CHAPTER 4

Integral Representations in Terms of Powers of the Helmholtz Operator

In this chapter, inspired by Chapter 2 together with their applications on boundary value problem presented in Chapter 3, we wish to develop these results for the $n-th$ power of the Helmholtz operator for $n \geq 1$.

As in Section 2 of Chapter 2, we again iterate both formulas (1.22) and (1.23) to obtain the representation of solutions to inhomogeneous power Helmholtz equation. Moreover, we also prove that the subspaces $\ker(\Delta+\alpha^2)\cap L_2(\Omega,\mathbb{H}(\mathbb{C}))$ and $(\Delta+\alpha^2)(W_2^2(\Omega,\mathbb{H}(\mathbb{C})))\cap \ker tr_\Gamma \cap \ker tr_\Gamma D_\alpha$ are orthogonal subspaces. By an inductive method, the space $L_2(\Omega,\mathbb{H}(\mathbb{C}))$ is decomposed as well into the orthogonal sum of the subspace of *poly-left (right)* $\alpha-$*hyperholomorphic functions* of arbitrary order $k \geq 1$ and its orthogonal complement as into the orthogonal sum of the subspace of *polymetaharmonic functions* of arbitrary order $k \geq 1$ and its orthogonal complement. In addition, the projections onto the subspace of metaharmonic functions are defined. Next, the general integral representation formulas for the $n-th$ power of the Helmholtz operator for $n \geq 2$ are proved. Finally, with the aid of these results, the close connection of these projections with boundary value problem for bimetaharmonic functions are outlined.

1. Integral representations for metaharmonic functions

As introduced in Chapter 1, all *left (right)* $\alpha-$*hyperholomorphic functions* are *metaharmonic functions* in all their coordinates. Therefore, it seems to be necessary to consider the links between *left (right)* $\alpha-$*hyperholomorphic functions* and *metaharmonic functions*. We begin with an integral representation formula for arbitrary C^2- functions. This formula can be obtained by iterating both formulas (1.22) and (1.23).

THEOREM 4.1. *Let* $f \in C^2(\Omega,\mathbb{H}(\mathbb{C}))\cap C^1(\overline{\Omega},\mathbb{H}(\mathbb{C}))$. *Then*

$$\begin{aligned} f(x) = -\int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)f(y)d\Gamma_y \;&+\; \int_\Gamma \vartheta(x-y)\vec{n}(y)D_{\alpha,y}f(y)d\Gamma_y \\ &-\; \int_\Omega \vartheta(x-y)D_{-\alpha,y}D_{\alpha,y}f(y)dy. \qquad (4.1)\end{aligned}$$

PROOF. Applying the quaternionic Cauchy-Pompeiu formula (1.23) for $D_{\alpha,y}f(y)$ we obtain

$$D_{\alpha,y}f(y) = -\int_\Gamma K_{-\alpha}^{(1)}(y-\tilde{y})\vec{n}(\tilde{y})D_{\alpha,\tilde{y}}f(\tilde{y})d\Gamma_{\tilde{y}} + \int_\Omega K_{-\alpha}^{(1)}(y-\tilde{y})D_{-\alpha,\tilde{y}}D_{\alpha,\tilde{y}}f(\tilde{y})d\tilde{y}.$$

This leads to

$$f(x) = -\int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)f(y)d\Gamma_y - \int_\Gamma \psi_(x,\tilde{y})\vec{n}(\tilde{y})D_{\alpha,\tilde{y}}f(\tilde{y})d\Gamma_y + \int_\Omega \psi(x,\tilde{y})D_{-\alpha,\tilde{y}}D_{\alpha,\tilde{y}}f(\tilde{y})d\tilde{y},$$

where $\psi(x,\tilde{y}) = \int_\Gamma K_\alpha^{(1)}(x-y)K_{-\alpha}^{(1)}(y-\tilde{y})dy$.
For $x, \tilde{y} \in \Omega$ with $x \neq \tilde{y}$ formula (1.22) yields $-\vartheta(x-\tilde{y}) = \psi(x,\tilde{y}) + \tilde{\psi}(x,\tilde{y})$ where

$$\tilde{\psi}(x,\tilde{y}) = \int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)\vartheta(y-\tilde{y})d\Gamma_y$$

Since $\int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)K_\alpha^{(1)}(y-\tilde{y})d\Gamma_y = 0$, we have $(D_{r,\tilde{y}}+\alpha)\tilde{\psi}(x,\tilde{y}) = 0$ and for $x \neq \tilde{y}$, the equalities (1.20, 1.21) now yields

$$\int_\Gamma \tilde{\psi}_(x,\tilde{y})\vec{n}(\tilde{y})D_{\alpha,\tilde{y}}f(\tilde{y})d\Gamma_y - \int_\Omega \tilde{\psi}(x,\tilde{y})D_{-\alpha,\tilde{y}}D_{\alpha,\tilde{y}}f(\tilde{y})d\tilde{y} = 0.$$

This leads to

$$f(x) = -\int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)f(y)d\Gamma_y + \int_\Gamma \vartheta(x-y)\vec{n}(y)D_{\alpha,y}f(y)dy - \int_\Omega \vartheta(x-y)D_{-\alpha,y}D_{\alpha,y}f(y)dy$$

□

From this desired expression we get immediately the following results.

Corollary 4.2. *Let $f \in C^2(\Omega,\mathbb{H}(\mathbb{C})) \cap C^1(\overline{\Omega},\mathbb{H}(\mathbb{C}))$ be a metaharmonic function. Then*

$$f(x) = -\int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)f(y)d\Gamma_y + \int_\Gamma \vartheta(x-y)\vec{n}(y)D_{\alpha,y}f(y)d\Gamma_y.$$

This corollary shows that a metaharmonic function allows a simple integral representation on the basis of its boundary values $f|_\Gamma$ and $D_\alpha f|_\Gamma$.

Corollary 4.3. *If $f \in C^2(\Omega,\mathbb{H}(\mathbb{C})) \cap C^1(\overline{\Omega},\mathbb{H}(\mathbb{C}))$ is $\alpha-$ hyperholomorphic, then*

$$f(x) = -\int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)f(y)d\Gamma_y.$$

This means that, we get from our integral representation directly the Cauchy formula for *$\alpha-$hyperholomorphic functions.*

We define an analogue of the Newtonian potential

$$T_{\Delta+\alpha^2}f(x) = \int_\Omega \vartheta(x-y)f(y)dy$$

where $\vartheta(x-y) = -\frac{e^{i\alpha|x-y|}}{4\pi|x-y|}$, $f \in L_1(\Omega,\mathbb{H}(\mathbb{C}))$, $x \in \Omega$.
Analogously, the acoustic single layer potential is defined by

$$V_\alpha f(x) = \int_\Gamma \vartheta(x-y)\vec{n}(y)f(y)d\Gamma_y.$$

By the same method as in [**29**, Chap. 8] we can prove that V_α is a bounded operator from $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ to $W_2^2(\Omega, \mathbb{H}(\mathbb{C}))$. Since Chapter 2, we also have $T_{\Delta+\alpha^2}$ is defined from $C(\overline{\Omega}, \mathbb{H}(\mathbb{C}))$ to $C(\overline{\Omega}, \mathbb{H}(\mathbb{C}))$.

