CHAPTER 5

Integral Representation of Solutions to the General Inhomogeneous Polynomial Equation

Using a very important process for constructing the fundamental solutions from [**80, 91**], we obtain integral representations of solutions to the general inhomogeneous polynomial equation $\prod_{\nu=1}^{j}(D+\alpha_\nu)^{k_\nu} f = g$ in Ω where Ω is a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega =: \Gamma$. In [**80, 91**] a fundamental solution for the operator $\prod_{\nu=1}^{j}(D+\alpha_\nu)^{k_\nu}$ in Ω has been well constructed step by step. We start with the fundamental solution $K_\alpha(x)^{(1)}$ for the operator $D_\alpha := D+\alpha$ as in formulas (1.14) and (1.15) and via an integration process the procedure from [**91**] gives us the polynomial kernel funtions. We then obtain a similar results for the general Cauchy - Pompeiu representation of polynomial order. For such representations in complex and Clifford analysis, (see [**9, 10, 11, 14, 16**]), these representations are related to powers of Dirac and Laplace operator. And the Helmholtz operator and its factors were investigated ealier in Chapter 2 and Chapter 4.

1. A fundamental solution for a general polynomial operator

Using the fundamental solution of the Helmholtz equation a fundamental solution can be constructed for the product of Helmholtz operators. In a similar way as in [**80, 91**], starting with the fundamental solution $K_\alpha^{(1)}(x)$ for the D_α operator, a fundamental solution for the operator $\prod_{\nu=1}^{j}(D+\alpha_\nu)^{k_\nu}$ in Ω is constructed.

Lemma 5.1. *Let $\alpha_1, \alpha_2, \dots, \alpha_j$ be mutually different complex constants. Then*

(i) *the function $K_{\alpha_1,\alpha_2}^{(1,1)}(x) := \frac{1}{\alpha_2-\alpha_1}(K_{\alpha_1}^{(1)}(x) - K_{\alpha_2}^{(1)}(x))$ is a fundamental solution for the operator $D_{\alpha_1}D_{\alpha_2}$.*

(ii) *if $K_{\alpha_1,\alpha_2,\dots,\alpha_{j-1},\alpha_j}^{(k_1,k_2,\dots,k_{j-1},k_j-1)}(x)$ and $K_{\alpha_1,\alpha_2,\dots,\alpha_{j-1},\alpha_j}^{(k_1,k_2,\dots,k_{j-1}-1,k_j)}(x)$ are fundamental solutions for the operators $D_{\alpha_1}^{k_1}D_{\alpha_2}^{k_2}\dots D_{\alpha_{j-1}}^{k_{j-1}}D_{\alpha_j}^{k_j-1}$ and $D_{\alpha_1}^{k_1}D_{\alpha_2}^{k_2}\dots D_{\alpha_{j-1}}^{k_{j-1}-1}D_{\alpha_j}^{k_j}$ respectively, then the function*

$$K_{\alpha_1,\alpha_2,\dots,\alpha_{j-1},\alpha_j}^{(k_1,k_2,\dots,k_{j-1},k_j)}(x) = \frac{1}{\alpha_{j-1}-\alpha_j}\left(K_{\alpha_1,\alpha_2,\dots,\alpha j-1,\alpha_j}^{(k_1,k_2,\dots,k_{j-1}-1,k_j)}(x) - K_{\alpha_1,\alpha_2,\dots,\alpha_{j-1},\alpha_j}^{(k_1,k_2,\dots,k_{j-1},k_j-1}(x)\right)$$

is a fundamental solution for the operator $D_{\alpha_1}^{k_1}D_{\alpha_2}^{k_2}\dots D_{\alpha_{j-1}}^{k_{j-1}}D_{\alpha_j}^{k_j}$.

(iii)

$$D_{\alpha_\nu}^{h_\nu}K_{\alpha_\ell,\alpha_{\ell+1},\dots,\alpha_\nu,\dots,\alpha_{m-1},\alpha_m}^{(k_\ell,k_{\ell+1},\dots,k_\nu,\dots,k_{m-1},k_m)}(x) = K_{\alpha_\ell,\alpha_{\ell+1},\dots,\alpha_\nu,\dots,\alpha_{m-1},\alpha_m}^{(k_\ell,k_{\ell+1},\dots,(k_\nu-h_\nu),\dots,k_{m-1},k_m)}(x)$$

in the distributional sense, where $\ell \le \nu \le m, 0 \le h_\nu \le k_\nu, k_\imath \in \mathbb{N}, \imath = 1,\dots,m$.

Proof. For the way to prove (i) and (ii) we refer the readers to [**91**]. In order to prove (iii) we also need the following notions: Since $D_r K_\alpha^{(k)}(x) = DK_\alpha^{(k)}(x)$ for all $k \in \mathbb{N}$, it is easy to see that

$$D^{h_\nu}_{r,x,\alpha_\nu} K^{(k_\ell,k_{\ell+1},\ldots,k_\nu,\ldots,k_{m-1},k_m)}_{\alpha_\ell,\alpha_{\ell+1},\ldots,\alpha_\nu,\ldots,\alpha_{m-1},\alpha_m}(x) = D^{h_\nu}_{x,\alpha_\nu} K^{(k_\ell,k_{\ell+1},\ldots,k_\nu,\ldots,k_{m-1},k_m)}_{\alpha_\ell,\alpha_{\ell+1},\ldots,\alpha_\nu,\ldots,\alpha_{m-1},\alpha_m}(x)$$

and for all $\phi \in C^\infty_c(\Omega, \mathbb{H}(\mathbb{C}))$ with the right $\mathbb{H}(\mathbb{C})$ - distributions, we have

