Introduction

The use of symmetry concepts in physics is as old as theoretical physics itself, and nowadays symmetries play a very important role in nearly every branch of modern theoretical physics. Traditionally symmetries have always been considered as being given (in some way or another) by a symmetry *group*, and only about twenty years ago more general algebraic objects, such as so–called *quantum groups* or certain generalizations thereof, started being considered as suitable symmetry objects. The invention and the use of quantum groups is surely one of the most fascinating developments in mathematical physics of the past fifteen years.

Originally, quantum groups emerged as a natural abstraction of certain basic ideas of the *quantum inverse scattering method* [STF79], which - roughly speaking - is a systematic method of exact solution of a broad class of two–dimensional quantum field theories or lattice models. They are intimately related to the famous Yang Baxter equation, which has already appeared before in other methods of exact solution. Meanwhile these new algebraic structures have been used in many different areas of theoretical physics, and also in mathematics, where they initiated new developments in the theory of link invariants and in low dimensional topology (see e.g. [Tur94, RT90]).

In this thesis we are interested in quantum groups as objects describing the global symmetry of a low–dimensional quantum field theory generalizing the concept of a global gauge group. These "new symmetries" go under the name *Quantum Symmetry*.

Let us shortly recall what is meant by a global symmetry. In quantum field theory one distinguishes two different types of symmetries. The *external symmetry* is given by a group of space-time transformations. For example the external symmetry of a relativistic quantum field theory always entails the group of Poincaré transformations. This kind of symmetry mostly fits into everybody's idea of a symmetry. The second type of symmetry playing a prominent role in quantum field theory is the so–called *internal symmetry* or *global symmetry* (gauge symmetry of 1. kind). These symmetries are somewhat more subtle to describe. One usually needs some non-observable quantities such as charged fields to "see how they act". The occurrence of global symmetries is connected with limitations of the superposition principle in quantum field theory, due to the existence of global conserved quantities, usually called charges. Roughly speaking, the global symmetry is expected to describe the superselection structure, i.e. the charge spectrum of the theory.

As an example one may consider the electric charge in quantum electrodynamics [WWW52]. The electric charge is supposed to be globally conserved, i.e. observables do not change the total charge of a given state. Thus the Hilbert space $\mathfrak{H}$ of physical states decomposes into a direct sum of subspaces $\mathfrak{H}_z$ ($\cong$ *superselection sectors*) consisting of states with charge $z \cdot e$, $z \in \mathbb{Z}$. Relative phases of states with different charges cannot be measured. The charge operator Q provides a unitary representation of the global gauge group $U(1)$ on the Hilbert space $\mathfrak{H}$ according to $\alpha \mapsto e^{i\alpha Q}$. The above decomposition of $\mathfrak{H}$ into charged sectors provides a diagonalization of the charge operator Q. This way every sector carries a multiple of an irreducible representation of $U(1)$: $e^{i\alpha Q}|\psi\rangle = e^{i\alpha z}|\psi\rangle$, $\psi \in \mathfrak{H}_z$. Since all irreducible representations of the global gauge group $U(1)$ appear, the superselection sectors ($\cong$ charge spectrum) may be labeled by the inequivalent irreducible representations of the global gauge group $U(1)$. Moreover an operator creating the charge z transforms under global gauge transformation according to $e^{i\alpha Q}\, F\, e^{-i\alpha Q} = e^{iz\alpha} F$.

The general theory of superselection sectors is most clearly formulated in the algebraic approach to local quantum field theory as initiated by [HK64] and further developed by [DHR69a, DHR69b, DHR71, DHR74], nowadays known as the Doplicher-Haag-Roberts superselection theory (DHR-theory). In this framework superselection sectors are interpreted as inequivalent representations of the (abstract) algebra of observables. In a second step one may then construct non-observable charged fields interpolating between different sectors and transforming under the action of the global symmetry. For the reader not being familiar with these concepts we give a short introduction into DHR-theory below.

But why - if at all - is there any need to talk about global (gauge) symmetries, which are not given by a group? Why is there no sign of these fancy symmetries in the most famous theories such as QED or the standard model, which have been so successful in describing the experiments in high energy physics? An answer to these questions is again provided by the general superselection theory. It was one of the main achievements of the algebraic approach to show [DR90] that under quite general assumptions in space time dimension $d = 4$ the global symmetry may in fact always be realized as a compact group (the global gauge group). This result relies heavily on the fact that in $d = 4$ particles obey Bose/Fermi statistics (permutation statistics). How highly nontrivial this result is becomes clear if one considers low dimensional QFTs ($d \leq 3$). Here the situation changes drastically. Due to the occurrence of braid group statistics instead of permutation statistics, one is forced to consider more general algebraic objects than groups as global symmetries. Somewhat simplified the reason for this lies in the fact that - in contrast to Bose/Fermi commutation relations - more general commutation relations (such as braiding relations) of charged fields cannot be compatible with transformations under a group action. This is the place where the search for "new symmetry concepts in QFT" takes place. Actually by now it is still an open problem to find the most general algebraic object describing the superselection structure of low dimensional QFT, substituting the global gauge group. This is called the "symmetry problem of low–dimensional QFT".

There are good reasons to be interested in low-dimensional quantum field theories. First of all one should be aware that - beyond formal perturbation theory - there are no nontrivial models in space time dimension $d = 4$. Conversely there exist a lot of nontrivial low–dimensional models, many of them even being exactly solvable. Therefore to investigate conceptual issues in concrete models, it may be of great use to pass to low-dimensional quantum field theories. Another motivation to investigate models with braid group statistics stems from the fact that particles with this kind of statistics (also called fractional statistics) may possibly be used to describe some effects in low-dimensional solid state physics, as e.g. the fractional quantum Hall effect [Sto92].

As has already been indicated above, ordinary groups are ruled out as candidates describing the global symmetry in models with braid group statistics. However quantum groups may be used! Quantum groups are algebras which have nearly the same properties as a group (i.e. as the associated group algebras). These properties assure that their representation theory is quite similar to the representation theory of ordinary groups. In particular there exists a trivial representation, a tensor product of representations, etc. This implies that a quantum group may act on field operators rather similar as a group. Moreover, the existence of a so-called R–matrix allows for a formulation of braiding relations of field operators, which are consistent with the quantum group action. This is, why during the last decade quantum groups have become the most fashionable candidates describing the global symmetry in low dimensional quantum field theories (QFT)[1] or lattice models[2]. We will give a short introduction to quantum groups below, also describing how they might act as a global symmetry.

However, it was soon realized that at least for rational theories (i.e. with a finite number of superselection sectors) not only groups but also ordinary quantum groups are ruled out, unless all sectors have integer statistical dimensions (see e.g. [FK93] for a review or [Nil93] for a specific discussion of q–dimensions in finite quantum groups), which is actually not the case in many models. Thus the search for generalizations went on. Based on the theory of quasi–quantum groups introduced by Drinfel'd [Dri90], G. Mack and V. Schomerus [MS92] have proposed the notion of *weak quasi–quantum groups* $\mathcal{G}$ as appropriate symmetry candidates, where "weak" means that the tensor product of two "physical" representations of $\mathcal{G}$ may also contain "unphysical" subrepresentations (i.e. of q–dimension ≤ 0), which have to be discarded. Examples are semisimple quotients of q-deformations of enveloping algebras $\mathcal{U}(\mathfrak{g})$, $\mathfrak{g}$ a semisimple Lie algebra, at roots of unity, $q^N = 1$ (also called truncated quantum groups at roots of unity).

In this way non-integer dimensions could successfully be incorporated. The price to pay was that now commutation relations of $\mathcal{G}$-covariant charged fields involve operator valued R-matrices and, more drastically, the operator product expansion for $\mathcal{G}$-covariant multiplets of charged fields involves non-scalar coefficients with values in $\mathcal{G}$. Thus, the analogue of the "would-be" DHR-field algebra $\mathcal{F}$ is no longer algebraically closed. Instead, Mack and Schomerus have proposed a new "covariant product" for charged fields, which does not lead outside of $\mathcal{F}$, but which is no longer associative. In [Sch95] Schomerus has analyzed this scenario somewhat more systematically in the framework of DHR-theory, showing that a weak quasi-Hopf algebra $\mathcal{G}$ and a field "algebra" $\mathcal{F}$ may always be constructed such that the combined algebra $\mathcal{F} \vee \mathcal{G}$ is associative and satisfies all desired properties, except that $\mathcal{F} \subset \mathcal{F} \vee \mathcal{G}$ is only a linear subspace but not a subalgebra. Technically, the reason for this lies in the fact that the dual $\hat{\mathcal{G}}$ of a quasi-quantum group is not an associative algebra.