Noting that $D_\alpha\vartheta(x) = -K_{-\alpha}(x)$, $D_{-\alpha}\vartheta(x) = -K_\alpha(x)$ and $\vartheta(x)$ has a singularity of order 1, using the methods as Theorem 2.20 as well as the formula (1.9), by a straightforward calculation

$$\begin{aligned} D_\alpha T_{\Delta+\alpha^2} f(x) &= -T_{-\alpha,1} f(x), \quad D_{-\alpha} T_{\Delta+\alpha^2} f(x) = -T_{\alpha,1} f(x), \\ D_\alpha V_\alpha f(x) &= -F_{-\alpha} f(x), \quad D_{-\alpha} V_\alpha f(x) = -F_\alpha f(x), \end{aligned} \tag{4.2}$$

follow. Moreover, applying the Stokes formula (1.20) we get

$$\int_\Omega [D_{r,-\alpha,y}\vartheta(x-y)f(y) + \vartheta(x-y)D_{\alpha,y}f(y)]\, dy = \int_\Gamma \vartheta(x-y)\vec{n}(y)f(y)d\Gamma_y.$$

This means that

$$V_\alpha f(x) = T_{-\alpha,1} f(x) + T_{\Delta+\alpha^2} D_\alpha f(x). \tag{4.3}$$

It is easily seen that

$$V_\alpha f(x) = T_{\alpha,1} f(x) + T_{\Delta+\alpha^2} D_{-\alpha} f(x). \tag{4.4}$$

if we apply the Stokes formula (1.21). The equations (4.2), (4.3) and (4.4) are valid in $C^\varepsilon(\Omega, \mathbb{H}(\mathbb{C}))$ and in Sobolev spaces $W_p^k(\Omega, \mathbb{H}(\mathbb{C}))$. The equation (4.3) and (4.4) connect operators from the classical potential theory with operators arising in complex quaternionic analysis.

Remark 4.4. Let us introduce the boundary operator

$$F_{\Delta+\alpha^2} f(x) = F_\alpha f(x) + V_\alpha D_\alpha f(x).$$

Then the Cauchy-Pompeiu formula (4.1) for the term of the Helmholtz equation can be rewritten as

$$f(x) = F_{\Delta+\alpha^2} Tr_\Gamma f(x) + T_{\Delta+\alpha^2}(\Delta + \alpha^2) f(x)$$

where

$$Tr_\Gamma f(x) = \begin{pmatrix} tr_\Gamma f \\ tr_\Gamma D_\alpha f \end{pmatrix}.$$

2. Orthogonal decomposition of $L_2(\Omega, \mathbb{H}(\mathbb{C}))$

As presented in Chapter 3, $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ is decomposed into the orthogonal sum of the subspace of *left $\alpha-$hyperholomorphic functions* of the first order and its proof is based on properties of the boundary projections P_α and Q_α. We refer the readers to [**13**] for another proof of decompositions of Sobolev spaces in Clifford analysis. In this paper, the orthogonal complement of the subspace of *poly-left monogenic functions* of arbitrary order $k \geq 1$ and of the subspace of *polyharmonic functions* of arbitrary order $k \geq 1$ are determined. The proofs are based on proper higher-order Cauchy-Pompeiu formulas and Green functions for powers of the Laplacian.

Recently, there has been an increasing interest in the orthogonal decomposition in complex quaternion-valued Hilbert spaces (see, e.g., [**46, 52**]) with respect to classical *left $\alpha-$hyperholomorphic functions*. Besides decompositions with respect to *left $\alpha-$hyperholomorphic functions,* decompositions are related to *poly-left $\alpha-$hyperholomorphic functions* as well as *polymetaharmonic functions* are available. As introduced in Chap. 1,

$\mathcal{M}_p^k(\Omega, \mathbb{H}(\mathbb{C}))$ is the set of all *poly-left* $\alpha-$*hyperholomorphic functions* of order $k \geq 1$. The subspace of *polymetaharmonic functions* of order k in $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ is denoted by

$$\mathbb{H}_{k,2} = \left\{ f,\ f \in L_2(\Omega, \mathbb{H}(\mathbb{C})),\ (\Delta + \alpha^2)^k f = 0 \text{ in } \Omega \right\}.$$

Our method to prove these decompositions is based on the Stokes' formulas and the existence of an inner product in $L_2(\Omega, \mathbb{H}(\mathbb{C}))$. We start with a decomposition of the space $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ with respect to *metaharmonic functions.*

THEOREM 4.5. *The space* $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ *allows the orthogonal decomposition*

$$\begin{aligned} L_2(\Omega, \mathbb{H}(\mathbb{C})) = \quad & \ker(\Delta + \alpha^2) \cap L_2(\Omega, \mathbb{H}(\mathbb{C})) \oplus \\ & \left[(\Delta + \alpha^2)\{W_2^2(\Omega, \mathbb{H}(\mathbb{C}))\} \cap \ker tr_\Gamma \cap \ker tr_\Gamma D_\alpha \right]. \end{aligned} \tag{4.5}$$

PROOF. Notice that for $u, \upsilon \in L_2(\Omega, \mathbb{H}(\mathbb{C}))$ we have

$$(D_\alpha u, \upsilon) = \int_\Omega \overline{D_\alpha u(x)} \upsilon(x) dx = -\int_\Omega D_{r,-\alpha} \overline{u(x)} \upsilon(x) dx.$$

Using the Stokes formula (1.20) we have

$$\int_\Gamma \overline{u(x)} \vec{n}(x) \upsilon(x) d\Gamma_x = \int_\Omega D_{r,-\alpha} \overline{u(x)} \upsilon(x) dx + \int_\Omega \overline{u(x)} D_\alpha \upsilon(x) dx.$$

This means that

$$(D_\alpha u, \upsilon) - (u, D_\alpha \upsilon) = -\int_\Gamma \overline{u(x)} \vec{n}(x) \upsilon(x) d\Gamma_x.$$

If we use the Stokes formula (1.21) we also get

$$(D_{-\alpha} u, \upsilon) - (u, D_{-\alpha} \upsilon) = -\int_\Gamma \overline{u(x)} \vec{n}(x) \upsilon(x) d\Gamma_x.$$

Now we look at

$$\begin{aligned} ((\Delta + \alpha^2) u, \upsilon) &= -(D_\alpha\{D_{-\alpha} u\}, \upsilon) = \int_\Gamma \overline{D_{-\alpha} u(x)} \vec{n}(x) \upsilon(x) d\Gamma_x - (D_{-\alpha} u, D_\alpha \upsilon) \\ &= \int_\Gamma \overline{D_{-\alpha} u(x)} \vec{n}(x) \upsilon(x) d\Gamma_x + \int_\Gamma \overline{u(x)} \vec{n}(x) D_\alpha \upsilon(x) d\Gamma_x - (u, D_{-\alpha} D_\alpha \upsilon) \\ &= \int_\Gamma \overline{D_{-\alpha} u(x)} \vec{n}(x) \upsilon(x) d\Gamma_x + \int_\Gamma \overline{u(x)} \vec{n}(x) D_\alpha \upsilon(x) d\Gamma_x + (u, (\Delta + \alpha^2) \upsilon). \end{aligned}$$