$$\begin{aligned} &< D^{h_\nu}_{r,x,\alpha_\nu} K^{(k_\ell,k_{\ell+1},\ldots,k_\nu,\ldots,k_{m-1},k_m)}_{\alpha_\ell,\ \alpha_{\ell+1},\ldots,\alpha_\nu,\ldots,\alpha_{m-1},\alpha_m}(x), \qquad \phi(x) > \\ = & \ - < D^{h_\nu-1}_{r,x,\alpha_\nu} K^{(k_\ell,k_{\ell+1},\ldots,k_\nu-1,\ldots,k_{m-1},k_m)}_{\alpha_\ell,\alpha_{\ell+1},\ldots,\alpha_\nu,\ldots,\alpha_{m-1},\alpha_m}(x), D_{-\alpha_\nu}\phi(x) > \end{aligned}$$

and using (ii) step by step then (*iii*) is seen. □

Remark 5.2. From (i) and (ii) in the above lemma by a straightforward computation we obtain

$$K^{(1,1,\ldots,1,1)}_{\alpha_1,\ \alpha_2,\ldots,\alpha_{k-1},\alpha_k}(x) = \sum_{i=1}^{k} \prod_{\substack{\nu=1\\ \nu\neq i}}^{k} \frac{1}{(\alpha_\nu - \alpha_i)} K^{(1)}_{\alpha_i}(x).$$

These results are used to prove Cauchy-Pompeiu type representation formulas in quaternionic analysis for the general polynomial operator $\prod_{\nu=1}^{j}(D+\alpha_\nu)^{k_\nu}$, where $\alpha_\nu \neq \alpha_\mu$ if $\nu \neq \mu$.

2. Representation for the general polynomial operator

In this section the representation formulas of solutions to the general inhomogeneous polynomial equation $\prod_{\nu=1}^{j}(D+\alpha_\nu)^{k_\nu} f = g$ in Ω is proved. This opens the door for investigating the boundary valued problem of classical Vekua type.

Theorem 5.3. *Let Ω be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega =: \Gamma$ and $f \in C^2(\Omega, \mathbb{H}(\mathbb{C})) \cap C^1(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$. Then*

$$\begin{aligned} f(x) = -\int_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)f(y)d\Gamma_y \ &- \int_\Gamma K^{(1,1)}_{\alpha_1,\alpha_2}(x-y)\vec{n}(y)D_{\alpha_1,y}f(y)d\Gamma_y \\ &+ \int_\Omega K^{(1,1)}_{\alpha_1,\alpha_2}(x-y)D_{\alpha_1,y}D_{\alpha_2,y}f(y)dy, \quad (5.1) \end{aligned}$$

where $K^{(1,1)}_{\alpha_1,\alpha_2}(x)$ is given in Lemma 5.1 and α_1, α_2 are different complex constants.

Proof. Notice that

$$D_{\alpha_2,x}D_{\alpha_1,x} = D_{\alpha_1,x}D_{\alpha_2,x} \text{ and } D_{r,-\alpha,y}K^{(1)}_\alpha(x-y) = -D_{r,\alpha,x}K^{(1)}_\alpha(x-y). \quad (5.2)$$

Applying the quaternionic Cauchy- Pompeiu formula (1.22) for $D_{\alpha_1,y}f(y)$ we obtain

$$D_{\alpha_1,y}f(y) = -\int_\Gamma K^{(1)}_{\alpha_2}(y-\tilde{y})\vec{n}(\tilde{y})D_{\alpha_1,\tilde{y}}f(\tilde{y})d\Gamma_{\tilde{y}} + \int_\Omega K^{(1)}_{\alpha_2}(y-\tilde{y})D_{\alpha_2,\tilde{y}}D_{\alpha_1,\tilde{y}}f(\tilde{y})d\tilde{y}.$$

This leads to

$$\begin{aligned} f(x) = -\int_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)f(y)d\Gamma_y \ &- \int_\Gamma \psi^{(1),\ (1)}_{(\alpha_1),(\alpha_2)}(x,\tilde{y})\vec{n}(\tilde{y})D_{\alpha_1,\tilde{y}}f(\tilde{y})d\Gamma_y \\ &+ \int_\Omega \psi^{(1),\ (1)}_{(\alpha_1),(\alpha_2)}(x,\tilde{y})D_{\alpha_1,\tilde{y}}D_{\alpha_2,\tilde{y}}f(\tilde{y})d\tilde{y}, \quad (5.3) \end{aligned}$$

where $\psi^{(1),\ (1)}_{(\alpha_1),(\alpha_2)}(x,\tilde{y}) = \int\limits_{\Omega} K^{(1)}_{\alpha_1}(x-y)K^{(1)}_{\alpha_2}(y-\tilde{y})dy$. On the other hand, we have in the distributional sense that

$$D_{\alpha_1}K^{(1)}_{\alpha_1}(x) = \delta(x), \quad D_{\alpha_2}K^{(1)}_{\alpha_2}(x) = \delta(x), \tag{5.4}$$

so that for any $\phi \in C^\infty_c(\Omega, \mathbb{H}(\mathbb{C}))$

$$\begin{aligned} &\langle (D+\alpha_1)K^{(1,1)}_{\alpha_1,\alpha_2}(x), \phi(x)\rangle = \frac{1}{\alpha_2-\alpha_1}\langle (D+\alpha_1)[K^{(1)}_{\alpha_1}(x) - K^{(1)}_{\alpha_2}(x)], \phi(x)\rangle \\ = &\frac{1}{\alpha_2-\alpha_1}\{\langle (D+\alpha_1)K^{(1)}_{\alpha_1}(x), \phi(x)\rangle - \langle (D+\alpha_2-\alpha_2+\alpha_1)K^{(1)}_{\alpha_2}(x), \phi(x)\rangle\} \\ = &\langle K^{(1)}_{\alpha_2}(x), \phi(x)\rangle. \end{aligned}$$