To study quantum symmetries on the lattice in an axiomatic approach, K. Szlachányi and P. Vecsernyés [SV93] have proposed an "amplified" version of the DHR-theory, which also applies to locally finite dimensional lattice models. This setting has been further developed by [NS97, NS95], where, based on the example of Hopf spin chains, the authors proposed the notion of a *universal localized cosymmetry* $\rho : \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$, $\mathcal{A}$ being the observable algebra, incorporating all sectors ρ_I of $\mathcal{A}$ via $\rho_I = (\mathrm{id}_{\mathcal{A}} \otimes \pi_I) \circ \rho,\ \pi_I \in \mathrm{Rep}\,\mathcal{G}$. In the specific example studied by [NS97, NS95] $\mathcal{G}$ was given by a so-called quantum double and the cosymmetry ρ was given by a *coaction* of $\mathcal{G}$ on $\mathcal{A}$. Related results have later been obtained for lattice current algebras [AFFS98], the later actually being a special case of the Hopf spin chains of [NS97] (see [Nil97] and Section 1.4). The analogue of a DHR–field algebra for these models is now given by the standard crossed product $\mathcal{F} \equiv \mathcal{A} \rtimes \hat{\mathcal{G}}$ [NS95], where $\hat{\mathcal{G}}$ is the Hopf algebra dual to $\mathcal{G}$.

Now the methods and results of these works were still restricted to ordinary quantum groups and therefore to integer statistical dimensions. To formulate lattice current algebras at roots of unity one may identify them with the boundary part of lattice Chern-Simons algebras [AGS95, AGS96, AS96] defined on a disk. Nevertheless, it remains unclear whether and how for $q =$ root of unity the structural results of [AFFS98] survive the truncation to the semi-simple ("physical") quotients. Similarly, the generalizations of the model, of the methods and of the results of [NS97] to weak quasi quantum groups are by no means obvious. In particular one would like to know whether and in what sense in such models universal localized cosymmetries $\rho : \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$ still

[1]see [BdWP94, BL93, Gaw91, DPR90, FGV94, FK93, MS90, MS92, MR89, Müg98, PS90, RS90, Szl94, Vec94]

[2]see [AFFS98, AFSV91, AFS92, BS98, Fad92, FG93, KS97, NS97, NS95, Pas88, PS89, SV93]

provide coactions and whether $\mathcal{G}$ would still be (an analogue of) a quantum double of a quasi-Hopf algebra, possibly in the sense recently described by Majid [Maj97].

In this thesis we construct and investigate quantum spin chains and lattice current algebras based on a weak quasi–quantum group and therefore in particular at 'roots of unity'.

Before we give an overview of this thesis and a summary of the main results we provide a short introduction into the basic concepts on which our models are based. In a first section we describe the DHR superselection theory. The following section is concerned with quantum groups and their role as symmetry algebras, and finally we sketch how the DHR-theory may be adapted to treat lattice models. Readers already being familiar with all concepts mentioned so far are invited to skip these sections.

DHR-superselection theory

We will give a rather non–technical introduction into the basic notions of algebraic QFT, especially into the Doplicher–Haag–Roberts (DHR) theory of superselection sectors. For more detailed introductions see [Haa92], [Kas90], [BW92], [Bau95]. A nice introduction with emphasis on the low–dimensional case is given in [Fre90].

The algebraic formulation of local quantum field theory - as given by [HK64] - uses the language of (C^*, von Neumann) algebras. This approach may be mainly characterized by its emphasis on observable quantities and its insistence in separating local from global properties. Local properties are encoded in the algebraic structure of the local observables, whereas global properties enter through states and the corresponding representation spaces. The algebraic framework is especially suited for describing and investigating superselection structures without relying on ad hoc introduced non observable quantities like charged fields. The latter are to be reconstructed from the "intrinsic" properties of the theory, i.e. from properties of the local observables. This also leads to an intrinsic notion of statistics, which is in particular useful when passing to low-dimensional quantum field theories, where one has to deal with generalizations of Bose/Fermi statistics.

In short, a quantum field theory is supposed to be completely defined by its local net of observables, i.e. an inclusion preserving map

$$\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}),$$

which assigns the C^*–algebra $\mathcal{A}(\mathcal{O})$ of observables measurable in $\mathcal{O}$ to every bounded region $\mathcal{O}$ in space–time. The local algebras $\mathcal{A}(\mathcal{O})$ may be thought as being generated by (bounded functions of) observable field operators (e.g. currents), which are "smeared" with test functions having support in $\mathcal{O}$. The C^*–algebra generated by all $\mathcal{A}(\mathcal{O})$ is called the algebra of quasi local observables and is again denoted by $\mathcal{A}$. Einstein causality is implemented by the requirement that the local algebras $\mathcal{A}(\mathcal{O}^1)$ and $\mathcal{A}(\mathcal{O}^2)$ associated to space-like separated regions $\mathcal{O}^1$ and $\mathcal{O}^2$ commute element-wise and relativistic covariance by demanding the Poincaré group to act on the observables by automorphisms $\alpha_{(\Lambda,a)}$ such that $\alpha_{(\Lambda,a)}(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\Lambda\mathcal{O}+a)$. Note that till now there is no Hilbert space of physical states specified, on which the observable algebra acts.

A first approach to superselection theory in the algebraic framework was given by S. Doplicher, R. Haag and J.E. Roberts in [DHR69a]. They still start with predefined non observable fields given as a net of local field algebras $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$ obeying Bose-Fermi commutation relations and acting irreducibly on a Hilbert space $\mathfrak{H}$. As above there is supposed to be a unitary representation of a compact group G of internal symmetries on $\mathfrak{H}$, which acts by its adjoint action on the field net, i.e. $U(g)\,\mathcal{F}(\mathcal{O})\,U(g)^{-1} = \mathcal{F}(\mathcal{O}), \quad \forall g \in \mathcal{G}$. The local observable algebra $\mathcal{A}(\mathcal{O})$ is defined as the gauge invariant subalgebra of $\mathcal{F}(\mathcal{O})$ (fix point algebra). Under some technical requirement (twisted duality, which implies Haag duality in every simple sector), [DHR69a] proved that the Hilbert space $\mathfrak{H}$ decomposes into orthogonal subspaces invariant under the action of the observables $\mathcal{A}$ and the action of the gauge group G as follows

$$\mathfrak{H} = \oplus_\mu \Big(\mathfrak{H}_\mu \otimes V_\mu \Big), \tag{0.1}$$

where the representations $\pi(\mathcal{A}) \equiv \mathcal{A}$ and $U(G)$ decompose according to

$$\pi = \oplus_\mu \Big(\pi_\mu \otimes \mathbf{1}_{V_\mu} \Big) \tag{0.2}$$

$$U = \oplus_\mu \Big(\mathbf{1}_{\mathfrak{H}_\mu} \otimes \tau_\mu \Big). \tag{0.3}$$

Here μ runs through the set of all inequivalent irreducible representations (τ_μ, V_μ) of G and the π_μ are inequivalent irreducible representations of $\mathcal{A}$. Thus, superselection sectors are labeled by the irreducible representations of G - the global symmetry G is "dual" to the superselection structure. The vacuum sector $\mathfrak{H}_0 \otimes \mathbb{C} \equiv \mathfrak{H}_0$ carries the trivial (one-dimensional) representation of G. Moreover all representations are "locally generated" out of the vacuum in the sense that they fulfill the DHR–criterion

$$\pi_\mu \restriction \mathcal{A}(\mathcal{O}') \cong \pi_0 \restriction \mathcal{A}(\mathcal{O}'), \quad \text{for some bounded region } \mathcal{O}, \tag{0.4}$$

where $\mathcal{O}'$ denotes the causal complement of $\mathcal{O}$ and $\mathcal{A}(\mathcal{O}')$ denotes the subalgebra of $\mathcal{A}$ generated by all local algebras $\mathcal{A}(\tilde{\mathcal{O}})$ satisfying $\tilde{\mathcal{O}} \subset \mathcal{O}'$.

