

Diagonal crossed products by duals of quantum groups

To strip off all technicalities from the main ideas, in this first chapter we restrict ourselves to strictly coassociative Hopf algebras (quantum groups) \mathcal{G} . After reviewing some basic notions on coactions and crossed products in Section 1.1 we introduce as a new construction the diagonal crossed product $\mathcal{M} \bowtie \hat{\mathcal{G}}$ of a unital algebra \mathcal{M} and the dual $\hat{\mathcal{G}}$ of a Hopf algebra \mathcal{G} in Section 1.2. Section 1.3 gives a reformulation of this construction in terms of generating matrices summarized in Theorem 1.13. We will see later on in Chapter 2 that this Theorem allows for a generalization to quasi-Hopf algebras given in Theorem 2.1 (and also to weak quasi-Hopf algebras given in Theorem 3.1), which may be viewed as the heart piece of this thesis. In order to carefully prepare the much more complicated quasi-coassociative scenario we deliberately present this construction in rather elementary steps.

In Section 1.4 we reformulate the Hopf spin chains of [NS97] and also the lattice current algebras of [AFFS98] - both models being based on a Hopf algebra \mathcal{G} - as iterated diagonal crossed products. This opens the way to generalize these models to (weak) quasi-Hopf algebras, thus covering the physically important case of truncated quantum groups at roots of unity, in Chapter 5. Although we do not establish any new results in Section 1.4, we think it to be quite illuminating that these two models may be based on the same algebraic construction. In particular the isomorphy of the two models - more exactly the second being obtained by imposing periodic boundary conditions on the first -, as already established in [Nil97], becomes rather obvious, as well as the role of the quantum double, describing the representation theory of both models.

We emphasize that all concepts and constructions given in the subsequent chapters already appear in this first chapter. Thus it may also serve as an overview and the reader is invited to frequently return to Chapter 1 when feeling to get lost in the much more complicated treatment of the quasi-coassociative case in the following chapters.

1.1. Coactions and crossed products

To fix our conventions and notations we start with shortly reviewing some basic notions on Hopf module actions, coactions and crossed products. For full textbook treatments see e.g. [Abe80], [Maj95], [Swe69]. We also introduce the “generating” matrix formalism. Throughout by an algebra we will mean an associative unital algebra over \mathbb{C} and unless stated differently all algebra morphisms are supposed to be unit preserving.

Let \mathcal{G} and $\hat{\mathcal{G}}$ be a dual pair of finite dimensional Hopf algebras. We denote elements of \mathcal{G} by Roman letters a, b, c, \dots and elements of $\hat{\mathcal{G}}$ by Greek letters $\varphi, \psi, \chi, \dots$. The units are denoted by $\mathbf{1} \in \mathcal{G}$ and $\hat{\mathbf{1}} \in \hat{\mathcal{G}}$. Identifying $\hat{\mathcal{G}} = \mathcal{G}$, the dual pairing $\mathcal{G} \otimes \hat{\mathcal{G}} \rightarrow \mathbb{C}$ is written as

$$\langle a | \psi \rangle \equiv \langle \psi | a \rangle \in \mathbb{C}, \quad a \in \mathcal{G}, \psi \in \hat{\mathcal{G}}.$$

We denote $\Delta : \mathcal{G} \rightarrow \mathcal{G} \otimes \mathcal{G}$ the coproduct, $\epsilon : \mathcal{G} \rightarrow \mathbb{C}$ the counit and $S : \mathcal{G} \rightarrow \mathcal{G}$ the antipode. Similarly, $\hat{\Delta}, \hat{\epsilon}$ and \hat{S} are the structural maps on $\hat{\mathcal{G}}$. We will use the Sweedler notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$, $(\Delta \otimes \text{id})(\Delta(a)) \equiv (\text{id} \otimes \Delta)(\Delta(a)) = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$, etc. where the summation symbol and the summation indices are suppressed. Together with \mathcal{G} we have the Hopf algebras \mathcal{G}_{op} , \mathcal{G}^{cop} and $\widehat{\mathcal{G}}_{op}^{cop}$, where “op” refers to opposite multiplication and “cop” to opposite comultiplication. Note that the antipode of \mathcal{G}_{op} and \mathcal{G}^{cop} is given by S^{-1} and the antipode of $\widehat{\mathcal{G}}_{op}^{cop}$ by S . Also, $\widehat{\mathcal{G}}_{op} = (\hat{\mathcal{G}})^{cop}$, $\widehat{\mathcal{G}}^{cop} = (\hat{\mathcal{G}})_{op}$ and $\widehat{\widehat{\mathcal{G}}}_{op}^{cop} = (\hat{\mathcal{G}})^{cop}$.

The notion of group actions on algebras and the associated crossed products generalize to Hopf algebras as follows: A *(left) Hopf module action* of \mathcal{G} on a unital algebra \mathcal{M} is a linear

map $\triangleright : \mathcal{G} \otimes \mathcal{M} \rightarrow \mathcal{M}$ satisfying for all $m, n \in \mathcal{M}$ and $a, b \in \mathcal{G}$

$$a \triangleright (b \triangleright m) = (ab) \triangleright m$$

$$a \triangleright (mn) = (a_{(1)} \triangleright m)(a_{(2)} \triangleright n)$$

$$\mathbf{1} \triangleright m = m, \quad a \triangleright \mathbf{1}_{\mathcal{M}} = \epsilon(a) \mathbf{1}_{\mathcal{M}}$$

Note that for group like elements $a \in \mathcal{G}$ (i.e. $\Delta(a) = a \otimes a$ and a invertible), $a \triangleright$ becomes an algebra automorphism of \mathcal{M} , which means that Hopf module actions generalize the notion of group actions. Right Hopf module actions $\triangleleft : m \otimes a \mapsto m \triangleleft a$ are defined analogously. There is also a dual version of Hopf module actions: A *right coaction* of \mathcal{G} on an algebra \mathcal{M} is an algebra map $\rho : \mathcal{M} \rightarrow \mathcal{M} \otimes \mathcal{G}$ satisfying

$$(\rho \otimes \text{id}) \circ \rho = (\text{id} \otimes \Delta) \circ \rho \tag{1.1}$$

$$(\text{id} \otimes \epsilon) \circ \rho = \text{id}. \tag{1.2}$$

Similarly, a *left coaction* λ is an algebra map $\lambda : \mathcal{M} \rightarrow \mathcal{G} \otimes \mathcal{M}$ satisfying

$$(\text{id} \otimes \lambda) \circ \lambda = (\Delta \otimes \text{id}) \circ \lambda \tag{1.3}$$

$$(\epsilon \otimes \text{id}) \circ \lambda = \text{id}. \tag{1.4}$$

Similarly as for coproducts we will use the suggestive notations

$$\begin{aligned} \rho(m) &= m_{(0)} \otimes m_{(1)} \\ (\rho \otimes \text{id})(\rho(m)) &\equiv (\text{id} \otimes \Delta)(\rho(m)) = m_{(0)} \otimes m_{(1)} \otimes m_{(2)} \\ \lambda(m) &= m_{(-1)} \otimes m_{(0)} \\ (\Delta \otimes \text{id}) \circ \lambda &\equiv (\text{id} \otimes \lambda)(\lambda(m)) = m_{(-2)} \otimes m_{(-1)} \otimes m_{(0)} \end{aligned} \tag{1.5}$$

etc., where again summation indices and a summation symbol are suppressed. In this way we will always have $m_{(i)} \in \mathcal{G}$ for $i \neq 0$ and $m_{(0)} \in \mathcal{M}$. The notions of actions and coactions are dual to each other in the sense that there is a one-to-one correspondence between right (left) coactions of \mathcal{G} on \mathcal{M} and left (right) Hopf module actions, respectively, of $\hat{\mathcal{G}}$ on \mathcal{M} given for $\psi \in \hat{\mathcal{G}}$ and $m \in \mathcal{M}$ by

$$\psi \triangleright m := (\text{id} \otimes \psi)(\rho(m)) \tag{1.6}$$

$$m \triangleleft \psi := (\psi \otimes \text{id})(\lambda(m)) \tag{1.7}$$

where $\varphi, \psi \in \hat{\mathcal{G}}$ and $m, n \in \mathcal{M}$. As a particular example we recall the case $\mathcal{M} = \mathcal{G}$ with $\rho = \lambda = \Delta$. In this case we denote the associated left and right actions of $\psi \in \hat{\mathcal{G}}$ on $a \in \mathcal{G}$ by $\psi \rightarrow a$ and $a \leftarrow \psi$, respectively. Analogously, choosing $\mathcal{M} = \mathcal{G}$ with $\rho = \lambda = \hat{\Delta}$, one arrives at left and right actions of \mathcal{G} on $\hat{\mathcal{G}}$, denoted by $a \rightarrow \psi$ and $\psi \leftarrow a$, respectively.