Therefore, we obtain

$$((\Delta + \alpha^2) u, \upsilon) = \int_\Gamma \overline{D_\alpha u(x)} \vec{n}(x) \upsilon(x) d\Gamma_x + \int_\Gamma \overline{u(x)} \vec{n}(x) D_{-\alpha} \upsilon(x) d\Gamma_x + (u, (\Delta + \alpha^2) \upsilon).$$

The above formula shows that the subspaces $\ker(\Delta + \alpha^2) \cap L_2(\Omega, \mathbb{H}(\mathbb{C}))$ and $[(\Delta + \alpha^2)\{W_2^2(\Omega, \mathbb{H}(\mathbb{C}))\} \cap \ker tr_\Gamma \cap \ker tr_\Gamma D_\alpha]$ are orthogonal subspaces. □

Notice that, if we define

$$\overset{0}{W}{}^{2}_{2,\Delta+\alpha^2}(\Omega, \mathbb{H}(\mathbb{C})) := \left\{ f : \quad f \in W_2^2(\Omega, \mathbb{H}(\mathbb{C})), \ tr_\Gamma f = 0, \ tr_\Gamma D_\alpha f = 0 \right\},$$

then the formula (4.5) can be rewritten as

$$L_2(\Omega, \mathbb{H}(\mathbb{C})) = \ker(\Delta + \alpha^2) \cap L_2(\Omega, \mathbb{H}(\mathbb{C})) \oplus (\Delta + \alpha^2)(\overset{0}{W}{}^{2}_{2,\Delta+\alpha^2}(\Omega, \mathbb{H}(\mathbb{C}))).$$

Now, we can denote the corresponding orthoprojections by

$$\begin{aligned} P_{\Delta+\alpha^2} : \ L_2(\Omega, \mathbb{H}(\mathbb{C})) \quad &\longrightarrow \quad \ker(\Delta + \alpha^2) \cap L_2(\Omega, \mathbb{H}(\mathbb{C})) \\ Q_{\Delta+\alpha^2} : \ L_2(\Omega, \mathbb{H}(\mathbb{C})) \quad &\longrightarrow \quad (\Delta + \alpha^2)(\overset{0}{W}{}^{2}_{2,\Delta+\alpha^2}(\Omega, \mathbb{H}(\mathbb{C}))). \end{aligned}$$

The following theorem can be considered as an extension of the results known in [**13**] for classical *polymetaharmonic functions*.

THEOREM 4.6. *The space $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ allows the orthogonal decomposition*

(i)

$$L_2(\Omega, \mathbb{H}(\mathbb{C})) = \mathcal{M}_2^k(\Omega, \mathbb{H}(\mathbb{C})) \oplus D_\alpha^k(\overset{0}{W}{}^{k}_{2,\alpha}(\Omega, \mathbb{H}(\mathbb{C}))),$$

where $\overset{0}{W}{}^{k}_{2,\alpha}(\Omega, \mathbb{H}(\mathbb{C})) := \{f, \ f \in L_2(\Omega, \mathbb{H}(\mathbb{C}), \ D_\alpha^\nu f = 0 \ \textit{on} \ \Gamma \ \textit{for} \ 0 \le \nu \le k-1\}$,

(ii)

$$L_2(\Omega, \mathbb{H}(\mathbb{C})) = \mathbb{H}_2^k(\Omega, \mathbb{H}(\mathbb{C})) \oplus (\Delta + \alpha^2)^k(\overset{0}{W}{}^{k}_{2,\Delta+\alpha^2}(\Omega, \mathbb{H}(\mathbb{C})),$$

where

$$\begin{aligned} \overset{0}{W}{}^{k}_{2,\Delta+\alpha^2}(\Omega, \mathbb{H}(\mathbb{C})) \ := \ \{f : \ f \in L_2(\Omega, \mathbb{H}(\mathbb{C})), \ (\Delta + \alpha^2)^\nu f = 0, \\ D_\alpha(\Delta + \alpha^2)^\nu f = 0 \ \textit{on} \ \Gamma \ \textit{for} \ \ 0 \le \nu \le k-1\}. \end{aligned}$$

PROOF. (i) For $u, \upsilon \in L_2(\Omega, \mathbb{H}(\mathbb{C}))$ and using the Stokes' formula (1.20) we have

$$\begin{aligned} (D_\alpha^k u, \upsilon) &= -\int_\Gamma \overline{D_\alpha^{k-1} u(x)} \vec{n}(x) \upsilon(x) d\Gamma_x + (D_\alpha^{k-1} u, D_\alpha \upsilon) \\ &= -\int_\Gamma \overline{D_\alpha^{k-1} u(x)} \vec{n}(x) \upsilon(x) d\Gamma_x - \int_\Gamma \overline{D_\alpha^{k-2} u(x)} \vec{n}(x) D_\alpha \upsilon(x) d\Gamma_x + (D_\alpha^{k-2} u, D_\alpha \upsilon) \\ &= \dots \\ &= -\int_\Gamma \overline{D_\alpha^{k-1} u(x)} \vec{n}(x) \upsilon(x) d\Gamma_x - \int_\Gamma \overline{D_\alpha^{k-2} u(x)} \vec{n}(x) D_\alpha \upsilon(x) d\Gamma_x \\ &\quad - \dots - \int_\Gamma \overline{D_\alpha u(x)} \vec{n}(x) D_\alpha^{k-2} \upsilon(x) d\Gamma_x - \int_\Gamma \overline{u(x)} \vec{n}(x) D_\alpha^{k-1} \upsilon(x) d\Gamma_x + (u, D_\alpha^k \upsilon). \end{aligned}$$

The above equality shows that the inner product of each element $\upsilon \in \overset{0}{W}{}^{k}_{2,\alpha}(\Omega, \mathbb{H}(\mathbb{C}))$ and any $u \in L_2(\Omega, \mathbb{H}(\mathbb{C}))$ equals zero if and only if $u \in \mathcal{M}_2^k(\Omega, \mathbb{H}(\mathbb{C}))$. This means that the subspaces $\mathcal{M}_2^k(\Omega, \mathbb{H}(\mathbb{C}))$ and $D_\alpha^k \overset{0}{W}{}^{k}_{2,\alpha}(\Omega, \mathbb{H}(\mathbb{C}))$ are orthogonal subspaces.

(ii)By the same method as in the above proofs of the assertion (i) and Theorem 4.5, the assertion (ii) of the theorem follows. ☐

Here, the more general decompositions of the complex quaternion-valued bimodul-Hilbert space open the door for the consideration of further classes of boundary value problems of partial differential equations.

3. Integral representations in terms of powers of the Helmholtz operator

The main purpose of this section is to prove the formulas for integral representations in terms of powers of the Helmholtz operator. They represent the solutions to the higher order inhomogenous Helmholtz equation $(\Delta + \alpha^2)^n f = g,\ n \geq 2$. Since the Helmholtz operator is the product of D_α and $D_{-\alpha}$,a representation fomula in terms of the Helmholtz operator can be obtained by interacting both formulas (1.22) and (1.23). To do so we need the following lemma which constructs the fundamental solution for higher order Helmholtz equations.