The last observation is most important since it provides a reasonable selection criterion for physically relevant states[3]. Starting with a local net of observables (without any predefined field net), given in a vacuum representation $(\mathfrak{H}_0, \pi_0)$, the DHR–superselection sectors are given by the equivalence classes of representations of $\mathcal{A}$ satisfying (0.4). A further analysis of DHR superselection sectors has been given in [DHR69b], [DHR71], [DHR74]. Starting point is the basic observation that there is a one-to-one correspondence between (equivalence classes of) representations π fulfilling the DHR-criterion (0.4) and (equivalence classes of) localized algebra endomorphisms $\rho : \mathcal{A} \to \mathcal{A}$ given by

$$\pi \cong \pi_0 \circ \rho, \tag{0.5}$$

where ρ is called localized in $\mathcal{O}$ if $\rho \restriction \mathcal{A}(\mathcal{O}') = \mathrm{id} \restriction \mathcal{A}(\mathcal{O}')$. This equivalence holds provided one requires the vacuum representation to fulfill Haag duality, i.e. $\pi_0(\mathcal{A}(\mathcal{O})) = \pi_0(\mathcal{A}(\mathcal{O}'))'$, for all doubles cones $\mathcal{O}$, which may be seen as some maximality condition on the local algebras (Einstein causality already implies the inclusion $\subset$). The importance of the possibility to describe superselection sectors in terms of endomorphisms instead of representations of $\mathcal{A}$ lies in the fact, that endomorphism may be composed (composition of charges), which provides the set Rep_{DHR} of DHR–sectors with a (monoidal) product. This opens the way to analyze the rich algebraic structure of Rep_{DHR} [4]:

(i) *Fusion rules*: Every ρ decomposes into a direct sum of finitely many irreducibles. In particular the product of two irreducible endomorphisms ρ_i, ρ_j, decomposes into irreducibles ρ_k according to

$$\rho_i \circ \rho_j = N_{ij}^k \, \rho_k \tag{0.6}$$

with multiplicities $N_{ij}^k \in \mathbb{N}$.

[3] i.e. relevant for elementary particle physics. Surely there are physically interesting states - e.g. thermal equilibrium states - which clearly violate (0.4). Also note that even in elementary particle physics the DHR–criterion is in general to restrictive. [BF82] formulate a weaker criterion, which at least covers all massive theories. In chiral conformal field theories one knows, that all positive energy representations are of DHR–type, see [BMT88]

[4] provided property B holds, which ensures the existence of subobjects and direct sums in Rep_{DHR}

(ii) *Conjugate sectors*: For every irreducible ρ there exists a unique conjugate $\bar{\rho}$ such that $\rho_0 = \mathrm{id}_{\mathcal{A}}$ (which represents the vacuum sector) appears in the decomposition of $\bar{\rho} \circ \rho$ according to (0.6) (with multiplicity one).

(iii) *Statistics:* There is an intrinsic notion of statistics of sectors given by the so-called statistics operators $\varepsilon_\rho \in \mathcal{A}$

Let us discuss (iii) in more detail, since here one finds the crucial structural differences between DHR-theory in space-time dimensions $d \leq 2$ and in higher dimensions. Given a localized endomorphism ρ the statistics operator ε_ρ is defined as $\varepsilon_\rho := U^{-1}\,\rho(U)$, where $U \in \mathcal{A}$ is a unitary such that $\hat{\rho} := AdU \circ \rho$ is localized space-like separated to ρ (the definition of ε_ρ does not depend on the special choice of U). The unitary ε_ρ has the following remarkable properties

$$\varepsilon_\rho\, \rho(\varepsilon_\rho)\, \varepsilon_\rho = \rho(\varepsilon_\rho)\, \varepsilon_\rho\, \rho(\varepsilon_\rho) \tag{0.7}$$

$$\varepsilon_\rho \in \rho^2(\mathcal{A})' \tag{0.8}$$

$$d \geq 3: \quad \varepsilon_\rho^2 = \mathbf{1}. \tag{0.9}$$

The reason for (0.9) being valid only in space time dimensions $d \geq 3$ lies in the fact that in $d \leq 2$ the causal complement of a bounded region is not connected. Now given a unitary ε_ρ satisfying (0.7)-(0.8) one may define a representation of the braid group B_∞. According to Artin [Art65] the braid group with n strings is generated by elements $\sigma_1 \ldots \sigma_{n-1}$, which satisfy the relations

$$\begin{aligned} \sigma_i\, \sigma_{i+1}\, \sigma_i = \sigma_{i+1}\, \sigma_i\, \sigma_{i+1}, \quad \forall i \leq n-2 \\ \sigma_i\, \sigma_j = \sigma_j\, \sigma_i, \quad \text{if} \quad |i-j| \geq 2. \end{aligned} \tag{0.10}$$

A representation of B_∞ is obtained by the assignment $\sigma_i \mapsto \rho^{i-1}(\varepsilon_\rho)$. Now it follows from (0.9) that in $d \geq 3$ dimensions also the relation $\sigma_i^2 = 1$ is respected in this representation and one therefore obtains a representation of the permutation group. Thus DHR-sectors obey permutation group statistics in $d \geq 3$ space time dimensions and braid group statistics in $d \leq 2$ dimensions. We finally remark that ϵ_ρ allows for the definition of two numerical invariants associated to the sector represented by ρ as follows. For every irreducible ρ there exists a so-called left inverse ϕ_ρ which is a positive linear map, such that $\phi_\rho \circ \rho = \mathrm{id}$ and $\rho \circ \phi_\rho : \mathcal{A} \to \rho(\mathcal{A})$ is a conditional expectation. It turns out that $\phi_\rho(\epsilon_\rho)$ is a multiple of the identity and by polar decomposition one obtains

$$\phi_\rho(\epsilon_\rho) = \frac{\omega_\rho}{d_\rho}, \quad |\omega_\rho| = 1,\, d_\rho > 0. \tag{0.11}$$

The number d_ρ is called the *statistical dimension* and ω_ρ the *statistical phase*. (There may also be sectors with infinite statistics $d_\rho = 0$, which are usually discarded.)

Phrased in precise mathematical terms, the set Rep_{DHR} of DHR-sectors is a strict rigid braided monoidal C^*-category with subobjects and direct sums, which in $d \geq 3$ space time dimensions is even symmetric. The objects are given by the local endomorphisms of $\mathcal{A}$ and the morphisms (arrows) between two objects by the set of intertwiners in $\mathcal{A}$.

It took another fifteen years until S. Doplicher and J.E. Roberts succeeded in proving that in the case of permutation statistics the scenario of $\mathcal{A}$ being the fix point algebra of a compact group G acting on a bigger field net is generic [DR89a, DR89b, DR90]. In particular they proved that one may reconstruct a compact group G and a field net $\mathcal{F}(\mathcal{O})$ obeying Bose/Fermi commutation relations such that (0.1-0.3) are fulfilled. This construction is unique up to isomorphism. They essentially used the rich algebraic structure of the set Rep_{DHR} of DHR–sectors. More precisely Doplicher and Roberts succeeded in proving a new duality theorem identifying Rep_{DHR} with the representation category of a compact group with the following correspondences:

irreduc. DHR-sectors	irreduc. representations of G
vacuum sector	trivial representation
conjugate sector	contragredient representation
composition of sectors	tensor product of representations
fusion rules	Clebsch-Gordan decomposition
statistics operator	permutation of tensor factors.

This categorical equivalence is the precise meaning of the statement that the gauge group G is the dual object of the superselection structure. We again point out that these findings essentially use the fact, that in dimensions $d \geq 3$ the intrinsic statistics defined with the help of the statistics operator is always permutation statistics. Until now there is no known analogue of the DR–reconstruction in the case of (non Abelian) braid group statistics.

The reconstructed local field algebras are spanned by the local observables and multiplets of field operators transforming covariantly under G as follows: Every irreducible endomorphism ρ_μ is a restriction of an inner endomorphism of the field algebra $\mathcal{F}$, i.e.

$$\rho_\mu(\cdot) = \sum_{i=1}^{d_\mu} \psi_i \cdot \psi_i^*, \tag{0.12}$$

where $\{\psi_i\}_{i=1,\ldots d_\mu}$ is a set of isometries with support one (d_μ is precisely the statistical dimension of ρ_μ, see (0.11)):

$$\psi_i^* \psi_j = \delta_{i,j}\, \mathbf{1}, \quad \sum_{i=1}^{d_\mu} \psi_i \psi_i^* = \mathbf{1}. \tag{0.13}$$

Moreover $\{\psi_i\}$ is a G-multiplet transforming according to the irreducible representation τ_μ, see (0.2). Also $\psi_i \in \mathcal{F}(\mathcal{O})$ if ρ_μ is localized in $\mathcal{O}$ and the operators ψ_i provide transitions between the vacuum sector and the charged sector π_μ. Thus they may be interpreted as *charged fields*[5].

As already noted, there is no general reconstruction of charged fields transforming under some global symmetry available in the case of braid group statistics. For partial results in this case see [FRS89, FRS92].

As a first step one may at least try to find some "symmetry algebra" $\mathcal{G}$, such that the representation category of $\mathcal{G}$ may be identified with Rep_{DHR}. First candidates would be semisimple quasitriangular C^*-Hopf algebras, whose representation categories have indeed all desired properties, see below. The trouble is that (at least in rational theories) they give only rise to integer statistical dimensions, see e.g. [FK93, Nil93]. As has already been indicated above, a way out of this is to also consider more general symmetry algebras such as quasi–Hopf algebras or weak quasi–Hopf algebras, which will be introduced in Chapter 2 and Chapter 3, respectively.

