On the Evolution of Hypersurfaces in Asymptotically Flat Riemannian Manifolds by their Inverse Null Mean Curvature

DISSERTATION
zur Erlangung des Grades eines Doktors der Naturwissenschaften
Fachbereich Mathematik und Informatik
der Freien Universität Berlin
Max-Planck-Institut für Gravitationsphysik
(Albert-Einstein-Institut)

vorgelegt von
KRISTEN MOORE
betreut von
Prof. Dr. GERHARD HUISKEN

August 2012

Tag der mündlichen Qualifikation: August 2012

1. Gutachter: Prof. Dr. Gerhard Huisken
2. Gutachter: Prof. Dr. Jan Metzger

Erklärung

Ich bestätige hiermit, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen verwendet habe.

Kristen Moore

Abstract

We consider Riemannian manifolds which arise naturally as asymptotically flat initial data sets in general relativity. We introduce a new evolution equation for hypersurfaces in such manifolds where the speed is given by the reciprocal of the null mean curvature. This PDE unites the theory of marginally trapped surfaces (MOTS) in Lorentzian spacetimes with the study of inverse mean curvature flow in asymptotically flat Riemannian manifolds. A theory of weak solutions to this new flow is developed using level-set methods and an appropriate variational principle. The key ingredient is the use of elliptic regularisation, which amounts to solving an elliptic PDE which can be interpreted as Jang's equation [J] with a gradient regularisation term. As noted by Metzger [M], the assumption of an appropriate sign on the mean curvature on the initial data set prevents solutions of Jang's equation from blowing up to negative infinity over marginally inner trapped surfaces (MITS) in the initial data set. For similar reasons, it is necessary to restrict to initial data sets with non-negative mean curvature in this work. We then prove existence of a weak solution of the flow under this curvature assumption. This new flow has a natural application as a variational-type approach to constructing MOTS, and and this work also gives new insights into the theory of weak solutions of inverse mean curvature flow.

Zusammenfassung

Wir untersuchen Riemannsche Mannigfaltigkeiten, welche in natürlicher Weise als asymptotisch flache Anfangsdaten in der Allgemeinen Relativitätstheorie auftreten. Dazu führen wir eine neue Evolutionsgleichung für Hyperflächen solcher Mannigfaltigkeiten ein. Hierbei ist die Geschwindigkeit als die reziproke mittlere Krümmung in Richtung des Lichtkegels gegeben. Diese partielle Differentialgleichung verbindet die Theorie der MOTS (engl. "marginally outer trapped surfaces") in Lorentzschen Raumzeiten mit dem Studium des inversen mittleren Krümmungsflusses in asymptotisch flachen Riemannschen Mannigfaltikeiten. Mit Hilfe der Niveaumengenmethode und einer geeigneten Variationsformulierung entwickeln wir eine Theorie schwacher Lösungen für diesen Fluss. Der Kernpunkt dabei ist die Benutzung der sog. elliptischen Regularisierung, welche das Problem auf das Lösen einer elliptischen partiellen Differentialgleichung zurückführt. Diese kann als die Jangsche Gleichung [J] mit einem Gradienten-Regularisierungsterm verstanden werden. Wie schon von Metzger [M] bemerkt, verhindert die Annahme eines Vorzeichens an die mittlere Krümmung, dass Lösungen der Jangschen Gleichung am Rand der MITS (engl. "marginally inner trapped surfaces") der Anfangsdaten nach minus Unendlich divergieren.

Aus ähnlichen Gründen ist es notwendig sich in diesem Kontext auf Anfangsdaten mit nicht-negativer mittlerer Krümmung zu beschränken. Unter diesen Annahmen beweisen wie die Existenz von schwachen Lösungen.

Wir skizzieren eine Anwendung dieses Flusses auf die Exisitenztheorie für MOTS, und zeigen zudem, dass wenn eine schwache Lösung als Limes von Lösungen der regularisierten Gleichung gewonnen wird, das Innere des Sprungregion durch glatte MOTS geblättert wird.

Contents

1 Introduction

In this work we introduce a new geometric evolution equation which unites the theory of marginally outer trapped surfaces in Lorentzian spacetimes with the study of inverse mean curvature flow in asymptotically flat Riemannian manifolds. In what follows we consider an initial data set (M^{n+1}, g, K) that arises as an asymptotically flat, spacelike hypersurface M^{n+1} in a spacetime (L^{n+2}, h), with induced metric g and second fundamental form K. Let $\vec{n}$ denote the future directed timelike unit normal vector field of $M \subset L$, and consider a 2-sided hypersurface $\Sigma^n \subset M^{n+1}$ with globally defined outer unit normal vector field ν in M. The mean curvature vector of Σ inside the spacetime L is then given by

$$\vec{H}_\Sigma := H\nu - P\vec{n},$$

where $H := \mathrm{div}_\Sigma(\nu)$ denotes the mean curvature of Σ in M, and $P := \mathrm{tr}_\Sigma K$ is the trace of K over the tangent space of Σ.

The new initial value problem is then defined as follows. Given a smooth hypersurface immersion $F_0 : \Sigma \to M$, the evolution of $\Sigma_0 = F_0(\Sigma)$ by inverse null mean curvature is the one-parameter family of smooth immersions $F : \Sigma \times [0, T) \to M$ satisfying

$$\begin{cases} \dfrac{\partial F}{\partial t}(p,t) = \dfrac{\nu}{H+P}(p,t), \quad p \in \Sigma, t \geq 0, \\ F(\cdot, 0) = F_0. \end{cases} \tag{$*$}$$

The quantity $H + P$ corresponds to the component of $\vec{H}_{\Sigma_t}$ in the future null direction $l^+ := \nu + \vec{n}$ of $\Sigma_t = F(\cdot, t)(\Sigma)$, and we assume that $(H+P)|_{\Sigma_0} > 0$ so that $(*)$ is parabolic and the surface Σ_t expands under the flow.

The aim of this thesis is to develop a theory of weak solutions of the classical flow $(*)$, and outline some key geometric properties and applications of such solutions. We begin with an informal discussion motivating the choice of this new flow, and then summarise the main results in Section 1.2.

1.1 Background

The main motivation for this work is the theory of inverse mean curvature flow, which corresponds to the special case $K \equiv 0$ of $(*)$. In particular, given a smooth immersion $F_0 : N^n \to M^{n+1}$ of a hypersurface $N_0 = F_0(N^n)$ in a Riemannian manifold M^{n+1}, the smooth family $F : N^n \times [0,T) \to M$ of hypersurfaces $N_t = F(\cdot,t)(N^n)$ solves inverse mean curvature flow if

$$\begin{cases} \dfrac{\partial F}{\partial t}(p,t) = \dfrac{\nu}{H}(p,t), \quad p \in N, t \geq 0, \\ F(\cdot,0) = F_0. \end{cases} \tag{IMCF}$$

We make the additional assumption that H is positive, and therefore that (IMCF) is parabolic.

Inverse mean curvature flow is characterised by the fundamental property that the area of the evolving surface grows exponentially in time,

$$|N_t| = |N_0|e^{t/n}. \tag{1.1}$$

Interest in the PDE was first sparked by Geroch's discovery [G] of the famous monotonicity formula governing the evolution of the Hawking mass under inverse mean curvature flow. In particular, the Hawking mass of a connected hypersurface in a manifold of non-negative scalar curvature is non-decreasing along solutions of inverse mean curvature flow. This yields a proof of the positive energy theorem for asymptotically flat 3-manifolds under the assumption that the manifold admits a smooth global solution of inverse mean curvature flow.

Jang and Wald [JW] showed that, when applied to an asymptotically flat 3-manifolds containing a single outermost minimal sphere, a similar approach using smooth inverse mean curvature flow and Geroch's monotonicity formula provides a rigorous proof of the so-called Riemannian Penrose Inequality.

In order to prove the general case of the Riemannian Penrose inequality, however, it is necessary to establish the existence theory for generalised solutions of inverse mean curvature flow. This program was carried out by Huisken and Ilmanen in [HI]. We note that an independent proof of the Riemannian Penrose inequality - which uses different techniques and which applies to the case of multiple horizons - was later provided by Bray [Br].

A classical PDE analysis of inverse mean curvature flow was first carried out by Gerhardt [Ge], who proved that the solution starting from any starshaped surface with positive mean curvature will exist for all time and become round in the limit; see also Urbas [U]. Existence and regularity relies on an upper bound on the mean curvature

$$\sup_{N_t} H \leq C \sup_{N_0} H \cdot e^{-\frac{t}{n}}, \qquad C = C(\sup_M |Ric|), \tag{1.2}$$

which is generated by the sign on the zero order term in the evolution

$$\frac{\partial H}{\partial t} = \frac{\Delta H}{H^2} - \frac{2|DH|^2}{H^3} - \frac{|A|^2}{H} - \frac{Ric(\nu,\nu)}{H}.$$

Huisken and Ilmanen [HI2] later characterised smoothness by the condtion that the mean curvature remains bounded away from zero.

Within the class of mean-convex surfaces, however, the solution of inverse mean curvature flow will in general become singular in finite time. For example, starting from a thin torus with positive mean curvature in $\mathbb{R}^3$, we have $\min_{N_t} H \to 0$ as t approaches some finite singular time $T < \infty$.[1] At this point the classical description breaks down and it is in fact clear that any appropriate weak definition of inverse mean curvature flow would need to allow for a change of topology in order to continue the evolution in this example.

Motivated by observations of this kind, Huisken and Ilmanen [HI] put forward an alternative level-set formulation of inverse mean curvature flow and a corresponding theory of weak solutions. Given an initial surface $N_0 = \partial\Sigma_0$, they interpret the surfaces N_t as level-sets of the scalar function $u : M\backslash\Omega_0 \to \mathbb{R}$ which maps each point $x \in M\backslash\Omega_0$ to the time t when $x \in N_t$. This transforms (IMCF) from a parabolic system to the following degenerate elliptic partial differential equation for the scalar function u,

$$\operatorname{div}\left(\frac{Du}{|Du|}\right) = |Du|. \tag{$\star$}$$

It follows from a direct computation that ($\star$) is equivalent to (IMCF) wherever u is smooth and $|\nabla u| \neq 0$. Furthermore, the level-set formulation inherently allows the evolving surface to change topology by jumping instantly across a positive volume, ie. the level-sets of u "jump" across regions where u is constant.

This level-set approach was inspired by work of Evans and Spruck [ES1] and Chen, Giga and Goto [CGG], who independently developed a theory of weak solutions to mean curvature flow using viscosity techniques. By contrast, however, Huisken and Ilmanen define a locally Lipschitz function u on M to be a weak solutions of ($\star$) if it satisfies a certain minimisation principle with respect the the energy functional

$$J_u(v) = \int_\Omega |\nabla v| + v|\nabla u|, \quad \{u \neq v\} \subset \Omega \subset\subset M. \tag{1.3}$$

It is this variational principle that determines how and when the surfaces jump. Moreover, at a heuristic level, this is the mechanism that can be seen to preserve the monotonicity of the Hawking mass in the weak setting. The complete proof of the Geroch monotonicity formula for the weak evolution is one of the most technical breakthroughs in [HI].

Carefully combining the classical argument with these new ideas, Huisken and Ilmanen were able to establish the first rigorous proof of the Riemannian Penrose inequality in General Relativity.

Subsequent (suprising) applications of Huisken and Ilmanen's weak inverse mean curvature flow were recently uncovered by Bray and Neves, and by Bray and Miao. In [BN], Bray and Neves used the weak solution of inverse mean curvature flow to prove the Poincaré conjecture for the class of three-manifolds with σ-invariant greater than that of $\mathbb{RP}^3$. In [BM], Bray and Miao derive an upper bound for the capacity of a surface in Riemannian 3-manifolds of non-negative scalar curvature in terms of the area and Willmore functional

[1] The minimum is attained on the inner hole of torus.

of the surface.

The motivation for introducing the genearlised inverse mean curvature type flow $(*)$ follows from the theory of marginally outer trapped surfaces in general relativity. Given a 2-sided hypersurface Σ^n in the initial data set (M^{n+1}, K, g), the null mean curvature or *null expansion* θ^+_Σ of Σ with respect to its future directed outward null vector field l^+ is given by

$$\theta^+_\Sigma := H_\Sigma + P_\Sigma.$$

Physically this outward null expansion measures the divergence of the outward directed light rays emanating from Σ. If θ^+ vanishes on all of Σ, then Σ is called a *marginally outer trapped surface*, or MOTS for short. MOTS play the role of apparent horizons, or quasi-local black hole boundaries in general relativity, and are particularly useful for numerically modelling the dynamics and evolution of black holes.

From a mathematical point of view, MOTS are the Lorentzian analogue of minimal surfaces. This is especially true with regard to stability, since many results for stable minimal surfaces carry over directly to their Lorentzian counterpart. However, since MOTS are not stationary solutions of an elliptic variational problem, the direct method of the calculus of variations is not a viable approach to the existence theory. One successful approach to proving existence of MOTS comes from studying the blow-up set of solutions of *Jang's equation*

$$\left(g^{ij} - \frac{\nabla^i u \nabla^j u}{|\nabla u|^2 + 1}\right)\left(\frac{\nabla_i \nabla_j u}{\sqrt{|\nabla u|^2 + 1}} + K_{ij}\right) = 0, \tag{1.4}$$

which was an essential ingredient in the Schoen-Yau proof of the positive mass theorem [SY][2]. In their analysis, Shoen and Yau showed that the boundary of the blow-up set of Jang's equation consists of marginally trapped surfaces. Building upon this work, existence of MOTS in compact data sets with two boundary components, such that the inner boundary is (outer) trapped and the outer boundary is (outer) untrapped, has been proven by Andersson and Metzger [AM], and subsequently by Eichmair [E], using a different approach. Despite the lack of a variational principle, a "minimising" property arises naturally for the MOTS constructed via the Perron method for Jang's equation in [E]. Similarly, we see below that Jang's equation also plays a key role in the existence theory for the evolution by inverse null mean curvature, and that the solution Σ_t also satisfies a minimising property at each time t.

As a final remark on the motivation for this work, we note that the *spacetime Penrose Inequality* generalizes the Riemannian Penrose Inequality to the Lorentzian codimension-2 setting. It was noted in [HI] that the Hawking mass is also monotone under smooth

[2] The equation proposed by Jang in [J] and studied by Schoen and Yau in [SY] is actually (1.4) with respect to the tensor $-K$, but we will refer to (1.4) as Jang's equation since this is the equation that is relevant to our work below.

inverse mean curvature flow of spacelike 2-surfaces in a Lorentzian 4-manifold, thus in theory it may be possible to generalize the approach of Huisken and Ilmanen in order to prove the full Penrose Conjecture. However additional analytic difficulties arise in this case since the embedding functions describing the flow along the inverse mean curvature vector satisfy a backward-forward parabolic system. There is no general theory which guarantees local existence of solutions to such systems, so any successful approach would require a clever utilisation of the special geometric properties of this flow. On the other hand, the evolution by inverse null mean curvature $(*)$ is a generalisation of (IMCF) that incorporates the extrinsic curvature K of M in the spacetime, whilst being an analytically tractable problem. It is therefore hoped that this flow can be used to gain insight into the full Penrose Inequality in general relativity.

1.2 Overview of main results

Smooth evolution by inverse null mean curvature. At the curvature level, the flow by inverse null mean curvature leads to a reaction-diffusion system – typical of nonlinear heat flows – controlling the evolution of the null mean curvature. We establish the following interior estimate for the null mean curvature under the smooth evolution.

Lemma 1.1 (Interior null mean curvature estimate) *Let Σ_t be a smooth solution of $(*)$ for $0 \leq s \leq t$. Then for each $p \in \Sigma_t$*

$$H(p,t) + P(p,t) \leq \max\left((H+P)_R, C\right), \tag{1.5}$$

where $(H+P)_R$ is the maximum of $H+P$ on the parabolic boundary of $\Sigma_t \cap B_R(p)$, and $C = C(n, R, \|K\|_{C_0}, \|K\|_{C^1}) = o\left(R^{-1}\right)$ as $R \to \infty$.

This estimate is the key to existence and regularity of weak solutions.

The level-set flow and elliptic regularisation. Rather than defining the evolving surfaces as smooth immersions, they can alternatively be represented as level-sets

$$\Sigma_t = \partial\{x \in M \backslash \Omega_0 \mid u(x) < t\} \tag{1.6}$$

of a scalar function $u : M\backslash\Omega_0 \to \mathbb{R}$, for some open, bounded set $\Omega_0 \subset M^n$ such that $\Sigma_0 = \partial\Omega_0$. Then whenever u is smooth and $\nabla u \neq 0$, $(*)$ is equivalent to the following degenerate elliptic scalar PDE

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \left(g^{ij} - \frac{\nabla^i u \nabla^j u}{|\nabla u|^2}\right) K_{ij} = |\nabla u|, \tag{$**$}$$

where the left hand side is the null mean curvature $\theta^+_{\Sigma_t}$ of the surface Σ_t and the right hand side is the inverse speed of the family of level-sets. Since $|\nabla u| = H + P$, (1.5) suggests that it is reasonable to expect locally Lipschitz solutions of $(**)$. However, in order for $u(x)$ to correspond to the time t when the evolving surface passes through the point x, the function u should be monotone non-decreasing, which implies in particular that the zero function must be a subsolution barrier for the Dirichlet problem $(**)$. For this reason, it only makes sense to study $(**)$ on initial data sets (M, g, K) satisfying $\mathrm{tr}_M K \geq 0$.

In order to solve $(**)$, we employ the method of *elliptic regularisation*, and study solutions of the following strictly elliptic equation

$$(**)_\varepsilon \qquad \operatorname{div}\left(\frac{\nabla u_\varepsilon}{\sqrt{|\nabla u_\varepsilon|^2+\varepsilon^2}}\right)+\left(g^{ij}-\frac{\nabla^i u_\varepsilon \nabla^j u_\varepsilon}{|\nabla u_\varepsilon|^2+\varepsilon^2}\right)K_{ij}=\sqrt{|\nabla u_\varepsilon|^2+\varepsilon^2}.$$

A notable feature of elliptic regularisation is that the downward translating graph

$$\tilde{\Sigma}_t^\varepsilon := \mathrm{graph}\left(\frac{u_\varepsilon}{\varepsilon}-\frac{t}{\varepsilon}\right) \tag{1.7}$$

solves the classical evolution $(*)$ in the product manifold $(M \times \mathbb{R}, \bar{g} = g \oplus dz^2)$ where we extend the given data K to be parallel in the z-direction. Therefore elliptic regularisation transforms the singular level-set formulation $(**)$ in M^{n+1} into a smooth, non-compact evolution one dimension higher in $M^{n+1} \times \mathbb{R}$.

Furthermore, this elliptic regularisation problem sheds new light on the study of Jang's equation (1.4), since a rescaling of $(**)_\varepsilon$ can be interpreted as Jang's equation (1.4) with a gradient regularisation term. We find that in order to solve $(**)_\varepsilon$, it is in fact *necessary* to restrict to initial data sets satisfying $\mathrm{tr}_M K \geq 0$. In Section 3 we compare this mean curvature condition for the gradient regularised Jang's equation, to the corresponding properties of capilarity-regularised solutions of Jang's equation, which were analysed in detail by Scheon and Yau in [SY].

Proposition 1.2 *Let $tr_M K \geq 0$. Then there exists a smooth solution u_ε of $(**)_\varepsilon$ such that u_ε possesses a uniform Lipschitz estimate on each compact subset of $M\backslash\Omega_0$.*

The Weak Solution. A variational principle is used to define weak solutions to $(**)$. This is motivated by the observation that freezing $|\nabla u| - \mathrm{tr}_{\Sigma_t} K = |\nabla u| - (g^{ij} - \nu^i_{\Sigma_t}\nu^j_{\Sigma_t})K_{ij}$ and treating it as a bulk term in the energy functional

$$\mathcal{J}^A_{u,\nu}(v) := \int_A |\nabla v| + v\left(|\nabla u| - \left(g^{ij}-\nu^i\nu^j\right)K_{ij}\right), \tag{1.8}$$

allows one to interpret $(**)$ as the *Euler-Lagrange equation* of (1.8). The special case $K \equiv 0$ corresponds to the functional employed by Huisken and Ilmanen [HI] to define weak solutions to inverse mean curvature flow. Here in the general case, $\nu = \dfrac{\nabla u}{|\nabla u|}$ is the unit normal to the surfaces Σ_t defined by (1.6). However, since $\dfrac{\nabla u}{|\nabla u|}$ is undefined on plateaus of the (locally Lipschitz) function u, that is, in *jump regions*, the concept of weak solution in this case requires an appropriate notion of normal vector in the jump region.

A suitable vector field can be constructed by taking an appropriate limit of the smooth translating solution $\tilde{\Sigma}_t^\varepsilon$ of $(*)$, where $\tilde{\Sigma}_t^\varepsilon$ was defined by (1.7). The level-set function describing $\tilde{\Sigma}_t^\varepsilon$ is defined by

$$U_\varepsilon(x,z) := u_\varepsilon(x) - \varepsilon z, \qquad (x,z) \in (M\backslash\Omega_0) \times \mathbb{R},$$

since $\tilde{\Sigma}_t^\varepsilon = \{U_\varepsilon = t\}$. The uniform local Lipschitz estimates for u_ε means that there exists a locally Lipschitz function u and a subsequence ε_i such that

$$u_i \to u, \tag{1.9}$$

locally uniformly on $M\backslash\Omega_0$. Then setting

$$U(x,z) := u(x), \tag{1.10}$$

we obtain that $U_i \to U$ locally uniformly on $(M\backslash\Omega_0) \times \mathbb{R}$. Since the null mean curvature of the surfaces $\tilde{\Sigma}_t^i = \{U_i = t\}$ is uniformly bounded, results of measure theory allow us to control these surfaces in $C^{1,\alpha}$. At a jump time t_0 of U, this approach then leads to a foliation of the interior $\tilde{\mathcal{K}}_{t_0}$ of the jump region $\{U = t_0\} = \{u = t_0\} \times \mathbb{R}$ by $C^{1,\alpha}$ hypersurfaces

$$\tilde{\Sigma}_{X_0}, \qquad X_0 \in \tilde{\mathcal{K}}_{t_0}, \tag{1.11}$$

with corresponding $C^{0,\alpha}$ unit normal vector field ν. Furthermore, this vector field ν is translation invariant in the $\mathbb{R}$ direction, since each jump region hypersurface $\tilde{\Sigma}$ is either (part of) a vertical cylinder, or is a graph over an open subset of $\tilde{\mathcal{K}}_{t_0}$ in the stack

$$\tilde{\Sigma} + z\,\mathbf{e}_{n+2}, \quad z \in \mathbb{R}, \tag{1.12}$$

of vertical translates of $\tilde{\Sigma}$. Projecting the normal vector field ν of this hypersurface foliation to TM produces a vector field that extends $\nabla u/|\nabla u|$ as a calibration across the jump region of u in $M\backslash\Omega_0$. However, since in general information will be lost by projecting the normal vector field ν of the graphical hypersurfaces (1.12) to TM, we study a formulation of weak solutions of $(**)$ one dimension higher in $M \times \mathbb{R}$, in terms of a translation invariant vector field $\nu \in C^{0,\alpha}_{\text{loc}}(T(M\backslash\Omega_0) \times \mathbb{R})$ and a translation invariant function $U(x,z) = u(x) \in C^{0,1}_{\text{loc}}(M \times \mathbb{R})$ satisfying

$$\mathcal{J}^A_{U,\nu}(U) \le \mathcal{J}^A_{U,\nu}(V), \tag{1.13}$$

for every $V \in C^{0,1}_{\text{loc}}(M \times \mathbb{R})$ such that $\{V \neq U\} \subset\subset (M\backslash\bar{\Omega}_0) \times \mathbb{R}$, and any compact set A containing $\{V \neq U\}$, where

$$\mathcal{J}_{U,\nu}(V) = \mathcal{J}^A_{U,\nu}(V) := \int_A |\bar{\nabla} V| + V\left(|\bar{\nabla} U| - (\bar{g}^{ij} - \nu^i\nu^j)K_{ij}\right). \tag{1.14}$$

The variational principle (1.13) has an equivalent formulation in terms of individual level-sets (which holds for general U and ν that are not necessarily translation invariant like we demand for the weak solution). Namely, let $\tilde{F} \subset (M \backslash \Omega_0) \times \mathbb{R}$ be a Caccioppoli set with reduced boundary $\partial^* \tilde{F}$, and consider the functional

$$\mathcal{J}_{U,\nu}(\tilde{F}) = \mathcal{J}^A_{U,\nu}(\tilde{F}) := |\partial^* \tilde{F} \cap A| - \int_{\tilde{F} \cap A} |\bar{\nabla} U| - (\bar{g}^{ij} - \nu^i \nu^j) K_{ij} dx. \tag{1.15}$$

The relationship between the two formulations can be explained as follows. The locally Lipschitz function U satisfies (1.13) on an open set Ω if and only if the open set $E_t := \{U < t\}$ minimises (1.15) in Ω for each t. That is, if

$$\mathcal{J}_{U,\nu}(E_t) \leq \mathcal{J}_{U,\nu}(\tilde{F}), \tag{1.16}$$

for each t and each $\tilde{F}$ that differs from E_t on a set compactly contained in Ω.

In Section 7 we show that each jump region hypersurface (1.11) forms part of the boundary of a Caccioppoli set that minimises (1.15). With this in mind, we call the pair (U, ν) a weak solution of $(**)$ with initial condition E_0 if $U \in C^{0,1}_{\text{loc}}(M \times \mathbb{R})$ and $\nu \in C^{0,\alpha}_{\text{loc}}(T(M \backslash E_0) \times \mathbb{R})$ satisfy

1. $U(x, z) = u(x)$ is translation invariant in the vertical direction.

2. The set $E_t = \{U < t\}$ minimises $J_{U,\nu}$ in $(M \backslash E_0) \times \mathbb{R}$ for each $t > 0$. At jump times t_0, each point $X_0 = (x_0, z_0)$ in the interior $\tilde{\mathcal{K}}_{t_0}$ of the jump region $\{U = t_0\}$ lies in the boundary of a Caccioppoli set E_{X_0} that minimises $J_{U,\nu}$ in $\tilde{\mathcal{K}}_{t_0}$.

3. ν is the translation invariant, unit vector field normal to ∂E_t at each point $X = (x, z) \in \partial E_t$, and normal to ∂E_{X_0} at each point $X = (x, z) \in \partial E_{X_0}$ at a jump time t_0.

The following weak existence theorem is the main result of this thesis.

Theorem 1.3 *Let (M, g, K) be a complete, connected, asymptotically flat initial data set satisfying $tr_M K \geq 0$, and let E_0 be any precompact, smooth open set in M. Then there exists a weak solution of $(**)$ in $M \times \mathbb{R}$ with initial condition E_0.*

Given a weak solution $(U(x, z) = u(x), \nu)$ on $M \times \mathbb{R}$, the pair $(u, \nu_M := \nu|_{TM})$ then satisfy

$$\mathcal{J}_{u,\nu_M}(u) \leq \mathcal{J}_{u,\nu_M}(v) \tag{1.17}$$

on $M \backslash E_0$, for every $v \in C^{0,1}_{\text{loc}}(M)$ such that $\{v \neq u\} \subset\subset M \backslash E_0$, where $\mathcal{J}_{u,\nu_M}$ is defined by (1.8).

The minimisation principle (1.16) also leads to a geometric characterisation of the jump regions. In particular, if $(U(x, z) := u(x), \nu)$ is a weak solution of $(**)$, (1.16) implies that the sets $E_t := \{u < t\}$ are *outward optimising (with respect to $\nu_M := \nu\big|_{TM}$)* for each $t > 0$, in the sense that

$$|\partial^* E_t| \leq |\partial^* F| + \int_{F \backslash E_t} (g^{ij} - \nu^i_M \nu^j_M) K_{ij} \tag{1.18}$$

for any set F containing E_t such that $F \backslash E_t \subset\subset M$. That is, each E_t minimises *"area plus bulk energy P"* on the outside. This leads to the following heuristic interpretation of the flow: E_t evolves by inverse null mean curvature until a set E_t' appears that contains E_t and gives equality in (1.18), then E_t jumps to the "outermost" such set E_t' and the flow continues.

Furthermore, the surfaces $\tilde{\Sigma}_{X_0}$ from (1.11) foliating the jump region $\tilde{\mathcal{K}}_{t_0}$ minimise area plus bulk energy P, and therefore satisfy a stronger variational principle than (1.16).

Proposition 1.4 *The surfaces (1.11) foliating the interior $\tilde{\mathcal{K}}_{t_0}$ of the jump region $\{u = t_0\} \times \mathbb{R}$ at a jump time t_0 consist of vertical cylinders and stacks of graphs (5.6), where each is a smooth MOTS in $\tilde{\mathcal{K}}_{t_0}$ (with respect to the normal vector ν).*

Since the surfaces $\tilde{\Sigma}_t := \partial\{U < t\}$ and $\tilde{\Sigma}_t^+ := \partial\{U > t\}$ bound Caccioppoli sets that minimise $\mathcal{J}_{U,\nu}$ in $(M \backslash E_0) \times \mathbb{R}$, together with Proposition 1.4, this implies that elliptic regularisation produces a family of surfaces in $(M \backslash E_0) \times \mathbb{R}$ such that each bounds a Caccioppoli set minimising $\mathcal{J}_{U,\nu}$. The pair $(u, \nu_{_M} := \nu|_{_{TM}})$ also have the following interpretation in $M \backslash E_0$. Namely, they satisfy

$$
\begin{aligned}
|\nu_{_M}| \leq 1, \quad \nabla u \cdot \nu_{_M} = |\nabla u| \text{ a.e.}, \\
\int_\Omega \nabla \xi \cdot \nu_{_M} + \xi \left(|\nabla u| - \left(g^{ij} - \nu_{_M}^i \nu_{_M}^j \right) K_{ij} \right) = 0 \quad \text{for all } \xi \in C_c^1(\Omega)
\end{aligned} \tag{1.19}
$$

Applications of the flow. The evolution by inverse null mean curvature has the following immediate application to the existence theory for MOTS. Consider an initial initial data set (M, g, K) satisfying $\mathrm{tr}_M K \geq 0$ that contains an outer trapped surface $\Sigma_0 = \partial E_0$ such that $\theta^+_{\Sigma_0} < 0$. If we take E_0 to be the initial condition for the flow, then the outward optimisation property (1.18) of the solution, together with the fact that Σ_0 is outer trapped, forces the weak solution to jump at time $t = 0$ to the vertical cylinder $\partial\{U > 0\} = \Sigma \times \mathbb{R}$ over a MOTS Σ in $M \backslash E_0$.