Crossed products. Given a right coaction $\rho : \mathcal{M} \rightarrow \mathcal{M} \otimes \mathcal{G}$ with dual left $\hat{\mathcal{G}}$ -action \triangleright one defines the (untwisted) *crossed product* (also called smash product) $\mathcal{M} \rtimes \hat{\mathcal{G}}$ to be the vector space $\mathcal{M} \otimes \hat{\mathcal{G}}$ with associative algebra structure given for $m, n \in \mathcal{M}$ and $\varphi, \psi \in \hat{\mathcal{G}}$ by

$$(m \rtimes \varphi)(n \rtimes \psi) = (m(\varphi_{(1)} \triangleright n) \rtimes \varphi_{(2)} \psi) \tag{1.8}$$

where we use the notation $m \rtimes \psi$ in place of $m \otimes \psi$ to emphasize the new algebraic structure. Then $\mathbf{1}_{\mathcal{M}} \rtimes \hat{\mathbf{1}}$ is the unit in $\mathcal{M} \rtimes \hat{\mathcal{G}}$ and $m \mapsto (m \rtimes \hat{\mathbf{1}})$, $\varphi \mapsto (\mathbf{1}_{\mathcal{M}} \rtimes \varphi)$ provide unital inclusions $\mathcal{M} \rightarrow \mathcal{M} \rtimes \hat{\mathcal{G}}$ and $\hat{\mathcal{G}} \rightarrow \mathcal{M} \rtimes \hat{\mathcal{G}}$, respectively. Similarly if $\lambda : \mathcal{M} \rightarrow \mathcal{G} \otimes \mathcal{M}$ is a left coaction with dual right action \triangleleft then $\hat{\mathcal{G}} \ltimes \mathcal{M}$ denotes the associative algebra structure on $\hat{\mathcal{G}} \otimes \mathcal{M}$ given by

$$(\varphi \ltimes m)(\psi \ltimes n) = (\varphi \psi_{(1)} \ltimes (m \triangleleft \psi_{(2)})n) \tag{1.9}$$

containing again \mathcal{M} and $\hat{\mathcal{G}}$ as unital subalgebras. If there are several coactions under consideration we will also write $\mathcal{M}_\rho \rtimes \hat{\mathcal{G}}$ and $\hat{\mathcal{G}} \ltimes_\lambda \mathcal{M}$, respectively. We note that (1.8) implies that as an algebra

$$\mathcal{M} \rtimes \hat{\mathcal{G}} = \mathcal{M}\hat{\mathcal{G}} = \hat{\mathcal{G}}\mathcal{M}$$

where we have identified $\mathcal{M} \equiv \mathcal{M} \rtimes \hat{\mathbf{1}}$ and $\hat{\mathcal{G}} \equiv \mathbf{1}_{\mathcal{M}} \rtimes \hat{\mathcal{G}}$. In fact using the antipode axioms one easily verifies from (1.8)

$$m \rtimes \varphi = (m \rtimes \hat{\mathbf{1}})(\mathbf{1}_{\mathcal{M}} \rtimes \varphi) = (\mathbf{1}_{\mathcal{M}} \rtimes \varphi_{(2)})(\hat{S}^{-1}(\varphi_{(1)}) \triangleright m) \rtimes \hat{\mathbf{1}} \tag{1.10}$$

Similar statements hold in $\hat{\mathcal{G}} \ltimes \mathcal{M}$. More generally we have

LEMMA 1.1. Let $\triangleright : \hat{\mathcal{G}} \otimes \mathcal{M} \rightarrow \mathcal{M}$ be a left Hopf module action and let \mathcal{A} be an algebra containing \mathcal{M} and $\hat{\mathcal{G}}$ as unital subalgebras. Then in \mathcal{A} the relations

$$\varphi m = (\varphi_{(1)} \triangleright m) \varphi_{(2)}, \quad \forall \varphi \in \hat{\mathcal{G}}, \forall m \in \mathcal{M} \quad (1.11)$$

$$m \varphi = \varphi_{(2)} (\hat{S}^{-1}(\varphi_{(1)}) \triangleright m), \quad \forall \varphi \in \hat{\mathcal{G}}, \forall m \in \mathcal{M} \quad (1.12)$$

are equivalent and if these hold then $\mathcal{M}\hat{\mathcal{G}} = \hat{\mathcal{G}}\mathcal{M} \subset \mathcal{A}$ is a subalgebra and

$$\mathcal{M} \rtimes \hat{\mathcal{G}} \ni (m \rtimes \varphi) \mapsto m \varphi \in \mathcal{M}\hat{\mathcal{G}},$$

is an algebra epimorphism.

The proof of Lemma 1.1 is obvious from the antipode axioms and therefore omitted. A similar statement of course holds for the crossed product $\hat{\mathcal{G}} \ltimes \mathcal{M}$.

Generating matrices. We conclude this introductory part by describing crossed products in terms of the “generating matrix” formalism as advocated by the St. Petersburg school. Our presentation will closely follow the review of [Nil97]. First we note that since \mathcal{G} is finite dimensional we may identify $\text{Hom}_{\mathbb{C}}(\hat{\mathcal{G}}, V) \cong \mathcal{G} \otimes V$ for any \mathbb{C} -vector space V . In particular, the relation

$$T(\varphi) = (\varphi \otimes \text{id})(\mathbf{T}), \quad \forall \varphi \in \hat{\mathcal{G}}, \quad (1.13)$$

provides a one-to-one correspondence between algebra maps $T : \hat{\mathcal{G}} \rightarrow \mathcal{A}$ into some target algebra \mathcal{A} and elements $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ satisfying

$$\mathbf{T}^{13} \mathbf{T}^{23} = (\Delta \otimes \text{id})(\mathbf{T}) \quad (1.14)$$

where (1.14) is to be understood as an identity in $\mathcal{G} \otimes \mathcal{G} \otimes \mathcal{A}$, the upper indices indicating the canonical embedding of tensor factors (e.g. $\mathbf{T}^{23} = \mathbf{1}_{\mathcal{G}} \otimes \mathbf{T}$, etc.). Throughout, we will call elements $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ *normal*, if

$$(\epsilon \otimes \text{id})(\mathbf{T}) = \mathbf{1}_{\mathcal{A}},$$

which in (1.13) is equivalent to $T : \hat{\mathcal{G}} \rightarrow \mathcal{A}$ being unit preserving. In what follows, the target algebra \mathcal{A} may always be arbitrary. In the particular case $\mathcal{A} = \text{End}(V)$ we would be talking of representations of $\hat{\mathcal{G}}$ on V , or more generally, as discussed in Lemma 1.3 below, of representations of $\mathcal{M} \rtimes \hat{\mathcal{G}}$ or $\hat{\mathcal{G}} \ltimes \mathcal{M}$, respectively, on V .

DEFINITION 1.2. Let $\lambda : \mathcal{M} \rightarrow \mathcal{G} \otimes \mathcal{M}$ be a left coaction and let $\gamma : \mathcal{M} \rightarrow \mathcal{A}$ be an algebra map. An *implementer* of λ in \mathcal{A} (with respect to γ) is an element $\mathbf{L} \in \mathcal{G} \otimes \mathcal{A}$ satisfying

$$[\mathbf{1}_{\mathcal{G}} \otimes \gamma(m)] \mathbf{L} = \mathbf{L} [(\text{id}_{\mathcal{G}} \otimes \gamma)(\lambda(m))] \quad (1.15)$$

for all $m \in \mathcal{M}$. Similarly, an *implementer* in \mathcal{A} of a right coaction $\rho : \mathcal{M} \rightarrow \mathcal{M} \otimes \mathcal{G}$ is an element $\mathbf{R} \in \mathcal{G} \otimes \mathcal{A}$ satisfying (denoting $\rho^{op} = \tau_{\mathcal{M} \otimes \mathcal{G}} \circ \rho$, τ being the permutation of tensor factors)

$$\mathbf{R} [\mathbf{1}_{\mathcal{G}} \otimes \gamma(m)] = [(\text{id} \otimes \gamma)(\rho^{op}(m))] \mathbf{R} \quad (1.16)$$

We now have

LEMMA 1.3. Under the conditions of Definition 1.2 the relations

$$\gamma_L(\varphi \ltimes m) := (\varphi \otimes \text{id})(\mathbf{L}) \gamma(m)$$

$$\gamma_R(m \rtimes \varphi) := \gamma(m) (\varphi \otimes \text{id})(\mathbf{R})$$

provide one-to-one correspondences between algebra maps $\gamma_L : \hat{\mathcal{G}} \ltimes \mathcal{M} \rightarrow \mathcal{A}$ ($\gamma_R : \mathcal{M} \rtimes \hat{\mathcal{G}} \rightarrow \mathcal{A}$) extending γ and normal λ -implementers $\mathbf{L} \in \mathcal{G} \otimes \mathcal{A}$ (normal ρ -implementers $\mathbf{R} \in \mathcal{G} \otimes \mathcal{A}$), respectively, satisfying

$$\mathbf{L}^{13} \mathbf{L}^{23} = (\Delta \otimes \text{id})(\mathbf{L})$$

$$\mathbf{R}^{13} \mathbf{R}^{23} = (\Delta \otimes \text{id})(\mathbf{R})$$

PROOF. Writing $R(\varphi) := (\varphi \otimes \text{id})(\mathbf{R}) \equiv \gamma_R(\mathbf{1}_{\mathcal{M}} \rtimes \varphi) \in \mathcal{A}$ and using $\text{Hom}_{\mathbb{C}}(\hat{\mathcal{G}}, \mathcal{A}) \cong \mathcal{G} \otimes \mathcal{A}$, the relation $\mathbf{R} \leftrightarrow \gamma_R$ is one-to-one. The implementer property (1.16) is then equivalent to $R(\varphi)\gamma(m) = \gamma(\varphi_{(1)} \triangleright m)R(\varphi_{(2)})$ and \mathbf{R} is normal iff γ_R is unit preserving. Together with the remarks (1.13 - 1.14) this is further equivalent to γ_R defining an algebra map, similarly as in Lemma 1.1. The argument for γ_L is analogous. \square