Quantum groups as symmetry algebras

As mentioned above quantum groups emerged as certain algebraic structures lying behind some general methods of exact solutions of certain low-dimensional quantum field theoretic models and two–dimensional lattice models of statistical mechanics. For an introduction to quantum groups from this point of view we highly recommend the lectures on quantum groups given by L.A. Takhtajan [Tak89]. The term quantum group was introduced by V. Drinfel'd in his seminal paper [Dri86] for a certain class of quasitriangular Hopf algebras obtained as 'deformations' of usual (classical) groups (see below). Meanwhile the term quantum group is often used in general for Hopf algebras as we will do here. Two well-known monographs on Hopf algebras are [Abe80, Swe69].

[5] Actually the field multiplet $\{\psi_i\}$ generates a unique C^*–algebra inside $\mathcal{F}$, the so–called Cuntz algebra, which plays a major role in the reconstruction of G and $\mathcal{F}$.

There are also several introductory books on quantum groups incorporating the more recent developments, e.g. [Kas95, Lus93, Maj95, CP94].

A Hopf algebra (quantum group) is an algebra $\mathcal{G}$ (over $\mathbb{C}$) with unit $\mathbf{1}$, having some further algebraic structures, which guarantee that its set of representations Rep $\mathcal{G}$ (which mathematically speaking is a category) has similar properties as the representations of a group. We remark that in quantum physics a symmetry (group, algebra) usually enters through its representations. More specifically for an algebra to be interpreted as a symmetry (acting on some Hilbert space of physical states) one should be able to formulate covariance properties of field multiplets and also invariance of a ground state (or of observables). As we shall see, a Hopf algebra perfectly fits these requirements. If $\mathcal{G}$ is additionally required to be quasitriangular, these transformation properties are consistent with local braid relations.

To begin with, a Hopf algebra admits unital algebra maps $\Delta : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ (the *coproduct*) and $\epsilon : \mathcal{G} \to \mathbb{C}$ (the *counit*), satisfying

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \qquad (\textit{coassociativity}) \tag{0.14}$$

$$(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta \qquad (\textit{counit axiom}). \tag{0.15}$$

This makes $\mathcal{G}$ into a *bialgebra*. In the following we will use the Sweedler notation

$$\Delta(a) = \sum_i a^i_{(1)} \otimes a^i_{(2)} \equiv a_{(1)} \otimes a_{(2)}.$$

The counit furnishes a distinguished one dimensional representation of $\mathcal{G}$ (the *trivial representation*), whereas the coproduct Δ allows for the definition of a product of representations. Indeed, given two representations π_I, π_J on vector spaces V_I and V_J, respectively, then $\pi_I \boxtimes \pi_J$ given by

$$(\pi_I \boxtimes \pi_J)(a) := (\pi_I \otimes \pi_J)\big(\Delta(a)\big) \equiv \pi_I(a_{(1)}) \otimes \pi_J(a_{(2)}), \quad a \in \mathcal{G} \tag{0.16}$$

defines a representation of $\mathcal{G}$ on the vector space $V_I \otimes V_J$. Moreover coassociativity of Δ implies associativity of the product $\boxtimes$, and the counit axiom (0.15) ensures that ϵ is a 'unit' with respect to $\boxtimes$, i.e. $\pi_I \boxtimes \epsilon = \pi_I = \epsilon \boxtimes \pi_I$. This justifies to call ϵ the trivial representation of $\mathcal{G}$.

EXAMPLE 0.1 (The group algebra). Let us pause for a moment to see how the definition of a bialgebra generalizes well-known properties of ordinary groups. To every (topological) group there is an associated algebra, the so–called *group algebra*, which - somewhat simplified - is the $\mathbb{C}$–vector space with basis given by the group elements and with product being defined on this basis by the group multiplication (which is then extended linearly). In general a group fixes uniquely its group algebra and vice versa. Let for simplicity G be a finite group with unit e. Then the group algebra of G is given by $\mathcal{G} := \mathbb{C}G = \mathrm{lin}\{g \in G\}$ with multiplication

$$\Big(\sum_i \alpha_i\, g_i\Big)\Big(\sum_i \beta_j\, g_j\Big) = \sum_{i,j} \alpha_i \beta_j\, g_i g_j, \quad \alpha_i, \beta_j \in \mathbb{C}.$$

Clearly $e \equiv \mathbf{1}$ provides a unit in $\mathcal{G}$. The algebra $\mathcal{G}$ becomes a bialgebra with coproduct Δ and counit ϵ defined on group elements $g \in G$ by

$$\Delta(g) = g \otimes g \tag{0.17}$$

$$\epsilon(g) = 1 \tag{0.18}$$

and extended linearly to the whole of $\mathcal{G}$. Thus in this case the counit ϵ is in fact given by the trivial representation of G and the product of representations as defined in (0.16) yields the well-known product of group representations, i.e. $(\pi_I \boxtimes \pi_J)(g) = \pi_I(g) \otimes \pi_J(g)$, $g \in G \subset \mathcal{G}$. Note that the coproduct in (0.17) is cocommutative, i.e $\Delta(a) = \Delta^{op}(a)$, where $\Delta^{op} := \tau \circ \Delta$, τ denoting the permutation of tensor factors.

One further requires the existence of an *antipode* S. This is an algebra antimorphism $S:\mathcal{G}\to\mathcal{G}$ satisfying

$$S(a_{(1)})\,a_{(2)} = a_{(1)}\,S(a_{(2)}) = \epsilon(a)\mathbf{1}, \quad a\in\mathcal{G}. \tag{0.19}$$

A *Hopf algebra* (or a *quantum group*) is a bialgebra with antipode. As one might guess from (0.19) and (0.17), the antipode generalizes the inverse in a group. Indeed, in the above example $\mathcal{G}:=\mathbb{C}G$ the antipode S is given on the basis by $S(g)=g^{-1}$. This clearly defines an anti algebra morphism satisfying (0.19), making the group algebra $\mathbb{C}G$ into a Hopf algebra. In this sense Hopf algebras (quantum groups) are generalizations of ordinary groups.

As in the group case, the antipode allows for the definition of (left) contragredient representations $\pi_{\bar{I}}$ on the dual space $V_{\bar{I}}$ of V_I by setting $\pi_{\bar{I}}(a) := \pi_I^t\big(S(a)\big)$, ^{t} denoting the transposed map. The antipode property (0.19) ensures that the trivial representation ϵ is always contained in the product $\pi_I\boxtimes\pi_{\bar{I}}$. Indeed, (0.19) implies that the dual pairing $\langle\cdot\mid\cdot\rangle : V_I\otimes V_{\bar{I}}\to\mathbb{C}$, $v\otimes\hat{w}\mapsto\langle v\mid\hat{w}\rangle := \hat{w}(v)$ intertwines the representations $\pi_I\boxtimes\pi_{\bar{I}}$ and ϵ.

Let us finally introduce the notion of quasitriangularity. A Hopf algebra is called *quasitriangular*, if there exists an invertible element $R\in\mathcal{G}\otimes\mathcal{G}$ (the *R-matrix*) intertwining Δ and Δ^{op}, i.e. (recall that $\Delta^{op}(a)=a_{(2)}\otimes a_{(1)}$)

$$R\,\Delta(a) = \Delta^{op}(a)\,R, \qquad \forall a\in\mathcal{G} \tag{0.20}$$

and obeying $(\Delta\otimes\mathrm{id})(R)=R^{13}R^{23}$, $(\mathrm{id}\otimes\Delta)(R)=R^{13}R^{12}$, which are identities in the threefold tensor product $\mathcal{G}\otimes\mathcal{G}\otimes\mathcal{G}$, the upper indices denoting the embedding of tensor factors, i.e. $R^{23}=\mathbf{1}\otimes R$, etc. . This implies that R satisfies the Yang Baxter equation

$$R^{23}\,R^{13}\,R^{12} = R^{12}\,R^{13}\,R^{23}. \tag{0.21}$$

Let us recall two consequences of the existence of an R–matrix. First (0.20) implies that the two representations $\pi_I\boxtimes\pi_J$ and $\pi_J\boxtimes\pi_I$ are equivalent with intertwiner given by $B_{IJ}=\tau_{IJ}\circ(\pi_I\otimes\pi_J)(R)$, τ_{IJ} denoting the permutation of tensor factors. Secondly (0.20) implies that an R-matrix defines a representation of the braid group B_n (see (0.10)) on the n-th tensor power $V_I\otimes V_I\otimes\cdots\otimes V_I$ of any $\mathcal{G}$-module V_I by the assignment

$$\sigma_i\mapsto B_{II}^{i(i+1)},$$

where again the upper indices mean that $B_{IJ}^{i(i+1)}$ is supposed to act nontrivially an the i-th and $(i+1)$-th tensor factor. In the case of the group algebra $\mathbb{C}G$ one may choose $R=\mathbf{1}\otimes\mathbf{1}$ which is clearly quasitriangular since $\mathbb{C}G$ is cocommutative. Hence $B_{II}^{i(i+1)}$ is just a transposition and yields a representation of the permutation group.

