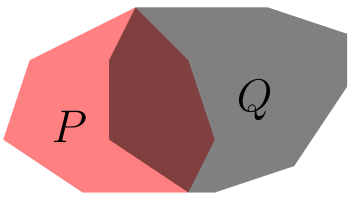
Fachbereich Mathematik und Informatik
der Freien Universität Berlin

Dissertation

On the Combinatorics of Valuations



$$\varphi(P \cup Q) = \varphi(P) + \varphi(Q) - \varphi(P \cap Q)$$

presented by

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Für meine Eltern.
To my parents.

Summary

This thesis deals with structural results for *translation invariant valuations* on polytopes and certain related *enumeration problems* together with geometric approaches to them.

The starting point of the first part are two theorems by Richard Stanley. The first one is his famous *Nonnegativity Theorem* [42] stating that the Ehrhart h^**-vector* of every lattice polytope has *nonnegative* integer entries. In [48] he further proves that the entries satisfy a *monotonicity* property. In Chapter 2 we consider the h^*-vector for arbitrary translation invariant valuations. Our main theorem states that monotonicity and nonnegativity of the h^*-vector are, in fact, equivalent properties and we give a simple characterization. In Chapter 3 we consider the h^*-vector of *zonotopes* and show that the entries of their h^*-vector form a *unimodal* sequence for all translation invariant valuations that satisfy the nonnegativity condition.

The second part deals with certain enumeration problems for *order preserving maps*. Given a suitable pair of finite *posets* $\mathfrak{A} \subseteq \mathfrak{P}$ and an order preserving map λ from $\mathfrak{A}$ to $[n]$ we consider the problem of enumerating order preserving extensions of λ to $\mathfrak{P}$. In Chapter 4 we show that their number is given by a piecewise multivariate polynomial. We apply our results to counting *extensions of graph colorings* and generalize a theorem by Herzberg and Murty [21]. We further apply our results to counting *monotone triangles*, which are closely related to *alternating sign matrices*, and give a short geometric proof of a reciprocity theorem by Fischer and Riegler [17]. In Chapter 5 we consider counting order preserving maps from $\mathfrak{P}$ to $[n]$ up to *symmetry.* We show that their number is given by a polynomial in n, thus, giving an order theoretic generalization of *Pólya's enumeration theorem* [33]. We further prove a reciprocity theorem and apply our results to counting graph colorings up to symmetry.

Chapters 2 and 4 are based on joint work with Raman Sanyal. Chapter 4 appeared in [23]. Chapter 3 is part of a joint project with Matthias Beck and Emily McCullough. The content of Chapter 5 is published in [22].

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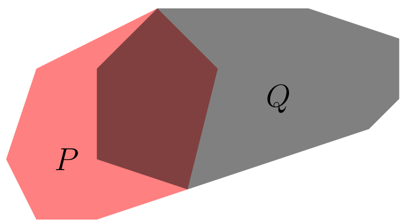
Introduction

The present thesis is located at the crossroads of *enumerative and algebraic combinatorics* and *geometric combinatorics*. By switching perspectives between these fields, new applications and structural results can be obtained in both directions. The "classical" setup is to model an enumerative problem as *counting lattice points* in a *polyhedral* object. If the parameter of the counting function corresponds to geometric dilation, then *Ehrhart theory* provides suitable geometric tools. On the other hand, the number of lattice points inside a polytope is, like the Euler characteristic and the volume, a "measure" on polytopes — a so-called *valuation* — and interesting combinatorics is involved in the behavior and structure of the valuations themselves.

The thesis is divided into two parts: The first part deals with combinatorial properties of *translation-invariant valuations*. The second is devoted to certain *enumeration problems* and new geometric approaches to order theoretic questions.

The background will be discussed in Chapter 1. We will provide rigorous definitions of all concepts in later chapters as needed.

A map φ on convex polytopes in $\mathbb{R}^d$ into an abelian group is a *valuation* if for polytopes P and Q such that $P \cup Q$ is convex, we obtain the value on $P \cup Q$ by adding the values on P and Q and subtracting the value on the intersection $P \cap Q$:



$$\varphi(P \cup Q) = \varphi(P) + \varphi(Q) - \varphi(P \cap Q).$$

The first example of a valuation that usually comes to mind is the volume.

Another simple example is the Euler characteristic which is constant 1 on non-empty polytopes. But there are far more exciting valuations, as we hope to convince the reader in the following.

Questions concerning the characterization of valuations with specific properties are of central interest in the theory of valuations. The volume and the Euler characteristic of a polytope, for example, have the additional property that they do not change when transforming the polytope by a rigid motion — they are *rigid motion invariant*. A fundamental result is *Hadwiger's theorem* [19], which characterizes real-valued continuous rigid-motion invariant valuations on convex bodies as forming a $(d+1)$-dimensional vector space spanned by the so-called *quermassintegrals*.

In the present work we will consider more general translation-invariant valuations on polytopes with vertices in an additive subgroup of $\mathbb{R}^d$. An example is given by the *lattice point enumerator* $\mathsf{Ehr}(P) = \left|P \cap \mathbb{Z}^d\right|$: restricted to the class of *lattice polytopes*, that is, polytopes with vertices in the integer lattice, $\mathsf{Ehr}(P)$ is a valuation which is invariant under translation by integer vectors. McMullen [28] showed that if φ is a translation-invariant valuation and P is a lattice polytope then the function $\varphi(nP)$ agrees with a polynomial $\varphi_P(n)$ in n of degree at most $\dim(P)$ for natural numbers $n \geq 0$. For the lattice point enumerator this is due to Ehrhart [14] and the counting function $\mathsf{Ehr}_P(n)$ is called the *Ehrhart polynomial*. Ehrhart polynomials appear all over enumerative and algebraic combinatorics. For some of their coefficients an interpretation can be given: The leading coefficient of $\mathsf{Ehr}_P(n)$ corresponds to the volume, the second highest coefficient is related to half the surface area, and the constant coefficient is the Euler characteristic of P. Still, a full understanding of Ehrhart polynomials is far out of sight. However, a groundbreaking step in the direction of a characterization was done by Stanley [42]. He showed that for an r-dimensional lattice polytope $P \subset \mathbb{R}^d$ the Ehrhart polynomial has only *nonnegative integers* as coefficients when written in the polynomial basis $\binom{n+r}{r}, \binom{n+r-1}{r}, \ldots, \binom{n}{r}$:

$$\mathsf{Ehr}_P(n) = h_0^*\binom{n+r}{r} + h_1^*\binom{n+r-1}{r} + \cdots + h_r^*\binom{n}{r}.$$

The coefficients form the so-called *h^*-vector* $(h_0^*, \ldots, h_d^*)$ of P (sometimes called δ-vector), where $h_i^* := 0$ for $i > r$. In [48] Stanley proved furthermore that the coefficients have a *monotone* behavior: For two lattice polytopes $P, Q \subseteq \mathbb{R}^d$ such that $P \subseteq Q$ he showed that $h_i^*(P) \leq h_i^*(Q)$ for all $0 \leq i \leq d$.

In Section 2.6 we consider more generally the expansion of $\varphi_P(n)$ in the polynomial basis $\binom{n+r}{r}, \binom{n+r-1}{r}, \ldots, \binom{n}{r}$ for arbitrary translation-invariant valuations φ. A valuation for which the coefficients are nonnegative for all P will be called h^**-nonnegative*. If the coefficients are monotone with respect to inclusion, then φ will be called h^**-monotone*. Our main theorem states that h^*-monotonicity and h^*-nonnegativity are, in fact, equivalent properties of φ and are characterized by a simple property. This enables us to give a short and simple proof of Stanley's results on the nonnegativity and monotonicity of the Ehrhart h^*-vector, and to reprove a result by Beck, Robins and Sam [3] on *solid-angle polynomials*. Moreover, we consider *Steiner polynomials*, which are closely related to Hadwiger's theorem, investigate a weak notion of h^*-monotonicity and illuminate other related properties of translation-invariant valuations.

In Section 2.7 we study the class of real-valued h^*-nonnegative valuations as a geometric set. We give a new characterization of volume in Section 2.7.1, namely as the unique (up to scaling) h^*-monotone translation-invariant valuation on all polytopes in $\mathbb{R}^d$. In Section 2.7.2 we consider *lattice-invariant valuations* on lattice polytopes: Betke and Kneser showed in [6] that they form a $(d+1)$-dimensional vector space. We give a full characterization of all lattice-invariant h^*-nonnegative valuations for $d \leq 2$ and show that they form a full-dimensional simplicial cone. For $d \geq 0$ we prove that they form a full-dimensional convex cone and conjecture that this cone is polyhedral.

As $\mathsf{Ehr}_P(n)$ agrees with a polynomial for $n \geq 0$, it is natural to ask whether there is a combinatorial interpretation for the evaluation at negative integers. The answer is given by *Ehrhart–Macdonald reciprocity* [26] which states that $(-1)^{\dim(P)}\mathsf{Ehr}_P(-n)$ counts the number of lattice points in the relative interior of nP. Polynomial counting functions that have a natural combinatorial meaning for negative integers — so-called *reciprocities* — occur quite often in combinatorics (see, for example, Stanley [41]). Many of these reciprocities can be seen as an incarnation of *Ehrhart–Macdonald reciprocity*. Ehrhart's result on the polynomiality of $\mathsf{Ehr}_P(n)$ has a multivariate generalization to Minkowski sums: The *Bernstein–McMullen theorem* [4, 28] states that for lattice polytopes $P_1, \dots, P_k$ and natural numbers $n_1, \dots, n_k$ the function $\mathsf{Ehr}_{P_1,\dots,P_k}(n_1, \dots, n_k)$ counting lattice points in $n_1P_1 + \cdots + n_kP_k$ agrees with a multivariate polynomial in $n_1, \dots, n_k$ for $n_1, \dots, n_k \geq 0$. We consider the same question that appears in the univariate case: *Is there an interpretation for the evaluation of* $\mathsf{Ehr}_{P_1,\dots,P_k}$ *at arbitrary integers* $n_1, \dots, n_k \in \mathbb{Z}$*?* Using a formal reciprocity for translation-invariant valuations by McMullen [28] we are able to give an interpretation as weighted enumeration of lattice points in Section 2.5, where the weights are given by the reduced Euler characteristic of certain polytopal complexes.

In Chapter 3 we consider the h^*-vector for a large class of polytopes: *zono-*

topes. Every zonotope $\mathcal{Z}$ with vertices in $\mathbb{Z}^d$ is *integrally closed*, i.e., for all $n \geq 1$ and all $p \in n\mathcal{Z} \cap \mathbb{Z}^d$ there are $p_1, \ldots, p_n \in \mathcal{Z} \cap \mathbb{Z}^d$ such that

$$p = p_1 + \cdots + p_n.$$

The simplest examples of integrally closed polytopes are *unimodular simplices*, i.e., simplices that affinely span $\mathbb{Z}^d$. A sequence $h_0^*, \ldots, h_d^*$ is said to be *unimodal*, if it is of the form

$$h_0^* \leq \cdots \leq h_i^* \geq \cdots \geq h_d^*, \qquad \text{for some } i.$$

Stanley conjectured in [46] that the entries of the Ehrhart h^*-vector of any integrally closed polytope form a unimodal sequence. Schepers and van Langenhoven recently showed in [36] that this is true for all lattice parallelepipeds. Towards Stanley's conjecture, we investigate the h^*-vector of arbitrary lattice zonotopes. Along the way we show unimodality for half-open unit cubes by giving the entries of the h^*-vector a combinatorial interpretation in terms of refined descent statistics of permutations. We then pass to half-open parallelepipeds and, in the end, show unimodality for lattice zonotopes by taking a suitable half-open decomposition as introduced by Köppe and Verdoolaege in [24].

In the second part of the thesis we consider specific enumeration problems. Many problems in enumerative combinatorics, such as counting graph colorings, can be translated into lattice point enumeration problems for certain polyhedral objects. We are concerned with order preserving maps from a finite *partially ordered set* $\mathfrak{P}$ into the chain $[n] = \{1, \ldots, n\}$. A classical theorem by Stanley [38] states that their number is given by a polynomial $\Omega_{\mathfrak{P}}(n)$ for $n \geq 1$ and that there exists a reciprocity, namely that $(-1)^{|\mathfrak{P}|}\Omega_{\mathfrak{P}}(-n)$ equals the number of *strictly* order preserving maps $\mathfrak{P} \to [n]$.

In Chapter 4 we consider a generalized version of this problem: Given a partially ordered set $\mathfrak{P}$, a suitable subposet $\mathfrak{A} \subseteq \mathfrak{P}$, and an order preserving map $\lambda\colon \mathfrak{A} \to \mathbb{Z}$, *what is the number of order preserving extensions of* λ *to* $\mathfrak{P}$*?* By passing to real-valued order preserving maps we can identify every order preserving extension with a lattice point in the corresponding *marked order polytope* introduced by Ardila, Bliem and Salazar in [1]. In Section 4.2 we study the arithmetic of counting lattice points in these polytopes and show that the counting function is given by a *piecewise multivariate polynomial* $\Omega_{\mathfrak{P},\mathfrak{A}}(\lambda)$ in the values of λ. By studying a specific subdivision into products of simplices we can give explicit regions of polynomiality. We further prove a reciprocity for the evaluation at order reversing maps.

In Section 4.4 we apply our results to counting extensions of *partial graph colorings*. We reprove a theorem by Herzberg and Murty [21] and we generalize Stanley's classical reciprocity theorem for graph colorings [40].

The main application of the arithmetic for marked order polytopes concerns *monotone triangles*, which are strongly connected to *alternating sign matrices*. An alternating sign matrix of size n is a $n \times n$-matrix with entries in $\{0, 1, -1\}$ such that the non-zero entries in each row and column alternate in sign and sum up to 1. There is a bijection between these matrices and states in the "square ice" model in statistical mechanics. Mathematical interest was fueled by a longstanding open conjecture of Mills, Robbins and Rumsey [30] from the early 1980s — *the alternating sign matrix conjecture* — which was finally proven by Zeilberger in 1995. In [16] Fischer gave a new proof of a refined version using *monotone triangles*. These are triangular arrays of integers such that the entries increase along the northeast and southeast direction and strictly increase in east direction. Fischer [15] showed that the number

$$\begin{array}{ccccccccc} & & & & 4 & & & & \\ & & & 3 & & 5 & & & \\ & & 2 & & 3 & & 5 & & \\ & 1 & & 3 & & 5 & & 8 & \\ 1 & & 3 & & 4 & & 7 & & 8 \end{array} \qquad \leftrightarrow$$

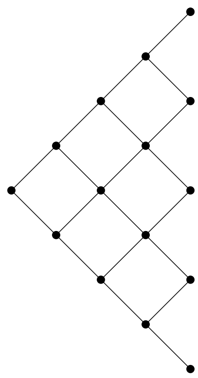


Figure 1: A monotone triangle and a Gelfand–Tsetlin poset of order n.

of monotone triangles with bottom row $\mathbf{k} = (k_1, \ldots, k_n)$ is given by a multivariate polynomial in the strictly increasing entries of k, and later, together with Riegler, she gave an interpretation for the evaluation of this polynomial at weakly decreasing entries [17]. Their proofs are purely algebraic in nature and use advanced methods from the calculus of finite differences. By interpreting monotone triangles as integer-valued order preserving maps from the *Gelfand–Tsetlin poset* we are able to apply our methods to give a short geometric proof of their results in Section 4.3.

Chapter 5 has a purely algebraic nature and deals with problems of enumeration up to symmetry. An important theorem in this context is *Pólya's enumeration theorem* [33]. It gives an explicit formula for the number of orbits of labelings of a set as a polynomial in the number of labels. In Chapter 5 we generalize Pólya's enumeration theorem in terms of order preserving maps. To that end, we consider a finite poset $\mathfrak{P}$ and a group G acting on $\mathfrak{P}$ by *automorphisms*. It turns out that the number of *orbits* of order preserving maps from $\mathfrak{P}$ into the n-chain agrees with a polynomial $\Omega_{\mathfrak{P},G}(n)$ for natural numbers $n \geq 1$. We call this polynomial the *orbital order polynomial*. Moreover, we show a combinatorial reciprocity theorem by giving an interpretation for the evaluation of $\Omega_{\mathfrak{P},G}$ at negative integers. We thus prove an orbital generalization of Stanley's polynomiality and reciprocity theorem for order preserving maps [38]. Further, we outline a generalization to counting orbits of $(\mathfrak{P}, \omega)$-partitions. Applying our results to the poset without relations, called antichain, we obtain Pólya's enumeration theorem.

We further consider orbits of graph colorings of a finite simple graph Γ under the action of a group G on Γ. The function $\chi_{\Gamma,G}(n)$ counting orbits of proper n-colorings is a polynomial — the *orbital chromatic polynomial* — which was first studied by Cameron and Kayibi [9]. We give a new proof for the polynomiality of $\chi_{\Gamma,G}$ by showing that $\chi_{\Gamma,G}$ is a sum of order polynomials. Moreover, we are able to interpret the values of $\chi_{\Gamma,G}$ at negative integers. Thus, we obtain an orbital generalization of Stanley's polynomiality and reciprocity results for graph colorings [40].

The results of Chapter 2 and 4 are based on joint work with Raman Sanyal. The content of Chapter 4 appeared in [23]. Chapter 3 is part of a joint project with Matthias Beck and Emily McCullough. The content of Chapter 5 was published in [22].

Contents

Chapter 1

Basics

The present thesis is situated within geometric, enumerative, and algebraic combinatorics. In this chapter we will introduce the main objects together with their basic properties. Our main sources which we recommend for further reading are Ziegler's lectures on polytopes [52], the book by Beck and Robins on integer point enumeration [2], the first volume of Stanley's book on enumerative combinatorics [49], and the hand book on convex and discrete geometry by Gruber [18]. Basic knowledge of linear algebra and combinatorics is assumed.

Polyhedra

Let $\mathbb{N}$ be the natural numbers $\{0, 1, 2, \ldots, \}$ and $[n] := \{1, \ldots, n\}$ for $n \in \mathbb{N}_{>0}$. Our geometric objects live in the Euclidean space $\mathbb{R}^d$ for some $d \in \mathbb{N}$.

A set $S \subseteq \mathbb{R}^d$ is **convex** if the **segment** $[x, y] = \{(1 - \lambda)x + \lambda y \colon 0 \leq \lambda \leq 1\}$ is contained in S whenever $x, y \in S$. The **convex hull** of S

$$\operatorname{conv}(S) := \left\{ \sum_{i=1}^{m} \lambda_i x_i \colon m \geq 1, x_1, \ldots, x_m \in S, \lambda_1, \ldots, \lambda_m \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

is the inclusion-wise minimal convex set in $\mathbb{R}^d$ containing S.

$P \subset \mathbb{R}^d$ is called a **polytope** if there are finitely many points $x_1, \ldots, x_m$ such that

$$P = \operatorname{conv}(\{x_1, \ldots, x_m\}).$$

The **affine hull** $\operatorname{aff}(S)$ of a set $S \subseteq \mathbb{R}^d$ is the inclusion-wise minimal affine space that contains S, or equivalently

$$\operatorname{aff}(S) := \left\{ \sum_{i=1}^{m} \lambda_i x_i \colon m \geq 1, x_1, \ldots, x_m \in S, \lambda_1, \ldots, \lambda_m \in \mathbb{R}, \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

A **hyperplane** $H \subseteq \mathbb{R}^d$ is an affine space of dimension $d-1$, i.e., there are $a \in \mathbb{R}^d$, $a \neq 0$, and $b \in \mathbb{R}$ such that $H = \{x \in \mathbb{R}^d \colon a^t x = b\}$. H divides $\mathbb{R}^d$ into two half spaces $H^{\geq} = \{x \in \mathbb{R}^d \colon a^t x \geq b\}$ and $H^{\leq} = \{x \in \mathbb{R}^d \colon a^t x \leq b\}$. H is called **supporting** for a set S if S is fully contained in $H^{\geq}$ or $H^{\leq}$.

A subset $P \subseteq \mathbb{R}^d$ is a **polyhedron** or **polyhedral** if it is the intersection of finitely many halfspaces, i.e., if there are $m \in \mathbb{N}$, $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$ such that

$$P = \{x \in \mathbb{R}^d \colon Ax \leq b\}.$$

The **dimension** of P is defined by $\dim(P) := \dim(\operatorname{aff}(P))$.

The *main theorem for polytopes* is the following:

Theorem 1.1 (Minkowksi–Weyl Theorem [52, Theorem 1.1])**.** *$P \subset \mathbb{R}^d$ is a polytope if and only if P is a bounded polyhedron.*

$F \subseteq P$ is a **face** of P if there is a supporting hyperplane H of P such that

$$F = P \cap H.$$

Further, $\varnothing$ and P itself are by definition faces as well. Faces of dimension $0, 1$ and $\dim(P)-1$ are called **vertices**, **edges**, and **facets** respectively. The empty face $\varnothing$ has dimension -1. A face F is called **proper** if $\dim(F) \leq \dim(P) - 1$. $P \subset \mathbb{R}^d$ is **full-dimensional** if $\dim(P) = d$.

The **boundary** ∂P of P is the set of points contained in a proper face. The **relative interior** of P is defined by

$$\operatorname{relint}(P) = P \setminus \partial P.$$

The next theorem collects some fundamental results about polyhedra:

Theorem 1.2 ([52, Propositions 2.2 and 2.3], [18, Theorem 4.4])**.** *Let $P \subset \mathbb{R}^d$ be a polyhedron. Then*

(i) $F \cap G$ is a face of P for all faces F, G of P;

(ii) the faces of a face F of P are exactly the faces of P contained in F;

(iii) if P is a polytope, then P is the convex hull of its vertices;

(iv) *for every point p outside a polyhedron P there is a hyperplane H such that $p \in H^{\geq} \setminus H$ and $P \subset H^{\leq} \setminus H$;*

(v) *for polyhedra P and Q such that* $\operatorname{relint}(P) \cap \operatorname{relint}(Q) = \varnothing$ *there is a hyperplane H such that $P \subseteq H^{\geq}$ and $Q \subseteq H^{\leq}$.*

A set $C \subseteq \mathbb{R}^d$ is a **cone** if for all $\lambda \in \mathbb{R}_{\geq 0}$ and $s \in C$ we also have $\lambda s \in C$. A **convex cone** is a subset $C \subseteq \mathbb{R}^d$ such that for $\lambda, \mu \in \mathbb{R}_{\geq 0}$ and $s, t \in C$ we have $\lambda s + \mu t \in C$. C is called **pointed** if it does not contain a linear subspace of positive dimension. The **conical hull** of a set $S \subseteq \mathbb{R}^d$ is defined by

$$\operatorname{cone}(S) := \left\{ \sum_{i=1}^{m} \lambda_i x_i \colon m \geq 1, x_1, \ldots, x_m \in S, \lambda_1, \ldots, \lambda_m \geq 0 \right\}.$$

The **homogenization** of an r-dimensional polytope $P \subseteq \mathbb{R}^d$ is defined by

$$\hom(P) = \{(x,t) \in \mathbb{R}^d \times \mathbb{R}_{\geq 0} \colon x \in tP\} \subseteq \mathbb{R}^{d+1}.$$

The set $\hom(P)$ is a pointed polyhedral cone of dimension $r+1$.

A polytope Δ with vertex set $v_1, \ldots, v_{r+1}$ is an r-dimensional **simplex** if $v_1, \ldots, v_{r+1}$ are affinely independent. The convex hull of any subset of vertices is a simplex itself and a face of Δ.

For an r-dimensional polytope P and a point p outside the affine hull of P, the **pyramid** $\operatorname{Pyr}_p(P)$ is the $(r+1)$-dimensional polytope defined by

$$\operatorname{Pyr}_p(P) = \operatorname{conv}(P \cup \{p\}).$$

Thus, an r-dimensional simplex can be obtained by applying the pyramid operation r times starting out with a single point.

For $n \in \mathbb{R}_{\geq 0}$ and a polyhedron P, the n-th **dilation** of P is defined by

$$nP = \{np \colon p \in P\}.$$

The **Minkowski sum** of two polyhedra P and Q is

$$P + Q = \{p + q \colon p \in P, q \in Q\}.$$

Both operations, taking the dilation or the Minkowski sum, yield again a polyhedron or a polytope if P and Q are bounded.

A polytope $\mathcal{Z}$ is a **zonotope** if it is the Minkowksi sum of finitely many segments. Equivalently, a zonotope is a translate of

$$\mathcal{Z} = \left\{ \sum_{i=1}^{m} \lambda_i u_i \colon 0 \leq \lambda_1, \ldots, \lambda_m \leq 1 \right\}$$

for some vectors $u_1, \ldots, u_m \in \mathbb{R}^d$. If $u_1, \ldots, u_m$ are linearly independent, then $\mathcal{Z}$ is a **parallelepiped**.

Subdivisions

A **polyhedral complex** is a finite set $\mathcal{C}$ of polyhedra in $\mathbb{R}^d$ such that

(i) $\varnothing \in \mathcal{C}$,

(ii) if $P \in \mathcal{C}$, then all faces of P are in $\mathcal{C}$ as well,

(iii) if $P, Q \in \mathcal{C}$, then $P \cap Q$ is a face both of P and Q.

The **dimension** of $\mathcal{C}$ is defined as the maximal dimension of a polyhedron in $\mathcal{C}$. The **underlying set** of $\mathcal{C}$ is the set $|\mathcal{C}| = \bigcup_{P \in \mathcal{C}} P \subseteq \mathbb{R}^d$. $|\mathcal{C}|$ can be partitioned into relatively open polyhedra

$$|\mathcal{C}| = \bigsqcup_{P \in \mathcal{C}} \operatorname{relint}(P).$$

A set $\mathcal{C}' \subseteq \mathcal{C}$ is a **subcomplex** of $\mathcal{C}$ if it is itself a polyhedral complex. $\mathcal{C}$ is called a **polytopal complex** if it contains only polytopes.

Let $f_i(\mathcal{C}) = |\{F \in \mathcal{C} \colon \dim(F) = i\}|$. An important topological invariant is the **Euler characteristic** defined by

$$\chi(\mathcal{C}) = f_0(\mathcal{C}) - f_1(\mathcal{C}) + f_2(\mathcal{C}) - \cdots .$$

The **reduced Euler characteristic** is $\tilde{\chi}(\mathcal{C}) = \chi(\mathcal{C}) - f_{-1}(\mathcal{C}) = \chi(\mathcal{C}) - 1$.

The collection of faces of a polyhedron $P \subseteq \mathbb{R}^d$ is by Theorem 1.2 a polyhedral complex with underlying set P. It is denoted by $\mathcal{L}(P)$ and called **face lattice**. The set of proper faces forms the **boundary complex** ∂P of P.

The **Euler–Poincaré formula** states that for every non-empty polytope P we have

$$\chi(P) := \chi(\mathcal{L}(P)) = 1.$$

A polyhedral complex $\mathcal{C}$ is a **polyhedral subdivision** of a polyhedron P if $|\mathcal{C}| = P$. Thus, $\mathcal{L}(P)$ is the trivial subdivision. $\mathcal{C}$ is **using no new vertices** if every vertex of a polytope in $\mathcal{C}$ is a vertex of P. If F is a face of P, then $\mathcal{C}$ induces a subdivision of F

$$\mathcal{C}_F = \{C \cap F : C \in \mathcal{C}\}.$$

$\mathcal{C}$ is called a **triangulation** if all polytopes in $\mathcal{C}$ are simplices.

Theorem 1.3 ([2, Theorem 3.1])**.** *Every polytope $P \subseteq \mathbb{R}^d$ has a triangulation $\mathcal{C}$ of P using no new vertices.*

Let $p \in \mathbb{R}^d$. A face F of a polyhedron P is **visible** from p if for all $q \in F$ we have $[p, q) = \{(1-\lambda)p + \lambda q : 0 \leq \lambda < 1\} \subseteq \mathbb{R}^d \setminus P$. Equivalently, by Theorem 1.2, F is visible if there is a supporting hyperplane H of P such that $P \cap H = F$, $P \subseteq H^{\leq}$ and $p \in H^{\leq} \setminus H$. The faces visible from p form a subcomplex of P called the **visibility complex** which is denoted by $\mathrm{Vis}_p(P)$. (See Figure 1.) More generally, if $\mathcal{C}$ is a subdivision of P, then $\mathrm{Vis}_p(P)$ induces a subcomplex $\mathrm{Vis}_p(\mathcal{C}) := \{C \cap F : C \in \mathcal{C}, F \in \mathrm{Vis}_p(P)\} \subseteq \mathcal{C}$.

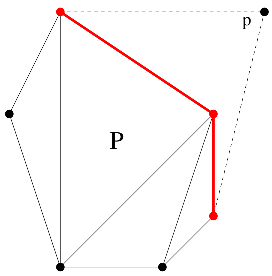


Figure 1.1: A triangulated polytope P and its visibility complex (red).

Theorem 1.3 can, for example, be shown using the **beneath-beyond algorithm** (see, e.g., [12, Section 4.3.1]):

Theorem 1.4 (Beneath-Beyond algorithm)**.** *Let P be a polytope given by $P = \mathrm{conv}(\{v_1, \ldots, v_m\})$. Then the algorithm*

$\mathcal{C} = \{\varnothing\}$;
FOR $i = 1, \ldots, m$ DO
  $\mathcal{C} = \mathcal{C} \cup \{\mathrm{conv}(F \cup \{v_i\}) : F \in \mathrm{Vis}_{v_i}(\mathcal{C}) \setminus \{\varnothing\}\}$;
RETURN $\mathcal{C}$;

returns a triangulation of P. If $v_1, \ldots, v_m$ are vertices of P, then $\mathcal{C}$ is a triangulation using no new vertices.

If $Q \subseteq P$ is a polytope with vertices $w_1, \ldots, w_l$ then by passing the points $v'_1 = w_1, \ldots, v'_l = w_l, v'_{l+1} = v_1, \ldots, v'_{l+m} = v_m$ to the algorithm we obtain a triangulation of Q and an extension to P using only vertices of P and Q.

Posets

A **partially ordered set**, or **poset** for short, is a set $\mathfrak{P}$ together with a binary relation $\preceq$ such that the following three conditions are satisfied:

(i) $x \preceq x$ for all $x \in \mathfrak{P}$ (reflexivity),

(ii) if $x \preceq y$ and $y \preceq z$, then $x \preceq z$ for $x, y, z \in \mathfrak{P}$ (transitivity), and

(iii) if $x \preceq y$ and $y \preceq x$, then $x = y$ for $x, y \in \mathfrak{P}$ (antisymmetry).

A poset $\mathfrak{A} \subseteq \mathfrak{P}$ with a binary relation $\preceq_{\mathfrak{A}}$ is a **subposet** of $\mathfrak{P}$ if $x \preceq_{\mathfrak{A}} y$ implies $x \preceq y$ in $\mathfrak{P}$ for all $x, y \in \mathfrak{A}$. Moreover, $\mathfrak{A}$ is called **induced** if $x \preceq y$ in $\mathfrak{P}$ implies $x \preceq_{\mathfrak{A}} y$ for all $x, y \in \mathfrak{A}$.

Two elements $x, y \in \mathfrak{P}$ are called **comparable** if $x \preceq y$ or $y \preceq x$; otherwise they are called **incomparable**. A set of pairwise incomparable elements is called an **antichain**; a set of elements $\{x_1 \preceq \cdots \preceq x_n\}$ is called a **chain**. $I \subseteq \mathfrak{P}$ is an **ideal** if $x \in I$ whenever there is a y with $x \preceq y$ and $y \in I$.

An **upper bound** of a set $S \subseteq \mathfrak{P}$ is an element $M \in \mathfrak{P}$ such that $s \preceq M$ for all s in S. Analogously, m is a **lower bound** if $m \preceq s$ for all $s \in S$. The **join** $x \vee y$ of two elements $x, y \in \mathfrak{P}$ is the unique least upper bound of x and y if it exists, i.e., $x \vee y \preceq M$ for every upper bound M of $\{x, y\}$. Analogously, the **meet** $x \wedge y$ is the unique largest lower bound of x and y if it exists. A poset in which every pair of elements x and y has a join and a meet is called a **lattice**. Thus, in a finite lattice there is a unique minimal element and a unique maximal element denoted by $\hat{0}$ and $\hat{1}$, respectively.

For $x, y \in \mathfrak{P}$ with $x \preceq y$ the set $[x, y] := \{z \in \mathfrak{P} \colon x \preceq z \preceq y\}$ is called an **interval**.

An element y **covers** x if $x \prec y$ and $[x, y] \setminus \{x, y\} = \varnothing$. The relation between x and y is called a **cover relation** and we write $x \prec\cdot\, y$.

In a poset $\mathfrak{P}$ with minimal element $\hat{0}$ and maximal element $\hat{1}$, all $a \in \mathfrak{P}$ with $\hat{0} \prec\cdot\, a$ are called **atoms**. Analogously, all $b \in \mathfrak{P}$ such that $b \prec\cdot\, \hat{1}$ are called **coatoms**.

The elements of a polyhedral complex $\mathcal{C}$, ordered by inclusion, form a poset. In fact, by Theorem 1.2, this is a **meet-semilattice**, i.e., a poset in which meets exist. If $\mathcal{C} = \mathcal{L}(P)$ for some polyhedron P, then it is even a lattice, which justifies the name face lattice for $\mathcal{L}(P)$.

The **Hasse diagram** of a poset $\mathfrak{P}$ is a graphical diagram, where the vertices correspond to the elements of $\mathfrak{P}$ and the edges to cover relations. If $x \prec\cdot$

y then y is drawn above x. Sometimes we will identify $\mathfrak{P}$ with its Hasse diagram.

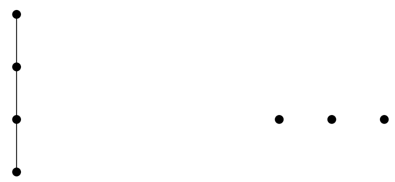


Figure 1.2: Hasse diagram of a chain (left) and of an antichain (right).

Möbius functions of posets

Let $S_1, \ldots, S_m$ be finite sets. The classical **inclusion-exclusion principle** allows us to calculate the number of elements in the union $\bigcup_{i=1}^m S_i$ as an alternating sum of the number of elements in all possible intersections of $S_1, \ldots, S_m$:

Theorem 1.5 (Inclusion-exclusion principle)**.** *Let $S_1, \ldots, S_m$ be finite sets. Then*

$$|S_1 \cup \cdots \cup S_m| = |S_1| + \cdots + |S_m| - |S_1 \cap S_2| - \cdots = \sum_{\varnothing \neq I \subseteq [m]} (-1)^{|I|-1} \left| \cap_{i \in I} S_i \right|. \tag{1.1}$$

Some of the intersections might be empty or occur several times. A priori, it is not clear how often each summand will appear on the right-hand side of equation (1.1). With the help of Möbius functions we can calculate the correct multiplicity of every non-empty intersection.

The **Möbius function** $\mu_\Pi \colon \mathfrak{P} \times \mathfrak{P} \to \mathbb{Z}$ of a finite poset $\mathfrak{P}$ is recursively defined by

$$\mu_{\mathfrak{P}}(x,y) = \begin{cases} 0 & \text{if } x \npreceq y, \\ 1 & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu_{\mathfrak{P}}(x,z) & \text{otherwise.} \end{cases}$$

Its importance stems from the following remarkable property:

Theorem 1.6 (Möbius inversion formula [49, Proposition 3.7.1])**.** *Let $\mathfrak{P}$ be a finite poset and let $f, g \colon \mathfrak{P} \to G$, where G is an abelian group. Then*

$$g(y) = \sum_{x \preceq y} f(x) \text{ for all } y \in \mathfrak{P}$$

if and only if

$$f(y) = \sum_{x \preceq y} g(x)\mu_{\mathfrak{P}}(x,y) \text{ for all } y \in \mathfrak{P}.$$

Let again $S_1, \ldots, S_m$ be finite sets. The **intersection poset** $\mathfrak{P}_\cap$ consists of all the intersections $\{\cap_{i \in I} S_i \colon \varnothing \neq I \subseteq m\}$ ordered by inclusion. Let $\hat{\mathfrak{P}}_\cap = \mathfrak{P}_\cap \cup \{\hat{1}\}$, where we define $S_1 \cup \cdots \cup S_m =: \hat{1}$. Then we obtain the following proposition (see, for example, [49, Section 3.7]).