Proposition 1.5 *Let $(U(x,z) = u(x), \nu)$ be a weak solution of $(**)$ with initial condition E_0 satisfying $\theta^+_{\partial E_0} < 0$. Then $\partial\{u > 0\}$ is a smooth MOTS in $M \backslash E_0$.*

If the mean curvature of the initial data set instead satisfies $\mathrm{tr}_M K \leq 0$, the corresponding existence result applies for the flow with inverse speed equal to $H - P$, with analogous interpretations of the solution in relation to marginally inner trapped surface (MITS) in the initial data set.

We make the final remark that applying this work to the special case $K \equiv 0$ provides a new point of view on weak solutions to inverse mean curvature flow. Namely, a family of $C^{1,1}_{\mathrm{loc}}$ hypersurfaces in $(M \backslash E_0) \times \mathbb{R}$ associated to the pair (U, ν), such that each surface satisfies the parametric variational principle (1.16), and the pair $(u, \nu|_{_{TM}})$ have a corresponding interpretation in $M\backslash E_0$ analogous to (1.19). Furthermore one obtains the following result for the hypersurfaces foliating in the jump region.

Corollary 1.6 *Let u be the weak solution to IMCF. Then the interior of the jump region $\{u = t_0\} \times \mathbb{R}$ at jump times t_0 is foliated by smooth area minimising surfaces.*

Acknowledgements. I would like to thank my advisor, Gerhard Huisken, for introducing me to this topic and for his insightful and enthusiastic guidance throughout the development of this work. Many thanks also to Klaus Ecker for his ongoing mathematical and moral support.

2 Evolution by inverse null mean curvature

In this work we introduce a new evolution equation for hypersurfaces inside asymptotically flat spacetime initial data sets. Let M^{n+1} be a complete, connected Riemannian manifold of dimension $n+1 \geq 3$, without boundary, which arises as a spacelike hypersurface in a Lorentzian manifold (L^{n+2}, h), with induced metric g and second fundamental form K. We further assume that the initial data set (M, g, K) is asymptotically flat, that is, there exists a compact set $\Omega \subset M$ such that $M \backslash \Omega$ consists of a finite number of components, each diffeomorphic to $\mathbb{R}^{n+1} \backslash \bar{B}(0,1)$ and such that under these diffeomorphisms

$$|g_{ij} - \delta_{ij}| \leq \frac{C}{|x|^{n-1}}, \quad |g_{ij,k}| \leq \frac{C}{|x|^{n}} \tag{2.1}$$

$$|K_{ij}| \leq \frac{C}{|x|^{n}}, \quad |K_{ij,k}| \leq \frac{C}{|x|^{n+1}}, \quad \Big|\sum_i K_{ii}\Big| \leq \frac{C}{|x|^{\frac{n+3}{2}}}. \tag{2.2}$$

We work in an adapted coordinate system where Greek indices run from 0 to n on the ambient space, and latin indices run from 1 to n on the evolving hypersurface.

Now consider the smooth immersion $F_0 : \Sigma \to M$ of a hypersurface $\Sigma_0 = F_0(\Sigma)$ in M. We say that the smooth family $F : \Sigma^n \times [0,T] \to M^{n+1}$ of hypersurfaces $\Sigma_t = F(\cdot,t)(\Sigma^n)$ evolves by inverse null mean curvature if

$$\begin{aligned} \frac{\partial F}{\partial t}(p,t) &= \frac{\nu}{H+P}(p,t), \quad p \in \Sigma, \quad t \in [0,T), \\ F(p,0) &= F_0, \quad p \in \Sigma, \end{aligned} \tag{$*$}$$

where $H = \mathrm{div}_\Sigma(\nu)$ is the mean curvature of Σ_t in M, $P := tr_{\Sigma_t}(K) = tr_M K - K(\nu,\nu)$, and we assume that $(H+P)|_{N_0} > 0$.

2.1 The smooth flow

Since the aim of this work is to develop the weak theory for the evolution by inverse null mean curvature, we will not discuss existence of solutions to the classical flow $(*)$, except to remark that the leading order term of the linearised equation is $\frac{1}{(H+P)^2}\Delta_{g_t}$, where Δ_{g_t} denotes the Laplace-Beltrami operator with respect to the metric at time t. This is an elliptic operator as long as $(H+P)^{-2}$ remains non-singular, so $(*)$ is parabolic so long as the null mean curvature of the evolving surface remains strictly positive.

In Chapter 2.3 we construct an explicit, non-compact solution of $(*)$, for which we require an upper null mean curvature bound. The objective of this section is therefore to derive the interior $H+P$ estimate (2.5) for smooth solutions of $(*)$ (which also holds for non-compact solutions). We begin by stating the evolution equations for some fundamental quantities. Let ∇ be the connection on (M,g) and let the induced connection and second fundamental form on Σ_t be denoted by D and $A=\{h_{ij}\}$ respectively.

Lemma 2.1 *Smooth solutions of $(*)$ with $H+P>0$ satisfy the following evolution equations.*

i) $\frac{d}{dt}H = \frac{1}{(H+P)^2}\Delta(H+P) - 2\frac{|D(H+P)|^2}{(H+P)^3} - \frac{1}{H+P}(|A|^2 + \bar{Ric}(\nu,\nu))$.

ii) $\frac{d}{dt}P = \frac{1}{H+P}\left(\nabla_\nu tr_M K - (\nabla_\nu K)(\nu,\nu)\right) - \frac{2}{(H+P)^2}D_i(H+P)K_{i\nu}$.

iii) $\frac{d}{dt}(d\mu) = \frac{H}{H+P}(d\mu)$.

iv) $\frac{d}{dt}g_{ij} = \frac{2}{H+P}h_{ij}$.

v) $\frac{d}{dt}\nu = -\nabla(H+P)$.

In the case where Σ_0 is closed, we also obtain

vi) $\frac{d}{dt}|\Sigma_t| + \int_{V(\Sigma_t)\setminus V(\Sigma_0)} P\,dV = |\Sigma_t|$, *where $V(\Sigma)$ denotes the volume enclosed by Σ.*

Proof. These are well known computations, except for the evolution of P which satisfies

$$\begin{aligned}\frac{d}{dt}P &= \frac{d}{dt}\mathrm{tr}_M K - \nu^i\nu^j\frac{d}{dt}K_{ij} - 2\nu^j K_{ij}\frac{d}{dt}\nu^i\\ &= \frac{1}{H+P}\nabla_\nu \mathrm{tr}_M K - \nu^i\nu^j\frac{\nabla_\nu K_{ij}}{H+P} - 2\nu^j K_{ij}D_i\left(\tfrac{1}{H+P}\right)\\ &= \frac{1}{H+P}\left(\nabla_\nu \mathrm{tr}_M K - (\nabla_\nu K)(\nu,\nu)\right) - \frac{2}{(H+P)^2}D_i(H+P)K_{i\nu}.\end{aligned}$$

□

Combining *i)* and *ii)* above, we obtain

$$\begin{aligned}\frac{d}{dt}(H+P) =&\frac{\Delta(H+P)}{(H+P)^2} - \frac{2|D(H+P)|^2}{(H+P)^3} - \frac{|A|^2+\bar{Ric}(\nu,\nu)}{H+P} + \frac{\nabla_\nu \mathrm{tr}_M K - (\nabla_\nu K)(\nu,\nu)}{H+P}\\ &- \frac{2D_i(H+P)K_{i\nu}}{(H+P)^2},\end{aligned} \tag{2.3}$$

and for the speed function $\psi := \dfrac{1}{H+P}$,

$$\frac{\partial\psi}{\partial t} = \psi^2(\Delta\psi + |A|^2\psi + Ric(\nu,\nu)\psi + (\nabla_\nu \mathrm{tr}_M K - (\nabla_\nu K)(\nu,\nu))\psi + 2D_i\psi K_{i\nu}). \tag{2.4}$$

Lemma 2.2 (Interior null mean curvature estimate.) *Let Σ_t be a smooth solution of $(*)$ on M for $0 \le s \le t$. Then for each $x \in \Sigma_t$ and $R < \sigma(x)$*

$$H(x,t) + P(x,t) \le \max\left((H+P)_R, \frac{\lambda}{R\left(\sqrt{\alpha^2 + 2n\lambda} - \alpha\right)}\right), \tag{2.5}$$

where $\lambda := 4\left(3n + (12+3n)\|K\|_{C^0}R + n\|K\|_{C^1}R^2\right)$, $\alpha := 12 + 4n\|K\|_{C^0}R$ and $(H+P)_R$ is the maximum of $H+P$ on $\boldsymbol{B}_R$, the parabolic boundary of $\Sigma_t \cap B_R(x)$.

Like in [HI], the supremum $\sigma(x)$ of radii r for which (2.5) holds is defined as follows.

Definition 2.3 Let d_x denote the distance to x. Then for any $x \in M$, we define $\sigma(x) \in (0,\infty]$ to be the supremum of radii R such that $B_R(x) \subset\subset M$, $Ric \ge -\frac{1}{100(n+1)R^2}$ in $B_R(x)$, and there exists $p \in C^2(B_R(x))$ such that

$$p(x) = 0,\ \ p \ge d_x^2 \quad \text{on } \partial B_R(x), \quad \text{yet} \quad |\nabla p| \le 3d_x \ \text{ and } \ \nabla^2 p \le 3g \quad \text{on } B_R(x).$$

Proof. We wish to construct a subsolution to (2.4). Since

$$|A|^2 \ge \frac{H^2}{n} = \frac{1}{n}\left((H+P)^2 - 2P(H+P) + P^2\right) \ge \frac{1}{n}\left((H+P)^2 - 2P(H+P)\right),$$

and $D\psi \le |\nabla\psi|$, $P \le n\|K\|_{C^0}$ and $\nabla_\nu P \le n\|K\|_{C^1}$, from (2.4) we obtain

$$\frac{\partial\psi}{\partial t} \ge \psi^2\Delta\psi + \frac{\psi}{n} - \frac{\psi^3}{100(n+1)R^2} - 2\|K\|_{C^0}\psi^2 - n\|K\|_{C^1}\psi^3 - 2|\nabla\psi|\,\|K\|_{C^0}\psi^2. \tag{2.6}$$

We allow Σ_t to have a smooth boundary $\partial\Sigma_t$, and define the parabolic boundary of the flow $\Sigma_t \cap B_R$ to be

$$\mathbf{B}_R = \mathbf{B}_R(x,t) := (B_R \cap \Sigma_0) \times \{0\} \cup (\cup_{0\le s\le t}(B_R \cap \partial\Sigma_s) \times \{s\}),$$

and

$$(H+P)_R = (H+P)_R(x,t) := \sup_{(y,s)\in P_R} H(y,s) + P(y,s).$$

Now consider the function

$$\phi = \phi_\delta(y) := \frac{C_\delta}{R}\big(R^2 - p(y)\big),$$

where $C_\delta := \big(\max\big(R(H+P)_R, \frac{\lambda}{(\sqrt{\alpha^2+2n\lambda}-\alpha)}\big)\big)^{-1} - \delta$, for $0 < \delta \ll 1$ and p as defined above. Note that

$$\Delta\phi = \mathrm{tr}_{\Sigma_t}(\nabla^2\phi) - H\langle\nabla\phi, \nu\rangle.$$

Then for $y \in \Sigma_t \cap B_R$, we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \phi^2\Delta\right)\phi &= \langle\nabla\phi, \frac{\partial y}{\partial t}\rangle - \phi^2\mathrm{tr}_{\Sigma_t}\nabla^2\phi + H\phi^2\langle\nabla\phi,\nu\rangle \\ &= -\frac{C_\delta}{R}\langle\nabla p,\nu\rangle\left(\psi + \frac{\phi^2}{\psi} - P\phi^2\right) + \phi^2\frac{C_\delta}{R}\mathrm{tr}_{\Sigma_t\cap B_R}(\nabla^2 p). \qquad (2.7)\end{aligned}$$

Since $\phi \le C_\delta R \le \dfrac{1}{(H+P)_R} - \delta R < \psi$, it follows that $\phi < \psi$ on $\mathbf{B}_R$. In order to obtain a contradiction, let $0 < s \le t$ denote the first time when $(\psi - \phi)(y, s) = 0$ for $y \in \Sigma_s \cap B_R(x)$. At this point

$$\left(\frac{\partial}{\partial t} - \phi^2\Delta\right)(\psi - \phi) \le 0.$$

On the other hand, since $\phi < R$, it follows from (2.6), (2.7) and the condtions on p defined above that at the point (y, s)

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \phi^2\Delta\right)(\psi-\phi) > \\ &> \phi\left(\frac{1}{2n} - 2\|K\|_{C^0}\phi - n\|K\|_{C^1}\phi^2 - 2|\nabla\phi|\,\|K\|_{C^0}\phi + \frac{C_\delta}{R}\langle\nabla p,\nu\rangle(2 - P\phi) - \phi\frac{C_\delta}{R}\mathrm{tr}_{\Sigma_t\cap B_R}(\nabla^2 p)\right) \\ &\ge \phi\left(\frac{1}{2n} - 2\|K\|_{C^0}\phi - n\|K\|_{C^1}\phi^2 - 2|\nabla\phi|\,\|K\|_{C^0}\phi - 3C_\delta\left(2 + n\|K\|_{C^0}\phi\right) - 3nC^2\right) \\ &\ge \phi\left(\frac{1}{2n} - 2C_\delta\Big(3 + n\|K\|_{C^0}R\Big) - C_\delta^2\big(3n + 12\|K\|_{C^0}R + n\|K\|_{C^1}R^2 + 3n\|K\|_{C^0}R\big)\right) \\ &= 0.\end{aligned}$$

Thus $\psi > \phi$ on all of $\Sigma_t \cap B_R(x)$. In particular $\psi(x,t) > \phi(x,t) = C_\delta R$, and as δ was arbitrary it follows that $\psi(x,t) \ge C_0 R$. This completes the proof of the lemma. □

In Chapter 2.3 we see that the null mean curvature upper bound given by Lemma 2.2 is the key to existence and regularity, and that this estimate remains true in the weak setting. On the other hand, the reaction term $-\frac{|A|^2}{H+P}$ in the evolution (2.3) of the null mean curvature in general leads to singularity formulation in finite time, analogous to inverse mean curvature flow. We therefore turn to the question of a weak formulation of solutions to the evolution by inverse null mean curvature.

3 Level-set description and elliptic regularisation

In this section we outline a level-set description of the evolution by inverse null mean curvature. We use the method of elliptic regularisation as a tool to approximate solutions of the level set problem by smooth solutions of a strictly elliptic equation. Studying the properties of the regularised solutions helps to guide us towards the optimal formulation for weak solutions of $(**)$, which we then define in Chapter 6.

Level-Set Formulation. The following ansatz lies at the foundation of the level-set formulation. We assume that the evolving surfaces are given by the level-sets of a scalar function $u : M \to \mathbb{R}$ via

$$E_t := \{x : u(x) < t\}, \qquad \Sigma_t := \partial E_t. \tag{3.1}$$

We call u the *time-of-arrival function*[1] for the evolution by null mean curvature. Then wherever u is smooth and $\nabla u \neq 0$, the normal vector to Σ_t is given by $\nu = \dfrac{\nabla u}{|\nabla u|}$ and the boundary value problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) + \left(g^{ij} - \dfrac{\nabla^i u \nabla^j u}{|\nabla u|^2}\right) K_{ij} = |\nabla u|, \\ \qquad\qquad\qquad\qquad\qquad u\Big|_{\partial E_0} = 0, \end{cases} \tag{$**$}$$

describes the evolution of the level-sets of u by inverse null mean curvature. In this smooth setting, the left hand side is the null mean curvature of Σ_t and the right hand side is the inverse speed of the family of level-sets. Since $|\nabla u| = H + P$, the local uniform estimate (2.5) for the null mean curvature suggests that it is reasonable to expect locally Lipschitz solutions of $(**)$. In order to interpret $(**)$ as the level-set formulation of the classical flow $(*)$, where u is the time-of-arrival function, it is necessary for the zero function be a subsolution barrier for the Dirichlet problem $(**)$. In particular, this suggests that it only makes sense to study $(**)$ on initial data sets (M, g, K) satisfying $\operatorname{tr}_M K \geq 0$. We see below that this mean curvature restriction also comes up when studying the elliptic regularisation problem.

[1] This term was coined by Brian White in [W].

Elliptic regularisation. In order to solve the degenerate elliptic problem $(**)$, we study solutions of the following strictly elliptic equation on the precompact domain $\Omega_L := F_L \backslash \bar{E}_0$,

$$(*)_\varepsilon \begin{cases} E^\varepsilon u_\varepsilon := \operatorname{div}\left(\dfrac{\nabla u_\varepsilon}{\sqrt{|\nabla u_\varepsilon|^2+\varepsilon^2}}\right) + \left(g^{ij} - \dfrac{\nabla^i u_\varepsilon \nabla^j u_\varepsilon}{|\nabla u_\varepsilon|^2+\varepsilon^2}\right) K_{ij} - \sqrt{|\nabla u_\varepsilon|^2+\varepsilon^2} = 0 & \text{in } \Omega_L, \\ u_\varepsilon = 0 & \text{on } \partial E_0, \\ u_\varepsilon = L-2 & \text{on } \partial F_L, \end{cases}$$

where $F_L := \{v < L\}$ for an appropriate comparison function v defined below. In this chapter we prove existence of a smooth solution of $(*)_\varepsilon$.

The remarkable utility of epsilon regularisation is revealed by rescaling $(*)_\varepsilon$ via $u_\varepsilon := \varepsilon \hat{u}_\varepsilon$, to obtain

$$(*)_{\hat{\varepsilon}} \qquad \operatorname{div}\left(\frac{\nabla \hat{u}_\varepsilon}{\sqrt{|\nabla \hat{u}_\varepsilon|^2+1}}\right) + \left(g^{ij} - \frac{\nabla^i \hat{u}_\varepsilon \nabla^j \hat{u}_\varepsilon}{|\hat{u}_\varepsilon|^2+1}\right) K_{ij} = \varepsilon\sqrt{|\nabla \hat{u}_\varepsilon|^2+1}.$$

Here we interpret the left hand side as the null mean curvature $\hat{H}_\varepsilon + \hat{P}_\varepsilon$ of the hypersurface $\operatorname{graph}(\hat{u}_\varepsilon)$ in the product manifold

$$(M^{n+1} \times \mathbb{R}, \bar{g}), \qquad \bar{g} := g \oplus dz^2, \tag{3.2}$$

where we extend the given data K to be constant in the z-direction. Then on the right hand side of $(*)_{\hat{\varepsilon}}$ we have $\sqrt{|\nabla \hat{u}_\varepsilon|^2+1} = -\langle \tau_{n+2}, \hat{\nu}_\varepsilon \rangle^{-1}$, where $\hat{\nu}_\varepsilon := \dfrac{(\nabla \hat{u}_\varepsilon, -1)}{\sqrt{1+|\nabla \hat{u}_\varepsilon|^2}}$ is the lower unit normal to $\operatorname{graph}(\hat{u}_\varepsilon)$. Thus $(*)_\varepsilon$ has the geometric interpretation that the downward translating graph

$$\tilde{\Sigma}_t^\varepsilon := \operatorname{graph}\left(\hat{u}_\varepsilon - \frac{t}{\varepsilon}\right), \tag{3.3}$$

solves $(*)$ smoothly in $\Omega_L \times \mathbb{R}$. This is equivalent to the statement that the function

$$U_\varepsilon(x,z) := u_\varepsilon(x) - \varepsilon z, \qquad (x,z) \in \Omega_L \times \mathbb{R},$$

solves $(**)$ in $\Omega_L \times \mathbb{R}$, since U_ε is the time-of-arrival function for the solution $\tilde{\Sigma}_t^\varepsilon$, that is

$$\tilde{\Sigma}_t^\varepsilon = \{U_\varepsilon = t\}. \tag{3.4}$$

Therefore elliptic regularisation allows one to approximate solutions of $(**)$ by smooth, noncompact solutions of $(*)$ one dimension higher. This observation proves to be useful in the forthcoming work, since results for the smooth evolution $(*)$ can be applied to the solution $\tilde{\Sigma}_t^\varepsilon$.

In fact, $(*)_{\hat{\varepsilon}}$ has the further interpretation as Jang's equation

$$\left(g^{ij} - \frac{\nabla^i u \nabla^j u}{|\nabla u|^2 + 1}\right)\left(\frac{\nabla_i \nabla_j u}{\sqrt{|\nabla u|^2 + 1}} + K_{ij}\right) = 0, \tag{3.5}$$

with a gradient regularisation term.[2] In [J], Jang used equation (3.5) to generalise Geroch's [G] approach to proving the positive mass theorem from the time symmetric case to the general case. He noted however that the equation cannot be solved in general, leaving the question of existence and regularity of solutions open. The analytical difficulty is the lack of an a priori estimate for $\sup|u|$ due to the presence of the zero order term $tr_M(K)$. In [SY], Schoen and Yau bypass this issue using a positive capillarity regularisation term

$$\begin{cases} \left(g^{ij} - \frac{\nabla^i u_\tau \nabla^j u_\tau}{|\nabla u_\tau|^2 + 1}\right)\left(\frac{\nabla_i \nabla_j u_\tau}{\sqrt{|\nabla u_\tau|^2 + 1}} + K_{ij}\right) = \tau u_\tau & \text{on } M, \\ u_\tau \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{3.6}$$

which provides a direct sup estimate via the maximum principle. They prove existence and regularity estimates for solutions u_τ to (3.6), whose graphs Σ_τ have a smooth, embedded sub-sequential limit Σ as $\tau \to 0$. The components of the limit Σ consist of graphical solutions of Jang's equation, as well as cylindrical components, whose cross-sections are apparent horizons in the initial data set.

In the case of the Dirichlet problem for Jang's equation, appropriate trapping assumptions must be placed on the boundaries in order to obtain the required boundary gradient estimates (see [AM], [E], and [AEM] for an overview).

In the case of the Dirichlet problem $(*)_\varepsilon$, we see below that the zero order term $tr_M(K)$ obstructs the existence of a subsolution barrier at the inner boundary. In order to obtain the required boundary gradient estimate at this inner boundary, we must impose the restriction that the ambient mean curvature $\mathrm{tr}_M(K) = g^{ij}K_{ij}$ is nonnegative on M. Similarly, restricting to $\mathrm{tr}_M K \geq 0$ in (3.6) makes the zero function a subsolution barrier for (3.6), which has the effect of preventing the solution from blowing-up to negative infinity over marginally inner trapped surfaces in the initial data set. This connection between the blow-up of solutions to Jang's equation and the mean curvature of the initial data set was observed by J. Metzger [M] in the context of solving the capilarity-regularised Jang's equation with zero Dirichlet boundary condition on initial data sets with non-empty boundary.

3.1 A priori estimates

As stated above, we will use a comparison function v to prescribe the outer boundary ∂F_L of the annulus domain Ω_L for the Dirichlet problem $(*)_\varepsilon$. Since M is asymptotically flat, outside some compact set $\Omega \subset M$ we can choose a radial coordinate chart such that for

[2] For the sake of convenience we call 3.5 Jang's equation, despite the wrong sign on the tensor K.

an appropriately chosen $\alpha > 0$, the function $v = \alpha \log r$ is a smooth subsolution of the following approximating level-set equation

$$Div\left(\frac{\nabla u}{|\nabla u|}\right) + s\left(g^{ij} - \frac{\nabla^i u \nabla^j u}{|\nabla u|^2}\right) K_{ij} = |\nabla u|, \tag{3.7}$$

for $s \in [0,1]$ in this asymptotic region $M \backslash \Omega$. To prove existence of solutions to the Dirichlet problem $(*)_\varepsilon$, we then consider solutions of the family of approximating equations

$$(*)_{\varepsilon,s} \begin{cases} \operatorname{div}\left(\frac{\nabla u_{\varepsilon,s}}{\sqrt{|\nabla u_{\varepsilon,s}|^2+\varepsilon^2}}\right) + s\left(g^{ij} - \frac{\nabla^i u_{\varepsilon,s}\nabla^j u_{\varepsilon,s}}{|\nabla u_{\varepsilon,s}|^2+\varepsilon^2}\right)K_{ij} - \sqrt{|\nabla u_{\varepsilon,s}|^2+\varepsilon^2} = 0 & \text{in } \Omega_L, \\ u_{\varepsilon,s} = 0 & \text{on } \partial E_0, \\ u_{\varepsilon,s} = s(L-2) & \text{on } \partial F_L, \end{cases}$$

for $s \in [0,1]$, where the subsolution $v = \alpha \log r$ to (3.7) prescribes the outer boundary $\partial F_L = \partial\{v < L\}$ for the Dirichlet problems $(*)_{\varepsilon,s}$ and $(*)_\varepsilon$.

Lemma 3.1 *For every $L > 0$, there exists $\varepsilon(L) > 0$ such that for $0 < \varepsilon < \varepsilon(L)$ and $s \in [0,1]$, a smooth solution of $(*)_{\varepsilon,s}$ on $\bar{\Omega}_L$ satisfies the following a priori estimates:*

$$u_{\varepsilon,s} \geq -\varepsilon \quad \text{in } \bar{\Omega}_L, \qquad u_{\varepsilon,s} \geq v + (s-1)(L-2) - 2 \quad \text{in } \bar{F}_L \backslash F_0, \tag{3.8}$$

$$u_\varepsilon \leq C(L, \|K\|_{C_0}) \qquad \text{in } \bar{\Omega}_L, \tag{3.9}$$

$$|\nabla u_{\varepsilon,s}| \leq H_+\varepsilon + n|p| \quad \text{on } \partial E_0, \qquad |\nabla u_{\varepsilon,s}| \leq C(L, \|K\|_{C^0}) \quad \text{on } \partial F_L, \tag{3.10}$$

$$|\nabla u_{\varepsilon,s}(x)| \leq \max_{\partial\Omega_L \cap B_r(x)} |\nabla u_{\varepsilon,x}| + \varepsilon + C, \quad x \in \bar{\Omega}_L, \tag{3.11}$$

$$|u_{\varepsilon,s}|_{C^{2,\alpha}(\bar{\Omega}_L)} \leq C(\varepsilon, L, n, \|K\|_{C^0}, \|K\|_{C^1}). \tag{3.12}$$

Proof. Let

$$E^{\varepsilon,s}u_{\varepsilon,s} := \operatorname{div}\left(\frac{\nabla u_{\varepsilon,s}}{\sqrt{|\nabla u_{\varepsilon,s}|^2+\varepsilon^2}}\right) + s\left(g^{ij} - \frac{\nabla^i u_{\varepsilon,s}\nabla^j u_{\varepsilon,s}}{|\nabla u_{\varepsilon,s}|^2+\varepsilon^2}\right)K_{ij} - \sqrt{|\nabla u_{\varepsilon,s}|^2+\varepsilon^2} = 0,$$

and let $|\lambda|$ denote the size of the largest eigenvalue of K_{ij} on $\bar{\Omega}_L$. Aside from the supersolution barrier at the outer boundary, the following a-priori estimates follow essentially as in [HI, Lemma 3.4].

Boundary Gradient Estimates

1. Subsolution barrier at the inner boundary.

We wish to construct a subsolution that bridges from E_0 to where v starts in the asymptotic region. We seek a a subsolution that allows for unrestricted jumps in the compact part of the manifold, and therefore consider perturbations of zero. Furthermore, as mentioned in Section 3, we want zero to be a subsolution for the level-set flow $(**)$ in the limit $\varepsilon \to 0$, which suggests the need to restrict to initial data sets (M, g, K) satisfying $\mathrm{tr}_M K \geq 0$.