Let us now describe how a Hopf algebra may act as a global symmetry. Given some action U of a Hopf algebra $\mathcal{G}$ on a Hilbert space $\mathfrak{H}$ of physical states, one can as in the group case define the *adjoint action* γ of $\mathcal{G}$ on the algebra of operators $\mathcal{F}\ni F:\mathfrak{H}\to\mathfrak{H}$ (field operators) by the formula

$$\gamma_a(F) := U(a_{(1)})\,F\,U(S(a_{(2)})), \quad a\in\mathcal{G} \tag{0.22}$$

Note that for $a\in\mathcal{G}$ group-like, i.e. $\Delta(a)=a\otimes a,\ S(a)=a^{-1}$ (see (0.17)) one recovers the usual adjoint action of group elements. The action γ satisfies

$$\gamma_a(FF') = \gamma_{a_{(1)}}(F)\,\gamma_{a_{(2)}}(F'). \tag{0.23}$$

(Actions obeying (0.23) are called *Hopf module actions.*) The importance of this relation lies in the fact that if $\{F_I^i\}$ and $\{F_J^j\}$ are multiplets for representations π_I and π_J, respectively, of $\mathcal{G}$, i.e. $\gamma_a(F_\alpha^k)=F_\alpha^l\,\pi_\alpha^{lk}(a)$, $\alpha=I,J$, then $F^{ij}:=F_I^i\,F_J^j$ is a $\pi_I\boxtimes\pi_J$-multiplet. We remark that for any π-multiplet $\{F^k\}$ the action γ given in (0.22) is equivalent to the following 'generalized commutation relations'

$$U(a)\, F^k = F^l\, \pi^{lk}(a_{(1)})\, U(a_{(2)}), \tag{0.24}$$

which is easily verified using the antipode property (0.19). This relation may - in contrast to (0.22) - be applied also in the case of (weak)–quasi Hopf algebras, see below.

The counit ϵ may be used to state invariance properties. The field operator F is called $\mathcal{G}$–invariant, if $\gamma_a(F) = \epsilon(a)F,\ \forall a \in \mathcal{G}$, which - using (0.24) and the counit axiom (0.15) - is equivalent to $U(a)\, F = F\, U(a),\ \forall a \in \mathcal{G}$. Thus as in the group case the 'gauge invariant' operators are precisely the ones commuting with $U(\mathcal{G})$.

Let us now also assume $\mathcal{G}$ to be quasitriangular. One may then propose the following local braiding relations for space like separated field multiplets $\{\psi_I^i\}$ and $\{\psi_J^j\}$ (see [Frö88]), where we assume ψ_I to be localized to the left of ψ_J.

$$\psi_I^i\, \psi_J^j = \psi_J^k\, \psi_I^l\, R_{IJ}^{lk,ij}, \tag{0.25}$$

where $R_{IJ}^{lk,ij} = (\pi_I^{lk} \otimes \pi_J^{ij})(R)$. The Yang-Baxter equation for R ensures consistency of these quadratic relations and the quasitriangularity of R assures consistency with the transformation properties of the field multiplets under the global symmetry $\mathcal{G}$.

Thus, summarizing the above scenario, the algebraic properties of a quasitriangular Hopf algebra allow them to act as a global symmetry on field multiplets obeying braid group statistics with braid relations given by (0.25). Let us conclude this section by giving at least one example of a genuine quantum group.

EXAMPLE 0.2 (The quantum enveloping algebra $\mathcal{U}_q(sl_2)$). The probably best-known and most prominent quantum groups are the so–called q-deformations $\mathcal{U}_q(sl(2)$ of the universal enveloping algebra of the three dimensional complex Lie algebra sl_2, first introduced in [KR81]. Recall that this Lie algebra is spanned by elements $X_\pm, H$ with Lie brackets

$$\begin{aligned} [H, X_\pm] &= \pm 2 X_\pm, \\ [X_+, X_-] &= H. \end{aligned} \tag{0.26}$$

It is the smallest simple Lie algebra and most of its properties (especially the below described deformation) generalizes to all complex simple Lie algebras. Also recall the universal enveloping algebra $\mathcal{U}(sl_2)$ defined as the (non-commutative) algebra generated by a unit $\mathbf{1}$ and the elements of sl_2 satisfying the relations (0.26) where the Lie bracket is replaced by the commutator, i.e. $[a, b] := ab - ba$. $\mathcal{U}(sl_2)$ is an infinite dimensional algebra. It uniquely determines the Lie algebra sl_2 (and vice versa). In particular it has the same representation theory as $\mathfrak{g}$. In fact every representation of sl_2 may be continued to a representation of $\mathcal{U}(sl_2)$ and since sl_2 may be viewed as a subset of $\mathcal{U}(sl_2)$ one also has the converse statement. The unital algebra $\mathcal{U}(sl_2)$ is a Hopf algebra with coproduct Δ, counit ϵ and antipode S defined on the generators $a \in \mathfrak{g}$ by

$$\begin{aligned} \Delta(a) &= a \otimes \mathbf{1} + \mathbf{1} \otimes a \\ \epsilon(a) &= 0, \quad S(a) = -a \end{aligned} \tag{0.27}$$

and extended multiplicatively (in the case of Δ and ϵ) or anti multiplicatively (in the case of S). The Hopf algebra $\mathcal{U}(sl_2)$ is non commutative but cocommutative as can be seen from (0.27).

Now let $q \in \mathbb{C},\ q \neq \pm 1$. Then the *quantum enveloping algebra* $\mathcal{U}_q(sl_2)$ is defined to be the unique algebra generated by $\mathbf{1}$ and elements X_+, X_-, K, K^{-1} with relations (again $[\cdot,\cdot]$ denoting the commutator)

$$\begin{aligned} K\, X_\pm\, K^{-1} &= q^{\pm 1}\, X_\pm \\ [X_+, X_-] &= \frac{K^2 - K^{-2}}{q - q^{-1}} \end{aligned} \tag{0.28}$$

Writing formally $K = q^{H/2}$, $K^{-1} = q^{-H/2}$ - which in fact makes sense in any finite dimensional representation of $\mathcal{U}_q(sl_2)$, the relations (0.28) may be written as

$$\begin{aligned} [H, X_\pm] &= \pm 2 X_\pm \\ [X_+, X_-] &= \frac{q^H - q^{-H}}{q - q^{-1}} \end{aligned} \tag{0.29}$$

which explains the name "q–deformed". The coproduct, counit and antipode may also be deformed by

$$\begin{aligned} \Delta(K^\pm) &= K^\pm \otimes K^\pm \\ \Delta(X_\pm) &= X_\pm \otimes K + K^{-1} \otimes X_\pm \\ \epsilon(K) &= 1, \quad \epsilon(X_\pm) = 0 \\ S(K) &= K^{-1}, \quad S(X_\pm) = -q^\pm X_\pm \end{aligned} \tag{0.30}$$

making $\mathcal{U}_q(sl_2)$ into a Hopf algebra. The Hopf algebra structure of $\mathcal{U}_q(sl_2)$ is due to [Skl85]. Quantum enveloping algebras $\mathcal{U}_q(\mathfrak{g})$ may be defined in a similar way for all complex semisimple Lie algebras $\mathfrak{g}$ as has been shown independently by [Dri85, Dri86] and [Jim85].

If q is a root of unity, i.e. $q^N = 1$, the algebras $\mathcal{U}_q(\mathfrak{g})$ are not semisimple. This may be cured by passing to semisimple quotients. The resulting algebras $\mathcal{U}_q^{tr}$ are finite dimensional semisimple quasitriangular weak quasi–Hopf algebras in the sense of [MS92], see Chapter 3. Their representation categories are strongly connected with the modular categories of [Tur94]. As algebras they are isomorphic to the direct sum of the irreducible representations of $\mathcal{U}_q(\mathfrak{g})$ with positive quantum dimensions. Thus they may be obtained by dividing $\mathcal{U}_q(\mathfrak{g})$ through the ideal which is annihilated by all physical representations (i.e. with positive quantum dimension). One may also use a general reconstruction theorem, see [Här95]. The importance of the algebras $\mathcal{U}_q^{tr}(\mathfrak{g})$ for applications in physics lies in the fact that they are not only semisimple but also C^*–algebras.

Lattice models and amplified DHR-theory

Let us shortly describe how the DHR-theory may be modified to deal with locally finite dimensional lattice models, i.e. lattice models where the local observable algebras are finite dimensional. We will in the following only deal with one dimensional lattices ("quantum chains"), which are considered as models of 1+1-dimensional quantum field theories.