We finally note that the equivalence (1.11) \Leftrightarrow (1.12) can be reformulated for implementers as follows

LEMMA 1.4. *Under the conditions of Definition 1.2 denote $\lambda(m) = m_{(-1)} \otimes m_{(0)}$ and $\rho(m) = m_{(0)} \otimes m_{(1)}$. Dropping the symbol γ we then have*

$$(1.16) \Leftrightarrow [\mathbf{1}_{\mathcal{G}} \otimes m] \mathbf{R} = [S^{-1}(m_{(1)}) \otimes \mathbf{1}_{\mathcal{A}}] \mathbf{R} [\mathbf{1}_{\mathcal{G}} \otimes m_{(0)}], \quad \forall m \in \mathcal{M} \quad (1.17)$$

$$(1.15) \Leftrightarrow \mathbf{L} [\mathbf{1}_{\mathcal{G}} \otimes m] = [\mathbf{1}_{\mathcal{G}} \otimes m_{(0)}] \mathbf{L} [S^{-1}(m_{(-1)}) \otimes \mathbf{1}_{\mathcal{A}}], \quad \forall m \in \mathcal{M} \quad (1.18)$$

PROOF. Suppose \mathbf{R} is an implementer of ρ . Then by (1.16)

$$[S^{-1}(m_{(1)}) \otimes \mathbf{1}_{\mathcal{A}}] \mathbf{R} [\mathbf{1}_{\mathcal{G}} \otimes m_{(0)}] = [S^{-1}(m_{(2)})m_{(1)} \otimes m_{(0)}] \mathbf{R} = [\mathbf{1}_{\mathcal{G}} \otimes m] \mathbf{R}$$

by (1.2) and the antipode axioms. Conversely, if \mathbf{R} satisfies the right equality in (1.17), then

$$\mathbf{R} [\mathbf{1}_{\mathcal{G}} \otimes m] = [m_{(2)}S^{-1}(m_{(1)}) \otimes \mathbf{1}_{\mathcal{A}}] \mathbf{R} [\mathbf{1}_{\mathcal{G}} \otimes m_{(0)}] = [m_{(1)} \otimes m_{(0)}] \mathbf{R}$$

proving (1.16). The equivalence (1.18) is proven analogously. \square

1.2. Two-sided coactions and diagonal crossed products

In Chapter 2 we will give a straightforward generalization of the notion of coactions to quasi-Hopf algebras. However, in general an associated notion of a crossed product extension $\mathcal{M} \rtimes \hat{\mathcal{G}}$ will not be well defined as an associative algebra, basically because in the quasi-Hopf case the natural product in $\hat{\mathcal{G}}$ is not associative. We are now going to provide a new construction of what we call a *diagonal crossed product* which will allow to escape this obstruction when generalized to the quasi-Hopf case. Our diagonal crossed products are always based on *two-sided coactions* or, equivalently, on pairs of commuting left and right coactions. These structures are largely motivated by the specific example $\mathcal{M} = \mathcal{G}$, where our methods reproduce the quantum double $\mathcal{D}(\mathcal{G})$.

DEFINITION 1.5. *A two-sided coaction of \mathcal{G} on an algebra \mathcal{M} is an algebra map $\delta : \mathcal{M} \rightarrow \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ satisfying*

$$(\text{id}_{\mathcal{G}} \otimes \delta \otimes \text{id}_{\mathcal{G}}) \circ \delta = (\Delta \otimes \text{id}_{\mathcal{M}} \otimes \Delta) \circ \delta \quad (1.19)$$

$$(\epsilon \otimes \text{id}_{\mathcal{M}} \otimes \epsilon) \circ \delta = \text{id}_{\mathcal{M}} \quad (1.20)$$

An example of a two-sided coaction is given by $\mathcal{M} = \mathcal{G}$ and $\delta := D \equiv (\Delta \otimes \text{id}) \circ \Delta$. More generally let $\lambda : \mathcal{M} \rightarrow \mathcal{G} \otimes \mathcal{M}$ and $\rho : \mathcal{M} \rightarrow \mathcal{M} \otimes \mathcal{G}$ be a left and a right coaction, respectively. We say that λ and ρ *commute*, if

$$(\lambda \otimes \text{id}) \circ \rho = (\text{id} \otimes \rho) \circ \lambda \quad (1.21)$$

It is straight forward to check that in this case

$$\delta := (\lambda \otimes \text{id}) \circ \rho \equiv (\text{id} \otimes \rho) \circ \lambda \quad (1.22)$$

provides a two-sided coaction. Conversely, given a two-sided coaction $\delta : \mathcal{M} \rightarrow \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ then $\lambda := (\text{id} \otimes \text{id} \otimes \epsilon) \circ \delta$ and $\rho := (\epsilon \otimes \text{id} \otimes \text{id}) \circ \delta$ provide a pair of commuting left and right coactions, respectively, obeying Eq. (1.22). Thus using the notation (1.5) we may write

$$\delta(m) = m_{(-1)} \otimes m_{(0)} \otimes m_{(1)} \quad (1.23)$$

etc., implying again the usual summation conventions. We remark that in the quasi-coassociative setting of Chapter 2 the relation between two-sided coactions and pairs (λ, ρ) of left and right coactions becomes more involved, justifying the treatment of two-sided coactions as distinguished objects on their own right also in the present setting. Next, in view of (1.22) we also have a one-to-one correspondence between two-sided coactions δ of \mathcal{G} on \mathcal{M} and pairs of mutually commuting left and right Hopf module actions, \triangleright and \triangleleft , of $\hat{\mathcal{G}}$ on \mathcal{M} , the relation being given by

$$(\varphi \otimes \text{id} \otimes \psi)(\delta(m)) = \psi \triangleright m \triangleleft \varphi, \quad (1.24)$$

where $\varphi, \psi \in \hat{\mathcal{G}}$ and $m \in \mathcal{M}$. This allows to construct as a new algebra the *right diagonal crossed product* $\mathcal{M} \rtimes \hat{\mathcal{G}}$ as follows.

PROPOSITION 1.6. Let $\delta = (\lambda \otimes \text{id}_{\mathcal{G}}) \circ \rho = (\text{id}_{\mathcal{G}} \otimes \rho) \circ \lambda$ be a two-sided coaction of \mathcal{G} on \mathcal{M} and let \triangleright and \triangleleft be the associated commuting pair of left and right actions of $\hat{\mathcal{G}}$ on \mathcal{M} . Define on $\mathcal{M} \otimes \hat{\mathcal{G}}$ the product

$$(m \bowtie \varphi)(n \bowtie \psi) := (m(\varphi_{(1)} \triangleright n \triangleleft \hat{S}^{-1}(\varphi_{(3)})) \bowtie \varphi_{(2)} \psi) \quad (1.25)$$

where we write $(m \bowtie \varphi)$ in place of $(m \otimes \varphi)$ to distinguish the new algebraic structure. Then with this product $\mathcal{M} \otimes \hat{\mathcal{G}}$ becomes an associative algebra with unit $(\mathbf{1}_{\mathcal{M}} \bowtie \hat{\mathbf{1}})$ containing $\mathcal{M} \equiv \mathcal{M} \bowtie \hat{\mathbf{1}}$ and $\hat{\mathcal{G}} \equiv \mathbf{1}_{\mathcal{M}} \bowtie \hat{\mathcal{G}}$ as unital subalgebras.

PROOF. For $m, m', n \in \mathcal{M}$ and $\varphi, \psi, \xi \in \hat{\mathcal{G}}$ we compute

$$\begin{aligned} [(m \bowtie \varphi)(m' \bowtie \psi)](n \bowtie \xi) &= [m(\varphi_{(1)} \triangleright m' \triangleleft \hat{S}^{-1}(\varphi_{(3)})) \bowtie \varphi_{(2)} \psi](n \bowtie \xi) \\ &= [m(\varphi_{(1)} \triangleright m' \triangleleft \hat{S}^{-1}(\varphi_{(5)}))(\varphi_{(2)} \psi_{(1)} \triangleright n \triangleleft \hat{S}^{-1}(\psi_{(3)}) \hat{S}^{-1}(\varphi_{(4)})) \bowtie (\varphi_{(3)} \psi_{(2)} \xi)] \\ &= m[\varphi_{(1)} \triangleright [m'(\psi_{(1)} \triangleright n \triangleleft \hat{S}^{-1}(\psi_{(3)}))] \triangleleft \hat{S}^{-1}(\varphi_{(3)})] \bowtie (\varphi_{(2)} \psi_{(2)} \xi) \\ &= (m \bowtie \varphi)[(m' \bowtie \psi)(n \bowtie \xi)], \end{aligned}$$

which proves the associativity. The remaining statements follow trivially from $\varphi \triangleright \mathbf{1}_{\mathcal{M}} = \mathbf{1}_{\mathcal{M}} \triangleleft \varphi = \epsilon(\varphi)\mathbf{1}_{\mathcal{M}}$ and the counit axioms. \square

We emphasize that while Proposition 1.6 still is almost trivial as it stands, its true power only appears when generalized to quasi-Hopf algebras \mathcal{G} .