Lemma 4.7. *Let $\vartheta(x)$ be a fundamental solution for the Helmholtz operator, i.e, a quaternionic function satisfying in distributional sense $(\Delta + \alpha^2)\vartheta(x) = \delta(x), \alpha \neq 0$ and $\vartheta(x)$ be infinitely often differentiable with respect to α. Then the functions $\vartheta^{(k)}(x), k \in \mathbb{N}$, determined by the following recurrence formulas*

$$\begin{aligned} \vartheta^{(1)}(x) &= \vartheta(x), \\ \vartheta^{(2)}(x) &= \frac{1}{2\alpha}\frac{\partial}{\partial\alpha}\vartheta^{(1)}(x), \\ \vartheta^{(k+1)}(x) &= \frac{1}{2k\alpha}\frac{\partial}{\partial\alpha}\vartheta^{(k)}(x) \end{aligned}$$

satisfy in distributional sense the equation

$$(D^2 - \alpha^2)\vartheta^{(n+1)}(x) = \vartheta^{(n)}(x) \text{ for all } 2 \leq n \in \mathbb{N}. \tag{4.6}$$

Proof. Since $D_r\vartheta^{(1)}(x) = D\vartheta^{(1)}(x)$ we have $D_r\vartheta^{(k)}(x) = D\vartheta^{(k)}(x)$. First of all, we prove this lemma for $n = 2$. For all $\phi \in C_c^\infty(\Omega, \mathbb{H}(\mathbb{C}))$ we have

$$\begin{aligned} & < -D_{-\alpha}D_\alpha\vartheta^{(1)}(x), \phi(x) >=< \delta(x), \phi(x) >= \phi(0), \\ \Leftrightarrow\ & \frac{\partial}{\partial\alpha} < -D_{-\alpha}D_\alpha\vartheta^{(1)}(x), \phi(x) >= \frac{\partial}{\partial\alpha}\phi(0), \\ \Leftrightarrow\ & < (D^2 - \alpha^2)\frac{\partial}{\partial\alpha}\vartheta^{(1)}(x) - 2\alpha\vartheta^{(1)}(x), \phi(x) >= 0, \\ \Leftrightarrow\ & \vartheta^{(1)}(x) = (D^2 - \alpha^2)\frac{1}{2\alpha}\frac{\partial}{\partial\alpha}\vartheta^{(1)}(x), \\ \Leftrightarrow\ & \vartheta^{(1)}(x) = (D^2 - \alpha^2)\vartheta^{(2)}(x), \end{aligned}$$

in distributional sense.
For $n = 2$ since $\vartheta^{(1)}(x) = (D^2 - \alpha^2)\vartheta^{(2)}(x)$ we have

$$\begin{aligned} & \frac{\partial}{\partial\alpha}\vartheta^{(1)}(x) = \frac{\partial}{\partial\alpha}(D^2 - \alpha^2)\vartheta^{(2)}(x), \\ \Leftrightarrow\ & \frac{1}{2\alpha}\frac{\partial}{\partial\alpha}\vartheta^{(1)}(x) = \frac{1}{2\alpha}(D^2 - \alpha^2)\frac{\partial}{\partial\alpha}\vartheta^{(2)}(x) - \frac{2\alpha}{2\alpha}\vartheta^{(2)}(x), \\ \Leftrightarrow\ & \vartheta^{(2)}(x) = (D^2 - \alpha^2)[\frac{1}{4\alpha}\vartheta^{(1)}(x)], \\ \Leftrightarrow\ & \vartheta^{(2)}(x) = (D^2 - \alpha^2)\vartheta^{(3)}(x). \end{aligned}$$

From $(D^2-\alpha^2)\vartheta^{(n-1)}(x)=\vartheta^{(n-2)}(x)$ we obtain

$$\Leftrightarrow \quad \frac{1}{2(n-1)\alpha}\frac{\partial}{\partial\alpha}\vartheta^{(n-2)}(x)=\frac{1}{2(n-1)\alpha}(D^2-\alpha^2)\frac{\partial}{\partial\alpha}\vartheta^{(n-1)}(x)-\frac{2\alpha}{2(n-1)\alpha}\vartheta^{(n-1)}(x),$$

$$\Leftrightarrow \quad \vartheta^{(n-1)}(x)=(D^2-\alpha^2)\frac{1}{2n\alpha}\frac{\partial}{\partial\alpha}\vartheta^{(n-1)}(x),$$

$$\Leftrightarrow \quad \vartheta^{(n-1)}(x)=(D^2-\alpha^2)\vartheta^{(n)}(x).$$

This is the formula (4.6) for n rather than for $n-1$. □

Now we can come to our main result of this section. These following representation formulas express the function through a combination of boundary values of proper lower-order derivatives and an area integral of the $n-th$ order derivative.

THEOREM 4.8. *Let $f\in C^{2n}(\Omega,\mathbb{H}(\mathbb{C}))\cap C^{2n-1}(\bar\Omega,\mathbb{H}(\mathbb{C}))$. Then*

$$\begin{aligned} f(x) &= \sum_{k=1}^{n}\big[-\int_\Gamma D_{\alpha,y}\vartheta^{(k)}(x-y)\vec n(y)(D_{-\alpha,y}D_{\alpha,y})^{k-1}f(y)d\Gamma_y \\ &\quad+\int_\Gamma \vartheta^{(k)}(x-y)\vec n(y)D_{\alpha,y}(D_{-\alpha,y}D_{\alpha,y})^{k-1}f(y)d\Gamma_y\big] \\ &\quad-\int_\Omega \vartheta^{(n)}(x-y)(D_{-\alpha,y}D_{\alpha,y})^{n}f(y)dy. \end{aligned} \tag{4.7}$$

PROOF. For $n=1$, the fomula (4.7) follows from Theorem 4.1 and the equality $D_{\alpha,y}\vartheta(x-y)=K_\alpha(x-y)$.

For $n=2$ applying (4.1) to $D_{-\alpha,y}D_{\alpha,y}f(y)$ we have

$$\begin{aligned} D_{-\alpha,y}D_{\alpha,y}f(y) &= -\int_\Gamma K_\alpha^{(1)}(y-\tilde y)\vec n(\tilde y)(D_{-\alpha,\tilde y}D_{\alpha,\tilde y})f(\tilde y)d\Gamma_{\tilde y} \\ &\quad+\int_\Gamma \vartheta^{(1)}(y-\tilde y)\vec n(\tilde y)D_{\alpha,\tilde y}(D_{-\alpha,\tilde y}D_{\alpha,\tilde y})f(\tilde y)d\Gamma_{\tilde y} \\ &\quad-\int_\Omega \vartheta^{(1)}(y-\tilde y)(D_{-\alpha,\tilde y}D_{\alpha,\tilde y})^2f(\tilde y)d\tilde y. \end{aligned}$$

