathrm{id})(\lambda(m)), \qquad \forall m \in \mathcal{M} \tag{2.35a}$$

$$(\mathbf{1}_{\mathcal{G}} \otimes \phi_\lambda)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi_\lambda)(\phi \otimes \mathbf{1}_{\mathcal{M}}) = (\mathrm{id} \otimes \mathrm{id} \otimes \lambda)(\phi_\lambda)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\lambda), \tag{2.35b}$$

$$(\epsilon \otimes \mathrm{id}) \circ \lambda = \mathrm{id} \tag{2.35c}$$

$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\phi_\lambda) = (\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\lambda) = \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{M}} \tag{2.35d}$$

Similarly a *right coaction* of $\mathcal{G}$ on $\mathcal{M}$ is an algebra morphism $\rho : \mathcal{M} \to \mathcal{M} \otimes \mathcal{G}$ together with $\phi_\rho \in \mathcal{M} \otimes \mathcal{G} \otimes \mathcal{G}$ such that

$$\phi_\rho\, (\rho \otimes \mathrm{id})(\rho(m)) = (\mathrm{id} \otimes \Delta)(\rho(m))\, \phi_\rho, \qquad \forall m \in \mathcal{M} \tag{2.36a}$$

$$(\mathbf{1}_{\mathcal{M}} \otimes \phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi_\rho)(\phi_\rho \otimes \mathbf{1}_{\mathcal{G}}) = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\phi_\rho)(\rho \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\rho), \tag{2.36b}$$

$$(\mathrm{id} \otimes \epsilon) \circ \rho = \mathrm{id} \tag{2.36c}$$

$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\phi_\rho) = (\mathrm{id} \otimes \mathrm{id} \otimes \epsilon)(\phi_\rho) = \mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\mathcal{G}} \tag{2.36d}$$

The triple $(\mathcal{M}, \lambda, \phi_\lambda)$ [$(\mathcal{M}, \rho, \phi_\rho)$] is called a left [right] *comodule algebra* over $\mathcal{G}$ also denoted ${}_\lambda\mathcal{M}$ [$\mathcal{M}_\rho$].

We remark, that of the two counit conditions in (2.35d) and (2.36d), respectively, actually either one of them already implies the other. Clearly, if $\mathcal{G}$ is a Hopf algebra, $\phi = \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}}$ and $\phi_\lambda = \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{M}}$, one recovers the definitions given in (1.1 - 1.4). Also, particular examples are given by $\mathcal{M} = \mathcal{G}$ and $\lambda = \rho = \Delta$, $\phi_\lambda = \phi_\rho = \phi$. In the general case equations (2.35b),(2.36b) may be understood as a generalized pentagon equation, whereas (2.35a),(2.36a) mean, that λ, ρ respect the quasi-coalgebra structure of $\mathcal{G}$. One should notice, that because of the pentagon equations (2.35b) and (2.36b), ϕ_λ and ϕ_ρ have to be nontrivial if ϕ is nontrivial (i.e. if $\mathcal{G}$ is not a Hopf algebra). On the other hand ϕ_λ or ϕ_ρ may be nontrivial even if $\phi = \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}}$, i.e. if $\mathcal{G}$ is a Hopf algebra. In fact, such a restricted setting has been investigated before, see [DT86], [BCM86], or [BM89]. In [BCM86, BM89] Eq. (2.36a) is called a "twisted module condition" and Eq. (2.36b) (for $\phi = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$) a "cocycle condition". It has been shown in [HNa], that the twisted crossed products considered in [DT86], [BCM86], [BM89] are in fact special types of our diagonal crossed products to be given in Definition 2.9 below.

As with Hopf algebras, a left coaction λ (a right coaction ρ) induces a map $\triangleleft : \mathcal{M}\otimes\hat{\mathcal{G}} \to \mathcal{M}$ ($\triangleright : \hat{\mathcal{G}}\otimes\mathcal{M} \to \mathcal{M}$) by

$$m \triangleleft \varphi := (\varphi\otimes\mathrm{id})(\lambda(m)), \tag{2.37}$$

$$\varphi \triangleright m := (\mathrm{id}\otimes\varphi)(\rho(m)), \quad \varphi\in\hat{\mathcal{G}},\ m\in\mathcal{M} \tag{2.38}$$

which by convenient abuse of notation and terminology we still call a "right action" ("left action") of $\hat{\mathcal{G}}$ on $\mathcal{M}$, despite of the fact that $\hat{\mathcal{G}}$ may not be an associative algebra.

Similarly as for the coproduct Δ there is a natural notion of *twist equivalence* for coactions of quasi–Hopf algebras.

LEMMA 2.4. *Let (ρ, ϕ_ρ) be a right coaction of a quasi–bialgebra $\mathcal{G}$ on $\mathcal{M}$ and let $U \in \mathcal{M}\otimes\mathcal{G}$ be invertible such that $(\mathrm{id}\otimes\epsilon)(U) = \mathbf{1}_\mathcal{M}$. Then the pair (ρ', ϕ'_ρ), given by*

$$\rho'(m) := U\,\rho(m)\,U^{-1}$$
$$\phi'_\rho := (\mathrm{id}_\mathcal{M}\otimes\Delta)(U)\,\phi_\rho\,(\rho\otimes\mathrm{id}_\mathcal{G})(U^{-1})\,(U^{-1}\otimes\mathbf{1}_\mathcal{G})$$

again defines a right coaction of $\mathcal{G}$ on $\mathcal{M}$ (with respect to the same quasi–bialgebra structure on $\mathcal{G}$).

The proof of Lemma 2.4 is straightforward and therefore omitted. A similar statement holds for left coactions λ, where one would have to take $U \in \mathcal{G}\otimes\mathcal{M}$ and

$$\phi'_\lambda = (\mathbf{1}_\mathcal{G}\otimes U)\,(\mathrm{id}_\mathcal{G}\otimes\lambda)(U)\,\phi_\lambda\,(\Delta\otimes\mathrm{id}_\mathcal{M})(U^{-1}).$$

Note that twisting indeed defines an equivalence relation for coactions. Similarly, if Δ_F and ϕ_F are given by (2.22) and (2.23), then any right (left) $\mathcal{G}$–coaction on $\mathcal{M}$ may also be considered as a coaction with respect to the F–twisted structures on $\mathcal{G}$ by putting $\rho_F = \rho$ ($\lambda_F = \lambda$) and

$$(\phi_\rho)_F := (\mathbf{1}_\mathcal{M}\otimes F)\,\phi_\rho, \quad (\phi_\lambda)_F := \phi_\lambda\,(F^{-1}\otimes\mathbf{1}_\mathcal{M}).$$

The reader is invited to check that with these definitions (2.35b) and (2.36b) are indeed also twist covariant. Note that in the case $\lambda = \rho = \Delta$, $U = F$, one recovers (2.23).

2.3. Two–sided coactions

As already mentioned before, the fact that the dual $\hat{\mathcal{G}}$ fails to be an associative algebra is the reason why there is no generalization of the definitions of ordinary crossed products to the quasi–Hopf algebra case. Nevertheless this will be possible for our diagonal crossed product constructed from two–sided coactions. First we need

DEFINITION 2.5. A *two–sided coaction* of a quasi–bialgebra $(\mathcal{G}, \Delta, \epsilon, \phi)$ on an algebra $\mathcal{M}$ is an algebra map $\delta : \mathcal{M} \to \mathcal{G}\otimes\mathcal{M}\otimes\mathcal{G}$ together with an invertible element $\Psi \in \mathcal{G}\otimes\mathcal{G}\otimes\mathcal{M}\otimes\mathcal{G}\otimes\mathcal{G}$ satisfying

$$(\mathrm{id}_\mathcal{G}\otimes\delta\otimes\mathrm{id}_\mathcal{G})(\delta(m))\,\Psi = \Psi\,(\Delta\otimes\mathrm{id}_\mathcal{M}\otimes\Delta)(\delta(m)), \quad \forall m\in\mathcal{M} \tag{2.39a}$$

$$\begin{aligned}(\mathbf{1}_\mathcal{G}\otimes\Psi\otimes\mathbf{1}_\mathcal{G})\,&(\mathrm{id}_\mathcal{G}\otimes\Delta\otimes\mathrm{id}_\mathcal{M}\otimes\Delta\otimes\mathrm{id}_\mathcal{G})(\Psi)\,(\phi\otimes\mathbf{1}_\mathcal{M}\otimes\phi^{-1})\\ &= (\mathrm{id}_\mathcal{G}\otimes\mathrm{id}_\mathcal{G}\otimes\delta\otimes\mathrm{id}_\mathcal{G}\otimes\mathrm{id}_\mathcal{G})(\Psi)\,(\Delta\otimes\mathrm{id}_\mathcal{G}\otimes\mathrm{id}_\mathcal{M}\otimes\mathrm{id}_\mathcal{G}\otimes\Delta)(\Psi)\end{aligned} \tag{2.39b}$$

$$(\epsilon\otimes\mathrm{id}_\mathcal{M}\otimes\epsilon)\circ\delta = \mathrm{id}_\mathcal{M} \tag{2.39c}$$

$$(\mathrm{id}_\mathcal{G}\otimes\epsilon\otimes\mathrm{id}_\mathcal{M}\otimes\epsilon\otimes\mathrm{id}_\mathcal{G})(\Psi) = (\epsilon\otimes\mathrm{id}_\mathcal{G}\otimes\mathrm{id}_\mathcal{M}\otimes\mathrm{id}_\mathcal{G}\otimes\epsilon)(\Psi) = \mathbf{1}_\mathcal{G}\otimes\mathbf{1}_\mathcal{M}\otimes\mathbf{1}_\mathcal{G}. \tag{2.39d}$$

The triple $(\mathcal{M}, \delta, \Psi)$ is called a *two–sided comodule algebra*, also denoted $\mathcal{M}_\delta$.

Again we remark, that either one of the two counit axioms in (2.39d) already implies the other. We also note, that two–sided coactions could of course be considered as right coactions of $\mathcal{G}\otimes\mathcal{G}^{cop}$, or left coactions of $\mathcal{G}^{cop}\otimes\mathcal{G}$, respectively. Moreover, if (δ, Ψ) is a two–sided coaction of $\mathcal{G}$ on $\mathcal{M}$, then (δ, Ψ^{-1}) is a two–sided coaction of $\mathcal{G}_{op}$ on $\mathcal{M}_{op}$ and (δ_{op}, Ψ_{op}) is a two–sided coaction of $\mathcal{G}^{cop}$ on $\mathcal{M}$, where

$$\delta_{op} := \delta^{321}, \quad \Psi_{op} := \Psi^{54321}. \tag{2.40}$$

An example of a two–sided coaction is given by $\mathcal{M} = \mathcal{G}$, $\delta = (\Delta\otimes\mathrm{id})\circ\Delta$ and

$$\Psi := [(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\phi)\otimes\mathbf{1}][\phi\otimes\mathbf{1}\otimes\mathbf{1}][(\delta\otimes\mathrm{id}\otimes\mathrm{id})(\phi^{-1})]. \tag{2.41}$$

Similarly we could choose $\delta' = (\mathrm{id} \otimes \Delta) \circ \Delta$ and

$$\Psi' := [\mathbf{1} \otimes (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi^{-1})][\mathbf{1} \otimes \mathbf{1} \otimes \phi^{-1}][(\mathrm{id} \otimes \mathrm{id} \otimes \delta')(\phi)]. \tag{2.42}$$

From this example one already realizes that in the present context the relation between two–sided coactions and pairs of commuting left and right coactions gets somewhat more involved as compared to Chapter 1, where we had $\delta = \delta'$. First, one easily checks that for any two–sided coaction (δ, Ψ) the definitions

$$\lambda := (\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon) \circ \delta \qquad \phi_\lambda : = (\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon \otimes \epsilon)(\Psi), \tag{2.43}$$

$$\rho := (\epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}}) \circ \delta \qquad \phi_\rho^{-1} : = (\epsilon \otimes \epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}})(\Psi) \tag{2.44}$$

provide us again with a left coaction (λ, ϕ_λ) and a right coaction (ρ, ϕ_ρ). Moreover, putting $\delta^{(2)} := (\mathrm{id} \otimes \delta \otimes \mathrm{id}) \circ \delta$ we have

$$\delta_l := (\lambda \otimes \mathrm{id}_{\mathcal{G}}) \circ \rho = (\epsilon \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon \otimes \mathrm{id}_{\mathcal{G}}) \circ \delta^{(2)} \tag{2.45}$$

$$\delta_r := (\mathrm{id}_{\mathcal{G}} \otimes \rho) \circ \lambda = (\mathrm{id}_{\mathcal{G}} \otimes \epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon) \circ \delta^{(2)} \tag{2.46}$$

However, due to the appearance of the reassociator Ψ in axiom (2.39a), the two expressions (2.45) and (2.46) are in general unequal, and neither one needs to coincide with δ. Indeed, defining

$$U_l := (\epsilon \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon \otimes \mathrm{id}_{\mathcal{G}})(\Psi) \tag{2.47}$$

$$U_r := (\mathrm{id}_{\mathcal{G}} \otimes \epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon)(\Psi) \tag{2.48}$$

$$\phi_{\lambda\rho} := U_r U_l^{-1}, \tag{2.49}$$

(2.43)-(2.46) and (2.39a)-(2.39d) immediately imply

$$\begin{gathered} \delta_l(m) = U_l\, \delta(m)\, U_l^{-1}, \quad \delta_r(m) = U_r\, \delta(m)\, U_r^{-1} \\ \phi_{\lambda\rho}\, (\lambda \otimes \mathrm{id})(\rho(m)) = (\mathrm{id} \otimes \rho)(\lambda(m))\, \phi_{\lambda\rho}. \end{gathered} \tag{2.50}$$

Moreover $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ provides a *quasi–commuting pair* of coactions in the following sense.