Let $\mathrm{Cut}(E_0)$ be the cut locus of E_0 in M. We construct a subsolution to $(*)_\varepsilon$ on $M \backslash \big(E_0 \cup \mathrm{Cut}(E_0)\big)$. Define $G_0 = E_0$, $G_d := \{x : dist(x, E_0) < d\}$ and choose d_L large enough that $G_{d_L} \supseteq F_L$. In general, for a surface moving in normal direction with speed f, the evolution of the mean curvature is given by

$$\frac{\partial H}{\partial t} = -\Delta f - |A|^2 f - Ric(\nu, \nu) f. \tag{3.13}$$

We can therefore estimate the mean curvature of the surfaces ∂G_d via

$$\frac{\partial H}{\partial d} = -|A|^2 - Ric(\nu, \nu) \leq C_1(L) \qquad \text{on } \partial G_d \backslash \mathrm{Cut}(E_0),\ 0 \leq d \leq d_L,$$

yielding

$$H_{\partial G_d} \leq \max_{\partial E_0} H_+ + C_1 d \leq C_2(L) \qquad \text{on } \partial G_d \backslash \mathrm{Cut}(E_0),\, 0 \leq d \leq d_L,$$

where $H_+ = \max(0, H)$. Consider the prospective subsolution

$$v_1(x) := f(d) = f(\mathrm{dist}(x, G_0)), \quad x \in \bar{G}_{d_L} \backslash E_0,\ f' < 0.$$

Since $\nabla v_1 = f'\nu$, we have $\nabla^2_{ij} v_1 = f'\langle \nabla_{e_i}\nu, e_j\rangle = f' h_{ij}$ and thus

$$(g^{ij} - \nu^i\nu^j)\nabla^2_{ij} v_1 = f' H_{\partial G_d} \geq f' C_2.$$

Hence

$$\begin{aligned}
\sqrt{f'^2+\varepsilon^2} E^{\varepsilon,s} v_1 &= \left(g^{ij} - \frac{f'^2\nu^i\nu^j}{f'^2+\varepsilon^2}\right)\left(\nabla^2_{ij} v_1 + s\sqrt{f'^2+\varepsilon^2} K_{ij}\right) - f'^2 - \varepsilon^2 \\
&= \left(g^{ij} - \nu^i\nu^j + \frac{\varepsilon^2\nu^i\nu^j}{f'^2+\varepsilon^2}\right)\nabla^2_{ij} v_1 + s\sqrt{f'^2+\varepsilon^2} g^{ij} K_{ij} - s\frac{f'^2}{\sqrt{f'^2+\varepsilon^2}} K_{\nu\nu} - f'^2 - \varepsilon^2 \\
&\geq -|f'| C_2 + \frac{\varepsilon^2 f''}{f'^2+\varepsilon^2} + s\sqrt{f'^2+\varepsilon^2} g^{ij} K_{ij} - |f'||K_{\nu\nu}| - f'^2 - \varepsilon^2.
\end{aligned}$$

If we restrict to initial data sets (M, g, K) with $g^{ij} K_{ij} \geq 0$, we can discard the bad term $s\sqrt{f'^2+\varepsilon^2} g^{ij} K_{ij}$ and use the following barrier

$$f(d) := \frac{\varepsilon}{A}(-1 + e^{-Ad}) \quad \text{on } 0 \leq d \leq d_L.$$

Then $|f'| \leq \varepsilon$ on $0 \leq d \leq d_L$, and if we restrict ε such that $\varepsilon \leq e^{-Ad_L}$, then $|f'| = \varepsilon e^{-Ad} \geq \varepsilon^2$ and $\varepsilon^2 \leq |f'| \leq \varepsilon$.
Thus we have

$$\begin{aligned}
(f'^2 + \varepsilon^2)(|f'| C_2 + |f'||K_{\nu\nu}| + f'^2 + \varepsilon^2) &\leq 2\varepsilon^2 (C_2 + |K_{\nu\nu}| + 2)|f'| \\
&\leq \varepsilon^2 f''
\end{aligned}$$

if we take $A := 2(C_2 + |\lambda| + 2)$.
This shows that, for sufficiently small ε, the function $v_{1,s}(x) := \frac{\varepsilon}{2(C_2 + |\lambda| + 2)}\left(e^{-(2C_2+|\lambda|+2)d} - 1\right)$ is a smooth subsolution for $E^{\varepsilon,s}$ on $G_{d_L}\backslash\left(E_0 \cup \text{Cut}(E_0)\right)$. Furthermore, $v_{1,s}$ is a viscosity subsolution of $E^{\varepsilon,s}$ on all of $G_{d_L}\backslash\bar{E}_0$. Since $u \geq v_1$ on the boundary, it follows by the maximum principle for viscosity solutions [CIL, Thm 3.3] that

$$u \geq v_1 \geq -\varepsilon \quad \text{in } \bar{\Omega}_L, \text{ and } \frac{\partial u}{\partial \nu} \geq \frac{\partial v_1}{\partial \nu} \geq -\varepsilon \quad \text{on } \partial E_0. \tag{3.14}$$

1b) Subsolution barrier at outer boundary.
We construct a subsolution on $\bar{F}_L\backslash F_0$. Assume $L > 1$ and consider the function

$$v_2 := \frac{L-1}{L}v - 1 + (s-1)(L-2).$$

Then

$$E^{0,s}v_2 = Div\left(\frac{\nabla v}{|\nabla v|}\right) + s\left(g^{ij} - \frac{\nabla^i v \nabla^j v}{|\nabla v|^2}\right)K_{ij} - \frac{L-1}{L}|\nabla v| > 0$$

on $\bar{F}_L\backslash F_0$. Since the domain is compact, for all sufficiently small ε we obtain $E^{\varepsilon,s}v_2 > 0$. From (3.14) we have that $u \geq -\varepsilon$ in $\bar{\Omega}_L$, thus $u \geq v_2$ on ∂F_0 and $u = s(L-2) = v_2$ on ∂F_L. It then follows from the maximum principle that $u \geq v_2 \geq v + (s-1)(L-2) - 2$ in $\bar{F}_L\backslash F_0$, thus

$$\frac{\partial u}{\partial \nu} \geq -C(L) \qquad \text{on } \partial F_L. \tag{3.15}$$

A rescaled version of v_2 provides the required barrier when $L \leq 1$.

2a) Supersolution barrier at outer boundary.
The zero order term $tr_M(K)$ prevents constant functions from being supersolutions to $(*)_{\varepsilon,s}$, like in [HI]. We therefore construct a linear supersolution to $(*)_\varepsilon$ on $F_L\backslash\left(E_0 \cup \text{Cut}(\partial F_L)\right)$, where $\text{Cut}(\partial F_L)$ is the cut locus of ∂F_L in $\bar{F}_L$. Consider $v_3(x) := f(d) = f(\text{dist}(x, G_0))$ where $G_0 := \partial F_L$, $G_d := \{x : dist(x, \partial F_L) < d\}$ and choose d_0 large enough that $G_{d_0} \supseteq \partial E_0$. From (3.13) we find

$$\frac{\partial H}{\partial d} = |A|^2 + Ric(\nu,\nu) \geq -C_1(L) \qquad \text{on } \partial G_d\backslash\text{Cut}(\partial F_L),\ 0 \leq d \leq d_0,$$

yielding

$$H_{\partial G_d} \geq -\max_{\partial F_L} H_- - C_1 d \geq -C_2(L) \qquad \text{on } \partial G_d\backslash\text{Cut}(\partial F_L),\ 0 \leq d \leq d_0,$$

where $H_- = -\min(H, 0)$. Setting $f(d) := s(L-2) + \left(m + \frac{2}{d_0}\right)d$, where $m > 0$ is to be chosen, we obtain

$$\begin{aligned}\sqrt{f'^2 + \varepsilon}E^{\varepsilon,s}f(d) &= -f'H_{\partial G_d} + \frac{\varepsilon^2 f''}{f'^2 + \varepsilon^2} + s\left(g^{ij} - \frac{f'^2\nu^i\nu^j}{f'^2 + \varepsilon^2}\right)\sqrt{f'^2 + \varepsilon^2}K_{ij} - (f'^2 + \varepsilon^2)\\ &\leq f'C_2 + (f' + \varepsilon)|g^{ij}K_{ij}| + f'|K_{\nu\nu}| - f'^2\\ &\leq f'(C_2 + 2|g^{ij}K_{ij}| + |K_{\nu\nu}| - f').\end{aligned}$$

Setting $m := C_2+2|g^{ij}K_{ij}|+|\lambda|$ ensures $\sqrt{f'^2+\varepsilon}E^{\varepsilon,s}f(d) \le 0$ for all sufficiently small ε (so that $\varepsilon \le f'$). Then $v_3(x)$ is a smooth supersolution on $G_{d_0}\setminus\big(E_0\cup\mathrm{Cut}(\partial F_L)\big)$. Furthermore, v_3 is a viscosity subsolution on all of $G_{d_0}\setminus\bar{E}_0$. Since $u = f$ on ∂F_L and $u < f$ on ∂E_0, it follows by the maximum principle for viscosity solutions that

$$u \le f \le L + (C_2 + 2|g^{ij}K_{ij}| + |p|)\, d_0 \qquad \text{in } \bar{\Omega}_L, \tag{3.16}$$

$$\frac{\partial u}{\partial \nu} \le C_2 + 2|g^{ij}K_{ij}| + |p| + \frac{2}{d_0} = C(L, \|K\|_{C^0}) \qquad \text{on } \partial F_L. \tag{3.17}$$

2b)Supersolution barrier along ∂E_0.
Choose a smooth function v_3 which vanishes on ∂E_0 such that

$$H_+ + n\|K\|_{C^0} < \frac{\partial v_4}{\partial \nu} \le H_+ + \varepsilon + n\|K\|_{C^0} \qquad \text{along } \partial E_0. \tag{3.18}$$

Let ν be the normal vector to ∂E_0, and τ be the tangent to ∂E_0 satisfying

$$\nu = \lambda\frac{\nabla v_4}{|\nabla v_4|} + \sqrt{\lambda^2-1}\tau, \quad \text{for some } \lambda \ge 1.$$

Then

$$\mathrm{div}\left(\frac{\nabla v_4}{|\nabla v_4|}\right) = \frac{1}{\lambda}\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) - \frac{\sqrt{\lambda^2-1}}{\lambda}\mathrm{div}(\tau) = \frac{1}{\lambda}H_{\partial E_0},$$

thus along ∂E_0 we obtain

$$E^{0,s}v_4 = \mathrm{div}\left(\frac{\nabla v_4}{|\nabla v_4|}\right) + s\left(g^{ij} - \frac{\nabla^i v_4 \nabla^j v_4}{|\nabla v_4|^2}\right)K_{ij} - |\nabla v_4| < \frac{1}{\lambda}H_{\partial E_0} + n\|K\|_{C^0} - \big(H_+ + n\|K\|_{C^0}\big) \le 0.$$

This implies that $E^0 v_4 < 0$ in the neighbourhood $U := \{0 \le v_4 \le \delta\}$, for sufficiently small $\delta > 0$.
Now define the scaled-up function

$$v_5 := \frac{v_4}{1 - v_4/\delta}, \qquad x \in U.$$

So $v_5 \to \infty$ for $v_4 \to \delta$, ie on $\partial U\setminus\partial E_0$, and

$$E^{0,s}v_5 = \mathrm{div}\left(\frac{\nabla v_4}{|\nabla v_4|}\right) + s\left(g^{ij} - \frac{\nabla^i v_4 \nabla^j v_4}{|\nabla v_4|^2}\right)K_{ij} - \frac{|\nabla v_4|}{(1 - v_4/\delta)^2} \le E^{0,s}v_4 < 0.$$

Since $E^{0,s}v_5 < 0$ for $v_5 \ge 0$, for ε sufficiently small (depending on L and m) we obtain that $E^{\varepsilon,s}v_5 < 0$ on the set $V := \{0 \le v_5 \le L + m\, d_0\}$. From (3.16) we have $u \le L + m\, d_0$ on $\bar{\Omega}_L$, thus $u \le v_5$ on ∂V.
Then by the maximum principle, $u \le v_5$ on V, and therefore

$$\frac{\partial u}{\partial \nu} \le \frac{\partial v_5}{\partial \nu} = \frac{\partial v_4}{\partial \nu} \le H_+ + \varepsilon + n|p| \qquad \text{on } \partial E_0 \tag{3.19}$$

for sufficiently small ε.

Interior Gradient Estimate
The desired gradient estimate can be obtained from the interior estimate for $H+P$ in Lemma 2.2. Since we can not apply the result directly to $(*)_{\varepsilon,s}$ (except when $s=1$), we instead rework the proof of Lemma 2.2 for the evolution equation

$$\frac{\partial F}{\partial t} = \frac{1}{H+sP}\nu, \qquad s\in[0,1], \tag*{$(*)_s$}$$

to obtain the corresponding estimate

$$H(x,t)+sP(x,t) \le \max\left((H+sP)_R, \frac{\lambda}{\left(\sqrt{\alpha^2+2n\lambda}-\alpha\right)}\right). \tag{3.20}$$

Here λ and α are as defined above, and $(H+sP)_R$ is the maximum of $H+sP$ on P_R, the parabolic boundary of $\Sigma_t^s \cap B_R(x)$.

The downward translating graph

$$\tilde{\Sigma}_t^{\varepsilon,s} := \operatorname{graph}\left(\frac{u_{\varepsilon,s}}{\varepsilon} - \frac{t}{\varepsilon}\right),$$

is a smooth solution of $(*)_s$, described by the level-set function

$$U_{\varepsilon,s}(x,z) := u_{\varepsilon,s}(x) - \varepsilon z,$$

since $\tilde{\Sigma}_t^{\varepsilon,s} = \{U_{\varepsilon,s}=t\}$. We then relate estimate (3.20) to $|\nabla u_{\varepsilon,s}|$ via $(*)_{\varepsilon,s}$, which asserts that

$$(H+sP)_{\tilde{\Sigma}_t^{\varepsilon,s}} = \sqrt{|\nabla u_{\varepsilon,s}|^2+\varepsilon^2}.$$

Now let $\mathbf{B} := B_R^{n+1}(x,z)$, be an (n+1)-dimensional ball in $M\times\mathbb{R}$. Since $\tilde{\Sigma}_t^{\varepsilon,s}$ is a translating solution to $(*_s)$, its parabolic boundary is just a translation of $\partial\Omega_L$ in time. Furthermore, as $|\nabla u_{\varepsilon,s}|$ is independent of z, applying (3.20) to $\tilde{\Sigma}_t^{\varepsilon,s}\cap\mathbf{B}$ yields

$$\begin{aligned}\sqrt{|\nabla u_{\varepsilon,s}|^2+\varepsilon^2} &\le \sup_t \max_{\partial\tilde{\Sigma}_t^{\varepsilon,s}\cap\mathbf{B}} \sqrt{|\nabla u_{\varepsilon,s}|^2+\varepsilon^2} + C\\ &\le \max_{\partial\Omega_L\cap B_R^n(x)} |\nabla u_{\varepsilon,s}| + \varepsilon + C,\end{aligned}$$

where $C := \frac{\lambda}{\left(\sqrt{\alpha^2+2n\lambda}-\alpha\right)}$ is defined in Lemma 2.2. For ε small enough, we obtain from the boundary gradient estimates

$$|\nabla u_{\varepsilon,s}(x)| \le \max_{\partial E_0\cap B_R(x)} H_+ + 2 + \tilde{C}(L,n,\|K\|_{C^1}) + C, \tag{3.21}$$

which leads to the Lipschitz estimate

$$|u_{\varepsilon,s}|_{C^{0,1}(\bar{\Omega}_L)} \leq C(L, n, \|K\|_{C^1}).$$

Then by reworking the proof of the Nash-Moser-De Giorgi estimate ([GT], Thm 13.2), we obtain

$$|u_{\varepsilon,s}|_{C^{1,\alpha}(\bar{\Omega}_L)} \leq C(\varepsilon, L, n, \|K\|_{C^1}),$$

for some $\alpha = \alpha(\Omega_L)$. This implies a bound on the Hölder modulus of continuity for the coefficients of $E^\varepsilon u$, so Schauder theory improves this estimate to $C^{2,\alpha}$

$$|u_{\varepsilon,s}|_{C^{2,\alpha}(\bar{\Omega}_L)} \leq C(\varepsilon, L, n, \|K\|_{C^1}). \tag{3.22}$$

□

3.2 Existence of solutions

Lemma 3.2 (Existence for the ε-problem) *A smooth solution of $(*)_\varepsilon$ exists.*

Proof. Recall,

$$(*)_{\varepsilon,s} \begin{cases} E^{\varepsilon,s}u := \operatorname{div}\left(\dfrac{\nabla u_\varepsilon}{\sqrt{|\nabla u_\varepsilon|^2+\varepsilon^2}}\right) + s g_\Sigma^{ij} K_{ij} - \sqrt{|\nabla u_\varepsilon|^2+\varepsilon^2} = 0 & \text{in } \Omega_L, \\ u_\varepsilon = 0 & \text{on } \partial E_0, \\ u_\varepsilon = s(L-2) & \text{on } \partial F_L. \end{cases}$$

We first prove there is a solution for $s=0$ and small ε. Let $\hat{u} = \frac{u_\varepsilon}{\varepsilon}$ and rewrite $(*)_{\varepsilon,0}$ as

$$(*)_{\hat{\varepsilon}} \begin{cases} F(\hat{u}) := \dfrac{1}{\sqrt{|\nabla \hat{u}|^2+1}} \operatorname{div}\left(\dfrac{\nabla \hat{u}}{\sqrt{|\nabla \hat{u}|^2+1}}\right) = \varepsilon & \text{in } \Omega_L \\ \hat{u} = 0 & \text{on } \partial\Omega_L. \end{cases}$$

The map

$$F : C_0^{2,\alpha}(\bar{\Omega}_L) \to C^\alpha(\bar{\Omega}_L),$$

is C^1, and possesses the solution $F(0) = 0$ for $\varepsilon = 0$. The linearisation of F at $\hat{u} = 0$ is

$$\mathcal{D}F|_0 = \Delta_g : C_0^{2,\alpha}(\bar{\Omega}_L) \to C^\alpha(\bar{\Omega}_L).$$

The Laplacian on M is an isomorphism, so by the Implicit Function Theorem there exists $\varepsilon_0 > 0$ such that $(*)_{\hat{\varepsilon}}$ has a unique solution for $0 \leq \varepsilon < \varepsilon_0$.

We now fix $\varepsilon \in (0, \varepsilon_0)$ and vary s. Let I be the set of s such that $(*)_{\varepsilon,s}$ has a solution $u_{\varepsilon,s} \in C^{2,\alpha}(\bar{\Omega}_L)$. We have shown that I contains 0. We first show that I is open. Let π be the boundary value map $u \mapsto u|_{\partial\Omega}$. Consider the map

$$G : C^{2,\alpha}(\bar{\Omega}_L) \times \mathbb{R} \to C^\alpha(\bar{\Omega}_L) \times C^{2,\alpha}(\partial\Omega_L),$$

defined by $G(w,s) := G^s(w) = \Big(E^{\varepsilon,s}(w), \pi(w) - s(L-2)\chi_{\partial F_L}\Big)$, so that $(*)_{\varepsilon,s}$ is equivalent to $G^s(w) = (0,0)$. $G^s(w)$ is C^1, and possesses the solution $G^0(u_0) = (0,0)$, where u_0 is the $C^{2,\alpha}_0(\bar{\Omega}_L)$ solution from above. The linearisation of G^0 at u_0 is the operator $\mathcal{D}G^0_{u_0}$ given by

$$\mathcal{D}G^0_{u_0} = \begin{pmatrix} A^{ij}\nabla_i\nabla_j + B^i\nabla_i \\ \pi \end{pmatrix} : C^{2,\alpha} \to C^\alpha(\bar{\Omega}_L) \times C^{2,\alpha}(\partial\Omega_L), \tag{3.23}$$

where

$$A^{ij} = \frac{1}{\sqrt{1+|\nabla u_0|^2}}\left(g^{ij} - \frac{\nabla^i u_0 \nabla^j u_0}{1+|\nabla u_0|^2}\right),$$
$$B^i = \nabla_j A^{ij} - \varepsilon\frac{\nabla^i u_0}{\sqrt{|\nabla u_0|^2+1}}.$$

Since $\mathcal{D}E^0_{u_0}(w)$ is a linear elliptic equation with Hölder continuous coefficients, we can apply Schauder theory (eg [GT], Thm 6.14) to deduce that $\mathcal{D}G^0_{u_0}$ is a bijective map with continuous inverse. It follows from the Implicit function theorem that G maps a neighbourhood of $(u_0, 0)$ onto a neighbourhood of $(0,0)$. Thus I is relatively open, which completes the proof of existence of $u_\varepsilon \in C^{2,\alpha}(\bar{\Omega})$ solving $(*)_\varepsilon$. Smoothness then follows from standard Schauder estimates. □

In view of the local uniform Lipschitz estimates for u_ε, by the Arzela Ascoli theorem there exist sequences $\varepsilon_i \to 0$, $L_i \to \infty$, a subsequence u_i and a locally Lipschitz function $u : M\backslash E_0 \to \mathbb{R}$ such that

$$u_i \to u \tag{3.24}$$

locally uniformly on $M\backslash E_0$, and from (3.21), u satisfies

$$|\nabla u(x)| \le \sup_{\partial E_0 \cap B_R(x)} H_+ + C(L, n, \|K\|_{C^1}). \tag{3.25}$$

Thus by setting

$$U(x,z) := u(x), \tag{3.26}$$

we obtain that $U_i \to U$ locally uniformly on $(M\backslash E_0) \times \mathbb{R}$, where $U_i(x,z) := u_i(x) - \varepsilon_i z$ is the time-of-arrival function for the smooth translating solution $\tilde{\Sigma}^i_t$ of $(*)$. Therefore U is the time of arrival function of the limit of the smooth flow $t \mapsto \tilde{\Sigma}^i_t$. Since U is independent of z, at regular times t the level-sets of U are vertical cylinders.

4 The limit of the translating ε-graphs $\tilde{\Sigma}_t^\varepsilon$

Our choice of variational formulation to define weak solutions to $(**)$, detailed in the next section, is motivated by:

1. The variational properties of smooth solutions of $(**)$,
2. The limiting behaviour of the family $\tilde{\Sigma}_t^\varepsilon$ of translating solutions of $(**)$ in $M \times \mathbb{R}$.

In particular, we show that the sets $E_t = \{u < t\}$, associated to a *smooth* solution u of $(**)$, minimise the following parametric energy functional

$$\mathcal{J}_{u,\nu}^A(F) := |\partial^* F \cap A| - \int_{F \cap A} |\nabla u| - \left(g^{ij} - \nu^i \nu^j\right) K_{ij}, \tag{4.1}$$

for each $t > 0$. That is

$$\mathcal{J}_{u,\nu}^A(E_t) \leq \mathcal{J}_{u,\nu}^A(F), \tag{4.2}$$

for each set F of locally finite perimeter that differs from the set E_t on a compact subset A of the domain. Here $P = \mathrm{tr}_{\Sigma_t} K = (g^{ij} - \nu^i \nu^j) K_{ij}$, where ν represents the unit normal $\nabla u / |\nabla u|$ to the surfaces $\Sigma_t = \partial E_t$. The functional (4.1), together with the minimisation principle (4.2), generalises the variational formulation employed by Huisken and Ilmanen in [HI], and accordingly allows the evolving surfaces to jump instantaneously over a positive volume at plateaus of the time-of-arrival function, u. However, in the weak setting, $\nabla u / |\nabla u|$ is undefined on plateaus of the locally Lipschitz function u, so in order to incorporate the extra P term for this new flow, we must define an appropriate notion of normal vector in these jump regions. In this section we show that such a vector field can be obtained by taking an appropriate limit of the translating graphs $\tilde{\Sigma}_t^\varepsilon$. Since the null mean curvature of these surfaces is uniformly bounded, results of measure theory allow us to control them in $C^{1,\alpha}$, which leads to a foliation of the interior of the jump region $\{U = t_0\} = \{u = t_0 \times \mathbb{R}\}$, at jump times t_0, by hypersurfaces satisfying the following result.

Proposition 4.1 *Let $U(x,z) = u(x)$, where $u \in C^{0,1}_{loc}(M \backslash E_0)$ is the limit of the solution u_ε of $(**)_\varepsilon$, as in (3.26). Then the interior, $\tilde{\mathcal{K}}_{t_0}$, of the jump region $\{u = t_0\} \times \mathbb{R} = \{U = t_0\}$, at jump times t_0 is foliated by hypersurfaces with local uniform $C^{1,\alpha}$ estimates, where each such hypersurface is either a vertical cylinder or a graph over an open subset of $\{u = t_0\}$. Furthermore, each hypersurface bounds a Caccioppoli set that minimises $J_{U,\tilde{\nu}}$ in $\tilde{\mathcal{K}}_{t_0}$, where $\tilde{\nu}$ denotes the $C^{0,\alpha}_{loc}$ normal vector field to the hypersurface foliation.*

The normal vector field $\tilde{\nu}$ to this foliation extends $\dfrac{\bar{\nabla}U}{|\bar{\nabla}U|} = \dfrac{(\nabla u, 0)}{|\nabla u|}$ across the jump region $\tilde{\mathcal{K}}_{t_0}$ in $M \times \mathbb{R}$, and this extended vector field helps motivate the definition of weak solutions to $(**)$ in the next section. In this context, hypersurfaces and sets in $M \times \mathbb{R}$ will be denoted by the $\sim$ superscript for the remainder of this work, unless otherwise stated, and $\bar{\nabla}$ denotes the connection on $(M \times \mathbb{R}, \bar{g})$.

To prove Proposition 4.1 we utilise the following compactness result for sequences of minimisers of (4.1).

Compactness Property 4.2 *Let $\tilde{\Omega} \subset M \times \mathbb{R}$, and let $\tilde{E}_i \subset \tilde{\Omega}$ be a sequence of sets with $C^{1,\alpha}_{loc}$ boundary such that $\partial\tilde{E}_i \to \partial\tilde{E}$, locally in $C^{1,\alpha}$, with outward unit normal $\nu_i \in C^{0,\alpha}_{loc}(T\tilde{\Omega})$ to $\partial\tilde{E}_i$ satisfying $\nu_i \to \nu$ locally uniformly. Let $U_i \in C^{0,1}_{loc}(\tilde{\Omega})$ satisfy $U_i \to U$ locally uniformly, and assume that for each $\tilde{A} \subset\subset \tilde{\Omega}$, $\sup_{\tilde{A}} |\bar{\nabla}U_i| \le C(\tilde{A})$ for large i.
If the sequence $\tilde{E}_i$ minimises $\mathcal{J}_{U_i,\nu_i}$ on $\tilde{\Omega}$, then $\tilde{E}$ minimises $\mathcal{J}_{U,\nu}$ in $\tilde{\Omega}$.*

Proof. We use the inequality

$$\mathcal{J}_{U_i,\nu_i}(\tilde{E}_1) + \mathcal{J}_{U_i,\nu_i}(\tilde{E}_2) \ge \mathcal{J}_{U_i,\nu_i}(\tilde{E}_1 \cup \tilde{E}_2) + \mathcal{J}_{U_i,\nu_i}(\tilde{E}_1 \cap \tilde{E}_2), \tag{4.3}$$

for an appropriate choice of Caccioppoli sets $\tilde{E}_1$ and $\tilde{E}_2$ such that $\tilde{E}_1 \Delta \tilde{E}_2$ is precompact.

We first prove that $\tilde{E}$ minimises $\mathcal{J}_{U,\nu}$ on the outside in $\tilde{\Omega}$. To this end, consider $\tilde{F} \supset \tilde{E}$ with $\tilde{F}\backslash\tilde{E} \subset\subset \tilde{\Omega}$ and a suitable compact set $G \subset \tilde{\Omega}$ containing $\tilde{F}\backslash\tilde{E}$. Since the boundary of G is not necessarily Lipschitz continuous, we consider a compact set $\bar{G} \subset \tilde{\Omega}$ with smooth boundary and $G \subset \mathrm{int}(\bar{G})$ such that

$$|\partial^*(\tilde{F} \cup \tilde{E}_i) \cap \partial\bar{G}| = |\partial^*(\tilde{F} \cap \tilde{E}_i) \cap \partial^*\bar{G}| = |\partial\tilde{E}_i \cap \partial\bar{G}| = 0,$$

for all i, with traces satisfying $\int_{\partial\bar{G}} |\varphi^-_{\tilde{F}\cup\tilde{E}_i} - \varphi^+_{\tilde{E}_i}| d\mathcal{H}^{n+1} \to 0$. This is possible because $\tilde{F} \cup \tilde{E}_i \to \tilde{E}$ and $\tilde{E}_i \to \tilde{E}$ in $L^1_{\mathrm{loc}}(\tilde{\Omega}\backslash G)$. Then setting $\tilde{F}_i := \tilde{E}_i \cup (\tilde{F} \cap \bar{G})$ we see that

$$|\partial^*\tilde{F}_i \cap \tilde{\Omega}| = |\partial^*\tilde{E}_i \cap (\tilde{\Omega}\backslash\bar{G})| + |\partial^*(\tilde{F} \cup \tilde{E}_i) \cap \bar{G}| + \int_{\partial\bar{G}} |\varphi^-_{\tilde{F}\cup\tilde{E}_i} - \varphi^+_{\tilde{E}_i}|.$$

Now, since $\tilde{F}_i$ is an appropriate comparison function for $\tilde{E}_i$, we have
$\mathcal{J}^{\bar{G}}_{U_i,\nu_i}(\tilde{E}_i) \le \mathcal{J}^{\bar{G}}_{U_i,\nu_i}(\tilde{F}_i)$, implying

$$\mathcal{J}^{\bar{G}}_{U_i,\nu_i}(\tilde{E}_i) \le \mathcal{J}^{\bar{G}}_{U_i,\nu_i}(\tilde{F} \cup \tilde{E}_i) + \int_{\partial\bar{G}} |\varphi^-_{\tilde{F}\cup\tilde{E}_i} - \varphi^+_{\tilde{E}_i}|.$$

Now inserting $\tilde{E}_1 = \tilde{E}_i$ and $\tilde{E}_2 = \tilde{F}$ into (4.3) we obtain

$$\mathcal{J}^{\bar{G}}_{U_i,\nu_i}(\tilde{F}) \ge \mathcal{J}^{\bar{G}}_{U_i,\nu_i}(\tilde{E}_i \cap \tilde{F}) - \int_{\partial\bar{G}} |\varphi^-_{\tilde{F}\cup\tilde{E}_i} - \varphi^+_{\tilde{E}_i}|. \tag{4.4}$$

Next we pass to limits. Since the trace term converges to zero, using lower semicontinuity we obtain

$$\mathcal{J}^{\bar{G}}_{U,\nu}(\tilde{F}) \geq \mathcal{J}^{\bar{G}}_{U,\nu}(\tilde{E}).$$

The fact that $\tilde{E}$ minimises $\mathcal{J}_{U,\nu}$ on the inside in $\tilde{\mathcal{G}}_{t_0}$ amongst competing sets $\tilde{F} \subset \tilde{E}$ satisfying $\tilde{E} \backslash \tilde{F} \subset\subset \tilde{\Omega}$ can similarly be proven by again constructing $\bar{G}$ and considering the comparison function $\tilde{F}_i := \tilde{E}_i \cap \tilde{F}$ for $i >> 1$ large enough. □

To prove Proposition 4.1 we will also draw upon regularity theory for obstacle problems of the type (4.6) below. In particular, if the set $E_t := \{u < t\}$ minimises $\mathcal{J}_{u,\nu}$, then it is almost minimal in the sense that

$$|\partial^* E_t \cap B_R| \leq |\partial^* F \cap B_R| + C(n, \|Du\|_\infty, \|K\|_{C^0}) R^{n+1}, \tag{4.5}$$

for $E_t \Delta F \subset\subset B_R$. This means we can apply partial regularity results of geometric measure theory to obtain higher regularity for the level-sets $\Sigma_t = \partial E_t$. Specifically, we consider the following $C^{1,\alpha}$ result (see for example [T]), as quoted in [HI].