DEFINITION 1.7. Under the setting of Proposition 1.6 we define the *right diagonal crossed product* $\mathcal{M}_{\delta} \bowtie \hat{\mathcal{G}} \equiv {}_{\lambda}\mathcal{M}_{\rho} \bowtie \hat{\mathcal{G}}$ to be the vector space $\mathcal{M} \otimes \hat{\mathcal{G}}$ with associative multiplication structure (1.25).

In cases where the two-sided coaction δ is unambiguously understood from the context we will also write $\mathcal{M} \bowtie \hat{\mathcal{G}}$. We emphasize already at this place that in Chapter 2 not every two-sided coaction will be given as $\delta = (\lambda \otimes \text{id}_{\mathcal{G}}) \circ \rho$ (or $\delta = (\text{id}_{\mathcal{G}} \otimes \rho) \circ \lambda$), in which case the notations $\mathcal{M}_{\delta} \bowtie \hat{\mathcal{G}}$ and ${}_{\lambda}\mathcal{M}_{\rho} \bowtie \hat{\mathcal{G}}$ will denote different (although still equivalent) extensions of \mathcal{M} . Here we freely use either one of them. If $\delta = \text{id}_{\mathcal{G}} \otimes \rho$ for some right coaction ρ then $\mathcal{M}_{\delta} \bowtie \hat{\mathcal{G}} = \mathcal{M}_{\rho} \bowtie \hat{\mathcal{G}}$. More generally, ${}_{\lambda}\mathcal{M}_{\rho} \bowtie \hat{\mathcal{G}}$ may be identified as a subalgebra of $\hat{\mathcal{G}} \times ({}_{\lambda}\mathcal{M}_{\rho} \bowtie \hat{\mathcal{G}}) \equiv (\hat{\mathcal{G}} \times {}_{\lambda}\mathcal{M}_{\rho}) \bowtie \hat{\mathcal{G}}$ using the injective algebra map

$$\begin{aligned} {}_{\lambda}\mathcal{M}_{\rho} \bowtie \hat{\mathcal{G}} \ni (m \bowtie \varphi) &\leftrightarrow (\hat{\mathbf{1}} \times m \times \hat{\mathbf{1}})(\varphi_{(2)} \times \mathbf{1}_{\mathcal{M}} \times \varphi_{(1)}) \\ &\equiv [\varphi_{(2)} \times (m \triangleleft \varphi_{(3)}) \times \varphi_{(1)}] \in \hat{\mathcal{G}} \times {}_{\lambda}\mathcal{M}_{\rho} \times \hat{\mathcal{G}} \end{aligned}$$

which we leave to the reader to check. This also motivates our choice of calling the crossed product $\mathcal{M} \bowtie \hat{\mathcal{G}}$ “diagonal”.

The quantum double. In the case $\mathcal{M} = \mathcal{G}$ and $\delta := D \equiv (\Delta \otimes \text{id}) \circ \Delta$, the formula (1.25) coincides with the multiplication rule in the quantum double $\mathcal{D}(\mathcal{G})$ [Dri86],[Maj90], i.e.

$$\mathcal{D}(\mathcal{G}) = \mathcal{G}_D \bowtie \hat{\mathcal{G}}. \quad (1.26)$$

It is well known, that $\mathcal{D}(\mathcal{G})$ is itself again a Hopf algebra with coproduct Δ_D given by

$$\Delta_D(a \bowtie_D \varphi) = (a_{(1)} \bowtie_D \varphi_{(2)}) \otimes (a_{(2)} \bowtie_D \varphi_{(1)}) \quad (1.27)$$

where $a \in \mathcal{G}$ and $\varphi \in \hat{\mathcal{G}}$. It turns out that this result generalizes to diagonal crossed products as follows

PROPOSITION 1.8. Let $\delta : \mathcal{M} \rightarrow \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ be a two-sided coaction. Then $\mathcal{M} \bowtie \hat{\mathcal{G}}$ admits a commuting pair of coactions $\lambda_D : \mathcal{M} \bowtie \hat{\mathcal{G}} \rightarrow \mathcal{D}(\mathcal{G}) \otimes (\mathcal{M} \bowtie \hat{\mathcal{G}})$ and $\rho_D : \mathcal{M} \bowtie \hat{\mathcal{G}} \rightarrow (\mathcal{M} \bowtie \hat{\mathcal{G}}) \otimes \mathcal{D}(\mathcal{G})$ given by

$$\begin{aligned} \lambda_D(m \bowtie \varphi) &= (m_{(-1)} \bowtie_D \varphi_{(2)}) \otimes (m_{(0)} \bowtie \varphi_{(1)}) \\ \rho_D(m \bowtie \varphi) &= (m_{(0)} \bowtie \varphi_{(2)}) \otimes (m_{(1)} \bowtie_D \varphi_{(1)}) \end{aligned}$$

where elements in $\mathcal{D}(\mathcal{G})$ are written as $(a \bowtie_D \varphi)$, $a \in \mathcal{G}$, $\varphi \in \hat{\mathcal{G}}$.

PROOF. In view of (1.27) the comodule axioms and the commutativity (1.22) are obvious. That λ_D and ρ_D provide algebra maps is shown by direct computation, which we leave to the reader. \square

Let us also recall the well known Hopf algebra identity $\mathcal{D}(\hat{\mathcal{G}}) \cong \mathcal{D}(\mathcal{G})^{cop}$, with algebra isomorphism given by

$$\mathcal{D}(\hat{\mathcal{G}}) \ni (\varphi \bowtie_{\hat{\mathcal{D}}} a) \mapsto (\mathbf{1} \bowtie_D \varphi)(a \bowtie_D \hat{\mathbf{1}}) \in \mathcal{D}(\mathcal{G}). \quad (1.28)$$

This generalizes to diagonal crossed products in the sense that they may equivalently be modeled on the vector space $\hat{\mathcal{G}} \otimes \mathcal{M}$.

COROLLARY 1.9. *We define the left diagonal crossed product $\hat{\mathcal{G}} \bowtie \mathcal{M}_\delta$ as the vector space $\hat{\mathcal{G}} \otimes \mathcal{M}$ with multiplication given by*

$$(\varphi \bowtie m)(\psi \bowtie n) := \varphi\psi_{(2)} \bowtie (\hat{S}^{-1}(\psi_{(1)}) \triangleright m \triangleleft \psi_{(3)})n \quad (1.29)$$

This defines an associative algebra, and the analog of (1.28) is given by

$$\hat{\mathcal{G}} \bowtie \mathcal{M} \ni \varphi \bowtie m \mapsto (\mathbf{1}_{\mathcal{M}} \bowtie \varphi)(m \bowtie \hat{\mathbf{1}}) \equiv \varphi_{(1)} \triangleright m \triangleleft \hat{S}^{-1}(\varphi_{(3)}) \bowtie \varphi_{(2)} \in \mathcal{M} \bowtie \hat{\mathcal{G}}, \quad (1.30)$$

which provides an isomorphism of algebras restricting to the identity on \mathcal{M} .

The proof of Corollary 1.9 is straight forward from the antipode axioms. The reader is invited to check that in the case $\mathcal{M} = \mathcal{G}$ and $\delta = (\Delta \otimes \text{id}) \circ \Delta$ we recover $\hat{\mathcal{G}} \bowtie \mathcal{M}_\delta = \mathcal{D}(\hat{\mathcal{G}})$.

1.3. Generating matrices

Similarly as in Lemma 1.3 we now describe the defining relations of diagonal crossed products in terms of a generating matrix \mathbf{T} . However, whereas in Lemma 1.3 the generating matrices \mathbf{L} and \mathbf{R} had to fulfill the *implementer* properties (1.15) or (1.16), respectively, the natural requirement here is that \mathbf{T} *intertwines* the left and right coactions associated with δ .

