CHAPTER 3

A Boundary Value Problem of the Helmholtz Equation

This chapter is devoted to the investigation of the Dirichlet problem for the Helmholtz equation in a bounded domain. In Section 1, we briefly explain why and how boundary valued problems for the Helmholtz equation are solved. Using the properties of $D_\alpha,\ T_\alpha,\ F_\alpha$ as well as the projections defined by the singular Cauchy operator the Dirichlet problem for classical $\alpha-$*hyperholomophic functions* are solved. This work is presented in Section 2. An orthogonal decomposition of $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ for a generalization of the Laplacian is considered in Section 3. Analogous works are done in Section 4 for classical $\alpha-$*metaharmonic functions* instead of $\alpha-$ *hyperholomophic functions* . Finally, by induction in Section 5, the existence and the unique solution to the boundary value problem for the $n-$th order Helmholtz equation are given.

1. History and Motivation

First of all, let us begin with a brief discussion of the physical background to the propagation of sound waves with small amplitudes in a homogeneous isotropic medium in $\mathbb{R}^3$ viewed as an inviscid fluid.

Let $\upsilon = \upsilon(x,t)$ be the velocity field and let $p = p(x,t),\ \rho = \rho(x,t)$ and $S = S(x,t)$ denote the pressure, density and specific entropy, respectively, of the fluid. The motion is then governed by *Euler's equation*

$$\frac{\partial \upsilon}{\partial t} + (\upsilon \mathrm{grad})\upsilon + \frac{1}{\rho}\mathrm{grad} p = 0,$$

the *equation of continuity*

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\upsilon) = 0,$$

the *state equation*

$$p = f(\rho, S),$$

and the *adiabatic hypothesis*

$$\frac{\partial S}{\partial t} + \upsilon \mathrm{grad} S = 0,$$

where f is a function depending on the nature of the fluid. We assume that $\upsilon,\ p,\ \rho$, and S are small perturbations of the static state $\upsilon_0 = 0,\ \rho_0 = const,\ p_0 = const$ and $S = const$ and linearize to obtain the linearized Euler equation

$$\frac{\partial \upsilon}{\partial t} + \frac{1}{\rho_0}\mathrm{grad} p = 0,$$

the linearized equation of continuity

$$\frac{\partial \rho}{\partial t} + \rho_0 \mathrm{div}\upsilon = 0,$$

and the linearized state equation

$$\frac{\partial p}{\partial t} = \frac{\partial f}{\partial \rho}(\rho_0, S_0)\frac{\partial \rho}{\partial t}.$$

From this we obtain the *wave equation*

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial^2 t} = \Delta p,$$

where the *speed of sound* c is defined by

$$c^2 = \frac{\partial f}{\partial \rho}(\rho_0, S_0).$$

From the linearized Euler equation, we observe that there exists a velocity potential $U = U(x,t)$ such that

$$v = \frac{1}{\rho_0}\mathrm{grad}U$$

and

$$p = -\frac{\partial U}{\partial t}.$$

Clearly, the velocity potential also satisfies the wave equation

$$\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = \Delta U.$$

For time-harmonic acoustic waves of the form

$$U(x,t) = \mathrm{Re}\{u(x)e^{-i\omega t}\}$$

with frequency $\omega > 0$, we deduce that the complex valued space dependent part u satisfies the *reduced wave equation* or *Helmholtz equation*

$$\Delta u + \alpha^2 u = 0$$

where the *wave number* α is given by the positive constant $\alpha = \omega/c$. This equation carries the name of the physicist Hermann Ludwig Ferdinand von Helmholtz (1821-1894) for his contributions to mathematical acoustics and electromagnetics. For a solid knowledge of some basic properties of the solutions to the Helmholtz equation and its physical background, we refer the readers to D.L. Colton's books, for instance [**28, 29**].

On the orther hand, a large number of interesting physical applications, for instance problems in elasticity theory of shells and in gas dynamics, lead to the so-called *Vekua type problems.* With methods of function theory of one complex variable a complete theory of the solutions to the equation $\bar{\partial}u + au + b\bar{u} = f$ is described in [**24, 86**]. Here let $\bar{\partial} = \partial_1 + i\partial_2$ and a, b be constant complex numbers, $u, f \in C(\Omega, \mathbb{C})$.

In 1944, Richard von Mises [**70**] wrote one of the first papers for this type of equations in higher dimensions. In quaternionic notation this problem can be rewritten in the form $Du + \alpha u = f$. Solutions of that system (see fomulas (1.9), (1.10)) are closely related as well to time harmonic electromagnetic fields, see [**61, 64, 65, 66**] as to time- harmonic spinor fields, see [**62, 63**]. As introduced in Chapter 1, it is known that $-(D+\alpha)(D-\alpha) = \Delta + \alpha^2$. D_α is called the Helmholtz operator. We refer to [**26, 30, 48, 64**] for detailed discussions on various examples in physical models which may be described by the Helmholtz equation.