DEFINITION 2.6. Let $(\mathcal{G}, \Delta, \epsilon, \phi)$ be a quasi–bialgebra. By a *quasi–commuting pair* of $\mathcal{G}$–coactions on an algebra $\mathcal{M}$ we mean a quintuple $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$, where (λ, ϕ_λ) and (ρ, ϕ_ρ) are left and right $\mathcal{G}$–coactions on $\mathcal{M}$, respectively, and where $\phi_{\lambda\rho} \in \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ is invertible and satisfies

$$\phi_{\lambda\rho}\, (\lambda \otimes \mathrm{id})(\rho(m)) = (\mathrm{id} \otimes \rho)(\lambda(m))\, \phi_{\lambda\rho}, \qquad \forall m \in \mathcal{M} \tag{2.51a}$$

$$(\mathbf{1}_{\mathcal{G}} \otimes \phi_{\lambda\rho})(\mathrm{id} \otimes \lambda \otimes \mathrm{id})(\phi_{\lambda\rho})(\phi_\lambda \otimes \mathbf{1}_{\mathcal{G}}) = (\mathrm{id} \otimes \mathrm{id} \otimes \rho)(\phi_\lambda)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi_{\lambda\rho}) \tag{2.51b}$$

$$(\mathbf{1}_{\mathcal{G}} \otimes \phi_\rho)(\mathrm{id} \otimes \rho \otimes \mathrm{id})(\phi_{\lambda\rho})(\phi_{\lambda\rho} \otimes \mathbf{1}_{\mathcal{G}}) = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\phi_{\lambda\rho})(\lambda \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\rho) \tag{2.51c}$$

Obviously, the conditions (2.51a)-(2.51c) apply to the case $\mathcal{M} = \mathcal{G}$, $\lambda = \rho = \Delta$ and $\phi_\lambda = \phi_\rho = \phi$. Also note, that acting with $(\epsilon \otimes \epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}})$ on (2.51b) and with $(\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon \otimes \epsilon)$ on (2.51c) and using the invertibility of $\phi_{\lambda\rho}$ one concludes the further identities

$$(\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon)(\phi_{\lambda\rho}) = \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{M}}, \quad (\epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}})(\phi_{\lambda\rho}) = \mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\mathcal{G}}. \tag{2.51d}$$

We also remark that quasi–commutativity is stable under twisting.

The fact that $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ given in (2.43), (2.44) and (2.49) provides a quasi–commuting pair is shown in detail in [HNa]. Conversely, one also verifies by direct computation, that every pair of quasi–commuting coactions $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ provides us with two–sided coactions (Ψ_l, δ_l) and (Ψ_r, δ_r) defined by

$$\delta_l := (\lambda \otimes \mathrm{id}) \circ \rho \tag{2.52a}$$

$$\Psi_l := (\mathrm{id}_{\mathcal{G}} \otimes \lambda \otimes \mathrm{id}_{\mathcal{G}}^{\otimes^2})\Big((\phi_{\lambda\rho} \otimes \mathbf{1}_{\mathcal{G}})(\lambda \otimes \mathrm{id}_{\mathcal{G}}^{\otimes^2})(\phi_\rho^{-1})\Big)\, [\phi_\lambda \otimes \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}}] \tag{2.52b}$$

$$\delta_r := (\mathrm{id} \otimes \rho) \circ \lambda \tag{2.52c}$$

$$\Psi_r := (\mathrm{id}_{\mathcal{G}}^{\otimes^2} \otimes \rho \otimes \mathrm{id}_{\mathcal{G}})\Big((\mathbf{1}_{\mathcal{G}} \otimes \phi_{\lambda\rho}^{-1})\, (\mathrm{id}_{\mathcal{G}}^{\otimes^2} \otimes \rho)(\phi_\lambda)\Big)\, [\mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}} \otimes \phi_\rho^{-1}] \tag{2.52d}$$

Note that (2.52a)-(2.52d) generalize the examples (2.41), (2.42).

Using this result one is now in the position to show that twist–equivalence classes of quasi–commuting pairs of coactions $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ are in one-to-one correspondence with twist equivalence classes of two–sided coactions (δ, Ψ), since by (2.50) up to twist equivalence any two–sided coaction is of the type $(\delta_{l/r}, \Psi_{l/r})$ given in (2.52a) - (2.52d). Here one uses that two–sided coactions (δ, Ψ) may be twisted in the same fashion as one–sided ones.

DEFINITION 2.7. Let (δ, Ψ) and (δ', Ψ') be two–sided coactions of $(\mathcal{G}, \Delta, \epsilon, \phi)$ on $\mathcal{M}$. Then (δ', Ψ') is called *twist equivalent* to (δ, Ψ), if there exists $U \in \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ invertible such that

$$\delta'(m) = U\,\delta(m)\,U^{-1} \tag{2.53a}$$

$$\Psi' = (\mathbf{1}_\mathcal{G} \otimes U \otimes \mathbf{1}_\mathcal{G})\,(\mathrm{id}_\mathcal{G} \otimes \delta \otimes \mathrm{id}_\mathcal{G})(U)\,\Psi\,(\Delta \otimes \mathrm{id}_\mathcal{M} \otimes \Delta)(U^{-1}) \tag{2.53b}$$

$$(\epsilon \otimes \mathrm{id}_\mathcal{M} \otimes \epsilon)(U) = \mathbf{1}_\mathcal{M} \tag{2.53c}$$

The reader is invited to check that for any two–sided coaction (δ, Ψ) and any invertible U satisfying (2.53c) the definitions (2.53a) and (2.53b) indeed produce another two–sided coaction (δ', Ψ'). It is also easy to see that twisting does provide an equivalence relation between two–sided coactions. Moreover, similarly as for one–sided coactions one readily verifies that if (δ, Ψ) is a two–sided coaction of $(\mathcal{G}, \Delta, \epsilon, \phi)$ on $\mathcal{M}$, then for any twist $F \in \mathcal{G} \otimes \mathcal{G}$ the pair (δ, Ψ_F) is a two–sided coaction of $(\mathcal{G}, \Delta_F, \epsilon, \phi_F)$ on $\mathcal{M}$, where Δ_F and ϕ_F are the twisted structures on $\mathcal{G}$ given by (2.22) and (2.23), and where

$$\Psi_F := \Psi\,(F^{-1} \otimes \mathbf{1}_\mathcal{M} \otimes F^{-1}) \tag{2.53d}$$

We summarize the connection between two-sided coactions and quasi–commuting pairs of coactions in the following Proposition.

PROPOSITION 2.8. *Twist–equivalence classes of quasi–commuting pairs of coactions* $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ *are in one-to-one correspondence with twist equivalence classes of two–sided coactions* (δ, Ψ). *In particular the elements* $U_{l/r}$ *defined in* (2.47)/(2.48) *provide a twist equivalence between* (δ, Ψ) *and* $(\delta_{l/r}, \Psi_{l/r})$ *given by* (2.43)-(2.46), (2.49) *and* (2.52a)-(2.52d).

The proof of Proposition 2.8, especially the detailed calculations that all pentagon equations are satisfied, is elementary but quite lengthy and is given in [HNa].

The importance of Proposition 2.8 stems from the fact that below the diagonal crossed products associated with twist–equivalent two–sided coactions will be shown to be isomorphic.

2.4. The algebras $\hat{\mathcal{G}} \bowtie \mathcal{M}$ and $\mathcal{M} \bowtie \hat{\mathcal{G}}$

Having developed our theory of two–sided $\mathcal{G}$–coactions δ for quasi–bialgebras $\mathcal{G}$ we are now in the position to generalize the construction of the left and right diagonal crossed products $\hat{\mathcal{G}} \bowtie \mathcal{M}_\delta$ and $\mathcal{M}_\delta \bowtie \hat{\mathcal{G}}$ to the quasi–coassociative setting. Before writing down the concrete multiplication rules we would like to draw the reader's attention to some important conceptual differences in comparison with the results of Chapter 1.

As already remarked, the natural "multiplication" $\hat{\mu} : \hat{\mathcal{G}} \otimes \hat{\mathcal{G}} \to \hat{\mathcal{G}}$ given as the transpose of the coproduct $\Delta : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ is *not* associative. Nevertheless, we will still write $\varphi\psi := \hat{\mu}(\varphi \otimes \psi)$, $\varphi, \psi \in \hat{\mathcal{G}}$, for details see the end of Section 2.1. This will imply the fact that although we will have $\hat{\mathcal{G}} \bowtie \mathcal{M}_\delta = \hat{\mathcal{G}} \otimes \mathcal{M}$ and $\mathcal{M}_\delta \bowtie \hat{\mathcal{G}} = \mathcal{M} \otimes \hat{\mathcal{G}}$ as linear spaces, the subspaces $\hat{\mathcal{G}} \otimes \mathbf{1}_\mathcal{M}$ and $\mathbf{1}_\mathcal{M} \otimes \hat{\mathcal{G}}$ will *not* be subalgebras in the diagonal crossed product. On the other hand, $\mathcal{M}$ will naturally be embedded as the unital subalgebra $\mathcal{M} \cong \hat{\mathbf{1}} \otimes \mathcal{M} \cong \mathcal{M} \otimes \hat{\mathbf{1}}$. We would also like to stress that $\hat{\mathcal{G}} \bowtie \mathcal{M}_\delta \cong \mathcal{M}_\delta \bowtie \hat{\mathcal{G}}$ will still be equivalent algebra extensions of $\mathcal{M}$. However the subspaces $\hat{\mathcal{G}} \bowtie \mathbf{1}_\mathcal{M}$ and $\mathbf{1}_\mathcal{M} \bowtie \hat{\mathcal{G}}$ will *not* be mapped onto each other under this isomorphism. (Recall that this was the case in (1.30)).

We now proceed to the details. Given a two–sided $\mathcal{G}$–coaction (δ, Ψ) on $\mathcal{M}$ we still write as before

$$\varphi \triangleright m \triangleleft \psi := (\psi \otimes \mathrm{id}_\mathcal{M} \otimes \varphi)(\delta(m)), \quad m \in \mathcal{M},\ \varphi, \psi \in \hat{\mathcal{G}}, \tag{2.54}$$

disregarding the fact that δ might be neither of the form (2.52a) nor (2.52c). We also introduce the element $\Omega_L \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G} \otimes \mathcal{G}$ built from the reassociator Ψ by

$$\Omega_L \equiv \Omega_L^1 \otimes \Omega_L^2 \otimes \Omega_L^3 \otimes \Omega_L^4 \otimes \Omega_L^5 := (\mathrm{id}_\mathcal{G} \otimes \mathrm{id}_\mathcal{G} \otimes \mathrm{id}_\mathcal{M} \otimes S^{-1} \otimes S^{-1})(\Psi^{-1}) \cdot h^{54}, \tag{2.55}$$

where $h \equiv (S^{-1} \otimes S^{-1})(f^{21}) \in \mathcal{G} \otimes \mathcal{G}$ has been introduced in (2.31). As before, we have dropped all summation symbols and summation indices.

DEFINITION 2.9. Let (δ, Ψ) be a two–sided coaction of a quasi–Hopf algebra $\mathcal{G}$ on an algebra $\mathcal{M}$. We define the *left diagonal crossed product* $\hat{\mathcal{G}} \bowtie \mathcal{M}_\delta$ to be the vector space $\hat{\mathcal{G}} \otimes \mathcal{M}$ equipped with the multiplication rule

$$(\varphi \bowtie m)(\psi \bowtie n) := \Big[(\Omega_L^1 \rightharpoonup \varphi \leftharpoonup \Omega_L^5)(\Omega_L^2 \rightharpoonup \psi_{(2)} \leftharpoonup \Omega_L^4)\Big] \bowtie \Big[\Omega_L^3(\hat{S}^{-1}(\psi_{(1)}) \triangleright m \triangleleft \psi_{(3)})\, n\Big] \tag{2.56}$$

In cases where the two–sided coaction is unambiguously understood from the context we also write $\hat{\mathcal{G}} \bowtie \mathcal{M}$. Note that (2.56) again implies

$$(\varphi \bowtie m) = (\varphi \bowtie \mathbf{1}_\mathcal{M})(\hat{\mathbf{1}} \bowtie m)\,. \tag{2.57}$$

and for $\Omega_{L/R} = \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{M} \otimes \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G}$ we recover the definition of Chapter 1. Also, in $\hat{\mathcal{G}} \bowtie \mathcal{M}$ we still have the "commutation relation"

$$m\,\psi = \psi_{(2)}\, \big(\hat{S}^{-1}(\psi_{(1)}) \triangleright m \triangleleft \psi_{(3)}\big).$$

But for the product $(\varphi \bowtie \mathbf{1}_\mathcal{M})(\psi \bowtie \mathbf{1}_\mathcal{M})$ to be consistent with this relation one has to inject the reassociator Ψ since $\delta^{(2)} \neq (\Delta \otimes \mathrm{id} \otimes \Delta) \circ \delta$ and also the twist h since $\Delta \circ S^{-1} \neq (S^{-1} \otimes S^{-1}) \circ \Delta^{op}$ in the quasi–coassociative case. We now formulate our first main result.