Inserting this equality into (4.1) gives

$$\begin{aligned} f(x) &= -\int_\Gamma K_\alpha^{(1)}(x-y)\vec n(y)f(y)d\Gamma_y+\int_\Gamma \vartheta^{(1)}(x-y)\vec n(y)D_{\alpha,y}f(y)d\Gamma_y \\ &\quad+\int_\Gamma \psi(x,\tilde y)\vec n(\tilde y)(D_{-\alpha,\tilde y}D_{\alpha,\tilde y})f(\tilde y)d\Gamma_{\tilde y} \\ &\quad-\int_\Gamma \varphi(x,\tilde y)\vec n(\tilde y)D_{\alpha,\tilde y}(D_{-\alpha,\tilde y}D_{\alpha,\tilde y})f(\tilde y)d\Gamma_{\tilde y} \\ &\quad+\int_\Omega \varphi(x,\tilde y)(D_{-\alpha,\tilde y}D_{\alpha,\tilde y})^2f(\tilde y)d\tilde y, \end{aligned}$$

where

$$\begin{aligned} \psi(x,\tilde y) &= \int_\Omega \vartheta^{(1)}(x-y)K_\alpha^{(1)}(y-\tilde y)dy, \\ \varphi(x,\tilde y) &= \int_\Omega \vartheta^{(1)}(x-y)\vartheta^{(1)}(y-\tilde y)dy. \end{aligned}$$

Applying (4.1) for $\vartheta^{(2)}(x-\tilde y)$ and $(D_{\tilde y}+\alpha)\vartheta^{(2)}(x-\tilde y)$ respectively, we have

$$\vartheta^{(2)}(x-\tilde y) = -\tilde\psi_1(x,\tilde y)+\tilde\varphi_1(x,\tilde y)-\phi(x,\tilde y),$$

$$(D_{\tilde{y}}+\alpha)\vartheta^{(2)}(x-\tilde{y}) \;=\; -\tilde{\psi}_2(x,\tilde{y})+\tilde{\varphi}_2(x,\tilde{y})-\psi(x,\tilde{y}),$$

where

$$\begin{aligned}
\tilde{\psi}_1(x,\tilde{y}) &= \int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)\vartheta^{(2)}(y-\tilde{y})d\Gamma_y,\\
\tilde{\psi}_2(x,\tilde{y}) &= \int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)D_{\alpha,\tilde{y}}\vartheta^{(2)}(y-\tilde{y})d\Gamma_y,\\
\tilde{\varphi}_1(x,\tilde{y}) &= \int_\Gamma \vartheta^{(1)}(x-y)\vec{n}(y)D_{\alpha,y}\vartheta^{(2)}(y-\tilde{y})d\Gamma_y,\\
\tilde{\varphi}_2(x,\tilde{y}) &= \int_\Gamma \vartheta^{(1)}(x-y)\vec{n}(y)D_{\alpha,y}D_{\alpha,\tilde{y}}\vartheta^{(2)}(y-\tilde{y})d\Gamma_y,\\
&= -\int_\Gamma \vartheta^{(1)}(x-y)\vec{n}(y)\vartheta^{(1)}(y-\tilde{y})d\Gamma_y.
\end{aligned}$$

It is straightforward to see that

$$\begin{aligned}
(D_{r,\tilde{y}}+\alpha)\tilde{\psi}_1(x,\tilde{y}) &= \tilde{\psi}_2(x,\tilde{y}),\\
(D_{r,\tilde{y}}+\alpha)\tilde{\varphi}_1(x,\tilde{y}) &= \tilde{\varphi}_2(x,\tilde{y}),\\
(D_{r,\tilde{y}}-\alpha)\tilde{\psi}_2(x,\tilde{y}) &= \int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)\vartheta^{(1)}(y-\tilde{y})d\Gamma_y,\\
(D_{r,\tilde{y}}-\alpha)\tilde{\varphi}_2(x,\tilde{y}) &= -\int_\Gamma \vartheta^{(1)}(x-y)\vec{n}(y)K_{-\alpha}^{(1)}(y-\tilde{y})d\Gamma_y.
\end{aligned}$$

Now we prove that

$$(D_{r,\tilde{y}}-\alpha)\tilde{\psi}_2(x,\tilde{y})-(D_{r,\tilde{y}}-\alpha)\tilde{\varphi}_2(x,\tilde{y})=0. \tag{4.8}$$

We recall that with $x\neq\tilde{y}$, $\Omega_\varepsilon=\Omega-\{y\in\Omega,|y-x|\leq\varepsilon \text{ or } |y-\tilde{y}|\leq\varepsilon\}$, $0<2\varepsilon<|x-\tilde{y}|$,

$$\begin{aligned}
\int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)\vartheta^{(1)}(y-\tilde{y})d\Gamma_y &= \int_{\Gamma_\varepsilon} K_\alpha^{(1)}(x-y)\vec{n}(y)\vartheta^{(1)}(y-\tilde{y})d\Gamma_y\\
&\quad+\int_{|y-x|=\varepsilon\cup|y-\tilde{y}|=\varepsilon} K_\alpha^{(1)}(x-y)\vec{n}(y)\vartheta^{(1)}(y-\tilde{y})d\Gamma_y,\\
\int_\Gamma \vartheta^{(1)}(x-y)\vec{n}(y)K_{-\alpha}^{(1)}(y-\tilde{y})d\Gamma_y &= \int_{\Gamma_\varepsilon} \vartheta^{(1)}(x-y)\vec{n}(y)K_{-\alpha}^{(1)}(y-\tilde{y})d\Gamma_y,\\
&\quad+\int_{|y-x|=\varepsilon\cup|y-\tilde{y}|=\varepsilon} \vartheta^{(1)}(x-y)\vec{n}(y)K_{-\alpha}^{(1)}(y-\tilde{y})d\Gamma_y.
\end{aligned}$$

By formula (1.20), (1.21) we obtain

$$\begin{aligned}
\int_{\Gamma_\varepsilon} K_\alpha^{(1)}(x-y)\vec{n}(y)\vartheta^{(1)}(y-\tilde{y})d\Gamma_y &= -\vartheta^{(1)}(x-\tilde{y})-\int_{\Omega_\varepsilon} K_\alpha^{(1)}(x-y)K_{-\alpha}^{(1)}(y-\tilde{y})dy,\\
\int_{\Gamma_\varepsilon} \vartheta^{(1)}(x-y)\vec{n}(y)K_{-\alpha}^{(1)}(y-\tilde{y})d\Gamma_y &= \vartheta^{(1)}(x-\tilde{y})+\int_{\Omega_\varepsilon} K_\alpha^{(1)}(x-y)K_{-\alpha}^{(1)}(y-\tilde{y})dy,
\end{aligned}$$

and

$$\lim_{\epsilon\to 0}\int_{|y-x|=\varepsilon\cup|y-\tilde{y}|=\varepsilon} K_\alpha^{(1)}(x-y)\vec{n}(y)\vartheta^{(1)}(y-\tilde{y})d\Gamma_y \;=\; -\vartheta^{(1)}(x-\tilde{y}),$$

$$\lim_{\epsilon\to 0} \int\limits_{|y-x|=\varepsilon \cup |y-\tilde{y}|=\varepsilon} \vartheta^{(1)}(x-y)\vec{n}(y)K^{(1)}_{-\alpha}(y-\tilde{y})d\Gamma_y \;=\; \vartheta^{(1)}(x-\tilde{y}).$$

Hence equality (4.8) holds. For $x \neq \tilde{y}$, from the equalities (1.20), (1.21) and (4.8) we have

$$\begin{aligned} &\int_\Gamma (\tilde{\psi}_1(x,\tilde{y}) - \tilde{\varphi}_1(x,\tilde{y}))\vec{n}(\tilde{y})D_{\alpha,\tilde{y}}(D_{-\alpha,\tilde{y}}D_{\alpha,\tilde{y}}f(\tilde{y}))d\Gamma_{\tilde{y}} \\ -&\int_\Gamma (\tilde{\psi}_2(x,\tilde{y}) - \tilde{\varphi}_2(x,\tilde{y}))\vec{n}(\tilde{y})D_{-\alpha,\tilde{y}}D_{\alpha,\tilde{y}}f(\tilde{y})d\Gamma_{\tilde{y}} \\ -&\int_\Omega (\tilde{\psi}_1(x,\tilde{y}) - \tilde{\varphi}_1(x,\tilde{y}))(D_{-\alpha,\tilde{y}}D_{\alpha,\tilde{y}})^2 f(\tilde{y}))d\tilde{y} = 0. \end{aligned}$$

Therefore equality (4.7) holds for $n = 2$.