A discrete local net of observables is an inclusion preserving map $I \mapsto \mathcal{A}(I)$ which assigns a (finite dimensional) C^*-algebra $\mathcal{A}(I)$ to every interval $I = (a, b) \subset \mathbb{Z}$. Defining the "space like complement" I' of I by $I' := \{z \in \mathbb{Z} \mid \mathrm{dist}(z, I) \geq 2\}$, locality means that if $J \subset I'$ then the algebras $\mathcal{A}(I)$ and $\mathcal{A}(J)$ commute (note that the definition of I' implies that one admits nontrivial relations between "next neighbors"). The quasilocal algebra of observables $\mathcal{A}$ is again defined as the C^*-algebra generated by all local observables.

Formulating an analog of the DHR-criterion one has to face the problem that every injective localized endomorphism of $\mathcal{A}$ would automatically be an automorphism due to the fact that the local observable algebras $\mathcal{A}(I)$ are finite dimensional. Thus taking the DHR-criterion (0.4) as it stands, all sectors would be Abelian. The following modification, which has been proposed by K. Szláchanyi and P. Vecsernyés, turns out to be appropriate. A representation π of $\mathcal{A}$ is said to fulfill the selection criterion, if there exists an interval I and $n \in \mathbb{Z}$, such that

$$\pi \restriction \mathcal{A}(I') \cong n \cdot \pi_0 \restriction \mathcal{A}(I'), \tag{0.31}$$

that is, when restricted to $\mathcal{A}(I')$, π is equivalent to a finite multiple of π_0. As before the (equivalence classes) of representations obeying (0.31) will be called DHR-sectors.

The role of the category of localized endomorphisms, which in the continuum case describe the DHR-sectors, is now played by the category of localized *amplimorphisms.* An amplimorphism (amplifying endomorphism) is an injective C^*-algebra map $\mu : \mathcal{A} \to \mathcal{A} \otimes \mathrm{End}(V) \cong M_n(\mathcal{A})$, where V is an n-dimensional Hilbert space and where $M_n(\mathcal{A})$ denotes the algebra of $n \times n$ - matrices with entries in $\mathcal{A}$. We also always assume μ to be unital. An amplimorphism is called localized in I if $\mu(A) = A \otimes \mathbf{1}_V$, $\forall A \in I'$.

Assuming algebraic Haag duality, there is a one-to-one correspondence between DHR-sectors and equivalence classes of localized amplimorphisms μ given by

$$\pi := (\pi_0 \otimes \mathrm{id}) \circ \mu.$$

Moreover, as for endomorphisms, amplimorphisms allow for the definition of a (monoidal) product of sectors given by $\mu \times \nu := (\mu \otimes \mathrm{id}) \circ \nu$. One proceeds as in the continuum case: The category of amplimorphisms (and therefore the category of DHR-sectors) admits subobjects and direct sums, fusion rules, conjugates, statistics operators, etc., making Rep_{DHR} into a strict braided rigid monoidal C^*-category, for details see [SV93].

In [NS97] the above scenario has been developed further by introducing the notion of a *universal cosymmetry.* Let us assume that the theory is rational, i.e. admits only a finite number of superselection sectors, given by amplimorphisms $\mu_1, \ldots, \mu_r$. An amplimorphism ρ is called universal, if ρ is equivalent to $\oplus_{i=1}^r \mu_i$. Denoting the C^*-algebra $\mathcal{G} := \oplus_i \mathrm{End}(V_i)$, ρ may be viewed as an algebra map $\rho : \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$ and the irreducible sectors of $\mathcal{A}$ may be recovered from ρ by applying the irreducible representations τ_i of $\mathcal{G}$: $\mu_i = (\mathrm{id} \otimes \tau_i) \circ \rho$. A universal ρ is called a universal *cosymmetry* if $\mathcal{G}$ is a bialgebra $(\mathcal{G}, \Delta, \epsilon)$ and if ρ is a coaction, i.e. $(\rho \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \Delta) \circ \rho$, $(\mathrm{id} \otimes \epsilon) \circ \rho = \mathrm{id}$. We also call $\mathcal{G}$ the universal cosymmetry. Moreover in this case one shows that $\mathcal{G}$ is even a quasitriangular Hopf algebra, where the antipode is recovered by studying the conjugate amplimorphism $\bar{\rho}$ and the R-matrix is determined by the statistics operator ϵ_ρ. The existence of a universal cosymmetry $\mathcal{G}$ in particular implies that the category of localized amplimorphisms (i.e. the category of DHR-sectors) may be identified with the representation category of $\mathcal{G}$ as rigid braided monoidal C^*-categories. Moreover in this case a field algebra may be reconstructed as an ordinary crossed product $\mathcal{F} = \mathcal{A} \rtimes \hat{\mathcal{G}}$, where $\hat{\mathcal{G}}$ is the Hopf algebra dual to $\mathcal{G}$. The Hopf spin models of [NS97] in fact possess a universal cosymmetry given by the quantum double of the underlying Hopf algebra.

Let us finally illustrate the above scenario by shortly reviewing G-spin models, G being a finite group, of [SV93]. Choosing $G = \mathbb{Z}(N)$, G-spin models reduce to the well known Ising and $\mathbb{Z}(N)$ models. Denote $Fun(G) =: \{f : G \to \mathbb{C}\}$ the algebra of functions on the group G and let $\mathfrak{H}_n$ be the tensor product of n copies of $Fun(G)$. The vectors $\{|\sigma\rangle \mid \sigma : \{1, \ldots, n\} \to G\}$ form an orthonormal basis in $\mathfrak{H}_n$. The full operator algebra on $\mathfrak{H}_n$ is generated by order parameters $F_{2i+1}(g)$ and disorder (or kink creating) operators $F_{2i}(g)$, $i \in \{1, \ldots, n\}$, $g \in G$, defined as follows:

$$\begin{aligned} F_{2i-1}(g)\,|\sigma\rangle &= \delta_{g,\sigma_i}\,|\sigma\rangle \\ F_{2i}(g)\,|\sigma\rangle &= |\sigma_1, \ldots, \sigma_{i-1}, g\sigma_i, \ldots g\sigma_n\rangle. \end{aligned} \tag{0.32}$$

One may now compute the multiplication and commutation relations of these operators and use them to define the abstract algebra $\mathcal{F}_{loc}$ as the algebra generated by $\mathbf{1}$ and elements $\{F_k(g) \mid k \in \mathbb{Z}, g \in G\}$ fulfilling these relations. The field algebra $\mathcal{F}$ is then obtained as the C^*-closure of $\mathcal{F}_{loc}$.

The algebra $\mathcal{A}$ of observables is obtained as the invariant subalgebra under the action of a generalized order $\times$ disorder symmetry which turns out to be given by the double $\mathcal{D}(G)$ of G [SV93]. Let us first recall how this is realized for Abelian ($\mathbb{Z}(N)$) spin models. Here the dual $\hat{G}$ consisting of all inequivalent irreducible representations

$\hat{g}$ of G is also a group, being isomorphic to G. The order symmetry $Q(g)$, $g \in G$ acts on the basis $\{|\sigma\rangle \mid \sigma : \mathbb{Z} \to G\}$ by the global spin rotation

$$Q(g)\,|\sigma\rangle = |\ldots, g\sigma_n, g\sigma_{n+1}, \ldots\rangle. \tag{0.33}$$

Next, if "kinks" or "solitons" are supposed to be stable, the space of physical states decomposes into inequivalent sectors labeled by the twist $\sigma_\infty \sigma_{-\infty}^{-1}$ in the boundary conditions. The corresponding projections are given by the operators $P(h)$, $h \in G$

$$P(h)\,|\sigma\rangle = \delta_{\sigma_\infty, h\sigma_{-\infty}}\,|\sigma\rangle. \tag{0.34}$$

Using the Abelianess of G one may build from the P's the unitary operators $\hat{Q}(\hat{g}) := \sum_{h\in G} \hat{g}(h)\, P(h)$, $\hat{g} \in \hat{G}$. In this way one arrives at a unitary representation $(g, \hat{g}) \mapsto Q(g)\hat{Q}(\hat{g})$ of the global order $\times$ disorder symmetry $G \times \hat{G}$ on the Hilbert space of physical states.