Proposition 1.7. *Let $S_1, \ldots, S_m$ be finite sets. Then*

$$|S_1 \cup \ldots \cup S_m| = -\sum_{S \in \mathfrak{P}_\cap} \mu_{\hat{\mathfrak{P}}_\cap}(S, \hat{1})\,|S|\,.$$

The Möbius function of a subdivision $\mathcal{C}$ of a polytope P has a simple form (see, e.g., [49, Proposition 3.8.9]):

Theorem 1.8. *Let P be an r-dimensional polytope and let $\mathcal{C}$ be a subdivision of P. Then $\mathfrak{P} = \mathcal{C} \cup \{\hat{1}\}$ is a lattice and*

$$\mu_{\mathfrak{P}}(F,G) = \begin{cases} 0 & \text{if } G = \hat{1} \text{ and } F \subseteq \partial P, \\ (-1)^{\dim(G)-\dim(F)} & \text{otherwise,} \end{cases}$$

where $\dim(\hat{1}) := r + 1$.

In particular, if F, G are faces of a polytope P and $F \subseteq G$, then $\mu_P(F,G) = (-1)^{\dim(G)-\dim(F)}$, where μ_P denotes the Möbius function of $\mathcal{L}(P)$.

One technique to determine the Möbius functions of lattices in general is the Crosscut Theorem:

Theorem 1.9 (Crosscut Theorem [25, Section 3.1.9])**.** *Let $\mathfrak{L}$ be a finite lattice and X be its set of atoms. Then*

$$\mu_{\mathfrak{L}}(\hat{0}, \hat{1}) = \sum_{k \geq 0} (-1)^k X_k,$$

where X_k is the number of k-element subsets of X with join $\hat{1}$.

Lattices and valuations

The word *lattice* has two different meanings in this thesis. Apart from the poset theoretic context, a **lattice** is a discrete additive subgroup of $\mathbb{R}^d$, i.e., a subgroup without limit points. An important example is $\mathbb{Z}^d$. In fact, every lattice is isomorphic to an integer lattice $\mathbb{Z}^l$ for some $0 \leq l \leq d$ (see, e.g., [32, Proposition 4.2]).

Throughout Λ will denote a lattice in $\mathbb{R}^d$ or a vector subspace over some subfield of $\mathbb{R}$. A polytope with vertices in Λ will be called a **Λ-polytope**, or **lattice polytope** if $\Lambda = \mathbb{Z}^d$. The class of all Λ-polytopes is denoted by $\mathcal{P}(\Lambda)$.

Let G be an abelian group. A **valuation** on Λ-polytopes is a map $\varphi\colon \mathcal{P}(\Lambda) \to G$ such that $\varphi(\varnothing) = 0$ and

$$\varphi(P \cup Q) = \varphi(P) + \varphi(Q) - \varphi(P \cap Q)$$

for all $P, Q \in \mathcal{P}(\Lambda)$ with $P \cup Q \in \mathcal{P}(\Lambda)$ and $P \cap Q \in \mathcal{P}(\Lambda)$[1]. φ is called **simple** if $\varphi(P) = 0$ for all Λ-polytopes with $\dim(P) < d$. φ is **homogeneous** of degree r if for all $n \in \mathbb{N}$ and for all $P \in \mathcal{P}(\Lambda)$ we have $\varphi(nP) = n^r \varphi(P)$.

A fundamental valuation is $\chi\colon \mathcal{P}(\Lambda) \to \mathbb{Z}$ with $\chi(P) = 1$ for every non-empty polytope P. As this valuation coincides with the Euler characteristic on every polytope, it is itself referred to as the **Euler characteristic**. An example of a simple and homogeneous valuation of degree d is the d-dimensional **volume** vol_d.

McMullen showed in [29] that every valuation $\mathcal{P}(\Lambda) \to G$ satisfies the so-called **inclusion-exclusion** property:

Theorem 1.10 (Inclusion-exclusion property)**.** *Let $\varphi\colon \mathcal{P}(\Lambda) \to G$ be a valuation and let $P_1, \ldots, P_m \in \mathcal{P}(\Lambda)$ such that*

(i) $P = P_1 \cup \cdots \cup P_m \in \mathcal{P}(\Lambda)$,

(ii) for all $\varnothing \neq I \subseteq [m]$ we have $\bigcap_{i \in I} P_i \in \mathcal{P}(\Lambda)$.

Then

$$\varphi(P) = \sum_{\varnothing \neq I \subseteq [m]} (-1)^{|I|-1} \varphi\left(\bigcap_{i \in I} P_i\right).$$

For real-valued Λ-valuations this was shown by Betke [5].

McMullen proved, in fact, something stronger, namely that in the context of valuations, *every Λ-polytope can be identified with its characteristic function.*

[1] The assumption $P \cap Q \in \mathcal{P}(\Lambda)$ is not necessary. In fact, $P \cap Q \in \mathcal{P}(\Lambda)$ already follows from $P, Q, P \cup Q \in \mathcal{P}(\Lambda)$.

The **characteristic function** of a polytope P is defined by

$$\mathbf{1}_P(x) = \begin{cases} 1 & \text{if } x \in P, \\ 0 & \text{otherwise.} \end{cases}$$

The precise statement is the following:

Theorem 1.11 ([29, Theorem 8.1])**.** *Let $\varphi\colon \mathcal{P}(\Lambda) \to G$ be a valuation. For Λ-polytopes $P_1, \dots, P_k$ and $\alpha_1, \dots, \alpha_k \in \mathbb{Z}$ we have*

$$\sum_{i=1}^{k} \alpha_i \cdot \varphi(P_i) = 0 \quad \textit{whenever} \quad \sum_{i=1}^{k} \alpha_i \cdot \mathbf{1}_{P_i} = 0.$$

Generating functions

Let G be an abelian group. Then $G^{\mathbb{Z}}$ denotes the set of function $\mathbb{Z} \to G$. Let $S\colon G^{\mathbb{Z}} \to G^{\mathbb{Z}}$, $(Sf)(n) = f(n+1)$ be the **shift operator** and let I denote the identity operator on $G^{\mathbb{Z}}$. The **difference operator** is defined by

$$\begin{aligned} \Delta\colon G^{\mathbb{Z}} &\to G^{\mathbb{Z}} \\ f &\mapsto (n \mapsto f(n+1) - f(n)), \end{aligned}$$

i.e., $\Delta = S - I$. A function $f\colon \mathbb{Z} \to G$ (or $f\colon \mathbb{N} \to G$) is a **polynomial** of degree at most d if $\Delta^{d+1} f = 0$, i.e., if f satisfies the recursion

$$0 = \sum_{i=0}^{d+1} (-1)^i \binom{d+1}{i} f(n+i) \quad \text{for all } n \in \mathbb{N}.$$

If the domain of f is $\mathbb{N}$, we say that f **agrees with a polynomial** and by using the recursion "backwards" we can uniquely expand f to a polynomial $\mathbb{Z} \to G$.

The values $f(0), \Delta f(0), \dots, \Delta^d f(0)$ uniquely determine f:

$$\begin{aligned} f(n) &= S^n f(0) \\ &= (I + \Delta)^n f(0) \\ &= \sum_{i=0}^{d} \binom{n}{i} \Delta^i f(0). \end{aligned}$$

On the other hand, f has a unique representation as $f(n) = \sum_{i=0}^{d} \binom{n}{i} a_i$ for all $n \in \mathbb{Z}$, namely $a_i = \Delta^i f(0)$ for $0 \le i \le d$.

The set of formal power series $G[[t]]$ is a $\mathbb{Z}[t]$-module. An element $F(t) = \sum_{n\geq 0} a_n t^n \in G[[t]]$ is **rational** if there are $h(t) \in G[t] \subset G[[t]]$ and $q(t) \in \mathbb{Z}[t] \setminus \{0\}$ such that $q(t) \cdot F(t) = h(t)$ and we write

$$\sum_{n\geq 0} a_n t^n = \frac{h(t)}{q(t)}.$$

We have the following characterization for polynomiality:

Theorem 1.12 ([49, Corollary 4.3.1]). *Let $f\colon \mathbb{Z} \to G$ and $d \in \mathbb{N}$. Then the following are equivalent:*

(i)
$$\sum_{n\geq 0} f(n)t^n = \frac{h(t)}{(1-t)^{d+1}}, \textit{ where } h(t) \in G[t] \textit{ and } \deg(h) \leq d,$$

(ii) $\Delta^{d+1} f = 0$, i.e., f is a polynomial of degree at most d.

Further, the following reciprocity holds:

Theorem 1.13 ([49, Proposition 4.2.3]). *Let $d \in \mathbb{N}$ and let $f\colon \mathbb{Z} \to G$ be a function such that*

$$f(n+d) + \alpha_1 f(n+d-1) + \cdots + \alpha_d f(n) = 0 \textit{ for all } n \in \mathbb{Z}.$$

Then $F(t) = \sum_{n\geq 0} f(n)t^n$ is rational as well as

$$\overline{F}(t) := \sum_{n\geq 1} f(-n)t^n,$$

and we have

$$\overline{F}\left(\frac{1}{t}\right) = -F(t)$$

as rational functions.

Part I

On the combinatorics of valuations

Chapter 2

On the combinatorics of valuations

2.1 Introduction

Enumerating lattice points in polytopes is a classical topic in geometric combinatorics. A classical theorem by Ehrhart [14] states that the function $\mathsf{Ehr}(nP)$ counting lattice points in the n-th dilate of an r-dimensional lattice polytope P in $\mathbb{R}^d$ agrees with a polynomial $\mathsf{Ehr}_P(n)$ of degree r for $n \geq 1$, the **Ehrhart polynomial** of P. It follows that the generating series — the **Ehrhart series** — is rational of the form

$$\mathsf{Ehr}_P(t) := 1 + \sum_{n\geq 1} \mathsf{Ehr}_P(n)t^n = \frac{h_0^*(P) + h_1^*(P)t + \cdots + h_r^*(P)t^r}{(1-t)^{r+1}}.$$

The vector $h^* = h^*(P) := (h_0^*(P), \ldots, h_d^*(P))$ is called the **h^*-vector** of P where $h_i^*(P) := 0$ for all $i > r$. The importance of the h^*-vector stems from the fact that it encodes the Ehrhart polynomial in a different polynomial basis:

$$\mathsf{Ehr}_P(n) = h_0^*(P)\binom{n+r}{r} + h_1^*(P)\binom{n+r-1}{r} + \cdots + h_r^*(P)\binom{n}{r}.$$

Although a complete classification of Ehrhart polynomials seems out of sight, there are non-trivial constraints on the set of Ehrhart polynomials. The starting point of this chapter are the following two fundamental results:

The first one is **Stanley's Nonnegativity Theorem** [42] stating that the entries of the h^*-vectors are nonnegative integers for all polytopes P. Stanley further showed in [48] that the h^*-vectors are **monotone** with respect to inclusion, i.e., for two lattice polytopes $P, Q \subset \mathbb{R}^d$ such that $P \subseteq Q$ we have $h_i^*(P) \leq h_i^*(Q)$ for $0 \leq i \leq d$.

The second one is **Ehrhart–Macdonald reciprocity** (see e.g. [2, Theorem 4.1]) which states that evaluating the Ehrhart polynomial at negative integers has a combinatorial meaning, namely $(-1)^{\dim P}\mathsf{Ehr}_P(-n)$ equals the number of points in the relative interior of nP. In particular, the sign of $\mathsf{Ehr}_P(n)$ is constant for $n \in \mathbb{Z}_{<0}$.

The **Bernstein–McMullen Theorem** (see e.g. [18, Theorem 19.4]) generalizes Ehrhart's theorem to counting lattice points in Minkowski sums of polytopes: It states that for lattice polytopes $P_1, \dots, P_k \subset \mathbb{R}^d$, the function counting lattice points in $n_1P_1 + \cdots + n_kP_k$ agrees with a multivariate polynomial for integers $n_1, \dots, n_k \geq 0$. Motivated by Ehrhart–Macdonald reciprocity, we ask for an interpretation for the evaluation of this multivariate polynomial at arbitrary integers $n_1, \dots, n_k \in \mathbb{Z}$. In Section 2.5 we give an interpretation as the weighted enumeration of lattice points where the weights are given by the reduced Euler characteristic of certain polytopal complexes.

We achieve these results by considering **translation-invariant valuations**. In the sequel let Λ be a discrete additive subgroup or a vector subspace of $\mathbb{R}^d$ over a subfield of $\mathbb{R}$. A **Λ-valuation**, or **translation-invariant valuation**, is a valuation $\varphi\colon \mathcal{P}(\Lambda) \to G$, where G is an abelian group, such that for all $t \in \Lambda$ and all $P \in \mathcal{P}(\Lambda)$

$$\varphi(P + t) = \varphi(P).$$

McMullen [28] showed that for a Λ-valuation φ the function $\varphi(nP)$ agrees with a polynomial for $n \geq 0$. He further showed a reciprocity. In Section 2.3 and Section 2.4 we will reproduce these theorems. A fundamental tool for our work are half-open decompositions of polyhedra as introduced by Köppe and Verdoolaege [24]. Polynomiality enables us to consider the coefficients of the numerator polynomial of the rational generating function $\sum_{n\geq 0} \varphi(nP)t^n$. A valuation is called **h^*-nonnegative** if all coefficients of the numerator are nonnegative, and **h^*-monotone** if the coefficients are monotone. In Section 2.6.1 we show that h^*-nonnegativity and -monotonicity are, in fact, equivalent properties of φ, and we give a simple characterization of this class of valuations. This allows us to give a short proof of Stanley's results on the monotonicity and nonnegativity of the Ehrhart h^*-vector. Further, we are able to reprove a result by Beck, Robins and Sam [3], which states h^*-nonnegativity and -monotonicity for **solid-angle polynomials**. We moreover give an example of a valuation that is not h^*-monotone related to **Steiner polynomials**.

In Section 2.6.2 we characterize all valuations that are h^*-monotone in a weak sense, such as the Euler characteristic, and in Section 2.6.3 we consider other related properties of valuations.

In Section 2.7 we investigate the geometry of h^*-nonnegative valuations. We show in Section 2.7.1 that there exists, up to scaling, only a single real-valued h^*-nonnegative $\mathbb{R}^d$-valuation — the d-dimensional volume. In Section 2.7.2 we consider real-valued $\mathbb{Z}^d$-valuations that are in addition invariant under **lattice transformations**. Betke and Kneser showed in [6] that they form a $(d+1)$-dimensional vector space. For $d \leq 2$ we completely determine the set of h^*-nonnegative valuations. We show that it is a simplicial cone and explicitly give its generators. In the general case $d \geq 0$ we prove that it is a full-dimensional convex cone and conjecture that it is polyhedral.

This chapter is based on joint work with Raman Sanyal.

2.2 Λ-valuations

Let $\varphi\colon \mathcal{P}(\Lambda) \to G$ be a Λ-valuation. The set of all Λ-valuation will be denoted by $\operatorname{Val}(\Lambda)$. Usually we will tacitly assume that $G = \mathbb{R}$ although if not stated otherwise the results are valid for arbitrary ordered abelian groups, i.e., groups G with a (partial) order $\preceq$ such that for all $a, b, c \in G$ we have $a + c \preceq b + c$ whenever $a \preceq b$.

By Theorem 1.11 we can extend our definition of φ to half-open polytopes: If $B \subseteq \partial P$ is the union of faces $F_1, \ldots, F_m$, then

$$\varphi(P \setminus B) := \varphi(P) - \sum_{\varnothing \neq I \subseteq [m]} (-1)^{|I|-1} \varphi\left(\bigcap_{i \in I} F_i\right). \tag{2.1}$$

In particular, for $B = \partial P$ we have by Theorem 1.8

$$\varphi(\operatorname{relint} P) = \sum_{F \in \mathcal{L}(P)} (-1)^{\dim(P)-\dim(F)} \varphi(F).$$

This can, for example, also be seen from the following results by Sallee, who showed in [34] that $\varphi(\operatorname{relint}(P))$ is, up to sign, a valuation:

Theorem 2.2.1 (Sallee [34]). *Let φ be a valuation. Then*

$$\varphi^*(P) = (-1)^{\dim(P)} \varphi(\operatorname{relint}(P))$$

defines a valuation.

Clearly, if φ is a Λ-valuation, then so is φ^*. Sallee furthermore proved the following:

Theorem 2.2.2 (Sallee [34])**.** *Let φ be a valuation. Then*

$$\varphi^{**} = \varphi.$$

An important tool are **half-open decompositions**. Köppe and Verdoolaege showed in [24] that every subdivision gives rise to a decomposition into half-open polytopes:

Theorem 2.2.3 ([24, Theorem 3])**.** *Let $P \subset \mathbb{R}^d$ be a polytope and $\mathcal{C}$ a subdivision of P. Let $P_1, \ldots, P_m$ be the elements of maximal dimension in $\mathcal{C}$ given by*

$$P_i = \{x \in \mathbb{R}^d \colon a_{i,j}^t x \le b_{i,j}, 1 \le j \le l_i\}$$

for certain facet defining $a_{i,j} \in \mathbb{R}^d$ and displacements $b_{i,j} \in \mathbb{R}$. Let $p \in \operatorname{aff}(P)$ outside any facet-defining hyperplane of $\mathcal{C}$, i.e.,

$$a_{i,j}^t p \neq b_{i,j}$$

for all $1 \le i \le m$ and $1 \le j \le l_i$. Let

$$\begin{aligned} \tilde{P}_i = \{x \in \mathbb{R}^d \colon\ & a_{i,j}^t x \le b_{i,j}, \text{ for } j \text{ such that } a_{i,j} p < b_{i,j}, \\ & a_{i,j}^t x < b_{i,j}, \text{ for } j \text{ such that } a_{i,j} p > b_{i,j}\}. \end{aligned}$$

Then $\tilde{P}_i = P_i \setminus |\mathrm{Vis}_p(P_i)|$, and

$$P \setminus |\mathrm{Vis}_p(P)| = \bigsqcup_{i=1}^m \tilde{P}_i$$

is a decomposition into half-open polytopes. In particular, by choosing $p \in P$, we obtain a half-open decomposition of P. If $\mathcal{C}$ is a triangulation, we obtain a decomposition into half-open simplices, where every half-open simplex arises from a simplex leaving out at most $\dim(P)$ many facets.

If P is a Λ-polytope and $\mathcal{C}$ is a subdivision of P using no new vertices, then every component $\tilde{P}_i$ is a half-open Λ-polytope. Thus, by Theorem 1.11, we can determine $\varphi(P \setminus |\mathrm{Vis}_p(P)|)$ from the half-open decomposition without using the inclusion-exclusion principle:

$$\varphi(P \setminus |\mathrm{Vis}_p(P)|) = \sum_{i=1}^m \varphi(\tilde{P}_i).$$

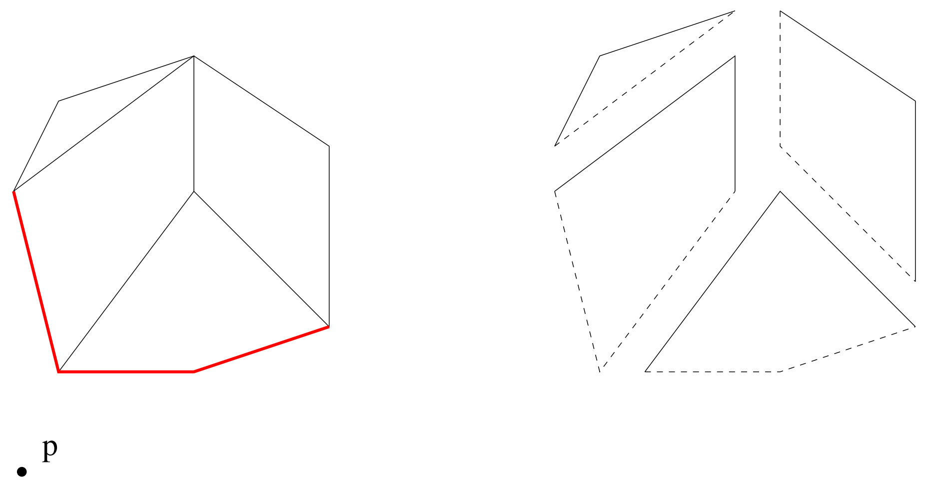


Figure 2.1: A subdivided polytope with visibility complex (red) and the corresponding half-open decomposition.

2.3 Polynomiality

For a polytope $P \in \mathcal{P}(\Lambda)$ and a Λ-valuation φ let $\varphi(nP)$ be the value of φ at the n-th dilate of P. It is due to McMullen [28] that $\varphi(nP)$ agrees with a polynomial $\varphi_P(n)$ for $n \geq 0$. Sometimes we will abuse notation and identify $\varphi(nP)$ with the polynomial it agrees with. Using the generating function

$$F_\varphi(P,t) := \sum_{n \geq 0} \varphi(nP) t^n$$

we will prove:

Theorem 2.3.1 ([28, Theorem 5])**.** *Let $P \in \mathcal{P}(\Lambda)$ be an r-dimensional polytope. Then*

$$F_\varphi(P,t) = \frac{h^*_{\varphi,0}(P) + h^*_{\varphi,1}(P)t + \cdots + h^*_{\varphi,r}(P)t^r}{(1-t)^{r+1}},$$

where $h^*_{\varphi,r}(P) = \varphi(\operatorname{relint}(-P))$. In particular, $\varphi(nP)$ agrees with a polynomial $\varphi_P(n)$ of degree at most r for all $n \geq 0$.

Proof. First, let $P \in \mathcal{P}(\Lambda)$ be a simplex with vertices $v_1, \ldots, v_{r+1}$. Its homogenization $\hom(P) = \{(p,t) \in \mathbb{R}^d \times \mathbb{R}_{\geq 0} \colon p \in tP\} \subset \mathbb{R}^{d+1}$ is a convex polyhedral cone generated by $\tilde{v}_1 := (v_1, 1), \ldots, \tilde{v}_{r+1} := (v_{r+1}, 1)$ and can be partitioned by translates of the half-open parallelepiped

$$\Pi = \{a_1\tilde{v}_1 + \cdots + a_{r+1}\tilde{v}_{r+1} \colon 0 \leq a_i < 1\} \subset \mathbb{R}^{d+1}.$$

In fact,

$$\hom(P) = \bigsqcup_{s_1,\dots,s_{r+1}\in\mathbb{Z}_{\geq 0}} (\Pi + s_1\tilde{v}_1 + \cdots + s_{r+1}\tilde{v}_{r+1}) .$$

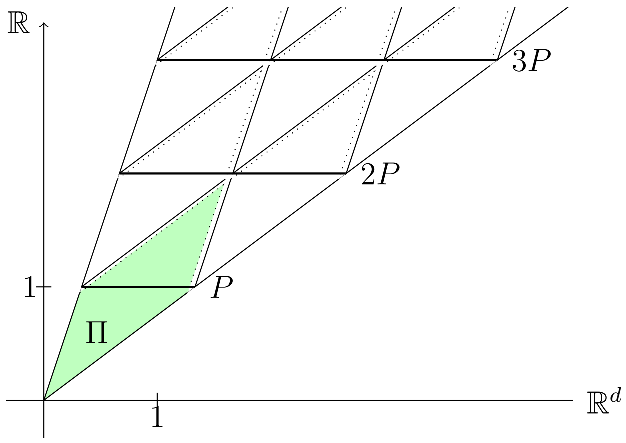


Figure 2.2: Partition of Hom(P) into translates of the half-open parallelepiped Π (in green).

For every $n \geq 0$ we identify $\{x \in \mathbb{R}^{d+1} \colon x_{d+1} = n\}$ with $\mathbb{R}^d$ by forgetting the last coordinate.

Then

$$F_\varphi(P,t) = \sum_{n\geq 0} \varphi(\hom(P) \cap \{x_{d+1} = n\})t^n. \tag{2.2}$$

We observe that $(\Pi + s_1\tilde{v}_1 + \cdots + s_{r+1}\tilde{v}_{r+1}) \cap \{x_{d+1} = n\}$ is a translate of $\Pi \cap \{x_{d+1} = n - s_1 - \cdots - s_{r+1}\}$ by an element in $\Lambda \times \mathbb{Z}$. It is empty if $n - s_1 - \cdots - s_{r+1}$ is negative or $> r$. Its closure is the hypersimplex

$$\mathrm{conv}(\{a_1\tilde{v}_1 + \cdots + a_{r+1}\tilde{v}_{r+1} \colon a_i \in \{0,1\}, \sum_{i=1}^{r+1} a_i = n - s_1 - \cdots - s_{r+1}\}) \in \mathcal{P}(\Lambda).$$

By the inclusion-exclusion property and by translation-invariance the right-hand side of equation (2.2) equals

$$\begin{aligned}
&\sum_{s_1,\ldots,s_{r+1}\geq 0}\sum_{n\geq 0}\varphi(\Pi\cap\{x_{d+1}=n-s_1-\cdots-s_{r+1}\})t^n \\
&=\sum_{s_1,\ldots,s_{r+1}\geq 0} t^{s_1+\cdots+s_{r+1}}\sum_{n=0}^{r}\varphi(\Pi\cap\{x_{d+1}=n\})t^n \\
&=\frac{1}{(1-t)^{r+1}}\sum_{n=0}^{r}\varphi(\Pi\cap\{x_{d+1}=n\})t^n.
\end{aligned}$$

Therefore, by Theorem 1.12, $\varphi(nP)$ is given by a polynomial for $n\geq 0$. We observe that

$$\Pi\cap\{x_{d+1}=r\}=\operatorname{relint}(-(P,1))+\tilde{v}_1+\cdots\tilde{v}_{r+1},$$

and therefore is a translate of $\operatorname{relint}(-P)$. Thus, $h^*_{\varphi,r}(P)=\varphi(\operatorname{relint}(-P))$.

For the general case, we consider a triangulation of P. By inclusion-exclusion we can represent $F_\varphi(P,t)$ by

$$F_\varphi(P,t)=\frac{g_0(P)+g_1(P)t+\cdots+g_d(P)t^d}{(1-t)^{d+1}}$$

for all $P\in\mathcal{P}(\Lambda)$. Then, $P\mapsto g_d(P)$ is a Λ-valuation and for every simplex $P\in\mathcal{P}(\Lambda)$ we have $g_d(P)=(-1)^{d-\dim(P)}\varphi(\operatorname{relint}(-P))$. Further, $P\to(-1)^{d-\dim(P)}\varphi(\operatorname{relint}(-P))$ is a Λ-valuation by Theorem 2.2.1. Thus,

$$h^*_{\varphi,\dim(P)}(P)=(-1)^{d-\dim(P)}g_d(P)=\varphi(\operatorname{relint}(-P))$$

for all $P\in\mathcal{P}(\Lambda)$, as every Λ-valuation is uniquely determined by its values on simplices in $\mathcal{P}(\Lambda)$ by the inclusion-exclusion property. □

There is a multivariate generalization of Theorem 2.3.1 by McMullen:

Theorem 2.3.2 ([28, Theorem 6]). *Let $\varphi\in\operatorname{Val}(\Lambda)$ and let $P_1,\ldots,P_k\in\mathcal{P}(\Lambda)$. Then $\varphi(n_1P_1+\cdots+n_kP_k)$ agrees with a polynomial in $n_1,\ldots,n_k$ for integers $n_1,\ldots,n_k\geq 0$. Its degree in n_i is at most $\dim(P_i)$ and its total degree does not exceed $\dim(P_1)+\cdots+\dim(P_k)$.*

The following lemma is needed:

Lemma 2.3.3 ([28, Lemma 2]). *Let $\varphi\in\operatorname{Val}(\Lambda)$ and $Q\in\mathcal{P}(\Lambda)$. Then*

$$\psi(P):=\varphi(P+Q)$$

defines a Λ-valuation.

Proof. We repeat the argument given in [28]: Let $P_1, P_2 \in \mathcal{P}(\Lambda)$ such that $P_1 \cup P_2 \in \mathcal{P}(\Lambda)$. Then the claim follows from

$$(P_1 \cup P_2) + Q = (P_1 + Q) \cup (P_2 + Q)$$

and

$$(P_1 \cap P_2) + Q = (P_1 + Q) \cap (P_2 + Q).$$

The first equation is immediate. For the second equation, observe that for $p \in (P_1 + Q) \cap (P_2 + Q)$ there are $p_1 \in P_1$, $p_2 \in P_2$ and $q_1, q_2 \in Q$ such that

$$p = p_1 + q_1 = p_2 + q_2.$$

Thus, for all $0 \leq \lambda \leq 1$

$$p = \lambda(p_1 + q_1) + (1 - \lambda)(p_2 + q_2) = \lambda p_1 + (1 - \lambda)p_2 + \underbrace{\lambda q_1 + (1 - \lambda)q_2}_{\in Q}.$$

As $P_1 \cup P_2$ is convex the segment $[p_1, p_2]$ is fully contained in $P_1 \cup P_2$, and therefore, there is a $\lambda \in [0, 1]$ such that $\lambda p_1 + (1 - \lambda)p_2 \in P_1 \cap P_2$, as $P_1 \cup P_2$ is connected. □

Proof of Theorem 2.3.2. We repeat McMullen's inductive argument here: If $k = 1$, then the statement follows from Theorem 2.3.1. For $k > 1$ we consider the function ψ defined by

$$\psi(P_{k+1}) := \varphi(n_1 P_1 + \cdots + n_k P_k + P_{k+1}).$$

Then, by Lemma 2.3.3, ψ is a Λ-valuation. By Theorem 2.3.1, $\psi(n_{k+1} P_{k+1})$ is given by a polynomial of degree at most $\dim(P_{k+1})$ in n_{k+1},

$$\psi_{P_{k+1}}(n_{k+1}) = \sum_{i=0}^{\dim(P_{k+1})} \binom{n_{k+1}}{i} \alpha_i(n_1 P_1 + \cdots + n_k P_k)$$

for $n_{k+1} \geq 0$. For fixed P_{k+1} we have $\alpha_i \in \mathrm{Val}(\Lambda)$. Therefore, by induction hypotheses, $\alpha_i(n_1 P_1 + \cdots + n_k P_k)$ agrees with a multivariate polynomial for $n_1, \ldots, n_k \geq 0$ with degree in n_i of at most $\dim(P_i)$ for $1 \leq i \leq k$.

For the total degree we introduce a new parameter m and consider

$$\varphi(m(n_1 P_1 + \cdots + n_k P_k)) = \varphi(m n_1 P_1 + \cdots + m n_k P_k).$$

Its degree as polynomial in m equals the total degree in $n_1, \ldots, n_k$ and is at most $\dim(n_1 P_1 + \cdots + n_k P_k)$ by Theorem 2.3.1. □

2.4 Reciprocity

Let $P \in \mathcal{P}(\Lambda)$ and $\varphi \in \mathrm{Val}(\Lambda)$. As $\varphi(nP)$ agrees with the polynomial $\varphi_P(n)$ for $n \geq 0$, it is natural to ask if there is an interpretation for the evaluation of φ_P at negative integers. An answer to this question is given by McMullen in [28]:

Theorem 2.4.1 (McMullen [28]). *Let $\varphi \in \mathrm{Val}(\Lambda)$ and let P be an r-dimensional Λ-polytope. Then*

$$\varphi_P(-n) = (-1)^r \varphi(\mathrm{relint}(-nP))$$

for $n \geq 1$.

Proof. Again, we first consider the case when P is a simplex with vertex set $\{v_1, \dots, v_{r+1}\}$. By Theorem 1.13 we have

$$F_\varphi\left(P, \frac{1}{t}\right) = -\sum_{n \geq 1} \varphi_P(-n) t^n$$

as rational functions. For $n \geq 0$ we identify $\{x \in \mathbb{R}^{d+1} \colon x_{d+1} = n\}$ with $\mathbb{R}^d$ by forgetting the last coordinate. Now we consider $\mathrm{relint}(\mathrm{hom}(-P))$ and observe

$$\varphi(\mathrm{relint}(\mathrm{hom}(-P)) \cap \{x_{d+1} = n\}) = \varphi(\mathrm{relint}(-nP)) \text{ for } n \geq 1.$$

We define

$$\tilde{\Pi} = \{a_1(-v_1, 1) + \cdots + a_{r+1}(-v_{r+1}, 1) \colon 0 < a_i \leq 1 \text{ for } 1 \leq i \leq r+1\}$$

and observe that $\mathrm{relint}(\mathrm{hom}(-P))$ can be partitioned by translates of $\tilde{\Pi}$. In fact,

$$\mathrm{relint}(\mathrm{hom}(-P)) = \bigsqcup_{s_1, \dots, s_{r+1} \geq 0} \left(\tilde{\Pi} + s_1(-v_1, 1) + \cdots + s_{r+1}(-v_{r+1}, 1)\right).$$

By analogous arguments as in the proof of Theorem 2.3.1 we see

$$F_\varphi^\circ(-P, t) := \sum_{n > 0} \varphi(\mathrm{relint}(-nP)) t^n = \frac{1}{(1-t)^{r+1}} \sum_{n=1}^{r+1} \varphi(\tilde{\Pi} \cap \{x_{d+1} = n\}) t^n.$$

By observing that $\Pi \cap \{x_{d+1} = i\}$ (Π defined as in the proof of Theorem 2.3.1) is a translate of $\tilde{\Pi} \cap \{x_{d+1} = r+1-i\}$ for $0 \leq i \leq r$, we obtain

$$\begin{aligned} F_\varphi(P, \frac{1}{t}) &= (-1)^{r+1} t^{r+1} \frac{1}{(1-t)^{r+1}} \sum_{n=0}^{r} \varphi(\Pi \cap \{x_{d+1} = n\}) \frac{1}{t^n} \\ &= (-1)^{r+1} F_\varphi^\circ(-P, t). \end{aligned}$$

The result follows by comparison of coefficients of the generating series $F_\varphi(P, \frac{1}{t})$ and $(-1)^{r+1} F^\circ_\varphi(-P, t)$.

If $P \in \mathcal{P}(\Lambda)$ is not a simplex, we triangulate P into simplices in $\mathcal{P}(\Lambda)$ and conclude with the inclusion-exclusion property, as $P \mapsto \varphi_P(-n)$ and $P \mapsto (-1)^{\dim(P)}\varphi(\operatorname{relint}(-nP))$ are Λ-valuation. □

Remark 2.4.2. The map $P \mapsto F_\varphi(P,t)$ is a Λ-valuation. We can therefore extend it to half-open polytopes $P \setminus B$ as given by equation (2.1). Then $\varphi(n(P \setminus B))$ agrees with a polynomial $\varphi_{P\setminus B}(n)$ for $n \geq 1$. However, for $n = 0$ the values $\varphi(n(P \setminus B))$ and $\varphi_{P\setminus B}(n)$ can be different: as $|P \setminus B| = \bigsqcup_{F \subset B face} \operatorname{relint}(F)$, we obtain from Theorem 2.4.1

$$\varphi_{P\setminus B}(0) = \varphi_P(0) - \sum_{\substack{F\in\mathcal{L}(P)\\ F\subset B}} (-1)^{\dim(F)}\varphi(\{0\}) = (1-\chi(B))\varphi(\{0\}),$$

while $\varphi(0(P \setminus B))$ always equals $\varphi(\{0\})$.

2.5 Multivariate Ehrhart–Macdonald reciprocity

This section is devoted to a specific valuation, namely to the **lattice point enumerator**. For a $\mathbb{Z}^d$-polytope or **lattice polytope** P the lattice point enumerator $\mathsf{Ehr}\colon \mathcal{P}(\mathbb{Z}^d) \to \mathbb{Z}$ counts the number of lattice points in P:

$$\mathsf{Ehr}(P) = \left|P \cap \mathbb{Z}^d\right|.$$

It is immediate that Ehr is a $\mathbb{Z}^d$-valuation.

2.5.1 Counting lattice points in Minkowski sums

Specializing Theorem 2.3.2 to the lattice point enumerator yields the so-called **Bernstein–McMullen Theorem** [4, 28]:

Theorem 2.5.1 (Bernstein–McMullen Theorem)**.** *For $P_1, \ldots, P_k \in \mathcal{P}(\Lambda)$ the function*

$$\mathsf{Ehr}_{P_1,\ldots,P_k}(n_1,\ldots,n_k) := |(n_1 \cdot P_1 + \cdots + n_k \cdot P_k) \cap \mathbb{Z}^d|$$

agrees with a multivariate polynomial for integers $n_1, \ldots, n_k \geq 0$.