Let the equality (4.7) hold for $n-1$. We now prove it for n. By inductive hypothesis we have

$$\begin{aligned} D_{-\alpha,y}D_{\alpha,y}f(y) \;=\; &\sum_{k=1}^{n-1}\big[-\int_\Gamma D_{\alpha,\tilde{y}}\vartheta^{(k)}(y-\tilde{y})\vec{n}(\tilde{y})(D_{-\alpha,y}D_{\alpha,\tilde{y}})^k f(\tilde{y})d\Gamma_{\tilde{y}} \\ &+\int_\Gamma \vartheta^{(k)}(y-\tilde{y})\vec{n}(\tilde{y})D_{\alpha,\tilde{y}}(D_{-\alpha,\tilde{y}}D_{\alpha,\tilde{y}})^k f(\tilde{y})d\Gamma_{\tilde{y}}\big] \\ &-\int_\Omega \vartheta^{(n-1)}(y-\tilde{y})(D_{-\alpha,\tilde{y}}D_{\alpha,\tilde{y}})^n f(\tilde{y})d\tilde{y}. \end{aligned}$$

This leads to

$$\begin{aligned} f(x) \;=\; &-\int_\Gamma D_{\alpha,y}\vartheta^{(1)}(x-y)\vec{n}(y)f(y)d\Gamma_y + \int_\Gamma \vartheta^{(1)}(x-y)\vec{n}(y)D_{\alpha,y}f(y)d\Gamma_y \\ &+\sum_{k=1}^{n-1}\big[\int_\Gamma \psi_k(x,\tilde{y})\vec{n}(\tilde{y})(D_{-\alpha,y}D_{\alpha,\tilde{y}})^k f(\tilde{y})d\Gamma_{\tilde{y}} \\ &-\int_\Gamma \varphi_k(x,\tilde{y})\vec{n}(\tilde{y})D_{\alpha,\tilde{y}}(D_{-\alpha,\tilde{y}}D_{\alpha,\tilde{y}})^k f(\tilde{y})d\Gamma_{\tilde{y}}\big] \\ &-\int_\Omega \varphi_{n-1}(x,\tilde{y})(D_{-\alpha,\tilde{y}}D_{\alpha,\tilde{y}})^n f(\tilde{y})d\tilde{y}, \end{aligned}$$

where

$$\begin{aligned} \varphi_k(x,\tilde{y}) \;&=\; \int_\Omega \vartheta^{(1)}(x-y)\vartheta^{(k)}(y-\tilde{y})dy, \\ \psi_k(x,\tilde{y}) \;&=\; \int_\Omega \vartheta^{(1)}(x-y)D_{\alpha,\tilde{y}}\vartheta^{(k)}(y-\tilde{y})dy, \end{aligned}$$

for $k = 1,..,n-1$. Applying the formula (4.1) and using(4.6) for the functions $\vartheta^{(k+1)}(x-\tilde{y})$ and $-D_{-\alpha,x}\vartheta^{(k+1)}(x-\tilde{y})$ for $k = 1,2,..,n-2$ shows

$$\begin{aligned} \varphi_k(x,\tilde{y}) \;&=\; -\vartheta^{(k+1)}(x-\tilde{y}) - \tilde{\psi}_{1,k}(x,\tilde{y}) + \tilde{\varphi}_{1,k}(x,\tilde{y}), \\ \psi_k(x,\tilde{y}) \;&=\; -D_{\alpha,\tilde{y}}\vartheta^{(k+1)}(x-\tilde{y}) - \tilde{\psi}_{2,k}(x,\tilde{y}) + \tilde{\varphi}_{2,k}(x,\tilde{y}), \end{aligned}$$

where

$$\tilde{\psi}_{1,k}(x,\tilde{y}) \;=\; \int_\Gamma D_{\alpha,y}\vartheta^{(1)}(x-y)\vec{n}(y)\vartheta^{(k+1)}(y-\tilde{y})d\Gamma_y,$$

$$\begin{aligned}\tilde{\psi}_{2,k}(x,\tilde{y}) &= \int_\Gamma D_{\alpha,y}\vartheta^{(1)}(x-y)\vec{n}(y)D_{\alpha,\tilde{y}}\vartheta^{(k+1)}(y-\tilde{y})d\Gamma_y,\\ \tilde{\varphi}_{1,k}(x,\tilde{y}) &= \int_\Gamma \vartheta^{(1)}(x-y)\vec{n}(y)D_{\alpha,y}\vartheta^{(k+1)}(y-\tilde{y})d\Gamma_y,\\ \tilde{\varphi}_{2,k}(x,\tilde{y}) &= \int_\Gamma \vartheta^{(1)}(x-y)\vec{n}(y)D_{\alpha,y}D_{\alpha,\tilde{y}}\vartheta^{(k+1)}(y-\tilde{y})d\Gamma_y.\end{aligned}$$

It is easy to see that

$$\begin{aligned}(D_{r,\tilde{y}}+\alpha)\tilde{\psi}_{1,k}(x,\tilde{y}) &= \tilde{\psi}_{2,k}(x,\tilde{y}),\\ (D_{r,\tilde{y}}+\alpha)\tilde{\varphi}_{1,k}(x,\tilde{y}) &= \tilde{\varphi}_{2,k}(x,\tilde{y}),\\ (D_{r,\tilde{y}}-\alpha)\tilde{\psi}_{1,k+1}(x,\tilde{y}) &= \tilde{\psi}_{1,k}(x,\tilde{y}),\\ (D_{r,\tilde{y}}-\alpha)\tilde{\varphi}_{2,k+1}(x,\tilde{y}) &= \tilde{\varphi}_{1,k}(x,\tilde{y}).\end{aligned}$$

From the equalities (1.20), (1.21) and (4.8) we have

$$\begin{aligned}&\sum_{k=1}^{n-1}\int_\Gamma \big[\tilde{\psi}_{1,k}(x,\tilde{y})-\tilde{\varphi}_{1,k}(x,\tilde{y})\big]\vec{n}(\tilde{y})D_{\alpha,\tilde{y}}(D_{-\alpha,\tilde{y}}D_{\alpha,\tilde{y}})^k f(\tilde{y}))d\Gamma_{\tilde{y}}\\ -&\sum_{k=1}^{n-1}\int_\Gamma \big[\tilde{\psi}_{2,k}(x,\tilde{y})-\tilde{\varphi}_{2,k}(x,\tilde{y})\big]\vec{n}(\tilde{y})(D_{-\alpha,\tilde{y}}D_{\alpha,\tilde{y}})^k f(\tilde{y}))d\Gamma_{\tilde{y}}\\ +&\int_\Omega \big[\tilde{\psi}_{1,n-2}(x,\tilde{y})-\tilde{\varphi}_{1,n-2}(x,\tilde{y})\big](D_{-\alpha,\tilde{y}}D_{\alpha,\tilde{y}})^n f(\tilde{y})d\tilde{y}=0.\end{aligned}$$