If the group G is non Abelian, the dual $\hat{G}$ and also the operator $\hat{Q}$ loose its meaning, but the algebra generated by $Q(g)$ and $P(h)$ may still be viewed as a symmetry algebra of the model. From (0.33),(0.34) one obtains the relations

$$\begin{gathered} Q(g_1)\,Q(g_2) = Q(g_1 g_2), \quad P(h_1)\,P(h_2) = \delta_{h_1,h_2}\,P(h_2) \\ Q(g)\,P(h) = P(ghg^{-1})\,Q(g), \end{gathered} \tag{0.35}$$

which are the defining relations of the double $\mathcal{D}(G)$. As a vector space $\mathcal{D}(G) = Fun(G) \otimes \mathbb{C}G$, with the identification $P(h)\,Q(g) = \delta_h \otimes g$, where $\delta_h \in Fun(G)$ is defined by $\delta_h(g) = \delta_{h,g}$. $D(G)$ is a quasitriangular Hopf algebra with $Fun(G)$ and $\mathbb{C}(G)$ being Hopf subalgebras. Using (0.22) and the formulas (0.33),(0.34) one obtains a definition of a (Hopf module) action γ of $\mathcal{D}(G)$ on the field algebra $\mathcal{F}$, for details see [SV93].

Defining the observable algebra $\mathcal{A}$ as $\mathcal{A} := \{F \in \mathcal{F} \mid \gamma_a(F) = \epsilon(a)F,\, \forall a \in \mathcal{D}(G)\}$, it turns out, that $\mathcal{A}$ is a C^*–algebra generated by a local net $\mathcal{A}(I)$, $I = (a, b) \subset \mathbb{Z}$ in the sense described above. The local algebras may be described as follows: Consider even (odd) integers to represent sites (links) of a one dimensional lattice and place a copy $\mathbb{C}G \cong \mathcal{A}_{2i}$ on each site and a copy of $Fun(G) \cong \mathcal{A}_{2i+1}$ on each link. Then $\mathcal{A}(I)$ is generated by $\{\mathcal{A}_n,\, n \in I\}$, where nontrivial commutation relations are postulated only on neighboring site link pairs, where they are given by

$$\begin{aligned} A_{2i+1}(\delta_h)\,A_{2i}(g) &= A_{2i}(g)\,A_{2i+1}(\delta_{g^{-1}h}) \\ A_{2i}(g)\,A_{2i-1}(\delta_h) &= A_{2i-1}(\delta_{hg^{-1}})\,A_{2i}(g). \end{aligned} \tag{0.36}$$

Here $A_{2i}(\cdot)$, $A_{2i+1}(\cdot)$ denote the embeddings $\mathbb{C}G \equiv \mathcal{A}_{2i} \subset \mathcal{A}(I)$ and $Fun(G) \equiv \mathcal{A}_{2i+1} \subset \mathcal{A}(I)$. On the basis $\{|\sigma\rangle \mid \sigma : \mathbb{Z} \to G\}$ these operators act as

$$\begin{aligned} A_{2i}(g)\,|\sigma\rangle &= |\ldots, \sigma_{i-1}, g\sigma_i, \sigma_{i+1}, \ldots\rangle \\ A_{2i+1}(\delta_h)\,|\sigma\rangle &= \delta_h(\sigma_i\sigma_{i+1}^{-1})\,|\sigma\rangle. \end{aligned}$$

Now suppose $\mathcal{A}$ is given in some Haag dual vacuum representation π_0. Then one can show that the DHR-sectors as described above are labeled by the irreducible representations $\{\tau_\alpha\}$ of $\mathcal{D}(G)$. The corresponding localized amplimorphisms μ_α may be realized as follows: Given τ_α, there exists a matrix multiplet $F_\alpha^{ij} \in \mathcal{F}$, $i, j \in \{1, \ldots, d_\alpha\}$, d_α denoting the dimension of the representation space V_α, such that

$$\gamma_a(F_\alpha^{ij}) = \sum_k F_\alpha^{ik}\,\tau_\alpha^{kj}(a), \quad a \in \mathcal{D}(\mathcal{G}) \tag{0.37}$$

$$\sum_i F_\alpha^{ik*}\,F_\alpha^{ik'} = \delta^{kk'}\,\mathbf{1} \tag{0.38}$$

$$\sum_k F_\alpha^{ik}\,F_\alpha^{jk*} = \delta^{ij}\,\mathbf{1}. \tag{0.39}$$

(The operators F_α^{ij} may be constructed as products of disorder and order operators. In the Ising model $G = \mathbb{Z}(2)$, these operators do anticommute. In the general case the F^{ij} obey braiding relations defined by the R-matrix of the double $\mathcal{D}(G)$.) Then

$$\mu_\alpha^{ij}(A) := \sum_k F_\alpha^{ik}\, A\, F_\alpha^{jk*}, \quad A \in \mathcal{A}$$

defines an amplimorphism $\mu_\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathrm{End}(V_\alpha)$. Relations (0.38), (0.39) generalize the Cuntz relations (0.13) to the amplified setting.

Choosing instead of $Fun(G)$ and $\mathbb{C}G$ any dual pair $\mathcal{G}$, $\hat{\mathcal{G}}$ of C^*-Hopf algebras one may formulate the corresponding commutation relations (0.36) yielding the Hopf Spin chains of [NS97], see Section 1.4.2.

Overview and summary of results

In this thesis we present an explicit construction of quantum chains based on a weak quasi–quantum group $\mathcal{G}$. This way we arrive at a generalization of the above mentioned Hopf spin models of [NS97] and also of the lattice current algebras of [AFFS98], both being based on ordinary quantum groups. The most important application is given by choosing $\mathcal{G}$ to be the semisimple quotient of a quantum enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ at roots of unity. The reason for studying these generalizations is to provide examples of lattice quantum field theories exhibiting a quantum symmetry with non–integer statistical dimensions. Moreover lattice current algebras have been invented as lattice regularizations of WZW-models and should therefore eventually be studied at roots of unity. Since the quantum chains based on ordinary quantum groups $\mathcal{G}$ the quantum symmetry is given by the quantum double $\mathcal{D}(\mathcal{G})$ we would also like to define the quantum double of a weak quasi–quantum group.

In approaching these aims, one has to face the problem that the dual $\hat{\mathcal{G}}$ of a quasi–quantum group is *not* an algebra (the multiplication is not associative). But since the dual $\hat{\mathcal{G}}$ appears as a subalgebra of the quantum double $\mathcal{D}(\mathcal{G})$ and also of quantum chains based on $\mathcal{G}$, it is not quite clear from the beginning if a generalization to quasi–quantum groups is possible at all. We were able to solve this problem by introducing a new construction of what we call the *diagonal crossed product* $\mathcal{M} \bowtie \hat{\mathcal{G}}$ of a unital algebra $\mathcal{M}$ and the dual $\hat{\mathcal{G}}$ of a (weak) quasi–Hopf algebra $\mathcal{G}$. In particular, $\mathcal{M} \bowtie \hat{\mathcal{G}}$ will always be an associative algebra extending $\mathcal{M} \equiv \mathcal{M} \bowtie \hat{\mathbf{1}}$. On the other hand, the linear subspace $\mathbf{1}_\mathcal{M} \bowtie \hat{\mathcal{G}}$ will in general not be a subalgebra of $\mathcal{M} \bowtie \hat{\mathcal{G}}$, unless $\mathcal{G}$ is an ordinary (i.e. coassociative) Hopf algebra. The diagonal crossed product will be the mathematical structure underlying all applications given later on. It also allows to formulate in a very elegant way an amplified version of the Mack Schomerus field "algebra".

The basic idea for this construction comes from generalizing the relations defining the quantum double. To this end we start from an algebra $\mathcal{M}$ equipped with a (quasi-) commuting pair of right and left $\mathcal{G}$-coactions, $\rho : \mathcal{M} \to \mathcal{M} \otimes \mathcal{G}$ and $\lambda : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M}$ and denote $\delta_l := (\lambda \otimes \mathrm{id}) \circ \rho$ and $\delta_r := (\mathrm{id} \otimes \rho) \circ \lambda$ as the associated equivalent *two-sided coactions*. In the simplest case of $\mathcal{G}$ being an ordinary Hopf algebra and (λ, ρ) being strictly commuting (i.e. $\delta_l = \delta_r$) this amounts to providing a commuting pair of left and right Hopf module actions $\triangleright : \hat{\mathcal{G}} \otimes \mathcal{M} \to \mathcal{M}$ (dual to ρ) and $\triangleleft : \mathcal{M} \otimes \hat{\mathcal{G}} \to \mathcal{M}$ (dual to λ) of the dual Hopf algebra $\hat{\mathcal{G}}$ on $\mathcal{M}$. In this case our diagonal crossed product $\mathcal{M} \bowtie \hat{\mathcal{G}}$ is defined to be generated by $\mathcal{M}$ and $\hat{\mathcal{G}}$ as unital subalgebras with commutation relations given by ($\hat{S} : \hat{\mathcal{G}} \to \hat{\mathcal{G}}$ being the antipode)

$$\varphi m = (\varphi_{(1)} \triangleright m \triangleleft \hat{S}^{-1}(\varphi_{(3)}))\, \varphi_{(2)}, \; m \in \mathcal{M}, \; \varphi \in \hat{\mathcal{G}} \; . \tag{0.40}$$