The case $k = 1$ goes back to Ehrhart (see e.g. [2, Theorem 3.8]).

Example 2.5.2. For a set of vectors $\{v_1, \ldots, v_k\} \subset \mathbb{Z}^d$ we define the zonotope

$$\mathcal{Z}(v_1, \ldots, v_k) := [0, v_1] + \cdots + [0, v_k].$$

In [47] Stanley gave an explicit formula for the number of lattice points in $\mathcal{Z}(v_1, \ldots, v_k)$.

Theorem 2.5.3 ([47, Lemma 2.2])**.** *Let $\{v_1, \ldots, v_k\} \subset \mathbb{Z}^d$. Then*

$$|\mathcal{Z}(v_1, \ldots, v_k) \cap \mathbb{Z}^d| = \sum_{I \subseteq [k]} \gcd(\{|m| : m \text{ maximal minor of } (v_i)_{i\in I} \in \mathbb{R}^{d\times|I|}\}),$$

where the sum is taken over all subsets I of $[k]$ such that $\{v_i\}_{i\in I}$ is linearly independent, and gcd *denotes the greatest common divisor.*

Since the minor of a matrix is a multilinear function in the column vectors, we can calculate the multivariate Ehrhart function for segments explicitly:

Corollary 2.5.4. *Let $v_1, \ldots, v_k \in \mathbb{Z}^d$. Then*

$$\mathsf{Ehr}_{[0,v_1],\ldots,[0,v_k]}(n_1, \ldots, n_k) = \sum_{I\subseteq[k]} n^I g_I,$$

where the sum is taken over all subsets I of $[k]$ such that $\{v_i\}_{i\in I}$ is linearly independent, and $g_I := \gcd(\{|m| : m \text{ maximal minor of } (v_i)_{i\in I} \in \mathbb{R}^{d\times|I|}\})$ and $n^I := \prod_{i\in I} n_i$.

For the Λ-valuation $\mathsf{Ehr}(P) = \left|P \cap \mathbb{Z}^d\right|$, Theorem 2.4.1 specializes to the classical **Ehrhart–Macdonald reciprocity** (see, e.g., [2, Theorem 4.1]):

Theorem 2.5.5 (Ehrhart–Macdonald reciprocity)**.** *Let P be a lattice polytope in $\mathbb{R}^d$. Then*

$$(-1)^{\dim P}\mathsf{Ehr}_P(-n) = |\operatorname{relint}(nP) \cap \mathbb{Z}^d|$$

for integers $n \geq 1$.

We aim for a general interpretation of evaluating $\mathsf{Ehr}_{P_1,\ldots,P_k}(n_1, \cdots, n_k)$ at arbitrary integers $n_1, \ldots, n_k$.

Observation 1. By introducing a parameter t and considering

$$|(tn_1 \cdot P_1 + \cdots + tn_k \cdot P_k) \cap \mathbb{Z}^d|$$

we see that $\mathsf{Ehr}_{P_1,\ldots,P_k}(-n_1, \ldots, -n_k)$ equals

$$(-1)^{\dim(P_1+\cdots+P_k)} \left|\operatorname{relint}(n_1 \cdot P_1 + \cdots + n_k \cdot P_k) \cap \mathbb{Z}^d\right|$$

by Ehrhart–Macdonald reciprocity.

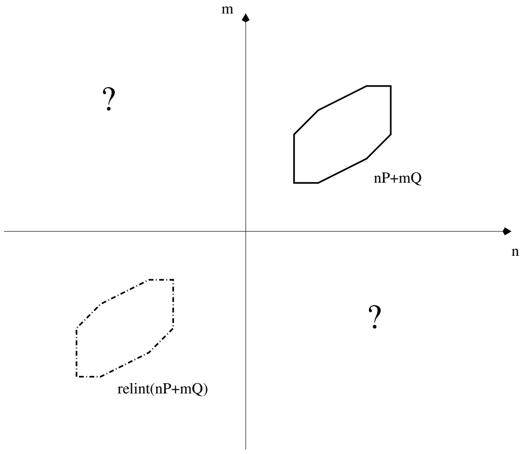


Figure 2.3: Interpretations of $\mathsf{Ehr}_{P,Q}(n,m)$ by orthants.

Observation 2. By introducing two parameters s and t and considering

$$|(tn_1 \cdot P_1 + \cdots + tn_l \cdot P_l + sn_{l+1} \cdot P_{l+1} + \cdots + sn_k P_k) \cap \mathbb{Z}^d|$$

we can conclude that $\mathsf{Ehr}_{P_1,\ldots,P_k}(-n_1,\ldots,-n_l,n_{l+1},\ldots,n_k)$ is equal to

$$\mathsf{Ehr}_{n_1P_1+\cdots+n_lP_l,n_{l+1}P_{l+1}+\cdots+n_kP_k}(-1,1).$$

From these observations we see that it suffices to restrict our considerations to two lattice polytopes P and Q and to give an interpretation for

$$\mathsf{Ehr}_{P,Q}(n,m)$$

for $n<0$ and $m \geq 0$. Theorem 2.5.5 asserts that $(-1)^{\dim P}\mathsf{Ehr}_P(-n)$ has the same sign for all $n>0$. However, in the multivariate case sign changes are possible, as the following considerations show.

For a graph $G=([m],E)$ without loops or multiple edges, let $V_G=\{v_{ij}=e_i-e_j : ij \in E, i<j\}$. The **graphical zonotope** is defined by

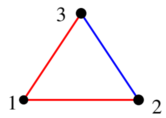
$$\mathcal{Z}(V_G)=\sum_{ij\in E}[0,v_{ij}].$$

It is easy to see that a subset $I \subseteq E$ is cycle-free, i.e., a forest, if and only if $\{v_{ij}=e_i-e_j : ij \in I, i<j\}$ is linearly independent. For a weight function $\omega\colon E \to \mathbb{Z}$, the weight w_I of a subgraph $I \subseteq E$ is defined by $w_I=\prod_{ij\in I} w(ij)$. From Corollary 2.5.4 we obtain the following:

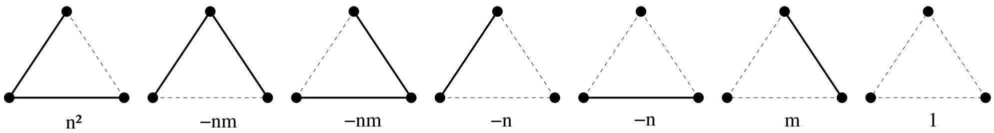
Corollary 2.5.6. *Let $\omega\colon E \to \mathbb{Z}$ be a weight. Let $E_r = \{ij \in E\colon \omega(ij) < 0\}$ consist of the red-colored edges, let $E_b = \{ij \in E\colon \omega(ij) > 0\}$ consist of the blue-colored edges and let $E_0 = \{ij \in E\colon \omega(ij) = 0\}$. Then*

$$\mathsf{Ehr}_{\{[0,v_{ij}]\colon ij\in E\}}((\omega(ij))_{ij\in E}) = \sum_{\substack{I\subseteq E\setminus E_0 \text{ forest} \\ |I\cap E_r| \text{even}}} |\omega_I| - \sum_{\substack{I\subseteq E\setminus E_0 \text{ forest} \\ |I\cap E_r| \text{odd}}} |\omega_I|\,.$$

Example 2.5.7. For $G = K_3$ with vertices $1, 2, 3$ we color the edges 12 and 13 with red and the edge 23 with blue:



K_3 contains 7 forests:



Therefore

$$\mathsf{Ehr}([0, v_{12}] + [0, v_{13}], -n; [0, v_{23}], m) = n^2 - 2nm + m - 2n + 1$$

for $n, m > 0$. If $n \gg m > 0$, then the right-hand side is > 0, however, if $m \gg n > 0$, then it is < 0.

2.5.2 Weighted enumeration

Example 2.5.7 shows that in general $\mathsf{Ehr}_{P,Q}(n, m)$ does not have a consistent sign for $n < 0$ and $m \geq 0$. Therefore, we cannot expect a combinatorial interpretation for $\mathsf{Ehr}_{P,Q}(n, m)$ similar to the one in the univariate case. Instead we will give an interpretation as weighted counting of lattice points.

For two polytopes P and Q we define the **Q-deletion** $\mathcal{C}_Q(P)$ of P as the set of faces of P that have empty intersection with Q:

$$\mathcal{C}_Q(P) = \{F \subseteq P \text{ face}\colon F \cap Q = \varnothing\}.$$

It is immediate that this is a polyhedral subcomplex of $\mathcal{L}(P)$.

We can give the following interpretation for $\mathsf{Ehr}_{P,Q}(-n, m)$:

Theorem 2.5.8. *Let P and Q be non-empty lattice polytopes. Then*

$$\mathsf{Ehr}_{P,Q}(-n,m) = -\sum_{p\in\mathbb{Z}^d} \tilde{\chi}(\mathcal{C}_{mQ}(nP+p)) \tag{2.3}$$

for $n > 0$ and $m \geq 0$.

Proof. By Lemma 2.3.3, $\varphi(P) = \mathsf{Ehr}(P+mQ)$ defines a valuation. Therefore, by Theorem 2.4.1,

$$\mathsf{Ehr}_{P,Q}(-n,m) = \varphi_P(-n) = \sum_{F\in\mathcal{L}(P)} (-1)^{\dim(F)} \left|(-nF+mQ)\cap\mathbb{Z}^d\right|.$$

For every $p \in \mathbb{Z}^d$ we have $p \in -nF+mQ$ if and only if $(nF+p)\cap mQ \neq \varnothing$. Thus,

$$\begin{aligned}\sum_{F\in\mathcal{L}(P)} (-1)^{\dim F} \mid (-nF+mQ)\cap\mathbb{Z}^d \mid &= \sum_{p\in\mathbb{Z}^d} \sum_{\substack{F\in\mathcal{L}(P):\\ (nF+p)\cap mQ\neq\varnothing}} (-1)^{\dim F} \\ &= \sum_{p\in\mathbb{Z}^d} (-\tilde{\chi}(\mathcal{C}_{mQ}(nP+p)),\end{aligned}$$

where the last equation follows from the Euler–Poincaré formula stating that $\tilde{\chi}(nP) = 0$. □

Theorem 2.5.8 allows us to give a geometric proof and interpretation for Ehrhart–Macdonald reciprocity:

Proof of Theorem 2.5.5. Let $Q = \{0\}$ and $p \in \mathbb{Z}^d$. Then for all $n \in \mathbb{Z}$ we have $\mathsf{Ehr}_P(n) = \mathsf{Ehr}_{P,Q}(n,1)$. Then $-p$ is contained in $\operatorname{relint}(nP)$ if and only if 0 is contained in $\operatorname{relint}(nP+p)$, and in this case $\mathcal{C}_Q(nP+p)$ is homeomorphic to a $(\dim(P)-1)$-sphere whose reduced Euler characteristic is $(-1)^{\dim(P)-1}$. Further, $-p$ is contained in $n\partial P$ if and only if 0 is contained in $\partial(nP+p)$. In this case, $\mathcal{C}_Q(nP+p)$ is contractible and therefore $\tilde{\chi}(\mathcal{C}_Q(nP+p)) = 0$. If $-p \notin nP$, then $\mathcal{C}_Q(nP+p) = nP$ and therefore $\tilde{\chi}(\mathcal{C}_Q(nP+p)) = 0$.

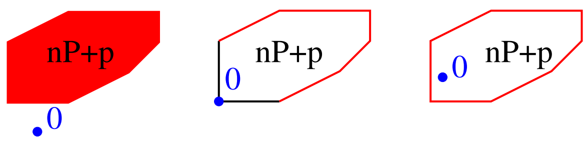


Figure 2.4: $\mathcal{C}_Q(nP+p)$ in red.

Therefore,

$$\mathsf{Ehr}_P(n) = (-1)^{\dim(P)} \left|\{p \in \mathbb{Z}^d \colon -p \in \operatorname{relint}(nP)\}\right| = (-1)^{\dim(P)} \left|\operatorname{relint}(nP)\right|.$$

□

Remark 2.5.9. It is natural to ask which weights can occur in equation (2.3). In fact, all possible Euler characteristics can occur: any simplicial complex $\mathcal{C}$ is isomorphic to the Q-deletion of a polytope P, where P and Q are lattice polytopes. To see this, assume that $\mathcal{C}$ is a simplicial complex on vertices $v_1, \dots, v_m$. Let $\Delta_{m-1} = \operatorname{conv}(\{e_1, \dots, e_m\})$ be the $(m-1)$-dimensional standard simplex in $\mathbb{R}^m$. For all $I \subseteq [m]$ let

$$w_I := \frac{1}{|I|} \sum_{i \in I} e_i$$

be the barycenter of the face $\operatorname{conv}(\{e_i \colon i \in I\})$ of Δ. Further, let $W = \operatorname{conv}(\{w_I \colon F_I \notin \mathcal{C}\})$. Then $\mathcal{C}_W(\Delta_{m-1})$ is isomorphic to $\mathcal{C}$, and by setting $P = m!\Delta_{m-1}$ and $Q = m!W$ we guarantee that $\mathcal{C}$ is isomorphic to the Q-deletion of P, where P and Q are both lattice polytopes.

2.6 Nonnegativity and monotonicity

2.6.1 h^*-nonnegativity and -monotonicity

Let φ be a Λ-valuation and P be an r-dimensional Λ-polytope in $\mathbb{R}^d$. We will study the numerator polynomial of the generating function

$$F_\varphi(P,t) = \sum_{n \geq 0} \varphi(nP) t^n = \frac{h^*_{\varphi,0}(P) + h^*_{\varphi,1}(P)t + \cdots + h^*_{\varphi,r}(P)t^r}{(1-t)^{r+1}}.$$

We call $h^*_\varphi(P) = (h^*_{\varphi,0}(P), \dots, h^*_{\varphi,d}(P))$ the **h^*-vector** of P (with respect to φ), where we define $h^*_{\varphi,i} = 0$ for all $r < i \leq d$. The polynomial $h^*_\varphi(P)(t) := \sum_{i=0}^d h^*_{\varphi,i}(P) t^i$ is called the **h^*-polynomial** of P (with respect to φ).

φ is called **h^*-nonnegative** if $h^*_{\varphi,i}(P) \geq 0$ for all Λ-polytopes P and all $0 \leq i \leq d$. Further, we call φ **h^*-monotone** if for all $P, Q \in \mathcal{P}(\Lambda)$ with $P \subseteq Q$

$$h^*_{\varphi,i}(P) \leq h^*_{\varphi,i}(Q) \quad \text{for } 0 \leq i \leq d.$$

Our main theorem is that both properties are, in fact, equivalent and characterized by a simple property:

Theorem 2.6.1. *Let φ be a Λ-valuation. Then the following are equivalent:*

(i) *φ is h^*-monotone,*

(ii) *φ is h^*-nonnegative,*

(iii) $\varphi(\mathrm{relint}(\Delta)) \geq 0$ *for all simplices* $\Delta \in \mathcal{P}(\Lambda)$.

Proof. (i) to (ii): Let φ be a h^*-monotone Λ-valuation. For every Λ-polytope P we have $\varnothing \subseteq P$. By h^*-monotonicity we therefore obtain

$$0 = h^*_{\varphi,i}(\varnothing) \leq h^*_{\varphi,i}(P)$$

for all $0 \leq i \leq d$.

(ii) to (iii): Let φ be a h^*-nonnegative Λ-valuation. By Theorem 2.3.1 we have for all r-dimensional simplices $\Delta \in \mathcal{P}(\Lambda)$

$$h^*_{\varphi,r}(-\Delta) = \varphi(\mathrm{relint}(\Delta))$$

which is nonnegative as φ is h^*-nonnegative.

(iii) to (i): We partition the proof into three steps.

Step 1: We show h^*-nonnegativity for half-open simplices. Recall that $P \mapsto F_\varphi(P,t)$ is a Λ-valuation and can therefore be extended to half-open simplices: In the sequel let $\Delta = \mathrm{conv}(\{v_1, \dots, v_{r+1}\})$ be an r-dimensional simplex in $\mathcal{P}(\Lambda)$, and for every subset $I \subseteq [r+1]$ let $F_I = \mathrm{conv}(\{v_j : j \in I\})$. Then $F_\varnothing = \varnothing$ and for $j^c := [r+1] \setminus \{j\}$ let F_{j^c} denote the unique facet not containing v_j. The generating function of a half-open simplex $H_I = \Delta \setminus \bigcup_{j \in I} F_{j^c}$ is

$$F_\varphi(H_I, t) = F_\varphi(\Delta, t) - \sum_{J \subseteq I} (-1)^{|J|-1} F_\varphi\left(\bigcap_{j \in J} F_{j^c}, t\right)$$

and can by Theorem 2.3.1 be represented by

$$\frac{h^*_{\varphi,0}(H_I) + h^*_{\varphi,1}(H_I)t + \cdots + h^*_{\varphi,r}(H_I)t^r}{(1-t)^{r+1}}.$$

We will show that the coefficients of the numerator polynomial — *the h^*-polynomial of H_I* — are nonnegative whenever $I \subsetneq [r+1]$. We define the homogenization of H_I by

$$\mathrm{hom}(H_I) = \mathrm{hom}(\Delta) \setminus \bigcup_{j \in I} \mathrm{hom}(F_{j^c}) \subset \mathbb{R}^{d+1}.$$

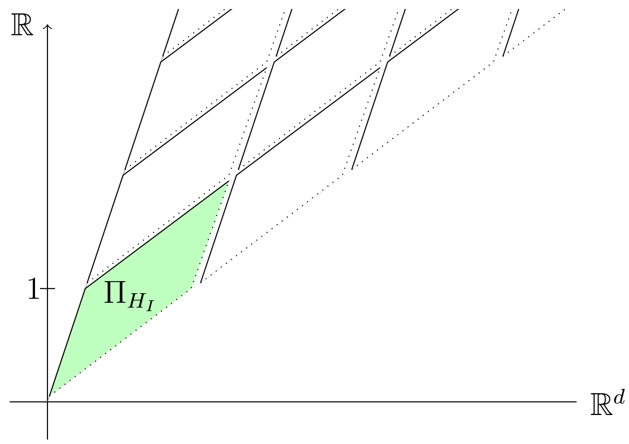


Figure 2.5: Partition of $\mathrm{Hom}(H_I)$ into translates of an half-open parallelepiped (in green) for H_I a half-open segment.

We identify the hyperplane $\{x \in \mathbb{R}^{d+1} \colon x_{d+1} = n\}$ with $\mathbb{R}^d$ by forgetting the last coordinate. Then we see that $\varphi_{H_I}(n)$ equals $\varphi(\mathrm{hom}(H_I) \cap \{x_{d+1} = n\})$ for $n > 1$. If $I = \varnothing$, then H_I is a closed simplex and $\varphi(\{0\}) = \varphi_{H_I}(0)$ by Theorem 2.3.1. For $\varnothing \neq I \subsetneq [r+1]$ we have that $\bigcup_{j \in I} F_{j^c}$ is contractible as $\bigcup_{j \in I} F_{j^c}$ is star-convex with respect to any point in the non-empty intersection $\bigcap_{j \in I} F_{j^c}$. Thus, $\varphi_{H_I}(0) = 0$ and equals $\varphi(\mathrm{hom}(H_I) \cap \{x_{d+1} = 0\}) = \varphi(\varnothing)$. Therefore, $\varphi_{H_I}(n) = \varphi(\mathrm{hom}(H_I) \cap \{x_{d+1} = n\})$ for all $n \geq 0$ and all $I \subsetneq [r+1]$.

We observe that $\mathrm{hom}(H_I)$ can be partitioned by translates of

$$\Pi_{H_I} = \{a_1 v_1 + \cdots + a_{r+1} v_{r+1} \colon 0 \leq a_j < 1 \text{ for } j \notin I, 0 < a_j \leq 1 \text{ for } j \in I\}.$$

Explicitly,

$$\mathrm{hom}(H_I) = \bigsqcup_{s_1, \ldots, s_{r+1} \geq 0} \left(\Pi_{H_I} + s_1(v_1, 1) + \cdots + s_{r+1}(v_{r+1}, 1)\right).$$

By arguments analogous to those in the proof of Theorem 2.3.1 we obtain

$$F_\varphi(H_I, t) = \frac{\sum_{i=0}^{r} \varphi(\Pi_{H_I} \cap \{x_{d+1} = i\}) t^i}{(1-t)^{r+1}}.$$

Thus, $\varphi(\Pi_{H_I} \cap \{x_{d+1} = i\}) = h^*_{\varphi,i}(P)$. Every $Q_i := \Pi_{H_I} \cap \{x_{d+1} = i\}$ is a half-open hypersimplex in $\mathcal{P}(\Lambda)$, i.e., a hypersimplex where certain faces are left out. Let $\mathcal{C}$ be a triangulation of the corresponding closed hypersimplex $\overline{Q_i}$ with simplices in $\mathcal{P}(\Lambda)$. Then

$$Q_i = \bigsqcup_{\Delta \in \mathcal{C} : \mathrm{relint}(\Delta) \subseteq Q_i} \mathrm{relint}(\Delta),$$

and therefore

$$h^*_{\varphi,i}(H_I) = \sum_{\Delta\in\mathcal{C}:\mathrm{relint}(\Delta)\subseteq Q_i} \varphi(\mathrm{relint}(\Delta)) \geq 0$$

by assumption.

Step 2: Now let $P \subseteq Q$ be Λ-polytopes. If $\dim P = \dim Q = r$, we triangulate P and extend it to a triangulation $\mathcal{C}$ of Q into simplices in $\mathcal{P}(\Lambda)$ using the beneath-beyond method. Taking a point in the relative interior of P outside any facet defining hyperplane of $\mathcal{C}$ yields a decomposition of $Q = H_1 \sqcup \ldots \sqcup H_m$ into half-open simplices with Theorem 2.2.3 such that $P = H_1 \sqcup \ldots \sqcup H_k$ for some $k \leq m$. We therefore obtain

$$h^*_{\varphi,i}(P) = \sum_{l=1}^{k} h^*_{\varphi,i}(H_l) \leq \sum_{l=1}^{k} h^*_{\varphi,i}(H_l) + \sum_{l=k+1}^{m} h^*_{\varphi,i}(H_l) = h^*_{\varphi,i}(Q), \quad 0 \leq i \leq r,$$

by Step 1.

Step 3: If $\dim P < \dim Q$, then using the beneath-beyond method we can construct Λ-polytopes $P = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_m \subseteq Q$ and vertices $p_0, \ldots, p_{m-1}$ of Q such that $P_{j+1} = \mathrm{Pyr}_{p_j}(P_j)$ for $0 \leq j \leq m-1$ and $\dim P_m = \dim Q$. By Step 2 it therefore suffices to show that $h^*_{\varphi,i}(P) \leq h^*_{\varphi,i}(\mathrm{Pyr}_p(P))$ for arbitrary $P \in \mathcal{P}(\Lambda)$ and $p \in \Lambda \setminus \mathrm{aff}(P)$.

We show more generally that for $p \in \Lambda \setminus \mathrm{aff}(\Delta)$, the h^*-vector of the half-open simplex H_I is componentwise smaller than the h^*-vector of the $(r+1)$-dimensional half-open simplex

$$\mathrm{Pyr}_p(H_I) := \mathrm{Pyr}_p(\Delta) \setminus \bigcup_{i\in I} \mathrm{Pyr}_p(F_{i^c}).$$

We observe that

$$\Pi_{\mathrm{Pyr}_p(H_I)} = \{x + \lambda(p,1) \colon x \in \Pi_{H_I}, 0 \leq \lambda < 1\}.$$

For $0 \leq i \leq r+1$ let $Q_i = \Pi_{\mathrm{Pyr}_p(H_I)} \cap \{x_{d+1} = i\}$ and $\tilde{Q}_i = \Pi_{H_I} \cap \{x_{d+1} = i\}$. Let $\mathcal{C}$ be a triangulation of the closed hypersimplex $\overline{Q_i}$ into Λ-simplices. Then $\mathcal{C}$ induces a triangulation of the face $\overline{\tilde{Q}_i}$ of $\overline{Q_i}$ and

$$Q_i = \bigsqcup_{\Delta\in\mathcal{C}:\mathrm{relint}(\Delta)\subseteq Q_i} \mathrm{relint}(\Delta) \qquad \text{and} \qquad \tilde{Q}_i = \bigsqcup_{\Delta\in\mathcal{C}:\mathrm{relint}(\Delta)\subseteq \tilde{Q}_i} \mathrm{relint}(\Delta).$$

Therefore

$$\begin{aligned} h^*_{\varphi,i}(H_I) &= \sum_{\Delta\in\mathcal{C}:\mathrm{relint}(\Delta)\subseteq\tilde{Q}_i} \varphi(\mathrm{relint}(\Delta)) \\ &\leq \sum_{\Delta\in\mathcal{C}:\mathrm{relint}(\Delta)\subseteq Q_i} \varphi(\mathrm{relint}(\Delta)) = h^*_{\varphi,i}(\mathrm{Pyr}_p(H_I)). \end{aligned}$$

Now let $P = \bigsqcup H$ be a decomposition into half-open simplices. Then

$$\mathrm{Pyr}_p(P) = \bigsqcup \mathrm{Pyr}_p(H)$$

is a decomposition into half-open simplices and by the considerations above

$$h^*_{\varphi,i}(P) = \sum h^*_{\varphi,i}(H) \leq \sum h^*_{\varphi,i}(\mathrm{Pyr}_p(H)) = h^*_{\varphi,i}(\mathrm{Pyr}_p(P))$$

which proves the result. $\square$

Remark 2.6.2. The condition that $\varphi(\mathrm{relint}(\Delta)) \geq 0$ for all simplices $\Delta \in \mathcal{P}(\Lambda)$ is clearly equivalent to the condition that $\varphi(\mathrm{relint}(P)) \geq 0$ for all Λ-polytopes P, as for every Λ-polytope P there is a triangulation $\mathcal{C}$ of P into simplices in $\mathcal{P}(\Lambda)$, and thus,

$$\varphi(\mathrm{relint}(P)) = \sum_{\substack{\Delta\in\mathcal{C} \\ \Delta\not\subseteq|\partial P|}} \varphi(\mathrm{relint}(\Delta)) \geq 0.$$

As a corollary of Theorem 2.6.1 we can give a short proof of Stanley's result on the nonnegativity and monotonicity of the Ehrhart h^*-vector:

Theorem 2.6.3 (Stanley [42],[48])**.** $\mathsf{Ehr}\colon \mathcal{P}(\mathbb{Z}^d) \to \mathbb{Z}$ *is h^*-monotone and h^*-nonnegative.*

Proof. Let P be a lattice polytope. Then

$$\mathsf{Ehr}(\mathrm{relint}(P)) = \sum_{F\in\mathcal{L}(P)} (-1)^{\dim(P)-\dim(F)}\mathsf{Ehr}(F) = \left|\mathrm{relint}(P)\cap\mathbb{Z}^d\right| \geq 0.$$

$\square$

We can apply our results also to solid-angles. The **solid-angle** $\omega(P,x)$ of a lattice point $x \in \mathbb{Z}^d$ with respect to a lattice polytope P is defined by

$$\omega(P,x) = \lim_{\varepsilon\to 0} \frac{\mathrm{vol}_d(P\cap\mathcal{B}^d_\varepsilon(x))}{\mathrm{vol}_d(\mathcal{B}^d_\varepsilon(x))},$$

where $\mathcal{B}^d_\varepsilon(x) = \{y \in \mathbb{R}^d : \|y-x\| \leq \varepsilon\}$.

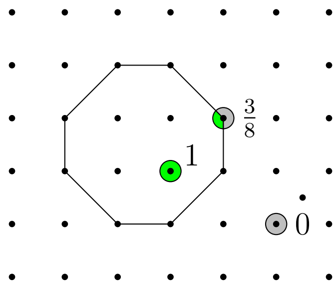


Figure: Solid-angles with respect to a lattice octagon.

For fixed x the map $\omega(.,x)\colon \mathcal{P}(\mathbb{Z}^d) \to \mathbb{R}$ is a valuation. Further, $\omega(.,x)$ is simple as the d-dimensional volume is simple. The **solid-angle sum** $A\colon \mathcal{P}(\mathbb{Z}^d) \to \mathbb{R}$ defined by

$$A(P) = \sum_{x \in \mathbb{Z}^d} \omega(P,x)$$

is a $\mathbb{Z}^d$-valuation, as $\omega(P+y, x+y) = \omega(P,x)$ for all $P \in \mathcal{P}(\mathbb{Z}^d)$ and $x, y \in \mathbb{Z}^d$. As a sum of simple valuations A is simple and therefore

$$A(\operatorname{relint}(P)) = A(P) \geq 0$$

for all lattice polytopes P. Together with Theorem 2.6.1 this reproves a theorem by Beck, Robins and Sam:

Theorem 2.6.4 (Beck, Robins, Sam [3])**.** *The solid-angle sum is h^*-monotone and h^*-nonnegative.*

An example of a non h^*-monotone valuation is given by Steiner polynomials:

Example 2.6.5. We consider the $\mathbb{R}^d$-valuation $\mathcal{S}\colon \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ defined by

$$\mathcal{S}(P) := \operatorname{vol}_d(P + \mathcal{B}^d),$$

where $\mathcal{B}^d := \mathcal{B}_1^d(0)$ denotes the unit ball. The polynomial $t^d\mathcal{S}_P(1/t) = \operatorname{vol}_d(P + t\mathcal{B}^d)$ is called **Steiner polynomial** and plays an important role in Hadwiger's theorem [19]. In fact, its coefficients — the **quermassintegrals** (up to scaling) — form a basis of the vector space of rigid-motion-invariant valuations on convex bodies. $\mathcal{S}$ is not h^*-monotone. To see that, let $P = [0, \alpha e_1]$ be a segment of length $\alpha > 0$ in $\mathbb{R}^d$. Then

$$\mathcal{S}(\operatorname{relint}(P)) = \operatorname{vol}_d(P + \mathcal{B}^d) - 2\operatorname{vol}_d(\mathcal{B}^d) = \alpha \operatorname{vol}_{d-1}(\mathcal{B}^{d-1}) - \operatorname{vol}_d(\mathcal{B}^d) < 0,$$

if α is sufficiently small.

2.6.2 Weak h^*-monotonicity

We call a Λ-valuation $\varphi \in \mathrm{Val}(\Lambda)$ **weakly h^*-monotone**, if $\varphi(\{0\}) \geq 0$ and if for all $P, Q \in \mathcal{P}(\Lambda)$ with $\dim(P) = \dim(Q)$

$$h^*_{\varphi,i}(P) \leq h^*_{\varphi,i}(Q) \text{ for all } i, \text{ whenever } P \subseteq Q.$$

Clearly, every h^*-monotone valuation is weakly h^*-monotone. However, the converse is not true in general:

Example 2.6.6. The Euler characteristic χ, which is $\chi(P) = 1$ for every non-empty polytope P, is weakly h^*-monotone but not h^*-monotone, as for every r-dimensional Λ-polytope P,

$$\sum_{n \geq 0} \chi(nP)t^n = \frac{(1-t)^r}{(1-t)^{r+1}}.$$

Thus, the h^*-polynomial of P has negative coefficients if $r > 0$ and therefore χ is not h^*-monotone by Theorem 2.6.1.

Remark 2.6.7. Expanding the generating series such that the denominator equals $(1-t)^{\dim(P)+1}$ more generally shows that for a h^*-nonnegative Λ-valuation φ with $\varphi(\{0\}) > 0$ the degree of $\varphi_P(n)$ is equal to $\dim(P)$ for all Λ-polytopes P.

The following theorem characterizes the class of weakly h^*-monotone valuations:

Theorem 2.6.8. *Let φ be a Λ-valuation. Then the following are equivalent:*

(i) *φ is weakly h^*-monotone,*

(ii) $\varphi(\mathrm{relint}(\Delta)) + \varphi(\mathrm{relint}(F)) \geq 0$ *for every simplex $\Delta \in \mathcal{P}(\Lambda)$ and every facet F of Δ.*

We will use the subsequent lemma:

Lemma 2.6.9. *Let φ be a Λ-valuation such that the condition (ii) of Theorem 2.6.8 is satisfied. Then for all $0 \leq l \leq r$, every r-dimensional simplex $\Delta \in \mathcal{P}(\Lambda)$, and every choice of l facets $F_1, \ldots, F_l$,*

$$\varphi\left(\Delta \setminus \bigcup_{i=1}^{l} F_i\right) \geq 0.$$

Proof. Let $\Delta \in \mathcal{P}(\Lambda)$ be an r-simplex with vertices $v_1, \ldots, v_{r+1}$ and let F_j be the facet not containing v_j for all $1 \leq j \leq r+1$. Then for $0 \leq l \leq r$

$$\Delta \setminus \bigcup_{j=1}^{l} F_j = \left\{ \sum_{j=1}^{r+1} \lambda_j v_j \colon \lambda_j > 0 \text{ for } j \leq l, \sum_{j=1}^{r+1} \lambda_j = 1, \lambda_j \geq 0 \text{ for all } j \right\}.$$

The right-hand side can be partitioned as

$$\bigsqcup_{[l]\subseteq I\subseteq [r]} \left\{ \sum_{j=1}^{r+1} \lambda_j v_j \colon \lambda_j > 0 \text{ if } j \in I, \lambda_j = 0 \text{ if } j \in [r]\setminus I, \sum_{j=1}^{r+1} \lambda_j = 1, \lambda_{r+1} \geq 0 \right\}$$

which equals

$$\bigsqcup_{[l]\subseteq I\subseteq [r]} \operatorname{relint}\left(\bigcap_{j\in[r]\setminus I} F_j \right) \sqcup \operatorname{relint}\left(\bigcap_{j\in[r]\setminus I} F_j \cap F_{r+1} \right).$$

As $\bigcap_{j\in I} F_j \cap F_{r+1}$ is a facet of $\bigcap_{j\in I} F_j$, the result follows. □

Proof of Theorem 2.6.8. (i) to (ii): Let Δ be an r-dimensional simplex with vertex set $\{v_1, \ldots, v_{r+1}\}$. Then (ii) holds by definition for $r = 0$ as $\varphi(\{0\}) \geq 0$. For $r > 0$ we can assume that $v_1 = 0$. Then the truncated pyramid $P := \overline{2\Delta \setminus \Delta}$ is contained in 2Δ and is of dimension r as well. As φ is weakly h^*-monotone we have $\varphi(\operatorname{relint}(P)) = h^*_{\varphi,r}(-P) \leq h^*_{\varphi,r}(-2\Delta) = \varphi(\operatorname{relint}(2\Delta))$. Therefore

$$\varphi(\operatorname{relint}(\Delta)) + \varphi(\operatorname{relint}(F_1)) = \varphi(\operatorname{relint}(2\Delta)) - \varphi(\operatorname{relint}(P)) \geq 0,$$

where F_j denotes the facet of Δ that does not contain v_j.
(ii) to (i): Let $P, Q \in \mathcal{P}(\Lambda)$ such that $P \subseteq Q$ and $\dim(P) = \dim(Q) = r$. If $r = 0$, then $P = Q$ and $h^*_{\varphi,0}(P) = h^*_{\varphi,0}(Q) = \varphi(\{0\})$. If $r > 0$, we take a triangulation of P and extend it to a triangulation of Q into simplices in $\mathcal{P}(\Lambda)$. Taking a point in the relative interior of P that does not lie on any hyperplane defining a facet in the triangulation, we obtain a partition of $Q \setminus P$ into half-open simplices by Theorem 2.2.3. The elements in the partition are of the form

$$H = \Delta \setminus \bigcup_{j=1}^{l} F_j, \quad 1 \leq l \leq r,$$

where F_j are facets of Δ, i.e., they arise from simplices excluding at least 1 and at most r facets. Let Δ have vertices $v_1, \ldots, v_{r+1}$ and let F_j be the facet of Δ not containing v_j. Let

$$\Pi_H = \{a_1 \tilde{v}_1 + \cdots + a_{r+1}\tilde{v}_{r+1} \colon 0 \leq a_j < 1 \text{ for } j > l, 0 < a_i \leq 1 \text{ for } j \leq l\},$$

where $\tilde{v}_j = (v_j, 1)$. As in the proof of Theorem 2.6.1 we obtain

$$h^*_{\varphi,i}(H) = \varphi(\Pi_H \cap \{x_{d+1} = i\})$$

for all $0 \leq i \leq r$. By Lemma 2.6.9 it therefore suffices to show that we can subdivide the half-open hypersimplex $Q_i := \Pi_H \cap \{x_{d+1} = i\}$ into half-open simplices, where at least one and at most r facets are left out.