Regularity Theorem 4.3 *Let f be a bounded measurable function on a domain Ω with smooth metric g and dimension $n+1 < 8$. Suppose E contains an open set A and minimises the functional*

$$|\partial^* F| + \int_F f \tag{4.6}$$

with respect to competitors F such that $F \supseteq A$, and $F \Delta E \subset\subset \Omega$. If ∂A is $C^{1,\alpha}$, $0 < \alpha \leq 1/2$, then ∂E is a $C^{1,\alpha}$ submanifold of Ω with $C^{1,\alpha}$ estimates depending only on the distance to $\partial\Omega$, ess sup$|f|$, $C^{1,\alpha}$ bounds for ∂A, and C^1 bounds (including positive lower bounds) for the metric g. When $n+1 \geq 8$, this remains true away from a closed singular set Z of dimension at most $n-7$ that is disjoint from $\bar{A}$.

Proof of Theorem 9.6: We break up the proof into the following Lemmata.

Lemma 4.4 *The level-sets $\tilde{\Sigma}^\varepsilon_t = \{U_\varepsilon = t\}$ are locally uniformly bounded in $C^{1,\alpha}$.*

Proof. Since $\tilde{\Sigma}^\varepsilon_t = \{U_\varepsilon = t\}$ is a smooth solution of $(*)$ on $(M \backslash E_0) \times \mathbb{R}$, with smooth normal vector field $\nu_\varepsilon = \frac{\bar{\nabla} U_\varepsilon}{|\bar{\nabla} U_\varepsilon|}$, the functional $J_{U_\varepsilon, \nu_\varepsilon}$ is well defined for sets $\tilde{F} \subset M \times \mathbb{R}$ of locally finite perimeter. Using ν_ε as a calibration and applying the divergence theorem exactly as in the proof of Smooth Flow Lemma 6.4 shows that $\tilde{E}^\varepsilon_t := \{U_\varepsilon < t\}$ minimises $\mathcal{J}_{U_\varepsilon, \nu_\varepsilon}$ in $\Omega := \tilde{E}^\varepsilon_b \backslash \tilde{E}^\varepsilon_a$ for $a \leq t \leq b$.

Now consider $\bar{x} = (x, x') \in (M \backslash \bar{E}_0) \times \mathbb{R}$, and $d = \mathrm{dist}(\bar{x}, \partial E_0 \times \mathbb{R}) = \mathrm{dist}(x, \partial E_0)$. Take L' large enough that $B^M_{2d}(x) \subset F_{L'}$. Then for $\varepsilon \leq \varepsilon' = \varepsilon(L')$, (3.21) provides a uniform bound for $|\nabla u_\varepsilon|$ (and thus also $|\bar{\nabla} U_\varepsilon| + P_{\tilde{\Sigma}^\varepsilon_t}$) on $B^{M\times\mathbb{R}}_d(\bar{x})$. It then follows from Theorem 4.3 that the surfaces $\tilde{\Sigma}^\varepsilon_t \cap B^{n+1}_d(x)$ are uniformly bounded in $C^{1,\alpha}$ in ε and t. □

Lemma 4.5 *Let $\tilde{\mathcal{K}}_{t_0}$ denote the interior of the jump region $\{U = t_0\}$, at a jump time t_0. Then each point $X_0 = (x_0, z_0) \in \tilde{\mathcal{K}}_{t_0}$ lies in a surface $\tilde{\Sigma}_{X_0} \subset \tilde{\mathcal{K}}_{t_0}$ that is locally uniformly bounded in $C^{1,\alpha}$, and is either a vertical cylinder or a graph over an open subset of $\tilde{\mathcal{K}}_{t_0}$.*

Proof. The sought-after surfaces are constructed using a pointwise approach similar to that used by Heidusch [He] to prove local uniform $C^{1,1}$ regularity estimates for the level-sets of the weak solution to inverse mean curvature flow. In particular, we fix a *target point*

$$X_0 = (x_0, z_0) \in \tilde{\mathcal{K}}_{t_0}, \tag{4.7}$$

and construct a surface containing that point.

Given the convergent sequence $\varepsilon_i \to 0$ that produces the limit u of the elliptic regularised solution u_ε as in (3.24), we consider the corresponding sequence of times, t_i, at which the surfaces $\tilde{\Sigma}_{t_i}^i = \operatorname{graph}\left(\frac{u_i}{\varepsilon_i} - \frac{t_i}{\varepsilon_i}\right)$ pass through the target point X_0. This is possible because the translating graphs $\tilde{\Sigma}_t^i$ for $-\infty < t < \infty$ foliate $\Omega_i \times \mathbb{R}$, thus for every i there is a unique t_i such that $X_0 \in \tilde{\Sigma}_{t_i}^i$.

In order to write each surface $\tilde{\Sigma}_{t_i}^i$ locally as a graph over its tangent space $T_{X_0}\tilde{\Sigma}_{t_i}^i$, we use the exponential map to work locally in normal coordinate charts on small Euclidean balls B^{n+2}. In particular, let $\iota(X_0)$ be the injectivity radius of X_0 in $M \backslash E_0 \times \mathbb{R}$, and set

$$d = d(X_0) = \min(\iota(X_0), \operatorname{dist}(X_0, \partial\tilde{\mathcal{K}}_{t_0})). \tag{4.8}$$

By Corollary 5.2 there exists $\varepsilon_0 > 0$ such that for all t and $\varepsilon \leq \varepsilon_0$, the surface pieces $\tilde{\Sigma}_{t_i}^i \cap B_d^{M\times\mathbb{R}}(X_0)$ are uniformly $C^{1,\alpha}$ bounded in t and ε. Now consider the exponential map

$$\exp_{X_0} = (\exp_{x_0}, id_{\mathbb{R}}) : T_{X_0}(M \times \mathbb{R}) \cap B_d^{n+2}(0, z_0) \to B_d^{M\times\mathbb{R}}(X_0), \tag{4.9}$$

and set

$$\hat{\Sigma}_{t_i}^i = \exp_{X_0}^{-1}(\tilde{\Sigma}_{t_i}^i \cap B_d^{M\times\mathbb{R}}(X_0)) \subset T_{X_0}(M \times \mathbb{R}). \tag{4.10}$$

In the $\mathbb{R}$-direction the exponential map is just the identity, thus each surface $\hat{\Sigma}_{t_i}^i$ translates downwards in exactly the same manner as $\tilde{\Sigma}_{t_i}^i$. Furthermore, the surfaces $\hat{\Sigma}_{t_i}^i$ are uniformly $C^{1,\alpha}$ bounded in t and ε.

Then there exists $R > 0$, depending only on the locally uniform $C^{1,\alpha}$ bound, such that $B_R^{n+2}(\hat{X}_0) \subseteq \hat{\Sigma}_{t_i}^i$ and thus the surface pieces $\hat{\Sigma}_{t_i}^i \cap B_R^{n+2}(\hat{X}_0)$ possess uniform $C^{1,\alpha}$ bounds. Here $\hat{X}_0 = (\hat{x}_0, \hat{z}_0) = \exp_q^{-1}(X_0)$ is our target point.

The corresponding normals $\hat{\nu}_i(\hat{X}_0)$ to $\hat{\Sigma}_{t_i}^i \cap B_R^{n+2}(\hat{X}_0)$ create a sequence, a subsequence of which converges uniformly to a vector $\hat{\nu}(\hat{X}_0)$. The normal space to $\hat{\nu}(\hat{X}_0)$ defines a hyperplane $\hat{T}$ containing $\hat{X}_0$. Then by taking $i \gg 1$ large enough, we can write the converging surfaces $\hat{\Sigma}_{t_i}^i \cap B_R^{n+2}(\hat{X}_0)$ as graphs of $C^{1,\alpha}$ functions $\hat{w}_i$ over $\hat{T}$. By reducing R, and taking $i \gg 1$ large enough, we can then write each $\hat{\Sigma}_{t_i}^i$ locally as the graph of $\hat{w}_i$ over $\hat{T} \cap B_R^{n+1}(\hat{x}_0)$.

By Arzela-Ascoli, there exists a further subsequence $\hat{w}_{i_j}$ and a C^1 function $\hat{w} : \hat{T} \cap B_R^{n+1}(\hat{x}_0) \to \mathbb{R}$ such that

$$\hat{w}_{i_j} \to \hat{w} \quad \text{in } C^1(\hat{T} \cap B_R^{n+1}(\hat{x}_0)). \tag{4.11}$$

Here $\hat{w}$ is locally the graph of a surface $\hat{\Sigma}_{\hat{X}_0}$ around $\hat{X}_0$, and $\hat{T} = T_{\hat{X}_0}\hat{\Sigma}_{X_0}$. Since the $C^{1,\alpha}$ bounds on $\hat{w}_i$ were independent of i, it follows that $\hat{w} \in C^{1,\alpha}(\hat{T} \cap B_R^{n+1}(\hat{x}_0))$, with the same uniform $C^{1,\alpha}$ bounds as $\hat{w}_i$. Thus $\exp_q(\hat{\Sigma}_{\hat{X}_0}) := \tilde{\Sigma}_{X_0} \cap B_d^{M\times\mathbb{R}}(X_0)$ is uniformly $C^{1,\alpha}$ bounded. By successively taking subsequences, the $\tilde{\Sigma}^i_{t_i}$ converge to a complete hypersurface that we will henceforth denote by $\tilde{\Sigma}_{X_0}$, since it coincides with $\tilde{\Sigma}_{X_0}$ near X_0.

Now $X_0 \in \{U = t_0\}$ where, by hypothesis, $t_0 := \lim_{i\to\infty} t_i$ is a jump time. In order to argue that $\tilde{\Sigma}_{X_0}$ is contained in the set $\{U = t_0\}$, we note that it is a consequence of the above construction that any $y \in \tilde{\Sigma}_{X_0}$ is the limit of a sequence $y_i \in \tilde{\Sigma}^i_{t_i}$. The local uniform convergence $u_i \to u$ implies that $\hat{u}_i \to \hat{u}$ uniformly on $B_d^{n+1}(0)$. Thus the uniform convergence $U_i \to U$ on $B_d^{M\times\mathbb{R}}(X_0)$, together with the fact that $\lim_{i\to\infty} y_i = y$ then implies that $U(y) = t_0$, since

$$|U_i(y_i) - U(y)| \le |U_i(y_i) - U_i(y)| + |U_i(y) - U(y)| \to 0,$$

and $\lim\limits_{i\to\infty} U^i(y_i) = \lim\limits_{i\to\infty} t_i = t_0$, thus $\tilde{\Sigma}_{X_0} \subset \{U = t_0\}$.

This approach enables one to choose any point X_0 in the jump region $\tilde{\mathcal{K}}_{t_0}$ and construct the corresponding surface $\tilde{\Sigma}_{X_0}$ containing X_0. Since each $\tilde{\Sigma}_{X_0}$ is the limit of the graphs $\tilde{\Sigma}^i_{t_i}$ with local uniform $C^{1,\alpha}$ bounds, it is clear that each $\tilde{X}_0$ is either a vertical cylinder or a graph over an open subset of $\tilde{\mathcal{K}}_{t_0} \cap M$. Therefore, let Ω_G denote the open region in $\tilde{\mathcal{K}}_{t_0} \cap M$ where $|\nabla \hat{u}_i|$ converges locally uniformly to a finite limit, and let Ω_C denote the region where $|\nabla \hat{u}_i|$ converges to infinity. Then the translating nature of $\tilde{\Sigma}^\varepsilon_t$ together with the above construction dictates that the $\tilde{\Sigma}^i_{t_i}$ converge to a graph $\tilde{\Sigma}_{X_0}$ over Ω_G, lying in a stack

$$\{\tilde{\Sigma}_{X_\alpha}\} = \tilde{\Sigma}_{X_0} + \alpha \mathbf{e}_{n+2}, \qquad \alpha \in \mathbb{R}, \tag{4.12}$$

of vertical translates of $\tilde{\Sigma}_{X_0}$. To see this, note that

$$X_0 = (x_0, z_0) \in \tilde{\Sigma}^{i_j}_{t_{i_j}} = \text{graph}\left(\frac{u_{i_j}}{\varepsilon_{i_j}} - \frac{t_{i_j}}{\varepsilon_{i_j}}\right) \to \text{graph}(w) = \tilde{\Sigma}_{X_0},$$

implies $X_\alpha := (x_0, z_0 + \alpha) \in \tilde{\Sigma}^{i_j}_{t_{i_j} - \alpha\varepsilon_{ij}}$, where

$$\tilde{\Sigma}^{i_j}_{t_{i_j} - \alpha\varepsilon_{ij}} = \text{graph}\left(\frac{u_{i_j}}{\varepsilon_{i_j}} - \frac{t_{i_j} - \alpha\varepsilon_{i_j}}{\varepsilon_{i_j}}\right) \to \text{graph}(w) + \alpha \mathbf{e}_z := \tilde{\Sigma}_{X_\alpha},$$

where $w := \exp_q(\hat{w})$. Therefore $\Omega_G \times \mathbb{R}$ is bounded by vertical cylinders, and filled by the stacks produced by the family $\{\tilde{\Sigma}_{X_\alpha}\}$ of vertical translations of each graph $\tilde{\Sigma}_{X_0}$. $\square$

The possibility of two surfaces $\tilde{\Sigma}_{P_1}$ and $\tilde{\Sigma}_{P_2}$ from Lemma 5.3 touching tangentially at one point P, such that the outward unit normals agree at P and $\tilde{\Sigma}_{P_1}$ lies outside $\tilde{\Sigma}_{P_2}$ (in the direction of the outward normal near P) is ruled out by the strong maximum principle. Furthermore, the intersection of two surfaces in the limit is ruled out by the translation invariance of the surfaces $\tilde{\Sigma}_t^\varepsilon$ and their local uniform $C^{1,\alpha}$ bounds.

We now argue that we can construct a "normal" vector field $\tilde{\nu}$ on $\tilde{\mathcal{K}}_{t_0}$ using the surfaces from the proof of Lemma 5.3. Since the limit surfaces are vertical cylinders or stacks of translation invariant graphs, the normal vector field $\tilde{\nu}$ to the family of surfaces in $\mathcal{K}_{t_0}$ is translation invariant, and we need only show that we can construct $\tilde{\Sigma}_{X_0}$ for each $X_0 \subset \tilde{\mathcal{K}}_{t_0} \cap M$. Therefore, choose a dense set of points in $\tilde{\mathcal{K}}_{t_0} \cap M$. This corresponds to a countable set of points $\{p_i\}$, and for each such $p_i \in \tilde{\mathcal{K}}_{t_0} \cap M$, we consider the convergent subsequence ε_i such that $\tilde{\Sigma}^i_{t_i}$ converges to the hypersurface $\tilde{\Sigma}_{P_i}$ in $\tilde{\mathcal{K}}_{t_0}$, where $P_i := (p_i, 0)$. Then by taking a diagonal subsequence ε_{i_*}, we obtain local convergence of $\tilde{\Sigma}^{i_*}_{t_{i_*}}$ to $\tilde{\Sigma}_{P_i}$ for every point p_i in the dense set.

Now consider a point $p_0 \in \Omega_G$ such that p_0 is not in $\{p_i\}$. We wish to argue that we obtain local convergence to $\tilde{\Sigma}_{P_0}$ via the convergent sequence ε_{i_*}. There exists a point p_i in the dense subset such that $\text{dist}(p_i, p_0) < d/10$. Let

$$d_P := \min(\iota(P), \text{dist}(P, \partial\tilde{\mathcal{K}}_{t_0})). \tag{4.13}$$

By Corollary 5.2, the surfaces $\tilde{\Sigma}_t^\varepsilon$ are uniformly bounded in $B^{M\times\mathbb{R}}_{d_{P_i}}(P_i)$. Then since $B_{d_{P_i}/10}(P_0) \subset B_{d_{P_i}}(P_i)$, the surfaces $\tilde{\Sigma}_t^\varepsilon \cap B_{d_{P_i}/10}(P_0)$ possess the same uniform $C^{1,\alpha}$ bounds and we can take a convergent subsequence of ε_{i_*} such that we obtain convergence to a limit surface $\tilde{\Sigma}_{P_0}$ in $B_{d_{P_i}/10}(P_0)$. Therefore this approach constructs a complete graph through each point $x_0 \in \Omega_G$, and we obtain the vector field $\tilde{\nu}$ in all of Ω_G.

Then given the uniform $C^{0,\alpha}$ normal vector field $\tilde{\nu}$ of the hypersurfaces constructed through the dense set of points $\{p_i\}$, we can extend the vector field $\tilde{\nu}$ to any points that have been missed in Ω_C. Then translating $\tilde{\nu}$ in the e_{n+2} direction, we obtain a normal vector field on the entire jump region $\tilde{\mathcal{K}}_{t_0}$. For the remainder of this work, let $\tilde{\nu}$ denote this translation invariant normal vector field to the surfaces $\tilde{\Sigma}_{X_0}$ foliating $\tilde{\mathcal{K}}_{t_0}$.

Lemma 4.6 *Let $\tilde{\nu}$ denote the normal vector field to the surfaces foliating the jump region $\tilde{\mathcal{K}}_{t_0}$, as above. Then each surface $\tilde{\Sigma}_{X_0}$ in the jump region bounds a Caccioppoli set that minimises $\mathcal{J}_{U,\tilde{\nu}}$ in $\tilde{\mathcal{K}}_{t_0}$.*

Proof. Consider the Caccioppoli set $\tilde{E}$ that is bounded by the limit hypersurface $\tilde{\Sigma}_{X_0}$, such that $\tilde{\nu}$ is the outward unit normal of the relative boundary $\partial\tilde{E}\cap\tilde{\mathcal{K}}_{t_0}$. The sets $\tilde{E}^i_{t_i}$ minimize the functional J_{U_i,ν_i} in $\tilde{\mathcal{K}}_{t_0}$, where $\nu_i = \frac{\bar{\nabla} U_i}{|\bar{\nabla} U_i|}$. Passing these sets to limits as in the proof of Lemma 5.3 to obtain the limit surface $\tilde{\Sigma}_{X_0}$, Theorem 4.2 then says that $\tilde{E}$ minimises $\mathcal{J}_{U,\tilde{\nu}}$ in $\tilde{\mathcal{K}}_{t_0}$. □

Collecting the above results, we obtain a family of $C^{1,\alpha}_{\text{loc}}$ hypersurfaces foliating $\Omega_G \times \mathbb{R}$, and by extending the family of cylindrical hypersurfaces in $\Omega_C \times \mathbb{R}$ to any missed points in

Ω_C, we obtain a foliation of the entire interior region $\tilde{\mathcal{K}}_{t_0}$. At each point $X_0 = (x_0, t_0)$, the corresponding leaf of the foliation passing through X_0 is constructed by taking the limit of the Σ_t^ε locally around X_0, as in Lemma 5.3. This completes the proof. □

5 The jump region in the elliptic regularisation limit

In this section we use the smooth epsilon-regularised solutions $\tilde{\Sigma}_t^\varepsilon = \{U_\varepsilon = t\}$ to study the jump regions of the limiting time-of-arrival function U defined by (3.26). In particular, we show that taking an appropriate limit of these surfaces at jump times t_0, leads to a foliation of the interior $\tilde{\mathcal{K}}_{t_0}$ of the jump region $\{U = t_0\}$ by hypersurfaces that minimise area plus bulk energy P, and in particular are smooth MOTS.

Theorem 5.1 *Let $U \in C^{0,1}_{loc}((M\backslash E_0) \times \mathbb{R})$ be the limit of U_ε given by (3.26). At a jump time t_0, the interior $\tilde{\mathcal{K}}_{t_0}$ of $\{U = t_0\}$ is foliated by smooth MOTS, each of which is either (part of) a vertical cylinder or a smooth graph over an open subset of $\tilde{\mathcal{K}}_{t_0}$.*

The normal vector field to this foliation extends $\dfrac{\bar{\nabla} U}{|\bar{\nabla} U|}$ as a calibration across the jump region, and this is then used in Chapter 2.5 to prove existence of a weak solution.

We break up the proof of Theorem 5.1 into the following Lemmata.

Lemma 5.2 *The level-sets $\tilde{\Sigma}_t^\varepsilon = \{U_\varepsilon = t\}$ are locally uniformly bounded in $C^{1,\alpha}$.*

Proof. Since $\tilde{\Sigma}_t^\varepsilon = \{U_\varepsilon = t\}$ is a smooth solution of $(*)$ on $(M\backslash E_0) \times \mathbb{R}$, with smooth normal vector field $\nu_\varepsilon = \frac{\bar{\nabla} U_\varepsilon}{|\bar{\nabla} U_\varepsilon|}$, the functional

$$J_{U_\varepsilon,\nu_\varepsilon}(\tilde{F}) = \mathcal{J}^{\tilde{A}}_{U_\varepsilon,\nu_\varepsilon}(\tilde{F}) := |\partial^* \tilde{F} \cap \tilde{A}| - \int_{\tilde{F}\cap\tilde{A}} |\bar{\nabla} U_\varepsilon| - \left(\bar{g}^{ij} - \nu_\varepsilon^i \nu_\varepsilon^j\right) K_{ij} dx.$$

is well defined, and Smooth Flow Lemma 6.4 implies that $\tilde{E}_t^\varepsilon$ minimises $\mathcal{J}_{U_\varepsilon,\nu_\varepsilon}$ in $\tilde{\Omega} := \tilde{E}_b^\varepsilon \backslash \tilde{E}_a^\varepsilon$ for $a \leq t \leq b$. It is then a corollary of Theorem 4.3 that the level-sets $\tilde{\Sigma}_t^\varepsilon$ are locally uniformly bounded in $C^{1,\alpha}$, with estimates depending on $\|K\|_{C_0}$ and the local Lipschitz bounds for U_ε. The $C^{1,\alpha}$ bound in the statement of Theorem 4.3 depends on the bulk energy f of the functional (**??**) and the distance to the boundary $\partial\Omega$. Consider $\bar{x} = (x, x') \in (M\backslash \bar{E}_0) \times \mathbb{R}$, and $d = \text{dist}(\bar{x}, \partial E_0 \times \mathbb{R}) = \text{dist}(x, \partial E_0)$. Take L' large enough that $B^M_{2d}(x) \subset F_{L'}$. Then for $\varepsilon \leq \varepsilon' = \varepsilon(L')$, (3.21) provides a uniform bound for $|\nabla u_\varepsilon|$ (and thus also $|\bar{\nabla} U_\varepsilon| + P_{\tilde{\Sigma}_t^\varepsilon}$) on $B^{M\times\mathbb{R}}_d(\bar{x})$. It then follows from Theorem 4.3 that the surfaces $\tilde{\Sigma}_t^\varepsilon \cap B^{n+1}_d(x)$ are uniformly bounded in $C^{1,\alpha}$ in ε and t. □

Lemma 5.3 *Let $\tilde{\mathcal{K}}_{t_0}$ denote the interior of the jump region $\{U = t_0\}$, at a jump time t_0. Then each point $X_0 = (x_0, z_0) \in \tilde{\mathcal{K}}_{t_0}$ lies in a surface $\tilde{\Sigma}_{X_0} \subset \tilde{\mathcal{K}}_{t_0}$ that is locally uniformly bounded in $C^{1,\alpha}$, and is either a vertical cylinder or a graph over an open subset of $\tilde{\mathcal{K}}_{t_0}$.*

Proof. The sought-after surfaces are constructed using a pointwise approach similar to that used by Heidusch [He] to prove local uniform $C^{1,1}$ regularity estimates for the level-sets of the weak solution to inverse mean curvature flow. In particular, we fix a *target point*

$$X_0 = (x_0, z_0) \in \tilde{\mathcal{K}}_{t_0}, \tag{5.1}$$

and construct a surface containing that point.

Given the convergent sequence $\varepsilon_i \to 0$ that produces the limit u of the elliptic regularised solution u_ε as in (3.24), we consider the corresponding sequence of times, t_i, at which the surfaces $\tilde{\Sigma}^i_{t_i} = \text{graph}\left(\frac{u_i}{\varepsilon_i} - \frac{t_i}{\varepsilon_i}\right)$ pass through the target point X_0. This is possible because the translating graphs $\tilde{\Sigma}^i_t$ for $-\infty < t < \infty$ foliate $\Omega_i \times \mathbb{R}$, thus for every i there is a unique t_i such that $X_0 \in \tilde{\Sigma}^i_{t_i}$.

In order to write each surface $\tilde{\Sigma}^i_{t_i}$ locally as a graph over its tangent space $T_{X_0}\tilde{\Sigma}^i_{t_i}$, we use the exponential map to work locally in normal coordinate charts on small Euclidean balls B^{n+2}. In particular, let $\iota(X_0)$ be the injectivity radius of X_0 in $M \backslash E_0 \times \mathbb{R}$, and set

$$d = d(X_0) = \min(\iota(X_0), \text{dist}(X_0, \partial\tilde{\mathcal{K}}_{t_0})). \tag{5.2}$$

By Lemma 5.2 there exists $\varepsilon_0 > 0$ such that for all t and $\varepsilon \le \varepsilon_0$, the surface pieces $\tilde{\Sigma}^i_{t_i} \cap B^{M\times\mathbb{R}}_d(X_0)$ are uniformly $C^{1,\alpha}$ bounded in t and ε. Now consider the exponential map

$$\exp_{X_0} = (\exp_{x_0}, id_{\mathbb{R}}) : T_{X_0}(M \times \mathbb{R}) \cap B^{n+2}_d(0, z_0) \to B^{M\times\mathbb{R}}_d(X_0), \tag{5.3}$$

and set

$$\hat{\Sigma}^i_{t_i} = \exp^{-1}_{X_0}(\tilde{\Sigma}^i_{t_i} \cap B^{M\times\mathbb{R}}_d(X_0)) \subset T_{X_0}(M \times \mathbb{R}). \tag{5.4}$$

In the $\mathbb{R}$-direction the exponential map is just the identity, thus each surface $\hat{\Sigma}^i_{t_i}$ translates downwards in exactly the same manner as $\tilde{\Sigma}^i_{t_i}$. Furthermore, the surfaces $\hat{\Sigma}^i_{t_i}$ are uniformly $C^{1,\alpha}$ bounded in t and ε.

Then there exists $R > 0$, depending only on the locally uniform $C^{1,\alpha}$ bound, such that $B^{n+2}_R(\hat{X}_0) \subseteq \hat{\Sigma}^i_{t_i}$ and thus the surface pieces $\hat{\Sigma}^i_{t_i} \cap B^{n+2}_R(\hat{X}_0)$ possess uniform $C^{1,\alpha}$ bounds. Here $\hat{X}_0 = (\hat{x}_0, \hat{z}_0) = \exp^{-1}_q(X_0)$ is our target point.

The corresponding normals $\hat{\nu}_i(\hat{X}_0)$ to $\hat{\Sigma}^i_{t_i} \cap B^{n+2}_R(\hat{X}_0)$ create a sequence, a subsequence of which converges uniformly to a vector $\hat{\nu}(\hat{X}_0)$. The normal space to $\hat{\nu}(\hat{X}_0)$ defines a hyperplane $\hat{T}$ containing $\hat{X}_0$. Then by taking $i \gg 1$ large enough, we can write the converging surfaces $\hat{\Sigma}^i_{t_i} \cap B^{n+2}_R(\hat{X}_0)$ as graphs of $C^{1,\alpha}$ functions $\hat{w}_i$ over $\hat{T}$. By reducing R, and taking $i \gg 1$ large enough, we can then write each $\hat{\Sigma}^i_{t_i}$ locally as the graph of $\hat{w}_i$ over $\hat{T} \cap B^{n+1}_R(\hat{x}_0)$.

By Arzela-Ascoli, there exists a further subsequence $\hat{w}_{i_j}$ and a C^1 function $\hat{w} : \hat{T} \cap B_R^{n+1}(\hat{x}_0) \to \mathbb{R}$ such that

$$\hat{w}_{i_j} \to \hat{w} \quad \text{in } C^1(\hat{T} \cap B_R^{n+1}(\hat{x}_0)). \tag{5.5}$$

Here $\hat{w}$ is locally the graph of a surface $\hat{\Sigma}_{\hat{X}_0}$ around $\hat{X}_0$, and $\hat{T} = T_{\hat{X}_0}\hat{\Sigma}_{X_0}$. Since the $C^{1,\alpha}$ bounds on $\hat{w}_i$ were independent of i, it follows that $\hat{w} \in C^{1,\alpha}(\hat{T} \cap B_R^{n+1}(\hat{x}_0))$, with the same uniform $C^{1,\alpha}$ bounds as $\hat{w}_i$. Thus $\exp_q(\hat{\Sigma}_{\hat{X}_0}) := \tilde{\Sigma}_{X_0} \cap B_d^{M\times\mathbb{R}}(X_0)$ is uniformly $C^{1,\alpha}$ bounded. By successively taking subsequences, the $\tilde{\Sigma}^i_{t_i}$ converge to a complete hypersurface that we will henceforth denote by $\tilde{\Sigma}_{X_0}$, since it coincides with $\tilde{\Sigma}_{X_0}$ near X_0.