Therefore, $D_{\alpha_1}K^{(1,1)}_{\alpha_1,\alpha_2}(x) = K^{(1)}_{\alpha_2}(x)$ in the sense of distributions.
For $x, \tilde{y} \in \Omega$ with $x \neq \tilde{y}$ the quaternionic Cauchy-Pompeiu formula yields

$$K^{(1,1)}_{\alpha_1,\alpha_2}(x-\tilde{y}) = \psi^{(1),\ (1)}_{(\alpha_1),(\alpha_2)}(x,\tilde{y}) - \tilde{\psi}^{(1),\ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}),$$

where

$$\tilde{\psi}^{(1),\ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}) = \int_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(1,1)}_{\alpha_1,\alpha_2}(y-\tilde{y})d\Gamma_y.$$

Inserting this equality into equality (5.3), we obtain

$$\begin{aligned} f(x) = &-\int\limits_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)f(y)d\Gamma_y - \int\limits_\Gamma K^{(1,1)}_{\alpha_1,\alpha_2}(x-y)\vec{n}(y)D_{\alpha_1,y}f(y)d\Gamma_y \\ + &\int_\Omega K^{(1,1)}_{\alpha_1,\alpha_2}(x-y)D_{\alpha_1,y}D_{\alpha_2,y}f(y)dy - \int\limits_\Gamma \tilde{\psi}^{(1),\ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y})\vec{n}(\tilde{y})D_{\alpha_1,\tilde{y}}f(\tilde{y})d\Gamma_{\tilde{y}} \\ + &\int\limits_\Omega \tilde{\psi}^{(1),\ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y})D_{\alpha_1,\tilde{y}}D_{\alpha_2,\tilde{y}}f(\tilde{y})d\tilde{y}. \end{aligned}$$

Applying Stokes' formula (1.20) again gives

$$\begin{aligned} &\int\limits_\Gamma \tilde{\psi}^{(1),\ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y})\vec{n}(\tilde{y})D_{\alpha_1,\tilde{y}}f(\tilde{y})d\Gamma_{\tilde{y}} - \int\limits_\Omega \tilde{\psi}^{(1),\ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y})D_{\alpha_2,\tilde{y}}D_{\alpha_1,\tilde{y}}f(\tilde{y})d\tilde{y} \\ = &\int\limits_\Omega D_{r,-\alpha_2,\tilde{y}}\tilde{\psi}^{(1),\ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y})D_{\alpha_1,\tilde{y}}f(\tilde{y})d\tilde{y}. \end{aligned}$$

Using the definition of $K^{(1,1)}_{\alpha_1,\alpha_2}(x)$ and the equalities (5.2) and (5.4) with note that $\Gamma \ni y \neq x \in \Omega,\ \Gamma \ni y \neq \tilde{y} \in \Omega$ we obtain

$$D_{r,-\alpha_2,\tilde{y}}\tilde{\psi}^{(1),\ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}) = -\int\limits_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(1)}_{\alpha_1}(y-\tilde{y})d\Gamma_y.$$

Since $\int\limits_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(1)}_{\alpha_1}(y-\tilde{y})d\Gamma_y = 0$, see the proof of Theorem 2.3 in Chapter 2, we have $D_{r,-\alpha_2,\tilde{y}}\tilde{\psi}^{(1),\ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}) = 0$. This leads to (5.1). □

REMARK 5.4. If $\alpha_2 = -\alpha_1$ we have the representation formula in terms of the Helmholtz operator as in Chapter 4, Theorem 4.1.

In oder to obtain the generalization of the above theorem, we need the following lemma.

LEMMA 5.5. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be mutually different complex constants. Then*

$$\sum_{i=1}^{n} \prod_{\substack{\nu=1\\ \nu\neq i}}^{n} \frac{1}{(\alpha_\nu - \alpha_i)}) = 0, \text{ for all } 2 \leq n \in \mathbb{N}.$$

PROOF. For $n = 2$, we have $\sum_{i=1}^{2} \prod_{\substack{\nu=1\\ \nu\neq i}}^{2} \frac{1}{(\alpha_\nu - \alpha_i)} = \frac{1}{\alpha_2-\alpha_1} + \frac{1}{\alpha_1-\alpha_2} = 0$. By direct caculation we also get $\sum_{i=1}^{3} \prod_{\substack{\nu=1\\ \nu\neq i}}^{3} \frac{1}{(\alpha_\nu - \alpha_i)} = 0$. In the case $n > 3$, we suppose that this lemma holds for some n. We now consider the function $f(x) = \sum_{i=1}^{n+1} (x - \alpha_i) \prod_{\substack{\nu=1\\ \nu\neq i}}^{n+1} \frac{1}{(\alpha_\nu - \alpha_i)}$. Note that

$$f(\alpha_{n+1}) = \sum_{i=1}^{n+1} (\alpha_{n+1} - \alpha_i) \prod_{\substack{\nu=1\\ \nu\neq i}}^{n+1} \frac{1}{(\alpha_\nu - \alpha_i)} = \sum_{i=1}^{n} \prod_{\substack{\nu=1\\ \nu\neq i}}^{n} \frac{1}{(\alpha_\nu - \alpha_i)} = 0$$

by inductive hypothesis. Similarly, $f(\alpha_j) = 0$ for all $1 \leq \ j \leq n$. Therefore, $f(x)$ has $(n+1)$ zeroes α_j, $1 \leq \ j \leq n+1$. However, it is a polynomical of degree one. Thus, $f(x) \equiv 0$. Then, we have $f'(x) \equiv 0$. In other words, $f'(x) = \sum_{i=1}^{n+1} \prod_{\substack{\nu=1\\ \nu\neq i}}^{n+1} \frac{1}{(\alpha_\nu - \alpha_i)} = 0$, i.e., the lemma holds. □