To make life a bit easier, we apply the linear transformation defined by

$$\begin{aligned} \operatorname{span}(\tilde{v}_1, \dots, \tilde{v}_{r+1}) &\to \mathbb{R}^{r+1} \\ \tilde{v}_j &\mapsto e_j, \quad 1 \leq j \leq r+1, \end{aligned}$$

where $e_1, \dots, e_{r+1}$ denote the standard vectors in $\mathbb{R}^{r+1}$.

We then identify Q_i with its image

$$[0,1]^{r+1} \cap \left\{ x \in \mathbb{R}^{r+1}_{\geq 0} \colon \sum_{j=1}^{r+1} x_j = i, x_j > 0 \text{ for } 1 \leq j \leq l, x_j < 1 \text{ for } j > l \right\}.$$

We now consider the set

$$V := \left\{ x \in \mathbb{R}^{r+1} \colon \sum_{j=1}^{r+1} x_j = i, x_j < 0 \text{ for } 1 \leq j \leq l, x_j > 1 \text{ for } j > l \right\}.$$

Then V has a non-empty relative interior for $1 \leq l \leq r$. We observe that for all $p \in \operatorname{relint}(V)$

$$Q_i = \overline{Q}_i \setminus \operatorname{Vis}_p(\overline{Q}_i),$$

i.e., Q_i arises from the closed hypersimplex $\overline{Q}_i$ by cutting out all faces visible from p. For a triangulation $\mathcal{C}$ of a $\overline{Q}_i$ and a point $p \in \operatorname{relint}(V)$ that lies outside every facet defining hyperplane of $\mathcal{C}$ we thus obtain, by Theorem 2.2.3, a partition of $Q_i = \bigsqcup G$ into half-open simplices. As p lies outside $\overline{Q}_i$, every half-open simplex G in this partition arises from an r-dimensional simplex by leaving out at least 1 and at most r facets. By Lemma 2.6.9 $\varphi(G) \geq 0$, and thus

$$h^*_{\varphi,i}(H) = \varphi(Q_i) = \sum \varphi(G) \geq 0.$$

$h^*_{\varphi,i}(H)$ is therefore nonnegative for every H in the partition of $Q \setminus P$ and we conclude that

$$h^*_{\varphi,i}(Q) = h^*_{\varphi,i}(P) + \sum h^*_{\varphi,i}(H) \geq h^*_{\varphi,i}(P).$$

□

2.6.3 Other properties of Λ-valuations

A valuation φ is called **monotone** if for all Λ-polytopes P and Q

$$\varphi(P) \leq \varphi(Q) \text{ whenever } P \subseteq Q,$$

and φ is called **nonnegative** if $\varphi(P) \geq 0$ for all $P \in \mathcal{P}(\Lambda)$. Every monotone Λ-valuation is nonnegative as $\varnothing \subseteq P$ for all Λ-polytopes P. However, the converse is not true in general, as we will see below.

As for h^*-monotonicity we could define a weak notion of monotonicity: φ is **weakly monotone** if $\varphi(\{0\}) \geq 0$ and if for all $P, Q \in \mathcal{P}(\Lambda)$ with $\dim(P) = \dim(Q)$ we have $\varphi(P) \leq \varphi(Q)$ whenever $P \subseteq Q$. However, weak monotonicity already implies monotonicity:

Proposition 2.6.10. *Let φ be a Λ-valuation. Then the following are equivalent:*

(i) φ is monotone,

(ii) φ is weakly monotone.

Proof. (i) to (ii) is immediate. For the other direction we argue that

$$\varphi(\operatorname{Pyr}_p(P)) \geq \varphi(P)$$

for all Λ-polytopes P and all $p \in \Lambda \setminus \operatorname{aff}(P)$. If $P = \varnothing$, then $\varphi(\operatorname{Pyr}_p(P)) = \varphi(\{p\}) \geq 0$ by definition. If $\dim(P) \geq 0$, then we can assume that $p = 0$. Then the truncated pyramid $2 \operatorname{Pyr}_p(P) \setminus (\operatorname{Pyr}_p(P) \setminus P))$ is contained in $2 \operatorname{Pyr}_p(P)$ and is of the same dimension as $\operatorname{Pyr}_p(P)$. Thus, $\varphi(\operatorname{Pyr}_p(P) \setminus P) \geq 0$ by weak monotonicity of φ.

By using the beneath-beyond method, for every pair of Λ-polytopes P and Q with $P \subseteq Q$ we can construct a chain of Λ-polytopes

$$P = P_1 \subseteq \cdots \subseteq P_n \subseteq Q,$$

where $P_{i+1} = \operatorname{Pyr}_{p_i}(P_i)$ for some $p_i \notin \operatorname{aff} P_i$ and $\dim(P_n) = \dim(Q)$. ☐

The next result shows that for a Λ-valuation being weakly h^*-monotone is a stronger property than being monotone:

Proposition 2.6.11. *Let φ be a weakly h^*-monotone Λ-valuation. Then φ is monotone.*

Proof. By Proposition 2.6.10 it suffices to prove that

$$\varphi(P) \le \varphi(Q)$$

for all $P, Q \in \mathcal{P}(\Lambda)$ with $\dim(P) = \dim(Q)$. Let $\mathcal{C}_P$ be a triangulation of P into Λ-polytopes. Then by the beneath-beyond method there exists a triangulation $\mathcal{C}_Q$ of Q that extends $\mathcal{C}_P$, i.e., $\mathcal{C}_P \subseteq \mathcal{C}_Q$. Let x be a generic point in the relative interior of P. Using Theorem 2.2.3 we obtain a partition $\bigsqcup H$ of $Q \setminus P$ into half-open simplices. Each half-open simplex H in the partition is of the form

$$H = \Delta \setminus \bigcup_{i=1}^{l} F_i,$$

where Δ is a simplex, $F_i \subset \Delta$ are facets, and $1 \le l \le \dim(P)$. By Lemma 2.6.9 we therefore have

$$\varphi(Q) = \varphi(P) + \sum \varphi(H) \ge \varphi(P).$$

$\square$

Figure 2.6 shows the hierarchy of properties of Λ-valuations discussed in this section. We will proceed by showing that the reverse implications do not hold in general.

h^*-nonnegative
$\Updownarrow$
h^*-monotone

$\Downarrow$

weakly
h^*-monotone

$\Downarrow$

monotone
$\Updownarrow$
weakly monotone

$\Downarrow$

nonnegative

Figure 2.6: Hierarchy of properties of Λ-valuations.

In [3] Beck, Robins and Sam considered $\mathbb{Z}^d$-valuations given by weights on lattice points. A **weight** in this context is a function $\nu\colon \mathcal{P}(\mathbb{Z}^d) \times \mathbb{Z}^d \to \mathbb{R}$ such that

$$\nu(P \cup Q, x) = \nu(P, x) + \nu(Q, x) - \nu(P \cap Q, x)$$

for all $P, Q \in \mathcal{P}(\mathbb{Z}^d)$ with $P \cup Q, P \cap Q \in \mathcal{P}(\mathbb{Z}^d)$. ν is called **nonnegative**, if it takes only nonnegative values. Further, it is **translation-invariant** if for all $P \in \mathcal{P}(\mathbb{Z}^d)$ and all $x, y \in \mathbb{Z}^d$,

$$\nu(P + y, x + y) = \nu(P, x).$$

The function

$$N_\nu(P) = \sum_{x \in \mathbb{Z}^d} \nu(P, x)$$

is a $\mathbb{Z}^d$-valuation, whenever $|N_\nu(P)| < \infty$ for all lattice polytopes P. We call it the **weight valuation** associated with the weight ν. For example, if $\omega(P, x)$ is the solid-angle of x with respect to P, then N_ω is the solid-angle sum. For $\nu(P, x) = |P \cap \{x\}|$ the valuation N_ν is the lattice point enumerator.

The following example shows that weak h^*-monotonicity does not imply h^*-monotonicity:

Example 2.6.12. Let $\nu\colon \mathcal{P}(\mathbb{Z}) \times \mathbb{Z} \to \mathbb{Z}$ the weight defined by

$$\nu(P, x) = \begin{cases} 2 & \text{if } \dim P = 0, \\ 2 & \text{if } \dim P = 1 \text{ and } x \text{ is the left endpoint of } P, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to check, that ν is, in fact, a weight and that N_ν is nonnegative and translation-invariant. Further

$$N_\nu(\operatorname{relint}([0,1])) = N_\nu([0,1]) - N_\nu(\{0\}) - N_\nu(\{1\}) = -1$$

and therefore, by Theorem 2.6.1, N_ν is not h^*-monotone. In addition

$$N([0,a]) - N(\{0\}) = a$$

for all $a \in \mathbb{Z}_{>0}$. Thus, by Theorem 2.6.8, N_ν is weakly h^*-monotone.

The next example shows that a monotone $\mathbb{Z}^d$-valuation is not necessarily weakly h^*-monotone:

Example 2.6.13. Let R be the polytope with vertex set $\{(0,0), (2,0), (2,1)\}$ and Q be the segment with endpoints $(0,0)$ and $(1,1)$ in $\mathbb{R}^2$ (see Figure 2.7). Let μ be the weight defined by

$$\mu(P, x) = \mathbf{1}_{P+Q}(x).$$

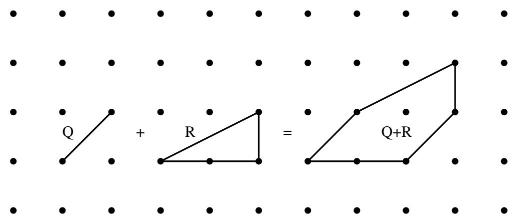


Figure 2.7: Q, R and $Q+R$ in Example 2.6.13.

Then N_μ is clearly monotone. It is easy to check that

$$N_\mu(\operatorname{relint}(R)) = \sum_{F \in \mathcal{L}(R)} (-1)^{\dim(F)} \left|F \cap \mathbb{Z}^2\right| = 7-4-4-6+2+2+2 = -1,$$

and $N_\mu(\operatorname{relint}(\operatorname{conv}(\{(2,0),(2,1)\}))$ is equal to

$$\sum_{F \in \mathcal{L}(R)} (-1)^{\dim(F)-1} \left|F \cap \mathbb{Z}^2\right| = 4-2-2 = 0.$$

Thus, $N_\mu(\operatorname{relint}(R)) + N_\mu(\operatorname{relint}(\operatorname{conv}(\{(2,0),(2,1)\})) = -1 < 0$ and therefore, by Theorem 2.6.8, N_μ is not weakly h^*-monotone.

This is also a counterexample to a theorem by Beck, Robins and Sam [3] which states that every nonnegative translation-invariant weight valuation is h^*-nonnegative and weakly h^*-monotone. In fact, nonnegativity does not even imply monotonicity as the following example shows:

Example 2.6.14. Let $\mu \colon \mathcal{P}(\mathbb{Z}^3) \times \mathbb{Z}^3 \to \mathbb{Z}$ be the weight defined by

$$\mu(P,x) = \mathbf{1}_P(x) + (-1)^{\dim(P)} \mathbf{1}_{\operatorname{relint}(P)}(x).$$

By Theorem 2.2.1 this defines a valuation. μ is nonnegative and for 3-dimensional polytopes N_μ counts the number of lattice points on the boundary. Let P and Q in $\mathbb{R}^2$, $P \subset Q$, as in Figure 2.8. Then N_μ is not monotone as $8 = N_\mu(P \times [0,1]) > N_\mu(Q \times [0,1]) = 6$.

2.7 The geometry of h^*-monotone Λ-valuations

In this section $\varphi \in \operatorname{Val}(\Lambda)$ will always be real-valued. Let $\operatorname{Val}^+(\Lambda)$ denote the set of h^*-nonnegative and, by Theorem 2.6.1, equivalently, h^*-monotone Λ-valuations. The set $\operatorname{Val}(\Lambda)$ of real-valued Λ-valuations is a vector space over $\mathbb{R}$. We will consider the geometry of $\operatorname{Val}^+(\Lambda)$ as a subset of $\operatorname{Val}(\Lambda)$. The set $\operatorname{Val}^+(\Lambda)$ forms a convex cone and by Theorem 2.6.1 it is given by the inequalities $\varphi(\operatorname{relint}(\Delta)) \geq 0$ for all simplices $\Delta \in \mathcal{P}(\Lambda)$ and all $\varphi \in \operatorname{Val}^+(\Lambda)$.

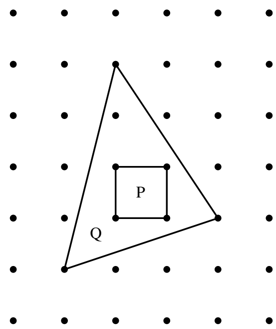


Figure 2.8: P and Q in Example 2.6.14.

2.7.1 $\mathbb{R}^d$-valuations

If $\Lambda = \mathbb{R}^d$, then the set $\mathrm{Val}^+(\mathbb{R}^d)$ has essentially one single element:

Theorem 2.7.1. *Let $\varphi \in \mathrm{Val}^+(\mathbb{R}^d)$. Then there is a $\lambda \geq 0$ such that*

$$\varphi = \lambda \operatorname{vol}_d .$$

The **Hausdorff metric** $d_H \colon \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is defined by

$$d_H(P, Q) = \min\left\{\varepsilon \geq 0 \colon P \subseteq Q + \varepsilon\mathcal{B}^d, Q \subseteq P + \varepsilon\mathcal{B}^d\right\}$$

for polytopes P, Q in $\mathbb{R}^d$. The Hausdorff metric naturally induces a topology and thus, we have the concepts of convergence and continuity of maps on polytopes.

To prove Theorem 2.7.1 we will use the following result by McMullen:

Theorem 2.7.2 ([28, Theorem 8])**.** *Every monotone $\mathbb{R}^d$-valuation is continuous with respect to the Hausdorff metric.*

Further, we will need the following lemma (see, e.g., [18, Chapter 16]):

Lemma 2.7.3. *Let φ be a simple monotone $\mathbb{R}^d$-valuation. Then there is a $\lambda \geq 0$ such that*

$$\varphi = \lambda \operatorname{vol}_d .$$

Proof. We find it instructive to give a proof here. Let $C^d = [0,1]^d$ the d-dimensional unit cube and $\lambda = \varphi(C^d)$. Then we have $\lambda \geq 0$ as φ is monotone by assumption. By tiling C^d into n^d axes-parallel cubes of side length $\frac{1}{n}$ we obtain $\varphi(\frac{1}{n}C^d) = \frac{\lambda}{n^d}$ for all $n \geq 1$ as φ is simple. In the same manner we see that $\varphi(\frac{m}{n}C^d) = \left(\frac{m}{n}\right)^d \lambda$ for $\frac{m}{n} \in \mathbb{Q}_{\geq 0}$. As φ is monotone, by Theorem 2.7.2, φ is continuous with respect to the Hausdorff metric and we can therefore conclude that $\varphi(aC^d) = a^d\lambda$ for all $a \in \mathbb{R}_{\geq 0}$.

Now let P be an arbitrary d-dimensional polytope. Then $P = \bigcup_{p\in\frac{1}{n}\mathbb{Z}^d}(p + \frac{1}{n}C^d)\cap P$. Let $A_n = \{p \in \frac{1}{n}\mathbb{Z}^d \colon p+\frac{1}{n}C^d \subseteq P\}$ and $P_n = \operatorname{conv}(\bigcup_{p\in A_n} p+\frac{1}{n}C^d)$. Then $P_n \subseteq P$. Let $B_n = \{p \in \frac{1}{n}\mathbb{Z}^d \colon (p + \frac{1}{n}C^d) \cap P \neq \varnothing\}$. Then

$$|A_n|\,\tfrac{\lambda}{n^d} = \sum_{p\in A_n} \varphi(p + \tfrac{1}{n}C^d) \leq \varphi(P_n) \leq \varphi(P) \leq \sum_{p\in B_n} \varphi(p + \tfrac{1}{n}C^d) = |B_n|\,\tfrac{\lambda}{n^d},$$

as φ is simple. We conclude by observing that $\frac{|A_n|}{n^d}$ and $\frac{|B_n|}{n^d}$ tend to $\operatorname{vol}_d(P)$ as $n \to \infty$. □

Proof of Theorem 2.7.1. As φ is h^*-nonnegative, it is also monotone by Proposition 2.6.11 and therefore, by Theorem 2.7.2, continuous with respect to the Hausdorff metric. By Lemma 2.7.3 it suffices to prove that φ is simple. For every polytope P in $\mathbb{R}^d$ let $g(P) = (g_0(P), g_1(P), \ldots, g_d(P))$ such that

$$F_\varphi(P,t) = \frac{g_0(P) + g_1(P)t + \cdots + g_d(P)t^d}{(1-t)^{d+1}}.$$

Then every g_i is a continuous nonnegative Λ-valuation for $0 \leq i \leq d$, and the numerator polynomial $g(P)(t) = \sum_{i=0}^d g_i(P)t^i$ equals $h^*_\varphi(P)(t)(1-t)^{d-r}$ if P is r-dimensional. In particular, $g_i(P) = h^*_{\varphi,i}(P)$ if $\dim(P) = d$ for $0 \leq i \leq d$.

Now let Q be a polytope such that $\dim Q = r < d$. We want to show that $\varphi(Q) = 0$. We can assume that $\operatorname{aff}(Q) = \{x \in \mathbb{R}^d \colon x_{r+1} = x_{r+2} = \cdots = x_d = 0\}$. For $n \geq 1$ let $Q_n = Q \times [0, \frac{1}{n}]^{d-r}$. Then $Q_n \to Q$ in the Hausdorff metric and therefore $h^*_{\varphi,i}(Q_n) = g_i(Q_n) \to g_i(Q)$. Thus $g_i(Q) \geq 0$ for all $0 \leq i \leq d$, as φ is h^*-nonnegative. On the other hand, $(1-t)|g(Q)(t)$ and therefore $\sum_{i=0}^d g_i(Q) = 0$. We conclude that $g_i(Q) = 0$ for all $0 \leq i \leq d$ and thus $\varphi(Q) = 0$. □

2.7.2 Lattice-invariant valuations

Let $\operatorname{Val}_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d) \subset \operatorname{Val}(\mathbb{Z}^d)$ be the set of real-valued $\mathbb{Z}^d$-valuations that are invariant under transformations in $\mathrm{GL}_d(\mathbb{Z})$, i.e., transformations that preserve the integer lattice $\mathbb{Z}^d$. We call the elements in $\operatorname{Val}_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d)$ **lattice-invariant valuations**. Betke and Kneser showed in [6] that every lattice-invariant valuation is uniquely determined by its value on the standard simplices:

Theorem 2.7.4 (Betke, Kneser [6]). *For $0 \leq i \leq d$ there is a unique valuation $\varphi_i \in \mathrm{Val}_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d)$ such that*

$$\varphi_i(\Delta_j) = \delta_{ij},$$

where $\Delta_j = \mathrm{conv}(0, e_1, \ldots, e_j)$ is the j-dimensional standard simplex in $\mathbb{R}^d$ and δ_{ij} denotes the Kronecker delta. $\varphi_i(P) = 0$ for all lattice polytopes P with $\dim(P) < i$. The set $\{\varphi_0, \ldots, \varphi_d\}$ forms a basis of $\mathrm{Val}_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d)$.

For our purposes we consider a different basis of $\mathrm{Val}_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d)$:

Proposition 2.7.5. *For $0 \leq i \leq d$ there is a unique lattice-invariant valuation $\varphi_i^\circ \in \mathrm{Val}_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d)$ such that*

$$\varphi_i^\circ(\mathrm{relint}(\Delta_j)) = \delta_{ij}.$$

The set $\{\varphi_0^\circ, \ldots, \varphi_d^\circ\}$ forms a basis of $\mathrm{Val}_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d)$.

Proof. For $0 \leq i \leq d$ we define

$$\varphi_i^\circ = (-1)^i \varphi_i^*.$$

Then $\varphi_i^\circ \in \mathrm{Val}_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d)$, and for all $0 \leq j \leq d$

$$\varphi_i^\circ(\mathrm{relint}(\Delta_j)) = (-1)^i \varphi_i^*(\mathrm{relint}(\Delta_j)) = (-1)^i(-1)^j \varphi_i^{**}(\Delta_j) = \delta_{ij},$$

by Theorem 2.2.2. Linear independence of $\{\varphi_0^\circ, \ldots, \varphi_d^\circ\}$ follows from the linear independence of $\{\varphi_0, \ldots, \varphi_d\}$ and Theorem 2.2.2. □

$\mathrm{Val}^+_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d) := \mathrm{Val}_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d) \cap \mathrm{Val}^+(\mathbb{Z}^d)$ denotes the class of h^*-nonnegative lattice-invariant valuations. A lattice simplex Δ is called **unimodular** if its vertices form an affine lattice basis of $\mathrm{aff}(\Delta) \cap \mathbb{Z}^d$. Every unimodular simplex can be bijectively mapped to any other unimodular simplex of the same dimension by an element in $\mathrm{GL}_d(\mathbb{Z})$ followed by a translation in $\mathbb{Z}^d$. A triangulation that consists of unimodular simplices is a **unimodular triangulation**.

For every $\varphi \in \mathrm{Val}_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d)$ we have

$$\varphi = \sum_{i=0}^{d} \varphi(\mathrm{relint}(\Delta_i))\varphi_i^\circ.$$

Therefore, by Theorem 2.6.1, we see that $\mathrm{Val}^+_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d) \subseteq \mathrm{cone}\{\varphi_0^\circ, \ldots \varphi_d^\circ\}$. For $d \leq 2$ equality holds:

Proposition 2.7.6. *For $d \in \{0,1,2\}$*

$$\mathrm{Val}^+_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d) = \mathrm{cone}\{\varphi_0^\circ, \dots, \varphi_d^\circ\}.$$

Proof. If $d \leq 2$, then every lattice polytope $P \subset \mathbb{R}^d$ admits a unimodular triangulation $\mathcal{C}$, and $\mathrm{relint}(P) = \bigsqcup_{\Delta \in \mathcal{C}\colon \Delta \not\subseteq \partial P} \mathrm{relint}(\Delta)$. Therefore

$$\varphi_i^\circ(\mathrm{relint}(P)) = |\{\Delta \in \mathcal{C} \colon \Delta \not\subseteq \partial P, \dim(\Delta) = i\}| \geq 0$$

for all $i \leq d$. Thus, by Theorem 2.6.1, we obtain $\varphi_i^\circ \in \mathrm{Val}^+_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d)$. $\square$

We conjecture the following:

Conjecture 2.7.7. *The cone $\mathrm{Val}^+_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d)$ is polyhedral for all $d \in \mathbb{N}$.*

Further, it would be interesting to see if $\mathrm{Val}^+_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d) = \mathrm{cone}\{\varphi_0^\circ, \dots, \varphi_d^\circ\}$ in all dimensions $d \geq 0$.

So far, we are able to show that $\mathrm{Val}^+_{\mathrm{GL}_d(\mathbb{Z})}$ is full-dimensional:

Theorem 2.7.8. *For $d \geq 0$*

$$\dim(\mathrm{Val}^+_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d)) = d + 1.$$

Proof. For $m \in \mathbb{N}$ we define $\mathsf{Ehr}_m(P) := \mathsf{Ehr}(mP)$ for all $P \in \mathcal{P}(\mathbb{Z}^d)$. Then Ehr_m is a $\mathbb{Z}^d$-valuation and

$$\mathsf{Ehr}_m(\mathrm{relint}(P)) = \left|\mathrm{relint}(mP) \cap \mathbb{Z}^d\right| \geq 0.$$

Thus, by Theorem 2.6.1, Ehr_m is h^*-monotone for all natural numbers $m \geq 0$. For pairwise distinct $m_1, \dots, m_{d+1} \in \mathbb{N}$, the valuations $\mathsf{Ehr}_{m_1}, \dots, \mathsf{Ehr}_{m_{d+1}}$ are linearly independent. To see this, assume that

$$\sum_{i=1}^{d+1} \lambda_i \mathsf{Ehr}_{m_i} = 0. \tag{2.4}$$

The evaluation of both sides of (2.4) at the k-dimensional unit cube $C^k = [0,1]^k \times \{0\}^{d-k}$ yields

$$\sum_{i=1}^{d+1} \lambda_i (m_i + 1)^k = 0$$

for all $0 \leq k \leq d$, and this is equivalent to

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ (m_1+1) & (m_2+1) & \cdots & (m_{d+1}+1) \\ \vdots & \vdots & \ddots & \vdots \\ (m_1+1)^d & (m_2+1)^d & \cdots & (m_{d+1}+1)^d \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{d+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.5}$$

For pairwise distinct $m_1, \ldots, m_{d+1}$, the Vandermonde matrix on the left hand side of (2.5) is invertible and thus we deduce that $\lambda_1 = \ldots = \lambda_{d+1} = 0$. Therefore, $\mathsf{Ehr}_{m_1}, \ldots, \mathsf{Ehr}_{m_{d+1}}$ are linearly independent and span a $(d+1)$-dimensional cone contained in $\mathrm{Val}^+_{\mathrm{GL}_d(\mathbb{Z})}(\mathbb{Z}^d)$, which completes the proof. □

Chapter 3

Unimodality of h^*-vectors for zonotopes

3.1 Introduction

One starting point of the previous chapter was Stanley's Nonnegativity Theorem [42] stating that the entries of the h^*-vectors of a lattice polytope are nonnegative. More recently, the question of unimodality became of general interest. A vector $(h_0^*, h_1^*, \ldots, h_d^*)$ is called **unimodal** if there exists a $k \in \{0, \ldots, d\}$ such that

$$h_0^* \leq \cdots \leq h_k^* \geq \cdots \geq h_d^*.$$

There are lattice polytopes with non-unimodal h^*-vector. Examples of such polytopes can, for instance, be found in [31]. However, no integrally closed examples are known. A lattice polytope $P \subset \mathbb{R}^d$ is **integrally closed** if for all $n \geq 1$ and all $p \in nP \cap \mathbb{Z}^d$ there are $p_1, \ldots, p_n \in P \cap \mathbb{Z}^d$ such that

$$p_1 + \cdots + p_n = p.$$

The simplest integrally closed polytope is a unimodular simplex. In particular, every polytope that has a unimodular triangulation is integrally closed. Stanley conjectured in [46] that every integrally closed lattice polytope has a unimodal h^*-vector. Schepers and Van Langenhoven recently proved this conjecture for lattice **parallelepipeds** [36]. We follow their line of argumentation and generalize their result to lattice **zonotopes**. These are integrally closed as well, as every lattice zonotope can be subdivided into lattice parallelepipeds, which are easily seen to be integrally closed. We streamline their exposition by giving a combinatorial interpretation to the h^*-vectors of lattice zonotopes. We do not only consider their Ehrhart h^*-vectors but, more generally, their h^*_φ-vectors for arbitrary translation-invariant valuations φ. We show that for every h^*-nonnegative translation-invariant valuation φ and every lattice zonotope $\mathcal{Z}$ the vector $h^*_\varphi(\mathcal{Z})$ is unimodal. More specifically, if $\mathcal{Z}$ is r-dimensional, then a largest coefficient of $h^*_\varphi(\mathcal{Z})$ is $h^*_{\varphi,\frac{r}{2}}(\mathcal{Z})$ if r is even, and $h^*_{\varphi,\frac{r-1}{2}}(\mathcal{Z})$ or $h^*_{\varphi,\frac{r+1}{2}}(\mathcal{Z})$ if r is odd.

We start in Section 3.3 by showing unimodality of the Ehrhart h^*-vector of certain half-open unit cubes. We are able to give a combinatorial meaning to their h^*-vectors in terms of **j-Eulerian numbers**, a refinement of Eulerian numbers, which we investigate in Section 3.2. We then consider **half-open lattice parallelepipeds** in Section 3.4. In Section 3.5 we prove unimodality of the h^*-vectors of arbitrary lattice zonotopes by using half-open decompositions as introduced by Köppe and Verdoolaege [24].

The results of this chapter are based on a joint project with Matthias Beck and Emily McCullough and were developed during a research stay at San Francisco State University.

3.2 Descent statistics

Let S_d denote the symmetric group on the set $[d]$ consisting of all **permutations** of the numbers $1,\dots,d$. For every permutation σ in S_d the **descent set** is defined by

$$\mathrm{Des}(\sigma) = \{i \in [d-1] \colon \sigma(i) > \sigma(i+1)\}.$$

The number of descents of σ is denoted by $\mathrm{des}(\sigma) = |\mathrm{Des}(\sigma)|$. The **Eulerian number** $A(d,k)$ counts the number of permutations in S_d with k descents:

$$A(d,k) = |\{\sigma \in S_d \colon \ \mathrm{des}(\sigma) = k\}|\,.$$

We consider a refinement of the descent statistic: the **j-Eulerian number**

$$A_j(d,k) = |\{\sigma \in S_d \colon \sigma(1) = j \text{ and } \mathrm{des}(\sigma) = k\}|$$

which counts the number of permutations σ with $\sigma(1) = j$ and k descents. The corresponding **j-Eulerian polynomial** is

$$A_j(d,t) = \sum_{k=0}^{d-1} A_j(d,k) t^k.$$

From the definition it is clear that $A_j(d,k) = 0$ for $k < 0$ and $k > d-1$.

A polynomial $f(t) = \sum_{i=0}^{m} a_i t^i$ is called **unimodal** if its coefficient vector $\mathbf{a} = (a_0, \ldots, a_m)$ is unimodal. If a_k is a largest coefficient, then we say that $f(t)$ and $\mathbf{a}$ have a **peak** at k.

The polynomial $f(t)$ is called **palindromic** with center of symmetry at $\frac{d}{2}$ if $t^d f(\frac{1}{t}) = f(t)$. If it is in addition unimodal, then the coefficients closest to the center of symmetry are maximal, i.e., $f(t)$ has a peak at $\frac{d}{2}$ if d is even, and at $\lfloor\frac{d}{2}\rfloor$ and $\lfloor\frac{d}{2}\rfloor + 1$ if d is odd.

Example 3.2.1. The polynomial

$$t^2 - t^3 + t^4$$

is palindromic with center of symmetry at 3. It is well-known that the **Eulerian polynomial**

$$A(d,t) := \sum_{\sigma \in S_d} t^{\operatorname{des}(\sigma)}$$

is palindromic with center of symmetry at $\frac{d-1}{2}$ and its coefficients form a unimodal sequence with peak at $\frac{d-1}{2}$ if d is odd, and peaks at $\frac{d}{2} - 1$ and $\frac{d}{2}$ if d is even (see, e.g., [10, p. 292]). The j-Eulerian polynomials are in general not palindromic, for example,

$$A_2(5,t) = 8t + 14t^2 + 2t^3.$$

However, their coefficients form a unimodal sequence, as the next theorem shows.

Our main theorem for j-Eulerian numbers, which can also be deduced from [35, Theorem 2.3] and the proof of [35, Theorem 1.1], is the following:

Theorem 3.2.2. *For $1 \le j \le d$ the coefficients of $A_j(d,t)$ are unimodal. If d is even, then*

$$\begin{aligned} &A_j(d,0) \le \ldots \le A_j(d,\tfrac{d}{2}-1) \ge \ldots \ge A_j(d,d-1) && \text{if } 1 \le j \le \tfrac{d}{2}, \\ &A_j(d,0) \le \ldots \le A_j(d,\tfrac{d}{2}) \qquad \ge \ldots \ge A_j(d,d-1) && \text{if } \tfrac{d}{2} < j \le d. \end{aligned}$$

If $d \ge 3$ is odd, then

$$\begin{aligned} &A_1(d,0) \le \ldots \le A_1(d,\lfloor\tfrac{d}{2}\rfloor - 1) = A_1(d,\lfloor\tfrac{d}{2}\rfloor) \ge \ldots \ge A_1(d,d-1), \\ &A_d(d,0) \le \ldots \le A_d(d,\lfloor\tfrac{d}{2}\rfloor) = A_d(d,\lfloor\tfrac{d}{2}\rfloor + 1) \ge \ldots \ge A_d(d,d-1), \end{aligned}$$

and

$$A_j(d,0) \le \ldots \le A_j(d,\lfloor\tfrac{d}{2}\rfloor) \ge \ldots \ge A_j(d,d-1) \qquad \text{if } 1 < j < d.$$

To prove Theorem 3.2.2, we will use the following two lemmata that can also be found in [8, Lemma 2]. We repeat their proofs given in there.

Lemma 3.2.3 ([8, Lemma 2])**.**

$$\begin{array}{rcl} A_j(d,k) & = & A_{d+1-j}(d,d-1-k), \textit{ equivalently} \\ A_j(d,t) & = & t^{d-1}A_{d+1-j}(d,\frac{1}{t}). \end{array}$$

Proof. Let $r \in S_d$ be the permutation defined by $r(i) = d+1-i$ for all $1 \leq i \leq d$. Then $\sigma \mapsto r \circ \sigma$ defines a bijection on S_d such that $\mathrm{Des}(r \circ \sigma) = [d-1] \setminus \mathrm{Des}(\sigma)$ and permutations starting with j are mapped to permutations starting with $d+1-j$. □

Lemma 3.2.4 ([8, Lemma 2])**.**

$$\begin{array}{rcl} A_j(d+1,k) & = & \sum_{l=1}^{j-1} A_l(d,k-1) + \sum_{l=j}^{d} A_l(d,k), \textit{ equivalently} \\ A_j(d+1,t) & = & t\sum_{l=1}^{j-1} A_l(d,t) + \sum_{l=j}^{d} A_l(d,t). \end{array}$$

Proof. For $1 \leq j \leq d+1$ let $\mathcal{S}^j = \{\sigma \in S_{d+1} \colon \sigma(1) = j\}$. Then

$$\mathcal{S}^j = \bigsqcup_{l=1}^{j-1} \mathcal{S}_l^j \cup \bigsqcup_{l=j+1}^{d+1} \mathcal{S}_l^j$$

where $\mathcal{S}_l^j = \mathcal{S}^j \cap \{\sigma \in S_{d+1} \colon \sigma(2) = l\}$. The restriction $\hat{\sigma}$ of a permutation $\sigma \in S_{d+1}$ to the domain $\{2, \ldots, d+1\}$ gives rise to a permutation $\pi_{\hat{\sigma}} \in S_d$ by defining $\pi_{\hat{\sigma}}(a-1) = b$ if $\sigma(a)$ is the b-th smallest number in $[d+1] \setminus \{j\}$. We conclude by observing that $\mathrm{des}(\pi_{\hat{\sigma}}) = \mathrm{des}(\sigma) - 1$ if $\sigma \in \mathcal{S}_l^j$ for $1 \leq l \leq j$ and $\mathrm{des}(\pi_{\hat{\sigma}}) = \mathrm{des}(\sigma)$ otherwise. □

Proof of Theorem 3.2.2. Our proof differs from the approach given in [35]. We argue by induction on d. The cases $d = 2$ and $d = 3$ are easily checked.

$d \to d+1$: Let $d+1$ be even. We then distinguish two cases:

Case $1 \leq j \leq \frac{d+1}{2}$: Then

$$A_j(d+1,t) = t\sum_{l=1}^{j-1} A_l(d,t) + \sum_{l=j}^{d+1-j} A_l(d,t) + \sum_{l=d+2-j}^{d} A_l(d,t)$$

by Lemma 3.2.4. The first and the third summand added give, by Lemma 3.2.3, a palindromic polynomial with center of symmetry at $\frac{d}{2}$ which, by induction, has unimodal coefficients with peaks at $\lfloor \frac{d}{2} \rfloor$ and $\lfloor \frac{d}{2} \rfloor + 1$. The second summand has by induction unimodal coefficients with peak at $\lfloor \frac{d}{2} \rfloor = \frac{d+1}{2} - 1$.