Note that for $\mathcal{M} = \mathcal{G}$ and $\rho = \lambda = \Delta$ the coproduct on $\mathcal{G}$, these are the defining relations of the quantum double $\mathcal{D}(\mathcal{G})$ [Dri86], and therefore $\mathcal{G} \bowtie \hat{\mathcal{G}} = \mathcal{D}(\mathcal{G})$. Introducing the "generating matrix"

$$\mathbf{\Gamma} := \sum_{\mu} e_{\mu} \otimes e^{\mu} \in \mathcal{G} \otimes \hat{\mathcal{G}} \subset \mathcal{G} \otimes (\mathcal{M} \bowtie \hat{\mathcal{G}}),$$

where $e_{\mu} \in \mathcal{G}$ is a basis with dual basis $e^{\mu} \in \hat{\mathcal{G}}$, (0.40) is equivalent to

$$\mathbf{\Gamma} \lambda(m) = \rho^{op}(m) \mathbf{\Gamma}, \quad \forall m \in \mathcal{M}. \tag{0.41}$$

Moreover, in this case $\hat{\mathcal{G}} \subset \mathcal{M} \bowtie \hat{\mathcal{G}}$ being a unital subalgebra is equivalent to

$$(\epsilon \otimes \mathrm{id})(\mathbf{\Gamma}) = \mathbf{1} \tag{0.42}$$

$$\mathbf{\Gamma}^{13} \mathbf{\Gamma}^{23} = (\Delta \otimes \mathrm{id})(\mathbf{\Gamma}) \tag{0.43}$$

where (0.43) is an identity in $\mathcal{G} \otimes \mathcal{G} \otimes (\mathcal{M} \bowtie \hat{\mathcal{G}})$, the indices denoting the embeddings of tensor factors. We call $\mathbf{\Gamma}$ the *universal normal and coherent $\lambda\rho$-intertwiner* in $\mathcal{G} \otimes (\mathcal{M} \bowtie \hat{\mathcal{G}})$, where normality is the property (0.42) and coherence is the property (0.43). Again, for $\mathcal{M} = \mathcal{G}$ and $\mathcal{M} \bowtie \mathcal{G} = \mathcal{D}(\mathcal{G})$, Equations (0.41)-(0.43) are precisely the defining relations for the generating matrix $\mathbf{D} \equiv \mathbf{\Gamma}_{\mathcal{D}(\mathcal{G})}$ of the quantum double (see e.g. [Nil97], Lem.5.2).

Inspired by the techniques of [AGS95, AGS96, AS96] we show how to generalize the notion of coherent $\lambda\rho$-intertwiners to the case of (weak) quasi-Hopf algebras $\mathcal{G}$, such that analogues of the Equations (0.41)-(0.43) still serve as the defining relations of an associative algebra extending $\mathcal{M} \equiv \mathcal{M} \bowtie \hat{\mathbf{1}}$. We also show that diagonal crossed products may equivalently be modeled on the linear spaces $\mathcal{M} \otimes \hat{\mathcal{G}}$ or $\hat{\mathcal{G}} \otimes \mathcal{M}$ (or – in the weak case – certain subspaces thereof). We point out, that most of our algebraic constructions are based on representation categorical concepts, as we discuss in more detail in Appendix A. Therefore they may also be visualized by graphical proofs, see Appendix B.

The basic model for this generalization is again given by $\mathcal{M} = \mathcal{G}$ with its natural two-sided $\mathcal{G}$-coactions $\delta_l := (\Delta \otimes \mathrm{id}) \circ \Delta$ and $\delta_r := (\mathrm{id} \otimes \Delta) \circ \Delta$. In this case our construction provides a definition of the quantum double $\mathcal{D}(\mathcal{G})$ for (weak) quasi-Hopf algebras $\mathcal{G}$, which is discussed in detail in Chapter 4. We show that the representation category $\mathrm{Rep}\, \mathcal{D}(\mathcal{G})$ coincides with what has been called the "double of the category" $\mathrm{Rep}\, \mathcal{G}$ in [Maj97]. Hence our definition provides a concrete realization of the abstract Tannaka-Krein like reconstruction of the quantum double given by [Maj97]. We then prove that $\mathcal{D}(\mathcal{G})$ is a quasitriangular (weak) quasi-Hopf algebra. We give explicit formulas for the coproduct, the antipode and the R-matrix. The most nontrivial part is the construction of the antipode. To this end, as a central technical result we establish a formula for $(S \otimes S)(R)$ and the relations between R^{-1}, $(S \otimes \mathrm{id})(R)$ and $(\mathrm{id} \otimes S^{-1})(R)$ for a quasitriangular $R \in \mathcal{G} \otimes \mathcal{G}$ in any quasi–Hopf algebra $\mathcal{G}$ in Appendix B. Recall, that in ordinary Hopf algebras the last three quantities coincide and therefore $(S \otimes S)(R) = R$. To prove these formulae and also the antipode properties of S_D, we use the graphical calculus developed by [RT90, Tur94, AC92b]. This will also allow to give nice intuitive interpretations of many of the identities derived in Chapter 2. In fact, without this graphical machinery we would have been lost in proving or even only trying to guess these formulas. In particular, a purely algebraic proof of the formulas for R^{-1} and $(S \otimes S)(R)$ in Theorem B.2 would most likely be unreadable and therefore also untrustworthy. As applications we discuss the twisted double $D^{\omega}(G)$ of [DPR90] and generalize the results of [Nil97] on the relation with the monodromy algebras of [AGS95, AGS96].

We are finally in the position to reach our original aim by applying our formalism to construct quantum chains based on a weak quasi–Hopf algebra as iterated diagonal crossed products in Chapter 5. Putting $\mathcal{M} = \mathcal{G} \otimes \mathcal{G}$ and choosing $\lambda = \rho = \Delta$, where ρ acts nontrivially on $\mathcal{G} \equiv \mathcal{G} \otimes \mathbf{1}$ and λ on $\mathcal{G} \equiv \mathbf{1} \otimes \mathcal{G}$, yields a commuting pair of coactions on $\mathcal{M}$. In this case $\mathcal{M} \bowtie \hat{\mathcal{G}} \cong \mathcal{G} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G}$ becomes a *two–sided crossed product.* We take this construction as building block of a quantum chain living on two neighboring sites (carrying the copies of $\mathcal{G}$) joined by a link (carrying the copy of $\hat{\mathcal{G}}$). We show how this construction iterates to provide a local net of associative algebras $\mathcal{A}(I)$ for any lattice interval I bounded by sites. Generalizing the methods of [NS97] we also construct localized coactions of the quantum double $\mathcal{D}(\mathcal{G})$ on such (weak) quasi-Hopf spin chains. Periodic boundary conditions for these models are again described as a diagonal crossed product of the open chain by a copy of $\hat{\mathcal{G}}$ sitting on the link joining the end points. In this way we arrive at a formulation of lattice current algebras at roots of unity by adjusting the transformation rules of [Nil97] to the quasi-coassociative setting. In particular the lattice current algebra consisting of one site and one link is seen to be isomorphic to the quantum double $\mathcal{D}(\mathcal{G})$.

Having constructed quantum chains based on a weak quasi–Hopf algebra $\mathcal{G}$, we investigate their representation theory. At this point we have to assume that $\mathcal{G}$ is semisimple, which is surely the case in all applications, in particular for the semisimple quotients of quantum enveloping algebras at roots of unity. We prove that the center of any finite quantum chain $\mathcal{A}(I)$ is isomorphic to the center of $\mathcal{G}$, whereas the center of any periodic quantum chain is isomorphic th the center of the quantum double $\mathcal{D}(\mathcal{G})$. This way we generalize the results of [AFFS98] on the representation theory of lattice current algebras, based on a quasitriangular (modular) Hopf algebra, to weak quasi–Hopf algebras.

Our main results are stated in Theorem 2.1 and Theorem 3.1 (diagonal crossed products by duals of quasi–Hopf algebras and weak quasi–Hopf algebras, respectively), Theorem 4.3 and Theorem 4.4 (the quantum double $\mathcal{D}(\mathcal{G})$, its quasitriangular quasi–Hopf structure and $\mathcal{D}(\mathcal{G})$–coactions) and Theorem 5.6 (representation theory of quantum chains).

We remark that our investigation of quantum chains is purely kinematical in the sense that we have not specified any dynamics. Also we do not introduce any $*$–structures. For lattice current algebras at roots of unity, C^*–structures may be introduced along the lines developed in [AGS95, AGS96]. For general quantum spin chains based on a C^*-quasi–Hopf algebra $\mathcal{G}$ we also believe this to be true, but the details still need to be worked out.

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