THEOREM 2.10.

(i) *The left diagonal crossed product $\hat{\mathcal{G}} \bowtie \mathcal{M}$ is an associative algebra with unit $\hat{\mathbf{1}} \bowtie \mathbf{1}_\mathcal{M}$.*
(ii) *$\mathcal{M} \equiv \hat{\mathbf{1}} \bowtie \mathcal{M} \subset \hat{\mathcal{G}} \bowtie \mathcal{M}$ is a unital algebra inclusions.*

We will give a detailed proof of Theorem 2.10 in Section 2.6. Let us shortly sketch the idea. Let $\mathbf{L} \in \mathcal{G} \otimes (\hat{\mathcal{G}} \bowtie \mathcal{M})$ be given by $\mathbf{L} = e_\mu \otimes (e^\mu \bowtie \mathbf{1}_\mathcal{M})$, where $\{e_\mu\}$ is a basis in $\mathcal{G}$ with dual basis $\{e^\mu\}$ in $\hat{\mathcal{G}}$. We also abbreviate our notation by identifying $m \equiv (\hat{\mathbf{1}} \bowtie m)$, $m \in \mathcal{M}$. The multiplication rule (2.56) implies

$$[\mathbf{1}_\mathcal{G} \otimes m]\,\mathbf{L} = [S^{-1}(m_{(1)}) \otimes \mathbf{1}_\mathcal{M}]\,\mathbf{L}\,[m_{(-1)} \otimes m_{(0)}], \quad \forall m \in \mathcal{M} \tag{2.58}$$

$$\mathbf{L}^{13}\mathbf{L}^{23} = [S^{-1}(\bar{\Psi}^5) \otimes S^{-1}(\bar{\Psi}^4) \otimes \mathbf{1}_\mathcal{M}]\,[(\Delta_L \otimes \mathrm{id})(\mathbf{L})]\,[\bar{\Psi}^1 \otimes \bar{\Psi}^2 \otimes \bar{\Psi}^3] \tag{2.59}$$

where we have introduced the notation $\delta(m) = m_{(-1)} \otimes m_{(0)} \otimes m_{(1)}$ and $\Psi^{-1} \equiv \bar{\Psi} = \bar{\Psi}^1 \otimes \bar{\Psi}^2 \otimes \bar{\Psi}^3 \otimes \bar{\Psi}^4 \otimes \bar{\Psi}^5$, and where $\Delta_L(a) := h\Delta(a)$, $a \in \mathcal{G}$, has been introduced in Corollary 2.2. With these relations the nontrivial associativity constraints to be shown are the following

$$\mathbf{L}^{14}(\mathbf{L}^{24}\mathbf{L}^{34}) \stackrel{!}{=} (\mathbf{L}^{14}\mathbf{L}^{24})\mathbf{L}^{34} \tag{2.60}$$

$$[\mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G} \otimes m](\mathbf{L}^{13}\mathbf{L}^{23}) \stackrel{!}{=} ([\mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G} \otimes m]\mathbf{L}^{13})\mathbf{L}^{23}, \tag{2.61}$$

where (2.60) is understood as an identity in $\mathcal{G}^{\otimes^3} \otimes (\hat{\mathcal{G}} \bowtie \mathcal{M})$ and (2.61) as an identity in $\mathcal{G}^{\otimes^2} \otimes (\hat{\mathcal{G}} \bowtie \mathcal{M})$. Now the identity (2.60) is shown by using the pentagon equation (2.39b) for Ψ, whereas (2.61) is implied by the intertwining properties (2.39a) of Ψ and (2.31) of h. The details are given in Section 2.6.

Before proceeding let us shortly discuss how in the present context one can see that ordinary crossed products $\hat{\mathcal{G}} \ltimes {}_\lambda\mathcal{M}$ in general cannot be defined as associative algebras any more. In the strictly coassociative setting of Chapter 1 these could be considered as special types of diagonal crossed products, where $\delta = \lambda \otimes \mathbf{1}_\mathcal{G}$. In the present setting it is not clear whether such δ's give well defined two–sided coactions, since in fact the map

$$\rho_0(m) := m \otimes \mathbf{1}_\mathcal{G}$$

need not even be a one–sided coaction. For this one would also need the existence of a reassociator ϕ_{ρ_0} satisfying the axioms of Definition 2.3 (note that the choice $\phi_{\rho_0} = \mathbf{1}_\mathcal{M} \otimes \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G}$ will in general not do the job due to the appearance of ϕ in the pentagon equation (2.36b)).

We conclude this section by giving the analog construction of a right diagonal crossed product $\mathcal{M} \bowtie \hat{\mathcal{G}}$, which in fact will be proven to be isomorphic to $\mathcal{M} \bowtie \hat{\mathcal{G}}$ in the next section.

DEFINITION 2.11. Given a two–sided coaction (δ, Ψ) of $\mathcal{G}$ on $\mathcal{M}$, and setting

$$\Omega_R \equiv \Omega_R^1 \otimes \Omega_R^2 \otimes \Omega_R^3 \otimes \Omega_R^4 \otimes \Omega_R^5 := (h^{-1})^{21} \cdot (S^{-1} \otimes S^{-1} \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}})(\Psi), \tag{2.62}$$

we define the *right diagonal crossed product* $\mathcal{M}_\delta \bowtie \hat{\mathcal{G}}$ to be the vector space $\mathcal{M} \otimes \hat{\mathcal{G}}$ with the multiplication rule

$$(m \bowtie \varphi)(n \bowtie \psi) := \Big[m\,(\varphi_{(1)} \triangleright n \triangleleft \hat{S}^{-1}(\varphi_{(3)}))\,\Omega_R^3\Big] \bowtie \Big[(\Omega_R^2 \rightharpoonup \varphi_{(2)} \leftharpoonup \Omega_R^4)(\Omega_R^1 \rightharpoonup \psi \leftharpoonup \Omega_R^5)\Big], \tag{2.63}$$

COROLLARY 2.12. *The right diagonal crossed product $\mathcal{M}_\delta \bowtie \hat{\mathcal{G}}$ is an associative algebra with unit $\mathbf{1}_{\mathcal{M}} \bowtie \hat{\mathbf{1}}$, containing $\mathcal{M} \equiv \mathcal{M} \bowtie \hat{\mathbf{1}} \subset \mathcal{M} \bowtie \hat{\mathcal{G}}$ as a unital subalgebra.*

PROOF. The proof goes along the same lines as the proof of Theorem 2.10 by noting that under the trivial permutation of tensor factors we have

$$(\mathcal{M}_\delta \bowtie \hat{\mathcal{G}})_{op} = \hat{\mathcal{G}}_{op}^{cop} \bowtie (\mathcal{M}_{op})_{\delta_{op}}$$

where $(\mathcal{M}_\delta \bowtie \hat{\mathcal{G}})_{op}$ denotes the diagonal crossed product with opposite multiplication, and where we recall our remark that with the definition (2.40) the pair $(\delta_{op}, \Psi_{op}^{-1})$ defines a two–sided coaction of $\mathcal{G}_{op}^{cop}$ on $\mathcal{M}_{op}$. ☐

2.5. Generating matrices

We now pass to a formulation of diagonal crossed products in terms of generating matrices similarly as in Section 1.3. As discussed in Section 2.3 the connection between two–sided coactions δ and (quasi–commuting) pairs of coactions (λ, ρ) becomes more involved in the quasi–coassociative setting. This will make it necessary to distinguish between $\lambda\rho$–intertwiners and what we call left and right δ–implementers, which all three coincide in the coassociative setting of Chapter 1 due to Lemma 1.11. The precise relation between these different generating matrices will be clarified in Proposition 2.19, which finally leads to a proof of the main Theorem 2.1. We would like to encourage the reader to frequently glance at Appendix A, where he will find a representation theoretic interpretation of the generating matrices together with their relationships expressed in terms of commuting diagrams.

2.5.1. Left and right diagonal δ–implementers. From the associativity proof of Theorem 2.10 in terms of the "generating matrix" $\mathbf{L}$ we immediately read off an analogue of Proposition 1.12 describing the conditions under which an algebra map $\gamma : \mathcal{M} \to \mathcal{A}$ into some target algebra $\mathcal{A}$ extends to an algebra map from the diagonal crossed products into $\mathcal{A}$. In view of (2.58) and (2.59) we are lead to the following

DEFINITION 2.13. Let $\gamma : \mathcal{M} \to \mathcal{A}$ be an algebra map into some target algebra $\mathcal{A}$ and let (δ, Ψ) be a two–sided $\mathcal{G}$–coaction on $\mathcal{M}$. A *left (right) diagonal δ– implementer* in $\mathcal{A}$ (with respect to γ) is an element $\mathbf{L} \in \mathcal{G} \otimes \mathcal{A}$ ($\mathbf{R} \in \mathcal{G} \otimes \mathcal{A}$) satisfying for all $m \in \mathcal{M}$, respectively,

$$[\mathbf{1}_{\mathcal{G}} \otimes \gamma(m)]\,\mathbf{L} = [S^{-1}(m_{(1)}) \otimes \mathbf{1}_{\mathcal{A}}]\,\mathbf{L}\,[m_{(-1)} \otimes \gamma(m_{(0)})] \tag{2.64}$$

$$\mathbf{R}\,[\mathbf{1}_{\mathcal{G}} \otimes \gamma(m)] = [m_{(1)} \otimes \gamma(m_{(0)})]\,\mathbf{R}\,[S^{-1}(m_{(-1)}) \otimes \mathbf{1}_{\mathcal{A}}] \tag{2.65}$$

A left δ–implementer $\mathbf{L}$ (right δ–implementer $\mathbf{R}$) is called *coherent* if, respectively,

$$\mathbf{L}^{13}\,\mathbf{L}^{23} = [\Omega_L^5 \otimes \Omega_L^4 \otimes \mathbf{1}_{\mathcal{A}}]\,(\Delta \otimes \mathrm{id})(\mathbf{L})\,[\Omega_L^1 \otimes \Omega_L^2 \otimes \gamma(\Omega_L^3)] \tag{2.66}$$

$$\mathbf{R}^{13}\,\mathbf{R}^{23} = [\Omega_R^4 \otimes \Omega_R^5 \otimes \gamma(\Omega_R^3)]\,(\Delta \otimes \mathrm{id})(\mathbf{R})\,[\Omega_R^2 \otimes \Omega_R^1 \otimes \mathbf{1}_{\mathcal{A}})], \tag{2.67}$$

where $\Omega_{L/R}$ have been defined in (2.55)/(2.62).

To unburden our terminology from now by a left (right) δ–implementer we will always mean a left (right) diagonal δ–implementer in the sense of the above definition. We trust that the reader will not be confused by this slight inconsistence of terminology (which arises in comparison with Definition 1.2, since two–sided coactions might also be looked upon as one–sided ones).

As before, we also call $\mathbf{L}/\mathbf{R}$ *normal*, if $(\epsilon \otimes \mathrm{id})(\mathbf{L}/\mathbf{R}) = \mathbf{1}_{\mathcal{A}}$. Note that in the coassociative setting of Lemma 1.11 left and right δ–implementers always coincide. In the present context we will still have one–to–one correspondences between left and right δ–implementers, however the identifications will not be the trivial ones.

Let us note the immediate

Corollary 2.14. *Let $(\mathcal{M}, \delta, \Psi)$ be a two–sided $\mathcal{G}$–comodule algebra and let $\gamma : \mathcal{M} \to \mathcal{A}$ be an algebra map into some target algebra $\mathcal{A}$. Then the relations*

$$\gamma_L(\varphi \bowtie m) = (\varphi \otimes \mathrm{id})(\mathbf{L})\, \gamma(m) \tag{2.68}$$

$$\gamma_R(m \bowtie \varphi) = \gamma(m)\, (\varphi \otimes \mathrm{id})(\mathbf{R}) \tag{2.69}$$

provide one–to–one correspondences between algebra maps $\gamma_L : \hat{\mathcal{G}} \bowtie \mathcal{M} \to \mathcal{A}$ ($\gamma_R : M \bowtie \hat{\mathcal{G}} \to \mathcal{A}$) extending γ and coherent left δ–implementers $\mathbf{L}$ (coherent right δ–implementers $\mathbf{R}$), respectively, where γ_L/γ_R is unital if and only if $\mathbf{L}/\mathbf{R}$ is normal.