CASE $\frac{d+1}{2} < j \leq d+1$. Then

$$A_j(d+1,t) = t \sum_{l=1}^{d+1-j} A_l(d,t) + t \sum_{l=d+2-j}^{j-1} A_l(d,t) + \sum_{l=j}^{d} A_l(d,t).$$

The first and the third summand added give a palindromic polynomial with center of symmetry at $\frac{d}{2}$, which has unimodal coefficients with peaks at $\lfloor \frac{d}{2} \rfloor$ and $\lfloor \frac{d}{2} \rfloor + 1$. In this case, the coefficients of the second summand form a unimodal sequence with peak at $\lfloor \frac{d}{2} \rfloor + 1 = \frac{d+1}{2}$.

If $d+1$ is odd the claim is easily seen for $j \in \{1, d+1\}$ by Lemma 3.2.3, Lemma 3.2.4 and induction. If $2 \leq j \leq d$ we distinguish again two cases:
CASE $1 \leq j < \frac{d+1}{2}$: By Lemma 3.2.4,

$$A_j(d+1,t) = t \sum_{l=1}^{j-1} A_l(d,t) + \sum_{l=j}^{d+1-j} A_l(d,t) + \sum_{l=d+2-j}^{d} A_l(d,t).$$

The second summand is, by induction and Lemma 3.2.3, a palindromic polynomial with unimodal coefficients and peaks at $\frac{d}{2} - 1$ and $\frac{d}{2}$. The coefficients of the first and third summand are unimodal with peak at $\frac{d}{2} = \lfloor \frac{d+1}{2} \rfloor$.

CASE $\frac{d+1}{2} < j \leq d+1$: We have

$$A_j(d+1,t) = t \sum_{l=1}^{d+1-j} A_l(d,t) + t \sum_{l=d+2-j}^{j-1} A_l(d,t) + \sum_{l=j}^{d} A_l(d,t).$$

As in the previous case, the coefficients of the summand in the middle are unimodal and palindromic, this time with peaks at $\frac{d}{2}$ and $\frac{d}{2} + 1$. The coefficients of the first and third summand form again a unimodal sequence with peak at $\frac{d}{2} = \lfloor \frac{d+1}{2} \rfloor$. □

Remark 3.2.5. Ehrenborg, Readdy and Steingrímmson showed in [13] that the j-Eulerian numbers have a geometric meaning as mixed volumes of hypersimplices. It would be interesting to see whether this yields a geometric proof of Theorem 3.2.2.

3.3 Half-open unit cubes

For $j \in \{0, \ldots, d\}$ we define the **half-open unit cube**

$$C_j^d = [0,1]^d \setminus \{x \in \mathbb{R}^d : x_1 = \cdots = x_j = 0\}.$$

The j-**descent set** $\mathrm{Des}_j(\sigma) \subseteq \{0,\dots,d-1\}$ of a permutation $\sigma \in S_d$ is

$$\mathrm{Des}_j(\sigma) := \begin{cases} \mathrm{Des}(\sigma) \cup \{0\} & \text{if } 1 \leq \sigma(1) \leq j, \\ \mathrm{Des}(\sigma) & \text{if } j < \sigma(1) \leq d, \end{cases}$$

and the j-**descent number** $\mathrm{des}_j(\sigma) = |\mathrm{Des}_j(\sigma)|$ counts the number of j-descents of σ.

Lemma 3.3.1. *Let $\sigma \in S_d$ be a permutation. Then*

$$|\{\sigma \in S_d \colon \mathrm{des}_j(\sigma) = k\}| = A_{j+1}(d+1,k).$$

Proof. The map

$$\begin{aligned} \psi\colon S_d &\to \{\sigma \in S_{d+1} \colon \sigma(1) = j+1\} \\ \sigma &\mapsto \left(i \mapsto \begin{cases} j+1 & \text{if } i = 1, \\ \sigma(i-1) & \text{if } i > 1 \text{ and } \sigma(i-1) \leq j, \\ \sigma(i-1)+1 & \text{otherwise} \end{cases} \right) \end{aligned}$$

defines a bijection. We conclude by observing that $\mathrm{des}(\psi(\sigma)) = \mathrm{des}_j(\sigma)$ for all $\sigma \in S_d$. $\square$

We can now give an explicit formula for the Ehrhart series of half-open unit cubes in terms of j-Eulerian numbers:

Theorem 3.3.2. *Let $0 \leq j \leq d$. Then*

$$\mathsf{Ehr}(C_j^d, t) = \delta_{j0} + \sum_{n \geq 1} \mathsf{Ehr}_{C_j^d}(n) t^n = \frac{A_{j+1}(d+1,t)}{(1-t)^{d+1}}.$$

In particular, the coefficients of the numerator polynomial form a unimodal sequence.

Proof. Let $\mathcal{C}$ be the subcomplex of the boundary of $C^d = [0,1]^d$ generated by the facets $F_1 := \{x_1 = 0\} \cap C^d, \dots, F_j := \{x_j = 0\} \cap C^d$. Then

$$\begin{aligned} \mathsf{Ehr}(C_j^d, t) &= \mathsf{Ehr}(C^d, t) - \sum_{\varnothing \neq I \subseteq [j]} (-1)^{|I|-1} \mathsf{Ehr}(\bigcap_{i \in I} F_i, t) \\ &= 1 - \chi(\mathcal{C}) + \sum_{n \geq 1} \mathsf{Ehr}_{C_j^d}(n) t^n \\ &= \delta_{j0} + \sum_{n \geq 1} \mathsf{Ehr}_{C_j^d}(n) t^n, \end{aligned}$$

as $\mathcal{C}$ is star-convex with respect to the origin for $j > 0$, and thus contractible. For all $\sigma \in S_d$ we define

$$C^d_{j,\sigma} = \{x \in C^d_j \colon x_{\sigma(1)} \leq \cdots \leq x_{\sigma(d)}, x_{\sigma(i)} < x_{\sigma(i+1)} \text{ for all } i \in \mathrm{Des}(\sigma)\}.$$

Then

$$C^d_j = \bigsqcup_{\sigma} C^d_{j,\sigma}$$

is a partition into half-open simplices.

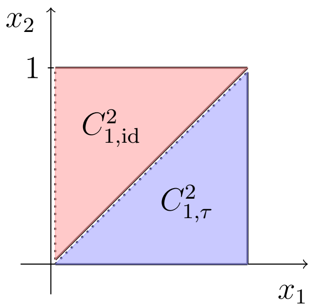


Figure 3.1: Decomposition of C^2_1 into $C^2_{1,\mathrm{id}}$ and $C^2_{1,\tau}$, where $\tau = (12)$.

In fact, we observe that for all $\sigma \in S_d$, there is an affine bijection between $nC^d_{j,\sigma} \cap \mathbb{Z}^d$ and the set

$$\mathcal{T}^\sigma_n := \left\{ y = (y_0, \ldots, y_d) \in \mathbb{N}^{d+1} \colon \sum_{i=0}^{d} y_i = n, y_i > 0 \text{ for all } i \in \mathrm{Des}_j(\sigma) \right\}$$

given by

$$x \mapsto y = (x_{\sigma(1)}, x_{\sigma(2)} - x_{\sigma(1)}, \ldots, x_{\sigma(d)} - x_{\sigma(d-1)}, n - x_{\sigma(d)}).$$

We observe that $\mathcal{T}^\sigma_0 = \varnothing$ unless σ is the identity and $j = 0$. Thus,

$$\begin{aligned}
\mathsf{Ehr}(C^d_{j,\sigma}, t) &= \sum_{n\geq 0} \sum_{y \in \mathcal{T}^\sigma_n} t^{\sum_{i=0}^d y_i} \\
&= \left(\prod_{i \in \mathrm{Des}_j(\sigma)} \sum_{y_i \geq 1} t^{y_i} \right) \cdot \left(\prod_{i \notin \mathrm{Des}_j(\sigma)} \sum_{y_i \geq 0} t^{y_i} \right) \\
&= \left(\prod_{i \in \mathrm{Des}_j(\sigma)} \frac{t}{1-t} \right) \cdot \left(\prod_{i \notin \mathrm{Des}_j(\sigma)} \frac{1}{1-t} \right) \\
&= \frac{t^{\mathrm{des}_j(\sigma)}}{(1-t)^{d+1}}.
\end{aligned}$$

Therefore, with Lemma 3.3.1 we conclude

$$\mathsf{Ehr}(C_j^d, t) = \sum_{\sigma \in S_d} \mathsf{Ehr}(C_{j,\sigma}^d, t) = \frac{\sum_{\sigma \in S_d} t^{\mathrm{des}_j(\sigma)}}{(1-t)^{d+1}} = \frac{A_{j+1}(d+1, t)}{(1-t)^{d+1}}.$$

$\square$

3.4 Half-open parallelepipeds

In the following let φ be a $\mathbb{Z}^d$-valuation. Let $v_1, \ldots, v_r \in \mathbb{Z}^d$ be fixed linearly independent vectors.

For every $I \subseteq [r]$ we define the **closed parallelepiped** generated by I

$$\Diamond(I) = \left\{ \sum_{i \in I} \lambda_i v_i \colon 0 \leq \lambda_i \leq 1 \text{ for all } i \in I\} \right\}$$

and the relatively **open parallelepiped**, or **box**, of I by

$$\Box(I) = \left\{ \sum_{i \in I} \lambda_i v_i \colon 0 < \lambda_i < 1 \text{ for all } i \in I \right\}.$$

We set $b_\varphi(I) := \varphi(\Box(I))$. We observe that if φ is h^*-nonnegative, then we obtain from Theorem 2.6.1 $b_\varphi(I) \geq 0$ for all $I \subseteq [r]$.

Further, for all $I \subseteq [r]$ we define the **half-open parallelepiped**

$$\diamondleftblack(I) = \left\{ \sum_{i=1}^{r} \lambda_i v_i \colon 0 < \lambda_i \leq 1 \text{ for all } i \in I, 0 \leq \lambda_i \leq 1 \text{ for all } i \notin I \right\}$$

and the **standard half-open parallelepiped** generated by I by

$$\Pi(I) = \left\{ \sum_{i \in I} \lambda_i v_i \colon 0 < \lambda_i \leq 1 \text{ for all } i \in I \right\}.$$

We observe that $\Diamond([r]) = \diamondleftblack(\varnothing)$ and that $\Diamond(\varnothing) = \Pi(\varnothing) = \{0\}$.

The following lemma of Schepers and van Langenhoven [36] was originally stated only for $\varphi(P) = \left|P \cap \mathbb{Z}^d\right|$. Their proof works as well for arbitrary $\mathbb{Z}^d$-valuations.

Lemma 3.4.1 ([36, Lemma 2.1]). *Let φ be a $\mathbb{Z}^d$-valuation and let $I \subseteq [r]$. Then*

$$\varphi(n\blacklozenge(I)) = \sum_{I \subseteq J} n^{|J|} \varphi(\Pi(J)).$$

Proof. To keep this chapter self-contained we give a (slightly modified) proof here. As $v_1, \ldots, v_r$ are linearly independent, for every $x \in \Diamond([r])$ there are unique $\lambda_1, \ldots, \lambda_r \in [0, 1]$ such that

$$x = \sum_{i=1}^{r} \lambda_i v_i.$$

Let $J_x = \{i \in [r] \colon \lambda_i > 0\}$. Then we have $x \in \Pi(J)$ if and only if $J_x = J$. We observe that $x \in \blacklozenge(I)$ if and only if $I \subseteq J_x$ and therefore we can partition

$$\blacklozenge(I) = \bigsqcup_{I \subseteq J} \Pi(J).$$

Further, for all $J \subseteq [r]$ and all $n \geq 1$ we can tile $n\Pi(J)$ with $n^{|J|}$ translates of $\Pi(J)$. Therefore, by translation-invariance of φ,

$$\varphi(n\blacklozenge(I)) = \sum_{I \subseteq J} \varphi(n\Pi(J)) = \sum_{I \subseteq J} n^{|J|} \varphi(\Pi(J)).$$

□

Applying Lemma 3.4.1 to the linearly independent standard basis vectors $e_1, \ldots, e_d$ and the lattice point enumerator we obtain the following corollary:

Corollary 3.4.2. *Let $0 \leq j \leq d$. Then the Ehrhart polynomial of the half-open unit cube C_j^d equals*

$$\mathsf{Ehr}_{C_j^d}(n) = \sum_{[j] \subseteq J} n^{|J|},$$

where we define $[0] := \varnothing$.

Lemma 3.4.3. *Let φ be a $\mathbb{Z}^d$-valuation. Then for all $I \subseteq [r]$*

$$\varphi(\Pi(I)) = \sum_{J \subseteq I} b_\varphi(J).$$

Proof. For $x \in \Pi(I)$ there are unique $\lambda_i \in (0,1]$ for all $i \in I$ such that

$$x = \sum_{i \in I} \lambda_i v_i.$$

Let $J_x = \{i \in I \colon \lambda_i = 1\} \subseteq I$. For all $J \subseteq I$ we have $J = J_x$ if and only if $x \in \Box(I \setminus J) + \sum_{i \in J} v_i$. Therefore

$$\Pi(I) = \bigsqcup_{J \subseteq I} \left(\Box(I \setminus J) + \sum_{i \in J} v_i \right)$$

and the result follows by the translation-invariance of φ. $\Box$

The following theorem generalizes [36, Proposition 2.2].

Theorem 3.4.4. *Let φ be a $\mathbb{Z}^d$-valuation. Then for all $I \subseteq [r]$ we obtain*

$$F_\varphi(\Diamond(I), t) = \frac{\sum_{K \subseteq [r]} b_\varphi(K) A_{|I \cup K|+1}(r+1, t)}{(1-t)^{r+1}}.$$

Proof. We follow the line of argumentation in [36, Proposition 2.2]. By Lemma 3.4.1 and Lemma 3.4.3,

$$\begin{aligned} F_\varphi(\Diamond(I), t) &= \sum_{n=0}^{\infty} t^n \sum_{J \colon I \subseteq J} n^{|J|} \varphi(\Pi(J)) \\ &= \sum_{n=0}^{\infty} t^n \sum_{J \colon I \subseteq J} n^{|J|} \sum_{K \colon K \subseteq J} b_\varphi(K) \\ &= \sum_{K \subseteq [r]} b_\varphi(K) \sum_{n=0}^{\infty} t^n \sum_{J \colon I \cup K \subseteq J} n^{|J|}. \end{aligned}$$

By Corollary 3.4.2,

$$\sum_{J \colon I \cup K \subseteq J} n^{|J|} = \mathsf{Ehr}_{C^r_{|I \cup K|}}(n).$$

Therefore, the claim follows by Theorem 3.3.2. $\Box$

As a corollary we obtain unimodality of the h^*-vectors of half-open parallelepipeds:

Corollary 3.4.5. *Let φ be a h^*-nonnegative $\mathbb{Z}^d$-valuation and let $I \subseteq [r]$. Let $h^*_\varphi(\Diamond(I)) = (h^*_{\varphi,0}, \ldots, h^*_{\varphi,r}, 0, \ldots, 0)$ be the h^*-vector of the half-open parallelepiped $\Diamond(I)$. Then*

$$h^*_{\varphi,0} \le \ldots \le h^*_{\varphi,\frac{r}{2}} \ge \ldots \ge h^*_{\varphi,r} \quad \text{if } r \text{ is even,}$$

and

$$h^*_{\varphi,0} \le \ldots \le h^*_{\varphi,\frac{r-1}{2}} \quad \text{and} \quad h^*_{\varphi,\frac{r+1}{2}} \ge \ldots \ge h^*_{\varphi,r} \quad \text{if } r \text{ is odd.}$$

In particular, $h^*_\varphi(\Diamond(I))$ is unimodal.

Proof. By Theorem 3.4.4

$$h^*_\varphi(\Diamond(I)) = \sum_{K \subseteq [r]} b_\varphi(K) A_{|I \cup K|+1}(r+1, t).$$

As φ is h^*-nonnegative we have $b_\varphi(K) \ge 0$ for all $K \subseteq [r]$. By Theorem 3.2.2 the coefficients of $A_{|I\cup K|+1}(r+1,t)$ form a unimodal sequence with peak at $\lfloor \frac{r+1}{2} \rfloor = \frac{r}{2}$ if r is even, and peak at $\frac{r+1}{2} - 1 = \frac{r-1}{2}$ or $\frac{r+1}{2}$ if r is odd, and so does any nonnegative linear combination. □

3.5 Zonotopes

Let Q be a parallelepiped and let $F_1, \ldots, F_m$ be a collection of facets of Q. We call $Q \setminus \bigcup_{i=1}^m F_i$ an **illuminated half-open parallelepiped** if $\bigcap_{i=1}^m F_i \neq \varnothing$.

Lemma 3.5.1. *Let $Q \setminus \bigcup_{i=1}^m F_i$ be an r-dimensional illuminated half-open parallelepiped. Then there are linearly independent vectors $v_1, \ldots, v_r \in \mathbb{Z}^d$ and a set $I \subseteq [r]$ such that $Q \setminus \bigcup_{i=1}^m F_i$ and $\Diamond(I)$ are equal up to translation in $\mathbb{Z}^d$.*

Proof. Let $w \in \bigcap_{i=1}^m F_i \neq \varnothing$ be a vertex of Q. As Q is simple the vertex figure at w is a simplex, i.e., every facet containing w is uniquely determined by the neighbor vertex of w it does not contain. For $1 \le i \le m$ let w_i be the neighbor of w which is not contained in F_i and let $w_{m+1}, \ldots, w_r$ be the other neighbors of w. Let $v_i := w_i - w$ for all $1 \le i \le r$. Then we set $I = \{1, \ldots, m\}$ and obtain

$$Q \setminus \bigcup_{i=1}^m F_i = \Diamond(I) + w.$$

□

We use the following well-known fact due to Shephard [37].

Theorem 3.5.2 ([37, Theorem 54]). *Every zonotope has a subdivision into parallelepipeds.*

Proof. We repeat Shephard's inductive argument. Let $\mathcal{Z}$ be an r-dimensional zonotope. We can assume that $\mathcal{Z} = \{\sum_{i=1}^{m} \lambda_i u_i \colon 0 \leq \lambda_i \leq 1\}$ for $u_1, \ldots, u_m \in \mathbb{R}^d$. We argue by induction on m and inductively define a fine mixed subdivision of $\mathcal{Z}$, i.e., a subdivision into parallelepipeds. If $m = r$, then $u_1, \ldots, u_m$ are linearly independent and $\mathcal{Z}$ is itself a parallelepiped. If $m > r$ then $\mathcal{Z}' = \{\sum_{i=1}^{m-1} \lambda_i u_i \colon 0 \leq \lambda_i \leq 1\}$ has by induction hypothesis a subdivision $\mathcal{C}$ into parallelepipeds. Now let $F_1, \ldots, F_l$ be the faces of $\mathcal{C}$ that are visible from a point tu_m for very large $t \gg 0$. Then $\mathcal{C} \cup \bigcup_{i=1}^{l} \{F_i + [0, u_m]\}$ is a subdivision of $\mathcal{Z}$ into parallelepipeds. □

Corollary 3.5.3. *Let $\mathcal{Z}$ be an r-dimensional zonotope. Then $\mathcal{Z}$ can be partitioned into r-dimensional illuminated half-open parallelepipeds.*

Proof. Let $\mathcal{C}$ be a subdivision of $\mathcal{Z}$ into parallelepipeds. Now we choose a point p in $\mathcal{Z}$ that does not lie on any facet defining hyperplane of $\mathcal{C}$ and obtain a decomposition $D = \bigsqcup H$ into half-open parallelepipeds using Theorem 2.2.3. Then every part H of D is illuminated because suppose H was not illuminated, then its closure $\overline{H}$ has two parallel facets F, F' such that $H \subseteq \overline{H} \setminus \{F, F'\}$. Thus, F and F' are both visible from p which is a contradiction as they are parallel. □

As a consequence we get the following theorem.

Theorem 3.5.4. *Let φ be a h^*-nonnegative $\mathbb{Z}^d$-valuation and let $\mathcal{Z}$ be an r-dimensional lattice zonotope with $h^*_\varphi(\mathcal{Z}) = (h^*_{\varphi,0}, \ldots, h^*_{\varphi,r}, 0, \ldots, 0)$. Then*

$$h^*_{\varphi,0} \leq \ldots \leq h^*_{\varphi,\frac{r}{2}} \geq \ldots \geq h^*_{\varphi,r} \text{ if } r \text{ is even}$$

and

$$h^*_{\varphi,0} \leq \ldots \leq h^*_{\varphi,\frac{r-1}{2}} \text{ and } h^*_{\varphi,\frac{r+1}{2}} \geq \ldots \geq h^*_{\varphi,r} \text{ if } r \text{ is odd.}$$

In particular, $h^*_\varphi(\mathcal{Z})$ is unimodal.

Proof. By the proof of Theorem 3.5.2 and Corollary 3.5.3 we can partition $\mathcal{Z}$ into illuminated r-dimensional half-open lattice parallelepipeds $\mathcal{Z} = \bigsqcup H$. For every H in D there are, by Lemma 3.5.1, linearly independent vectors $v_1, \ldots, v_r \in \mathbb{Z}^d$ and an index set $I \subseteq [r]$ such that $H = \lozenge(I) + t$ with $t \in \mathbb{Z}^d$. By Corollary 3.4.5, H has a unimodal h^*-vector with peak at $\frac{r}{2}$ if r is even and peak at $\frac{r-1}{2}$ or at $\frac{r+1}{2}$ if r is even. As $P \to F_\varphi(P,t)$ is a $\mathbb{Z}^d$-valuation we have $F_\varphi(\mathcal{Z},t) = \sum F_\varphi(H,t)$, and thus, $h^*_\varphi(\mathcal{Z}) = \sum h^*_\varphi(H)$. Therefore $h^*_\varphi(\mathcal{Z})$ is a positive linear combination and unimodal as well, as all summands satisfy the same condition on the position of a peak. □

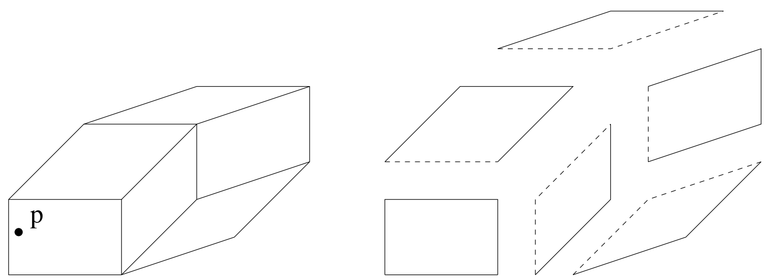


Figure 3.2: A zonotope and one of its half-open decomposition into parallelepipeds.

Part II

Order preserving maps

Chapter 4

Arithmetic of marked order polytopes

4.1 Introduction

Posets are among the most fundamental objects in combinatorics. For a finite poset $\mathfrak{P}$, Stanley [38] considered the problem of counting (strictly) order preserving maps from $\mathfrak{P}$ into n-chains and showed that many problems in combinatorics can be cast into this form. Here, a map $\lambda : \mathfrak{P} \to [n]$ into the n-chain is **order preserving** if $\lambda(p) \leq \lambda(q)$ whenever $p \prec_{\mathfrak{P}} q$ and the inequality is strict for **strict** order preservation. In [38] it is shown that the number of order preserving maps into a chain of length n is given by a polynomial $\Omega_{\mathfrak{P}}(n)$ in the positive integer n and the number of strictly order preserving maps is related to $\Omega_{\mathfrak{P}}(n)$ by a combinatorial reciprocity (see Section 4.2.5).

In this paper we consider the problem of counting the number of order preserving extensions of a map $\lambda : \mathfrak{A} \to \mathbb{Z}$ from an induced subposet $\mathfrak{A} \subseteq \mathfrak{P}$ to $\mathfrak{P}$. Clearly, this number is finite only when $\mathfrak{A}$ comprises all minimal and maximal elements of $\mathfrak{P}$ and we tacitly assume this throughout. It is also obvious that no extension exists unless λ is order preserving for $\mathfrak{A}$ and we define $\Omega_{\mathfrak{P},\mathfrak{A}}(\lambda)$ as the number of order preserving maps $\widehat{\lambda} : \mathfrak{P} \to \mathbb{Z}$ such that $\widehat{\lambda}|_{\mathfrak{A}} = \lambda$. By adjoining a minimum and maximum to $\mathfrak{P}$, we can see that $\Omega_{\mathfrak{P},\mathfrak{A}}(\lambda)$ generalizes the order polynomial.

The function $\Omega_{\mathfrak{P}}(n)$ can be studied from a geometric perspective by relating it to the Ehrhart function of the **order polytope** [45], the set of order preserving maps $\mathfrak{P} \to [0, 1]$. The finiteness of $\mathfrak{P}$ asserts that this is indeed a convex polytope in the finite-dimensional real vector space $\mathbb{R}^{\mathfrak{P}}$. The order polytope is a lattice polytope whose facial structure is intimately related to the structure of $\mathfrak{P}$ and which has a canonical unimodular triangulation, again described in terms of the combinatorics of $\mathfrak{P}$. Standard facts from Ehrhart theory (see, for example, [2]) then assert that $\Omega_{\mathfrak{P}}(n)$ is a polynomial of degree $|\mathfrak{P}|$. We pursue this geometric route and study the **marked order polytope**

$$\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda) \ := \ \left\{ \ \widehat{\lambda} : \mathfrak{P} \to \mathbb{R} \text{ order preserving } \ : \ \widehat{\lambda}(a) = \lambda(a) \text{ for all } a \in \mathfrak{A} \ \right\}$$

in $\mathbb{R}^{\mathfrak{P}}$. Marked order polytopes were considered (and named) by Ardila, Bliem, and Salazar [1] in connection with representation theory. In the case that $\mathfrak{A}$ is a chain, the polytopes already appear in [43]; see Section 4.2.4. The set $\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ defines a polyhedron for any choice of $\mathfrak{A} \subseteq \mathfrak{P}$ but it is a polytope precisely when $\min(\mathfrak{P}) \cup \max(\mathfrak{P}) \subseteq \mathfrak{A}$. It follows that $\Omega_{\mathfrak{P},\mathfrak{A}}(\lambda) = \left|\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda) \cap \mathbb{Z}^{\mathfrak{P}}\right|$. In Section 4.2 we elaborate on the geometric-combinatorial properties of $\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ and we show that $\Omega_{\mathfrak{P},\mathfrak{A}}(\lambda)$ is a piecewise polynomial over the space of integer-valued order preserving maps $\lambda : \mathfrak{A} \to \mathbb{Z}$. We give an explicit description of the polyhedral domains for which $\Omega_{\mathfrak{P},\mathfrak{A}}(\lambda)$ is a polynomial and we give a combinatorial interpretation for $\Omega_{\mathfrak{P},\mathfrak{A}}(-\lambda)$. We close by "transferring" our results to the marked chain polytopes of [1].

In Section 4.3, we use our results to give a geometric interpretation of a combinatorial reciprocity for **monotone triangles** that was recently described by Fischer and Riegler [17]. A monotone triangle is a triangular array of numbers such as

$$\begin{array}{ccccccccc} & & & & 5 & & & & \\ & & & 4 & & 5 & & & \\ & & 3 & & 5 & & 7 & & \\ & 3 & & 4 & & 7 & & 8 & \\ 1 & & 4 & & 6 & & 8 & & 9 \end{array} \tag{4.1}$$

with fixed bottom row such that the entries along the directions $\searrow$ and $\nearrow$ are weakly increasing and strictly increasing in direction $\to$; a more formal treatment is deferred to Section 4.3. Monotone triangles arose initially in connection with alternating sign matrices [30] and a significant amount of work regarding their enumerative behavior was done in [15]. In particular, it was shown that the number of monotone triangles is a polynomial in the strictly increasing bottom row. In [17] a (signed) interpretation is given for the evaluation of this polynomial at weakly decreasing arguments in terms of **decreasing monotone triangles**. In our language, monotone triangles are extensions of order preserving maps over posets known as **Gelfand-Tsetlin patterns** plus some extra conditions. These extra conditions can be interpreted as excluding the lattice points in a natural subcomplex of the boundary of $\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$. We investigate the combinatorics of this subcomplex and give a geometric interpretation for the combinatorial reciprocity of monotone triangles.

Finally, a well-known result of Stanley [40] gives a combinatorial interpretation for the evaluation of the chromatic polynomial χ_Γ of a graph Γ at negative integers in terms of acyclic orientations. We give a combinatorial reciprocity for the situation of counting extensions of partial colorings which was considered by Herzberg and Murty [21].

The results of this chapter are joint work with Raman Sanyal and appeared in [23][1].

4.2 Marked order polytopes

Marked order polytopes as defined in the introduction naturally arise as sections of a polyhedral cone, the **order cone**, which parametrizes order preserving maps from a finite poset $\mathfrak{P}$ to $\mathbb{R}$. The order cone is the "cone-analog" of the order polytope which was thoroughly studied in [45] and whose main geometric results we reproduce before turning to marked order polytopes. For a finite set S we identify $\mathbb{R}^S$ with the vector space of real-valued functions $S \to \mathbb{R}$.

4.2.1 Order cones

The order cone is the set $\mathcal{L}(\mathfrak{P}) \subseteq \mathbb{R}^{\mathfrak{P}}$ of order preserving maps $\mathfrak{P} \to \mathbb{R}$

$$\mathcal{L}(\mathfrak{P}) \;=\; \{\phi \in \mathbb{R}^{\mathfrak{P}} \;:\; \phi(p) \;\leq\; \phi(q) \text{ for all } p \preceq_{\mathfrak{P}} q\}.$$

This is a closed convex cone and the finiteness of $\mathfrak{P}$ ensures that $\mathcal{L}(\mathfrak{P})$ is polyhedral (i.e., bounded by finitely many halfspaces). The cone is not pointed and the lineality space of $\mathcal{L}(\mathfrak{P})$ is spanned by the indicator functions of the connected components of $\mathfrak{P}$. Said differently, the largest linear subspace contained in $\mathcal{L}(\mathfrak{P})$ is spanned by the functions $\chi : \mathfrak{P} \to \{0, 1\}$ that satisfy $\chi(p) = \chi(q)$ whenever there is a sequence $p = p_0 p_1 \dots p_{k-1} p_k = q$ such that $p_i p_{i+1}$ are comparable in $\mathfrak{P}$.

[1] First Published in SIAM Journal on Discrete Mathematics 28 (3), published by the Society of Industrial and Applied Mathematics (SIAM).

The cone $\mathcal{L}(\mathfrak{P}) \subseteq \mathbb{R}^{\mathfrak{P}}$ is of full dimension $|\mathfrak{P}|$ and its facet-defining equations are given by $\phi(p) = \phi(q)$ for every cover relation $p \prec\!\cdot_{\mathfrak{P}} q$. Every face $F \subseteq \mathcal{L}(\mathfrak{P})$ gives rise to a (in general not induced) subposet $\mathfrak{G}(F)$ of $\mathfrak{P}$ whose Hasse diagram is given by those $p \prec\!\cdot_{\mathfrak{P}} q$ for which $\phi(p) = \phi(q)$ for all $\phi \in F$. Such a subposet $\mathfrak{G}(F)$ arising from a face $F \subseteq \mathcal{L}(\mathfrak{P})$ is called a **face partition**. The following characterization of face partitions is taken from [45].

Proposition 4.2.1. *A subposet $\mathfrak{G} \subseteq \mathfrak{P}$ is a face partition if and only if for every $p, q \in \mathfrak{G}$ with $p \preceq_{\mathfrak{G}} q$ we have $[p, q]_{\mathfrak{P}} \subseteq \mathfrak{G}$.*

Equivalently, the directed graph obtained from the Hasse diagram of $\mathfrak{P}$ by contracting the cover relations in $\mathfrak{G}$ is an acyclic graph and, after removing transitive edges, is the Hasse diagram of a poset that we denote by $\mathfrak{P}/\mathfrak{G}$. Note that $\mathfrak{G}$ is typically not a connected poset. The face corresponding to such a graph $\mathfrak{G}$ is then

$$F_{\mathfrak{P}}(\mathfrak{G}) \;=\; \{\phi \in \mathcal{L}(\mathfrak{P}) \;:\; \phi \text{ is constant on every connected component of } \mathfrak{G}\}$$

and $F_{\mathfrak{P}}(\mathfrak{G})$ is isomorphic to $\mathcal{L}(\mathfrak{P}/\mathfrak{G})$ by a linear and lattice preserving map.

The order cone has a canonical subdivision into unimodular cones that stems from refinements of $\mathfrak{P}$ induced by elements of $\mathcal{L}(\mathfrak{P})$. To describe the constituents of the subdivision, recall that $I \subseteq \mathfrak{P}$ is an **order ideal** if $p \preceq_{\mathfrak{P}} q$ and $q \in I$ implies $p \in I$. Let $\phi \in \mathcal{L}(\mathfrak{P})$ be an order preserving map with range $\phi(\mathfrak{P}) = \{t_0 < t_1 < \cdots < t_k\}$. Then ϕ induces a chain of order ideals

$$I_\bullet^{\mathfrak{P}} \;:\; I_0 \;\subsetneq\; I_1 \;\subsetneq\; I_2 \;\subsetneq\; \cdots \;\subsetneq\; I_k \;=\; \mathfrak{P}$$

by setting $I_j = \{p \in \mathfrak{P} \;:\; \phi(p) \le t_j\}$. If the poset $\mathfrak{P}$ is clear from the context, we drop the superscript and simply write $I_\bullet$. Conversely, a given chain of order ideals $I_\bullet$ is induced by $\phi \in \mathcal{L}(\mathfrak{P})$ if and only if ϕ is constant on $I_j \setminus I_{j-1}$ for $j = 0, 1, \ldots, k$ (with $I_{-1} = \varnothing$) and

$$\phi(I_0) \;<\; \phi(I_1 \setminus I_0) \;<\; \phi(I_2 \setminus I_1) \;<\; \cdots \;<\; \phi(I_k \setminus I_{k-1}).$$

This defines the relative interior of a $(k+1)$-dimensional simplicial cone in $\mathcal{L}(\mathfrak{P})$ whose closure we denote by $F(I_\bullet)$. Chains of order ideals are ordered by refinement and the maximal elements correspond to saturated chains of order ideals or, equivalently, **linear extensions** of $\mathfrak{P}$. For a saturated chain $I_\bullet$, we have $I_j \setminus I_{j-1} = \{p_j\}$ for $j = 0, 1, \ldots, m = |\mathfrak{P}| - 1$ and $p_i \prec_{\mathfrak{P}} p_j$ implies $i < j$. In this case

$$F(I_\bullet) \;=\; \left\{\phi \in \mathbb{R}^{\mathfrak{P}} \;:\; \phi(p_0) \;\le\; \phi(p_1) \;\le\; \cdots \;\le\; \phi(p_{m-1})\right\}.$$

Modulo lineality space, this is a unimodular simplicial cone spanned by the characteristic functions $\phi^0, \phi^1, \ldots, \phi^{m-1} : \mathfrak{P} \to \{0,1\}$ with $\phi^k(p_j) = 1$ if and only if $j \geq k$. Faces of $F(I_\bullet)$ correspond to the coarsenings of $I_\bullet$ and since every $\phi \in \mathcal{L}(\mathfrak{P})$ induces a unique $I_\bullet = I_\bullet(\phi)$, this proves the following result, which was first shown by Stanley [45] for the order polytope $\mathcal{L}(\mathfrak{P}) \cap [0,1]^{\mathfrak{P}}$.

Proposition 4.2.2. *Let $\mathfrak{P}$ be a finite poset. Then*

$$\mathcal{T}_{\mathfrak{P}} \;=\; \left\{ F(I_\bullet^{\mathfrak{P}}) \;:\; I_\bullet^{\mathfrak{P}} \text{ chain of order ideals in } \mathfrak{P} \right\}$$

is a subdivision of $\mathcal{L}(\mathfrak{P})$ into unimodular simplicial cones.