Therefore the equality (4.7) holds for n. □

REMARK 4.9. Since $-D_{-\alpha}D_\alpha f(x) = (\Delta+\alpha^2)f(x),\ x\in\Omega$, the formula

$$\begin{aligned}f(x) &= \sum_{k=1}^{n}(-1)^k\big[-\int_\Gamma D_{\alpha,y}\vartheta^{(k)}(x-y)\vec{n}(y)(\Delta+\alpha^2)^{k-1}f(y)d\Gamma_y\\ &\quad+\int_\Gamma \vartheta^{(k)}(x-y)\vec{n}(y)D_{\alpha,y}(\Delta+\alpha^2)^{k-1}f(y)d\Gamma_y\big]\\ &\quad(-1)^{n+1}\int_\Omega \vartheta^{(n)}(x-y)g(y)dy.\end{aligned}$$

represents a solution to the problem $(\Delta+\alpha^2)^n f(x) = g(x)$. If the solvability of the boudary values problems to these equations are guaranteed then these representation formulas may be used for representing the solutions by some method, for instant as shown in Chapter 3.

4. Dirichlet problem for bimetaharmonic function

As in Chapter 3, we study the bimetaharmonic problem by the help of the self-contained theory given in the preceding sections. Questions of existence of the solutions is answered in this section as well as an important convenient representation of the solutions. We hope these integral representation formulas are adapted to the necessary numerical evaluation of the solutions which can be connected with mathematical models. To that purpose, we first assert the origin of the projection onto *metaharmonic functions.*

Looking at the formula (4.7), each bimetaharmonic function, i.e., satistyfing the equation $(\Delta+\alpha^2)^2 f(x)=0$, has the integral representation

$$\begin{aligned} f(x) = & - \int_\Gamma D_{\alpha,y}\vartheta^{(1)}(x-y)\vec{n}(y)f(y)d\Gamma_y + \int_\Gamma \vartheta^{(1)}(x-y)\vec{n}(y)D_{\alpha,y}f(y)d\Gamma_y \\ & + \int_\Gamma D_{\alpha,y}\vartheta^{(2)}(x-y)\vec{n}(y)(\Delta+\alpha^2)f(y)d\Gamma_y \\ & + \int_\Gamma \vartheta^{(2)}(x-y)\vec{n}(y)D_{\alpha,y}(\Delta+\alpha^2)f(y)d\Gamma_y \\ = & F_\alpha f(x) + V_\alpha D_\alpha f(x) + \int_\Gamma D_{\alpha,y}\vartheta^{(2)}(x-y)\vec{n}(y)(\Delta+\alpha^2)f(y)d\Gamma_y \\ & - \int_\Gamma \vartheta^{(2)}(x-y)\vec{n}(y)D_{\alpha,y}(\Delta+\alpha^2)f(y)d\Gamma_y. \end{aligned}$$

Applying formula (4.7) for $D_\alpha f$ gives

$$D_\alpha f(x) = F_{-\alpha}(D_\alpha f(x)) - V_\alpha(\Delta+\alpha^2)f(x) + \int_\Gamma D_{-\alpha,y}\vartheta^{(2)}(x-y)\vec{n}(y)D_{\alpha,y}(\Delta+\alpha^2)f(y)d\Gamma_y.$$

If we take the traces of f and $D_\alpha f$ then we obtain the following boundary integral equation, $x_0\in\Gamma$,

$$\begin{aligned} f(x_0) = & \frac{1}{2}\left(f(x_0)+S_\alpha f(x_0)\right) + V_\alpha D_\alpha f(x_0) \\ & + \int_\Gamma D_{\alpha,y}\vartheta^{(2)}(x_0-y)\vec{n}(y)(\Delta+\alpha^2)f(y)d\Gamma_y \\ & - \int_\Gamma \vartheta^{(2)}(x_0-y)\vec{n}(y)D_{\alpha,y}(\Delta+\alpha^2)f(y)d\Gamma_y. \\ D_\alpha f(x_0) = & \frac{1}{2}\left(D_\alpha f(x_0)+S_{-\alpha}D_\alpha f(x_0)\right) - V_\alpha(\Delta+\alpha^2)f(x_0) \\ & + \int_\Gamma D_{-\alpha,y}\vartheta^{(2)}(x_0-y)\vec{n}(y)D_{\alpha,y}(\Delta+\alpha^2)f(y)d\Gamma_y. \end{aligned}$$

If we now are looking for all bimetaharmonic functions f, which are also metaharmonic then we get the following conditions

$$\begin{aligned} f(x_0) &= \frac{1}{2}\left(f(x_0)+S_\alpha f(x_0)\right) + V_\alpha D_\alpha f(x_0), \\ D_\alpha f(x_0) &= \frac{1}{2}\left(D_\alpha f(x_0)+S_{-\alpha}D_\alpha f(x_0)\right), \end{aligned}$$

or with orther words

$$\begin{pmatrix} f \\ D_\alpha f \end{pmatrix} = \begin{pmatrix} P_\alpha & V_\alpha \\ 0 & P_{-\alpha} \end{pmatrix} \begin{pmatrix} f \\ D_\alpha f \end{pmatrix}.$$

From this we can derive the projections

$$P_{\Delta+\alpha^2} = \begin{pmatrix} P_\alpha & V_\alpha \\ 0 & P_{-\alpha} \end{pmatrix}, \quad Q_{\Delta+\alpha^2} = \begin{pmatrix} Q_\alpha & -V_\alpha \\ 0 & Q_{-\alpha} \end{pmatrix}.$$

These operators as well as the orthogonal decomposition of $L_2(\Omega,\mathbb{H}(\mathbb{C}))$ given in the preceding section will open the door for the consideration of further classes of boundary value problems for bimetaharmonic functions.

We now come to rewrite the form of the Dirichlet problem for the bimetaharmonic equation for sake of convenience. With the notation

$$Tr_\Gamma f(x) = \begin{pmatrix} tr_\Gamma f \\ tr_\Gamma D_\alpha f \end{pmatrix},$$

then the Dirichlet problem for the inhomogeneous bimetaharmonic equation can be written as

$$\begin{cases} (\Delta+\alpha^2)^2 u &= f \qquad \text{in } \Omega, \\ Tr_\Gamma u &= g \qquad \text{on } \Gamma, \end{cases} \tag{4.9}$$

In connection with Theorem 3.9 and Theorem 4 in [**43**] (see also [**47**, Theorem 4.57]) we are able to state the following result.