Now $X_0 \in \{U = t_0\}$ where, by hypothesis, $t_0 := \lim_{i\to\infty} t_i$ is a jump time. In order to argue that $\tilde{\Sigma}_{X_0}$ is contained in the set $\{U = t_0\}$, we note that it is a consequence of the above construction that any $y \in \tilde{\Sigma}_{X_0}$ is the limit of a sequence $y_i \in \tilde{\Sigma}^i_{t_i}$. The local uniform convergence $u_i \to u$ implies that $\hat{u}_i \to \hat{u}$ uniformly on $B_d^{n+1}(0)$. Thus the uniform convergence $U_i \to U$ on $B_d^{M\times\mathbb{R}}(X_0)$, together with the fact that $\lim_{i\to\infty} y_i = y$ then implies that $U(y) = t_0$, since

$$|U_i(y_i) - U(y)| \leq |U_i(y_i) - U_i(y)| + |U_i(y) - U(y)| \to 0,$$

and

$$\lim_{i\to\infty} U^i(y_i) = \lim_{i\to\infty} t_i = t_0,$$

thus $\tilde{\Sigma}_{X_0} \subset \{U = t_0\}$.

This approach enables one to choose any point X_0 in the jump region $\tilde{\mathcal{K}}_{t_0}$ and construct the corresponding surface $\tilde{\Sigma}_{X_0}$ containing X_0. Since each $\tilde{\Sigma}_{X_0}$ is the limit of the graphs $\tilde{\Sigma}^i_{t_i}$ with local uniform $C^{1,\alpha}$ bounds, it is clear that each $\tilde{X}_0$ is either a vertical cylinder or a graph over an open subset of $\tilde{\mathcal{K}}_{t_0} \cap M$. Therefore, let Ω_G denote the open region in $\tilde{\mathcal{K}}_{t_0} \cap M$ where $|\nabla \hat{u}_i|$ converges locally uniformly to a finite limit, and let Ω_C denote the region where $|\nabla \hat{u}_i|$ converges to infinity. Then the translating nature of $\tilde{\Sigma}^\varepsilon_t$ together with the above construction dictates that the $\tilde{\Sigma}^i_{t_i}$ converge to a graph $\tilde{\Sigma}_{X_0}$ over Ω_G, lying in a stack

$$\{\tilde{\Sigma}_{X_\alpha}\} = \tilde{\Sigma}_{X_0} + \alpha \mathbf{e}_{n+2}, \qquad \alpha \in \mathbb{R}, \tag{5.6}$$

of vertical translates of $\tilde{\Sigma}_{X_0}$. To see this, note that

$$X_0 = (x_0, z_0) \in \tilde{\Sigma}^{i_j}_{t_{i_j}} = \text{graph}\left(\frac{u_{i_j}}{\varepsilon_{i_j}} - \frac{t_{i_j}}{\varepsilon_{i_j}}\right) \to \text{graph}(w) = \tilde{\Sigma}_{X_0},$$

implies

$$X_\alpha := (x_0, z_0 + \alpha) \in \tilde{\Sigma}^{i_j}_{t_{i_j} - \alpha\varepsilon_{ij}} = \text{graph}\left(\frac{u_{i_j}}{\varepsilon_{i_j}} - \frac{t_{i_j} - \alpha\varepsilon_{i_j}}{\varepsilon_{i_j}}\right) \to \text{graph}(w) + \alpha \mathbf{e}_z := \tilde{\Sigma}_{X_\alpha},$$

where $w := \exp_q(\hat{w})$.

Therefore $\Omega_G \times \mathbb{R}$ is bounded by vertical cylinders, and filled by the stacks produced by the family $\{\tilde{\Sigma}_{X_\alpha}\}$ of vertical translations of each graph $\tilde{\Sigma}_{X_0}$.

□

The possibility of two surfaces $\tilde{\Sigma}_{P_1}$ and $\tilde{\Sigma}_{P_2}$ from Lemma 5.3 touching tangentially at one point P, such that the outward unit normals agree at P and $\tilde{\Sigma}_{P_1}$ lies outside $\tilde{\Sigma}_{P_2}$ (in the direction of the outward normal near P) is ruled out by the strong maximum principle. Furthermore, the intersection of two surfaces in the limit is ruled out by the translation invariance of the surfaces $\tilde{\Sigma}^{\varepsilon}_t$ and their local uniform $C^{1,\alpha}$ bounds.

We now argue that we can construct a "normal" vector field $\tilde{\nu}$ on $\tilde{\mathcal{K}}_{t_0}$ using the surfaces from the proof of Lemma 5.3. Since the limit surfaces are vertical cylinders or stacks of translation invariant graphs, the normal vector field $\tilde{\nu}$ to the family of surfaces in $\mathcal{K}_{t_0}$ is translation invariant, and we need only show that we can construct $\tilde{\Sigma}_{X_0}$ for each $X_0 \subset \tilde{\mathcal{K}}_{t_0} \cap M$. Therefore, choose a dense set of points in $\tilde{\mathcal{K}}_{t_0} \cap M$. This corresponds to a countable set of points $\{p_i\}$, and for each such $p_i \in \tilde{\mathcal{K}}_{t_0} \cap M$, we consider the convergent subsequence ε_i such that $\tilde{\Sigma}^i_{t_i}$ converges to the hypersurface $\tilde{\Sigma}_{P_i}$ in $\tilde{\mathcal{K}}_{t_0}$, where $P_i := (p_i, 0)$. Then by taking a diagonal subsequence ε_{i_*}, we obtain local convergence of $\tilde{\Sigma}^{i_*}_{t_{i_*}}$ to $\tilde{\Sigma}_{P_i}$ for every point p_i in the dense set.

Now consider a point $p_0 \in \Omega_G$ such that p_0 is not in $\{p_i\}$. We wish to argue that we obtain local convergence to $\tilde{\Sigma}_{P_0}$ via the convergent sequence ε_{i_*}. There exists a point p_i in the dense subset such that $\operatorname{dist}(p_i, p_0) < d/10$. Let

$$d_P := \min(\iota(P), \operatorname{dist}(P, \partial\tilde{\mathcal{K}}_{t_0})). \tag{5.7}$$

By Corollary 5.2, the surfaces $\tilde{\Sigma}^{\varepsilon}_t$ are uniformly bounded in $B^{M\times\mathbb{R}}_{d_{p_i}}(P_i)$. Then since $B_{d_{P_i}/10}(P_0) \subset B_{d_{P_i}}(P_i)$, the surfaces $\tilde{\Sigma}^{\varepsilon}_t \cap B_{d_{P_i}/10}(P_0)$ possess the same uniform $C^{1,\alpha}$ bounds and we can take a convergent subsequence of ε_{i_*} such that we obtain convergence to a limit surface $\tilde{\Sigma}_{P_0}$ in $B_{d_{P_i}/10}(P_0)$. Therefore this approach constructs a complete graph through each point $x_0 \in \Omega_G$, and we obtain the vector field $\tilde{\nu}$ in all of Ω_G.

Then given the uniform $C^{0,\alpha}$ normal vector field $\tilde{\nu}$ of the hypersurfaces constructed through the dense set of points $\{p_i\}$, we can extend the vector field $\tilde{\nu}$ to any points that have been missed in Ω_C. Then translating $\tilde{\nu}$ in the e_{n+2} direction, we obtain a normal vector field on the entire jump region $\tilde{\mathcal{K}}_{t_0}$.

For the remainder of this work, let $\tilde{\nu}$ denote this translation invariant normal vector field to the surfaces $\tilde{\Sigma}_{X_0}$ foliating $\tilde{\mathcal{K}}_{t_0}$.

Lemma 5.4 *Let $\tilde{\nu}$ denote the normal vector field to the surfaces foliating the jump region $\tilde{\mathcal{K}}_{t_0}$, as above. Then each surface $\tilde{\Sigma}_{X_0}$ in the jump region bounds a Caccioppoli set that minimises $\mathcal{J}_{U,\tilde{\nu}}$ in $\tilde{\mathcal{K}}_{t_0}$.*

Proof. Consider the Caccioppoli set $\tilde{E}$ that is bounded by the limit hypersurface $\tilde{\Sigma}_{X_0}$, such that $\tilde{\nu}$ is the outward unit normal of the relative boundary $\partial\tilde{E} \cap \tilde{\mathcal{K}}_{t_0}$. The sets $\tilde{E}^i_{t_i}$ minimize the functional J_{U_i,ν_i} in $\tilde{\mathcal{K}}_{t_0}$, where $\nu_i = \frac{\bar{\nabla} U_i}{|\bar{\nabla} U_i|}$. Passing these sets to limits as in the proof of Lemma 5.3 to obtain the limit surface $\tilde{\Sigma}_{X_0}$, Theorem 4.2 then says that $\tilde{E}$ minimises $\mathcal{J}_{U,\tilde{\nu}}$ (defined by (**??**)) in $\tilde{\mathcal{K}}_{t_0}$. □

Lemma 5.5

$$|\bar{\nabla} U_i| \to 0 \quad in\ L^1_{loc}(\tilde{\mathcal{K}}_{t_0}). \tag{5.8}$$

Proof. Recall d defined by (5.2), consider a target point $X_0 = (x_0, z_0)$ such that $z_0 > 2d+1$ and select a cutoff function $\phi \in C^2_c(\mathbb{R})$ such that $\phi \geq 0$ and $\operatorname{spt}\phi \subseteq [z_0 - 2d, z_0 + 2d]$. Then let $T_0 = z_0 - 2d - 1$, fix an arbitrary time $T > T_0$, and consider $T_0 \leq t \leq T$ and $L \geq T + 3 + z_0 + 2d$.

We wish to show that

$$\liminf_{i\to\infty} \int_{\tilde{\Sigma}^i_{t_i} \cap B^{M\times\mathbb{R}}_d(X_0)} |D(H+P)|^2 < \infty.$$

To this end, we calculate

$$\begin{aligned}
\frac{d}{dt}\int_{\tilde{\Sigma}^\varepsilon_t} \phi(z)(H+P)^2 =& \int_{\tilde{\Sigma}^\varepsilon_t} 2\phi(H+P)\frac{\partial}{\partial t}(H+P) + (H+P)^2 \frac{\partial\phi}{\partial z}\cdot\frac{\nu_\varepsilon}{H+P} + \phi H(H+P) \\
=& -2\int_{\tilde{\Sigma}^\varepsilon_t} \phi\Bigg((H+P)\Delta\left(\frac{1}{H+P}\right) + |A|^2 + \bar{Ric}(\nu_\varepsilon,\nu_\varepsilon) - \bar{\nabla}_{\nu_\varepsilon} P \\
&+ \frac{2D_i(H+P)}{H+P}K_{i\nu_\varepsilon}\Bigg) + (H+P)\frac{\partial\phi}{\partial z}\cdot\nu + \phi H(H+P) \\
=& \int_{\tilde{\Sigma}^\varepsilon_t} \phi\Bigg(-2\frac{|D(H+P)|^2}{(H+P)^2} - 2|A|^2 - 2\bar{Ric}(\nu,\nu) + H(H+P) + 2\bar{\nabla}_\nu P \\
&- 4\frac{D_i(H+P)}{H+P}K_{i\nu}\Bigg) - 2\frac{\phi}{\partial z}\cdot\frac{D(H+P)}{H+P} + (H+P)\frac{\partial\phi}{\partial z}\cdot\nu
\end{aligned} \tag{5.9}$$

In view of the sup estimates (3.8) and (3.16) for u_ε, there is $R(T) > 0$ depending only on the subsolution v and K_{ij} such that

$$\tilde{\Sigma}^\varepsilon_t \cap (M \times \operatorname{spt}\phi) \subseteq S(T) := (B_{R(T)}\backslash E_0) \times [z_0 - 2d, z_0 + 2d], \quad T_0 \leq t \leq T.$$

The Outward Optimising Property 7.1, applied to $\tilde{E}^\varepsilon_t$ compared to the perturbation $\tilde{E}^\varepsilon_t \cup S(T)$, then provides the area estimate

$$|\tilde{\Sigma}^\varepsilon_t \cap (M \times \operatorname{spt}\phi)| \leq C(T) + \int_{S(T)\backslash\tilde{E}^\varepsilon_t} P \leq C(T, \|K\|_{C^0}), \qquad T_0 \leq t \leq T. \tag{5.10}$$

Together with the interior estimate (2.5), and the boundary gradient estimates for u^ε, this shows

$$|H + P| \leq C(T, \|K\|_{C^1}) \qquad \text{on } \tilde{\Sigma}^\varepsilon_t \cap (M \times \operatorname{spt}\phi), \quad T_0 \leq t \leq T.$$

It follows that

$$\int_{\tilde{\Sigma}^\varepsilon_t} \phi|H|(H+P) + \phi(H+P)^2 + |(H+P)\bar{\nabla}\phi\cdot\nu| \leq C(T, \|K\|_{C^1}), \quad T_0 \leq t \leq T.$$

We estimate the $D\phi$ and $K_{i\nu}$ terms via

$$\left|2D\phi\cdot\frac{D(H+P)}{H+P}\right|\leq 2\frac{|D\phi|^2}{\phi}+\frac{\phi}{2}\frac{|D(H+P)|^2}{(H+P)^2}\leq C+\frac{\phi}{2}\frac{|D(H+P)|^2}{(H+P)^2},$$
$$\left|4\phi\frac{D_i(H+P)}{H+P}K_{i\nu}\right|\leq 8\phi\|K\|^2_{C^0}+\frac{\phi}{2}\frac{|D(H+P)|^2}{(H+P)^2}.$$

Thus (7.12) becomes

$$\frac{d}{dt}\int_{\tilde{\Sigma}^\varepsilon_t}\phi(H+P)^2\leq\int_{\tilde{\Sigma}^\varepsilon_t}-\phi\frac{|D(H+P)|^2}{(H+P)^2}+C(T,\|K\|_{C^1}),\tag{5.11}$$

and integrating gives

$$\int_{T_0}^T\int_{\tilde{\Sigma}^\varepsilon_t\cap(M\times[z_0-2d,z_0+2d])}\frac{|D(H+P)|^2}{(H+P)^2}\leq C(T,\|K\|_{C^1}),\tag{5.12}$$

using a ϕ such that $\phi=1$ on $[z_0-2d,z_0+2d]$.
Applying Fatou's Lemma, for any sequence $\varepsilon_i\to 0$ we obtain

$$\liminf_{i\to\infty}\int_{\tilde{\Sigma}^i_t\cap(M\times[z_0-2d,z_0+2d])}\frac{|D(H+P)|^2}{(H+P)^2}<\infty,\qquad\text{a.e. } t\geq T_0.\tag{5.13}$$

Now consider the subsequence $\varepsilon_{i_j}\to 0$ from (5.5) such that $\tilde{\Sigma}^{i_j}_{t_{i_j}}\to\tilde{\Sigma}_0$ in $C^1(T\cap B^{n+1}_R(X_0))$, where $T=T_{X_0}\tilde{\Sigma}_{X_0}$. We write $i=i_j$ henceforth. Since (7.16) only holds for a.e. $t\geq T_0$, it will take more work to argue that $\liminf_{i\to\infty}\int_{\tilde{\Sigma}^i_{t_i}\cap B^{M\times\mathbb{R}}_d(X_0)}|D(H+P)|^2<\infty$. To this end, we pick a sequence $\hat{t}_i$ such that $\hat{t}_i\to t_0+\delta$ for some $|\delta|>0$, $|\hat{t}_i-t_i|\leq\varepsilon_i d$ and

$$\liminf_{i\to\infty}\int_{\tilde{\Sigma}^i_{\hat{t}_i}\cap(M\times[z_0-2d,z_0+2d])}|D(H+P)|^2<\infty.\tag{5.14}$$

Define $\hat{z}_i:=\dfrac{u_i(x_0)}{\varepsilon_i}-\dfrac{\hat{t}_i}{\varepsilon_i}$ and $\delta_i:=\hat{z}_i-z_0$. Then the fact that $\tilde{\Sigma}^i_{\hat{t}_i}$ is just a translation of $\tilde{\Sigma}^i_{t_i}$ by δ_i in the z-direction implies that

$$\int_{\tilde{\Sigma}^i_{t_i}\cap B^{M\times\mathbb{R}}_d(X_0)}|D(H+P)|^2=\int_{\tilde{\Sigma}^i_{\hat{t}_i}\cap B^{M\times\mathbb{R}}_d(x_0,z_0+\delta_i)}|D(H+P)|^2,$$

for each i. Furthermore, the condition $|\hat{t}_i-t_i|\leq\varepsilon_i d$ implies that $|\delta_i|=|\hat{z}_i-z_0|\leq d$, which ensures that $\tilde{\Sigma}^i_{\hat{t}_i}\cap B^{M\times\mathbb{R}}_d(x_0,z_0+\delta_i)\subset M\times[z_0-2d,z_0+2d]$, and thus from (7.17) we obtain that

$$\liminf_{i\to\infty}\int_{\tilde{\Sigma}^i_{\hat{t}_i}\cap B^{M\times\mathbb{R}}_d(x_0,z_0+\delta_i)}|D(H+P)|^2\leq\int_{\tilde{\Sigma}^i_{\hat{t}_i}\cap(M\times[z_0-2d,z_0+2d])}|D(H+P)|^2<\infty,$$

from which our desired estimate follows

$$\liminf_{i\to\infty} \int_{\tilde{\Sigma}^i_{t_i} \cap B_d^{M\times\mathbb{R}}(X_0)} |D(H+P)|^2 < \infty. \tag{5.15}$$

As in the proof of Lemma (5.3), the converging surfaces $\tilde{\Sigma}^i_{t_i}$ can be written locally, via the exponential map, as graphs of $C^{1,\alpha}_{\text{loc}}$ functions w_i over the hyperplane T. This local $C^{1,\alpha}$ convergence of the hypersurfaces, together with the first variation of area formula and the Riesz Representation Theorem then implies that $H_{\tilde{\Sigma}_{X_0}}$ exists weakly as a locally L^1 function, with the weak convergence

$$\int_{\tilde{\Sigma}^i_{t_i}} H_{\tilde{\Sigma}^i_{t_i}} \nu_{\tilde{\Sigma}^i_{t_i}} \cdot X \to \int_{\tilde{\Sigma}_{X_0}} H_{\tilde{\Sigma}_{X_0}} \nu_{\tilde{\Sigma}_{X_0}} \cdot X, \qquad X \in C^0_c(T(M\backslash E_0 \times \mathbb{R})). \tag{5.16}$$

Then by (7.18) and Rellich's theorem there exists a subsequence (again denoted by i) such that

$$(H+P)_{\tilde{\Sigma}^i_{t_i}} \to (H+P)_{\tilde{\Sigma}_{X_0}} \qquad \text{in } L^2(T \cap B_R^{n+1}(X_0))). \tag{5.17}$$

Now the level-sets $\tilde{\Sigma}^i_{t_i} = \{U_i = t_i\}$ smoothly solve $(*)$ in $\Omega_i \times \mathbb{R}$, thus

$$(H+P)_{\tilde{\Sigma}^i_{t_i}} = |\bar{\nabla} U_i|,$$

and

$$\int_{\tilde{\Sigma}^i_{t_i}} |\bar{\nabla} U_i|^2 = \int_{\tilde{\Sigma}^i_{t_i}} (H+P)^2 \to \int_{\tilde{\Sigma}_{X_0}} (H+P)^2. \tag{5.18}$$

To proceed, we consider the special behaviour of the solution in the jump region. Let us first consider the case where the limit surface $\tilde{\Sigma}_{X_0}$ given by Lemma 5.3 is not a vertical cylinder. Then it is a graph, which means that $|\nabla \hat{u}_i|$ converges locally uniformly to something finite, and therefore that $|\nabla u_i| = \varepsilon_i |\nabla \hat{u}_i|$ converges locally uniformly to 0. In the other case the surface $\tilde{\Sigma}_{X_0}$ given by Lemma 5.3 is a vertical cylinder. We know from (7.18) and (7.21) that $|\bar{\nabla} U_{\bar{i}}|$ converges in L^2 to something finite. However this limit can only be zero since $U_i \to U$ locally uniformly, and U is constant in the jump region (namely $U = t_0$ on $\tilde{\mathcal{K}}_{t_0}$ by hypothesis). Furthermore, since the local uniform convergence of $U_i \to t_0$ holds for the entire sequence i, we must have L^2 convergence of the entire sequence $|\bar{\nabla} U_i|$, namely

$$\int_{\tilde{\Sigma}^i_{t_i}} |\bar{\nabla} U_i| \to 0.$$

□

Lemma 5.3 and Lemma 7.4 imply the following corollary.

Corollary 5.6 *Let $\tilde{\nu}$ denote the normal vector field to the $C^{1,\alpha}_{loc}$ hypersurface folation of $\tilde{\mathcal{K}}_{t_0}$, as constructed above. Then each surface $\tilde{\Sigma}_{X_0}$ in the jump region bounds a Caccioppoli set that minimises area plus bulk energy $(\bar{g}^{ij} - \tilde{\nu}^i \tilde{\nu}^j)\, K_{ij}$ in $\tilde{\mathcal{K}}_{t_0}$.*

To complete the proof of Theorem 5.1, it remains to show that each surface $\tilde{\Sigma}_{X_0}$ in $\tilde{\mathcal{K}}_{t_0}$ is in fact a smooth MOTS. To proceed, we use the well known connection between parametric and non-parametric variational problems, that follows from the relationship between a function $w \in BV_{\mathrm{loc}}(\Omega)$ and its subgraph

$$W = \{(x,t) \in \Omega \times \mathbb{R} : t < w(x)\}. \tag{5.19}$$

In particular, let φ_W denote the characteristic function of the subgraph (7.22). Then Theorem 14.6 in [Gi] states

$$\int_\Omega \sqrt{1+|\nabla w|^2} = \int_{\Omega\times\mathbb{R}} |\nabla \varphi_W|. \tag{5.20}$$

In (5.5) we established that at each point $X_0 \in \tilde{\mathcal{K}}_{t_0}$, there exists a subsequence i_j and a function $\hat{w} \in C^{1,\alpha}(\hat{B}_R(\hat{x}_0))$ such that

$$\hat{w}_{i_j} \to \hat{w} \qquad \text{in } C^1(\hat{B}_R(\hat{x}_0)),$$

where $\hat{B}_R(\hat{x}_0) := \hat{T} \cap B_R^{n+1}(\hat{x}_0)$, where

$$\mathrm{graph}(\hat{w}) = \hat{\Sigma}_{X_0} = \exp_q^{-1}\left(\tilde{\Sigma}_{X_0} \cap B_R^{M\times\mathbb{R}}(X_0)\right) \tag{5.21}$$

Then Corollary 5.6 establishes that each surface $\tilde{\Sigma}_{X_0}$ bounds a Caccioppoli set E in $\tilde{\mathcal{K}}_{t_0}$ that minimises area plus bulk energy $(\bar{g}^{ij} - \tilde{\nu}^i\tilde{\nu}^j)K_{ij}$ in $\tilde{\mathcal{K}}_{t_0}$, where by construction $\tilde{\nu}$ is the outward unit normal vector to the relative boundary $\partial\tilde{E} \cap \tilde{\mathcal{K}}_{t_0}$.

Therefore, writing the Caccioppoli set E locally as the subgraph of $w := \exp_q^{-1}(\hat{w})$, we find from (7.23) that w minimises the functional

$$\mathring{J}_{\tilde{\nu}}^{B_R(x_0)}(w) := \int\limits_{B_R(x_0)} \sqrt{1+|\bar{\nabla} w|^2}dx + \int\limits_{B_R(x_0)} \int_0^{w(x)} \mathrm{tr}_M K_{ij}(x,\tau) - K_{ij}(x,\tau)\tilde{\nu}^i(x,\tau)\tilde{\nu}^j(x,\tau)d\tau dx \tag{5.22}$$

in $B_R(x_0) := \exp_q(\hat{B}_R\hat{x}_0))$, whose Euler-Lagrange equation is the MOTS equation

$$\mathrm{div}\left(\frac{\bar{\nabla} w}{\sqrt{1+|\bar{\nabla} w|^2}}\right) + \left(\bar{g}^{ij} - \tilde{\nu}^i\tilde{\nu}^j\right)K_{ij} = 0, \tag{5.23}$$

and by construction $\tilde{\nu} = \dfrac{(\bar{\nabla} w, -1)}{\sqrt{1+|\bar{\nabla} w|^2}}$. The left hand side of (7.25) is an elliptic operator of the form

$$Aw = a^{ij}(\bar{\nabla} w)\left(\bar{\nabla}_i\bar{\nabla}_j w + \sqrt{1+|\bar{\nabla} w|^2}K_{ij}\right),$$

where

$$a^{ij}(p) := \frac{1}{\sqrt{1+|p|^2}}\left(\bar{g}^{ij} - \frac{p^i p^j}{1+|p|^2}\right).$$

Since $w \in C^{1,\alpha}(B_R(x_0))$, $a^{ij} \in C^{0,\alpha}(B_R(x_0))$, Aw is strictly elliptic on $B_R(x_0)$. Schauder theory then implies that $w \in C^{2,\alpha}(B_R(x_0))$, and by bootstrapping further we obtain $w \in C^{\infty}(B_R(x_0))$. Using a suitable partition of unity, we obtain that each surface $\tilde{\Sigma}_{X_0}$ is a smooth MOTS in $\tilde{\mathcal{K}}_{t_0}$.

Collecting the above results, we obtain a family of smooth MOTS foliating $\Omega_G \times \mathbb{R}$, and by extending the family of smooth, cylindrical MOTS in $\Omega_C \times \mathbb{R}$ to any missed points in Ω_C, we obtain a foliation of the entire interior region $\tilde{\mathcal{K}}_{t_0}$. At each point $X_0 = (x_0, t_0)$, the corresponding leaf of the foliation passing through X_0 is constructed by taking the limit of the Σ_t^{ε} locally around X_0, as in Lemma 5.3. This completes the proof of Theorem 5.1. □

6 Variational formulation of weak solutions

By freezing $|\nabla u| - \mathrm{tr}_{\Sigma_t} K$ and treating it as a bulk term, one may interpret $(**)$ as the Euler-Lagrange equation of the functional

$$\mathcal{J}_{u,\nu}(v) := \int |\nabla v| + v \left(|\nabla u| - \left(g^{ij} - \nu^i \nu^j \right) K_{ij} \right) dx. \tag{6.1}$$

For a smooth family of solutions of $(*)$, we will see below that the corresponding time-of-arrival function u defined by (3.1) satisfies

$$\mathcal{J}_{u,\nu}(u) \leq \mathcal{J}_{u,\nu}(v), \tag{6.2}$$

among competing locally Lipschitz functions v, that differ from u on a compact subset of $M \backslash \bar{E}_0$. The relationship between the variational formulation (6.2) and the functional (4.1) is then given by the following Lemma.

Lemma 6.1 *Let u be a locally Lipschitz function in the open set Ω, and ν a measurable vector field on $T\Omega$. Then u satisfies (6.2) on Ω if and only if for each t, $E_t := \{u < t\}$ minimizes (4.1) in Ω.*

Proof. This follows exactly as in [HI, Lemma 1.1], with $|\nabla u| - (g^{ij} - \nu^i \nu^j) K_{ij}$ replacing the bulk term $|\nabla u|$.

□

This equivalence between the two variational formulations also extends to the initial value problem

$$\begin{aligned} u \in & C^{0,1}_{\mathrm{loc}}(M), \ \nu \text{ a measurable vector field on } T(M \backslash E_0), \\ & E_0 = \{u < 0\}, \text{ and } u \text{ satisfies (6.2) in } M \backslash E_0. \end{aligned} \tag{6.3}$$

To see this equivalence, let E_t be a nested family of open sets in M, closed under ascending union, and define u as in the statement of Lemma 6.1 by the characterisation $E_t = \{u < t\}$. Then using Lemma 6.1 and approximating up to the boundary, we see that (6.3) is equivalent to

$$\begin{aligned} u \in C^{0,1}_{\mathrm{loc}} M, \ \nu & \text{ a measurable vector field on } T(M \backslash E_0) \\ \text{and } E_t & \text{ minimises } J_{u,\nu} \text{ in } M \backslash E_0 \text{ for each } t > 0. \end{aligned} \tag{6.4}$$

Lastly, by approximating $s \searrow t$, we see that (6.3) and (6.4) are equivalent to

$$\begin{aligned} &u \in C^{0,1}_{\mathrm{loc}}(M),\ \nu \text{ a measurable vector field on } T(M\backslash E_0) \\ &\text{and } \{u \le t\} \text{ minimises } J_{u,\nu} \text{ in } M\backslash E_0 \text{ for each } t \ge 0. \end{aligned} \tag{6.5}$$

We now present the precise definition of weak solutions to $(**)$. In the previous section we highlighted the need to define the normal vector field ν in jump regions in order to incorporate the $P = (g^{ij} - \nu^i\nu^j)K_{ij}$ term into a variational formulation of weak solutions to $(**)$. We showed that taking an appropriate limit of the smooth translating solutions $\tilde{\Sigma}^\varepsilon_t = \{U_\varepsilon = t\}$ of $(*)$, provides a constructive method of foliating the interior $\tilde{\mathcal{K}}_{t_0}$ of the jump region $\{U = t_0\} = \{u = t_0\} \times \mathbb{R}$, one dimension higher, by $C^{1,\alpha}_{\mathrm{loc}}$ hypersurfaces $\tilde{\Sigma}_{X_0}$ in $M \times \mathbb{R}$ with uniform $C^{0,\alpha}_{\mathrm{loc}}$ unit normal vector field $\tilde{\nu}$. Each such hypersurface $\tilde{\Sigma}_{X_0}$ in the foliation is either (part of) a vertical cylinder, or is a smooth graph over an open subset of $\tilde{\mathcal{K}}_{t_0}$, in the stack

$$\tilde{\Sigma} + \alpha\,\mathbf{e}_{n+2}, \quad \alpha \in \mathbb{R}, \tag{6.6}$$

of vertical translates of $\tilde{\Sigma}$. The normal vector field $\tilde{\nu}$ to each vertical cylinder is perpendicular to the z-direction, and could therefore be projected to M without loss of information. However, in the case of the graphical hypersurfaces (6.6), information would be lost if one were to define the vector field ν in (4.1) to be the projection of $\tilde{\nu}$ to TM.