THEOREM 5.6. *Let $f \in C^n(\Omega, \mathbb{H}(\mathbb{C})) \cap C^{n+1}(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$. Then*

$$\begin{aligned} f(x) = & - \int_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y) f(y) d\Gamma_y \\ & - \sum_{j=2}^{n} \int_\Gamma K^{(1,1,\ldots,1,1)}_{\alpha_1,\alpha_2,\ldots,\alpha_{j-1},\alpha_j}(x-y)\vec{n}(y) \prod_{k=1}^{j-1} D_{\alpha_k,y} f(y) d\Gamma_y \\ & + \int_\Omega K^{(1,1,\ldots,1,1)}_{\alpha_1,\alpha_2,\ldots,\alpha_{n-1},\alpha_n}(x-y) \prod_{k=1}^{n} D_{\alpha_k,y} f(y) dy, \end{aligned} \tag{5.5}$$

where $K^{(1,1,\ldots,1,1)}_{\alpha_1,\alpha_2,\ldots,\alpha_{k-1},\alpha_k}(x)$ is given in Remark 5.2 and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are mutually different complex constants.

PROOF. For $n = 1$ formula (5.5) coincides with the Cauchy- Pompeiu representation. For the case $n = 2$, we have already shown (5.5) in Theorem 5.3. In order to prove this formula for any $n > 2$ assume it holds for $n - 1$. By inductive hypothesis we have

$$\begin{aligned} f(x) = & - \int_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y) f(y) d\Gamma_y \\ & - \sum_{j=2}^{n-1} \int_\Gamma K^{(1,1,\ldots,1,1)}_{\alpha_1,\alpha_2,\ldots,\alpha_{j-1},\alpha_j}(x-y)\vec{n}(y) \prod_{k=1}^{j-1} D_{\alpha_k,y} f(y) d\Gamma_y \end{aligned}$$

$$+ \int_{\Omega} K^{(1,1,\ldots,1,1)}_{\alpha_1,\alpha_2,\ldots,\alpha_{n-1}}(x-y)\prod_{k=1}^{n-1} D_{\alpha_k,y} f(y)dy,$$

Applying the Cauchy-Pompeiu formula (1.22) to $\prod_{k=1}^{n-1} D_{\alpha_k,y} f(y)$ gives

$$\prod_{k=1}^{n-1} D_{\alpha_k,y} f(y) = - \int_{\Gamma} K^{(1)}_{\alpha_n}(y-\tilde{y})\vec{n}(\tilde{y})\prod_{k=1}^{n-1} D_{\alpha_k,\tilde{y}} f(\tilde{y})d\Gamma_{\tilde{y}} + \int_{\Omega} K^{(1)}_{\alpha_n}(y-\tilde{y})\prod_{k=1}^{n} D_{\alpha_k,\tilde{y}} f(\tilde{y})d\tilde{y}.$$

It follows

$$\begin{aligned} f(x) = & - \int_{\Gamma} K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)f(y)d\Gamma_y \\ & - \sum_{j=2}^{n-1}\int_{\Gamma} K^{(1,1,\ldots,1,1)}_{\alpha_1,\alpha_2,\ldots,\alpha_{j-1},\alpha_j}(x-y)\vec{n}(y)\prod_{k=1}^{j-1} D_{\alpha_k,y} f(y)d\Gamma_y \\ & - \int_{\Gamma} \psi^{(1,1,\ldots,1,1),\ (1)}_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1}),(\alpha_n)}(x,\tilde{y})\vec{n}(\tilde{y})\prod_{k=1}^{n-1} D_{\alpha_k,\tilde{y}} f(\tilde{y})d\Gamma_{\tilde{y}} \\ & + \int_{\Omega} \psi^{(1,1,\ldots,1,1),\ (1)}_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1}),\ (\alpha_n)}(x,\tilde{y})\prod_{k=1}^{n} D_{\alpha_k,\tilde{y}} f(\tilde{y})d\tilde{y}, \end{aligned}$$

where $\psi^{(1,1,\ldots,1,1),\ (1)}_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1}),\ (\alpha_n)}(x,\tilde{y}) = \int_{\Omega} K^{(1,1,\ldots,1)}_{\alpha_1,\alpha_2,\ldots,\alpha_{n-1}}(x-y)K^{(1)}_{\alpha_n}(y-\tilde{y})dy$. By inductive hypothesis, applying it for $K^{(1,1,\ldots,1,1)}_{\alpha_1,\alpha_2,\ldots,\alpha_{n-1},\alpha_n}(x-\tilde{y})$ as well as using the assertion (iii) of Theorem 5.1 step by step, we obtain

$$\psi^{(1,1,\ldots,1,1),\ (1)}_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1}),(\alpha_n)}(x,\tilde{y}) = K^{(1,1,\ldots,1,1)}_{\alpha_1,\alpha_2,\ldots,\alpha_{n-1},\alpha_n}(x-\tilde{y}) + \sum_{k=1}^{n-1}\tilde{\psi}^{(1,1,\ldots,1),\ (1,1,\ldots,1)}_{(\alpha_1,\alpha_2,\ldots,\alpha_k),(\alpha_k,\alpha_{k+1},\ldots,\alpha_n)}(x,\tilde{y}),$$