DEFINITION 1.10. Let (λ, ρ) be a commuting pair of left and right \mathcal{G} -coactions on \mathcal{M} and let $\gamma : \mathcal{M} \rightarrow \mathcal{A}$ be an algebra map into some target algebra \mathcal{A} . Then a $\lambda\rho$ -*intertwiner* in \mathcal{A} (with respect to γ) is an element $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ satisfying

$$\mathbf{T}\lambda_{\mathcal{A}}(m) = \rho_{\mathcal{A}}^{op}(m)\mathbf{T}, \quad \forall m \in \mathcal{M}, \quad (1.31)$$

where $\lambda_{\mathcal{A}} \equiv (\gamma \otimes \text{id}) \circ \lambda$ and $\rho_{\mathcal{A}} \equiv (\text{id} \otimes \gamma) \circ \rho$. A $\lambda\rho$ -intertwiner is called *coherent* if in $\mathcal{G} \otimes \mathcal{G} \otimes \mathcal{A}$ it satisfies

$$\mathbf{T}^{13}\mathbf{T}^{23} = (\Delta \otimes \text{id})(\mathbf{T}) \quad (1.32)$$

Similarly as in Lemma 1.4 we then have

LEMMA 1.11. *Let (\mathcal{M}, δ) be a two-sided \mathcal{G} -comodule algebra with associated commuting left and right \mathcal{G} -coactions (λ, ρ) , and let $\gamma : \mathcal{M} \rightarrow \mathcal{A}$ be an algebra map. Then for $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ the following properties are equivalent:*

- i) \mathbf{T} is a $\lambda\rho$ -intertwiner
- ii) $\mathbf{T} [\mathbf{1}_{\mathcal{G}} \otimes \gamma(m)] = [m_{(1)} \otimes \gamma(m_{(0)})] \mathbf{T} [S^{-1}(m_{(-1)}) \otimes \mathbf{1}_{\mathcal{A}}]$
- iii) $[\mathbf{1}_{\mathcal{G}} \otimes \gamma(m)] \mathbf{T} = [S^{-1}(m_{(1)}) \otimes \mathbf{1}_{\mathcal{A}}] \mathbf{T} [m_{(-1)} \otimes \gamma(m_{(0)})]$

PROOF. Suppose \mathbf{T} is a $\lambda\rho$ -intertwiner. Then

$$[m_{(1)} \otimes \gamma(m_{(0)})] \mathbf{T} [S^{-1}(m_{(-1)}) \otimes \mathbf{1}_{\mathcal{A}}] = \mathbf{T} [m_{(-1)} S^{-1}(m_{(-2)}) \otimes \gamma(m_{(0)})] = \mathbf{T} [\mathbf{1}_{\mathcal{G}} \otimes \gamma(m)]$$

by the antipode axiom. Conversely, if \mathbf{T} satisfies (ii) then

$$\mathbf{T} [m_{(-1)} \otimes \gamma(m_{(0)})] = [m_{(1)} \otimes \gamma(m_{(0)})] \mathbf{T} [S^{-1}(m_{(-1)})m_{(-2)} \otimes \mathbf{1}_{\mathcal{A}}] = [m_{(1)} \otimes \gamma(m_{(0)})] \mathbf{T}$$

proving (i) \Leftrightarrow (ii). The equivalence (i) \Leftrightarrow (iii) follows similarly. \square

We now conclude similarly as in Lemma 1.3.

PROPOSITION 1.12. *Let (\mathcal{M}, δ) be a two-sided \mathcal{G} -comodule algebra with associated commuting pair of coactions (λ, ρ) , and let $\gamma : \mathcal{M} \rightarrow \mathcal{A}$ be an algebra map. Then the relation*

$$\gamma_T(m \bowtie \varphi) = \gamma(m) (\varphi \otimes \text{id})(\mathbf{T}) \quad (1.33)$$

provides a one-to-one correspondence between normal coherent $\lambda\rho$ -intertwiners \mathbf{T} and unital algebra maps $\gamma_T : \lambda\mathcal{M}_\rho \bowtie \hat{\mathcal{G}} \rightarrow \mathcal{A}$ extending γ .

PROOF. Let $T(\varphi) := (\varphi \otimes \text{id})(\mathbf{T})$. Then (1.32) together with normality is equivalent to $\hat{\mathcal{G}} \ni \varphi \mapsto T(\varphi) \equiv \gamma_T(\mathbf{1}_{\mathcal{M}} \bowtie \varphi) \in \mathcal{A}$ being a unital algebra morphism and the correspondence $\mathbf{T} \leftrightarrow \gamma_T|_{\mathbf{1}_{\mathcal{M}} \bowtie \hat{\mathcal{G}}}$ is one-to-one. Clearly, γ_T extends γ and Lemma 1.11 ii) implies

$$T(\varphi) \gamma(m) = \gamma(\varphi_{(1)} \triangleright m \triangleleft S^{-1}(\varphi_{(3)})) T(\varphi_{(2)}), \quad \forall \varphi \in \hat{\mathcal{G}}, m \in \mathcal{M}$$

and therefore γ_T is an algebra map. Conversely, since $(m \bowtie \varphi) = (m \bowtie \hat{\mathbf{1}})(\mathbf{1}_{\mathcal{M}} \bowtie \varphi)$, any algebra map $\gamma : \lambda \mathcal{M}_\rho \bowtie \hat{\mathcal{G}} \rightarrow \mathcal{A}$ is of the form (1.33). \square

We remark that one could equivalently have chosen to work with

$$\gamma_T^{op}(\varphi \bowtie m) := (\varphi \otimes \text{id})(\mathbf{T}) \gamma(m) \tag{1.34}$$

to obtain algebra maps $\gamma_T^{op} : \hat{\mathcal{G}} \bowtie \lambda \mathcal{M}_\rho \rightarrow \mathcal{A}$. Note that by applying $(\varphi \otimes \text{id})$ to both sides the equivalence of (ii) and (iii) in Lemma 1.11 ensures that (1.30) is an isomorphism.

Applying the above formalism to the case $\mathcal{M} = \mathcal{G}$ and $\delta = D \equiv (\Delta \otimes \text{id}) \circ \Delta$ we realize that (1.31) becomes (suppressing the symbol γ)

$$\mathbf{T} \Delta(a) = \Delta_{op}(a) \mathbf{T}, \quad \forall a \in \mathcal{A} \tag{1.35}$$

In this special case we call \mathbf{T} a Δ -flip operator. As already remarked, in this case $\mathcal{G}_D \bowtie \hat{\mathcal{G}} \equiv \mathcal{D}(\mathcal{G})$ is the quantum double of \mathcal{G} , in which case Proposition 1.12 coincides with [Nil97], Lemma 5.2 describing $\mathcal{D}(\mathcal{G})$ as the unique algebra generated by \mathcal{G} and the entries of a generating Matrix $\mathbf{D} \equiv \mathbf{T}_{\mathcal{D}(\mathcal{G})} \in \mathcal{G} \otimes \mathcal{D}(\mathcal{G})$ satisfying (1.32) and (1.35).

More generally every diagonal crossed product $\mathcal{M} \bowtie \hat{\mathcal{G}}$ may be described as the unique algebra generated by \mathcal{M} and the entries of a generating matrix $\mathbf{\Gamma} \in \mathcal{G} \otimes (\mathcal{M} \bowtie \hat{\mathcal{G}})$ satisfying (1.31) and (1.32), by choosing $\mathcal{A} = \mathcal{M} \bowtie \hat{\mathcal{G}}$ in Proposition 1.12. This construction of diagonal crossed products in terms of generating matrices is summarized in the following Theorem, which we state in this explicit form, since it will allow a generalization to (weak) quasi-Hopf algebras in the next chapter.

THEOREM 1.13. *Let $(\mathcal{G}, \Delta, \epsilon, S)$ be a finite dimensional Hopf algebra and let (λ, ρ) be a commuting pair of (left and right) \mathcal{G} -coactions on an associative algebra \mathcal{M} .*

1. *Then there exists a unital associative algebra extension $\mathcal{M}_1 \supset \mathcal{M}$ together with a linear map $\gamma : \hat{\mathcal{G}} \rightarrow \mathcal{M}_1$ satisfying the following universal property:
 \mathcal{M}_1 is algebraically generated by \mathcal{M} and $\gamma(\hat{\mathcal{G}})$ and for any algebra map $\gamma : \mathcal{M} \rightarrow \mathcal{A}$ into some target algebra \mathcal{A} the relation*

$$\gamma_T(\gamma(\varphi)) = (\varphi \otimes \text{id})(\mathbf{T}), \quad \varphi \in \hat{\mathcal{G}} \tag{1.36}$$

provides a one-to-one correspondence between algebra maps $\gamma_T : \mathcal{M}_1 \rightarrow \mathcal{A}$ extending γ and elements $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ satisfying $(\epsilon \otimes \text{id}_{\mathcal{A}})(\mathbf{T}) = \mathbf{1}_{\mathcal{A}}$ and

$$\mathbf{T} \lambda_{\mathcal{A}}(m) = \rho_{\mathcal{A}}^{op}(m) \mathbf{T}, \quad \forall m \in \mathcal{M} \tag{1.37}$$

$$\mathbf{T}^{13} \mathbf{T}^{23} = (\Delta \otimes \text{id}_{\mathcal{A}})(\mathbf{T}), \tag{1.38}$$

where $\lambda_{\mathcal{A}}(m) := (\text{id}_{\mathcal{G}} \otimes \gamma)(\lambda(m))$ and $\rho_{\mathcal{A}}(m) := (\gamma \otimes \text{id}_{\mathcal{G}})(\rho(m))$.