This motivates the choice to formulate the weak solution to $(**)$ one dimension higher, in terms of a translation invariant function $U(x, z) = u(x) \in C^{0,1}_{\mathrm{loc}}(M \times \mathbb{R})$, and a translation invariant vector field $\bar{\nu} \in C^{0,\alpha}_{\mathrm{loc}}(T((M\backslash E_0) \times \mathbb{R}))$ that extends $\bar{\nabla} U/|\bar{\nabla} U|$ across the jump region. One then considers the analogously defined functionals $\mathcal{J}_{U,\bar{\nu}}$ to (6.1) and (4.1) for such pairs $(U, \bar{\nu})$ in $M \times \mathbb{R}$, and we remark that Lemma 6.1 and the initial value problem equivalences (6.3)-(6.5) hold in $M\times\mathbb{R}$ (for general U and $\bar{\nu}$ that are not necessarily translation invariant, like we will demand for the weak solution of $(**)$).

In Lemma 5.4 we showed that each of the surfaces $\tilde{\Sigma}_{X_0}$ foliating the jump region $\tilde{\mathcal{K}}_{t_0}$ bounds a Caccioppoli set that minimises $J_{U,\tilde{\nu}}$ in the jump region $\tilde{\mathcal{K}}_{t_0}$. Together with Lemma 6.1, this motivates the restriction in Definition 6.2 below that at each point $X \in (M\backslash \bar{E}_0) \times \mathbb{R}$, $\tilde{\nu}(X)$ be the normal vector to a $C^{1,\alpha}$ hypersurface that bounds a Caccioppoli set minimising $J_{U,\bar{\nu}}$ in $(M\backslash E_0) \times \mathbb{R}$.

Definition 6.2 Let $E_0 \subset M$ be a precompact, open set with C^2 boundary $\Sigma_0 = \partial E_0$. We call the pair $(U, \bar{\nu})$ a weak solution of $(**)$ with initial condition E_0 if $U \in C^{0,1}_{\mathrm{loc}}(M \times \mathbb{R})$ and $\bar{\nu} \in C^{0,\alpha}_{\mathrm{loc}}(T(M\backslash E_0) \times \mathbb{R})$ satisfy

(i) U is translation invariant in the vertical direction. In particular, there exists a locally Lipschitz function $u : M \to \mathbb{R}$ such that $U(x, z) = u(x)$ and u satisfies

$$\cdot\, u(x) \ge 0\ \forall\, x \in M\backslash E_0,$$
$$\cdot\, u\big|_{\partial E_0} = 0, \quad u(x) < 0 \quad \forall x \in E_0,$$
$$\cdot\, u(x) \to +\infty \text{ as } \mathrm{dist}(x, E_0) \to \infty.$$

(ii) The set $\tilde{E}_t = \{U < t\}$ minimises $J_{U,\bar{\nu}}$ in $(M\backslash E_0) \times \mathbb{R}$ for each $t > 0$. At jump times t_0, each point $X_0 = (x_0, z_0)$ in the interior $\tilde{\mathcal{K}}_{t_0}$ of the jump region $\{U = t_0\}$ lies in the boundary $\partial\tilde{E}_{X_0} \in C^{1,\alpha}_{\text{loc}}$ of a Caccioppoli set $\tilde{E}_{X_0}$ that minimises $J_{U,\bar{\nu}}$ in $\tilde{\mathcal{K}}_{t_0}$.

(iii) $\bar{\nu}$ is a translation invariant, unit vector field such that

$\cdot\ \bar{\nu}(X + \alpha\, \mathbf{e}_z) = \bar{\nu}(X) \quad \forall\, X \in (M\backslash E_0) \times \mathbb{R},\ \alpha \in \mathbb{R},$

$\cdot\ \bar{\nu}(X)$ is the normal vector to $\partial\tilde{E}_t$ at each point $X \in \partial\tilde{E}_t$,

$\cdot\ \bar{\nu}(X)$ is the normal vector to $\partial\tilde{E}_{X_0}$ at each point $X \in \partial\tilde{E}_{X_0}$,

at jump times t_0.

Remarks 1. Unlike in the weak formulation of inverse mean curvature flow, which asks only that $E_t = \{u < t\}$ minimise $J_{u,\nu}$ for each $t > 0$, we require the variational principle (4.2) for $J_{U,\bar{\nu}}$ to be satisfied *everywhere*, in particular in the interior of the jump region.
2. By Lemma 6.1, any weak solution $(U(x,z) := u(x), \nu)$ of $(**)$ satisfies (6.3) on $(M\backslash\bar{E}_0)\times\mathbb{R}$. Furthermore, we find that $(u, \nu_M := \bar{\nu}\big|_{TM})$ satisfies (6.2) in $M\backslash\bar{E}_0$.

Lemma 6.3 *Let $(U(x,z) := u(x), \bar{\nu})$ be a weak solution of $(**)$ with initial condition E_0. Then the pair $(u, , \nu_M)$ satisfies (6.2) on $M\backslash\bar{E}_0$, and $E_t = \{u < t\}$ minimises J_{u,ν_M} for each $t > 0$, where $\nu_M := \bar{\nu}\big|_{TM}$.*

Proof. Since the tensor K is extended trivially in the z-direction, we find

$$\left(\bar{g}^{ij} - \bar{\nu}^i\bar{\nu}^j\right) K_{ij} = \left(g^{ij} - \nu^i_M\nu^j_M\right) K_{ij}, \tag{6.7}$$

where $\nu_M := \bar{\nu}\big|_{TM}$. Let $B_{u,\nu_M} := |\nabla u| - \left(g^{ij} - \nu^i_M\nu^j_M\right) K_{ij}$ denote the bulk term of $\mathcal{J}_{u,\nu_M}$. Let v be a locally Lipschitz function such that $\{v \neq u\} \subset A \subset\subset M\backslash\bar{E}_0$. Let $\phi(z)$ be a cutoff function such that:

$$|\phi_z| \leq 2, \quad \phi = 1 \text{ on } [0,s] \text{ and } \phi = 0 \text{ on } \mathbb{R}\backslash(-1, s+1).$$

Then $V(x,z) := \phi(z)v(x) + (1-\phi(z))u(x)$ is an appropriate comparison function for U, and letting $\tilde{A} := A \times [-1, s+1]$, we obtain from (6.2)

$$\begin{aligned}
\int_{\tilde{A}} |\nabla u| + u\, B_{u,\nu_M} &= \int_{\tilde{A}} |\bar{\nabla} U| + U\left(|\bar{\nabla} U| - \left(\bar{g}^{ij} - \bar{\nu}^i\bar{\nu}^j\right) K_{ij}\right) \\
&\leq \int_{\tilde{A}} |\bar{\nabla} V| + V\left(|\bar{\nabla} U| - \left(\bar{g}^{ij} - \bar{\nu}^i\bar{\nu}^j\right) K_{ij}\right) \\
&\leq \int_{\tilde{A}} \phi\left(|\nabla v| + v\, B_{u,\nu_M}\right) + (1-\phi)\left(|\nabla u| + u\, B_{u,\nu_M}\right) \\
&\quad + |\phi_z|||v - u|.
\end{aligned}$$

This implies

$$
\begin{aligned}
s \cdot \mathcal{J}^A_{u,\nu_M}(u) &= s \int_A |\nabla u| + u\, B_{u,\nu_M} \\
&\leq \int_{\tilde{A}} \phi\,(|\nabla u| + u\, B_{u,\nu_M}) \\
&\leq \int_{\tilde{A}} \phi\,(|\nabla v| + v\, B_{u,\nu_M}) + |\phi_z||v-u| \\
&\leq (s+2) \int_A |\nabla v| + v\, B_{u,\nu_M} + \int_{A\times([-1,0]\subset[1,2])} |\phi_z||v-u| \\
&\leq (s+2)\mathcal{J}^A_{u,\nu_M}(v) + 4\int_A |v-u|.
\end{aligned}
$$

Dividing by s and passing $s \to \infty$ proves that the pair (u, ν_M) satisfies (6.2). Lemma 6.1 then implies that the sets $E_t := \{u < t\}$ minimise J_{u,ν_M} for each $t > 0$. □

We now state some further properties of weak solutions of $(**)$. We begin by showing that smooth solutions of the flow $(*)$ are weak solutions in the domain they foliate. This follows as in [HI, Lemma 2.3].

Smooth Flow Lemma 6.4 *Let $(\Sigma_t)_{a\leq t<b}$ be a smooth solution of $(*)$ on M. Let $U = t$ on $\Sigma_t \times \mathbb{R}$, $U < a$ in the region bounded by $\Sigma_a \times \mathbb{R}$, and $\tilde{E}_t := \{U < t\}$. Then $\tilde{E}_t$ minimises $\mathcal{J}_{U,\bar{\nu}}$ in $\tilde{E}_b \backslash \tilde{E}_a$ for $a \leq t < b$, where $\bar{\nu}$ is the smooth normal to the vertical cylinder $\Sigma_t \times \mathbb{R}$, given by $\bar{\nu} = (\nu_{\Sigma_t}, 0) = \dfrac{\bar{\nabla} U}{|\bar{\nabla} U|}$.*

Proof. We use the smooth normal $\tilde{\nu} = \dfrac{\bar{\nabla} U}{|\bar{\nabla} U|}$ as a calibration and apply the divergence theorem to relate $\mathcal{J}_{U,\bar{\nu}}(\tilde{E}_t)$ to $\mathcal{J}_{U,\bar{\nu}}(\tilde{F})$ for a competing set $\tilde{F}$ of finite perimeter with $\tilde{F}\Delta\tilde{E}_t \subset\subset \tilde{\Omega}$. Let $B_{U,\bar{\nu}} := |\bar{\nabla} U| - (\bar{g}^{ij} - \bar{\nu}^i\bar{\nu}^j)K_{ij}$ denote the bulk energy term in $\mathcal{J}_{U,\bar{\nu}}$.

$$
\begin{aligned}
\mathcal{J}_{U,\bar{\nu}}(\tilde{E}_t) =& |\partial\tilde{E}_t| - \int_{\tilde{E}_t} B_{U,\bar{\nu}}\, dx = \int_{\partial\tilde{E}_t} \nu_{\partial\tilde{E}_t} \cdot \bar{\nu} d\mathcal{H}^{n-1} - \int_{E_t} B_{U,\bar{\nu}}\, dx \\
=& \int_{\partial\tilde{E}_t \cap \bar{\tilde{F}}} \nu_{\partial_{\tilde{E}_t}} \cdot \bar{\nu} d\mathcal{H}^{n-1} + \int_{\partial\tilde{E}_t\backslash\tilde{F}} \nu_{\partial\tilde{E}_t} \cdot \bar{\nu} d\mathcal{H}^{n-1} - \int_{\tilde{E}_t} B_{U,\bar{\nu}}\, dx \\
=& \int_{\partial^* F \cap \bar{\tilde{E}}_t} \nu_{\partial^*\tilde{F}} \cdot \bar{\nu} d\mathcal{H}^{n-1} + \int_{\tilde{E}_t\backslash\tilde{F}} B_{U,\bar{\nu}}\, dx - \int_{\tilde{E}_t} B_{U,\bar{\nu}}\, dx \\
&+ \int_{\partial^*\tilde{F}\backslash\tilde{E}_t} \nu_{\partial^*\tilde{F}} \cdot \bar{\nu} d\mathcal{H}^{n-1} - \int_{\tilde{F}\backslash\tilde{E}_t} B_{U,\bar{\nu}}\, dx \\
=& \int_{\partial^*\tilde{F}} \nu_{\partial^*\tilde{F}} \cdot \bar{\nu} d\mathcal{H}^{n-1} - \int_{\tilde{F}} B_{U,\bar{\nu}} dx \leq |\partial^*\tilde{F}| - \int_{\tilde{F}} B_{U,\bar{\nu}}\, dx = \mathcal{J}_{U,\bar{\nu}}(\tilde{F}).
\end{aligned}
$$

□

Weak Mean Curvature. In view of the local $C^{1,\alpha}$ estimates given by Regularity Theorem 4.3, we can consider the weak mean curvature of the surfaces $\tilde{\Sigma}_t = \partial\{U < t\}$.

Let $\tilde{N}$ be a C^1 hypersurface in $M \times \mathbb{R}$. Then a locally integrable function H on $\tilde{N}$ is called the weak mean curvature provided

$$\int_{\tilde{N}} \operatorname{div}_{\tilde{N}} X d\mu = \int_{\tilde{N}} H\nu \cdot X d\mu, \qquad \forall X \in C_c^\infty(T(M \times \mathbb{R})). \tag{6.8}$$

Lemma 6.5 *Let $\tilde{E}_t := \{U < t\}$ minimise $J_{U,\bar{\nu}}$ in $\tilde{A} := \tilde{E}_b \backslash \tilde{E}_a$, for $U \in C^{0,1}_{loc}(\tilde{A})$. Then the surfaces $\tilde{\Sigma}_t = \partial\tilde{E}_t$ have weak mean curvature H satisfying $H = |\bar{\nabla}U| - P$ for a.e. $x \in \tilde{\Sigma}_t$ and a.e. $t \in (a,b)$, where $P = (\bar{g}^{ij} - \bar{\nu}^i\bar{\nu}^j)K_{ij}$.*

Proof. Let X be a compactly supported vector field defined on M, and $(\Phi_s)_{-\varepsilon<s<\varepsilon}$ the flow of diffeomorphisms generated by X with $\Phi_0 = id_M$. For minimisers of $\mathcal{J}_{U,\nu}$, we use the area formula, the dominated convergence theorem and the co-area formula in the form

$$\int_{\mathbb{R}^{n+2}} |\bar{\nabla} f| dx = \int_{-\infty}^{\infty} \int_{\{f=t\}} dt$$

to obtain

$$\begin{aligned}
0 &= \frac{d}{ds}\bigg|_{s=0} \mathcal{J}_{U,\bar{\nu}}(U \circ \Phi_s^{-1}) \\
&= \frac{d}{ds}\bigg|_{s=0} \left(\int_{\tilde{W}} |\nabla(U \circ \Phi_s^{-1})| + (U \circ \Phi_s^{-1}) \left(|\bar{\nabla}U| - (\bar{g}^{ij} - \bar{\nu}^i\bar{\nu}^j)K_{ij}\right) dx \right) \\
&= \frac{d}{ds}\bigg|_{s=0} \left(\int_{-\infty}^{\infty} \int_{\tilde{\Sigma}_t \cap \tilde{W}} |\det d\Phi_s(x)| d\mathcal{H}^n(x) dt \right) \\
&\quad - \int_{\tilde{W}} \bar{\nabla}U \cdot X \left(|\bar{\nabla}U| - (\bar{g}^{ij} - \bar{\nu}^i\bar{\nu}^j)K_{ij}\right) dx \\
&= \int_{-\infty}^{\infty} \int_{\tilde{\Sigma}_t \cap \tilde{W}} \operatorname{div}_{\tilde{\Sigma}_t} X d\mathcal{H}^n dt - \int_{\tilde{W}} \bar{\nu} \cdot X |\bar{\nabla}U| \left(|\bar{\nabla}U| - (\bar{g}^{ij} - \bar{\nu}^i\bar{\nu}^j)K_{ij}\right) dx,
\end{aligned}$$

since $\bar{\nu} = \dfrac{\bar{\nabla}U}{|\bar{\nabla}U|}$ when $\bar{\nabla}U \neq 0$, and $\bar{\nu}|\bar{\nabla}U| = 0$ when $\bar{\nabla}U = 0$. Then by the co-area formula, we obtain

$$0 = \int_{-\infty}^{\infty} \int_{\tilde{\Sigma}_t \cap \tilde{W}} \left(\operatorname{div}_{\tilde{\Sigma}_t} X + (P - |\bar{\nabla}U|)\bar{\nu}\right) \cdot X d\mathcal{H}^{n+1} dt.$$

Lebesgue differentiation and comparison with (6.8) yields the result. □

Exactly as in the proof of [HI, Theorem 2.1], we also obtain the following compactness theorem for the time-of-arrival function.

Compactness Property 6.6 *Let $U_i \in C^{0,1}_{loc}(\tilde{\Omega}_i)$ and $\bar{\nu}_i \in C^{0,\alpha}_{loc}(T\tilde{\Omega}_i)$ be a sequence of solutions of (6.2) on open sets $\tilde{\Omega}_i \subset M \times \mathbb{R}$, such that*

$$\tilde{\Omega}_i \to \tilde{\Omega}, \quad U_i \to U, \quad \bar{\nu}_i \to \bar{\nu}, \tag{6.9}$$

locally uniformly, and such that for each $\tilde{A} \subset\subset \tilde{\Omega}$, $\sup_{\tilde{A}} |\bar{\nabla} U_i| \leq C(\tilde{A})$, for large i, where $C(\tilde{A})$ is independent of i. Then $(U, \bar{\nu})$ solves (6.2) on $\tilde{\Omega}$. In the special case where $(U_i, \bar{\nu}_i)$ is a sequence of weak solutions of $(**)$ satisfying Definition 6.2, then the limit $(U, \bar{\nu})$ is a weak solution of $(**)$.

7 Geometric characterisation of jump regions

In this section we introduce the concept of outward optimisation in order to give a geometric characterisation of the criterion selecting jump times. Since weak solutions $(U(x,z) = u(x), \bar{\nu})$ of $(**)$ are translation invariant and the level sets of U are vertical cylinders, this characterisation follows from the parametric variational formulation (4.2) for $(u, \nu_M := \bar{\nu}|_{TM})$.

Let Ω be an open set in M. Then we call a set E *outward optimising* (*in* Ω) *with respect to* ν, if E minimises 'area plus bulk energy P' on the outside in Ω. That is, if

$$|\partial^* E \cap A| \leq |\partial^* F \cap A| + \int_{F \setminus E} (g^{ij} - \nu^i \nu^j) K_{ij}, \tag{7.1}$$

for any F containing E such that $F \setminus E \subset\subset \Omega$, and any compact set A containing $F \setminus E$. Here ν is a measurable vector field on $F \setminus E$. The set E is then called *strictly outward optimising* (*in* Ω) if equality in (7.1) implies that $F \cap \Omega = E \cap \Omega$ a.e.

We use this concept to define the strictly outward optimising hull of a measurable set $E \subset \Omega$. Specifically, we define $E' = E'_\Omega$ to be the intersection of the Lebesgue points of all the strictly outward optimising sets in Ω that contain E. We call E' the *strictly outward optimising hull of* E (*in* Ω). Up to a set of measure zero, E' may be realised by a countable intersection, so E' is strictly outward optimising, and open.

We then obtain the following interpretation of the variational formulation.

Outward Optimising Lemma 7.1 *Suppose that* $(U(x,z) := u(x), \bar{\nu})$ *is a weak solution of* $(**)$ *with initial condition* E_0*, and that* M *has no compact components. Then:*

(*i*) *For* $t > 0$*,* E_t *is outward optimising in* M *with respect to* $\nu_M := \bar{\nu}\big|_{TM}$.

(*ii*) *For* $t \geq 0$*,* E_t^+ *is strictly outward optimising in* M *with respect to* ν_M.

(*iii*) *For* $t \geq 0$*,* $E'_t = E_t^+$*, provided* E_t^+ *is precompact.*

(*iv*) *For* $t > 0$*,* $|\partial E_t| = |\partial E_t^+| + \int_{E_t^+ \setminus E_t} (g^{ij} - \nu_M^i \nu_M^j) K_{ij}$*, provided that* E_t^+ *is precompact. This extends to* $t = 0$ *precisely if* E_0 *is outward optimising.*

Furthermore, for general $(U, \bar{\nu})$ satisfying (6.3) in $M \times \mathbb{R}$, the analogous statements hold on compact sets $\tilde{\Omega} \subset M \times \mathbb{R}$ with

$$|\partial^* \tilde{E} \cap \tilde{A}| \leq |\partial^* \tilde{F} \cap \tilde{A}| + \int_{\tilde{F}\setminus\tilde{E}} (\bar{g}^{ij} - \bar{\nu}^i \bar{\nu}^j) K_{ij}, \tag{7.2}$$

replacing (7.1) in the definition of outward optimising.

To prove Outward Optimising Property 7.1, we will need the following Lemma.

Lemma 7.2 *Let $(U, \bar{\nu})$ satisfy (6.2) on $\tilde{\Omega}$. Then U has no strict local maxima or minima on $\tilde{\Omega}$.*

Proof. First assume that U possesses a strict local maximum so that there is a connected, precompact component $\tilde{E}$ of $\{U > t\}$ for some t. Define the Lipschitz function V_k by

$$V_k := \begin{cases} k & \text{on } \hat{E}_k := \tilde{E}_k \cap \tilde{E}, \\ U & \text{on } \tilde{\Omega}\setminus\hat{E}_k, \end{cases} \tag{7.3}$$

for $0 < k < \sup_{\tilde{E}} U$ and $\tilde{E}_k := \{U > k\}$. Then (6.2) and Hölder's inequality yield

$$\int_{\hat{E}_k} |\bar{\nabla} U|(1 + U - k) \leq \int_{\hat{E}_k} (U - k) C_0 \leq C_0 \left(\int_{\hat{E}_k} (U - k)^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} |\hat{E}_k|^{\frac{1}{n}}, \tag{7.4}$$

where $C_0 = (n + 1)|\lambda|$ and $|\lambda|$ is the size of the largest eigenvalue of K on $\tilde{\Omega}$. Then using the Sobolev inequality on the left hand side we obtain

$$\int_{\hat{E}_k} |\bar{\nabla} U|(1 + U - k) \geq \int_{\hat{E}_k} |\bar{\nabla} U| = \int_{\hat{E}_k} |\bar{\nabla}(U - k)| \geq \left(\int_{\hat{E}_k} (U - k)^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}. \tag{7.5}$$

Combining (7.4) and (7.5) we find $1 \leq C_0 |\hat{E}_k|^{1/n}$, which leads to a contradiction since $|\hat{E}_k|$ can be made arbitrarily small by choosing k close to $\sup_{\tilde{E}} U$.

Now assume that U possesses a strict local minimum and let $\tilde{E}$ be a connected, precompact component of $\{U < t\}$ for some t, and again consider the function V_k defined by (7.3), where this time $k > \inf_{\tilde{E}} U$ and $\tilde{E}_k := \{U < k\}$. Then as above, (6.2) and Hölder's inequality yield

$$\int_{\tilde{E}_k} |\bar{\nabla} U|(1 + U - k) \leq C_0 \left(\int_{\tilde{E}_k} (U - k)^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} |\tilde{E}_k|^{\frac{1}{n}}, \tag{7.6}$$

and by restricting to k small enough that $1 + U - k \geq \frac{1}{2}$ on $\tilde{E}_k$, we obtain

$$\int_{\tilde{E}_k} |\bar{\nabla} U|(1 + U - k) \geq \frac{1}{2} \left(\int_{\tilde{E}_k} (U - k)^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}. \tag{7.7}$$

Combining (7.6) and (7.7) we find $1/2 \leq C_0|\tilde{E}_k|^{1/n}$, which leads to a contradiction since $|\tilde{E}_k|$ can be made arbitrarily small by choosing k close to $\inf_E U$.
In the case where $(U(x,z) := u(x), \bar{\nu})$ is a weak solution of $(**)$, repeating the above calculation for u on M, using (6.3), yields the desired result. □

Proof of Outward Optimising Property 7.1: Refer to [HI, Minimising Hull Property 1.4].
(i) This follows immediately from Lemma 6.3.
(ii) From (6.5) we obtain for suitable A

$$|\partial^* E_t^+ \cap A| \leq |\partial^* F \cap A| + \int\limits_{F\setminus E_t^+} (g^{ij} - \nu_M^i \nu_M^j) K_{ij} - |\nabla u| dx, \tag{7.8}$$

for any $t \geq 0$, any F with $F \Delta E_t^+ \subset\subset M \setminus E_t^+$, proving that E_t^+ is outward optimising.
To prove strictly minimising, suppose F contains E_t^+ and

$$|\partial E_t^+ \cap A| - |\partial^* F \cap A| = \int_{F\setminus E_t^+} (g^{ij} - \nu_M^i \nu_M^j) K_{ij}.$$

Then by (7.8), $\nabla u = 0$ a.e. on $F \setminus E_t^+$. Since F is also outward optimising, and the Lebesgue points of an outward optimising set form an open set in M, by a measure zero modification we may assume F is open. Then u is constant on each connected component of the open set $F \setminus \{u \leq t\}$. Since M has no compact components, Lemma 7.2 (i) means that no connected component of F can have closure disjoint from $\bar{E}_t^+$, therefore $u = t$ on $F \setminus E_t$ and $F \subseteq E_t^+$. This proves that E_t^+ is strictly outward optimising.
(iii) It is clear from part (ii) and the definition of E_t' that $E_t' \subseteq E_t^+$. Assume E_t^+ is precompact. Then if

$$|E_t' \cap A| = |E_t^+ \cap A| + \int_{E_t^+ \setminus E_t'} (g^{ij} - \nu_M^i \nu_M^j) K_{ij},$$

strict outward optimisation implies that $E_t' = E_t^+$. Otherwise

$$|\partial E_t' \cap A| < |\partial E_t^+ \cap A| + \int_{E_t^+ \setminus E_t'} (g^{ij} - \nu_M^i \nu_M^j) K_{ij},$$

contradicting (7.8).
(iv) In view of (i), we can use E_t^+ as a competitor to obtain

$$|\partial E_t \cap A| \leq |\partial E_t^+ \cap A| + \int\limits_{E_t^+ \setminus E_t} (g^{ij} - \nu_M^i \nu_M^j) K_{ij} dx, \tag{7.9}$$

for $t > 0$, and for $t = 0$ if E_0 happens to be outward optimising itself. Since E_t^+ is precompact, strict inequality in (7.9) would contradict (iii), implying equality in (7.9), which proves (iv).
The proof for general $(U, \bar{\nu})$ satisfying (6.3) in $\tilde{\Omega} \subset M \times \mathbb{R}$ follows exactly as above. □

Outward Optimising Lemma 7.1 implies that ∂E_t satisfies the obstacle problem minimising "area plus bulk energy P", with E_t as the obstacle. This leads to a heuristic interpretation of the minimisation principle (6.2). Namely, as long as E_t remains strictly outward optimising, it evolves by inverse null mean curvature, and when this condition is violated, E_t jumps to E_t' and continues. This implies that the null mean curvature is nonnegative on the weak solution after time zero. Furthermore, part (*iv*) of Lemma 7.1 implies that the monotonicity property

$$\frac{d}{dt}|\Sigma_t| + \int\limits_{E_t \setminus E_0} P = |\Sigma_t| \tag{7.10}$$

derived in Lemma 2.1, also holds in the weak setting, as long as Σ_t remains compact.

The outward optimising property also implies a stronger result for the surfaces foliating the jump region in Proposition 9.6, namely we see that each $\tilde{\Sigma}_{X_0}$ is a smooth MOTS in $\tilde{\mathcal{K}}_{t_0}$.

Proposition 7.3 *Each surface $\tilde{\Sigma}_{X_0}$ from Proposition 4.1 in the foliation of the interior $\tilde{\mathcal{K}}_{t_0}$ of the jump region in $M \times \mathbb{R}$ is a smooth MOTS.*

To prove Proposition 7.3, we require the following Lemma.

Lemma 7.4

$$|\bar{\nabla} U_i| \to 0 \quad in\ L^1_{loc}(\tilde{\mathcal{K}}_{t_0}). \tag{7.11}$$

Proof. Recall d defined by (5.2), consider a target point $X_0 = (x_0, z_0)$ such that $z_0 > 2d+1$ and select a cutoff function $\phi \in C^2_c(\mathbb{R})$ such that $\phi \geq 0$ and $\operatorname{spt}\phi \subseteq [z_0 - 2d, z_0 + 2d]$. Then let $T_0 = z_0 - 2d - 1$, fix an arbitrary time $T > T_0$, and consider $T_0 \leq t \leq T$ and $L \geq T + 3 + z_0 + 2d$.