where $\tilde{\psi}^{(1,1,\ldots,1),\ (1,1,\ldots,1)}_{(\alpha_1,\alpha_2,\ldots,\alpha_k),(\alpha_k,\alpha_{k+1},\ldots,\alpha_n)}(x,\tilde{y}) = \int_{\Gamma} K^{(1,1,\ldots,1)}_{\alpha_1,\alpha_2,\ldots,\alpha_k}(x-y)\vec{n}(y)K^{(1,1,\ldots,1)}_{\alpha_k,\alpha_{k+1},\ldots,\alpha_n}(y-\tilde{y})d\Gamma_y$.
Hence,

$$\begin{aligned} f(x) = & - \int_{\Gamma} K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)f(y)d\Gamma_y \\ & - \sum_{j=2}^{n-1}\int_{\Gamma} K^{(1,1,\ldots,1,1)}_{\alpha_1,\alpha_2,\ldots,\alpha_{j-1},\alpha_j}(x-y)\vec{n}(y)\prod_{k=1}^{j-1} D_{\alpha_k,y} f(y)d\Gamma_y \\ & - \int_{\Gamma} K^{(1,1,\ldots,1,1)}_{\alpha_1,\alpha_2,\ldots,\alpha_{n-1},\alpha_n}(x-y)\vec{n}(y)\prod_{k=1}^{n-1} D_{\alpha_k,y} f(y)d\Gamma_y \\ & + \int_{\Omega} K^{(1,1,\ldots,1,1)}_{\alpha_1,\alpha_2,\ldots,\alpha_{n-1},\alpha_n}(x-y)\prod_{k=1}^{n} D_{\alpha_k,y} f(y)dy \end{aligned}$$

$$
\begin{aligned}
& - \int_{\Gamma} \left[\sum_{k=1}^{n-1} \tilde{\psi}^{(1,1,\ldots,1),\quad (1,1,\ldots,1)}_{(\alpha_1,\alpha_2,\ldots,\alpha_k),(\alpha_k,\alpha_{k+1},\ldots,\alpha_n)}(x,\tilde{y}) \right] \vec{n}(\tilde{y}) \prod_{j=1}^{n-1} D_{\alpha_j,\tilde{y}} f(\tilde{y}) d\Gamma_{\tilde{y}} \\
& + \int_{\Omega} \left[\sum_{k=1}^{n-1} \tilde{\psi}^{(1,1,\ldots,1),\quad (1,1,\ldots,1)}_{(\alpha_1,\alpha_2,\ldots,\alpha_k),(\alpha_k,\alpha_{k+1},\ldots,\alpha_n)}(x,\tilde{y}) \right] \prod_{j=1}^{n} D_{\alpha_j,\tilde{y}} f(\tilde{y}) d\tilde{y}. \qquad (5.6)
\end{aligned}
$$

Applying the quaternionic Stokes' formula (1.20) yields

$$
\begin{aligned}
& \int_{\Gamma} \left[\sum_{k=1}^{n-1} \tilde{\psi}^{(1,1,\ldots,1),\quad (1,1,\ldots,1)}_{(\alpha_1,\alpha_2,\ldots,\alpha_k),(\alpha_k,\alpha_{k+1},\ldots,\alpha_n)}(x,\tilde{y}) \right] \vec{n}(\tilde{y}) \prod_{j=1}^{n-1} D_{\alpha_j,\tilde{y}} f(\tilde{y}) d\Gamma_{\tilde{y}} \\
- & \int_{\Omega} \left[\sum_{k=1}^{n-1} \tilde{\psi}^{(1,1,\ldots,1),\quad (1,1,\ldots,1)}_{(\alpha_1,\alpha_2,\ldots,\alpha_k),(\alpha_k,\alpha_{k+1},\ldots,\alpha_n)}(x,\tilde{y}) \right] \prod_{j=1}^{n} D_{\alpha_j,\tilde{y}} f(\tilde{y}) d\tilde{y} \\
= & \int_{\Omega} \left[\sum_{k=1}^{n-1} D_{-\alpha_n,\tilde{y}} \tilde{\psi}^{(1,1,\ldots,1),\quad (1,1,\ldots,1)}_{(\alpha_1,\alpha_2,\ldots,\alpha_k),(\alpha_k,\alpha_{k+1},\ldots,\alpha_n)}(x,\tilde{y}) \right] \prod_{j=1}^{n-1} D_{\alpha_j,\tilde{y}} f(\tilde{y}) d\tilde{y}.
\end{aligned}
$$

Using the assertion (iii) of Lemma 5.1 and Remark 5.2 we get

$$
\begin{aligned}
& \sum_{k=1}^{n-1} D_{-\alpha_n,\tilde{y}} \tilde{\psi}^{(1,1,\ldots,1),\quad (1,1,\ldots,1)}_{(\alpha_1,\alpha_2,\ldots,\alpha_k),(\alpha_k,\alpha_{k+1},\ldots,\alpha_n)}(x,\tilde{y}) \\
= & -\sum_{k=1}^{n-1} \int_{\Gamma} K^{(1,1,\ldots,1)}_{\alpha_1,\alpha_2,\ldots,\alpha_k}(x-y)\vec{n}(y) K^{(1,1,\ldots,1)}_{\alpha_k,\alpha_{k+1},\ldots,\alpha_{n-1}}(y-\tilde{y}) d\Gamma_y \\
= & -\sum_{k=1}^{n-1} \left[\Big(\sum_{i=1}^{k} \prod_{\substack{\nu=1 \\ \nu\neq i}}^{k} \frac{1}{(\alpha_\nu-\alpha_i)}\Big)\Big(\sum_{j=k}^{n-1} \prod_{\substack{\mu=k \\ \mu\neq j}}^{n-1} \frac{1}{(\alpha_\mu-\alpha_j)}\Big) \int_{\Gamma} K^{(1)}_{\alpha_i}(x-y)\vec{n}(y) K^{(1)}_{\alpha_j}(y-\tilde{y}) d\Gamma_y \right] \\
= & \ 0
\end{aligned}
$$

by Lemma 5.5. Inserting this into equality (5.6) we obtain equality (5.5), i.e Theorem 5.6 is proved. □