Proof. This follows immediately from (2.58) and (2.59) and the analogue relations in $\mathcal{M}_\delta \bowtie \hat{\mathcal{G}}$ (Define $\mathbf{L} := \sum_\mu e_\mu \otimes \gamma_L(e^\mu)$, whith $\{e_\mu\} \subset \mathcal{G}$, $\{e^\mu\} \subset \hat{\mathcal{G}}$ being a pair of dual bases). □

Next, we show that the diagonal crossed products associated with twist equivalent two–sided coactions are equivalent algebra extensions.

Proposition 2.15.

1. *Let (δ, Ψ) and (δ', Ψ') be twist equivalent two–sided $\mathcal{G}$–coactions on $\mathcal{M}$. Then the diagonal crossed products $\mathcal{M}_\delta \bowtie \hat{\mathcal{G}}$ and $\mathcal{M}_{\delta'} \bowtie \hat{\mathcal{G}}$ are equivalent extensions of $\mathcal{M}$.*
2. *Let (δ, Ψ) be a two–sided $\mathcal{G}$–coaction on $\mathcal{M}$ with respect to the coproduct $\Delta : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$, and let (δ, Ψ_F) be the two–sided coaction with respect to a twist equivalent coproduct Δ_F on $\mathcal{G}$, see* (2.53d)*. Denote the associated diagonal crossed products by $\mathcal{M} \bowtie \hat{\mathcal{G}}$ and $\mathcal{M} \bowtie \hat{\mathcal{G}}_F$, respectively. Then $\mathcal{M} \bowtie \hat{\mathcal{G}} = \mathcal{M} \bowtie \hat{\mathcal{G}}_F$ with trivial identification.*

Proof. 1. Let $U \in \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ be a normal twist transformation from (δ, Ψ) to (δ', Ψ') and let $\mathbf{R} \in \mathcal{G} \otimes (\mathcal{M}_\delta \bowtie \hat{\mathcal{G}})$ and $\mathbf{R}' \in \mathcal{G} \otimes (\mathcal{M}_{\delta'} \bowtie \hat{\mathcal{G}})$ be the generating matrices. By Corollary 2.14, to provide a homomorphism

$$f : \mathcal{M}_\delta \bowtie \hat{\mathcal{G}} \to \mathcal{M}_{\delta'} \bowtie \hat{\mathcal{G}}$$

restricting to the identity on $\mathcal{M}$ we have to find a coherent normal right δ–implementer $\tilde{\mathbf{R}} \in \mathcal{G} \otimes (\mathcal{M}_{\delta'} \bowtie \hat{\mathcal{G}})$. We claim that the canonical choice (writing $U^{-1} \equiv \bar{U}^1 \otimes \bar{U}^2 \otimes \bar{U}^3$, where summation symbols are suppressed)

$$\tilde{\mathbf{R}} := [\bar{U}^3 \otimes \gamma(\bar{U}^2)]\, \mathbf{R}'\, [S^{-1}(\bar{U}^1) \otimes \mathbf{1}_\mathcal{A}] \tag{2.70}$$

will do the job. Indeed, $\tilde{\mathbf{R}}$ obviously is a normal right δ–implementer and one is left with checking the coherence condition with respect to (δ, Ψ). Using (2.53b) this is straight forward and is left to the reader.

To prove part 2. we note that $\Psi_F = \Psi\, (F^{-1} \otimes \mathbf{1}_\mathcal{M} \otimes F^{-1})$ implies by (2.62) $(\Omega_R)_F = F^{21}\, \Omega_R\, (F^{-1})^{45}$ since the element $h \in \mathcal{G} \otimes \mathcal{G}$ transforms under a twist according to $h_F = (S^{-1} \otimes S^{-1})(F_{op}^{-1})\, h\, F^{-1}$. Hence, by Definition 2.13, $\mathbf{R}$ is coherent with respect to (δ, Ψ, Δ) if and only if it is coherent with respect to $(\delta, \Psi_F, \Delta_F)$. □

Of course, analogous statements hold for the left diagonal crossed products.

Remark 2.16. In view of Proposition 2.8 we may from now on restrict ourselves to two–sided coactions of the form $(\delta, \Psi) = (\delta_{l/r}, \Psi_{l/r})$ for a quasi–commuting pair $(\lambda, \phi_\lambda, \rho, \phi_\rho, \phi_{\lambda\rho})$, where $\delta_l = (\lambda \otimes \mathrm{id}) \circ \rho$ and $\delta_r = (\mathrm{id} \otimes \rho) \circ \lambda$, see (2.52a - 2.52d). In this light it will also be appropriate to introduce as an alternative notation consistent with (2.6),(2.7)

$$\hat{\mathcal{G}} \bowtie {}_\lambda\mathcal{M}_\rho := \hat{\mathcal{G}} \bowtie \mathcal{M}_{\delta_l} \tag{2.71}$$

$${}_\lambda\mathcal{M}_\rho \bowtie \hat{\mathcal{G}} := \mathcal{M}_{\delta_r} \bowtie \hat{\mathcal{G}} \tag{2.72}$$

By Proposition 2.15(1.) we also have $\mathcal{M}_{\delta_r} \bowtie \hat{\mathcal{G}} \cong \mathcal{M}_{\delta_l} \bowtie \hat{\mathcal{G}}$ and $\hat{\mathcal{G}} \bowtie \mathcal{M}_{\delta_l} \cong \hat{\mathcal{G}} \bowtie \mathcal{M}_{\delta_r}$ since δ_l and δ_r are twist equivalent. As will be shown below, also $\hat{\mathcal{G}} \bowtie \mathcal{M}_{\delta_l}$ and $\mathcal{M}_{\delta_r} \bowtie \hat{\mathcal{G}}$ are equivalent extensions of $\mathcal{M}$. Thus we get four equivalent versions of diagonal crossed products associated with any quasi–commuting pair $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ of $\mathcal{G}$–coactions on $\mathcal{M}$, all of which will be shown to be a realization of the abstract algebra $\mathcal{M}_1$ in Theorem 2.1.

2.5.2. Coherent $\lambda\rho$–intertwiners. In this subsection we are going to generalize Lemma 1.11 by providing a normality and coherence preserving one–to–one correspondence between right δ_r–implementers $\mathbf{R}$ or left δ_l–implementers $\mathbf{L}$, respectively, and $\lambda\rho$–intertwiners $\mathbf{T}$, where $\delta_r := (\mathrm{id}\otimes\rho)\circ\lambda$ and $\delta_l := (\lambda\otimes\mathrm{id})\circ\rho$. This will finally lead to a proof of Theorem 2.1. As a Corollary we get that the left and right diagonal crossed products $\hat{\mathcal{G}}\bowtie\mathcal{M}_\delta$ and $\mathcal{M}_\delta\bowtie\hat{\mathcal{G}}$ are equivalent algebra extensions of $\mathcal{M}$. We start with a generalization of Definition 1.10

DEFINITION 2.17. Let $(\lambda,\phi_\lambda,\rho,\phi_\rho,\phi_{\lambda\rho})$ be a quasi–commuting pair of $\mathcal{G}$–coactions on $\mathcal{M}$ and let $\gamma:\mathcal{M}\otimes\mathcal{A}$ be a unital algebra map into some target algebra $\mathcal{A}$. A *$\lambda\rho$–intertwiner* in $\mathcal{A}$ (with respect to γ) is an element $\mathbf{T}\in\mathcal{G}\otimes\mathcal{A}$ satisfying

$$\mathbf{T}\,\lambda_{\mathcal{A}}(m)=\rho^{op}_{\mathcal{A}}(m)\,\mathbf{T},\quad \forall m\in\mathcal{A} \tag{2.73}$$

A $\lambda\rho$–intertwiner is called *normal* if $(\epsilon\otimes\mathrm{id})(\mathbf{T})=\mathbf{1}_{\mathcal{A}}$ and it is called *coherent*, if

$$(\phi_\rho^{312})_{\mathcal{A}}\,\mathbf{T}^{13}\,(\phi_{\lambda\rho}^{-1})^{132}_{\mathcal{A}}\,\mathbf{T}^{23}\,(\phi_\lambda)_{\mathcal{A}}=(\Delta\otimes\mathrm{id})(\mathbf{T}) \tag{2.74}$$

where the index $\mathcal{A}$ refers to the image $\gamma(\mathcal{M})\subset\mathcal{A}$, see also (2.2) and (2.3).

We first point out that (2.74) is consistent with (2.73) in the following sense

LEMMA 2.18. *Under the conditions of Definition 2.17 let $\mathbf{T}$ be a $\lambda\rho$–intertwiner in $\mathcal{A}$ and define $B\in\mathcal{G}\otimes\mathcal{G}\otimes\mathcal{A}$ by*

$$B:=(\phi_\rho^{312})_{\mathcal{A}}\,\mathbf{T}^{13}\,(\phi_{\lambda\rho}^{-1})^{132}_{\mathcal{A}}\,\mathbf{T}^{23}\,(\phi_\lambda)_{\mathcal{A}} \tag{2.75}$$

Then we have for all $m\in\mathcal{M}$

$$B\,(\Delta\otimes\mathrm{id}_{\mathcal{A}})\big(\lambda(m)_{\mathcal{A}}\big)=(\Delta\otimes\mathrm{id}_{\mathcal{A}})\big(\rho^{op}(m)_{\mathcal{A}}\big)\,B \tag{2.76}$$

PROOF. This is straightforward from the intertwiner properties of $\mathbf{T}$ and of $\phi_\lambda,\phi_{\lambda\rho}$ and ϕ_ρ, see (2.73), (2.35a), (2.36a), and (2.51a). □

To provide a bijective map between coherent δ–implementers and coherent $\lambda\rho$–intertwiners we first need a generalization of formulas like $m_{(0)}\otimes S^{-1}(m_{(2)})m_{(1)}=m\otimes\mathbf{1}_{\mathcal{G}}$, which are not valid any more due to quasi–coassociativity and the more complicated antipode axioms (2.15/2.16). Recall that formulas of this type have been used to prove Lemma 1.11. Associated with any left $\mathcal{G}$–coaction (λ,ϕ_λ) on $\mathcal{M}$ we define elements $p_\lambda,q_\lambda\in\mathcal{G}\otimes\mathcal{M}$ by

$$p_\lambda:=\phi_\lambda^2S^{-1}(\phi_\lambda^1\beta)\otimes\phi_\lambda^3,\quad\text{where}\quad\phi_\lambda=\phi_\lambda^1\otimes\phi_\lambda^2\otimes\phi_\lambda^3, \tag{2.77}$$

$$q_\lambda:=S(\bar\phi_\lambda^1)\alpha\bar\phi_\lambda^2\otimes\bar\phi_\lambda^3,\quad\text{where}\quad\phi_\lambda^{-1}=\bar\phi_\lambda^1\otimes\bar\phi_\lambda^2\otimes\bar\phi_\lambda^3, \tag{2.78}$$

and where as before we have dropped summation indices and summation symbols. Here $\alpha,\beta\in\mathcal{G}$ are the elements introduced in (2.15). In the case $\mathcal{M}=\mathcal{G}$ and $(\lambda,\phi_\lambda)=(\rho,\phi_\rho)=(\Delta,\phi)$ analogues of these elements have also been considered by [Dri90], [Sch95]. Denoting $\lambda(m)\equiv m_{(-1)}\otimes m_{(0)}$ they satisfy

$$\begin{aligned}\lambda(m_{(0)})\,p_\lambda\,[S^{-1}(m_{(-1)})\otimes\mathbf{1}_{\mathcal{M}}]&=p_\lambda\,[\mathbf{1}_{\mathcal{G}}\otimes m]\\ [S(m_{(-1)})\otimes\mathbf{1}_{\mathcal{M}}]\,q_\lambda\,\lambda(m_{(0)})&=[\mathbf{1}_{\mathcal{G}}\otimes m]\,q_\lambda\\ \lambda(q_\lambda^2)\,p_\lambda\,[S^{-1}(q_\lambda^1)\otimes\mathbf{1}_{\mathcal{M}}]&=\mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{M}}\\ [S(p_\lambda^1)\otimes\mathbf{1}_{\mathcal{M}}]\,q_\lambda\,\lambda(p_\lambda^2)&=\mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{M}}.\end{aligned}$$

Note that the first two equalities provide a substitute for the non–valid formula $m_{(0)}\otimes S^{-1}(m_{(2)})m_{(1)}=m\otimes\mathbf{1}_{\mathcal{G}}$, whereas the second pair state some kind of invertibility property of the elements p_λ,q_λ.