2. *If $\mathcal{M} \subset \tilde{\mathcal{M}}_1$ and $\tilde{\gamma} : \hat{\mathcal{G}} \rightarrow \tilde{\mathcal{M}}_1$ satisfy the same universality property as in part 1.), then there exists a unique algebra isomorphism $f : \mathcal{M}_1 \rightarrow \tilde{\mathcal{M}}_1$ restricting to the identity on \mathcal{M} , such that $\tilde{\gamma} = f \circ \gamma$,*
3. *The linear maps*

$$\mu_L : \hat{\mathcal{G}} \otimes \mathcal{M} \ni (\varphi \otimes m) \mapsto \gamma(\varphi) m \in \mathcal{M}_1 \tag{1.39}$$

$$\mu_R : \mathcal{M} \otimes \hat{\mathcal{G}} \ni (m \otimes \varphi) \mapsto m \gamma(\varphi) \in \mathcal{M}_1 \tag{1.40}$$

provide isomorphisms of vector spaces.

PROOF. Putting $\mathcal{M}_1 = \mathcal{M} \bowtie \hat{\mathcal{G}}$, $\mu_R := \text{id}_{\mathcal{M} \otimes \hat{\mathcal{G}}}$ and μ_L the map given in (1.39), part 1.) and 3.) follow from Proposition 1.12 and Corollary 1.9. The uniqueness of \mathcal{M}_1 up to equivalence follows by standard arguments. \square

Putting $\Gamma := e_\mu \otimes (e^\mu) \in \mathcal{G} \otimes \mathcal{M}_1$ Theorem 1.13 implies that Γ itself satisfies the defining relations (1.37) and (1.38). We call Γ the *universal $\lambda\rho$ -intertwiner* in \mathcal{M}_1 . We again emphasize that once being stated Theorem 1.13 almost appears trivial. Its true power only arises when generalized to the quasi-coassociative setting in Chapter 2. Note that part 2. of Theorem 1.13 implies that the algebraic structures induced on $\hat{\mathcal{G}} \otimes \mathcal{M}$ and $\mathcal{M} \otimes \hat{\mathcal{G}}$ via $\mu_{L/R}^{-1}$ from \mathcal{M}_1 are uniquely fixed. They are given by the left- and right diagonal crossed products $\hat{\mathcal{G}} \bowtie \mathcal{M}$ and $\mathcal{M} \bowtie \hat{\mathcal{G}}$, respectively, defined above in Proposition 1.6 and Corollary 1.9.

1.4. Quantum group spin chains and lattice current algebras

We are now in the position to reformulate the Hopf spin chains of [NS97] and the lattice current algebras of [AFFS98] as iterated diagonal crossed products, thereby also reviewing the relationship between the two models. It will turn out to be convenient to use the notion of two-sided crossed products, which we will now introduce as a special type of diagonal crossed products.

1.4.1. Two-sided crossed products. A simple recipe to produce two-sided \mathcal{G} -comodule algebras (\mathcal{M}, δ) is by taking a right \mathcal{G} -comodule algebra (\mathcal{A}, ρ) and a left \mathcal{G} -comodule algebra (\mathcal{B}, λ) and define $\mathcal{M} = \mathcal{A} \otimes \mathcal{B}$ and

$$\delta(A \otimes B) := B_{(-1)} \otimes (A_{(0)} \otimes B_{(0)}) \otimes A_{(1)} \quad (1.41)$$

where $A \in \mathcal{A}$, $B \in \mathcal{B}$, $\rho(A) = A_{(0)} \otimes A_{(1)}$ and $\lambda(B) = B_{(-1)} \otimes B_{(0)}$. In terms of the $\hat{\mathcal{G}}$ -actions \triangleright on \mathcal{A} and \triangleleft on \mathcal{B} dual to ρ and λ , respectively, the $\hat{\mathcal{G}}$ -actions $\triangleright_{\mathcal{M}}$ and $\triangleleft_{\mathcal{M}}$ dual to (1.41) are given by

$$\varphi \triangleright_{\mathcal{M}} (A \otimes B) \triangleleft_{\mathcal{M}} \psi = (\varphi \triangleright A \otimes B \triangleleft \psi), \quad \varphi, \psi \in \hat{\mathcal{G}}. \quad (1.42)$$

Hence, we may construct the diagonal crossed product $\mathcal{M} \bowtie \mathcal{G}$ as before. It turns out that this example may be presented differently as a so-called *two-sided crossed product*.

PROPOSITION 1.14. *Let $\triangleright : \hat{\mathcal{G}} \otimes \mathcal{A} \rightarrow \mathcal{A}$ and $\triangleleft : \mathcal{B} \otimes \hat{\mathcal{G}} \rightarrow \mathcal{B}$ be a left and a right Hopf module action, respectively, with dual \mathcal{G} -coactions ρ, λ . Define the “two-sided crossed product” $\mathcal{A}_\rho \times \hat{\mathcal{G}} \times \lambda \mathcal{B}$ to be the vector space $\mathcal{A} \otimes \hat{\mathcal{G}} \otimes \mathcal{B}$ with multiplication structure*

$$(A \times \varphi \times B) (A' \times \psi \times B') = A(\varphi_{(1)} \triangleright A') \times \varphi_{(2)} \psi_{(1)} \times (B \triangleleft \psi_{(2)}) B'. \quad (1.43)$$

Then $\mathcal{A}_\rho \times \hat{\mathcal{G}} \times \lambda \mathcal{B}$ becomes an associative algebra with unit $\mathbf{1}_{\mathcal{A}} \times \hat{\mathbf{1}} \times \mathbf{1}_{\mathcal{B}}$ and

$$f : \mathcal{A} \times \hat{\mathcal{G}} \times \mathcal{B} \ni A \times \varphi \times B \mapsto ((A \otimes \mathbf{1}_{\mathcal{B}}) \bowtie \varphi) ((\mathbf{1}_{\mathcal{A}} \otimes B) \bowtie \hat{\mathbf{1}}) \in (\mathcal{A} \otimes \mathcal{B}) \bowtie \hat{\mathcal{G}} \quad (1.44)$$

provides an algebra isomorphism with inverse given by

$$f^{-1}((A \otimes B) \bowtie \varphi) = (\mathbf{1}_{\mathcal{A}} \times \hat{\mathbf{1}} \times B)(A \times \varphi \times \mathbf{1}_{\mathcal{B}}) \quad (1.45)$$

Instead of giving a direct proof, let us reformulate the above Proposition in terms of generating matrices. Setting $\mathbf{T} := \sum_{\mu} e_{\mu} \otimes (\mathbf{1}_{\mathcal{A}} \times e^{\mu} \times \mathbf{1}_{\mathcal{B}})$, where as before e_{μ} denotes a basis of \mathcal{G} with dual basis $e^{\mu} \in \hat{\mathcal{G}}$, the multiplication rule (1.43) is equivalent to \mathbf{T} satisfying

$$\begin{aligned} (\Delta \otimes \text{id})(\mathbf{T}) &= \mathbf{T}^{13} \mathbf{T}^{23} \\ [\mathbf{1} \otimes B] \mathbf{T} &= \mathbf{T} \lambda(B), \quad B \in \mathcal{B} \\ \mathbf{T} [\mathbf{1} \otimes A] &= \rho^{op} \mathbf{T}, \quad A \in \mathcal{A}, \end{aligned} \quad (1.46)$$

where we identify $\mathcal{A} \equiv \mathcal{A} \otimes \mathbf{1}_{\mathcal{B}}$, $\lambda \equiv \lambda \otimes \text{id}_{\mathcal{B}}$, etc. . Thus \mathbf{T} is a $\lambda\rho$ -intertwiner and since $\mathcal{A} \times \hat{\mathcal{G}} \times \mathcal{B}$ is generated by $\mathcal{M} = \mathcal{A} \otimes \mathcal{B}$ and the matrix entries $(\varphi \otimes \text{id})(\mathbf{T})$, $\varphi \in \hat{\mathcal{G}}$, it has to be isomorphic to $(\mathcal{A} \otimes \mathcal{B}) \bowtie \hat{\mathcal{G}}$. Denoting the $\lambda\rho$ -intertwiner in $\mathcal{G} \otimes ((\mathcal{A} \otimes \mathcal{B}) \bowtie \hat{\mathcal{G}})$ by Γ , one verifies that

$$(\text{id} \otimes f)(\mathbf{T}) = \Gamma, \quad (\text{id} \otimes f^{-1})(\Gamma) = \mathbf{T},$$

which by Proposition 1.12 implies that f is an isomorphism. We leave the details to the reader.