THEOREM 4.10. *Suppose that Ω is a domain with sufficiently smooth boundary Γ. For each pair of functions $\omega_1 \in W_2^{k+3/2}(\Gamma, \mathbb{H}(\mathbb{C}))$, $\omega_2 \in W_2^{k+1/2}(\Gamma, \mathbb{H}(\mathbb{C}))$ there exists an extension $h \in W_2^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))$ with $h|_\Gamma = \omega_1$, and $(D+\alpha)h|_\Gamma = \omega_2$, where α is a complex constant.*

PROOF. By [**43**, Theorem 4], there exists a $W_2^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))-$extension h such that $h|_\Gamma = \omega_1$, and $Dh|_\Gamma = \omega_2 - \alpha\omega_1$. This yields Theorem 4.10. □

We now can look for solutions of the problem (4.9). To do this, we start with the following problem

$$\begin{cases} (\Delta+\alpha^2)^2 u &= 0 \qquad \text{in } \Omega, \\ Tr_\Gamma u &= g \qquad \text{on } \Gamma. \end{cases} \tag{4.10}$$

THEOREM 4.11. *If $k \in \mathbb{N}$, $g \in W_2^{k+3/2}(\Gamma, \mathbb{H}(\mathbb{C})) \times W_2^{k+1/2}(\Gamma, \mathbb{H}(\mathbb{C}))$ then the problem (4.10) has the solution $u \in W_2^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))$, which may be written as*

$$u = F_{\Delta+\alpha^2} g + T_{\Delta+\alpha^2} P_{\Delta+\alpha^2} (\Delta+\alpha^2) h$$

where h is the $W_2^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))-$extension of g appearing in Theorem 4.10.

PROOF. Using Theorem 4.10, if $g \in W_2^{k+3/2}(\Gamma, \mathbb{H}(\mathbb{C})) \times W_2^{k+1/2}(\Gamma, \mathbb{H}(\mathbb{C}))$ then there exists a $h \in W_2^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))$ such that $Tr_\Gamma h = g$. Let $u = \upsilon + h$ then our boundary value problem (4.10) has the form

$$\begin{cases} (\Delta+\alpha^2)^2 \upsilon &= -(\Delta+\alpha^2)^2 h \quad \text{in } \Omega, \\ Tr_\Gamma \upsilon &= 0 \qquad\qquad\qquad \text{on } \Gamma, \end{cases} \tag{4.11}$$

We now are looking for solutions of problem (4.11). Noting that $\vartheta^{(1)}$ has a singularity of order 1 and using Theorem 2.20 we get $(\Delta+\alpha^2) T_{\Delta+\alpha^2} = Id$ where Id is the identity operator and

$$T_{\Delta+\alpha^2} \; : \; W_2^k(\Gamma, \mathbb{H}(\mathbb{C})) \longrightarrow W_2^{k+2}(\Gamma, \mathbb{H}(\mathbb{C})).$$

Moreover, the validity of Theorem 4.5 and the corresponding orthoprojections $Q_{\Delta+\alpha^2}$ show that there exists an $\mathbb{H}(\mathbb{C})-$valued function $\upsilon \in \overset{0}{W}{}^{2}_{2,\Delta+\alpha^2}(\Omega, \mathbb{H}(\mathbb{C}))$ such that

$$(\Delta+\alpha^2)\upsilon = -Q_{\Delta+\alpha^2} T_{\Delta+\alpha^2} (\Delta+\alpha^2)^2 h.$$

Using the Cauchy-Pompeiu formula for the term of the Helmholtz equation in Remark 4.4, by the same methods as in Theorem 3.8 for $\upsilon \in W^2_{2,\Delta+\alpha^2}(\Omega, \mathbb{H}(\mathbb{C}))$ and $Tr_\Gamma \upsilon = 0$ we obtain

$$\upsilon = -T_{\Delta+\alpha^2} Q_{\Delta+\alpha^2} T_{\Delta+\alpha^2} (\Delta + \alpha^2)^2 h.$$

Therefore,

$$\begin{aligned}(\Delta+\alpha^2)^2 \upsilon &= -(\Delta+\alpha^2)^2 T_{\Delta+\alpha^2} Q_{\Delta+\alpha^2} T_{\Delta+\alpha^2} (\Delta+\alpha^2)^2 h \\ &= -(\Delta+\alpha^2) Q_{\Delta+\alpha^2} T_{\Delta+\alpha^2} (\Delta+\alpha^2)^2 h \\ &= -(\Delta+\alpha^2)(I - P_{\Delta+\alpha^2}) T_{\Delta+\alpha^2} (\Delta+\alpha^2)^2 h \\ &= -(\Delta+\alpha^2)^2 h,\end{aligned}$$

because of the definition of the orthoprojections $P_{\Delta+\alpha^2}$, $\mathrm{im} P_{\Delta+\alpha^2} \subset \ker(\Delta+\alpha^2)$.
Hence, using again the Cauchy-Pompeiu type formula for the Helmholtz equation we get

$$\begin{aligned}\upsilon &= -T_{\Delta+\alpha^2} Q_{\Delta+\alpha^2} \left[(\Delta+\alpha^2)h - F_{\Delta+\alpha^2}(\Delta+\alpha^2)h\right] \\ &= -T_{\Delta+\alpha^2} Q_{\Delta+\alpha^2} (\Delta+\alpha^2)h + T_{\Delta+\alpha^2} Q_{\Delta+\alpha^2} F_{\Delta+\alpha^2} (\Delta+\alpha^2)h.\end{aligned}$$

On the other hand, by the orthoprojections $\mathrm{im} P_{\Delta+\alpha^2} = \ker Q_{\Delta+\alpha^2}$, and because $F_{\Delta+\alpha^2}$ maps onto $\mathrm{im} P_{\Delta+\alpha^2}$ then we get $Q_{\Delta+\alpha^2} F_{\Delta+\alpha^2} (\Delta+\alpha^2)h = 0$.
Thus,

$$\begin{aligned}\upsilon &= -T_{\Delta+\alpha^2} Q_{\Delta+\alpha^2} (\Delta+\alpha^2)h \\ &= -T_{\Delta+\alpha^2} (\Delta+\alpha^2)h + T_{\Delta+\alpha^2} P_{\Delta+\alpha^2} (\Delta+\alpha^2)h.\end{aligned}$$

Meanwhile, the Cauchy-Pompeiu representation for the of Helmholtz equation shows that

$$T_{\Delta+\alpha^2}(\Delta+\alpha^2)h = h - F_{\Delta+\alpha^2} h.$$

This leads to

$$\upsilon = -h + F_{\Delta+\alpha^2} h + T_{\Delta+\alpha^2} P_{\Delta+\alpha^2} (\Delta+\alpha^2)h.$$

Consequently, it may be observed that

$$\upsilon + h = F_{\Delta+\alpha^2} h + T_{\Delta+\alpha^2} P_{\Delta+\alpha^2} (\Delta+\alpha^2)h.$$

Hence, $u = F_{\Delta+\alpha^2} h + T_{\Delta+\alpha^2} P_{\Delta+\alpha^2} (\Delta+\alpha^2)h$. □

Combining the problem (4.10) with (4.11) we obtain immediately the following proposition.

PROPOSITION 4.12. *Under the assumption of the above theorem, the problem (4.9) has a solution of the form*

$$u = F_{\Delta+\alpha^2} g + T_{\Delta+\alpha^2} P_{\Delta+\alpha^2} (\Delta+\alpha^2)h + T_{\Delta+\alpha^2} Q_{\Delta+\alpha^2} T_{\Delta+\alpha^2} f$$

where h denotes a $W_2^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))-$extension of g.