In order to obtain the generalized representation of solution for $\prod_{\nu=1}^{j}(D+\alpha_\nu)^{k_\nu} f = g$, we now as an example construct the representation of solutions for the inhomogeneous equation such that $D^3_{\alpha_1} D_{\alpha_2} f(x) = g(x)$ in a bounded domain Ω, where $\alpha_1 \neq \alpha_2$. The integral representation formulas for higher order D_α equations in Chapter 2, Theorem 5.6 and Lemma 5.1 are used.

THEOREM 5.7. *Let $f \in C^4(\Omega, \mathbb{H}(\mathbb{C})) \cap C^3(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$, and $\alpha_1 \neq \alpha_2,\ \alpha_1, \alpha_2 \in \mathbb{C}$. Then*

$$
\begin{aligned}
f(x) = & - \sum_{k=1}^{3} \int_{\Gamma} K^{(k)}_{\alpha_1}(x-y)\vec{n}(y) D^{k-1}_{\alpha_1,y} f(y) d\Gamma_y - \int_{\Gamma} K^{(3,1)}_{\alpha_1,\alpha_2}(x-y)\vec{n}(y) D^3_{\alpha_1,y} f(y) d\Gamma_y \\
& + \int_{\Omega} K^{(3,1)}_{\alpha_1,\alpha_2}(x-y) D^3_{\alpha_1,y} D_{\alpha_2,y} f(y) dy.
\end{aligned}
$$

Proof. Applying the quaternionic Cauchy - Pompeiu formula for $D^3_{\alpha_1,y}f(y)$ we get

$$D^3_{y,\alpha_1}f(y) = -\int_\Gamma K^{(1)}_{\alpha_2}(y-\tilde{y})\vec{n}(\tilde{y})D^3_{\alpha_1,\tilde{y}}f(\tilde{y})d\Gamma_{\tilde{y}} + \int_\Omega K^{(1)}_{\alpha_2}(y-\tilde{y})D_{\alpha_2,\tilde{y}}D^3_{\alpha_1,\tilde{y}}f(\tilde{y})d\tilde{y}.$$

It follows

$$f(x) = -\sum_{k=1}^{3}\int_\Gamma K^{(k)}_{\alpha_1}(x-y)\vec{n}(y)D^{k-1}_{\alpha_1,y}f(y)d\Gamma_y$$

$$- \int_\Gamma \psi^{(3),(0,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y})\vec{n}(\tilde{y})D^3_{\alpha_1,\tilde{y}}f(\tilde{y})d\Gamma_{\tilde{y}} + \int_\Omega \psi^{(3),(0,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y})D_{\alpha_2,\tilde{y}}D^3_{\alpha_1,\tilde{y}}f(\tilde{y})d\tilde{y}, \quad (5.7)$$

where $\psi^{(3),(0,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}) = \int\limits_\Omega K^{(3)}_{\alpha_1}(x-y)K^{(1)}_{\alpha_2}(y-\tilde{y})dy$, and it is easy to see

$$\begin{aligned}
K^{(1,1)}_{\alpha_1,\alpha_2}(x) &= \frac{1}{\alpha_2-\alpha_1}\left(K^{(1)}_{\alpha_1}(x) - K^{(1)}_{\alpha_2}(x)\right),\\
K^{(2,1)}_{\alpha_1,\alpha_2}(x) &= \frac{1}{\alpha_2-\alpha_1}\left(K^{(2)}_{\alpha_1}(x) - K^{(1,1)}_{\alpha_1,\alpha_2}(x)\right),\\
K^{(3,1)}_{\alpha_1,\alpha_2}(x) &= \frac{1}{\alpha_2-\alpha_1}\left(K^{(3)}_{\alpha_1}(x) - K^{(2,1)}_{\alpha_1,\alpha_2}(x)\right),\\
D^\nu_{\alpha_1}K^{(3,1)}_{\alpha_1,\alpha_2}(x) &= K^{(3-\nu,1)}_{\alpha_1,\alpha_2}(x),
\end{aligned}$$

in the sense of distribution for $\nu = 1,2,3$, because of Lemma 2.1 and Lemma 5.1.
Applying the representaion for higher order powers of D_α in Theorem 2.4 for $K^{(3,1)}_{\alpha_1,\alpha_2}(x-\tilde{y})$ gives

$$\psi^{(3),(0,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}) = K^{(3,1)}_{\alpha_1,\alpha_2}(x-\tilde{y}) + \sum_{\nu=1}^{3}\tilde{\psi}^{(\nu),(4-\nu,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}),$$

where $\tilde{\psi}^{(\nu),(4-\nu,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}) = \int\limits_\Gamma K^{(\nu)}_{\alpha_1}(x-y)\vec{n}(y)K^{(4-\nu,1)}_{\alpha_1,\alpha_2}(y-\tilde{y})d\Gamma_y$.
Substituting it into (5.7), then applying the Stokes' formula again with $x \neq y$, shows that

$$\begin{aligned}
f(x) = & - \sum_{k=1}^{3}\int_\Gamma K^{(k)}_{\alpha_1}(x-y)\vec{n}(y)D^{k-1}_{\alpha_1,y}f(y)d\Gamma_y - \int_\Gamma K^{(3,1)}_{\alpha_1,\alpha_2}(x-y)\vec{n}(y)D^3_{\alpha_1,y}f(y)d\Gamma_y\\
& + \int_\Omega K^{(3,1)}_{\alpha_1,\alpha_2}(x-y)D^3_{\alpha_1,y}D_{\alpha_2,y}f(y)dy\\
& - \int_\Omega \sum_{\nu=1}^{3} D_{-\alpha_2,\tilde{y}}\tilde{\psi}^{(\nu),(4-\nu,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y})D^3_{r,\alpha_1,\tilde{y}}D_{\alpha_2,\tilde{y}}f(\tilde{y})d\tilde{y}.
\end{aligned}$$