As a particular example of the setting of Proposition 1.14 we may choose $\mathcal{A} = \mathcal{B} = \mathcal{G}$ with its canonical left and right $\hat{\mathcal{G}}$ -action. It turns out that in this case the two-sided crossed product $\mathcal{G} \times \hat{\mathcal{G}} \times \mathcal{G} \equiv (\mathcal{G} \otimes \mathcal{G}) \bowtie \hat{\mathcal{G}}$ is isomorphic to the iterated crossed product $(\mathcal{G} \times \hat{\mathcal{G}}) \times \mathcal{G}$. More generally we have

PROPOSITION 1.15. *Let \mathcal{A} be a right \mathcal{G} -comodule algebra and consider the iterated crossed product $(\mathcal{A} \rtimes \hat{\mathcal{G}}) \rtimes \mathcal{G}$, where \mathcal{G} acts on $\mathcal{A} \rtimes \hat{\mathcal{G}}$ in the usual way by $a \triangleright (A \rtimes \varphi) := A \rtimes (a \rightarrow \varphi)$, $A \in \mathcal{A}$, $a \in \mathcal{G}$, $\varphi \in \hat{\mathcal{G}}$. Then as an algebra $(\mathcal{A} \rtimes \hat{\mathcal{G}}) \rtimes \mathcal{G} = \mathcal{A} \rtimes \hat{\mathcal{G}} \rtimes \mathcal{G}$ with trivial identification.*

PROOF. The claim follows from

$$A(\varphi_{(1)} \triangleright A') \rtimes \varphi_{(2)} \psi_{(1)} \rtimes (a \leftarrow \psi_{(2)}) b = A(\varphi_{(1)} \triangleright A') \rtimes \varphi_{(2)} (a_{(1)} \rightarrow \psi) \rtimes a_{(2)} b$$

as an identity in $\mathcal{A} \otimes \hat{\mathcal{G}} \otimes \mathcal{G}$, where we have used

$$\psi_{(1)} \otimes (a \leftarrow \psi_{(2)}) = \psi_{(1)} \langle a_{(1)} | \psi_{(2)} \rangle \otimes a_{(2)} = (a_{(1)} \rightarrow \psi) \otimes a_{(2)}$$

as an identity in $\hat{\mathcal{G}} \otimes \mathcal{G}$. \square

It will be shown in Chapter 5 that being a particular example of a two-sided (and therefore of a diagonal) crossed product the analogue of $(\mathcal{A} \rtimes \hat{\mathcal{G}}) \rtimes \mathcal{G} \equiv \mathcal{A} \rtimes \hat{\mathcal{G}} \rtimes \mathcal{G}$ may also be constructed for quasi-Hopf algebras \mathcal{G} . However, in this case $\mathcal{A} \rtimes \hat{\mathcal{G}}$ (if defined to be the linear subspace $\mathcal{A} \otimes \hat{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}}$) will no longer be a subalgebra of $\mathcal{A} \rtimes \hat{\mathcal{G}} \rtimes \mathcal{G}$. We will see in Chapter 5 that this fact is very much analogous to what happens in the field algebra constructions with quasi-Hopf symmetry as given by V. Schomerus [Sch95].

1.4.2. Hopf spin chains. Next, we point out that Proposition 1.14 and Proposition 1.15 also apply to the construction of Hopf algebraic quantum chains [NS97] as introduced in section 3 of the introduction. To see this let us shortly review the model of [NS97], where one considers even (odd) integers to represent the sites (links) of a one-dimensional lattice and where one places a copy of $\mathcal{G} \cong \mathcal{A}_{2i}$ on each site and a copy of $\hat{\mathcal{G}} \cong \mathcal{A}_{2i+1}$ on each link.

$$\begin{array}{c} \mathcal{G} \qquad \qquad \hat{\mathcal{G}} \qquad \qquad \mathcal{G} \\ \bullet \qquad \qquad \bullet \qquad \qquad \bullet \\ \hline 2i \qquad 2i+1 \qquad 2i+2 \end{array}$$

Non-vanishing commutation relations are then postulated only on neighboring site-link pairs, where one requires

$$\begin{aligned} A_{2i}(a) A_{2i-1}(\varphi) &= A_{2i-1}(a_{(1)} \rightarrow \varphi) A_{2i}(a_{(2)}) \\ A_{2i+1}(\varphi) A_{2i}(a) &= A_{2i}(\varphi_{(1)} \rightarrow a) A_{2i+1}(\varphi_{(2)}) \end{aligned} \tag{1.47}$$

Here $\mathcal{G} \ni a \mapsto A_{2i}(a) \in \mathcal{A}_{2i} \subset \mathcal{A}$ and $\hat{\mathcal{G}} \ni \varphi \mapsto A_{2i+1}(\varphi) \in \mathcal{A}_{2i+1} \subset \mathcal{A}$ denote the embedding of the single site (link) algebras into the global quantum chain \mathcal{A} . Denoting $\mathcal{A}_{i,j} \subset \mathcal{A}$ as the subalgebra generated by \mathcal{A}_{ν} , $i \leq \nu \leq j$, we clearly have from (1.47)

$$\begin{aligned} \mathcal{A}_{i,j+1} &= \mathcal{A}_{i,j} \rtimes \mathcal{A}_{j+1} \\ \mathcal{A}_{i-1,j} &= \mathcal{A}_{i-1} \rtimes \mathcal{A}_{i,j} \end{aligned} \tag{1.48}$$

Hence, by Proposition 1.14, we recognize the two-sided crossed products

$$\mathcal{A}_{2i,2j+2} \equiv (\mathcal{A}_{2i,2j} \rtimes \hat{\mathcal{G}}) \rtimes \mathcal{G} = \mathcal{A}_{2i,2j} \rtimes \hat{\mathcal{G}} \rtimes \mathcal{G} \tag{1.49}$$

More generally for all $i \leq \nu \leq j \Leftrightarrow 1$ we have

$$\mathcal{A}_{2i,j} = \mathcal{A}_{2i,2\nu} \rtimes \hat{\mathcal{G}} \rtimes \mathcal{A}_{2\nu+2,j} \tag{1.50}$$

where $\hat{\mathcal{G}} \equiv \mathcal{A}_{2\nu+1}$. The advantage of looking at it in this way again comes from the fact that the constructions (1.49) and (1.50) generalize to quasi-Hopf algebras \mathcal{G} whereas (1.48) do not. This observation will be needed to formulate a theory of Hopf spin models and lattice current algebras at roots of unity, see Chapter 5.

Next, we remark that the identifications (1.49),(1.50) may be iterated in the obvious way. This observation also generalizes to the situation where in Proposition 1.14 \mathcal{A} and \mathcal{B} are both two-sided \mathcal{G} -comodule algebras with dual $\hat{\mathcal{G}}$ actions denoted $\triangleright_{\mathcal{A}}$, $\triangleleft_{\mathcal{A}}$, $\triangleright_{\mathcal{B}}$, $\triangleleft_{\mathcal{B}}$, respectively. Then in the multiplication rule (1.43) only $\triangleright_{\mathcal{A}}$ and $\triangleleft_{\mathcal{B}}$ appear and one easily checks, that for $\varphi, \psi \in \hat{\mathcal{G}}$ and $A \in \mathcal{A}$, $B \in \mathcal{B}$ the definitions

$$\begin{aligned} \varphi \triangleright (A \rtimes \psi \rtimes B) &:= A \rtimes \psi \rtimes (\varphi \triangleright_{\mathcal{B}} B) \\ (A \rtimes \psi \rtimes B) \triangleleft \varphi &:= (A \triangleleft_{\mathcal{A}} \varphi) \rtimes \psi \rtimes B \end{aligned}$$

again define a two-sided \mathcal{G} -comodule structure on $\mathcal{A} \rtimes \hat{\mathcal{G}} \rtimes \mathcal{B}$. Hence, we have a multiplication law on two-sided \mathcal{G} -comodule algebras which is in fact associative, i.e. as a two-sided \mathcal{G} -comodule algebra

$$(\mathcal{A} \rtimes \hat{\mathcal{G}} \rtimes \mathcal{B}) \rtimes \hat{\mathcal{G}} \rtimes \mathcal{C} = \mathcal{A} \rtimes \hat{\mathcal{G}} \rtimes (\mathcal{B} \rtimes \hat{\mathcal{G}} \rtimes \mathcal{C}) \quad (1.51)$$

which the reader will easily check. Obviously, one may also consider mixed cases, e.g. where in (1.51) \mathcal{A} is only a right \mathcal{G} -comodule algebra, but \mathcal{B} and \mathcal{C} are two-sided, in which case (1.51) would be an identity between right \mathcal{G} -comodule algebras.