Similarly, associated with any right $\mathcal{G}$–coaction (ρ,ϕ_ρ) on $\mathcal{M}$ we define elements $p_\rho,q_\rho\in\mathcal{M}\otimes\mathcal{G}$ by

$$p_\rho:=\bar\phi_\rho^1\otimes\bar\phi_\rho^2\beta S(\bar\phi_\rho^3),\quad\text{where}\quad\phi_\rho^{-1}=\bar\phi_\rho^1\otimes\bar\phi_\rho^2\otimes\bar\phi_\rho^3 \tag{2.79}$$

$$q_\rho:=\phi_\rho^1\otimes S^{-1}(\alpha\phi_\rho^3)\phi_\rho^2,\quad\text{where}\quad\phi_\rho=\phi_\rho^1\otimes\phi_\rho^2\otimes\phi_\rho^3 \tag{2.80}$$

They obey a similar set of equations. All these equalities together with some kind of "coherence" property are proven below in Lemma 2.21 and Lemma 2.22 in Section 2.6. Again the reader may find it helpful to consult the representation theoretic interpretation of the elements $q_\lambda,p_\lambda,q_\rho,p_\rho$ given in Appendix A, starting with (A.8).

We now state the generalization of Lemma 1.11. Throughout, by a convenient abuse of notation, we are going to omit the symbol γ.

PROPOSITION 2.19. *Under the conditions of Definition 2.17*

1. *Let* $\delta_r := (\mathrm{id}\otimes\rho)\circ\lambda$ *and* $\Psi_r \in \mathcal{G}\otimes\mathcal{G}\otimes\mathcal{M}\otimes\mathcal{G}\otimes\mathcal{G}$ *as in* (2.52d) *and let* $p_\lambda, q_\lambda \in \mathcal{G}\otimes\mathcal{M}$ *be given by* (2.77),(2.78). *Then the assignments (omitting the symbol* γ*)*
$$\mathbf{T} \longmapsto \mathbf{R} := \mathbf{T}\, p_\lambda \tag{2.81}$$
$$\mathbf{R} \longmapsto \mathbf{T} := \rho^{op}(q_\lambda^2)\,\mathbf{R}\,[S^{-1}(q_\lambda^1)\otimes \mathbf{1}_{\mathcal{A}}] \tag{2.82}$$
provide mutually inverse normality and coherence preserving isomorphisms between the space of $\lambda\rho$*–intertwiners and the space of right* δ_r*–implementers.*
2. *Similarly let* $\delta_l := (\lambda\otimes\mathrm{id})\circ\rho$ *and* $\Psi_l \in \mathcal{G}\otimes\mathcal{G}\otimes\mathcal{M}\otimes\mathcal{G}\otimes\mathcal{G}$ *as in* (2.52b), *and let* $p_\rho, q_\rho \in \mathcal{M}\otimes\mathcal{G}$ *be given by* (2.79),(2.80). *Then the assignments*
$$\mathbf{T} \longmapsto \mathbf{L} := q_\rho^{op}\,\mathbf{T} \tag{2.83}$$
$$\mathbf{L} \longmapsto \mathbf{T} := [S^{-1}(p_\rho^2)\otimes \mathbf{1}_{\mathcal{A}}]\,\mathbf{L}\,\lambda(p_\rho^1) \tag{2.84}$$
provide mutually inverse normality and coherence preserving isomorphisms between the space of $\lambda\rho$*–intertwiners and the space of left* δ_l*–implementers.*

The proof of Proposition 2.19 is given in Section 2.6. The content of the above Proposition may also be expressed in terms of commuting diagrams, see (A.17) in Appendix A. We state the immediate

COROLLARY 2.20. *The left and the right diagonal crossed products* $\hat{\mathcal{G}} \bowtie \mathcal{M}_\delta$ *and* $\mathcal{M}_\delta \bowtie \hat{\mathcal{G}}$ *defined in Def. 2.9 and Def. 2.11 are isomorphic algebra extensions of* $\mathcal{M}$.

PROOF. First note that by Remark 2.16 we have $\hat{\mathcal{G}} \bowtie \mathcal{M}_\delta \cong \hat{\mathcal{G}} \bowtie \mathcal{M}_{\delta_l}$ and $\mathcal{M}_\delta \bowtie \hat{\mathcal{G}} \cong \mathcal{M}_{\delta_r} \bowtie \hat{\mathcal{G}}$. Now let $\mathbf{R}_{\delta_r} := e_\mu \otimes (\mathbf{1}_{\mathcal{M}} \bowtie e^\mu)$ be the coherent δ_r–implementer in $\mathcal{M}_{\delta_r} \bowtie \hat{\mathcal{G}}$. By (2.82) and (2.83)
$$\tilde{\mathbf{L}}_{\delta_l} := q_\rho^{op}\,\rho(q_\lambda^2)\,\mathbf{R}_{\delta_r}\,[S^{-1}(q_\lambda^1)\otimes \mathbf{1}_{\mathcal{A}}]$$
is a coherent δ_l–implementer. Thus, by Corollary 2.14 we get an algebra map $f : \hat{\mathcal{G}} \bowtie \mathcal{M}_{\delta_l} \to \mathcal{M}_{\delta_r} \bowtie \hat{\mathcal{G}}$ by setting
$$f(\varphi \bowtie m) := (\varphi\otimes\mathrm{id})(\tilde{\mathbf{L}}_{\delta_l})\,(m \bowtie \hat{\mathbf{1}}) \tag{2.85}$$
Using (2.81) and (2.84) one shows analogously that f is invertible. □

We are finally in the position to proof the main Theorem 2.1 stated in the introduction to Chapter 2.

PROOF OF THEOREM 2.1. To prove the existence of $\mathcal{M}_1$ we choose $\mathcal{M}_1 := \mathcal{M}_{\delta_r} \bowtie \hat{\mathcal{G}}$ and $\Gamma : \hat{\mathcal{G}} \to \mathcal{M}_1$,
$$\Gamma(\varphi) = (\varphi\otimes\mathrm{id}_{\mathcal{M}_1})(\mathbf{\Gamma}), \quad \mathbf{\Gamma} = \rho^{op}(q_\lambda^2)\,\mathbf{R}_{\delta_r}\,[S^{-1}(q_\lambda^1)\otimes\mathbf{1}_{\mathcal{A}}], \tag{2.86}$$
where $q_\lambda \in \mathcal{G}\otimes\mathcal{M}$ is given by (2.78) and where $\mathbf{R}_{\delta_r} := e_\mu \otimes (\mathbf{1}_{\mathcal{M}} \bowtie e^\mu) \in \mathcal{G}\otimes\mathcal{M}_1$ is the canonical coherent normal δ_r–implementer. Hence, by Proposition 2.19, $\mathbf{\Gamma}$ is a normal coherent $\lambda\rho$–intertwiner in $\mathcal{M}_1$. Moreover, we obtain for the map μ_R in (2.5)
$$\begin{aligned}\mu_R(m\otimes\varphi) :&= (m \bowtie \hat{\mathbf{1}})\,\Gamma(\varphi_{(1)})\,(\varphi_{(2)}\otimes\mathrm{id})(p_\lambda)\\ &= (m \bowtie \hat{\mathbf{1}})\,(\varphi\otimes\mathrm{id})(\mathbf{\Gamma}\,p_\lambda)\\ &= (m \bowtie \hat{\mathbf{1}})\,(\mathbf{1}_{\mathcal{M}} \bowtie \varphi)\\ &\equiv (m \bowtie \varphi)\end{aligned} \tag{2.87}$$
by Proposition 2.19 1.), and therefore $\mu_R : \mathcal{M}\otimes\hat{\mathcal{G}} \to \mathcal{M}_1$ becomes the identity map. This also shows that $\mathcal{M}_1$ is algebraically generated by $\mathcal{M} \equiv (\mathcal{M} \bowtie \hat{\mathbf{1}})$ and $\Gamma(\hat{\mathcal{G}})$.

The universality property follows from Corollary 2.14 - providing a one-to-one correspondence between algebra extensions $\mathcal{M}_1 \to \mathcal{A}$ and δ_r–implementers - and Proposition 2.19 - providing a one-to-one correspondence between δ_r-implementers $\mathbf{R}$ and $\lambda\rho$-intertwiners $\mathbf{T}$.

The uniqueness of $\mathcal{M}_1$ (up to equivalence) follows by standard arguments from the universality property stated in part 1 and the fact that $\mathcal{M}_1$ is generated by $\mathcal{M}$ and $\Gamma(\hat{\mathcal{G}})$.

We are left with showing that with $q_\rho \in \mathcal{M}\otimes\mathcal{G}$ given by (2.80) also $\mu_L : \hat{\mathcal{G}}\otimes\mathcal{M} \to \mathcal{M}_1$ given in (2.4) provides a linear isomorphism, which under the identification $\hat{\mathcal{G}}\otimes\mathcal{M} = \hat{\mathcal{G}} \bowtie \mathcal{M}_{\delta_l}$ in fact becomes an algebra map. This is seen by realizing, that $\mu_L \equiv f$, where $f : \hat{\mathcal{G}} \bowtie \mathcal{M}_{\delta_l} \to \mathcal{M}_{\delta_r} \bowtie \hat{\mathcal{G}}$ is the isomorphism defined in (2.85). □

2.6. Proofs

In this section we have collected the proofs omitted in the previous sections.

Proof of Theorem 2.10. One trivially checks the unit properties in part (i) and also the identity

$$(\varphi \bowtie m)(\hat{\mathbf{1}} \bowtie n) = (\varphi \bowtie mn) \tag{2.88}$$

for all $m, n \in \mathcal{M}$ and all $\varphi \in \hat{\mathcal{G}}$, thereby proving part (ii).

We now prove the associativity of the product in $\hat{\mathcal{G}} \bowtie \mathcal{M}$. First note that (2.56) and (2.88) immediately imply

$$[XY](\hat{\mathbf{1}} \bowtie m) = X[Y(\hat{\mathbf{1}} \bowtie m)] \tag{2.89}$$

for all $X, Y \in \hat{\mathcal{G}} \bowtie \mathcal{M}$ and all $m \in \mathcal{M}$. Next we show that

$$[X(\hat{\mathbf{1}} \bowtie m)]Y = X[(\hat{\mathbf{1}} \bowtie m)Y], \quad \forall X, Y \in \hat{\mathcal{G}} \bowtie \mathcal{M},\ m \in \mathcal{M} \tag{2.90}$$

To this end we use $(\mathrm{id}\otimes\epsilon)(h) = \mathbf{1}_\mathcal{G}$ and therefore $(\epsilon\otimes\mathrm{id}_\mathcal{G}\otimes\mathrm{id}_\mathcal{M}\otimes\mathrm{id}_\mathcal{G}\otimes\epsilon)(\Omega_L) = \mathbf{1}_\mathcal{G}\otimes\mathbf{1}_\mathcal{M}\otimes\mathbf{1}_\mathcal{G}$ to conclude for $m, m', n \in \mathcal{M}$ and $\psi \in \hat{\mathcal{G}}$

$$(\hat{\mathbf{1}} \bowtie m)(\psi \bowtie n) = \psi_{(2)} \bowtie (S^{-1}(\psi_{(1)}) \triangleright m \triangleleft \psi_{(3)})n \tag{2.91}$$

and hence also

$$\begin{aligned}(\hat{\mathbf{1}} \bowtie m')\big[(\hat{\mathbf{1}} \bowtie m)(\psi \bowtie n)\big] &= \psi_{(3)} \bowtie \big[(S^{-1}(\psi_{(2)}) \triangleright m' \triangleleft \psi_{(4)})(S^{-1}(\psi_{(1)}) \triangleright m \triangleleft \psi_{(5)})n\big] \\ &= \psi_{(2)} \bowtie (S^{-1}(\psi_{(1)}) \triangleright m'm \triangleleft \psi_{(3)}) \\ &= (\hat{\mathbf{1}} \bowtie m'm)(\psi \bowtie n)\end{aligned} \tag{2.92}$$

where we have used that δ is an algebra map. Moreover, (2.91) also implies for all $\varphi \in \hat{\mathcal{G}}$

$$\begin{aligned}&(\varphi \bowtie \mathbf{1}_\mathcal{M})\big[(\hat{\mathbf{1}} \bowtie m)(\psi \bowtie n)\big] \\ &\quad = \big[(\Omega_L^1 \rightharpoonup \varphi \leftharpoonup \Omega_L^5)(\Omega_L^2 \rightharpoonup \psi_{(2)} \leftharpoonup \Omega_L^4)\big] \bowtie \big[\Omega_L^3(S^{-1}(\psi_{(1)}) \triangleright m \triangleleft \psi_{(3)})n\big] \\ &\quad = (\varphi \bowtie m)(\psi \bowtie n)\end{aligned} \tag{2.93}$$

Putting (2.88), (2.89), (2.92) and (2.93) together, we have proven (2.90).