Namely, for arbitrary fixed x and $\tilde{y}$, the functions $K^{(k)}_{\alpha_1}(x-y),\ K^{(k)}_{\alpha_1}(y-\tilde{y}), k = 1,2$, are C^1 - functions in the whole domain Ω except for the two points x and $\tilde{y}$.
Therefore, for $\Omega_{x,\varepsilon} = \Omega - \{y \in \Omega, |y-x| \leq \varepsilon\}$, $\Omega_{\tilde{y},\varepsilon} = \Omega - \{y \in \Omega, |y-\tilde{y}| \leq \varepsilon\}$, $\Omega_\varepsilon = \Omega - \{y \in \Omega, |y-x| \leq \varepsilon \textit{ and } |y-\tilde{y}| \leq \varepsilon\}$ with $0 < \varepsilon$ small enough,

$$\int_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(3)}_{\alpha_1}(y-\tilde{y})d\Gamma_y = \int_{\partial\Omega_{x,\varepsilon}} K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(3)}_{\alpha_1}(y-\tilde{y})d\Gamma_y$$

$$+ \int\limits_{|y-x|=\varepsilon} K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(3)}_{\alpha_1}(y-\tilde{y})d\Gamma_y$$

$$\int\limits_{\Gamma} K^{(3)}_{\alpha_1}(x-y)\vec{n}(y)K^{(1)}_{\alpha_1}(y-\tilde{y})d\Gamma_y = \int\limits_{\partial\Omega_{\tilde{y},\varepsilon}} K^{(3)}_{\alpha_1}(x-y)\vec{n}(y)K^{(1)}_{\alpha_1}(y-\tilde{y})d\Gamma_y + \int\limits_{|y-\tilde{y}|=\varepsilon} K^{(3)}_{\alpha_1}(x-y)\vec{n}(y)K^{(1)}_{\alpha_1}(y-\tilde{y})d\Gamma_y,$$

$$\int\limits_{\Gamma} K^{(2)}_{\alpha_1}(x-y)\vec{n}(y)K^{(2)}_{\alpha_1}(y-\tilde{y})d\Gamma_y = \int\limits_{\partial\Omega_{\varepsilon}} K^{(2)}_{\alpha_1}(x-y)\vec{n}(y)K^{(2)}_{\alpha_1}(y-\tilde{y})d\Gamma_y + \int\limits_{|y-x|=\varepsilon} K^{(2)}_{\alpha_1}(x-y)\vec{n}(y)K^{(2)}_{\alpha_1}(y-\tilde{y})d\Gamma_y + \int\limits_{|y-\tilde{y}|=\varepsilon} K^{(2)}_{\alpha_1}(x-y)\vec{n}(y)K^{(2)}_{\alpha_1}(y-\tilde{y})d\Gamma_y.$$

Aplying Stokes' formula for $\Omega_{x,\varepsilon}$ and $\Omega_{\tilde{y},\varepsilon}$ respectively gives

$$\lim_{\varepsilon\to 0}\int\limits_{\partial\Omega_{x,\varepsilon}} K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(3)}_{\alpha_1}(y-\tilde{y})d\Gamma_y = -K^{(3)}_{\alpha_1}(x-\tilde{y}) + \lim_{\varepsilon\to 0}\int\limits_{\Omega_{x,\varepsilon}} K^{(1)}_{\alpha_1}(x-y)K^{(2)}_{\alpha_1}(y-\tilde{y})dy,$$

$$\lim_{\varepsilon\to 0}\int\limits_{\partial\Omega_{\tilde{y},\varepsilon}} K^{(3)}_{\alpha_1}(x-y)\vec{n}(y)K^{(1)}_{\alpha_1}(y-\tilde{y})d\Gamma_y = K^{(3)}_{\alpha_1}(x-\tilde{y}) - \lim_{\varepsilon\to 0}\int\limits_{\Omega_{\tilde{y},\varepsilon}} K^{(2)}_{\alpha_1}(x-y)K^{(1)}_{\alpha_1}(y-\tilde{y})dy.$$

Now, applying Stokes' formula for Ω_ε and observing

$$D_{r,-\alpha_1,y}K^{(2)}_{\alpha_1}(x-y) = -D_{\alpha_1,x}K^{(2)}_{\alpha_1}(x-y) = K^{(1)}_{\alpha_1}(x-y)$$

shows

$$\int\limits_{\partial\Omega_{\varepsilon}} K^{(2)}_{\alpha_1}(x-y)\vec{n}(y)K^{(2)}_{\alpha_1}(y-\tilde{y})d\Gamma_y = -\int\limits_{\Omega_{\varepsilon}} K^{(1)}_{\alpha_1}(x-y)K^{(2)}_{\alpha_1}(y-\tilde{y})dy + \int\limits_{\Omega_{\varepsilon}} K^{(2)}_{\alpha_1}(x-y)K^{(1)}_{\alpha_1}(y-\tilde{y})dy.$$