Let us now formulate the algebraic properties of Hopf Spin chains in terms of generating matrices, using the relations (1.46). Defining the generating ‘‘link operators’’ $\mathbf{L}_{2i+1} := \sum_{\mu} e_{\mu} \otimes A_{2i+1}(e^{\mu})$, $\mathcal{A}_{2i,2j}$ is the unique algebra generated by $\mathcal{G}^{\otimes j-i} \equiv \mathcal{A}_{2i} \otimes \mathcal{A}_{2i+2} \otimes \cdots \otimes \mathcal{A}_{2j}$ and the entries of generating matrices $\mathbf{L}_{2\nu+1} \in \mathcal{G} \otimes \mathcal{A}_{2i,2j}$, $i \leq \nu \leq j$ obeying the relations

$$\mathbf{L}_{2k+1}^{13} \mathbf{L}_{2l+1}^{23} = \mathbf{L}_{2l+1}^{23} \mathbf{L}_{2k+1}^{13}, \quad \forall k \neq l \quad (1.52a)$$

$$[\mathbf{1} \otimes A_{2k}(a)] \mathbf{L}_{2l+1} = \mathbf{L}_{2l+1} [\mathbf{1} \otimes A_{2k}(a)], \quad \forall k \neq l, l+1 \quad (1.52b)$$

$$\mathbf{L}_{2k+1}^{13} \mathbf{L}_{2k+1}^{23} = (\Delta \otimes \text{id})(\mathbf{L}_{2k+1}) \quad (1.52c)$$

$$[\mathbf{1} \otimes A_{2k}(a)] \mathbf{L}_{2k-1} = \mathbf{L}_{2k-1} [a_{(1)} \otimes A_{2k}(a_{(2)})] \quad (1.52d)$$

$$\mathbf{L}_{2k+1} [\mathbf{1} \otimes A_{2k}(a)] = [a_{(2)} \otimes A_{2k}(a_{(1)})] \mathbf{L}_{2k+1} \quad (1.52e)$$

Let us shortly comment on these relations for the sake of getting a better understanding of the ‘‘language’’ of generating matrices. Eqs. (1.52a) and (1.52b) express locality in the sense that they give nontrivial commutation relations only on neighboring site link pairs. Eq. (1.52c) may be viewed as an operator product expansion. Provided \mathcal{G} is quasitriangular with R-matrix $R \in \mathcal{G} \otimes \mathcal{G}$, it implies the braiding relations

$$R^{12} \mathbf{L}^{13} \mathbf{L}^{23} = \mathbf{L}^{23} \mathbf{L}^{13} R^{12}.$$

Finally, (1.52d) and (1.52e) express covariance properties of the link operators \mathbf{L} .

We finish our discussion of Hopf spin chains by noting that the identification (1.50) together with Proposition 1.14 and Proposition 1.8 immediately imply that quantum chains of the type (1.47) admit localized commuting left and right coactions of the quantum double $\mathcal{D}(\mathcal{G})$, which is precisely the result of Theorem 4.1 of [NS97]. In fact, applied to the example in Proposition 1.14, Proposition 1.8 gives

COROLLARY 1.16. *Under the setting of Proposition 1.14 we have a commuting pair of left and right coactions $\rho_D : \mathcal{A} \rtimes \hat{\mathcal{G}} \rtimes \mathcal{B} \rightarrow (\mathcal{A} \rtimes \hat{\mathcal{G}} \rtimes \mathcal{B}) \otimes \mathcal{D}(\mathcal{G})$ and $\lambda_D : \mathcal{A} \rtimes \hat{\mathcal{G}} \rtimes \mathcal{B} \rightarrow \mathcal{D}(\mathcal{G}) \otimes (\mathcal{A} \rtimes \hat{\mathcal{G}} \rtimes \mathcal{B})$ given by*

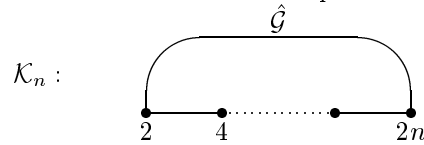
$$\begin{aligned} \rho_D(\mathcal{A} \rtimes \varphi \rtimes \mathcal{B}) &= (A_{(0)} \rtimes \varphi_{(2)} \rtimes \mathcal{B}) \otimes (A_{(1)} \bowtie_D \varphi_{(1)}) \\ \lambda_D(\mathcal{A} \rtimes \varphi \rtimes \mathcal{B}) &= (B_{(-1)} \bowtie_D \varphi_{(2)}) \otimes (\mathcal{A} \rtimes \varphi_{(1)} \rtimes B_{(0)}) \end{aligned}$$

This implies the existence of right coactions ρ_D^{2i} of the quantum double $\mathcal{D}(\mathcal{G})$ on the quantum chain, which are ‘‘localized’’ (i.e. act nontrivial only) in $\mathcal{A}_{2i,2i+1}$, where they are given by (using generating matrix notation)

$$\begin{aligned} \rho_D^{2i}(A_{2i}(a)) &= A_{2i}(a_{(1)}) \otimes (a_{(2)} \bowtie_D \hat{\mathbf{1}}) \\ (\text{id} \otimes \rho_D^{2i})(\mathbf{L}_{2i+1}) &= \mathbf{D}^{13} \mathbf{L}_{2i+1}^{12} \end{aligned} \quad (1.53)$$

Here $\mathbf{D} \in \mathcal{G} \otimes \mathcal{D}(\mathcal{G})$ denotes the universal Δ -flip operator $\mathbf{D} = \sum e_{\mu} \otimes (\mathbf{1} \bowtie_D e^{\mu})$. Analogously one may define localized left coactions λ_D^{2i} which are immediately shown to commute with ρ_D^{2i} .

1.4.3. Lattice current algebras. Diagonal crossed products also appear when formulating periodic boundary conditions for the quantum chain (1.47). In this case, starting with the open chain $\mathcal{A}_{2,2n}$ localized on $[2, 2n] \cap \mathbb{Z}$ one would like to add another copy of $\hat{\mathcal{G}}$ sitting on the link $2n+1 \equiv 1$ joining the sites $2n$ and 2 to form a periodic lattice.



Algebraically this means that $\mathcal{A}_1 (\cong \mathcal{A}_{2n+1}) \cong \hat{\mathcal{G}}$ should have non-vanishing commutation relations with $\mathcal{A}_{2n} \cong \mathcal{G}$ and $\mathcal{A}_2 \cong \mathcal{G}$ in analogy with (1.47), i.e.

$$\begin{aligned} A_1(\varphi) A_{2n}(a) &= A_{2n}(\varphi_{(1)} \rightharpoonup a) A_1(\varphi_{(2)}) \\ A_2(a) A_1(\varphi) &= A_1(\varphi_{(1)}) A_2(a \leftarrow \varphi_{(2)}) \end{aligned} \tag{1.54}$$

Written in this way Eqs. (1.54) are precisely the relations in

$$\mathcal{K}_n := \mathcal{A}_{2,2n} \bowtie \hat{\mathcal{G}}$$

where $\delta : \mathcal{A}_{2,2n} \rightarrow \mathcal{G} \otimes \mathcal{A}_{2,2n} \otimes \mathcal{G}$ is the two-sided coaction given by $\delta \upharpoonright_{\mathcal{A}_2} = \Delta \otimes \mathbf{1}_{\mathcal{G}}$, $\delta \upharpoonright_{\mathcal{A}_{2n}} = \mathbf{1}_{\mathcal{G}} \otimes \Delta$ and $\delta \upharpoonright_{\mathcal{A}_{3,2n-1}} = \mathbf{1}_{\mathcal{G}} \otimes \text{id} \otimes \mathbf{1}_{\mathcal{G}}$. Hence, the periodic quantum chain appears as a diagonal crossed product of the open lattice chain by a copy of $\hat{\mathcal{G}}$ sitting on the link joining the end points. Again we remark that this observation will be needed to give a generalization to (weak) quasi-Hopf algebras. A similar remark applies to the lattice current algebra of [AFFS98] defined below.

We also conclude from (1.26) that the ‘‘periodic chain’’ \mathcal{K}_1 consisting of one point and one link is given by the quantum double $\mathcal{D}(\mathcal{G})$

$$\mathcal{K}_1 = \mathcal{G} \bowtie \hat{\mathcal{G}} \equiv \mathcal{D}(\mathcal{G}) : \bullet \circlearrowright \tag{1.55}$$

Let us finally review the lattice current algebras of [AFFS98]¹, which appear as special examples of periodic Hopf spin chains. We follow the review of [Nil97], where the relation with the model of [NS97] has been clarified.

Suppose \mathcal{G} to be quasitriangular with R -matrix $R \in \mathcal{G} \otimes \mathcal{G}$ and define the generating *lattice currents*

$$\mathbf{J}_{2i+1} := (\text{id} \otimes A_{2i})(R^{op}) \mathbf{L}_{2i+1}. \tag{1.56}$$

Using (1.52), these are immediately verified to satisfy the lattice current algebra of [AFFS98]

$$\begin{aligned} [\mathbf{1}_{\mathcal{G}} \otimes A_{2i}(a)] \mathbf{J}_{2i-1} &= \mathbf{J}_{2i-1} [a_{(1)} \otimes A_{2i}(a_{(2)})], \quad \forall a \in \mathcal{G} \\ [a_{(1)} \otimes A_{2i}(a_{(2)})] \mathbf{J}_{2i+1} &= \mathbf{J}_{2i+1} [\mathbf{1}_{\mathcal{G}} \otimes A_{2i}(a)] \\ \mathbf{J}_{2i+1}^{13} \mathbf{J}_{2i+1}^{23} &= R^{12} (\Delta \otimes \text{id})(\mathbf{J}) \\ \mathbf{J}_{2i-1}^{13} R^{12} \mathbf{J}_{2i+1}^{23} &= \mathbf{J}_{2i+1}^{23} \mathbf{J}_{2i-1}^{13} \end{aligned}$$

Hence under the additional requirement of \mathcal{G} being quasitriangular, the lattice algebras of [NS97] and [AFFS98] are isomorphic.

¹For earlier versions of lattice current algebras see also [AFSV91, AFS92, FG93]