We wish to show that

$$\liminf_{i\to\infty} \int_{\tilde{\Sigma}^i_{t_i} \cap B^{M\times\mathbb{R}}_d(X_0)} |D(H+P)|^2 < \infty.$$

To this end, we calculate

$$\begin{aligned}
&\frac{d}{dt}\int_{\tilde{\Sigma}_t^\varepsilon}\phi(z)(H+P)^2 \qquad (7.12)\\
=&\int_{\tilde{\Sigma}_t^\varepsilon}2\phi(H+P)\frac{\partial}{\partial t}(H+P)+(H+P)^2\frac{\partial\phi}{\partial z}\cdot\frac{\nu_\varepsilon}{H+P}+\phi H(H+P)\\
=&-2\int_{\tilde{\Sigma}_t^\varepsilon}\phi\Bigg((H+P)\Delta\left(\frac{1}{H+P}\right)+|A|^2+\bar{Ric}(\nu_\varepsilon,\nu_\varepsilon)-\bar\nabla_{\nu_\varepsilon}P\\
&+\frac{2D_i(H+P)}{H+P}K_{i\nu_\varepsilon}\Bigg)+(H+P)\frac{\partial\phi}{\partial z}\cdot\nu_\varepsilon+\phi H(H+P)\\
=&\int_{\tilde{\Sigma}_t^\varepsilon}\phi\Bigg(-2\frac{|D(H+P)|^2}{(H+P)^2}-2|A|^2-2\bar{Ric}(\nu_\varepsilon,\nu_\varepsilon)+H(H+P)\\
&+2\bar\nabla_{\nu_\varepsilon}P-4\frac{D_i(H+P)}{H+P}K_{i\nu_\varepsilon}\Bigg)-2\frac{\phi}{\partial z}\cdot\frac{D(H+P)}{H+P}+(H+P)\frac{\partial\phi}{\partial z}\cdot\nu_\varepsilon
\end{aligned}$$

In view of the sup estimates (3.8) and (3.16) for u_ε, there is $R(T)>0$ depending only on the subsolution v and K_{ij} such that

$$\tilde{\Sigma}_t^\varepsilon\cap(M\times\mathrm{spt}\phi)\subseteq S(T):=(B_{R(T)}\backslash E_0)\times[z_0-2d,z_0+2d],\quad T_0\le t\le T.$$

The Outward Optimising Property 7.1, applied to $\tilde{E}_t^\varepsilon$ compared to the perturbation $\tilde{E}_t^\varepsilon\cup S(T)$, then provides the area estimate

$$|\tilde{\Sigma}_t^\varepsilon\cap(M\times\mathrm{spt}\phi)|\le C(T)+\int_{S(T)\backslash\tilde{E}_t^\varepsilon}P\le C(T,\|K\|_{C^0}),\quad T_0\le t\le T. \qquad (7.13)$$

Together with the interior estimate (2.5), and the boundary gradient estimates for u^ε, this shows

$$|H+P|\le C(T,\|K\|_{C^1})\qquad \text{on } \tilde{\Sigma}_t^\varepsilon\cap(M\times\mathrm{spt}\phi),\quad T_0\le t\le T.$$

It follows that

$$\int_{\tilde{\Sigma}_t^\varepsilon}\phi|H|(H+P)+\phi(H+P)^2+|(H+P)\bar\nabla\phi\cdot\nu_\varepsilon|\le C(T,\|K\|_{C^1}),\quad T_0\le t\le T.$$

We estimate the $D\phi$ and $K_{i\nu_\varepsilon}$ terms via

$$\begin{aligned}
\left|2D\phi\cdot\frac{D(H+P)}{H+P}\right|&\le 2\frac{|D\phi|^2}{\phi}+\frac{\phi}{2}\frac{|D(H+P)|^2}{(H+P)^2}\le C+\frac{\phi}{2}\frac{|D(H+P)|^2}{(H+P)^2},\\
\left|4\phi\frac{D_i(H+P)}{H+P}K_{i\nu_\varepsilon}\right|&\le 8\phi\|K\|_{C^0}^2+\frac{\phi}{2}\frac{|D(H+P)|^2}{(H+P)^2}.
\end{aligned}$$

Thus (7.12) becomes

$$\frac{d}{dt}\int_{\tilde{\Sigma}_t^\varepsilon}\phi(H+P)^2 \le \int_{\tilde{\Sigma}_t^\varepsilon} -\phi\frac{|D(H+P)|^2}{(H+P)^2} + C(T,\|K\|_{C^1}), \tag{7.14}$$

and integrating gives

$$\int_{T_0}^{T}\int_{\tilde{\Sigma}_t^\varepsilon\cap(M\times[z_0-2d,z_0+2d])}\frac{|D(H+P)|^2}{(H+P)^2} \le C(T,\|K\|_{C^1}), \tag{7.15}$$

using a ϕ such that $\phi = 1$ on $[z_0-2d, z_0+2d]$.
Applying Fatou's Lemma, for any sequence $\varepsilon_i \to 0$ we obtain

$$\liminf_{i\to\infty}\int_{\tilde{\Sigma}_t^i\cap(M\times[z_0-2d,z_0+2d])}\frac{|D(H+P)|^2}{(H+P)^2} < \infty, \qquad \text{a.e. } t \ge T_0. \tag{7.16}$$

Now consider the subsequence $\varepsilon_{i_j} \to 0$ from (5.5) such that $\tilde{\Sigma}^{i_j}_{t_{i_j}} \to \tilde{\Sigma}_0$ in $C^1(T\cap B_R^{n+1}(X_0))$, where $T = T_{X_0}\tilde{\Sigma}_{X_0}$. We write $i = i_j$ henceforth. Since (7.16) only holds for a.e. $t \ge T_0$, it will take more work to argue that $\liminf_{i\to\infty}\int_{\tilde{\Sigma}^i_{t_i}\cap B_d^{M\times\mathbb{R}}(X_0)}|D(H+P)|^2 < \infty$. To this end, we pick a sequence $\hat{t}_i$ such that $\hat{t}_i \to t_0+\delta$ for some $|\delta| > 0$, $|\hat{t}_i - t_i| \le \varepsilon_i d$ and

$$\liminf_{i\to\infty}\int_{\tilde{\Sigma}^i_{\hat{t}_i}\cap(M\times[z_0-2d,z_0+2d])}|D(H+P)|^2 < \infty. \tag{7.17}$$

Define $\hat{z}_i := \dfrac{u_i(x_0)}{\varepsilon_i} - \dfrac{\hat{t}_i}{\varepsilon_i}$ and $\delta_i := \hat{z}_i - z_0$. Then the fact that $\tilde{\Sigma}^i_{\hat{t}_i}$ is just a translation of $\tilde{\Sigma}^i_{t_i}$ by δ_i in the z-direction implies that

$$\int_{\tilde{\Sigma}^i_{t_i}\cap B_d^{M\times\mathbb{R}}(X_0)}|D(H+P)|^2 = \int_{\tilde{\Sigma}^i_{\hat{t}_i}\cap B_d^{M\times\mathbb{R}}(x_0,z_0+\delta_i)}|D(H+P)|^2,$$

for each i. Furthermore, the condition $|\hat{t}_i - t_i| \le \varepsilon_i d$ implies that $|\delta_i| = |\hat{z}_i - z_0| \le d$, which ensures that $\tilde{\Sigma}^i_{\hat{t}_i}\cap B_d^{M\times\mathbb{R}}(x_0, z_0+\delta_i) \subset M\times[z_0-2d, z_0+2d]$, and thus from (7.17) we obtain that

$$\liminf_{i\to\infty}\int_{\tilde{\Sigma}^i_{\hat{t}_i}\cap B_d^{M\times\mathbb{R}}(x_0,z_0+\delta_i)}|D(H+P)|^2 \le \int_{\tilde{\Sigma}^i_{\hat{t}_i}\cap(M\times[z_0-2d,z_0+2d])}|D(H+P)|^2 < \infty,$$

from which our desired estimate follows

$$\liminf_{i\to\infty}\int_{\tilde{\Sigma}^i_{t_i}\cap B_d^{M\times\mathbb{R}}(X_0)}|D(H+P)|^2 < \infty. \tag{7.18}$$

As in the proof of Lemma (5.3), the converging surfaces $\tilde{\Sigma}^i_{t_i}$ can be written locally, via the exponential map, as graphs of $C^{1,\alpha}_{\mathrm{loc}}$ functions w_i over the hyperplane T. This local $C^{1,\alpha}$ convergence of the hypersurfaces, together with the first variation of area formula and the Riesz Representation Theorem then implies that $H_{\tilde{\Sigma}_{X_0}}$ exists weakly as a locally L^1 function, with the weak convergence

$$\int_{\tilde{\Sigma}^i_{t_i}} H_{\tilde{\Sigma}^i_{t_i}} \nu_{\tilde{\Sigma}^i_{t_i}} \cdot X \to \int_{\tilde{\Sigma}_{X_0}} H_{\tilde{\Sigma}_{X_0}} \nu_{\tilde{\Sigma}_{X_0}} \cdot X, \qquad X \in C^0_c(T(M\backslash E_0 \times \mathbb{R})). \tag{7.19}$$

Then by (7.18) and Rellich's theorem there exists a subsequence (again denoted by i) such that

$$(H+P)_{\tilde{\Sigma}^i_{t_i}} \to (H+P)_{\tilde{\Sigma}_{X_0}} \qquad \text{in } L^2(T \cap B^{n+1}_R(X_0))). \tag{7.20}$$

Now the level-sets $\tilde{\Sigma}^i_{t_i} = \{U_i = t_i\}$ smoothly solve $(*)$ in $\Omega_i \times \mathbb{R}$, thus

$$(H+P)_{\tilde{\Sigma}^i_{t_i}} = |\bar{\nabla} U_i|,$$

and

$$\int_{\tilde{\Sigma}^i_{t_i}} |\bar{\nabla} U_i|^2 = \int_{\tilde{\Sigma}^i_{t_i}} (H+P)^2 \to \int_{\tilde{\Sigma}_{X_0}} (H+P)^2. \tag{7.21}$$

To proceed, we consider the special behaviour of the solution in the jump region. Let us first consider the case where the limit surface $\tilde{\Sigma}_{X_0}$ given by Lemma 5.3 is not a vertical cylinder. Then it is a graph, which means that $|\nabla \hat{u}_i|$ converges locally uniformly to something finite, and therefore that $|\nabla u_i| = \varepsilon_i |\nabla \hat{u}_i|$ converges locally uniformly to 0. In the other case the surface $\tilde{\Sigma}_{X_0}$ given by Lemma 5.3 is a vertical cylinder. We know from (7.18) and (7.21) that $|\bar{\nabla} U_{\bar{i}}|$ converges in L^2 to something finite. However this limit can only be zero since $U_i \to U$ locally uniformly, and U is constant in the jump region (namely $U = t_0$ on $\tilde{\mathcal{K}}_{t_0}$ by hypothesis). Furthermore, since the local uniform convergence of $U_i \to t_0$ holds for the entire sequence i, we must have L^2 convergence of the entire sequence $|\bar{\nabla} U_i|$, which implies

$$\int_{\tilde{\Sigma}^i_{t_i}} |\bar{\nabla} U_i| \to 0.$$

□

Proof of Proposition 7.3: Proposition 4.1 and Lemma 7.4 imply that each surface $\tilde{\Sigma}_{X_0}$ in the jump region bounds a Caccioppoli set that minimises area plus bulk energy $(\bar{g}^{ij} - \tilde{\nu}^i \tilde{\nu}^j)\, K_{ij}$ in $\tilde{\mathcal{K}}_{t_0}$. To complete the proof of Proposition 4.1, it remains to show that each surface $\tilde{\Sigma}_{X_0}$ in $\tilde{\mathcal{K}}_{t_0}$ is in fact a smooth MOTS. To proceed, we use the connection between parametric and non-parametric variational problems, that follows from the relationship between a function $w \in BV_{\mathrm{loc}}(\Omega)$ and its subgraph

$$W = \{(x,t) \in \Omega \times \mathbb{R} : t < w(x)\}. \tag{7.22}$$

In particular, let φ_W denote the characteristic function of the subgraph (7.22). Then Theorem 14.6 in [Gi] states

$$\int_\Omega \sqrt{1+|\nabla w|^2} = \int_{\Omega\times\mathbb{R}} |\nabla \varphi_W|. \tag{7.23}$$

In (5.5) we established that at each point $X_0 \in \tilde{\mathcal{K}}_{t_0}$, there exists a subsequence i_j and a function $\hat{w} \in C^{1,\alpha}(\hat{B}_R(\hat{x}_0))$ such that

$$\hat{w}_{i_j} \to \hat{w} \qquad \text{in } C^1(\hat{B}_R(\hat{x}_0)),$$

where $\hat{B}_R(\hat{x}_0) := \hat{T} \cap B_R^{n+1}(\hat{x}_0)$, where

$$\operatorname{graph}(\hat{w}) = \hat{\Sigma}_{X_0} = \exp_q^{-1}\left(\tilde{\Sigma}_{X_0} \cap B_R^{M\times\mathbb{R}}(X_0)\right) \tag{7.24}$$

Then Lemma 5.4 establishes that each surface $\tilde{\Sigma}_{X_0}$ bounds a Caccioppoli set E in $\tilde{\mathcal{K}}_{t_0}$ that minimises area plus bulk energy $(\bar{g}^{ij} - \tilde{\nu}^i\tilde{\nu}^j)K_{ij}$ in $\tilde{\mathcal{K}}_{t_0}$, where by construction $\tilde{\nu}$ is the outward unit normal vector to the relative boundary $\partial E \cap \tilde{\mathcal{K}}_{t_0}$.

Therefore, writing the Caccioppoli set E locally as the subgraph of $w := \exp_q^{-1}(\hat{w})$, we find from (7.23) that w minimises the functional

$$\begin{aligned}\mathring{J}_{\tilde{\nu}}^{B_R(x_0)}(w) := &\int_{B_R(x_0)} \sqrt{1+|\bar{\nabla} w|^2}dx \\ &+ \int_{B_R(x_0)} \int_0^{w(x)} \operatorname{tr}_M K_{ij}(x,\tau) - K_{ij}(x,\tau)\tilde{\nu}^i(x,\tau)\tilde{\nu}^j(x,\tau)d\tau dx\end{aligned}$$

in $B_R(x_0) := \exp_q(\hat{B}_R\hat{x}_0))$, whose Euler-Lagrange equation is the MOTS equation

$$\operatorname{div}\left(\frac{\bar{\nabla} w}{\sqrt{1+|\bar{\nabla} w|^2}}\right) + \left(\bar{g}^{ij} - \tilde{\nu}^i\tilde{\nu}^j\right)K_{ij} = 0, \tag{7.25}$$

and by construction $\tilde{\nu} = \dfrac{(\bar{\nabla} w, -1)}{\sqrt{1+|\bar{\nabla} w|^2}}$. The left hand side of (7.25) is an elliptic operator of the form

$$Aw = a^{ij}(\bar{\nabla} w)\left(\bar{\nabla}_i\bar{\nabla}_j w + \sqrt{1+|\bar{\nabla} w|^2}K_{ij}\right),$$

where

$$a^{ij}(p) := \frac{1}{\sqrt{1+|p|^2}}\left(\bar{g}^{ij} - \frac{p^ip^j}{1+|p|^2}\right).$$

Since $w \in C^{1,\alpha}(B_R(x_0))$, $a^{ij} \in C^{0,\alpha}(B_R(x_0))$, Aw is strictly elliptic on $B_R(x_0)$. Schauder theory then implies that $w \in C^{2,\alpha}(B_R(x_0))$, and by bootstrapping further we obtain $w \in C^\infty(B_R(x_0))$. Using a suitable partition of unity, we obtain that each surface $\tilde{\Sigma}_{X_0}$ is a smooth MOTS in $\tilde{\mathcal{K}}_{t_0}$. □

8 Existence of weak solutions

In this section we use the normal vector field $\tilde{\nu}$ of the hypersurface foliation of the jump region $\tilde{\mathcal{K}}_{t_0}$ from Proposition 4.1 to extend $\bar{\nu} = \frac{\bar{\nabla} U}{|\bar{\nabla} U|}$ across the jump region, thereby constructing a globally defined normal vector field $\bar{\nu}$. Existence of weak solutions is then proven by taking the limit of the translating graphs $\tilde{\Sigma}_t^{\varepsilon}$, using Compactness Property 4.2.

Theorem 8.1 (Existence of weak solutions) *Let (M^{n+1}, g, K) be a complete, connected, asymptotically flat initial data set without boundary, that satisfies $tr_M K \geq 0$. Then for any nonempty, precompact, open set $E_0 \subset M$ with C^2 boundary, there exists a weak solution of $(**)$ with initial condition E_0.*

Proof. Let U be the limit of U_ε as given by (3.26). We construct the vertical cylinders $\tilde{\Sigma}_t := \partial\{U < t\}$ and $\tilde{\Sigma}_t^+ := \partial\{U > t\}$ with local uniform $C^{1,\alpha}$ bounds and unit normal vector field ν with local $C^{0,\alpha}$ bounds. Then using Theorem (4.2), we show that $\{U < t\}$ minimises $\mathcal{J}_{U,\bar{\nu}}$ in $(M \backslash E_0) \times \mathbb{R}$ for each t, where $\bar{\nu}$ is extended in the jump regions $\tilde{\mathcal{K}}_{t_0}$ by the normal vector field $\tilde{\nu}$ to the family of smooth MOTS $\{\tilde{\Sigma}_{X_0}\}_{X_0 \in \tilde{\mathcal{K}}_{t_0}}$.

i) In the case where $\tilde{\Sigma}_t = \tilde{\Sigma}_t^+$, the surface $\tilde{\Sigma}_t$ is constructed by fixing a point $X_0 = (x_0, z_0) \in \tilde{\Sigma}_t$ and considering the sequence of times t_i such that $X_0 \in \tilde{\Sigma}_{t_i}^i$ for each i. It then follows exactly as in the proof of Lemma 5.3 that $\tilde{\Sigma}_{t_i}^i$ converges locally uniformly to $\tilde{\Sigma}_t$. Since $\tilde{\Sigma}_t = \tilde{\Sigma}_t^+$ is a vertical cylinder, convergence holds for the full sequence, and the unit normal vector field $\tilde{\nu}$ is equal to $\frac{\bar{\nabla} U}{|\bar{\nabla} U|}$.

ii) We use a slightly different pointwise approach to construct $\tilde{\Sigma}_t$ and $\tilde{\Sigma}_t^+$ when $\tilde{\Sigma}_t \neq \tilde{\Sigma}_t^+$. To this end, let $X_0 \in \tilde{\Sigma}_{t_0}^+$ at a jump time t_0. Since there are only countably many such t_0, there exists a sequence of points $X_i \in \tilde{\Sigma}_{t_i}$ with $t_i > t_0$, satisfying $\lim_{i\to\infty} X_i = X_0$ and $\lim_{i\to\infty} t_i = t_0$. For $i \gg 1$ large enough, we can assume that $\tilde{\Sigma}_{t_i} = \tilde{\Sigma}_{t_i}^+$, and as above each surface piece $\tilde{\Sigma}_{t_i} \cap B_R^{M\times\mathbb{R}}(X_i)$ can therefore be written via the exponential map (denoted by the hat superscript) as the graph of a $C^{1,\alpha}$ function $\hat{w}_i$ over $T_{\hat{X}_i}\hat{\Sigma}_{t_i}$, where

$$\hat{\Sigma}_{t_i} := \exp_{X_i}^{-1}(\tilde{\Sigma}_{t_i} \cap B_d^{M\times\mathbb{R}}(X_i)).$$

Now consider the sequence $\hat{\nu}_i$ of normal vectors to $\hat{\Sigma}_{t_i}$ at $\hat{X}_i$. Since the $\hat{\nu}_i(\hat{X}_i)$ are uniformly bounded in $C^{0,\alpha}$, there exists a subsequence $\hat{\nu}_{i_j}$ and a unit vector field $\hat{\nu}$ such that $\hat{\nu}_{i_j} \to \hat{\nu}$

uniformly on $B^{n+2}_{\hat{R}}(\hat{X}_i)$. Let $\hat{T}$ denote the hyperplane containing $\hat{X}_0$ and orthogonal to $\hat{\nu}(\hat{X}_0)$. For $i \gg 1$ large enough, we can then write each surface $\hat{\Sigma}_{t_i}$ locally as the graph of a $C^{1,\alpha}$ function $\hat{w}_i$ over $\hat{T} \cap B^{n+2}_{\hat{R}}(\hat{X}_0)$. By Arzela-Ascoli, there exists a further subsequence $\hat{w}_{i_j}$ and a $C^{1,\alpha}$ function $\hat{w} : \hat{T} \cap B^{n+1}_{\hat{R}}(\hat{X}_i)$ such that

$$\hat{w}_i \to \hat{w} \qquad \text{in } C^1(\hat{T} \cap B^{n+1}_{\hat{R}}(\hat{X}_i)),$$

where $\hat{X}_0 \in \operatorname{graph} \hat{w}$ and $\hat{T} = T_X \operatorname{graph} \hat{w}$. In order to recognise $\operatorname{graph} \hat{w}$ as a piece of $\hat{\Sigma}^+_{t_0}$ and $\hat{T}$ as $T_{\hat{X}_0}\hat{\Sigma}^+_{t_0}$, we consider a point $Y \in \operatorname{graph} \hat{w}$. Then there exists a sequence $Y_i \in \operatorname{graph} \hat{w}_i \subset \hat{\Sigma}_{t_i}$ such that $Y_i \to Y$, and thus $\hat{U}(Y_i) = t_i$ implies $\hat{U}(Y) = t_0$, where $\hat{U} := U \circ \exp$.

In order to obtain a contradiction, assume that $Y \in \hat{E}^+_{t_0}$. Then there exists $\delta > 0$ such that $B^{n+2}_{\delta}(Y) \in \hat{E}^+_{t_0}$, however this contradicts the fact that $Y_i \in \hat{\Sigma}_{t_i}$ for $t_i > t_0$. Thus we deduce that $\operatorname{graph} \hat{w} \in \hat{\Sigma}^+_{t_0}$ as required. The case where $X_0 \in \tilde{\Sigma}_{t_0}$ for $\tilde{\Sigma}_{t_0} \neq \Sigma^+_{t_0}$ follows analogously.

In Proposition 4.1 we constructed a family of surfaces $\{\tilde{\Sigma}_{X_0}\}_{X_0 \in \tilde{\mathcal{K}}_{t_0}}$ foliating the jump region $\tilde{\mathcal{K}}_{t_0}$ of U. This foliation has a $C^{0,\alpha}_{\text{loc}}$ normal vector field $\tilde{\nu}$, which extends the vector field of the surfaces $\tilde{\Sigma}_{t_0}$ and $\tilde{\Sigma}^+_{t_0}$ as a calibration across the jump region at jump times t_0 via the definition

$$\bar{\nu}(x) := \begin{cases} \dfrac{\bar{\nabla} U}{|\bar{\nabla} U|}(x) & \text{if } x \in \Sigma_t \text{ at regular times } t, \\ \tilde{\nu} & \text{if } x \in \tilde{\mathcal{K}}_{t_0} \text{ at a jump time } t_0, \\ \lim\limits_{i\to\infty} \dfrac{\bar{\nabla} U}{|\bar{\nabla} U|}(x_i) & \text{if } x \in \tilde{\Sigma}_{t_0}, \text{ where } x_i \in \tilde{\Sigma}_{t_i} \text{ for } x_i \to x, t_i \nearrow t_0, \\ \lim\limits_{i\to\infty} \dfrac{\bar{\nabla} U}{|\bar{\nabla} U|}(x_i) & \text{if } x \in \tilde{\Sigma}^+_{t_0}, \text{ where } x_i \in \tilde{\Sigma}_{t_i} \text{ for } x_i \to x, t_i \searrow t_0. \end{cases}$$

This global interpretation of the normal vector field $\bar{\nu}$ in $M \backslash \bar{E}_0 \times \mathbb{R}$ means the functional $\mathcal{J}_{U,\bar{\nu}}$ is well defined on $M \backslash \bar{E}_0 \times \mathbb{R}$, and it follows from Compactness Property 4.2 that the sets $\{U < t\}$ minimises $\mathcal{J}_{U,\bar{\nu}}$ in $M \backslash \bar{E}_0 \times \mathbb{R}$ for each t. The result then follows from Lemma 5.4. □

Proposition 8.2 *Let $(U(x,z) := u(x), \nu)$ be a weak solution to $(**)$ that is obtained via a limit of elliptic regularisation, as in the proof of Theorem 8.1, where ν is the vector field defined above.*

Projecting the normal vector ν of the weak solution to TM produces a vector field $\nu_{_M}$ that extends $\nabla u / |\nabla u|$ as a calibration across the jump region. The pair $(u, \nu_{_M})$ satisfy

$$|\nu_{_M}| \leq 1, \quad \nabla u \cdot \nu_{_M} = |\nabla u| \text{ a.e.},$$

$$\int_{M \backslash E_0} \nabla \hat{\xi} \cdot \nu_{_M} + \hat{\xi} \left(|\nabla u| - (g^{ij} - \nu^i_{_M} \nu^j_{_M}) K_{ij} \right) = 0 \quad \text{for all } \hat{\xi} \in C^1_c(M \backslash E_0). \tag{$\diamond$}$$

Proof. Since $U(x,z)$ is independent of z, the level-sets of U are vertical cylinders for a.e. t. Our approach will be to show that U solves

$$\int_{(M\backslash E_0)\times\mathbb{R}} \bar{\nabla}\xi\cdot\nu + \xi\left(|\bar{\nabla}U| - (\bar{g}^{ij}-\nu^i\nu^j)K_{ij}\right) = 0, \tag{8.1}$$

for ξ such that

$$\xi(x,z) = \hat{\xi}(x)\phi(z), \qquad \xi\in C^1_c((M\backslash\bar{E}_0)\times\mathbb{R}^+), \tag{8.2}$$

and then infer that u satisfies ($\diamond$). Since the level-sets $\tilde{\Sigma}^i_t = \{U_i = t\}$ smoothly solve ($*$) on $\Omega_i\times\mathbb{R}$ with $H+P>0$ for $-\infty<t<\infty$, the unit normal

$$\nu_i := \frac{\bar{\nabla}U_i}{|\bar{\nabla}U_i|},$$

to $\tilde{\Sigma}^i_t$ is a smooth unit vector field on $\Omega_i\times\mathbb{R}$ satisfying

$$\mathrm{div}(\nu_i) + \left(\bar{g}^{ij}-\nu^i_i\nu^j_i\right)K_{ij} - |\bar{\nabla}U_i| = 0, \tag{8.3}$$

where $\bar{\nabla}$ denotes the connection on $M\times\mathbb{R}$.

Multiply (8.3) by ξ satisfying (8.2), where ϕ is a cutoff function the z-direction satisfying $\phi\geq 0$, $\operatorname{spt}\phi\subset(0,\lambda]$ for some $0<\lambda<\infty$ and $\int\phi(z)dz = 1$. Integrating by parts we obtain

$$\int_{\tilde{\Sigma}^i_t} \bar{\nabla}\xi\cdot\nu_i + \xi(|\bar{\nabla}U_i| - (\bar{g}^{ij}-\nu^i_i\nu^j_i)K_{ij}) = 0. \tag{8.4}$$

To show convergence of the second term, we use the fact that $|\bar{\nabla}U_i| = (H+P)_{\tilde{\Sigma}^i_{t_i}}$ and follow the approach of Huisken and Ilmanen in [HI, Ch. 5] to obtain L^2 convergence of the null mean curvature. In particular, we fix a time $T>0$ and consider $0\leq t\leq T$ and $L\geq T+2+\lambda$, $\varepsilon<1$ so that $\partial\tilde{\Sigma}^\varepsilon_t$ is disjoint from sptξ for $0\leq t\leq T$ and the boundary term disappears when we perform the integration by parts in (7.12). Reworking the proof of Lemma 7.4 in this scenario, we obtain the area estimate

$$|\tilde{\Sigma}^i_t\cap\operatorname{spt}\xi| \leq C(T), \qquad 0\leq t\leq T,$$

analogous to (7.13). Together with the interior estimate (2.5), and the boundary gradient estimates for u^ε, this shows

$$|H|\leq C(T,\|K\|_{C^1}) \qquad \text{on } \tilde{\Sigma}^\varepsilon_t\cap\operatorname{spt}\xi, \quad 0\leq t\leq T. \tag{8.5}$$

As in the proof of Lemma 7.4, we obtain the following uniform L^2 estimate for $D(H+P)$

$$\liminf_{i\to\infty}\int_{\tilde{\Sigma}^i_t\cap(M\times[0,\lambda])} |D(H+P)|^2 < \infty, \qquad \text{a.e. } t\geq 0. \tag{8.6}$$

For t satisfying (8.6), we have

$$\tilde{\Sigma}_t^i \to \tilde{\Sigma}_t = \Sigma_t \times \mathbb{R} \qquad \text{locally in } C^1,$$

where $\Sigma_t := \partial\{u < t\}$. Specifically, the converging surfaces $\tilde{\Sigma}_t^i$ can be written locally as graphs of $C^{1,\alpha}$ functions w_i over the fixed hyperplane $\tilde{T} = T_{X_0}\tilde{\Sigma}_t$, for some $X_0 \in \tilde{\Sigma}_t$. The local C^1 convergence of these hypersurfaces, together with the first variation of area formula and the Riesz Representation Theorem then implies that $H_{\tilde{\Sigma}_t^i}$ exists weakly as a locally L^1 function, with the weak convergence

$$\int_{\tilde{\Sigma}_t^i} H_{\tilde{\Sigma}_t^i} \nu_{N_t^i} \cdot X \to \int_{\tilde{\Sigma}_t} H_{\tilde{\Sigma}_t} \nu_{\tilde{\Sigma}_t} \cdot X, \qquad X \in C_c^0(T(M \backslash E_0 \times \mathbb{R})), \tag{8.7}$$

analogous to (7.19). Then by Rellich's theorem together with (8.6) and (8.7), there exists a subsequence i_j such that

$$(H+P)_{\tilde{\Sigma}_t^{i_j}} \to (H+P)_{\tilde{\Sigma}_t} \qquad \text{in } L^2(\tilde{T} \cap (M \times [0,\lambda])). \tag{8.8}$$

To establish that the full sequence converges, we note that (7.14) (reworked for the function ξ) implies that the function $\int_{\tilde{\Sigma}_t^i} \xi(H+P)^2 - C(T < \|K\|_{C^0}, \|K\|_{C^1})t$ is monotone for $0 \le t \le T$. Then using growth control, we can take a diagonal subsequence $\hat{\imath}$ such that

$$\lim_{\hat{\imath}\to\infty} \int_{\tilde{\Sigma}_t^{\hat{\imath}}} \xi(H+P)^2 \text{ exists}, \qquad \text{a.e. } t \ge 0, \tag{8.9}$$

from which it follows that the full sequence i converges, that is

$$\int_{\tilde{\Sigma}_t^i} \xi(H+P)^2 \to \int_{\tilde{\Sigma}_t} \xi(H+P)^2 \qquad \text{a.e. } t \ge 0. \tag{8.10}$$

We now use Lemma (6.8) to identify $\int_{\tilde{\Sigma}_t}(H+P)$ with $\int_{\tilde{\Sigma}_t} |\bar{\nabla} U|$. It then follows from (8.4) and (8.10) that

$$\int_{\tilde{\Sigma}_t^i} \xi|\bar{\nabla} U_i|^2 = \int_{\tilde{\Sigma}_t^i} \xi(H+P)^2 \to \int_{\tilde{\Sigma}_t} \xi(H+P)^2 = \int_{\tilde{\Sigma}_t} \xi|\bar{\nabla} U|^2 \qquad \text{a.e. } t \ge 0. \tag{8.11}$$

Now for the first and third terms in (8.4), the local uniform convergence of the normal vector $\nu_i \to \nu$ implies

$$\int_{\tilde{\Sigma}_t^i} \bar{\nabla}\xi \cdot \nu_i - \xi\left(\bar{g}^{ij} - \nu_i^i \nu_i^j\right) K_{ij} \to \int_{\tilde{\Sigma}_t} \bar{\nabla}\xi \cdot \nu - \xi\left(\bar{g}^{ij} - \nu^i \nu^j\right) K_{ij}, \qquad \text{a.e. } t \ge 0, \tag{8.12}$$

and we obtain

$$0 = \int\limits_{\tilde{\Sigma}_t^i} \bar{\nabla}\xi \cdot \nu_i + \xi\left(|\bar{\nabla} U_i| - \left(\bar{g}^{ij} - \nu_i^i \nu_i^j\right) K_{ij}\right) \to \int\limits_{\tilde{\Sigma}_t} \bar{\nabla}\xi \cdot \nu + \xi\left(|\bar{\nabla} U| - \left(\bar{g}^{ij} - \nu^i \nu^j\right) K_{ij}\right). \tag{8.13}$$

for a.e. $t \geq 0$.