4.2.2 Marked order polytopes

In the following let $\mathfrak{A} \subseteq \mathfrak{P}$ always denote an induced subposet of a finite poset $\mathfrak{P}$, and let $\min(\mathfrak{P}) \cup \max(\mathfrak{P}) \subseteq \mathfrak{A}$. For an order preserving map $\lambda : \mathfrak{A} \to \mathbb{R}$, the marked order polytope

$$\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda) \;=\; \left\{ \widehat{\lambda} \in \mathcal{L}(\mathfrak{P}) \;:\; \widehat{\lambda}(a) = \lambda(a) \text{ for all } a \in \mathfrak{A} \right\} \;=\; \mathcal{L}(\mathfrak{P}) \cap \operatorname{Ext}_{\mathfrak{P},\mathfrak{A}}(\lambda)$$

is the intersection of the order cone with the affine space $\operatorname{Ext}_{\mathfrak{P},\mathfrak{A}}(\lambda) = \{\widehat{\lambda} \in \mathbb{R}^{\mathfrak{P}} \colon \widehat{\lambda}|_{\mathfrak{A}} = \lambda\}$. Every face of $\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ is a section of a face H of $\mathcal{L}(\mathfrak{P})$ with $\operatorname{Ext}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ and is itself a marked order polytope. We denote the dependence of H on λ by $H(\lambda)$. We can describe them in terms of face partitions.

Proposition 4.2.3. *Let $\mathfrak{G}$ be a face partition of $\mathfrak{P}$ and let $\lambda : \mathfrak{A} \to \mathbb{R}$ be an order preserving map for an induced subposet $\mathfrak{A} \subseteq \mathfrak{P}$. Then $\operatorname{Ext}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ meets $F_{\mathfrak{P}}(\mathfrak{G})$ in the relative interior if and only if the following holds for all $a, b \in \mathfrak{A}$: Let $\mathfrak{G}_a, \mathfrak{G}_b \subseteq \mathfrak{P}$ be the connected components of $\mathfrak{G}$ containing a and b, respectively.*

i) *If $\lambda(a) < \lambda(b)$, then*

$$\bigcup_{p \in \mathfrak{G}_a} \mathfrak{P}_{\preceq p} \;\cap\; \bigcup_{q \in \mathfrak{G}_b} \mathfrak{P}_{\succeq q} \;=\; \varnothing.$$

ii) *If $\lambda(a) = \lambda(b)$ and there are comparable elements $\tilde{a} \in \mathfrak{G}_a$ and $\tilde{b} \in \mathfrak{G}_b$, then $\mathfrak{G}_a = \mathfrak{G}_b$.*

In this case, $F_{\mathfrak{P}}(\mathfrak{G}) \cap \operatorname{Ext}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ is linearly isomorphic to $\mathcal{O}_{\mathfrak{P}/\mathfrak{G},\mathfrak{A}/\mathfrak{G}}(\lambda_{\mathfrak{G}})$, where $\lambda_{\mathfrak{G}} : \mathfrak{A}/\mathfrak{G} \to \mathbb{R}$ is the well-defined map on the quotient.

Proof. Let $\mathfrak{P}/\mathfrak{G}$ be the quotient poset associated to the face partition $\mathfrak{G}$. The quotient $\mathfrak{A}/\mathfrak{G}$ is a subposet of $\mathfrak{P}/\mathfrak{G}$ and $\lambda_{\mathfrak{G}} : \mathfrak{A}/\mathfrak{G} \to \mathbb{R}$ is a well-defined map if condition i) holds. Moreover, the induced map $\lambda_{\mathfrak{G}}$ is order preserving for $\mathfrak{A}/\mathfrak{G}$ if condition i) holds, and in fact strictly if ii) holds. Thus $F_{\mathfrak{P}}(\mathfrak{G}) \cap \mathrm{Ext}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ is linearly isomorphic to $\mathcal{O}_{\mathfrak{P}/\mathfrak{G},\mathfrak{A}/\mathfrak{G}}(\lambda_{\mathfrak{G}})$ which is of maximal dimension. □

We call a face partition **compatible** with λ if it satisfies the conditions above. In particular, taking the intersection of all compatible face partitions of $\mathfrak{P}$, we obtain $\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ as an improper face.

Corollary 4.2.4. *Let $\mathfrak{P}$ be a finite poset, $\mathfrak{A} \subseteq \mathfrak{P}$ be an induced subposet, and $\lambda : \mathfrak{A} \to \mathbb{R}$ an order preserving map. Then $\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ is a convex polytope of dimension*

$$\dim \mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda) \;=\; |\mathfrak{P} \setminus \{p \in \mathfrak{P} \;:\; a \preceq p \preceq b \text{ for } a, b \in \mathfrak{A} \text{ with } \lambda(a) = \lambda(b)\}|\,.$$

Proof. The presentation as the affine section of a cone marks $\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ as a convex polyhedron. As every element of $\mathfrak{P}$ has by assumption a lower and an upper bound in $\mathfrak{A}$, it follows that $\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ is a polytope. The right-hand side is exactly the number of elements of $\mathfrak{P}$ whose values are not yet determined by λ and $\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ has at most this dimension. On the other hand, Lemma 4.2.5 shows the existence of a subpolytope of exactly this dimension. □

4.2.3 Induced subdivisions and arithmetic

Intersecting every cell of the canonical subdivision $\mathcal{T}_{\mathfrak{P}}$ of $\mathcal{L}(\mathfrak{P})$ with the affine space $\mathrm{Ext}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ induces a subdivision of $\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ that we can explicitly describe. To describe the cells in the intersection, let $I_\bullet$ be a chain of order ideals of $\mathfrak{P}$. For $a \in \mathfrak{P}$ we denote by $i(I_\bullet, a)$ the smallest index j for which $a \in I_j$. We call a chain of order ideals $I_\bullet$ of $\mathfrak{P}$ **compatible** with λ if

$$i(I_\bullet, a) < i(I_\bullet, b) \quad \text{if and only if} \quad \lambda(a) < \lambda(b)$$

for all $a, b \in \mathfrak{A}$. The crucial observation is that relint $F(I_\bullet) \cap \mathrm{Ext}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ is not empty if and only if $I_\bullet$ is compatible with λ and in this case $F(I_\bullet) \cap \mathrm{Ext}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ is of a particularly nice form.

Lemma 4.2.5. *Let $\mathfrak{P}$ be a finite poset, $\mathfrak{A} \subseteq \mathfrak{P}$ be an induced subposet and $\lambda : \mathfrak{A} \to \mathbb{R}$ an order preserving map. If $I_\bullet$ is a chain of order ideals of $\mathfrak{P}$ compatible with λ, then the induced cell $F(I_\bullet) \cap \mathrm{Ext}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ is a Cartesian product of simplices.*

Proof. Let $\lambda(\mathfrak{A}) = \{t_0 < t_1 < \cdots < t_r\}$ be the range of λ and pick elements $a_0, a_1, \ldots, a_r \in \mathfrak{A}$ with $\lambda(a_i) = t_i$. Let $i_j = i(I_\bullet, a_j)$ for $j = 0, 1, \ldots, r$ and, since $I_\bullet$ is compatible with λ, we have $0 = i_0 < i_1 < \cdots < i_r = k$. It follows that $F(I_\bullet) \cap \mathrm{Ext}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ is the set of all $\phi \in \mathbb{R}^{\mathfrak{P}}$ such that ϕ is constant on $I_h \setminus I_{h-1}$ for $h = 0, 1, \ldots, k$ (with $I_{-1} = \varnothing$) and

$$\begin{array}{ccccccccccc} \phi(I_0) & \le & \phi(I_1 \setminus I_0) & \le & \cdots & \le & \phi(I_{i_1} \setminus I_{i_1 - 1}) & \le & \cdots & \le & \phi(I_k \setminus I_{k-1}). \\ \| & & & & & & \| & & & & \| \\ \lambda(a_0) & & & & & & \lambda(a_1) & & & & \lambda(a_r) \end{array}$$

Thus, $F(I_\bullet) \cap \mathrm{Ext}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ is linearly isomorphic to $F_0 \times F_1 \times \cdots \times F_{r-1}$, where, by setting $s_j = \phi(I_j \setminus I_{j-1})$,

$$F_j \;=\; \{\; \lambda(a_j) \;\le\; s_{i_j+1} \;\le\; s_{i_j+2} \;\le\; \cdots \;\le\; s_{i_{j+1}-1} \;\le\; \lambda(a_{j+1}) \;\} \tag{4.2}$$

is a simplex of dimension $d_j = i_{j+1} - i_j - 1$. □

Thus the canonical subdivision of $\mathcal{L}(\mathfrak{P})$ induces a subdivision of $\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ into products of simplices indexed by compatible chains of order ideals. This is the key observation for the following result.

Theorem 4.2.6. *Let $\mathfrak{P}$ be a finite poset and let $\mathfrak{A} \subseteq \mathfrak{P}$ be an induced subposet with $\min(\mathfrak{P}) \cup \max(\mathfrak{P}) \subseteq \mathfrak{A}$. For integer-valued order preserving maps $\lambda : \mathfrak{A} \to \mathbb{Z}$, the function*

$$\Omega_{\mathfrak{P},\mathfrak{A}}(\lambda) \;=\; |\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda) \cap \mathbb{Z}^{\mathfrak{P}}|$$

is a piecewise polynomial over the order cone $\mathcal{L}(\mathfrak{A})$. The cells of the canonical subdivision of $\mathcal{L}(\mathfrak{A})$ refine the domains of polynomiality of $\Omega_{\mathfrak{P},\mathfrak{A}}(\lambda)$. In other words, $\Omega_{\mathfrak{P},\mathfrak{A}}(\lambda)$ is a polynomial restricted to any cell $F(I^{\mathfrak{A}}_\bullet)$ of the subdivision of $\mathcal{L}(\mathfrak{A})$.

Proof. Lemma 4.2.5 shows that for fixed $\lambda : \mathfrak{A} \to \mathbb{Z}$ every maximal cell in the induced subdivision of $\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ is a product of simplices and the proof actually shows that, after taking successive differences, the simplices F_j of (4.2) are lattice isomorphic to

$$(\lambda(a_{j+1}) - \lambda(a_j)) \cdot \Delta_{d_j} \;=\; \left\{ y \in \mathbb{R}^{d_j}_{\ge 0} \;:\; y_1 + y_2 + \cdots + y_{d_j} \;\le\; \lambda(a_{j+1}) - \lambda(a_j) \right\}. \tag{4.3}$$

Elementary counting then shows that

$$|F(I_\bullet) \cap \mathrm{Ext}_{\mathfrak{P},\mathfrak{A}}(\lambda) \cap \mathbb{Z}^{\mathfrak{P}}| = \prod_{j=0}^{r-1} |F_j \cap \mathbb{Z}^{\mathfrak{P}}| = \prod_{j=0}^{r-1} \binom{\lambda(a_{j+1}) - \lambda(a_j) + d_j}{d_j}, \tag{4.4}$$

which is a polynomial in λ of degree $d_0 + d_1 + \cdots + d_{r-1} = \dim F(I_\bullet) \cap \mathrm{Ext}_{\mathfrak{P},\mathfrak{A}}(\lambda)$. Applying the inclusion-exclusion principle to the induced subdivision of $\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ shows that $\Omega_{\mathfrak{P},\mathfrak{A}}$ is the evaluation of a polynomial at the given λ. To complete the proof, note that $\lambda, \lambda' : \mathfrak{A} \to \mathbb{R}$ have the same collections of compatible chains of order ideals whenever $\lambda, \lambda' \in \operatorname{relint} C$ for some cell C in the canonical subdivision $\mathcal{T}_{\mathfrak{A}}$ of $\mathcal{L}(\mathfrak{A})$. $\square$

A weaker version of Theorem 4.2.6 can also be derived from the theory of partition functions [11, Chapter 13]. It can be seen that over $\mathcal{L}(\mathfrak{A})$, the marked order polytope is of the form

$$\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda) \;=\; \{x \in \mathbb{R}^n \;:\; Bx \;\leq\; c(\lambda)\}\,,$$

where $B \in \mathbb{Z}^{M\times n}$ is a fixed matrix with $n = |\mathfrak{P}|$ and $c : \mathbb{R}^{\mathfrak{A}} \to \mathbb{R}^M$ is an affine map. Moreover, B is unimodular. It follows from the theory of partition functions that the function $\Phi_B : \mathbb{Z}^M \to \mathbb{Z}$ given by

$$g \;\mapsto\; |\{x \in \mathbb{Z}^n : Bx \leq g\}|$$

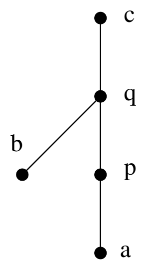
is a piecewise polynomial over the cone $C_B \subset \mathbb{R}^M$ of (real-valued) g such that the polytope above is non-empty. The domains of polynomiality are given by the **type cones** for B; see McMullen [27]. Consequently, we have $\Omega_{\mathfrak{P},\mathfrak{A}}(\lambda) = \Phi_B(c(\lambda))$. It follows that $\mathcal{L}(\mathfrak{A})$ is linearly isomorphic to a section of C_B and the canonical subdivision $\mathcal{T}_{\mathfrak{A}}$ is a refinement of the induced subdivision by type cones. It is generally difficult to give an explicit description of the subdivision of C_B by type, not to mention the sections of type cones by the image of $c(\lambda)$. So, an additional benefit of the proof presented here is the explicit description of the domains of polynomiality.

In the context of representation theory, the lattice points of certain marked order polytopes bijectively correspond to basis elements of irreducible representations; cf. the discussion in [1, 7]. Bliem [7] used partition functions of *chopped and sliced cones* to show that in the marking λ, the dimension of the corresponding irreducible representation is given by a piecewise quasipolynomial. Theorem 4.2.6 strengthens his result to piecewise polynomials. Bliem [7, Warning 1] remarks that his 'regions of quasi-polynomiality' might be too fine in the sense that the quasi-polynomials for adjacent regions might coincide. This also happens for the piecewise polynomial described in Theorem 4.2.6. In the simplest case $\mathfrak{A} = \mathfrak{P}$ and $\Omega_{\mathfrak{P},\mathfrak{A}} \equiv 1$.

Question 1. *What is the coarsest subdivision of $\mathcal{L}(\mathfrak{A})$ for which $\Omega_{\mathfrak{P},\mathfrak{A}}(\lambda)$ is a piecewise polynomial?*

For this it would be necessary to give a combinatorial condition when two adjacent cells of $\mathcal{T}_{\mathfrak{A}}$ carry the same polynomial.

Example 4.2.7. Consider the following poset $\mathfrak{P}$ given by its Hasse diagram:



Let $\mathfrak{A} = \{a, b, c\}$ and let $\lambda\colon \mathfrak{A} \to \mathbb{Z}$ be an order preserving map. If $\lambda(a) < \lambda(b) < \lambda(c)$, then there are two compatible linear extensions of $\mathfrak{P}$:

$$\begin{array}{ccccccccc} a & \prec & b & \prec & p & \prec & q & \prec & c \\ a & \prec & p & \prec & b & \prec & q & \prec & c \end{array}$$

The number of lattice points in the corresponding maximal cells of $\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ are $\binom{\lambda(c)-\lambda(b)+2}{2}$ and $(\lambda(c) - \lambda(b) + 1)(\lambda(b) - \lambda(a) + 1)$, respectively. Taking into account overcounting we have to subtract the number of order preserving extensions of λ to $\mathfrak{P}$ for which p and b have the same value. These correspond to lattice points in the cell given by the chain of order ideals

$$\{a\} \subset \{a, b, p\} \subset \{a, b, p, q\} \subset \{a, b, p, q, c\}$$

and their number is $\lambda(c) - \lambda(b) + 1$. Therefore in total $\Omega_{\mathfrak{P},\mathfrak{A}}(\lambda)$ equals

$$\binom{\lambda(c) - \lambda(b) + 2}{2} + (\lambda(c) - \lambda(b) + 1)(\lambda(b) - \lambda(a) + 1) - (\lambda(c) - \lambda(b) + 1).$$

If $\lambda(b) < \lambda(a) < \lambda(c)$, then the only compatible linear extension is

$$b \prec a \prec p \prec q \prec c$$

and thus

$$\Omega_{\mathfrak{P},\mathfrak{A}}(\lambda) = \binom{\lambda(c) - \lambda(a) + 2}{2}.$$

4.2.4 Chains and Cayley cones

Let us consider the special case in which $\mathfrak{A} \subseteq \mathfrak{P}$ is a chain. It turns out that in this case the relation between $\mathcal{L}(\mathfrak{P})$ and $\mathcal{L}(\mathfrak{A})$ is very special. A pointed polyhedral cone $K \subset \mathbb{R}^n$ is called a **Cayley cone** over L if there is a linear projection $\pi : K \to L$ onto a pointed simplicial cone L such that every ray of K is injectively mapped to a ray of L. In case K is not pointed, then $K \cong K' \times U$, where K' is pointed and U is a linear space and we require $L \cong L' \times U$ and π is an isomorphism on U. Cayley cones are the "cone-analogs" of Cayley configurations/polytopes [12, Sect. 9.2] which are precisely the preimages under π of bounded hyperplane sections $L \cap H$.

Proposition 4.2.8. *Let $\mathfrak{P}$ be a finite poset. If $\mathfrak{A} \subseteq \mathfrak{P}$ is a chain and $\min(\mathfrak{P}) \cup \max(\mathfrak{P}) \subseteq \mathfrak{A}$, then $\mathcal{L}(\mathfrak{P})$ is a Cayley cone over $\mathcal{L}(\mathfrak{A})$.*

Proof. The restriction map $\pi(\phi) = \phi|_{\mathfrak{A}}$ for $\phi \in \mathcal{L}(\mathfrak{P})$ is a surjective linear projection. Since $\mathfrak{A}$ is a chain and $\min(\mathfrak{P}) \cup \max(\mathfrak{P}) \subseteq \mathfrak{A}$, $\mathfrak{A}$ and $\mathfrak{P}$ are connected posets. The lineality spaces are spanned by $1_{\mathfrak{A}}$ and $1_{\mathfrak{P}}$, respectively, and π is an isomorphism on lineality spaces. Moreover, $\mathcal{L}_0(\mathfrak{A}) = \mathcal{L}(\mathfrak{A})/(\mathbb{R}\cdot 1_{\mathfrak{A}})$ is linear isomorphic to the cone of order preserving maps $\mathfrak{A} \to \mathbb{R}_{\geq 0}$ which map $\min(\mathfrak{A}) = \{a_0\}$ to 0, which shows that $\mathcal{L}_0(\mathfrak{A})$ is simplicial.

Thus, we only need to check that $\pi : \mathcal{L}_0(\mathfrak{P}) \to \mathcal{L}_0(\mathfrak{A})$ maps rays to rays. It follows from the description of face partitions (Proposition 4.2.1) that the rays of $\mathcal{L}_0(\mathfrak{P})$ are spanned by indicator functions of proper filters. Let ϕ be such an indicator function. Then $\phi|_{\mathfrak{A}} : \mathfrak{A} \to \{0, 1\}$ is also an indicator function of a proper filter of $\mathfrak{A}$ which proves the claim. □

Here is the main property of Cayley cones that make them an indispensable tool in the study of mixed subdivisions and mixed volumes.

Proposition 4.2.9. *Let K be a pointed Cayley cone over L. Let $r_1, \dots, r_k$ be linearly independent generators of L and let $K_i = \pi^{-1}(r_i)$ be the fiber over the generator r_i. Then for every point $p \in L$,*

$$\pi^{-1}(p) \;=\; \mu_1 K_1 \;+\; \mu_2 K_2 \;+\; \cdots \;+\; \mu_r K_r,$$

where $\mu_1, \mu_2, \dots, \mu_r \geq 0$ are the unique coefficients such that $p = \sum_i \mu_i r_i$.

Proof. Let $\{s_{ij} \in K \;:\; 1 \leq i \leq k, 1 \leq j \leq m_i\}$ be a minimal generating set of K such that $\pi(s_{ij}) = r_i$. It follows that $K_i = \operatorname{conv}\{s_{ij} \;:\; 1 \leq j \leq m_i\}$. Thus, if $\mu_{ij} \geq 0$ are such that

$$\sum_{i,j} \mu_{ij} s_{ij} \;\in\; \pi^{-1}(p),$$

then, by the uniqueness of the μ_i, we have $\sum_j \mu_{ij} = \mu_i$ and $\sum_j \mu_{ij} s_{ij} \in \mu_i K_i$. $\square$

If $\mathfrak{A} = \{a_0 \prec_{\mathfrak{P}} a_1 \prec_{\mathfrak{P}} \cdots \prec_{\mathfrak{P}} a_k\}$ is a chain, recall that $\phi^0, \phi^1, \ldots, \phi^k : \mathfrak{A} \to \{0, 1\}$ with $\phi^i(a_j) = 1$ if and only if $j \geq i$ is a minimal generating set of $\mathcal{L}(\mathfrak{A})$. If $\lambda : \mathfrak{A} \to \mathbb{R}$ is an order preserving map, then unique coordinates of $\lambda \in \mathcal{L}(\mathfrak{A})$ with respect to $\{\phi^i\}$ are given by $\mu_0 = \lambda(a_0)$ and $\mu_i = \lambda(a_i) - \lambda(a_{i-1})$ for $1 \leq i \leq r$.

Corollary 4.2.10. *Let $\mathfrak{P}$ be a finite poset and $\mathfrak{A} \subseteq \mathfrak{P}$ a chain such that $\min(\mathfrak{P}) \cup \max(\mathfrak{P}) \subseteq \mathfrak{A}$. Let $\Phi_i = \mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\phi^i)$ for $i = 1, 2, \ldots, k$. Then for any order preserving map $\lambda : \mathfrak{A} \to \mathbb{R}$,*

$$\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda) \;=\; \mu_0 1_{\mathfrak{P}} \;+\; \mu_1 \Phi_1 \;+\; \mu_2 \Phi_2 \;+\; \cdots \;+\; \mu_k \Phi_k. \tag{4.5}$$

This was already observed by Stanley [43, Theorem 3.2] and used to show that the number of order preserving maps extending a given map on a chain $\mathfrak{A} \subset \mathfrak{P}$ satisfies certain log-concavity conditions. This is done by identifying the numbers as mixed volumes which are calculated from the Cayley polytope.

In particular, $\Omega_{\mathfrak{P},\mathfrak{A}}(\lambda)$ counts the number of lattice points in the Minkowski sum (4.5). By Theorem 2.3.2 we have that $\Omega_{\mathfrak{P},\mathfrak{A}}(\lambda)$ is a multivariate polynomial in $\mu_1, \ldots, \mu_k$, where the degree in μ_i does not exceed $\dim(\Phi_i)$ for $1 \leq i \leq k$. We can even say more: It follows from Theorem 4.2.6 and equation (4.4) that over a maximal cell $C \in \mathcal{T}_{\mathfrak{A}}$, the function $\Omega_{\mathfrak{P},\mathfrak{A}}(\lambda)$ can be written as a polynomial $f(\mu)$ in the coordinates $\mu = (\mu_1, \ldots, \mu_k)$. The degree of $f(\mu)$ in every variable μ_i is given by

$$\deg_{\mu_i} f(\mu) \;=\; \dim \Phi^i \;=\; |\mathfrak{P} \setminus (\mathfrak{P}_{\preceq a_{i-1}} \cup \mathfrak{P}_{\succeq a_i})|.$$

The degree in λ_i is more difficult to determine.

Question 2. *What is $\deg_{\lambda_i} \Omega_{\mathfrak{P},\mathfrak{A}}(\lambda)$ in terms of the combinatorics of $\mathfrak{P}$?*

If $\mathfrak{A} \subseteq \mathfrak{P}$ is a chain with minimum a_0 and maximum a_k, then the degree of λ_0 and λ_k agrees with μ_1 and μ_k. A related situation is implicitly treated in Fischer [15]: The number $\alpha(n; k_1, k_2, \ldots, k_n)$ of monotone triangles with bottom row $\mathbf{k} = (k_1 \leq k_2 \leq \cdots \leq k_n)$ is a polynomial in $\mathbf{k}$ and is of degree $n-1$ in every variable k_i. In Section 4.3, it is shown that $\alpha(n; \mathbf{k})$ is essentially the number of integer-valued order preserving extensions from a particular poset with some extra conditions (i.e., certain faces of the marked order polytope are excluded). However, it appears that these extra conditions do not influence the degree.

4.2.5 Combinatorial reciprocity

For a special choice of $\mathfrak{A}$, we recover the classical order polytope.

Example 4.2.11 (Order polytopes)**.** Let $\mathfrak{P}'$ be the result of adjoining a minimum $\hat{0}$ and maximum $\hat{1}$ to a finite poset $\mathfrak{P}$. Let $\mathfrak{A} = \{\hat{0}, \hat{1}\}$ and for $n > 0$ let $\lambda_n : \mathfrak{A} \to \mathbb{Z}$ be the order preserving map with $\lambda_n(\hat{0}) = 1$ and $\lambda_n(\hat{1}) = n$. Then $\Omega_{\mathfrak{P}',\mathfrak{A}}(\lambda_n) = \Omega_{\mathfrak{P}}(n)$ is the order polynomial of $\mathfrak{P}$ which counts the number of order preserving maps from $\mathfrak{P}$ to $[n]$. Equivalently, $\Omega_{\mathfrak{P}',\mathfrak{A}}(\lambda_n)$ equals the Ehrhart polynomial of the order polytope $\mathcal{L}(\mathfrak{P}) \cap [0,1]^{\mathfrak{P}}$ evaluated at $n-1$. Ehrhart–Macdonald reciprocity (Theorem 2.5.5) then yields that

$$(-1)^{|\mathfrak{P}|}\,\Omega_{\mathfrak{P}}(-n) \;=\; (-1)^{\dim\,\mathcal{O}_{\mathfrak{P}',\mathfrak{A}}(\lambda_n)}\,\Omega_{\mathfrak{P}',\mathfrak{A}}(\lambda_{-n})$$

equals the number of strictly order preserving maps into $[n]$. This is a classical result by Stanley [38].

We wish to extend this combinatorial reciprocity to our more general setting. We say that an extension $\widehat{\lambda} : \mathfrak{P} \to \mathbb{R}$ of λ is **strict** if $\widehat{\lambda}(p) = \widehat{\lambda}(q)$ and $p \prec q$ implies that $a \preceq p \prec q \preceq b$ for some $a, b \in \mathfrak{A}$ with $\lambda(a) = \lambda(b)$.

Theorem 4.2.12. *Let $\mathfrak{P}$ be a finite poset and let $\mathfrak{A} \subseteq \mathfrak{P}$ be an induced subposet with* $\min(\mathfrak{P}) \cup \max(\mathfrak{P}) \subseteq \mathfrak{A}$*. If $\lambda : \mathfrak{A} \to \mathbb{Z}$ is an order preserving map, then*

$$(-1)^{\dim\,\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)}\,\Omega_{\mathfrak{P},\mathfrak{A}}(-\lambda)$$

equals the number of strict order preserving extensions of λ.

Note that if $F(I_\bullet^{\mathfrak{A}})$ is the unique cell of the subdivision of $\mathcal{L}(\mathfrak{A})$ that contains λ in the relative interior, then $\Omega_{\mathfrak{P},\mathfrak{A}}(\lambda)$ is the evaluation of a polynomial and it is this polynomial that is evaluated at $-\lambda$ in the course of Theorem 4.2.12. From the geometric point of view, $(-1)^{\dim\,\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)}\,\Omega_{\mathfrak{P},\mathfrak{A}}(-\lambda)$ counts the number of lattice points in the relative interior of $\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$. This is reminiscent of Ehrhart–Macdonald reciprocity and in fact follows from it.

Proof. For fixed λ, let $I_\bullet^{\mathfrak{A}}$ such that $\lambda \in \operatorname{relint} F(I_\bullet^{\mathfrak{A}})$. Then $\Omega_{\mathfrak{P},\mathfrak{A}}$ restricted to $\operatorname{relint} F(I_\bullet^{\mathfrak{A}})$ is given by some polynomial $p(\mathbf{x}) \in \mathbb{R}[x_a : a \in \mathfrak{A}]$. For $n \in \mathbb{Z}_{>0}$, we have that $n\lambda \in \operatorname{relint} F(I_\bullet^{\mathfrak{A}})$ and thus $\Omega_{\mathfrak{P},\mathfrak{A}}(n\lambda) = p(n\lambda)$. As $\Omega_{\mathfrak{P},\mathfrak{A}}(n\lambda)$ equals the number of lattice points in $n\,\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$, it follows that $p(n\lambda)$ is the Ehrhart polynomial of $\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$. Now, Ehrhart–Macdonald reciprocity implies that the number of points in the relative interior of $\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ equals

$$(-1)^d\,\mathsf{Ehr}(\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda), -1) \;=\; (-1)^d p(-\lambda) \;=\; (-1)^d\Omega_{\mathfrak{P},\mathfrak{A}}(-\lambda),$$

where $d = \dim\,\mathcal{O}_{\mathfrak{P},\mathfrak{A}}(\lambda)$. □

4.2.6 Marked chain polytopes

Let us close by transferring our results to the marked chain polytopes of Ardila, Bliem, and Salazar [1]. To that end we write $\phi(C) = \sum\{\phi(c) : c \in C\}$ for a subset $C \subseteq \mathfrak{P}$ and $\phi : \mathfrak{P} \to \mathbb{R}$. For a pair of posets $\mathfrak{A} \subset \mathfrak{P}$ with $\min(\mathfrak{P}) \cup \max(\mathfrak{P}) \subseteq \mathfrak{A}$ and an order preserving map $\lambda : \mathfrak{A} \to \mathbb{R}$, the **marked chain polytope** is the convex polytope

$$\mathcal{C}_{\mathfrak{P},\mathfrak{A}}(\lambda) = \left\{ \begin{array}{ll} \phi \in \mathbb{R}^{\mathfrak{P}}_{\geq 0} : & \phi(C) \ \leq \ \lambda(b) - \lambda(a) \text{ for all } a, b \in \mathfrak{A} \\ & \text{and every chain } C \subseteq [a,b] \end{array} \right\}.$$

The unmarked version of the chain polytope was introduced in [45] to show that certain invariants of $\mathfrak{P}$ (such as $\Omega_{\mathfrak{P}}(n)$) depend only on the comparability graph of $\mathfrak{P}$. The marked chain polytopes were introduced in [1] in connection with representation theory. Stanley defined a lattice preserving, piecewise linear map from the order polytope to the chain polytope and this transfer map was extended in [1] to relate the arithmetic of marked order polytope and marked chain polytopes. Thus, appealing to of [1, Theorem 3.4] proves the following.

Corollary 4.2.13. *For a finite poset $\mathfrak{P}$ and an induced subposet $\mathfrak{A} \subseteq \mathfrak{P}$ with $\min(\mathfrak{P}) \cup \max(\mathfrak{P}) \subseteq \mathfrak{A}$, the function*

$$\lambda \ \mapsto \ |\mathcal{C}_{\mathfrak{P},\mathfrak{A}}(\lambda) \ \cap \ \mathbb{Z}^{\mathfrak{P}}|$$

is a piecewise polynomial over $\mathcal{L}(\mathfrak{A}) \cap \mathbb{Z}^{\mathfrak{A}}$ and evaluating at $-\lambda$ equals the number of lattice points in the relative interior of $\mathcal{C}_{\mathfrak{P},\mathfrak{A}}(\lambda)$ times $(-1)^{\dim \mathcal{C}_{\mathfrak{P},\mathfrak{A}}(\lambda)}$.

4.3 Monotone triangle reciprocity

A **monotone triangle** of order n, as exemplified in (4.1), is a triangular array of integers $a = (a_{i,j})_{1 \leq j \leq i \leq n} \in \mathbb{Z}$ such that the entries

(M1) weakly increase along the northeast direction: $a_{i,j} \leq a_{i-1,j}$ for all $1 \leq j < i \leq n$;

(M2) weakly increase along the southeast direction: $a_{i,j} \leq a_{i+1,j+1}$ for all $1 \leq j \leq i < n$; and

(M3) strictly increase in the rows: $a_{i,j} < a_{i,j+1}$ for all $1 \leq j < i < n$.

The number of monotone triangles with fixed bottom row $\mathbf{k} = (k_1 \leq k_2 \leq \cdots \leq k_n)$ is finite and denoted by $\alpha(n; k_1, k_2, \ldots, k_n)$. Monotone triangles originated in the study of alternating sign matrices [30], where it was shown

$$
\begin{array}{ccccccccc}
 & & & & 3 & & & & \\
 & & & \multicolumn{3}{c}{\underline{3 \qquad 3}} & & & \\
 & & 4 & & 3 & & 3 & & \\
 & \multicolumn{3}{c}{\underline{4 \qquad 4}} & & 3 & & 2 & \\
\multicolumn{3}{c}{\underline{4 \qquad 4}} & & \multicolumn{3}{c}{\underline{3 \qquad 3}} & & 1
\end{array}
$$

Figure 4.1: An example of a decreasing monotone triangle.

that alternating sign matrices of order n exactly correspond to monotone triangles with bottom row $(1, 2, \ldots, n)$. The study of enumerative properties of monotone triangles with general bottom row was initiated in [15], where it was shown that $\alpha(n; k_1, k_2, \ldots, k_n)$ is a polynomial in the strictly increasing arguments. Note that our definition of a monotone triangle slightly differs from that of Fischer [15] in that we do not require that the bottom row is strictly increasing.

More precisely, there is a polynomial that agrees with $\alpha(n; \mathbf{k})$ for increasing $\mathbf{k} = (k_1 \leq k_2 \leq \cdots \leq k_n)$ and, by abuse of notation, we identify $\alpha(n; \mathbf{k})$ with this polynomial. As a polynomial, $\alpha(n; \mathbf{k})$ admits evaluations at arbitrary $\mathbf{k} \in \mathbb{Z}^n$ and it is natural to ask if there are domains for which the values $\alpha(n; \mathbf{k})$ have combinatorial significance. An interpretation for the values of α at weakly decreasing arguments was given by Fischer and Riegler [17] in terms of signed enumeration of so called decreasing monotone triangles. A **decreasing monotone triangle** (DMT) is again a triangular array $b = (b_{i,j})_{1 \leq j \leq i \leq n} \in \mathbb{Z}$ such that

(W1) the entries weakly decrease along the northeast direction: $b_{i,j} \geq b_{i-1,j}$ for $1 \leq j < i \leq n$;
(W2) the entries weakly decrease along the southeast direction: $b_{i,j} \geq b_{i+1,j+1}$ for $1 \leq j \leq i < n$;
(W3) there are no three identical entries per row; and
(W4) two consecutive rows do not contain the same integer exactly once.

The collection of DMTs with bottom row $\mathbf{k} = (k_1 \geq k_2 \geq \cdots \geq k_n) \in \mathbb{Z}^n$ is denoted by $\mathcal{W}_n(\mathbf{k})$. For a DMT b, two adjacent and identical elements in a row are called a **duplicate-descendant** if either they are in the last row or the row below contains exactly the same pair. In the example, the duplicate-descendants are underlined. The number of duplicate-descendants of b is denoted by $\mathrm{dd}(b)$.

The precise reciprocity statement now is as follows.

Theorem 4.3.1 ([17, Theorem 1]). *For weakly decreasing integers* $\mathbf{k} = (k_1 \geq k_2 \geq \cdots \geq k_n)$,

$$\alpha(n; k_1, k_2, \ldots, k_n) \;=\; (-1)^{\binom{n}{2}} \sum_{b \in \mathcal{W}_n(\mathbf{k})} (-1)^{\mathrm{dd}(b)}.$$

In this section we give a geometric proof of Theorem 4.3.1 by relating (decreasing) monotone triangles to special order preserving maps. A **Gelfand–Tsetlin poset** GT_n of order n is the poset on $\{(i,j) \in \mathbb{Z}^2 \;:\; 1 \leq j \leq i \leq n\}$ with order relation

$$(i,j) \preceq_{\mathrm{GT}_n} (k,l) \quad :\Longleftrightarrow \quad k-i \leq l-j \text{ and } j \leq l.$$

The Hasse diagram for GT_n is given in Figure 4.2. Throughout, we let

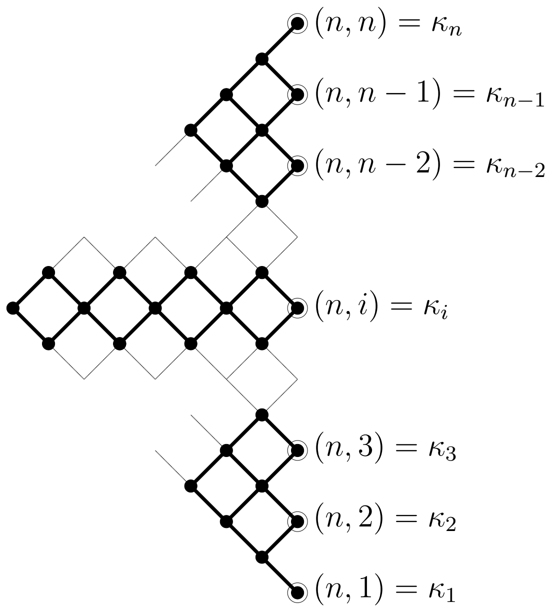


Figure 4.2: Hasse diagram for the Gelfand–Tsetlin poset of order n (in solid black).