In view of (2.88), (2.89) and (2.90), to finish the proof of associativity we are now left with proving the following two identities

$$(\varphi \bowtie \mathbf{1}_\mathcal{M})\big[(\psi \bowtie \mathbf{1}_\mathcal{M})(\chi \bowtie \mathbf{1}_\mathcal{M})\big] = \big[(\varphi \bowtie \mathbf{1}_\mathcal{M})(\psi \bowtie \mathbf{1}_\mathcal{M})\big](\chi \bowtie \mathbf{1}_\mathcal{M}) \tag{2.94}$$

$$(\hat{\mathbf{1}} \bowtie m)\big[(\varphi \bowtie \mathbf{1}_\mathcal{M})(\psi \bowtie \mathbf{1}_\mathcal{M})\big] = \big[(\hat{\mathbf{1}} \bowtie m)(\varphi \bowtie \mathbf{1}_\mathcal{M})\big](\psi \bowtie \mathbf{1}_\mathcal{M}) \tag{2.95}$$

for all $\varphi, \psi, \chi \in \hat{\mathcal{G}}$ and all $m \in \mathcal{M}$. To prove these remaining identities we rewrite them using the generating matrix formalism. Let $\mathbf{L} \in \mathcal{G}\otimes(\hat{\mathcal{G}} \bowtie \mathcal{M})$ be given by $\mathbf{L} = e_\mu \otimes (e^\mu \bowtie \mathbf{1}_\mathcal{M})$, where $\{e_\mu\}$ is a basis in $\mathcal{G}$ with dual basis $\{e^\mu\}$ in $\hat{\mathcal{G}}$. We also abbreviate our notation by identifying $m \equiv (\hat{\mathbf{1}} \bowtie m)$, $m \in \mathcal{M}$. Then Eqs. (2.94) and (2.95) are equivalent, respectively, to

$$\mathbf{L}^{14}(\mathbf{L}^{24}\mathbf{L}^{34}) = (\mathbf{L}^{14}\mathbf{L}^{24})\mathbf{L}^{34} \tag{2.96}$$

$$[\mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G} \otimes m](\mathbf{L}^{13}\mathbf{L}^{23}) = ([\mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G} \otimes m]\mathbf{L}^{13})\mathbf{L}^{23}, \tag{2.97}$$

where (2.96) is understood as an identity in $\mathcal{G}^{\otimes^3} \otimes (\hat{\mathcal{G}} \bowtie \mathcal{M})$ and (2.97) as an identity in $\mathcal{G}^{\otimes^2} \otimes (\hat{\mathcal{G}} \bowtie \mathcal{M})$. We now use that (2.56) and (2.57) imply

$$[\mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{M}]\,\mathbf{L} = \mathbf{L} \tag{2.98}$$

$$[\mathbf{1}_\mathcal{G} \otimes m]\,\mathbf{L} = [S^{-1}(m_{(1)}) \otimes \mathbf{1}_\mathcal{M}]\,\mathbf{L}\,[m_{(-1)} \otimes m_{(0)}], \quad \forall m \in \mathcal{M} \tag{2.99}$$

$$\mathbf{L}^{13}\mathbf{L}^{23} = [S^{-1}(\bar{\Psi}^5) \otimes S^{-1}(\bar{\Psi}^4) \otimes \mathbf{1}_\mathcal{M}]\ [(\Delta_L \otimes \mathrm{id})(\mathbf{L})]\,[\bar{\Psi}^1 \otimes \bar{\Psi}^2 \otimes \bar{\Psi}^3] \tag{2.100}$$

where we have introduced the notation $\delta(m) = m_{(-1)} \otimes m_{(0)} \otimes m_{(1)}$ and $\Psi^{-1} \equiv \bar{\Psi} = \bar{\Psi}^1 \otimes \bar{\Psi}^2 \otimes \bar{\Psi}^3 \otimes \bar{\Psi}^4 \otimes \bar{\Psi}^5$, and where $\Delta_L(a) := h\Delta(a)$, $a \in \mathcal{G}$, has been introduced in Corollary 2.2. To prove (2.97) we use (2.99) twice together with (2.90) to get for the r.h.s. of (2.97)

$$([\mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G} \otimes m]\, \mathbf{L}^{13})\mathbf{L}^{23} = [S^{-1}(m_{(2)}) \otimes S^{-1}(m_{(1)}) \otimes \mathbf{1}_\mathcal{M}]\, \mathbf{L}^{13}\mathbf{L}^{23}\, [m_{(-2)} \otimes m_{(-1)} \otimes m_{(0)}],$$

where we have used the notation $(\mathrm{id} \otimes \delta \otimes \mathrm{id}) \circ \delta(m) = m_{(-2)} \otimes m_{(-1)} \otimes m_{(0)} \otimes m_{(1)} \otimes m_{(2)}$. On the other hand, by the intertwiner property (2.31) together with (2.99),(2.100) the l.h.s. of (2.97) gives

$$\begin{aligned}&[\mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G} \otimes m]\, (\mathbf{L}^{13}\mathbf{L}^{23}) \\ &\quad = [(S^{-1} \otimes S^{-1})\big(\Delta^{op}(m_{(1)})(\bar{\Psi}^5 \otimes \bar{\Psi}^4)\big) \otimes \mathbf{1}_\mathcal{M}]\, [(\Delta_L \otimes \mathrm{id})(\mathbf{L})]\, [\Delta(m_{(-1)}) \otimes m_{(0)}]\, [\bar{\Psi}^1 \otimes \bar{\Psi}^2 \otimes \bar{\Psi}^3].\end{aligned}$$

Using again (2.100) to rewrite the r.h.s. of this formula, Eq. (2.97) follows from the defining property (2.39a) of $\bar{\Psi} \equiv \Psi^{-1}$.

To prove (2.96), we use (2.100) to compute for the l.h.s (writing $\hat{\bar{\Psi}}$ for another copy of $\bar{\Psi}$)

$$\begin{aligned}\mathbf{L}^{14}(\mathbf{L}^{24}\mathbf{L}^{34}) &= [\mathbf{1}_\mathcal{G} \otimes S^{-1}(\bar{\Psi}^5) \otimes S^{-1}(\bar{\Psi}^4) \otimes \mathbf{1}_\mathcal{M}][(\mathrm{id} \otimes \Delta_L \otimes \mathrm{id})(\mathbf{L}^{13}\mathbf{L}^{23})][\mathbf{1}_\mathcal{G} \otimes \bar{\Psi}^1 \otimes \bar{\Psi}^2 \otimes \bar{\Psi}^3] \\ &= [(S^{-1} \otimes S^{-1} \otimes S^{-1})\Big((\hat{\bar{\Psi}}^5 \otimes \Delta^{op}(\hat{\bar{\Psi}}^4))(\mathbf{1}_\mathcal{G} \otimes \bar{\Psi}^5 \otimes \bar{\Psi}^4)\Big) \otimes \mathbf{1}_\mathcal{M}] \qquad (2.101) \\ &\quad \times [(\mathrm{id} \otimes \Delta_L \otimes \mathrm{id}) \circ (\Delta_L \otimes \mathrm{id})(\mathbf{L})]\, [\hat{\bar{\Psi}}^1 \otimes \Delta(\hat{\bar{\Psi}}^2) \otimes \hat{\bar{\Psi}}^3]\, [\mathbf{1}_\mathcal{G} \otimes \bar{\Psi}^1 \otimes \bar{\Psi}^2 \otimes \bar{\Psi}^3],\end{aligned}$$

where for the second equality we have used the identity

$$\Delta_L(S^{-1}(a)bc) = (S^{-1} \otimes S^{-1})(\Delta^{op}(a))\Delta_L(b)\Delta(c)$$

following from (2.31). For the r.h.s. of (2.96) we get:

$$\begin{aligned}&(\mathbf{L}^{14}\mathbf{L}^{24})\mathbf{L}^{34} \\ &= [S^{-1}(\bar{\Psi}^5) \otimes S^{-1}(\bar{\Psi}^4) \otimes S^{-1}(\bar{\Psi}^3_{(1)}) \otimes \mathbf{1}_\mathcal{M}][\Delta_L \otimes \mathrm{id} \otimes \mathrm{id})(\mathbf{L}^{13}\mathbf{L}^{23})][\bar{\Psi}^1 \otimes \bar{\Psi}^2 \otimes \bar{\Psi}^3_{(-1)} \otimes \bar{\Psi}^3_{(0)}] \\ &= [(S^{-1} \otimes S^{-1} \otimes S^{-1})\Big((\Delta^{op}(\hat{\bar{\Psi}}^5) \otimes \hat{\bar{\Psi}}^4)(\bar{\Psi}^5 \otimes \bar{\Psi}^4 \otimes \bar{\Psi}^3_{(1)})\Big) \otimes \mathbf{1}_\mathcal{M}] \qquad (2.102) \\ &\quad \times [(\Delta_L \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\Delta_L \otimes \mathrm{id})(\mathbf{L})]\, [\Delta(\hat{\bar{\Psi}}^1) \otimes \hat{\bar{\Psi}}^2 \otimes \hat{\bar{\Psi}}^3][\bar{\Psi}^1 \otimes \bar{\Psi}^2 \otimes \bar{\Psi}^3_{(-1)} \otimes \bar{\Psi}^3_{(0)}],\end{aligned}$$

where for the first equality we have used (2.99) to move $\bar{\Psi}^3$ to the right of $\mathbf{L}^{34}$ and in the second equality again (2.31). Now we use that by Corollary 2.1

$$(\mathrm{id} \otimes \Delta_L)(\Delta_L(a)) = (S^{-1} \otimes S^{-1} \otimes S^{-1})(\phi^{321})\, ((\Delta_L \otimes \mathrm{id})(\Delta_L(a))\, \phi^{-1}, \quad \forall a \in \mathcal{G}.$$

Hence (2.101) and (2.102) are equal due to the pentagon identity (2.39b) for Ψ, which proves (2.96). This concludes the proof of parts (i) and (ii) of Theorem 2.10. ☐

Properties of the elements $p_\lambda, q_\lambda, p_\rho, q_\rho$.

LEMMA 2.21.

1. *Let (λ, ϕ_λ) be a left $\mathcal{G}$–coaction on $\mathcal{M}$ and let p_λ, q_λ be given by (2.77),(2.78). Then the following identities hold for all $m \in \mathcal{M}$, where $\lambda(m) \equiv m_{(-1)} \otimes m_{(0)}$*

$$\lambda(m_{(0)})\, p_\lambda\, [S^{-1}(m_{(-1)}) \otimes \mathbf{1}_\mathcal{M}] = p_\lambda\, [\mathbf{1}_\mathcal{G} \otimes m] \qquad (2.103a)$$

$$[S(m_{(-1)}) \otimes \mathbf{1}_\mathcal{M}]\, q_\lambda\, \lambda(m_{(0)}) = [\mathbf{1}_\mathcal{G} \otimes m]\, q_\lambda \qquad (2.103b)$$

$$\lambda(q^2_\lambda)\, p_\lambda\, [S^{-1}(q^1_\lambda) \otimes \mathbf{1}_\mathcal{M}] = \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{M} \qquad (2.103c)$$

$$[S(p^1_\lambda) \otimes \mathbf{1}_\mathcal{M}]\, q_\lambda\, \lambda(p^2_\lambda) = \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{M} \qquad (2.103d)$$

Moreover, with $f, h \in \mathcal{G}\otimes\mathcal{G}$ being the twists given by (2.26),(2.31), the following identities are valid

$$\begin{aligned}&\phi_\lambda^{-1}\, (\mathrm{id}_\mathcal{G} \otimes \lambda)(p_\lambda)\, (\mathbf{1}_\mathcal{G} \otimes p_\lambda) \\ &\quad = (\Delta \otimes \mathrm{id}_\mathcal{M})\big(\lambda(\phi^3_\lambda)p_\lambda\big)\, [h^{-1} \otimes \mathbf{1}_\mathcal{M}]\, [S^{-1}(\phi^2_\lambda) \otimes S^{-1}(\phi^1_\lambda) \otimes \mathbf{1}_\mathcal{M}] \qquad (2.103e)\end{aligned}$$

$$\begin{aligned}&(\mathbf{1}_\mathcal{G} \otimes q_\lambda)\, (\mathrm{id}_\mathcal{G} \otimes \lambda)(q_\lambda)\, \phi_\lambda \\ &\quad = [S(\bar{\phi}^2_\lambda) \otimes S(\bar{\phi}^1_\lambda) \otimes \mathbf{1}_\mathcal{M}]\, [f \otimes \mathbf{1}_\mathcal{M}]\, (\Delta \otimes \mathrm{id}_\mathcal{M})\big(q_\lambda\lambda(\bar{\phi}^3_\lambda)\big) \qquad (2.103f)\end{aligned}$$

2. *Similarly, let* (ρ, ϕ_ρ) *be a right* $\mathcal{G}$*–coaction on* $\mathcal{M}$ *and let* p_ρ, q_ρ *be given by* (2.79) *and* (2.80). *Then the following identities hold for all* $m \in \mathcal{M}$, *where* $\rho(m) \equiv m_{(0)} \otimes m_{(1)}$.