On the orther hand

$$\int\limits_{\Omega_{x,\varepsilon}} K^{(1)}_{\alpha_1}(x-y)K^{(2)}_{\alpha_1}(y-\tilde{y})dy = \int\limits_{\Omega_{\varepsilon}} K^{(1)}_{\alpha_1}(x-y)K^{(2)}_{\alpha_1}(y-\tilde{y})dy + \int\limits_{|y-\tilde{y}|<\varepsilon} K^{(1)}_{\alpha_1}(x-y)K^{(2)}_{\alpha_1}(y-\tilde{y})dy,$$

$$\int\limits_{\Omega_{\tilde{y},\varepsilon}} K^{(2)}_{\alpha_1}(x-y)K^{(1)}_{\alpha_1}(y-\tilde{y})dy = \int\limits_{\Omega_{\varepsilon}} K^{(2)}_{\alpha_1}(x-y)K^{(1)}_{\alpha_1}(y-\tilde{y})dy + \int\limits_{|y-x|<\varepsilon} K^{(2)}_{\alpha_1}(x-y)K^{(1)}_{\alpha_1}(y-\tilde{y})dy.$$

Using the Cauchy-Pompeiu representation formulas (1.22) and (1.23) for D^n_α, *with* $n = 1, 2$, for $K^{(3)}_{\alpha_1}(x-\tilde{y})$ as well as

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int\limits_{|y-\tilde{y}|=\varepsilon} K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(3)}_{\alpha_1}(y-\tilde{y})d\Gamma_y &= 0,\\
\lim_{\varepsilon\to 0}\int\limits_{|y-x|=\varepsilon} K^{(2)}_{\alpha_1}(x-y)\vec{n}(y)K^{(2)}_{\alpha_1}(y-\tilde{y})d\Gamma_y &= 0,\\
\lim_{\varepsilon\to 0}\int\limits_{|y-\tilde{y}|=\varepsilon} K^{(2)}_{\alpha_1}(x-y)\vec{n}(y)K^{(2)}_{\alpha_1}(y-\tilde{y})d\Gamma_y &= 0,\\
\lim_{\varepsilon\to 0}\int\limits_{|y-x|=\varepsilon} K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(3)}_{\alpha_1}(y-\tilde{y})d\Gamma_y &= -K^{(3)}_{\alpha_1}(x-\tilde{y}),\\
\lim_{\varepsilon\to 0}\int\limits_{|y-\tilde{y}|=\varepsilon} K^{(3)}_{\alpha_1}(x-y)\vec{n}(y)K^{(1)}_{\alpha_1}(y-\tilde{y})d\Gamma_y &= K^{(3)}_{\alpha_1}(x-\tilde{y}),
\end{aligned}$$

we obtain

$$\lim_{\varepsilon\to 0}\int\limits_{|y-\tilde{y}|<\varepsilon} K^{(1)}_{\alpha_1}(x-y)K^{(2)}_{\alpha_1}(y-\tilde{y})dy = 0, \quad \lim_{\varepsilon\to 0}\int\limits_{|y-x|<\varepsilon} K^{(2)}_{\alpha_1}(x-y)K^{(1)}_{\alpha_1}(y-\tilde{y})dy = 0.$$

This leads to

$$\sum_{\nu=1}^{3} D_{\tilde{y},-\alpha_2}\tilde{\psi}^{(\nu),\ (4-\nu,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}) = -\sum_{\nu=1}^{3}\int\limits_{\Gamma} K^{(\nu)}_{\alpha_1}(x-y)\vec{n}(y)K^{(4-\nu)}_{\alpha_1}(y-\tilde{y})d\Gamma_y = 0.$$

Hence Theorem 5.7 is proved. □

In a similar way as in the proof of Theorem 5.7 and using the representation formulas for higher order D_α operators in [**49**, Theorem 3.2] as well as Theorem 5.6, by induction we can also prove the next result.

THEOREM 5.8. *Let* $f \in C^n(\Omega,\mathbb{H}(\mathbb{C})) \cap C^{n-1}(\bar{\Omega},\mathbb{H}(\mathbb{C}))$ *and* $\sum\limits_{\nu=1}^{j} k_\nu = n$. *Then*

$$\begin{aligned}
f(x) = &- \sum_{\nu_1=1}^{k_1}\int\limits_{\Gamma} K^{(\nu_1)}_{\alpha_1}(x-y)\vec{n}(y)D^{\nu_1-1}_{\alpha_1,y}f(y)d\Gamma_y\\
&- \sum_{\nu_2=1}^{k_2}\int\limits_{\Gamma} K^{(k_1,\ \nu_2)}_{\alpha_1,\ \alpha_2}(x-y)\vec{n}(y)D^{\nu_2-1}_{\alpha_2,y}D^{k_1}_{\alpha_1,y}f(y)d\Gamma_y\\
&- \cdots - \sum_{\nu_j=1}^{k_j}\int\limits_{\Gamma} K^{(k_1,k_2,\ \dots,\ k_{j-1},\nu_j)}_{\alpha_1,\alpha_2,\dots,\alpha_{j-1},\ \alpha_j}(x-y)\vec{n}(y)D^{\nu_j-1}_{\alpha_{k_j},y}\prod_{\mu=1}^{j-1}D^{k_\mu}_{\alpha_\mu,y}f(y)d\Gamma_y\\
&+ \int\limits_{\Omega} K^{(k_1,k_2,\ \dots,\ k_{j-1},k_j)}_{\alpha_1,\alpha_2,\dots,\alpha_{j-1},\ \alpha_j}(x-y)\prod_{\nu=1}^{j}D^{k_\nu}_{\alpha_\nu,y}f(y)dy,
\end{aligned}$$

where $\alpha_1,\alpha_2,\dots,\alpha_{j-1},\alpha_j$ *are mutually different complex constants.*