$\mathfrak{A} = \{\kappa_1, \kappa_2, \ldots, \kappa_n\} \subset \mathrm{GT}_n$ be the n-chain of elements $\kappa_j = (n,j)$ with $1 \leq j \leq n$, depicted by the circled elements in Figure 4.2. An increasing sequence $k = (k_1 \leq k_2 \leq \cdots \leq k_n)$ corresponds to an order preserving map $k : \mathfrak{A} \to \mathbb{Z}$ by setting $k(\kappa_i) = k_i$. We call an order preserving map $a : \mathrm{GT}_n \to \mathbb{Z}$ a **weak monotone triangle** (also known as a **Gelfand–Tsetlin pattern**). Here is the main observation.

Observation 3. Monotone triangles $a = (a_{ij})_{1 \le j \le i \le n} \in \mathbb{Z}^{\binom{n+1}{2}}$ for a given bottom row $\mathbf{k} = (k_1 \le k_2 \le \cdots \le k_n) \in \mathbb{Z}^n$ bijectively correspond to integer-valued order preserving maps $a : \mathrm{GT}_n \to \mathbb{Z}$ extending $\mathbf{k} : \mathfrak{A} \to \mathbb{Z}$ and such that $a_{i,j} < a_{i,j+1}$ for all $1 \le j < i < n$.

To put this initial observation to good use, we pass to **real-valued** order preserving maps and we call an order preserving map $a : \mathrm{GT}_n \to \mathbb{R}$ extending $\mathbf{k}$ a **monotone triangle** if it satisfies (M3). Hence, the monotone triangles with bottom row $\mathbf{k}$ form a special subset of the marked order polytope for GT_n,

$$\mathcal{GT}_n(\mathbf{k}) \; := \; \mathcal{O}_{\mathrm{GT}_n,\mathfrak{A}}(\mathbf{k}).$$

Let us denote $\mathcal{B}_n = \{(i,j) \; : \; 1 \le j < i < n\}$ and for $(i,j) \in \mathcal{B}_n$ define

$$Q_{ij} \;=\; \{a \in \mathcal{L}(\mathrm{GT}_n) \; : \; a_{i,j} = a_{i,j+1}\},$$

the set of real-valued weak monotone triangles that fail (M3) nonexclusively at position (i,j). The Hasse diagram of the face partition $\mathfrak{G}_{ij} = \mathfrak{G}(Q_{ij})$ of Q_{ij} is a **diamond** in GT_n:

$$\mathfrak{G}_{ij} = \begin{tikzcd} & (i,j+1) \arrow[dl] \arrow[dr] & \\ (i-1,j) \arrow[dr] & & (i+1,j+1) \arrow[dl] \\ & (i,j) & \end{tikzcd}$$

It is easy to see that $\mathfrak{G}_{ij}$ is a compatible face partition for any strictly increasing bottom row $\mathbf{k}$ and together with a count of parameters we have the following geometric result.

Proposition 4.3.2. *Let* $\mathbf{k} = (k_1 < k_2 < \cdots < k_n)$. *For* $(i,j) \in \mathcal{B}_n$, *the set* $Q_{ij}(\mathbf{k}) \subseteq \mathcal{GT}_n(\mathbf{k})$ *is a face of codimension* 3.

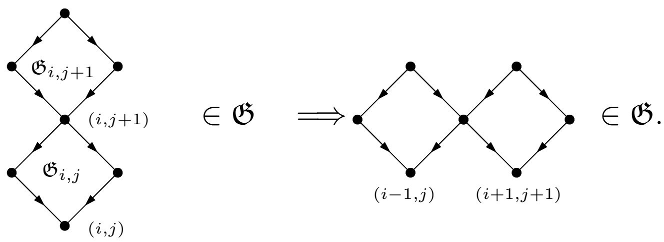
This yields a geometric perspective on monotone triangles.

Corollary 4.3.3. *For* $\mathbf{k} = (k_1 \le k_2 \le \cdots \le k_n)$, *the monotone triangles with bottom row* $\mathbf{k}$ *are precisely the lattice points in*

$$\mathcal{GT}_n(\mathbf{k}) \;\setminus \bigcup_{(i,j) \in \mathcal{B}_n} Q_{ij}(\mathbf{k}). \tag{4.6}$$

Notice that if $\mathbf{k}$ contains three identical elements $k_j = k_{j+1} = k_{j+2}$, then $\mathcal{GT}_n(\mathbf{k}) \subseteq Q_{n-1,j}(\mathbf{k})$ and the above set is empty. Hence, the number of monotone triangles with bottom row $\mathbf{k}$ can be nonzero only if $\mathbf{k}$ contains at most pairs of identical elements.

Corollary 4.3.3 allows us to write $\alpha(n;\mathbf{k})$ as a polynomial by inclusion-exclusion on the set of faces $\{Q_{ij}(\mathbf{k}) \,:\, (i,j) \in \mathcal{B}_n\}$. More refined, we will consider the poset of non-empty intersections of faces of the form Q_{ij} and obtain $\alpha(n;\mathbf{k})$ as a polynomial by Möbius inversion on that poset. This will be relatively easy, once we have a characterization of the face partitions of such finite intersections. Towards this goal we call a subposet $\mathfrak{G} \subseteq \mathrm{GT}_n$ a **diamond poset** if the Hasse diagram of $\mathfrak{G}$ is a union of graphs $\mathfrak{G}_{i,j}$. In addition, we call a diamond poset **closed** if $\mathfrak{G}_{i,j}, \mathfrak{G}_{i,j+1} \subset \mathfrak{G}$ implies $\mathfrak{G}_{i-1,j}, \mathfrak{G}_{i+1,j+1} \subset \mathfrak{G}$. That is,



Lemma 4.3.4. *Let $F \subseteq \mathcal{L}(\mathrm{GT}_n)$ be a non-empty face. Then*

$$F \;=\; \bigcap_{(i,j)\in I} Q_{ij}$$

for some $I \subseteq \mathcal{B}_n$ if and only if $\mathfrak{G}(F)$ is a closed diamond poset.

Proof. The face F is exactly the intersection of all facets for which the corresponding cover relation is in $\mathfrak{G}(F)$. If $\mathfrak{G}(F)$ is a closed diamond poset, then every cover relation is contained in at least one diamond and hence F is exactly the intersection of all Q_{ij} for which $\mathfrak{G}_{ij} \subseteq \mathfrak{G}(F)$.

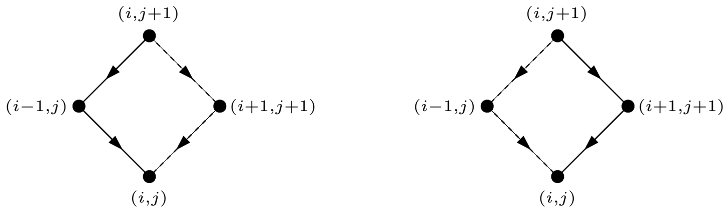
For the converse, we can assume that $\mathfrak{G} = \mathfrak{G}(F)$ is connected and we let $\mathfrak{G}' = \bigcup\{\mathfrak{G}_{ij} : F \subseteq Q_{ij}\}$ be the largest diamond poset contained in $\mathfrak{G}$. If $\mathfrak{G} \neq \mathfrak{G}'$, then by Proposition 4.2.1 there is a nontrivial directed path $\mathcal{P} = p_0 p_1 \ldots p_k$ that meets $\mathfrak{G}'$ only in a connected component containing p_0 and p_k. In particular no edge of $\mathcal{P}$ is contained in a diamond of $\mathfrak{G}$ and, furthermore, $\mathcal{P}$ cannot contain vertices (i,j) and $(i,j+1)$. Indeed, by Proposition 4.2.1, this would imply that $\mathfrak{G}_{ij} \subset \mathfrak{G}'$, which contradicts $\mathcal{P} \cap \mathfrak{G}' = \{p_0, p_k\}$. It follows that $c = p_{i+1} - p_i \in \mathbb{Z}^2$ is a constant direction for all $i = 0, 1, \ldots, k-1$.

Let us assume that $c = (1,0)$. Thus, every vertex p_h along $\mathcal{P}$ has constant second coordinate $\ell = (p_h)_2$. Let $\mathcal{R}$ be an undirected(!) path connecting p_0

and p_k in $\mathfrak{G}'$ such that

$$\rho(\mathcal{R}) \;=\; \sum_{r\in\mathcal{R}} |r_2 - \ell|$$

is minimal. Such a path exists, as p_0 and p_k are in the same connected component of the underlying undirected graph of $\mathfrak{G}'$, and $\rho(\mathcal{R}) > 0$. (Indeed, we have $\rho(\mathcal{R}) = 0$ if and only if $\mathcal{R} = \mathcal{P}$ after orienting edges.) But then $\mathcal{R}$ contains a sequence of vertices $(i,j),(i-1,j),(i,j+1)$ with $j < l$ or $(i,j),(i+1,j+1),(i,j+1)$ with $j \geq l$, and the value of $\rho(\mathcal{R})$ can be reduced by rerouting along $\mathfrak{G}_{i,j}$:



Hence, by contradiction, $\mathcal{R} = \mathcal{P}$ and $\mathfrak{G} = \mathfrak{G}'$. $\square$

Let us define $\mathcal{Q}$ as the set of all closed diamond subposets of GT_n ordered by reverse inclusion. In light of the above lemma,

$$\mathcal{Q} \;\cong\; \left\{ \bigcap_{(i,j)\in I} Q_{ij} \;:\; I \subseteq \mathcal{B}_n \right\}$$

is a meet-semilattice with greatest element $\hat{1} = \hat{1}_{\mathcal{Q}} := \varnothing$ corresponding to $\mathcal{L}(\mathrm{GT}_n)$. The Möbius function of $\mathcal{Q}$ can now be described in the language of diamond posets. Let us write

$$I(\mathfrak{G}) \;=\; \{(i,j) \in \mathcal{B}_n \;:\; \mathfrak{G}_{ij} \subseteq \mathfrak{G}\}$$

for $\mathfrak{G} \in \mathcal{Q}$.

Lemma 4.3.5. *Let $\mathfrak{G} \in \mathcal{Q}$ and $I = I(\mathfrak{G})$. Then*

$$\mu_{\mathcal{Q}}(\mathfrak{G}, \hat{1}) \;=\; \begin{cases} 0 & \textit{if } (i,j),(i,j+1) \in I, \\ (-1)^{|I|} & \textit{otherwise.} \end{cases}$$

Proof. Let $\mathcal{A}$ be the collection of atoms of the interval $[\mathfrak{G}, \hat{1}]_{\mathcal{Q}}$, that is, the elements of $\mathcal{Q}$ covering $\mathfrak{G}$. To prove the first claim, we will use the Crosscut Theorem 1.9:

$$\mu_{\mathcal{Q}}(\mathfrak{G}, \hat{1}) \;=\; N_0 \;-\; N_1 \;+\; \cdots \;+ (-1)^i N_i,$$

where N_k is the number of k-element subsets $S \subseteq \mathcal{A}$ such that $\hat{1}$ is the smallest joint upper bound for the elements in S. Now if there is some $Q \prec \hat{1}_{\mathcal{Q}}$ such that every $H \in \mathcal{A}$ is smaller than Q, then this implies $N_k = 0$ for all k and the claim follows.

To that end, let $(i_0, j_0) \in I(\mathfrak{G})$ with $(i_0+1, j_0), (i_0+1, j_0+1) \in I(\mathfrak{G})$ and i_0 minimal. We claim that $(i_0, j_0) \in I(H)$ for every $H \in \mathcal{A}$. Indeed, assume that $(i_0, j_0) \notin I(H)$. By Lemma 4.3.4, we have that $H \cup \mathfrak{G}_{i_0,j_0}$ is a diamond poset but not closed, as $H \in \mathcal{A}$ by assumption. This forces $\mathfrak{G}_{i_0,j_0-1}$ or $\mathfrak{G}_{i_0,j_0+1}$ to be in $\mathfrak{G}$, and establishing then the closedness condition has to introduce some $\mathfrak{G}_{i,j} \subseteq \mathfrak{G}$ with $i < i_0$. However, this contradicts the choice of i_0 and we can take $Q = Q_{i_0 j_0}$.

If there is no index pair $(i,j) \in I$ such that $(i, j+1) \in I$, we observe that the closedness condition for $\mathfrak{G}$ is vacuous. This stays true for every diamond subposet which is in bijection to the subsets of $I(\mathfrak{G})$. Hence $[\mathfrak{G}, \hat{1}]_{\mathcal{Q}}$ is isomorphic to the boolean lattice on $|I(\mathfrak{G})|$ elements. $\square$

Lemma 4.3.5 yields a partial explanation of condition (W3): A weak monotone triangle $a : \mathrm{GT}_n \to \mathbb{R}$ with strictly increasing bottom row satisfies (W3) and (W4) if and only if $a \in \operatorname{relint} F$ for some face F with $\mathfrak{G} = \mathfrak{G}(F) \in \mathcal{Q}$ and $\mu_{\mathcal{Q}}(F, \hat{1}) \neq 0$. For that reason, let us define the **essential** subposet of $\mathcal{Q}$ as

$$\mathcal{Q}_{\mathrm{ess}} \;=\; \left\{ \mathfrak{G} \in \mathcal{Q} \;:\; \mu_{\mathcal{Q}}(\mathfrak{G}, \hat{1}) \neq 0 \right\}.$$

Hence, we can identify $\mathcal{Q}_{\mathrm{ess}}$ with the collection of closed diamond posets $\mathfrak{G}$ of GT_n such that $\mathfrak{G}_{i,j} \cup \mathfrak{G}_{i,j+1} \not\subseteq \mathfrak{G}$. In particular, $\hat{1} \in \mathcal{Q}_{\mathrm{ess}}$ and from the definition of Möbius functions it follows that $\mu_{\mathcal{Q}_{\mathrm{ess}}}(\mathfrak{G}, \hat{1}) = \mu_{\mathcal{Q}}(\mathfrak{G}, \hat{1})$ for all $\mathfrak{G} \in \mathcal{Q}_{\mathrm{ess}}$.

We can now write the number of lattice points in (4.6) as a polynomial in $\mathbf{k}$. For clarity, let us emphasize that the combinatorics of $\mathcal{Q}_{\mathrm{GT}_n,\mathfrak{A}}(\mathbf{k})$ is independent of the actual choice of a strictly order preserving map $\mathbf{k} : \mathfrak{A} \to \mathbb{R}$. In this case, every $\mathfrak{G} \in \mathcal{Q}_{\mathrm{ess}}$ is a compatible face partition of a distinct face of $\mathcal{GT}_n(\mathbf{k})$ which we can identify with the marked order polytope $\mathcal{O}_{\mathrm{GT}_n/\mathfrak{G},\mathfrak{A}/\mathfrak{G}}(\mathbf{k})$.

Theorem 4.3.6. *For $\mathbf{k} = (k_1 \leq k_2 \leq \cdots \leq k_n)$, the number of monotone triangles with bottom row $\mathbf{k}$ is given by*

$$\alpha(n; \mathbf{k}) \;=\; \sum_{\mathfrak{G} \in \mathcal{Q}_{\mathrm{ess}}} (-1)^{|I(\mathfrak{G})|}\, \Omega_{\mathrm{GT}_n/\mathfrak{G},\mathfrak{A}/\mathfrak{G}}(\mathbf{k})$$

and thus is a polynomial. In particular, $\alpha(n; \mathbf{k}) = 0$ whenever $k_j = k_{j+1} = k_{j+2}$.

Proof. If $\mathbf{k}$ is strictly order preserving, then the above formula is exactly Möbius inversion (see Theorem 1.7) of the function $f_{\mathfrak{G}}(\mathbf{k}) = \Omega_{\mathrm{GT}_n/\mathfrak{G},\mathfrak{A}/\mathfrak{G}}(\mathbf{k})$ for $\mathfrak{G} \in \mathcal{Q}_{\mathrm{ess}}$, by Corollary 4.3.3 and Lemmata 4.3.4 and 4.3.5.

If $\mathbf{k}$ has two but not three identical entries, then $\mathfrak{G} \in \mathcal{Q}_{\mathrm{ess}}$ is not compatible with $\mathbf{k}$ but can be completed to a compatible face partition $\bar{\mathfrak{G}}$. It is easy to see that $\bar{\mathfrak{G}}$ arises from $\mathfrak{G}$ by adding the cover relations $(n, j) \prec_{\mathrm{GT}_n} (n-1, j)$ and $(n-1, j) \prec_{\mathrm{GT}_n} (n, j+1)$ for every $1 \le j < n$ with $k_j = k_{j+1}$. The map $\mathfrak{G} \mapsto \bar{\mathfrak{G}}$ is injective on $\mathcal{Q}_{\mathrm{ess}}$ and the image is a poset under reverse inclusion isomorphic to $\mathcal{Q}_{\mathrm{ess}}$. Hence, the above formula counts the number of lattice points in (4.6).

If $\mathbf{k}$ has three identical entries, then (4.6) is the empty set and $\alpha(n; \mathbf{k}) = 0$. Consequently, we have to show that the right-hand side is also identically zero for all such $\mathbf{k}$. It suffices to assume that $\mathbf{k}$ has exactly three identical entries as every bottom row with more than three identical elements belongs to the boundary of some cell for which the interior consists of bottom rows with exactly three identical elements. So, let us assume that $k_j = k_{j+1} = k_{j+2}$ are the only equalities for $\mathbf{k}$. Let $\mathfrak{G} \in \mathcal{Q}_{\mathrm{ess}}$ and $\bar{\mathfrak{G}}$ its completion to a face partition compatible with $\mathbf{k}$. Then $\Omega_{\mathrm{GT}_n/\bar{\mathfrak{G}},\mathfrak{A}/\bar{\mathfrak{G}}}(\mathbf{k})$ appears in the sum on the right-hand side with multiplicity

$$\sum \left\{ (-1)^{|I(H)|} \ : \ H \in \mathcal{Q}_{\mathrm{ess}}, \bar{H} = \bar{\mathfrak{G}} \right\}.$$

For any such H, let $(i, j) \in \mathcal{B}_n$ be the lexicographic smallest pair such that $\mathfrak{G}_{i+1,j} \cup \mathfrak{G}_{i+1,j+1} \subseteq \bar{H} = \bar{\mathfrak{G}}$. (Existence follows from $k_j = k_{j+1} = k_{j+2}$.) Hence $\mathfrak{G}_{i,j} \subseteq \bar{H}$ by closedness. We distinguish two cases:

1. If $\mathfrak{G}_{ij} \subseteq H$, then the largest diamond subposet $H' \subset H$ not containing $\mathfrak{G}_{ij}$ is closed as $H \in \mathcal{Q}_{\mathrm{ess}}$, and $\bar{H}' = \bar{\mathfrak{G}}$ as $\mathfrak{G}_{i+1,j} \cup \mathfrak{G}_{i+1,j+1} \subseteq \bar{H}$.
2. If $\mathfrak{G}_{ij} \nsubseteq H$, then set $H' = H \cup \mathfrak{G}_{ij}$. By the minimality of (i, j) we have that H' is closed diamond and $\bar{H}' = \bar{\mathfrak{G}}$.

This defines a perfect matching on $\left\{ H \in \mathcal{Q}_{\mathrm{ess}} \ : \ \bar{H} = \bar{\mathfrak{G}} \right\}$, and $|I(H)| = |I(H')| \pm 1$ shows that the multiplicity of $\Omega_{\mathrm{GT}_n/\bar{\mathfrak{G}},\mathfrak{A}/\bar{\mathfrak{G}}}(\mathbf{k})$ is zero. □

Coming back to the reciprocity statement for monotone triangles, we note that $b = (b_{ij})_{1 \le j \le i \le n}$ is a DMT if and only if $-b : \mathrm{GT}_n \to \mathbb{R}$ is a weak monotone triangle satisfying (W3) and (W4).

Proposition 4.3.7. *Let $a = (a_{ij})_{1 \le j \le i \le n} \in \mathbb{Z}^{\binom{n+1}{2}}$ be a weak monotone triangle with bottom row $\mathbf{k} = (k_1 \le k_2 \le \cdots \le k_n)$ with no three identical elements. Then $-a$ is a DMT with bottom row $-\mathbf{k}$ if and only if there is a unique $\mathfrak{G} \in \mathcal{Q}_{\mathrm{ess}}$ with corresponding face $F \subseteq \mathcal{GT}_n(\mathbf{k})$ such that $a \in \operatorname{relint} F$.*

Proof. Let F be the face of $\mathcal{GT}_n(\mathbf{k})$ that has a in the relative interior and let $\mathfrak{G}' = \mathfrak{G}(F)$ be its compatible face partition. If $\mathbf{k}$ is not strictly increasing, then $\mathfrak{G}'$ contains cover relations that reach into $\mathfrak{A}$. Let $\mathfrak{G} \subseteq \mathfrak{G}'$ be the subposet that arises by deleting those not contained in a diamond. Then $\mathfrak{G}$ is a face partition and $\mathrm{Ext}_{\mathrm{GT}_n,\mathfrak{A}}(\lambda) \cap F_{\mathrm{GT}_n}(\mathfrak{G}) = F$.

Now (W4) is equivalent to the condition that every cover relation in $\mathfrak{G}$ is contained in a diamond. Otherwise there are index pairs $(i,j),(i+1,k) \in \mathcal{B}_n$ with $k \in \{j, j+1\}$ such that $b_{i,j} = b_{i+1,k}$ and $b_{i,j-1} < b_{i,j} < b_{i,j+1}$ and $b_{i,k-1} < b_{i,k} < b_{i,k+1}$, which contradicts (W4). Since $\mathbf{k}$ does not contain three identical elements, $\mathfrak{G}$ is the unique diamond poset that gives rise to F. Moreover, $\mathfrak{G} \in \mathcal{Q}_{\mathrm{ess}}$ if and only if every point in the relative interior of F satisfies (W3). □

Let us extend the notion of duplicate-descendants to real-valued weak monotone triangles satisfying (W3) and define $\mathrm{dd}(F)$ for a non-empty face $F \subseteq \mathcal{GT}_n(\mathbf{k})$ as the number of duplicate-descendants for an arbitrary $a \in \operatorname{relint} F$.

Lemma 4.3.8. *Let* $\mathbf{k} = (k_1 \le k_2 \le \cdots \le k_n)$ *with no three identical elements and let* m *be the number of pairs of identical elements. Let* $\mathfrak{G} \in \mathcal{Q}_{\mathrm{ess}}$ *with corresponding face* $F \subseteq \mathcal{GT}_n(\mathbf{k})$*. Then*

$$|I(\mathfrak{G})| \ + \ \operatorname{codim} F + m \ \equiv \ \mathrm{dd}(F) \qquad \mod 2.$$

Proof. We induct on $l = |I(\mathfrak{G})|$. For $l = 0$, we have $F = \mathcal{GT}_n(\mathbf{k})$, which is of codimension 0 and $\mathrm{dd}(F) = m$ by definition.

For $l > 0$ there is a diamond $\mathfrak{G}_{ij} \subseteq \mathfrak{G}$ that shares at most one edge with another diamond or a "half-diamond" coming from a pair of equal numbers at the bottom row. Let $\mathfrak{G}' \subset \mathfrak{G}$ be the largest diamond poset not containing $\mathfrak{G}_{ij}$ and let F' be the corresponding face. By induction, the claim holds for $\mathfrak{G}'$ and $|I(\mathfrak{G})| = |I(\mathfrak{G}')| + 1$.

If $\mathfrak{G}_{ij} \cap \mathfrak{G}(F')$ does not contain an edge, then $\mathrm{dd}(F) = \mathrm{dd}(F')$ and $\operatorname{codim} F = \operatorname{codim} F' + 3$. In the remaining case, $\mathfrak{G}_{ij}$ shares exactly one edge with $\mathfrak{G}(F')$ and thus $\mathrm{dd}(F) = \mathrm{dd}(F') + 1$. On the other hand, adding $\mathfrak{G}_{ij}$ to $\mathfrak{G}(F')$ binds two degrees of freedom and $\operatorname{codim} F = \operatorname{codim} F' + 2$. □

Proof of Theorem 4.3.1. By Theorem 4.3.6, $\alpha \equiv 0$ restricted to the set of order preserving maps $-\mathbf{k} : \mathfrak{A} \to \mathbb{Z}$ with three identical entries. As α is a polynomial, it follows that this extends to $\alpha(n; \mathbf{k})$. This proves the claim in this case as $\mathcal{W}_n(\mathbf{k}) = \varnothing$.

Let us assume that $\mathbf{k}$ has m pairs of identical elements. Then $\dim \mathcal{GT}_n(-\mathbf{k}) = \binom{n}{2} - m$. For $\mathfrak{G} \in \mathcal{Q}_{\text{ess}}$ let us denote by $F_{\mathfrak{G}}(-\mathbf{k})$ the corresponding non-empty face of $\mathcal{GT}_n(-\mathbf{k})$. By Theorems 4.3.6 and 4.2.12,

$$\alpha(n;\mathbf{k}) \;=\; (-1)^{\binom{n}{2}} \sum_{\mathfrak{G} \in \mathcal{Q}_{\text{ess}}} (-1)^{|I(\mathfrak{G})|+m+\operatorname{codim} F_{\mathfrak{G}}(-\mathbf{k})} |\operatorname{relint} F_{\mathfrak{G}}(-\mathbf{k}) \cap \mathbb{Z}^{\mathrm{GT}_n}|,$$

where we use $\operatorname{codim} F_{\mathfrak{G}}(-\mathbf{k}) = \binom{n}{2} - m - \dim F_{\mathfrak{G}}(-\mathbf{k})$. The claim now follows from Proposition 4.3.7 and Lemma 4.3.8. □

4.4 Extending partial graph colorings

Let $\Gamma = (V, E)$ be a graph and k a positive integer. A **k-coloring** of Γ is simply a map $c\colon V \to [k]$. The coloring is called **proper** if $c(u) \neq c(v)$ for every edge $uv \in E$. It is well known that the number of proper k-colorings of Γ is given by a polynomial in k, the **chromatic polynomial** $\chi_\Gamma(k)$. Generalizing these notions, Murty and Herzberg [21] considered the problem of counting extensions of partial colorings of Γ. For a given subset $A \subseteq V$ and a partial coloring $c : A \to [k]$, an **extension** of c of size n is an n-coloring $\widehat{c} : V \to [n]$ such that $\widehat{c}(a) = c(a)$ for all $a \in A$. If $\widehat{c}$ is moreover a proper coloring, then $\widehat{c}$ is called a **proper extension**. Such extensions only exist for $n \geq k$.

Theorem 4.4.1 ([21, Theorem 1])**.** *Let $\Gamma = (V, E)$ be a graph and $c : A \to [k]$ a partial coloring for $A \subseteq V$. Then either there are no proper extensions or there is a polynomial $\chi_{\Gamma,c}(n)$ of degree $|V| - |A|$ such that*

$$\chi_{\Gamma,c}(n) = |\{\widehat{c} : V \to [n] : \widehat{c} \textit{ proper coloring with } \widehat{c}(a) = c(a) \textit{ for all } a \in A\}|$$

for all $n \geq k$.

We give an alternative proof of their result and a combinatorial interpretation for $\chi_{\Gamma,c}(-n)$ extending the combinatorial reciprocity of Stanley [40] for the ordinary chromatic polynomial. Recall that an **orientation** σ of Γ assigns every edge e a head and a tail. An orientation is **acyclic** if there are no directed cycles. An orientation σ is **weakly compatible** with a given coloring $c : V \to [n]$ if σ orients an edge $e = uv$ along its color gradient, that is, from u to v whenever $c(u) < c(v)$.

Theorem 4.4.2. *Let $\Gamma = (V, E)$ be a graph and let $c : A \to [k]$ be a partial coloring for $A \subseteq V$. Let $A_1, A_2, \ldots, A_k$ be the partition of A into color classes* *induced by c. For $n \geq k$ we have that $(-1)^{|V \setminus A|} \chi_{\Gamma,c}(-n)$ is the number of pairs $(\widehat{c}, \sigma)$, where $\widehat{c} : V \to [n]$ is a coloring extending c, and σ is a weakly compatible acyclic orientation such that there is no directed path with both endpoints in A_i for some $i = 1, 2, \dots, k$.*

In the case that no two vertices of A get the same color, the result simplifies.

Corollary 4.4.3. *Let $\Gamma = (V, E)$ be a graph and $A \subseteq V$. If $c : A \to [k]$ is injective and $n \geq k$, then $|\chi_{\Gamma,c}(-n)|$ equals the number of pairs $(\widehat{c}, \sigma)$, where $\widehat{c}$ is an n-coloring extending c, and σ is an acyclic orientation weakly compatible with $\widehat{c}$.*

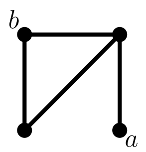
It is also possible to give an interpretation for the evaluations at $-n$ for $n < k$. Here, we constrain ourselves to one particularly interesting evaluation.

Corollary 4.4.4. *Let $\Gamma = (V, E)$ be a graph and $c : A \to [k]$ a partial coloring for $A \subseteq V$. Then $|\chi_{\Gamma,c}(-1)|$ equals the number of acyclic orientations of Γ for which there is no directed path from a to b whenever $a, b \in A$ with $c(a) \geq c(b)$.*

Furthermore, choosing $A = \varnothing$, we see that $\chi_{\Gamma,c} = \chi_\Gamma$ and the above theorem specializes to the classical reciprocity for chromatic polynomials due to Stanley [40] .

Corollary 4.4.5 ([40, Theorem 1.2])**.** *For a graph Γ, $|\chi_\Gamma(-n)|$ equals the number of pairs (c, σ) for which c is an n-coloring and σ is a weakly compatible acyclic orientation. In particular, $|\chi_\Gamma(-1)|$ is the number of acyclic orientations of Γ.*

Example 4.4.6. Consider the following graph Γ with $A = \{a, b\}$:



Let c: $A \to [k]$ be a coloring. If $c(a) = c(b)$, then for all $n \geq k$ the number of extensions of c to a proper n-coloring of Γ is

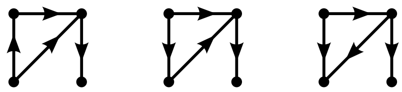
$$\chi_{\Gamma,c}(n) = (n-1)(n-2)$$

and $(-1)^2\chi_{\Gamma,c}(-1) = 6$ is the number of acyclic orientations of Γ where there is no directed path between a and b:

If $c(a) > c(b)$, then

$$\chi_{\Gamma,c}(n) = (n-2)(n-3) + (n-2) = (n-2)^2$$

and $(-1)^2\chi_{\Gamma,c}(-1) = 9$ counts the number of acyclic orientations where there is no directed path from a to b, i.e., there are three additional acyclic orientations:



The case $c(a) < c(b)$ is clearly analogous.

Proofs. First observe that we may assume that no two vertices of A are assigned the same color by c. Indeed, assume that $c(a) = c(b)$ for some $a, b \in A$. If ab is an edge of Γ, then no proper coloring can extend c and $\chi_{\Gamma,c} \equiv 0$. Moreover, in any orientation of Γ there is a directed path between a and b. If $ab \notin E$, let Γ_{ab} be obtained from Γ by identifying a and b. Then c descends to a partial coloring c_{ab} on Γ_{ab} and it is easy to see that there is a bijective correspondence between extensions of size n of c and c_{ab}. As for acyclic orientations, note that an acyclic orientation of Γ yields an acyclic orientation of Γ_{ab} if and only if there is no directed path between a and b. So, henceforth we assume that $c : A \to [k]$ is injective.

Let Γ' be the **suspension** of Γ, that is, the graph Γ with two additional vertices $\hat{0}, \hat{1}$ that are connected to all vertices of Γ. For $n \geq k$, let us consider all extensions of c to proper colorings $\widehat{c} : V' \to \{0, 1, \ldots, n+1\}$ such that $\widehat{c}(\hat{0}) = 0$ and $\widehat{c}(\hat{1}) = n+1$. Every such coloring $\widehat{c}$ gives rise to a unique compatible acyclic orientation σ by directing every edge along its color gradient. By definition, $\hat{0}$ is a source and $\hat{1}$ is a sink. The acyclicity of σ implies that we can define a partially ordered set Γ^σ on V' by setting $u \preceq_{\Gamma^\sigma} v$ if there is directed path from u to v. Extending A to $A' = A \cup \{\hat{0}, \hat{1}\}$ and c to c'_n by

$$c'_n(a) \;=\; \begin{cases} 0 & \text{if } a = \hat{0}, \\ n+1 & \text{if } a = \hat{1}, \text{ and} \\ c(a) & \text{otherwise,} \end{cases}$$

it follows that every proper coloring $\widehat{c}$ of Γ' that extends c'_n and induces σ is a strict order preserving map $\widehat{c} : \Gamma^\sigma \to \{0, 1, \ldots, n+1\}$ extending c'_n and vice versa. By Theorem 4.2.12

$$\chi_{\Gamma,c}(n) \;=\; \sum_\sigma (-1)^{|V \setminus A|} \Omega_{\Gamma^\sigma, A'}(-c'_n), \tag{4.7}$$

where the sum is over all acyclic orientations of Γ' such that for every $a, b \in A'$ there is no directed path from a to b whenever $c(a) > c(b)$. This shows that $\chi_{\Gamma,c}(n)$ is a sum of polynomials in n with positive leading coefficients. For n sufficiently large, there is an extension of c such that every vertex $V \setminus A$ gets a color $> k$. For the corresponding poset Γ^σ, the summand $\Omega_{\Gamma^\sigma,A'}(-c'_n)$ is of degree $|V| - |A|$ in n, which completes the proof of Theorem 4.4.1.

Let $A' = \{\hat{0} = a_0, a_1, \ldots, a_{r-1}, a_r = \hat{1}\}$ so that $i < j$ implies $c'_n(a_i) < c'_n(a_j)$. That is, c'_n is a strictly order preserving map for the chain A' with $c'_n(\hat{0}) = 0$ and $c'_n(\hat{1}) = n + 1$. Hence, we can consider the right-hand side of (4.7) as a polynomial in the colors $(0 = c_0 < c_1 < c_2 < \cdots < c_r = n + 1)$ of A'. However, the number of proper extensions of c is independent of the actual values of $c : A \to [k]$ for all $k \le n$. Indeed, if $k' \le n$ and $d : A \to [k']$ is a different injective partial coloring, then there is a permutation $\pi : [n] \to [n]$ that takes c to d, and π defines a bijection between the proper extensions of c and the proper extensions of d. It follows that the right-hand side of (4.7) is a polynomial independent of $c_1, \ldots, c_{r-1}$ and

$$(-1)^{|V \setminus A|}\chi_{\Gamma,c}(-n) \;=\; \sum_\sigma \Omega_{\Gamma^\sigma,A'}(-c'_{-n}) \;=\; \sum_\sigma \Omega_{\Gamma^\sigma,A'}(c'_{n-2} - \chi_A),$$

where $\chi_A : A \to \{0, 1\}$ is the characteristic function on A. Every summand is the number of order preserving maps $\Gamma^\sigma \to \{0, 1, \ldots, n-1\}$ extending $c'_{n-2} - \chi_A$. Translating back, this is exactly the number of pairs of (not necessarily proper) extensions of c and a weakly compatible acyclic orientations of Γ which yields Theorem 4.4.2. As the right-hand side of (4.7) is independent of $c_1, \ldots, c_{r-1}$,

$$(-1)^{|V \setminus A|}\chi_{\Gamma,c}(-1) \;=\; \sum_\sigma \Omega_{\Gamma^\sigma,A'}(-c'_{-1}) \;=\; \sum_\sigma \Omega_{\Gamma^\sigma,A'}(\mathbf{0}).$$

Here every summand is one, so the right-hand side counts the number of acyclic orientations such that for every $a, b \in A$ there is no directed path from a to b whenever $c(a) > c(b)$, which proves Corollary 4.4.4. $\square$

Chapter 5

Counting modulo symmetry

5.1 Introduction

Counting objects up to symmetry is a basic problem of enumerative combinatorics. A fundamental result in this context is Pólya's enumeration theorem which is concerned with counting labelings of a set of objects modulo symmetry. Here a **labeling** of a set X is defined as a map $f\colon X \to Y$ where Y is the set of labels. If G is a group acting on X then G also acts on the set of labelings $Y^X := \{f\colon X \to Y\}$. Pólya's enumeration theorem now states:

Theorem 5.1.1 (Pólya's enumeration theorem [33]). *Let G be a finite group acting on a finite set X and let Y be a finite set of $n = |Y|$ labels. Then*

$$\left|Y^X/G\right| = \frac{1}{|G|}\sum_{g\in G} n^{c(g)}$$

where Y^X/G is the collection of orbits of Y^X and $c(g)$ is the number of cycles of g as permutation of X.