$$\rho(m_{(0)})\, p_\rho\, [\mathbf{1}_\mathcal{M} \otimes S(m_{(1)})] = p_\rho\, [m \otimes \mathbf{1}_\mathcal{G}] \tag{2.104a}$$

$$[\mathbf{1}_\mathcal{M} \otimes S^{-1}(m_{(1)})]\, q_\rho\, \rho(m_{(0)}) = [m \otimes \mathbf{1}_\mathcal{G}]\, q_\rho \tag{2.104b}$$

$$\rho(q_\rho^1)\, p_\rho\, [\mathbf{1}_\mathcal{M} \otimes S(q_\rho^2)] = \mathbf{1}_\mathcal{M} \otimes \mathbf{1}_\mathcal{G} \tag{2.104c}$$

$$[\mathbf{1}_\mathcal{M} \otimes S^{-1}(p_\rho^2)]\, q_\rho\, \rho(p_\rho^1) = \mathbf{1}_\mathcal{M} \otimes \mathbf{1}_\mathcal{G} \tag{2.104d}$$

$$\begin{aligned}&\phi_\rho\, (\rho \otimes \mathrm{id}_\mathcal{G})(p_\rho)\, (p_\rho \otimes \mathbf{1}_\mathcal{G}) \\ &\quad = (\mathrm{id}_\mathcal{M} \otimes \Delta)\big(\rho(\bar\phi_\rho^1) p_\rho\big)\, [\mathbf{1}_\mathcal{M} \otimes f^{-1}]\, [\mathbf{1}_\mathcal{M} \otimes S(\bar\phi_\rho^3) \otimes S(\bar\phi_\rho^2)]\end{aligned} \tag{2.104e}$$

$$\begin{aligned}&(q_\rho \otimes \mathbf{1}_\mathcal{G})\, (\rho \otimes \mathrm{id}_\mathcal{G})(q_\rho)\, \phi_\rho^{-1} \\ &\quad = [\mathbf{1}_\mathcal{M} \otimes S^{-1}(\phi_\rho^3) \otimes S^{-1}(\phi_\rho^2)]\, [\mathbf{1}_\mathcal{M} \otimes h]\, (\mathrm{id}_\mathcal{M} \otimes \Delta)\big(q_\rho \rho(\phi_\rho^1)\big).\end{aligned} \tag{2.104f}$$

PROOF. Note that part 2. of Lemma 2.21 is functorially equivalent to part 1., since (ρ, ϕ_ρ) is a right $\mathcal{G}$–coaction if and only if $(\rho^{op}, (\phi_\rho^{-1})^{321})$ is a left $\mathcal{G}^{cop}$–coaction. Also, considering (ρ, ϕ_ρ^{-1}) as a right $\mathcal{G}_{op}$–coaction on $\mathcal{M}_{op}$, the roles of q_ρ and p_ρ interchange, which makes it enough to just prove Eqs. (2.104b), (2.104d) and (2.104f) or the corresponding sets of equations in part 1.

Let us begin with (2.104b). Denoting the multiplication in $\mathcal{G}^{op}$ by μ^{op} one computes

$$\begin{aligned}[\mathbf{1}_\mathcal{M} \otimes S^{-1}(m_{(1)})]\, q_\rho\, \rho(m_{(0)}) &= [\phi_\rho^1 \otimes S^{-1}(\alpha \phi_\rho^3 m_{(1)}) \phi_\rho^2]\, \rho(m_{(0)}) \\ &= (\mathrm{id}_\mathcal{M} \otimes \mu^{op}) \circ (\mathrm{id}_\mathcal{M} \otimes \mathrm{id}_\mathcal{G} \otimes S^{-1})\Big([\mathbf{1}_\mathcal{M} \otimes \mathbf{1}_\mathcal{G} \otimes \alpha]\, \phi_\rho\, (\rho \otimes \mathrm{id}_\mathcal{G})(\rho(m))\Big) \\ &= (\mathrm{id}_\mathcal{M} \otimes \mu^{op}) \circ (\mathrm{id}_\mathcal{M} \otimes \mathrm{id}_\mathcal{G} \otimes S^{-1})\Big([\mathbf{1}_\mathcal{M} \otimes \mathbf{1}_\mathcal{G} \otimes \alpha]\, (\mathrm{id}_\mathcal{M} \otimes \Delta)(\rho(m))\, \phi_\rho\Big) \\ &= [m \otimes \mathbf{1}_\mathcal{G}]\, q_\rho,\end{aligned}$$

where we have plugged in the definition (2.80) of q_ρ and used the intertwiner property (2.36a) of ϕ_ρ and the antipode property (2.15). This proves (2.104b).

To prove (2.104d) we introduce for $a, b, c \in \mathcal{G}$ the notation $\sigma(a \otimes b \otimes c) := c\, S^{-1}(\alpha b \beta)\, a$, to compute for the l.h.s.

$$\begin{aligned}[\mathbf{1}_\mathcal{M} \otimes S^{-1}(p_\rho^2)]\, q_\rho\, \rho(p_\rho^1) &\equiv [\phi_\rho^1 \otimes \bar\phi_\rho^3\, S^{-1}(\alpha \phi_\rho^3 \bar\phi_\rho^2 \beta) \phi_\rho^2]\, \rho(\bar\phi_\rho^1) \\ &= (\mathrm{id}_\mathcal{M} \otimes \sigma)\Big([\phi_\rho \otimes \mathbf{1}_\mathcal{G}]\, (\rho \otimes \mathrm{id}_\mathcal{G} \otimes \mathrm{id}_\mathcal{G})(\phi_\rho^{-1})\Big) \\ &= (\mathrm{id}_\mathcal{M} \otimes \sigma)\Big((\mathrm{id}_\mathcal{M} \otimes \Delta \otimes \mathrm{id}_\mathcal{G})(\phi_\rho^{-1})\, [\mathbf{1}_\mathcal{M} \otimes \phi^{-1}]\, (\mathrm{id}_\mathcal{M} \otimes \mathrm{id}_\mathcal{G} \otimes \Delta)(\phi_\rho)\Big) \\ &= \mathbf{1}_\mathcal{M} \otimes \bar\phi^3 S^{-1}(\alpha \bar\phi^2 \beta) \bar\phi^1 = \mathbf{1}_\mathcal{M} \otimes \mathbf{1}_\mathcal{G},\end{aligned}$$

where we have used the pentagon identity (2.36b), then the two antipode properties (2.15) together with $(\mathrm{id} \otimes \mathrm{id} \otimes \epsilon)(\phi_\rho) = \mathbf{1}_\mathcal{M} \otimes \mathbf{1}_\mathcal{G}$ to drop the reassociators ϕ_ρ and ϕ_ρ^{-1} and finally (2.16).

The proof of (2.104f) is more complicated. First we rewrite

$$\text{l.h.s. } (2.104f) = \omega(X),$$

where

$$X = [\phi_\rho^1 \otimes \phi_\rho^2 \otimes \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G} \otimes \phi_\rho^3]\, [(\rho \otimes \mathrm{id}_\mathcal{G} \otimes \mathrm{id}_\mathcal{G})(\phi_\rho) \otimes \mathbf{1}_\mathcal{G}]\, [\phi_\rho^{-1} \otimes \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G}] \tag{2.105}$$

and where the map $\omega : \mathcal{M} \otimes \mathcal{G}^{\otimes^2} \to \mathcal{M} \otimes \mathcal{G}^{\otimes^2}$ is given by

$$\omega(m \otimes a \otimes b \otimes c \otimes d) := m \otimes S^{-1}(\alpha d)\, a \otimes S^{-1}(\alpha c)\, b$$

To rewrite the r.h.s. of (2.104f) in the same fashion we first use the identities (2.31) and (2.33) and the definition (2.80) of q_ρ to compute

$$\begin{aligned}[\mathbf{1}_\mathcal{M} \otimes h]\, (\mathrm{id}_\mathcal{M} \otimes \Delta)(q_\rho) &= [\mathbf{1}_\mathcal{M} \otimes h]\, [\phi_\rho^1 \otimes \Delta\big(S^{-1}(\alpha \phi_\rho^3) \phi_\rho^2\big)] \\ &= [\phi_\rho^1 \otimes (S^{-1} \otimes S^{-1})(\Delta^{op}(\phi_\rho^3))]\, [\mathbf{1}_\mathcal{M} \otimes h]\, [\mathbf{1}_\mathcal{M} \otimes \Delta(S^{-1}(\alpha) \phi_\rho^2)] \\ &= [\mathbf{1}_\mathcal{M} \otimes (S^{-1} \otimes S^{-1})\big(\gamma^{op} \Delta^{op}(\phi_\rho^3)\big)]\, [\phi_\rho^1 \otimes \Delta(\phi_\rho^2)].\end{aligned}$$

Now we use the formula (2.24) for γ implying

$$(S^{-1}\otimes S^{-1})(\gamma^{op}) = S^{-1}(\alpha\bar\phi^3\phi^3_{(2)})\phi^1\otimes S^{-1}(\alpha\bar\phi^2\phi^3_{(1)})\bar\phi^1\phi^2$$

to obtain

$$\text{r.h.s. } (2.104\text{f}) = \omega(Y),$$

where

$$Y = [\mathbf{1}_{\mathcal M}\otimes\mathbf{1}_{\mathcal G}\otimes\phi^{-1}]\,(\mathrm{id}_{\mathcal M}\otimes\mathrm{id}_{\mathcal G}\otimes\mathrm{id}_{\mathcal G}\otimes\Delta)\big([\mathbf{1}_{\mathcal M}\otimes\phi]\,(\mathrm{id}_{\mathcal M}\otimes\Delta\otimes\mathrm{id}_{\mathcal G})(\phi_\rho)\big)$$
$$((\mathrm{id}\otimes\Delta)\circ\rho\otimes\mathrm{id}_{\mathcal G}\otimes\mathrm{id}_{\mathcal G})(\phi_\rho) \tag{2.106}$$

Using the pentagon eq. (2.36b) to replace the second and third reassociator in (2.106) yields

$$\begin{aligned} Y &= [\mathbf{1}_{\mathcal M}\otimes\mathbf{1}_{\mathcal G}\otimes\phi^{-1}]\,(\mathrm{id}_{\mathcal M}\otimes\mathrm{id}_{\mathcal G}^{\otimes^2}\otimes\Delta)\Big((\mathrm{id}^{\otimes^2}\otimes\Delta)(\phi_\rho)\,(\rho\otimes\mathrm{id}^{\otimes^2})(\phi_\rho)\,[\phi_\rho^{-1}\otimes\mathbf{1}_{\mathcal G}]\Big)\\ &\qquad((\mathrm{id}\otimes\Delta)\circ\rho\otimes\mathrm{id}^{\otimes^2})(\phi_\rho)\\ &= (\mathrm{id}^{\otimes^2}\otimes(\Delta\otimes\mathrm{id})\circ\Delta)(\phi_\rho)\,(\rho\otimes\mathrm{id}\otimes\mathrm{id})\Big([\mathbf{1}_{\mathcal M}\otimes\phi^{-1}]\,(\mathrm{id}^{\otimes^2}\otimes\Delta)(\phi_\rho)\,(\rho\otimes\mathrm{id}^{\otimes^2})(\phi_\rho)\Big)\\ &\qquad[\phi_\rho^{-1}\otimes\mathbf{1}_{\mathcal G}\otimes\mathbf{1}_{\mathcal G}]\\ &= (\mathrm{id}^{\otimes^2}\otimes(\Delta\otimes\mathrm{id})\circ\Delta)(\phi_\rho)\,(\rho\otimes\Delta\otimes\mathrm{id})(\phi_\rho)\,[(\rho\otimes\mathrm{id}^{\otimes^2})(\phi_\rho)\otimes\mathbf{1}_{\mathcal G}]\,[\phi_\rho^{-1}\otimes\mathbf{1}_{\mathcal G}\otimes\mathbf{1}_{\mathcal G}] \end{aligned} \tag{2.107}$$

where in the second equation we have used (2.8) and (2.36a) to shift the reassociators ϕ^{-1} and ϕ_ρ^{-1} by one step to the right, and in the third equation again the pentagon identity (2.36b). Hence, when computing $\omega(Y)$, the second factor in (2.107) may be dropped due to the antipode property (2.15) and the two coproducts in the first factor disappear by the same reason. Comparing with (2.105) proves, that $\omega(X) = \omega(Y)$ and therefore both sides of (2.104f) are equal. □

There are also some additional identities in the case where $(\lambda,\phi_\lambda,\rho,\phi_\rho,\phi_{\lambda\rho})$ is a quasi–commuting pair of coactions.