To determine the behaviour in the interior $\tilde{\mathcal{K}}_{t_0}$ of the jump region at a jump time t_0, we use the pointwise approach of Chapter 7. As in the proof of Lemma (5.3), by taking a diagonal subsequence, we obtain

$$\tilde{\Sigma}^i_{t_i} \to \tilde{\Sigma}_{X_0} \qquad \text{locally in } C^1,$$

for each point X_0 in the interior $\tilde{\mathcal{K}}_{t_0}$ of $\{U = t_0\} = \{u = t_0\} \times \mathbb{R}$. Then using Lemma (7.4) we obtain

$$0 = \int\limits_{\tilde{\Sigma}^i_{t_i} \cap B_d(X_0)} \bar{\nabla}\xi \cdot \nu_i + \xi \left(|\bar{\nabla} U_i| - (\bar{g}^{ij} - \nu^i_i \nu^j_i) K_{ij} \right) \to \int\limits_{\tilde{\Sigma}_{X_0} \cap B_d(X_0)} \bar{\nabla}\xi \cdot \tilde{\nu} - \xi (\bar{g}^{ij} - \tilde{\nu}^i \tilde{\nu}^j) K_{ij}, \tag{8.14}$$

where $B_d(X_0) = B^{M \times \mathbb{R}}_d(X_0)$ for d defined by (5.2).

Since the surfaces $\tilde{\Sigma}_{t_0}$ and $\tilde{\Sigma}^+_{t_0}$ at the jump time t_0 have zero $(n+2)$-dimensional Hausdorff measure, and there are only countably many jump times, combining the above we obtain (8.1),

$$\int_{(M \setminus \bar{E}_0) \times \mathbb{R}} \bar{\nabla}\xi \cdot \nu + \xi \left(|\bar{\nabla} U| - (\bar{g}^{ij} - \nu^i \nu^j) K_{ij} \right) = 0.$$

We now reduce this to $(\diamond)$. Since $\tilde{\Sigma}_t = \Sigma_t \times \mathbb{R}$ is a cylinder for a.e t and $U(x, z) = u(x)$, the integral of each geometric quantity $A(x, z) = A(x)$ breaks up as

$$\int_{\tilde{\Sigma}_t} \xi A \, d\mu_{\tilde{\Sigma}_t} = \int_0^\lambda \phi dz \int_{\Sigma_t} \hat{\xi} A \, d\mu_{\Sigma_t} = \int_{\Sigma_t} \hat{\xi} A \, d\mu_{\Sigma_t}, \qquad \text{a.e.} t \geq 0. \tag{8.15}$$

Using the fact that K_{ij} was extended trivially in the z-direction, and that $\nu = (\nu_M, 0) = \left(\frac{\nabla u}{|\nabla u|}, 0 \right)$ is perpendicular to $\bar{\nabla}\phi$, we obtain

$$\begin{aligned}
\int_{\tilde{\Sigma}_t} \xi |\bar{\nabla} U|^2 d\mu_{\tilde{\Sigma}_t} &= \int_0^\lambda \phi dz \int_{\Sigma_t} \hat{\xi} |\nabla u|^2 d\mu_{\Sigma_t} = \int_{\Sigma_t} \hat{\xi} |\nabla u|^2 d\mu_{\Sigma_t}, \\
\int_{\tilde{\Sigma}_t} \bar{\nabla}\xi \cdot \nu d\mu_{\tilde{\Sigma}_t} &= \int_0^\lambda \phi dz \int_{\Sigma_t} \nabla\hat{\xi} \cdot \nu_M d\mu_{\Sigma_t} = \int_{\Sigma_t} \nabla\hat{\xi} \cdot \nu_M d\mu_{\Sigma_t}, , \\
\int_{\tilde{\Sigma}_t} \xi \bar{g}^{ij} K_{ij} d\mu_{\tilde{\Sigma}_t} &= \int_0^\lambda \phi dz \int_{\Sigma_t} g^{ij} K_{ij} d\mu_{\Sigma_t} = \int_{\Sigma_t} g^{ij} K_{ij} d\mu_{\Sigma_t}, \\
\int_{\tilde{\Sigma}_t} \xi \nu^i \nu^j K_{ij} d\mu_{\tilde{\Sigma}_t} &= \int_0^\lambda \phi dz \int_{\Sigma_t} \hat{\xi} \nu^i_M \nu^j_M K_{ij} d\mu_{\Sigma_t} = \int_{\Sigma_t} \hat{\xi} \nu^i_M \nu^j_M K_{ij} d\mu_{\Sigma_t},
\end{aligned}$$

Therefore we obtain for a.e. $t \geq 0$

$$\int\limits_{\tilde{\Sigma}_t} \bar{\nabla}\xi \cdot \nu + \xi \left(|\bar{\nabla} U_i| - (\bar{g}^{ij} - \nu^i \nu^j) K_{ij} \right) = \int\limits_{\Sigma_t} \nabla\hat{\xi} \cdot \nu_M + \hat{\xi} \left(|\nabla u| - \left(g^{ij} - \nu^i_M \nu^j_M \right) K_{ij} \right). \tag{8.16}$$

We next consider the interior $\tilde{\mathcal{K}}_{t_0}$ of the jump region $\{u = t_0\} \times \mathbb{R}$. When $\tilde{\Sigma}_{X_0} = \Sigma_{X_0} \times \mathbb{R}$ is (part of) a vertical cylinder, then $\tilde{\nu} = (\nu_M, 0)$ is perpendicular to the z-direction and the right hand side decouples as in (8.15) above to obtain

$$\int\limits_{\tilde{\Sigma}_{X_0} \cap B_d(X_0)} \bar{\nabla}\xi \cdot \tilde{\nu} - \xi(\bar{g}^{ij} - \tilde{\nu}^i\tilde{\nu}^j)K_{ij} = \int\limits_{\Sigma_{X_0} \cap B_d(X_0)} \nabla\hat{\xi} \cdot \nu_M - \hat{\xi}(g^{ij} - \nu_M^i\nu_M^j)K_{ij}. \tag{8.17}$$

In the other case $\tilde{\nu} = (\nu_M, \nu^z)$, and, by reducing d if necessary, we can choose $\phi(z) \geq 0$ such that $\operatorname{spt}\phi \subseteq [0, \lambda]$, $\int \phi(z)dz = 1$ and $\phi(z) = 1$ on $[z_0 - d, z_0 + d]$. There exists $0 < R \leq d$ such that $\tilde{\Sigma}_{X_0} \cap \tilde{B}_R = \tilde{\Sigma}_{X_0} \cap (B_R^M(x_0) \times [z_0 - d, z_0 + d]) \subseteq \tilde{\Sigma}_{X_0} \cap B_d$, where $\tilde{B}_R := (B_R^M(x_0) \times \mathbb{R})$. The translation invariance of the surfaces $\tilde{\Sigma}_{X_\alpha}$ in the stack (5.6) means that $\tilde{\nu}(X_0) = \tilde{\nu}(X_0 + \alpha\mathbf{e}_z) = (\nu_M, \nu_z)$ and we find

$$\begin{aligned}\int\limits_{\tilde{\Sigma}_{X_0} \cap \tilde{B}_R} \bar{\nabla}\xi \cdot \tilde{\nu} - \xi(\bar{g}^{ij} - \tilde{\nu}^i\tilde{\nu}^j)K_{ij} &= \int\limits_{\tilde{\Sigma}_{X_0} \cap \tilde{B}_R} \phi\nabla\hat{\xi} \cdot \nu_M + \hat{\xi}\frac{\partial\phi}{\partial z} \cdot \nu^z - \phi\hat{\xi}(g^{ij} - \nu_M^i\nu_M^j)K_{ij} \\ &= \int\limits_{B_R^M(x_0)} \nabla\hat{\xi} \cdot \nu_M - \hat{\xi}(g^{ij} - \nu_M^i\nu_M^j)K_{ij},\end{aligned} \tag{8.18}$$

since K was extended trivially in the z-direction.

Combining (8.16), (8.17) and (8.18) we obtain ($\diamond$), thereby completing the proof of Proposition (8.2). □

We remark that one can show that a solution (u, ν_M) to ($\diamond$) satisfies (6.2).

Lemma 8.3 *If the pair* $(u, \nu_{_M})$ *satisfy* ($\diamond$) *in* $\Omega \subset M$, *then* $\mathcal{J}_{u,\nu_{_M}}(u) \leq \mathcal{J}_{u,\nu_{_M}}(v)$ *for every locally Lipschitz function* v *such that* $\{u \neq v\} \subset\subset \Omega$.

Proof. The locally Lipschitz function $\xi := v - u$ has compact support in Ω for any such v satisfying $\{v \neq u\} \subset\subset \Omega$. Using approximation by C^1 functions, for every $\delta > 0$ there exists a C^1_c function $\bar{\xi} : \Omega \to \mathbb{R}$ that differs from ξ on a set of measure $\delta > 0$. Then from ($\diamond$)

$$\begin{aligned}0 &= \int_\Omega \nabla\bar{\xi} \cdot \nu_{_M} + \bar{\xi}|\nabla u|\left(|\nabla u| - P\right)\,dx \\ &\leq \int_\Omega \nabla(v - u) \cdot \nu_{_M} + (v - u)\left(|\nabla u| - P\right)\,dx + \delta\left(|\nabla(\bar{\xi} - \xi)| + (\bar{\xi} - \xi)\left(|\nabla u| - P\right)\right) \\ &\leq \int_\Omega -|\nabla u| - u\left(|\nabla u| - P\right) + |\nabla v| + v\left(|\nabla u| - P\right)\,dx + \delta\left(C\operatorname{Lip}(\xi) + (\bar{\xi} - \xi)\left(|\nabla u| - P\right)\right).\end{aligned}$$

Since δ can be made arbitrarily small, we obtain

$$\int_\Omega |\nabla u| + u\left(|\nabla u| - P\right) \leq \int_\Omega |\nabla v| + v\left(|\nabla u| - P\right)$$

where $P = \left(g^{ij} - \nu^i_{_M}\nu^j_{_M}\right)K_{ij}$. □

9 Applications of weak solutions

In this section we highlight the natural applications of weak solutions of $(**)$ to the existence theory for MOTS and to the theory of weak solutions of IMCF.

9.1 Applications to MOTS

In this section we present an application of the evolution by inverse null mean curvature to the existence theory for MOTS, and compare this to the following existence theorem combining [AM] and [E], as stated in [AEM].

Theorem 9.1 ([AM, E]) *Let (M, g, K) be an initial data set of dimension $n+1 \leq 7$ and let $\Omega \subset M$ be a connected bounded open subset with smooth embedded boundary $\partial\Omega$. Assume this boundary consists of two non-empty closed hypersurfaces $\partial_+\Omega$ and $\partial_-\Omega$, possibly consisting of several components, such that*

$$H_{\partial_+\Omega} - tr_{\partial_+\Omega}K > 0 \quad \text{and } H_{\partial_+\Omega} + tr_{\partial_+\Omega}K > 0, \tag{9.1}$$

where the mean curvature scalar is computed as the tangential divergence of the unit normal vector field that is pointing out of Ω. Then there exists a smooth closed embedded hypersurface $\Sigma \subset \Omega$ homogolous to $\partial_-\Omega$ such that $H_\Sigma + tr_\Sigma K = 0$ (where H_Σ is computed with respect to the unit normal pointing towards $\partial_-\Omega$). Σ is stable in the sense of MOTS and it is λ minimising in Ω for $\lambda = 2(n+1)\kappa$, where κ denotes the largest eigenvalue of K with respect to g across Ω.

Here λ-*minimising in* Ω means that the surface Σ arises as (a relative boundary of) a subset $E \subset \Omega$ with perimeter Σ in Ω such that

$$|\partial E \cap W| \leq |\partial F \cap W| + \lambda \mathcal{L}^{n+1}(E \Delta F), \tag{9.2}$$

for every $F \subset \Omega$ such that $E\Delta F \subset\subset W \subset\subset \Omega$, for the constant $\lambda := (n+1)\kappa$, where κ denotes the largest eigenvalue of K with respect to g across Ω. A detailed analysis of such λ-minimising boundaries is carried out in [DS]. We say that the set E is λ-minimising *on the outside/inside* in Ω if E satisfies (9.2) for every F such that $E\Delta F \subset\subset W$, where $F \subseteq E$, $F \supseteq E$ respectively.

We point out the following relationship between the variational principle (4.2) and the λ-minimising property (9.2).

Proposition 9.2 *Let κ denote the size of the largest eigenvalue of K on $\Omega \backslash E_0$, let $\lambda := (n+1)\kappa$ and suppose the Caccioppoli set E satisfies (4.2) in $\Omega \backslash E_0$, then E is λ-minimising on the outside in $\Omega \backslash E_0$.*

In particular, let $U \in C^{0,1}_{loc}(M \times \mathbb{R})$ satisfy (??) in $M \backslash E_0 \times \mathbb{R}$, then the sets $\tilde{E}_t = \{U < t\}$ and $\{U \leq t\}$ are λ-minimising on the outside for each $t > 0$, $t \geq 0$ respectively. Furthermore, the surfaces $\tilde{\Sigma}_{X_0}$ foliating the interior $\tilde{\mathcal{K}}_{t_0}$ of the jump region are λ-minimising in $\tilde{\mathcal{K}}_{t_0}$.

The Outward Optimising Property 7.1 of solutions to (6.3), together with the Regularity Theorem 4.3 implies the following existence theorem for MOTS in initial data sets (M, g, K) containing an outer trapped surface $\partial\Omega_0$ such that $\theta^+_{\partial\Omega_0} < 0$.

Proposition 9.3 *Let (M^{n+1}, g, K) be an asymptotically flat initial data set of dimension $n+1 \leq 7$ satisfying $tr_M K \geq 0$, and let E_0 be any nonempty, precompact, smooth open set in M satisfying $\theta^+_{\partial E_0} < 0$ with respect to the unit normal pointing out of E_0. Then the level set $\partial\{U > 0\}$ of the locally Lipschitz solution U of (6.3) is a vertical cylinder $\Sigma \times \mathbb{R}$ over a smooth, outward optimising MOTS in $\Sigma \subset (M \backslash E_0)$ with respect to the outward normal vector of $\partial\{u > 0\}$.*

Proposition 9.3 exploits the fact that the solution must jump at $t = 0$ wherever the null mean curvature of ∂E_0 is strictly negative. In the special case where the outermost MOTS $\Sigma = \partial E$ in $M \backslash E_0$ satisfies

$$|\partial E| \leq |\partial F| - (n+1)\kappa \mathcal{L}^{n+1}(E \backslash F), \tag{9.3}$$

for every MOTS ∂F in $M \backslash E_0$, then the weak solution to (∗∗) will jump to the cylinder $\Sigma \times \mathbb{R}$ over the outermost MOTS Σ at $t = 0$. In general, given any initial data, and any initial condition E_0 satisfying $\theta^+_{\partial E_0} < 0$, the surface can't jump beyond the outmost MOTS at $t = 0$, and the MOTS $\partial\{u > 0\}$ is a barrier for the outermost MOTS Σ in $M \backslash E_0$.

9.2 Applications to IMCF

In this section we discuss the above results in the context of the work of Huisken and Ilmanen [HI] on inverse mean curvature flow, which motivated and guided this thesis. In particular, when applied to the special case $K \equiv 0$, Definition 9.7 provides a new perspective on weak solutions to inverse mean curvature flow. Furthermore, the work of Chapter 7 carries over to prove the analogous results for the jump region.

Recall from Chapter 1.1 that the boundary value problem

$$\begin{cases} \operatorname{div}_M\left(\frac{Du}{|Du|}\right) = |Du|, \\ u\Big|_{\partial E_0} = 0, \end{cases} \tag{$\star$}$$

describes inverse mean curvature flow of the level-sets of the scalar function $u : M\backslash E_0 \to \mathbb{R}$ wherever $|\nabla u| \neq 0$. Here $E_0 \subset M$ is an open bounded set such that the initial surface N_0 of the flow satisfies $N_0 = \partial E_0$. In [HI], Huisken and Ilmanen define a locally Lipschitz function $u \in C^{0,1}_{loc}(M)$ to be a weak solution of ($\star$) with initial condition E_0 if $E_0 = \{u < 0\}$ and

$$J_u(u) \leq J_u(v), \tag{9.4}$$

for every locally Lipschitz function v with $\{v \neq u\} \subset\subset M\backslash E_0$, where the integral is performed over any compact set $K \supseteq \{u \neq v\}$. It then follows that u is a weak solution if and only if the open set $E_t := \{u < t\}$ minimizes the parametric energy functional

$$J_u(F) = |\partial^* F| - \int_F |\nabla u|, \tag{9.5}$$

in $M\backslash E_0$ for each $t > 0$. This weak formulation has the following properties, and the subsequent existence result is obtained.

Theorem 9.4 (Properties of weak solutions, [HI])

(i) Compactness: u_i be a sequence of minimisers of (9.4) on $\Omega_i \subset M$ such that $u_i \to u$ and $\Omega_i \to \Omega$ locally uniformly, and such that for each $K \subset\subset \Omega$ and large i, $\sup_K |\nabla u_i| \leq C(K)$. Then u minimises (9.4) on Ω.

(ii) Uniqueness: If $(E_t)_{t>0}$ and $(F_t)_{t>0}$ minimise (9.5) in M with initial conditions satisfying $E_0 \subseteq F_0$, then $E_t \subseteq F_t$ provided E_t is precompact in M. In particular, there exists at most one weak solution $(E_t)_{t>0}$ for a given E_0 such that each E_t is precompact.

(iii) Smooth solutions of (IMCF) are weak solutions to ($\star$) on the domain foliated by the smooth solution.

Theorem 9.5 (Existence of weak solutions, [HI]) *Let M be a complete, connected Riemannian manifold without boundary. Suppose there exists a proper, locally Lipschitz, weak subsolution v of (9.4) with a precompact initial condtion.[1]*
Then for any nonempty, precompact, smooth open set E_0 in M, there exists a proper, locally Lipschitz weak solution u of ($\star$) with initial condition E_0, which is unique on $M\backslash E_0$.

[1] If M is asymptotically flat, then as in Chapter 2.3.1 there exists $\alpha > 0$ large enough such that $\alpha \log r$ is a subsolution. In part 2 of the proof of Theorem 3.1 in [HI], Huisken and Ilmanen show that it is sufficient that M be only asymptotically conic.

Furthermore, the gradient of u *satisfies*

$$|\nabla u(x)| \leq \sup_{\partial E_0 \cap B_r(x)} H_+ + \frac{C(n)}{r}, \qquad a.e.\ x \in M \backslash E_0, \tag{9.6}$$

for each $0 < r \leq \sigma(x)$*, where* σ *is defined in Definition 2.3, and* $H_+ := \max(0, H_{\partial E_0})$*.*

The jump region

The theory of weak solutions to inverse mean curvature flow as laid out by Huisken and Ilmanen in [HI] does not include a complete analysis of the jump regions. In [He], Heidusch proved optimal $C^{1,1}_{\text{loc}}$ regularity for the surfaces $N_{t_0} = \partial\{u < t_0\}$ and $N^+_{t_0} = \partial\{u > t_0\}$ enveloping the jump region. Applying Chapter 7 in this special case where $K \equiv 0$, we obtain a foliation of the interior of the jump region $\{u = t_0\} \times \mathbb{R}$ by area minimising hypersurfaces, a result which was left open in [HI].

Corollary 9.6 *Let* u *be the weak solution of* $(\star)$ *given by Theorem 9.5. At a jump time* t_0*, the interior* $\tilde{\mathcal{K}}_{t_0}$ *of the region* $\{u = t_0\} \times \mathbb{R}$ *is foliated by smooth area minimising surfaces, each of which is either a vertical cylinder or a smooth graph over an open subset of* $\tilde{\mathcal{K}}_{t_0}$*.*

Weak solutions of IMCF

In this section we utilise the jump region hypersurfaces of Corollary 9.6 to present a new perspective on weak solutions of inverse mean curavture flow. In particular, we show that by instead considering the weak solution to be a family of hypersurfaces one dimension higher in $M \times \mathbb{R}$, we obtain a richer notion of weak solution.

Definition 9.7 (Alternative weak formulation) Let u be the unique, locally Lipschtiz weak solution to $(\star)$ on $M \backslash E_0$ given by Theorem 9.5, and define the locally Lipshitz function $U(x, z) := u(x)$ on $(M \backslash E_0) \times \mathbb{R}$. The weak solution to $(\star)$ is defined to be the pair (U, ν), where ν is a unit length, translation invariant extension of $\dfrac{\bar{\nabla} U}{|\bar{\nabla} U|}$ in the jump regions such that at each point $x \in \tilde{\mathcal{K}}_{t_0}$, $\nu(x)$ is the normal vector to a $C^{1,\alpha}$ hypersurface passing through x, which bounds a Caccioppli set that minimises $J_{U,\nu}$ in $\tilde{\mathcal{K}}_{t_0}$.

Remark 9.8 This weak solution (U, ν) has the following interpretation in $M \backslash E_0$. Projecting the normal vector ν to TM produces a vector field $\nu_{_M}$ that extends $\nabla u / |\nabla u|$ as a calibration across the jump region. The pair $(u, \nu_{_M})$ then have the following interpretation in $M \backslash E_0$. Namely, there exists a measurable vector field $\nu_{_M}$ such that

$$\begin{gathered} |\nu_{_M}| \leq 1, \quad \nabla u \cdot \nu_{_M} = |\nabla u| \text{ a.e.}, \\ \int_\Omega \nabla \xi \cdot \nu_{_M} + \xi |\nabla u| = 0 \quad \text{for all } \xi \in C^1_c(M \backslash E_0). \end{gathered} \tag{$\diamond$}$$

We obtain the following weak existence result as a corollary of Theorem 8.1.

Corollary 9.9 (Existence of weak solutions) *Let M be a complete, connected Riemannian n-manifold without boundary. Suppose there exists a proper, locally Lipschitz, weak subsolution of (9.4) with a precompact initial condition.*

Then for any nonempty, precompact, smooth open set E_0 in M, there exists a weak solution satisfying Definition 9.7 in $M \backslash E_0 \times \mathbb{R}$ with initial condition E_0. Furthermore, the locally Lipschitz function u associated to this weak solution uniquely solves ($\diamond$) in $M \backslash E_0$.

Uniqueness of the function u solving ($\diamond$) follows from the observation that solutions of ($\diamond$) satisfy Huisken and Ilmanen's variational formulation (see proof of Lemma 8.3), which enjoys the Uniqueness Property 9.4 *ii)*.

Corollary 9.10 *If the locally Lipschitz function u satisfies ($\diamond$) in Ω, then*

$$J_u(u) \leq J_u(v)$$

for every locally Lipschitz function v such that $\{v \neq u\} \subset\subset \Omega$.

We conclude therefore that the formulation ($\diamond$) is equivalent to Huisken and Ilmanen's variational formulation under the hypothesis of Theorem 8.1. We further remark that taking $\nu \equiv 0$ in the jump regions is also a solution to ($\diamond$), thus one can only expect uniqueness for the function u solving ($\diamond$), and not for the pair (u, ν_M). This observation that information is lost by projecting the normal vector field $\tilde{\nu}$ to TM in the jump region highlights why it is preferable to formulate the weak solution to inverse mean curvature flow one dimension higher (as in Corollary 9.9), where one obtains a family of surfaces satisfying the variational principle *everywhere*.

Bibliography

[AEM] Andersson, L., Eichmair, M., Metzger, J., Jang's equation and it's applications to marginally outer trapped surfaces. *Contemporary Mathematics 554:Complex Analysis and Dynamical Systems IV* (2011)

[AM] Andersson, L., Metzger, J., The Area of Horizons and the Trapped Region. *Commun. Math. Phys.* **290**, 941-972 (2009)

[Br] Bray, H.L., Proof of the Riemannian Penrose inequality using the positive mass theorem. *J. Diff. Geom* **59**, 177-267 (2001)

[BM] Bray, H.L., Miao, P., On the capacity of surfaces in manifolds with nonnegative scalar curvature. *Invent. Math.* **172**, 459-475 (2008)

[BN] Bray, H.L., Neves, A., Classification of prime 3-manifolds with Yamabe invariant greater than $\mathbb{RP}^3$. *Ann. of Math.* **159**, 407-424 (2004)

[CGG] Chen, Y.G., Giga, Y., Goto, S., Uniqueness and existence of viscosity solutions of generalized mean curvature flow equation. *J. Diff. Geom.* **33**, 749-786 (1991)

[CIL] Crandall, M., Ishii, H., Lions, P-L., User's guide to viscosity solution of second order partial differential equations. *Bull. Amer. Math. Soc.* **27**, 1-67 (1992)

[DS] Duzaar, F., Steffen, K., λ-minimising currents. *Manuscripta Math.* **80**, 403-447 (1993)

[E] Eichmair, M., The Plateau problem for marginally outer trapped surfaces. *J. Diff. Geom.* **83**, 551-584 (2009)

[ES1] Evans, L.C., Spruck, J., Motion of level sets by mean curvature, I. *J. Diff. Geom.* **33**, 635-681 (1991) *J. Geom. Anal.* **5**, 77-114 (1995)

[Ge] Gerhardt, C., Flow of Nonconverx Hypersurfaces into Spheres. *J. Diff. Geom.* **32**, 299-314 (1990)

[G] Geroch, R., Energy Extraction. *Ann. New York Acad. Sci.* **224**, 108-117 (1973)

[GT] Gilbarg, D., Trudinger, N.S., Elliptic Partial Differential Equations of Second Order. *Springer* (2001)

[He] Heidusch, M., Zur Regularität des inversen mittleren Krümmungsflusses. *PhD thesis, Eberhard-Karls-Universität Tübingen.* (2001)

[HI] Huisken, G., Ilmanen, T., The inverse mean curvature flow and the Riemannian Penrose inequality. *J. Diff. Geom.* **59**, 353-437 (2001)

[HI2] Huisken, G., Ilmanen, T., Higher regularity of the inverse mean curvature flow. *J. Diff. Geom.* **80**, 433-451 (2008)

[J] Jang, P.S., On the positivity of energy in general relativity. *J. Math. Phys.*, **19(5)**, 1152-1155 (1978)

[JW] Jang, P.S., Wald, R., The Positive Energy Conjecture and the Cosmic Censor Hypothesis. *J. Math. Phys.* **18**, 41-44 (1977)

[M] Metzger, J., Blowup of Jangs equation at outermost marginally trapped surfaces. *Comm. Math. Phys.* **294**, 61-72 (2010)

[SY] Schoen, R., Yau, S.T., Proof of the Positive Mass Theorem. II. *Commun. Math. Phys.* **79**, 231-260 (1981)

[T] Tamanini, I., Regularity results for almost minimal oriented hypersurfaces in $\mathbb{R}^n$. *Quaderni del Dipartimento di Matmatica dell'Universita di Lecce.* (1984)

[W] White, B., Subsequent Singularities in Mean-convex Mean Curvature Flow. Preprint arXiv:1103.1469v1, (2011)

[U] Urbas, J., On the Expansion of Starshaped Hypersurfaces by Symmetric Functions of their Principal Curvatures. *Math. Z.* **205** 355-372 (1990)