In this chapter we give a new perspective on this theorem by generalizing it in terms of posets and order preserving maps. More precisely, we consider a finite poset $\mathfrak{P}$ and a group G acting on $\mathfrak{P}$ by automorphisms. Then G acts in a natural way also on the set of all order preserving maps $\mathrm{Hom}(\mathfrak{P},[n])$ from $\mathfrak{P}$ into the n-chain $[n] = \{1 < \cdots < n\}$. We show that the number of orbits of $\mathrm{Hom}(\mathfrak{P},[n])$ is given by a polynomial $\Omega_{\mathfrak{P},G}(n)$ which we call the **orbital order polynomial**. Pólya's enumeration theorem then follows by specializing this result to antichains. Further we give a combinatorial interpretation for $\Omega_{\mathfrak{P},G}(-n)$ in terms of orbits of strictly order preserving maps. This naturally generalizes the classical polynomiality and reciprocity theorems for order preserving maps due to Stanley [38]. These results can be furthermore generalized to counting $(\mathfrak{P}, \omega)$-partitions up to symmetry.

The results can be applied to graph colorings. We consider a finite group G acting by automorphisms on a finite simple graph $\Gamma = (V, E)$ and the function $\chi_{\Gamma,G}(n)$ counting proper colorings $c\colon V \to [n]$ up to group action. Cameron and Kayibi [9] seem to be the first who considered this function which they called the **orbital chromatic polynomial**. Previously, Hanlon [20] treated the case of G being the automorphism group of Γ. It is easy to see that $\chi_{\Gamma,G}(n)$ indeed agrees with a polynomial for all $n \geq 1$. We further give a representation as a sum of order polynomials.

We also give a combinatorial interpretation for evaluating this polynomial at negative integers in terms of acyclic orientations and compatible colorings. This naturally generalizes Stanley's reciprocity theorem for graph colorings [40].

The content of this chapter appeared in [22].

5.2 Order preserving Pólya enumeration

5.2.1 Groups

Let G be a finite group with identity element e and X be a finite set. A **group operation** of G on X is a map $\cdot\colon G \times X \to X$ such that $g \cdot (h \cdot x) = (gh) \cdot x$ and $e \cdot x = x$ for all $g, h \in G$ and $x \in X$. We say that G operates or **acts** on X. For every $g \in G$ we denote by X^g the **fixpoints** of g, i.e., $X^g = \{x \in X\colon g \cdot x = x\}$. For an element $x \in X$ we denote by $Gx = \{g \cdot x\colon g \in G\}$ the **orbit** of x. The set of all orbits partitions X and is called X/G.

Burnside's lemma (see e.g. [51, Theorem 10.5]) gives a formula for the number of orbits in terms of fixpoints:

Theorem 5.2.1 (Burnside's lemma)**.** *Let G be a finite group acting on a finite set X. Then*

$$|X/G| \;=\; \frac{1}{|G|}\sum_{g\in G} |X^g|.$$

For an element $x \in X$, the **stabilizer** of x is $\mathrm{Stab}(x) = \{g \in G\colon g \cdot x = x\}$. The operation can be restricted to any subgroup $H \subseteq G$. For $g \in G$ we denote by $\langle g\rangle = \{e, g, g^2, \ldots\}$ the cyclic subgroup generated by g. The orbit of x under the action of $\langle g\rangle$ is denoted by $[x]_g$ and is called a **cycle** of g. In particular, $c(g) = |X/\langle g\rangle|$ is the number of cycles of g. Identifying g with the corresponding permutation on X gives the usual notion of cycles.

Example 5.2.2. Let $G = S_n$ be the symmetric group acting on $[n] = \{1, \ldots, n\}$ in the usual way. Every permutation $\sigma \in S_n$ can be written as product of disjoint cycles in S_n and this representation is unique up to interchanging the order of the cycles in the product. In this case $[x]_\sigma = \{y \in [n] \colon y \text{ and } x \text{ are in the same cycle}\}$ and $c(\sigma)$ is the number of cycles in the unique representation as product of disjoint cycles.

If a set X has additional structure we say that a group G acts by **automorphisms** on X if the group operation respects the structure, that is, for all $g \in G$ the map $x \mapsto g \cdot x$ is a **structure preserving bijection**.

Let $\mathfrak{P}$ be a finite poset. Then a group G acts on $\mathfrak{P}$ by automorphisms if for all $g \in G$ and for all p and q in $\mathfrak{P}$ we have $g \cdot p \prec g \cdot q$ whenever $p \prec q$.

For a finite simple graph $\Gamma = (V, E)$, an action of a group G on V respects the structure of Γ if for all edges $uv \in E$ there is an edge between $g \cdot u$ and $g \cdot v$ for all $g \in G$.

The operation of G on X induces an operation on Y^X. This **induced operation** is defined by $(g \cdot f)(x) = f(g^{-1}x)$ for all $g \in G$, $f \in Y^X$ and $x \in X$.

5.2.2 Order preserving maps

Let $\mathfrak{P}$ be a finite poset. We denote the set of all order preserving maps from $\mathfrak{P}$ to $[n]$ by $\mathrm{Hom}(\mathfrak{P}, [n])$. The subset of strictly order preserving maps is denoted by $\mathrm{Hom}^\circ(\mathfrak{P}, [n])$. Their cardinalities are given by the **order polynomials** $\Omega_{\mathfrak{P}}(n)$ and $\Omega^\circ_{\mathfrak{P}}(n)$ respectively. We recall Stanley's classical result discussed in Example 4.2.11:

Theorem 5.2.3 (Stanley [38])**.** *For a finite poset $\mathfrak{P}$ the function $\Omega_{\mathfrak{P}}(n)$ agrees with a polynomial of degree $|\mathfrak{P}|$ for all $n \geq 1$, and*

$$\Omega^\circ_{\mathfrak{P}}(n) = (-1)^{|\mathfrak{P}|}\Omega_{\mathfrak{P}}(-n).$$

For every finite group G acting on $\mathfrak{P}$ by automorphisms we define a partial order on $\mathfrak{P}/G$ by defining $Gx \prec Gy$ whenever there are $\tilde{x} \in Gx$ and $\tilde{y} \in Gy$ such that $\tilde{x} \prec \tilde{y}$.
This, in fact, yields a poset, the **quotient poset** (see, e.g., [44]).

Lemma 5.2.4. *Let $\mathfrak{P}$ be a finite poset and G a finite group acting by automorphisms on $\mathfrak{P}$. Then $\mathfrak{P}/G$ is a poset.*

Proof. For irreflexivity we observe that every orbit Gx is an antichain. To see this, suppose $gx \prec hx$ for some $g, h \in G$ and consequently $x \prec g^{-1}hx$. Then it follows that $x \prec g^{-1}hx \prec (g^{-1}h)^2 x \prec \cdots \prec x$, as $g^{-1}h$ has finite order which is a contradiction as $\mathfrak{P}$ is a poset.

For transitivity, let $\tilde{x} \in Gx$, $\tilde{y}, \bar{y} \in Gy$ and $\bar{z} \in Gz$ with $\tilde{x} \prec \tilde{y}$ and $\bar{y} \prec \bar{z}$. Then there exists a $g \in G$ with $g\tilde{y} = \bar{y}$ and we have $g\tilde{x} \prec g\tilde{y} = \bar{y} \prec \bar{z}$. □

For $g \in G$ we define $\mathfrak{P}_g = \mathfrak{P}/\langle g \rangle$. Then $\mathfrak{P}_g$ is a poset with $c(g)$ elements.

An order preserving action of G on $\mathfrak{P}$ induces an action on $\mathrm{Hom}(\mathfrak{P}, [n])$ and $\mathrm{Hom}^\circ(\mathfrak{P}, [n])$ as subsets of $[n]^{\mathfrak{P}}$. We define the **orbital order polynomials** $\Omega_{\mathfrak{P},G}(n) = |\mathrm{Hom}(\mathfrak{P}, [n])/G|$ and $\Omega^\circ_{\mathfrak{P},G}(n) = |\mathrm{Hom}^\circ(\mathfrak{P}, [n])/G|$ for $n \geq 1$. The following main theorem states that $\Omega_{\mathfrak{P},G}(n)$ and $\Omega^\circ_{\mathfrak{P},G}(n)$ are indeed polynomials for $n \geq 1$ and gives formulas in terms of order polynomials:

Theorem 5.2.5. *Let G be a finite group acting by automorphisms on a finite poset $\mathfrak{P}$. Then*

$$\Omega_{\mathfrak{P},G}(n) = \frac{1}{|G|} \sum_{g \in G} \Omega_{\mathfrak{P}_g}(n), \tag{5.1}$$

$$\Omega^\circ_{\mathfrak{P},G}(n) = \frac{1}{|G|} \sum_{g \in G} \Omega^\circ_{\mathfrak{P}_g}(n) \tag{5.2}$$

for $n \geq 1$. In particular, $\Omega_{\mathfrak{P},G}(n)$ and $\Omega^\circ_{\mathfrak{P},G}(n)$ agree with polynomials of degree $|\mathfrak{P}|$ for $n \geq 1$.

Proof. We only show equation (5.1) as the argument for equation (5.2) is analogous. By Theorem 5.2.1,

$$|\mathrm{Hom}(\mathfrak{P}, [n])/G| = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Hom}(\mathfrak{P}, [n])^g| .$$

By definition, $\phi \in \mathrm{Hom}(\mathfrak{P}, [n])^g$ if and only if $\phi(g^{-1}x) = \phi(x)$ for all $x \in \mathfrak{P}$. But this is equivalent to ϕ being constant on $[x]_g$. Therefore

$$\begin{aligned} \mathrm{Hom}(\mathfrak{P}, [n])^g &\rightarrow \mathrm{Hom}(\mathfrak{P}_g, [n]) \\ \phi &\mapsto ([x]_g \mapsto \phi(x)) \end{aligned}$$

is a one-to-one correspondence. Further, observe that $\deg \Omega_{\mathfrak{P}_g}(n) = c(g) = |\mathfrak{P}|$ if and only if g acts trivially on $\mathfrak{P}$, and $c(e) = |\mathfrak{P}|$. □

By applying Theorem 5.2.5 to antichains, we get Pólya's enumeration theorem in the language of posets:

Corollary 5.2.6. *Let G be a finite group acting by automorphisms on a finite antichain A, and let $Y = [n]$. Then*

$$\left|Y^A/G\right| = \frac{1}{|G|}\sum_{g\in G} n^{c(g)}.$$

Proof. As A is an antichain we have $\mathrm{Hom}(A,[n]) = Y^A$. The result follows by observing that A_g is an antichain with $c(g)$ elements for all $g \in G$ and $\Omega_{A_g}(n) = n^{|A_g|}$. □

Example 5.2.7. Let S_k be the symmetric group acting on an antichain $A = \{x_1,\ldots,x_k\}$ on k elements by permuting indices, and let $Y = [n]$. Then

$$\{\phi \in Y^A \colon 1 \le \phi(x_1) \le \cdots \le \phi(x_k) \le n\}$$

is a set of representatives of $\mathrm{Hom}(A,Y)/S_k$ and therefore

$$\Omega_{A,S_k}(n) = \Omega_{[k]}(n) = \binom{n+k-1}{k}.$$

5.2.3 Combinatorial reciprocity

Let G be a finite group acting on a finite poset $\mathfrak{P}$ by automorphisms. As the number of orbits $|\mathrm{Hom}(\mathfrak{P},[n])/G|$ agrees with a polynomial by Theorem 5.2.5, we ponder the question if there is a combinatorial interpretation for evaluating this polynomial at negative integers. To this end, we have to consider a certain class of order preserving maps. The **sign** $\mathrm{sgn}(g)$ of an element $g \in G$ is defined as the sign of g as a permutation of $\mathfrak{P}$ and is equal to $(-1)^{|\mathfrak{P}|+c(g)}$. An order preserving map $\phi \in \mathrm{Hom}(\mathfrak{P},[n])$ is called **even** if for all $g \in \mathrm{Stab}(\phi)$ we have $\mathrm{sgn}(g) = 1$. The set of all even order preserving maps is denoted by $\mathrm{Hom}_+(\mathfrak{P},[n])$, and we define $\mathrm{Hom}_+^\circ(\mathfrak{P},[n]) := \mathrm{Hom}_+(\mathfrak{P},[n]) \cap \mathrm{Hom}^\circ(\mathfrak{P},[n])$ to be the set of even strictly order preserving maps. One observes that the action of G on $\mathrm{Hom}(\mathfrak{P},[n])$ restricts to an action on $\mathrm{Hom}_+(\mathfrak{P},[n])$. For these notions the following reciprocities hold:

Theorem 5.2.8. *Let G be a finite group acting by automorphisms on a finite poset $\mathfrak{P}$. Then*

$$\Omega_{\mathfrak{P},G}(-n) = (-1)^{|\mathfrak{P}|}\left|\mathrm{Hom}_+^\circ(\mathfrak{P},[n])/G\right|, \tag{5.3}$$

$$\Omega^\circ_{\mathfrak{P},G}(-n) = (-1)^{|\mathfrak{P}|}\left|\mathrm{Hom}_+(\mathfrak{P},[n])/G\right|. \tag{5.4}$$

Proof. Again, we only show equation (5.3) as equation (5.4) follows by analogous arguments. By equation (5.1) and Theorem 5.2.3,

$$\Omega_{\mathfrak{P},G}(-n) = \frac{1}{|G|}\sum_{g\in G}(-1)^{|\mathfrak{P}_g|}\Omega^{\circ}_{\mathfrak{P}_g}(n). \tag{5.5}$$

We observe that $|\mathfrak{P}_g| = c(g)$ is the number of orbits under the action of $\langle g\rangle$. Therefore equation (5.5) becomes

$$\begin{aligned}\Omega_{\mathfrak{P},G}(-n) &= (-1)^{|\mathfrak{P}|}\frac{1}{|G|}\sum_{g\in G}\operatorname{sgn}(g)\,|\mathrm{Hom}^{\circ}(\mathfrak{P},[n])^g| \\ &= (-1)^{|\mathfrak{P}|}\frac{1}{|G|}\sum_{\phi\in\mathrm{Hom}^{\circ}(\mathfrak{P},[n])}\ \sum_{g\in\mathrm{Stab}(\phi)}\operatorname{sgn}(g).\end{aligned}$$

For $\phi\in\mathrm{Hom}^{\circ}(\mathfrak{P},[n])$ and $g_0\in\mathrm{Stab}(\phi)$ such that $\operatorname{sgn}(g_0)=-1$ there is a bijection

$$\begin{array}{ccc}\{g\in\mathrm{Stab}(\phi)\colon\ \operatorname{sgn}(g)=1\} & \longrightarrow & \{g\in\mathrm{Stab}(\phi)\colon\ \operatorname{sgn}(g)=-1\} \\ g & \mapsto & g_0g.\end{array}$$

Hence, $\sum_{g\in\mathrm{Stab}(\phi)}\operatorname{sgn}(g) = 0$ whenever ϕ is not even. Therefore the right-hand side of equation (5.5) equals

$$(-1)^{|\mathfrak{P}|}\frac{1}{|G|}\sum_{g\in G}\left|\mathrm{Hom}^{\circ}_{+}(\mathfrak{P},[n])^g\right|$$

which equals $(-1)^{|\mathfrak{P}|}\left|\mathrm{Hom}^{\circ}_{+}(\mathfrak{P},[n])/G\right|$ by Theorem 5.2.1. □

In the setting of Pólya's enumeration theorem the statement simplifies:

Corollary 5.2.9. *Let G be a finite group acting on a finite antichain A, and let $Y=[n]$. Then*

$$\Omega_{A,G}(-n) = (-1)^{|A|}\left|\mathrm{Hom}_{+}(A,[n])/G\right|.$$

Proof. This follows from the fact that every order preserving map from an antichain is automatically strictly order preserving. □

Example 5.2.10. In Example 5.2.7 we have $\phi\in\mathrm{Hom}_{+}(A,[n])$ if and only if ϕ is injective. Therefore

$$\{\phi\in Y^A\colon 1\le\phi(x_1)<\cdots<\phi(x_k)\le n\}$$

is a set of representatives for $\mathrm{Hom}_{+}(A,[n])/S_k$. Therefore

$$|\mathrm{Hom}_{+}(A,[n])/S_k| = \Omega^{\circ}_{[k]}(n) = \binom{n}{k}.$$

Remark 5.2.11. An alternative, geometric route is by way of Ehrhart theory of order polytopes. Geometrically the setting can be translated into counting lattice points in order polytopes where the action of the group is given by permuting coordinates. This complements results by Stapledon [50] who considers lattice preserving group actions and counts lattice points inside stable rational polytopes.

5.2.4 $(\mathfrak{P},\omega)$-partitions

Theorems 5.2.5 and 5.2.8 hold, in fact, in greater generality for $(\mathfrak{P},\omega)$-partitions which were first considered by Stanley in [39]. Let $\mathfrak{P}$ be a poset and let $\omega\colon \mathfrak{P}\to\mathbb{R}$, such that $\omega(p)\neq\omega(q)$ whenever p and q are comparable. An order preserving map $f\in\operatorname{Hom}(\mathfrak{P},[n])$ is a **$(\mathfrak{P},\omega)$-partition** if for all $p,q\in\mathfrak{P}$

$$p\prec q \quad\text{and}\quad \omega(p)>\omega(q)\Longrightarrow f(p)<f(q).$$

The pairs $\{(p,q)\,:\,p\prec q,\omega(p)>\omega(q)\}$ are called **inversions**. Therefore, a $(\mathfrak{P},\omega)$-partition is an order preserving map which is strict on inversions given by ω. Let $\operatorname{Hom}^{\omega}(\mathfrak{P},[n])$ be the set of all $(\mathfrak{P},\omega)$-partitions $\mathfrak{P}\to[n]$. We observe that if ω is order preserving then $\operatorname{Hom}^{\omega}(\mathfrak{P},[n])$ simply equals $\operatorname{Hom}(\mathfrak{P},[n])$. If ω is order reversing then we have $\operatorname{Hom}^{\omega}(\mathfrak{P},[n]) = \operatorname{Hom}^{\circ}(\mathfrak{P},[n])$.

Stanley considered in [39] the **$(\mathfrak{P},\omega)$-polynomial** $\Omega^{\omega}_{\mathfrak{P}}(n)=|\operatorname{Hom}^{\omega}(\mathfrak{P},[n])|$ and showed the following generalization of Theorem 5.2.3:

Theorem 5.2.12 (Stanley [39])**.** *Let $\mathfrak{P}$ be a finite poset and $\omega\colon\mathfrak{P}\to\mathbb{R}$ be a map, such that $\omega(p)\neq\omega(q)$ whenever p and q are comparable. Then $\Omega^{\omega}_{\mathfrak{P}}(n)$ agrees with a polynomial of degree $|\mathfrak{P}|$ for $n\geq 1$ and*

$$\Omega^{\omega}_{\mathfrak{P}}(-n)=(-1)^{|\mathfrak{P}|}\Omega^{-\omega}_{\mathfrak{P}}(n).$$

Now let G be a group acting on $\mathfrak{P}$ by automorphisms that preserve inversions, i.e. for all $g\in G$ we have $\omega(p)<\omega(q)\Leftrightarrow\omega(gp)<\omega(gq)$ for all comparable p and q. For $g\in G$ we define $\omega_g\colon\mathfrak{P}_g\to\mathbb{R}$ by $\omega_g([x]_g)=\frac{1}{|\langle g\rangle|}\sum_{\tilde{x}\in[x]_g}\omega(\tilde{x})$. It is easy to see that ω_g takes different values on comparable elements in $\mathfrak{P}_g$. Analogously to the case of ordinary order preserving maps, we define the **orbital $(\mathfrak{P},\omega)$-polynomial** $\Omega^{\omega}_{\mathfrak{P},G}(n)=|\operatorname{Hom}^{\omega}(\mathfrak{P},[n])/G|$, and $\operatorname{Hom}^{\omega}_{+}(\mathfrak{P},[n])=\operatorname{Hom}^{\omega}(\mathfrak{P},[n])\cap\operatorname{Hom}_{+}(\mathfrak{P},[n])$. By very similar arguments as in the proofs of Theorem 5.2.5 and Theorem 5.2.8 we have the following generalization:

Theorem 5.2.13. *Let $\mathfrak{P}$ be a finite poset and $\omega\colon \mathfrak{P} \to \mathbb{R}$ be a map, such that $\omega(p) \neq \omega(q)$ whenever p and q are comparable. Let G be a finite group that acts on $\mathfrak{P}$ by inversion preserving automorphisms. Then*

$$\begin{aligned} \Omega^{\omega}_{\mathfrak{P},G}(n) &= \frac{1}{|G|}\sum_{g\in G}\Omega^{\omega_g}_{\mathfrak{P}_g}(n), \\ \Omega^{\omega}_{\mathfrak{P},G}(-n) &= (-1)^{|\mathfrak{P}|}\left|\mathrm{Hom}^{-\omega}_{+}(\mathfrak{P},[n])/G\right|. \end{aligned}$$

5.3 Graphs

Let $\Gamma = (V, E)$ be a finite simple graph and let G be a finite group acting on Γ by automorphisms. Recall from Section 4.4 that an n-coloring of Γ is a map $c\colon V \to [n]$. The coloring is called proper if $c(v) \neq c(w)$ whenever there is an edge between v and w. The action of G on Γ induces an action on the set of all colorings, and also on the set of all proper colorings which we denote by $\mathrm{Col}_n(\Gamma)$. The **orbital chromatic polynomial** $\chi_{\Gamma,G}$ is defined by $\chi_{\Gamma,G}(n) = |\mathrm{Col}_n(\Gamma)/G|$ for all $n \geq 1$. Further recall that an orientation $\sigma\colon E \to V$ of Γ assigns to every edge e a vertex of e called its head. An orientation is acyclic if there are no directed cycles. Every acyclic orientation σ induces a partial order on the vertex set of Γ by defining $v \prec_\sigma w$ if there is a directed path from v to w. For the corresponding poset we write Γ^σ. G acts on the set Σ of all acyclic orientations of Γ: For an edge uv we define $(g \cdot \sigma)(uv) = g \cdot \sigma(g^{-1} \cdot uv)$. The next theorem gives us an expression of $\chi_{\Gamma,G}(n)$ in terms of order polynomials. In particular, $\chi_{\Gamma,G}(n)$ is a polynomial for all $n \geq 1$.

Theorem 5.3.1. *Let Γ be a graph and let G be a group acting on Γ. Then G acts on $\mathrm{Col}_n(\Gamma)$ and*

$$\chi_{\Gamma,G}(n) = \frac{1}{|G|}\sum_{g\in G}\sum_{\sigma\in\Sigma^g}\Omega^{\circ}_{\Gamma^\sigma_g}(n)$$

for all $n \geq 1$. In particular, $\chi_{\Gamma,G}(n)$ agrees with a polynomial of degree $|\Gamma|$ for all $n \geq 1$.

Proof. By Theorem 5.2.1,

$$|\mathrm{Col}_n(\Gamma)/G| = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Col}_n(\Gamma)^g|.$$

Let ϕ be an element of $\mathrm{Col}_n(\Gamma)^g$ and let σ be the acyclic orientation induced by the coloring ϕ, i.e., an edge $e = uv$ is oriented from u to v whenever $\phi(u) < \phi(v)$. Then ϕ is a strictly order preserving map from Γ^σ into $[n]$ and σ is fixed by g, because for every edge $vw \in E$ we have $v \prec_\sigma w$ by definition if and only if $\phi(v) < \phi(w)$, and as ϕ is fixed by g, this implies $\phi(gv) < \phi(gw)$ which is equivalent to $gv \prec_\sigma gw$, i.e., $\sigma \in \Sigma^g$. $\square$

Example 5.3.2. Let $k > 2$. We consider a cycle C^k on k vertices $\{x_i\}_{i\in\mathbb{Z}_k}$. Then its symmetry group is the dihedral group

$$D_k \;=\; \langle r, s \mid r^k = 1, s^2 = 1, srs^{-1} = r^{-1}\rangle$$

which acts on C^k by

$$\begin{aligned} r \cdot x_i &= x_{i+1} \\ s \cdot x_i &= x_{-i}. \end{aligned}$$

Then

$$\chi_{C^k,D_k}(n) = \frac{1}{2k}\left(\sum_{l=1}^{k}\left|\mathrm{Col}_n(C^k)^{r^l}\right| + \sum_{l=1}^{k}\left|\mathrm{Col}_n(C^k)^{sr^l}\right|\right).$$

Let $c \in \mathrm{Col}_n(C^k)$. If $l = 2q$ is even, then $sr^{2q}\cdot c = c \Leftrightarrow (r^q\cdot c) = s\cdot(r^q\cdot c)$ and therefore

$$\left|\mathrm{Col}_n(C^k)^{sr^l}\right| = \left|\mathrm{Col}_n(C^k)^s\right| = \begin{cases} \left|\mathrm{Col}_n([\frac{k}{2}+1])\right| & \text{if } k \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

If $l = 2q+1$ is odd, then $sr^{2q+1}\cdot c = c \Leftrightarrow (r^{q+1}\cdot c) = rs\cdot(r^{q+1}\cdot c)$ and therefore

$$\left|\mathrm{Col}_n(C^k)^{s\cdot r^l}\right| = \left|\mathrm{Col}_n(C^k)^{rs}\right| = 0,$$

as $rs\cdot x_0 = x_1$, and x_0 and x_1 are connected by an edge. Further, for all $1 \le l \le k$,

$$\left|\mathrm{Col}_n(C^k)^{r^l}\right| = \begin{cases} |\mathrm{Col}_n(C^m)| & \text{if } m = \gcd(l,k) \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $k > 1$ is odd, we therefore get

$$\chi_{C^k,D_k}(n) = \frac{1}{2}\chi_{C^k,\mathbb{Z}_k}(n),$$

with $\mathbb{Z}_k := \mathbb{Z}/k\mathbb{Z} = \langle r\rangle \subset D_k$. If $k = p > 2$ is a prime this simplifies even further:

$$\left|\mathrm{Col}_n(C^p)/D_p\right| = \frac{1}{2p}\left|\mathrm{Col}_n(C^p)\right|.$$

This is reminiscent of counting necklaces with colored beads (see, e.g., [51, Chapter 35]).

A pair (c, σ) consisting of a coloring $c\colon V \to [n]$ and an acyclic orientation $\sigma\colon E \to V$ is called **weakly compatible** if for every edge $e = uv$ we have $\sigma(uv) = v$ whenever $c(u) < c(v)$. We define

$$\Sigma_n(\Gamma) = \{(c, \sigma) \in [n]^V \times \Sigma\colon \text{weakly compatible}\}$$

If G acts on Γ by automorphisms, it also acts on $\Sigma_n(\Gamma)$ by $g\cdot(c, \sigma) = (g\cdot c, g\cdot\sigma)$ for all $(c, \sigma) \in \Sigma_n(\Gamma)$ and $g \in G$. An element $(c, \sigma) \in \Sigma_n(\Gamma)$ is called **even** if for all $g \in \operatorname{Stab}((c, \sigma))$ we have $\operatorname{sgn}(g) = 1$ as permutation of the vertices. We denote the set of all even elements of $\Sigma_n(\Gamma)$ by $\Sigma_{n,+}(\Gamma)$. The action of G restricts to an action on $\Sigma_{n,+}(\Gamma)$. We get the following reciprocity statement:

Theorem 5.3.3. *Let Γ be a graph and G a group acting on Γ. Then*

$$\chi_{\Gamma,G}(-n) = (-1)^{|\Gamma|}\,|\Sigma_{n,+}(\Gamma)/G|$$

Proof. By Theorem 5.3.1 and Theorem 5.2.3 and $\operatorname{sgn}(g) = (-1)^{|\Gamma|+c(g)}$,

$$\chi_{\Gamma,G}(-n) = (-1)^{|\Gamma|}\frac{1}{|G|}\sum_{g\in G}\operatorname{sgn}(g)\sum_{\sigma\in\Sigma^g}\Omega_{\Gamma_g^\sigma}(n).$$

As in the proof of Theorem 5.2.5, we see $\Omega_{\Gamma_g^\sigma}(n) = |\operatorname{Hom}(\Gamma^\sigma, [n])^g|$, and we observe

$$|\Sigma_n(\Gamma)^g| = \sum_{\sigma\in\Sigma^g}|\operatorname{Hom}(\Gamma^\sigma, [n])^g|. \tag{5.6}$$

Now we argue the same way as in the proof of Theorem 5.2.8: By equation (5.6) we obtain that $\chi_{\Gamma,G}(-n)$ equals

$$(-1)^{|\Gamma|}\frac{1}{|G|}\sum_{g\in G}\operatorname{sgn} g\,|\Sigma_n(\Gamma)^g| = (-1)^{|\Gamma|}\frac{1}{|G|}\sum_{(c,\sigma)\in\Sigma_n(\Gamma)}\;\sum_{g\in\operatorname{Stab}((c,\sigma))}\operatorname{sgn}(g). \tag{5.7}$$

For $(c, \sigma) \in \Sigma_n(\Gamma)$ and $g_0 \in \operatorname{Stab}((c, \sigma))$ such that $\operatorname{sgn} g_0 = -1$ as a permutation of the vertices, there is a bijection

$$\begin{array}{ccc} \{g \in \operatorname{Stab}((c, \sigma))\colon \operatorname{sgn}(g) = 1\} & \longrightarrow & \{g \in \operatorname{Stab}((c, \sigma))\colon \operatorname{sgn}(g) = -1\} \\ g & \mapsto & g_0 g. \end{array}$$

Hence, $\sum_{g\in\operatorname{Stab}((c,\sigma))}\operatorname{sgn}(g) = 0$ whenever (c, σ) is not even. Therefore the right-hand side of equation (5.7) equals

$$(-1)^{|\Gamma|}\frac{1}{|G|}\sum_{g\in G}|\Sigma_{n,+}(\Gamma)^g|$$

which by Theorem 5.2.1 equals $(-1)^{|\Gamma|}\,|\Sigma_{n,+}(\Gamma)/G|$. $\square$

An easy interpretation can be given in the case of $G = \mathbb{Z}_2$:

Corollary 5.3.4. *Let Γ be a graph and let $\mathbb{Z}_2 = \{e, \tau\}$ act on Γ by automorphisms such that* $\operatorname{sgn} \tau = -1$. *Then*

$$\chi_{\Gamma,\mathbb{Z}_2}(-1) = (-1)^{|\Gamma|} \frac{|\Sigma_+|}{2}$$

where $\Sigma_+ = \Sigma_{1,+}(\Gamma)$ is the set of even acyclic orientations of Γ.

For G acting trivially on Γ we again recover Stanley's reciprocity theorem of the chromatic polynomial (see Corollary 4.4.5):

Corollary 5.3.5 ([40, Theorem 1.2])**.** *Let Γ be a graph and χ_Γ its chromatic polynomial. Then $|\chi_\Gamma(-n)|$ equals the number of weakly compatible pairs (c, σ) consisting of a n-coloring c and an acyclic orientation σ of Γ. In particular, $|\chi_\Gamma(-1)|$ is the number of acyclic orientations of Γ.*

Similarly to Theorem 5.2.8 there is a twin reciprocity in the case of graph colorings. We say that an n-coloring c of Γ is even if for all $g \in \operatorname{Stab}(c)$ we have $\operatorname{sgn} g = 1$ and define $\operatorname{Col}_{n,+}(\Gamma)$ as the set of all even proper n-colorings of Γ. Then the action of G on $\operatorname{Col}_n(\Gamma)$ restricts to an action on $\operatorname{Col}_{n,+}(\Gamma)$. We further define $\chi^+_{\Gamma,G}(n) = |\operatorname{Col}_{n,+}(\Gamma)/G|$ as the function counting the number of orbits of even proper n-colorings for $n \geq 1$. By arguments similar to those in Theorem 5.3.1 and Theorem 5.3.3, we then have the following:

Proposition 5.3.6. *Let Γ be a graph and G a group acting on Γ by automorphisms. Then $\chi^+_{\Gamma,G}(n)$ agrees with a polynomial of degree $|\Gamma|$ for $n \geq 1$ and*

$$\chi^+_{\Gamma,G}(-n) = (-1)^{|\Gamma|} \left|\Sigma_n(\Gamma)/G\right| .$$

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Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit der Struktur *translationsinvarianter Bewertungen* auf Polytopen und damit im Zusammenhang stehenden *Abzählproblemen* mit geometrischen Lösungsansätzen.

Den Ausgangspunkt des ersten Kapitels bilden zwei Resultate von Richard Stanley: Zum einen sein bahnbrechendes *Nonnegativity Theorem* [42], welches aussagt, dass der Ehrhart h^*-*Vektor* jedes Gitterpolytops nur *nichtnegative* ganze Zahlen enthält. Zum anderen zeigt er in [48], dass dieser Vektor eine *Monotonieeigenschaft* besitzt. In Kapitel 2 untersuchen wir allgemein translationsinvariante Bewertungen auf diese Eigenschaften hin. Unser Hauptresultat ist, dass Nichtnegativität und Monotonie äquivalent sind, und wir geben eine einfache Charakterisierung an. In Kapitel 3 zeigen wir, dass der h^*-Vektor eines *Zonotops unimodal* ist, falls die entsprechende translationsinvariante Bewertung die Monotoniebedingung erfüllt.

Der zweite Teil der Arbeit behandelt Abzählprobleme für *ordnungserhaltenden Abbildungen.* Für gegebene *partielle Ordnungen* $\mathfrak{A} \subseteq \mathfrak{P}$ und eine ordnungserhaltende Abbildung $\lambda\colon \mathfrak{A} \to [n]$ zeigen wir in Kapitel 4, dass die Anzahl der Fortsetzungen von λ nach $\mathfrak{P}$ durch ein stückweise multivariates Polynom gegeben ist. Angewandt auf das Zählen von *Fortsetzungen von Graphenfärbungen* verallgemeinert dies einen Satz von Herzberg und Murty [21]. Zudem enumerieren wir *Monotone Triangles*, welche in engem Zusammenhang mit *Alternating Sign Matrices* stehen, und können einen kurzen geometrischen Beweis einer Reziprozität von Fischer und Riegler [17] angeben. In Kapitel 5 zählen wir ordnungserhaltende Abbildungen von $\mathfrak{P}$ nach $[n]$ bis auf *Symmetry.* Wir zeigen, dass die Zählfunktion ein Polynom in n ist und beweisen eine ordnungstheoretische Verallgemeinerung von *Pólya's Enumeration Theorem* [33]. Zudem zeigen wir eine Reziprozität und wenden unsere Resultate darauf an Graphenfärbungen bis auf Symmetrie zu zählen.

Kapitel 2 und 4 basieren auf der Zusammenarbeit mit Raman Sanyal. Kapitel 4 erschien in [23]. Kapitel 3 ist Teil eines gemeinsamen Projekts mit Matthias Beck and Emily McCullough. Kapitel 5 wurde in [22] veröffentlicht.

Eidesstattliche Erklärung

Gemäß §7 (4) der Promotionsordnung des Fachbereichs Mathematik und Informatik der Freien Universität Berlin versichere ich hiermit, dass ich alle Hilfsmittel und Hilfen angegeben und auf dieser Grundlage die Arbeit selbständig verfasst habe. Des Weiteren versichere ich, dass ich diese Arbeit nicht schon einmal zu einem früheren Promotionsverfahren eingereicht habe.

Berlin, den

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