LEMMA 2.22. *Let $(\lambda,\phi_\lambda,\rho,\phi_\rho,\phi_{\lambda\rho})$ be a quasi–commuting pair of $\mathcal G$–coactions on $\mathcal M$ and let $p_{\lambda/\rho}$, $q_{\lambda/\rho}$ be given by Eqs. (2.77) - (2.80). Then putting $\bar\phi_{\lambda\rho}\equiv\phi_{\lambda\rho}^{-1}$*

$$\phi_{\lambda\rho}^{-1}\,(\mathrm{id}_{\mathcal G}\otimes\rho)(p_\lambda) = [\lambda(\phi^2_{\lambda\rho})p_\lambda\otimes\phi^3_{\lambda\rho}]\,[S^{-1}(\phi^1_{\lambda\rho})\otimes\mathbf{1}_{\mathcal M}\otimes\mathbf{1}_{\mathcal G}] \tag{2.108a}$$
$$(\mathrm{id}_{\mathcal G}\otimes\rho)(q_\lambda)\,\phi_{\lambda\rho} = [S(\bar\phi^1_{\lambda\rho})\otimes\mathbf{1}_{\mathcal M}\otimes\mathbf{1}_{\mathcal G}]\,[q_\lambda\lambda(\bar\phi^2_{\lambda\rho})\otimes\bar\phi^3_{\lambda\rho}] \tag{2.108b}$$
$$\phi_{\lambda\rho}\,(\lambda\otimes\mathrm{id}_{\mathcal G})(p_\rho) = [\bar\phi^1_{\lambda\rho}\otimes\rho(\phi^2_{\lambda\rho})p_\rho]\,[\mathbf{1}_{\mathcal G}\otimes\mathbf{1}_{\mathcal M}\otimes S(\bar\phi^3_{\lambda\rho})] \tag{2.108c}$$
$$(\lambda\otimes\mathrm{id}_{\mathcal G})(q_\rho)\,\phi_{\lambda\rho}^{-1} = [\mathbf{1}_{\mathcal G}\otimes\mathbf{1}_{\mathcal M}\otimes S^{-1}(\phi^3_{\lambda\rho})]\,[\phi^1_{\lambda\rho}\otimes q_\rho\rho(\phi^2_{\lambda\rho})] \tag{2.108d}$$

PROOF OF LEMMA 2.22. Again we remark that for functorial reasons Eqs. (2.108a) - (2.108d) are all equivalent, see the arguments in the proof of Lemma 2.21.

We prove the identity (2.108d). Introducing for $a,b\in\mathcal G$ the map $\nu(a\otimes b) := S^{-1}(\alpha b)\,a$ and using the formula (2.80) for q_ρ we compute

$$\begin{aligned} (\lambda\otimes\mathrm{id})(q_\rho)\,\phi_{\lambda\rho}^{-1} &\equiv [\lambda(\phi^1_\rho)\otimes S^{-1}(\alpha\phi^3_\rho)\phi^2_\rho]\,\phi_{\lambda\rho}^{-1}\\ &= (\mathrm{id}_{\mathcal G}\otimes\mathrm{id}_{\mathcal M}\otimes\nu)\Big((\lambda\otimes\mathrm{id}\otimes\mathrm{id})(\phi_\rho)\,[\phi_{\lambda\rho}^{-1}\otimes\mathbf{1}_{\mathcal G}]\Big)\\ &= (\mathrm{id}_{\mathcal G}\otimes\mathrm{id}_{\mathcal M}\otimes\nu)\Big((\mathrm{id}^{\otimes^2}\otimes\Delta)(\phi_{\lambda\rho}^{-1})[\mathbf{1}_{\mathcal G}\otimes\phi_\rho]\,(\mathrm{id}\otimes\rho\otimes\mathrm{id})(\phi_{\lambda\rho})\Big)\\ &= [\mathbf{1}_{\mathcal G}\otimes\mathbf{1}_{\mathcal G}\otimes S^{-1}(\phi^3_{\lambda\rho})]\,[\mathbf{1}_{\mathcal G}\otimes q_\rho]\,[\phi^1_{\lambda\rho}\otimes\rho(\phi^2_{\lambda\rho})], \end{aligned}$$

Here we have plugged in the pentagon equation (2.51c) and used the fact that $\phi_{\lambda\rho}$ may be dropped due to (2.51d) and the antipode property (2.15). This proves (2.108d) and therefore Lemma 2.22. 2.22). □

Proof of Proposition 2.19. We only need to prove part 1, since part 2 is functorially equivalent, see the proof of Lemma 2.21. If $\mathbf{T}$ is a $\lambda\rho$–intertwiner and $\mathbf{R}$ given by (2.81), then

$$\begin{aligned}\rho^{op}(m_{(0)})\,\mathbf{R}\,[S^{-1}(m_{(-1)})\otimes\mathbf{1}_{\mathcal{A}}] &\equiv \rho^{op}(m_{(0)})\,\mathbf{T}\,p_\lambda\,[S^{-1}(m_{(-1)})\otimes\mathbf{1}_{\mathcal{A}}]\\ &= \mathbf{R}\,[\mathbf{1}_{\mathcal{G}}\otimes m]\end{aligned}$$

by (2.73) and (2.103a), and therefore $\mathbf{R}$ is a right δ_r–implementer. Moreover, (2.103c) implies

$$\begin{aligned}\rho^{op}(q_\lambda^2)\,\mathbf{T}\,p_\lambda\,[S^{-1}(q_\lambda^1)\otimes\mathbf{1}_{\mathcal{M}}] &= \mathbf{T}\,\lambda(q_\lambda^2)\,p_\lambda\,[S^{-1}(q_\lambda^1)\otimes\mathbf{1}_{\mathcal{M}}]\\ &= \mathbf{T}\end{aligned}$$

Conversely if $\mathbf{R}$ is a right δ_r–implementer and $\mathbf{T}$ given by (2.82), then

$$\begin{aligned}\rho^{op}(m)\,\mathbf{T} &\equiv \rho^{op}(m\,q_\lambda^2)\,\mathbf{R}\,[S^{-1}(q_\lambda^1)\otimes\mathbf{1}_{\mathcal{A}}]\\ &= \rho^{op}(q_\lambda^2\,m_{(0,0)})\,\mathbf{R}\,[S^{-1}(q_\lambda^1\,m_{(0,-1)})m_{(-1)}\otimes\mathbf{1}_{\mathcal{A}}]\\ &= \rho^{op}(q_\lambda^2)\,\mathbf{R}\,[S^{-1}(q_\lambda^1)m_{(-1)}\otimes m_{(0)}]\\ &= \mathbf{T}\,\lambda(m)\end{aligned}$$

where in the second line we have used (2.103b) and in the third line the right δ_r–implementer property (2.65) of $\mathbf{R}$. Hence $\mathbf{T}$ is a $\lambda\rho$–intertwiner. Moreover

$$\begin{aligned}\rho^{op}(q_\lambda^2)\,\mathbf{R}\,[S^{-1}(q_\lambda^1)\otimes\mathbf{1}_{\mathcal{A}}]\,p_\lambda &= \rho^{op}(q_\lambda^2\,p^2_{\lambda(0)})\,\mathbf{R}\,\left[S^{-1}(q_\lambda^1\,p^2_{\lambda(-1)})\,p_\lambda^1\otimes\mathbf{1}_{\mathcal{A}}\right]\\ &= \mathbf{R},\end{aligned}$$

where we have used the δ_r-implementer property of $\mathbf{R}$ and then (2.103d). Thus the correspondence $\mathbf{T}\leftrightarrow\mathbf{R}$ is one–to–one, and since q_λ and p_λ are normal it is clearly normality preserving.

To prove that it is also coherence preserving assume now that the $\lambda\rho$–intertwiner $\mathbf{T}$ satisfies the coherence condition (2.74) and let $\mathbf{R}=\mathbf{T}\,p_\lambda$. Then

$$(\Delta\otimes\mathrm{id})(\mathbf{R})\,[h^{-1}\otimes\mathbf{1}_{\mathcal{M}}] = \phi_\rho^{312}\,\mathbf{T}^{13}\,(\phi_{\lambda\rho}^{-1})^{132}\,A \tag{2.109}$$

where $A\in\mathcal{G}\otimes\mathcal{G}\otimes\mathcal{M}$ is given by

$$\begin{aligned}A &= \mathbf{T}^{23}\,\phi_\lambda\,(\Delta\otimes\mathrm{id}_{\mathcal{M}})(p_\lambda)\,[h^{-1}\otimes\mathbf{1}_{\mathcal{M}}]\\ &= \mathbf{T}^{23}\,(\mathrm{id}_{\mathcal{G}}\otimes\lambda)\big(\lambda(\bar\phi_\lambda^3)\,p_\lambda\big)\,[\mathbf{1}_{\mathcal{G}}\otimes p_\lambda]\,[S^{-1}(\bar\phi_\lambda^2)\otimes S^{-1}(\bar\phi_\lambda^1)\otimes\mathbf{1}_{\mathcal{M}}]\\ &= (\mathrm{id}_{\mathcal{G}}\otimes\rho^{op})\big(\lambda(\bar\phi_\lambda^3)\,p_\lambda\big)\,\mathbf{R}^{23}\,[S^{-1}(\bar\phi_\lambda^2)\otimes S^{-1}(\bar\phi_\lambda^1)\otimes\mathbf{1}_{\mathcal{M}}]\end{aligned} \tag{2.110}$$

Here we have used (2.103e) in the second line and the intertwining property (2.73) of $\mathbf{T}$ in the third line. Using the intertwiner property (2.51a) of $\phi_{\lambda\rho}$ and (2.108a) we further compute

$$\begin{aligned}&\mathbf{T}^{13}\,(\phi_{\lambda\rho}^{-1})^{132}\,(\mathrm{id}_{\mathcal{G}}\otimes\rho^{op})\big(\lambda(\bar\phi_\lambda^3)\,p_\lambda\big)\\ &\quad= \Big[(\rho^{op}\otimes\mathrm{id})\big(\rho(\bar\phi_\lambda^3)\,(\phi_{\lambda\rho}^2\otimes\phi_{\lambda\rho}^3)\big)\,(\mathbf{R}\otimes\mathbf{1}_{\mathcal{M}})\Big]^{132}\,[S^{-1}(\phi_{\lambda\rho}^1)\otimes\mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{M}}]\end{aligned} \tag{2.111}$$

Putting (2.109) - (2.111) together we finally conclude

$$(\Delta\otimes\mathrm{id})(\mathbf{R})\,[h^{-1}\otimes\mathbf{1}_{\mathcal{M}}] = [\bar\Psi_r^4\otimes\bar\Psi_r^5\otimes\bar\Psi_r^3]\,\mathbf{R}^{13}\,\mathbf{R}^{23}\,[S^{-1}(\bar\Psi_r^2)\otimes S^{-1}(\bar\Psi_r^1)\otimes\mathbf{1}_{\mathcal{M}}] \tag{2.112}$$

where $\bar\Psi_r\in\mathcal{G}\otimes\mathcal{G}\otimes\mathcal{M}\otimes\mathcal{G}\otimes\mathcal{G}$ is given by

$$\bar\Psi_r := (\mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{G}}\otimes\phi_\rho)(\mathrm{id}_{\mathcal{G}}\otimes\mathrm{id}_{\mathcal{G}}\otimes\rho\otimes\mathrm{id}_{\mathcal{G}})\Big((\mathrm{id}_{\mathcal{G}}\otimes\mathrm{id}_{\mathcal{G}}\otimes\rho)(\Phi_\lambda^{-1})\,(\mathbf{1}_{\mathcal{G}}\otimes\phi_{\lambda\rho})\Big)$$

By (2.52d) we have $\bar\Psi_r=\Psi_r^{-1}$ and therefore (2.112) is equivalent to the coherence condition (2.67) for $\mathbf{R}$ as a right (δ_r,Ψ_r)–implementer.

Conversely, assume now that $\mathbf{R}$ is a coherent right (δ_r,Ψ_r)–implementer and let $\mathbf{T}$ be given by (2.82). To prove that $\mathbf{T}$ is coherent we have to show that

$$(\Delta\otimes\mathrm{id})(\mathbf{T}) = B$$

where B is given by (2.75). Now writing $\mathbf{R}=\mathbf{T}\,p_\lambda$ and going backwards through the derivation (2.109)←(2.112) we conclude

$$(\Delta\otimes\mathrm{id})(\mathbf{T}\,p_\lambda) = B\,(\Delta\otimes\mathrm{id})(p_\lambda) \tag{2.113}$$

Thus, if p_λ were invertible we could immediately conclude that $\mathbf{T}$ is coherent. It turns out that we may use (2.103c) as a substitute for the invertibility of p_λ, since it implies

$$\begin{aligned}
(\Delta \otimes \mathrm{id})(\mathbf{T}) &\equiv (\Delta \otimes \mathrm{id})\Big(\rho^{op}(q_\lambda^2)\,\mathbf{T}\,p_\lambda)\,(S^{-1}(q_\lambda^1) \otimes \mathbf{1}_{\mathcal{M}})\Big) \\
&= (\Delta \otimes \mathrm{id})\big(\rho^{op}(q_\lambda^2)\big)\,B\,(\Delta \otimes \mathrm{id})\Big(p_\lambda\,(S^{-1}(q_\lambda^1) \otimes \mathbf{1}_{\mathcal{M}})\Big) \\
&= B\,(\Delta \otimes \mathrm{id})\Big(\lambda(q_\lambda^2)\,p_\lambda\,(S^{-1}(q_\lambda^1) \otimes \mathbf{1}_{\mathcal{M}})\Big) \\
&= B
\end{aligned}$$

Here we have used (2.113) in the second line, (2.76) in the third line and again (2.103c) in the last line. Thus $\mathbf{T}$ is a coherent $\lambda\rho$–intertwiner, which concludes the proof of Proposition 2.19.

□