It is known that in the plane the boundary value problems for the Cauchy - Riemann $\bar{\partial}-$operator have deep implications to potential theory (see [**39, 87**]). In 1971, the Yukawa potential was developed by R. J. Duffin [**35**] to investigate the equation $(\Delta + \alpha^2)u = 0$ where $\alpha \in \mathbb{R}$, and u is a complex valued function. Of course, the important property of Yukawa potentials is that they approach those of Newton potential as α approaches zero. We also refer the readers to [**67**] for solving the Dirichlet problem for the Helmholtz equation in bounded plane domains. The existence of a classical solution is proved by potential theory. Generalizing these ideas to the Clifford algebra frameword and connecting with the Helmholtz operator is undertaken by M. Mitrea in 1996, see [**71**]. We also refer to [**20, 44, 56**] for studying classical elliptic equations in bounded and unbounded domains which include the Helmholtz equation. Of course this development was inspired by physicists with their interests in first and second order partial differential equations.

On the other hand, in particular for the unit disc and the unit polydisc in $\mathbb{C}$, $\mathbb{C}^n$, the existence of the explicit form of the Green function as well as the higher order Green functions corresponding to the Laplacian operator, as given in [**4**], is a very wonderful event. They lead to orthogonal polyanalytic and polyharmonic decompositions of the Hilbert space $L_2(\mathbb{D}, \mathbb{C})$, see [**13**]. The existence of the explicit form of the Green function with respect to the Laplacian and to the higher order Lapace operators together with Begehr's method becomes an excellent way to solve some boudary value problems for analytic, polyanalytic function etc., in the unit disc and the unit polydisc, e.g., [**7, 12, 15**].

However, in quaternionic analysis we can only prove the existence of the Green function for the Helmholtz equation. The proof can be given following the idea in [**72**, Chap. 8]. The explicit form of the Green function for the Helmholtz equation, even for some particular domains remains still unknown in quaternionic analysis.

Therefore, our method uses the fundamental solution of the Helmholtz equation in order to build a systematical theory for the *metaharmonic functions* introduced and developed in Chapter 1 and Chapter 2 to study the Dirichlet problem for the Helmholtz equation. We will investigate the problem

$$\begin{cases} \Delta u + \alpha^2 u & = f \qquad \text{in } \Omega, \\ u & = g \qquad \text{on } \Gamma, \end{cases} \tag{3.1}$$

where g is a function defined on the boundary, by the help of a self-contained theory. Questions of existence, uniqueness, and regularity are included in this theory and as an important advantage also convenient representations of the solutions. To do this, we have to divide this work into several steps. Now, we come to study the Dirichlet problem for the classical *left $\alpha-$hyperholomophic functions*. The *right $\alpha-$hyperholomophic functions* are treated in the same way.

2. The Dirichlet problem for the operator D_α

In this section we investigate the problem

$$\begin{cases} D_\alpha u & = f \qquad \text{in } \Omega, \\ u & = g \qquad \text{on } \Gamma. \end{cases} \tag{3.2}$$

To this purpose, we start with the *quaternionic Plemelj-Sokhotski formulas*. This theorem is taken from [**60**, Chap. 2, Theorem 8].

THEOREM 3.1. *(Quaternionic Plemelj-Sokhotski formulas)*
Let Γ be a closed Liapunov surface, $f \in C^{0,\varepsilon}(\Gamma, \mathbb{H}(\mathbb{C})),\ 0 < \varepsilon \leq 1$. Then everywhere on Γ the following limits exist satisfying

$$\lim_{\Omega^{+} \ni x \to \tau \in \Gamma} F_\alpha f(x) = P_\alpha f(\tau),$$

$$\lim_{\Omega^{-} \ni x \to \tau \in \Gamma} F_\alpha f(x) = -Q_\alpha f(\tau),$$

where $\Omega^{+} = \Omega,\ \Omega^{-} := \mathbb{R}^3 - \overline{\Omega}$.

In the book [**25**, p. 177], (see also [**46**, Theorem 2.5.10], [**47**, Theorem 3.64]), the authors have proven the theorem in the case $\alpha = 0$. For the case $\alpha \neq 0$ the proof can be found in [**60**, Theorem 8].

If we denote tr_Γ as the operator of restriction onto the boundary Γ, and using the trace theorem for $g \in W_p^{k-1/p}(\Gamma, \mathbb{H}(\mathbb{C}))$ then there exists an $\mathbb{H}(\mathbb{C})-$valued function $u \in W_p^k(\Gamma, \mathbb{H}(\mathbb{C}))$ such that $tr_\Gamma u = g$. Using the Cauchy-Pompeiu formula and Remark 2.21

$$u - T_{\alpha,1} D_\alpha u \in W_p^k(\Gamma, \mathbb{H}(\mathbb{C}))$$

follows. This means that the following Proposition holds.

PROPOSITION 3.2. *With F_α as defined in Chapter1, we have*

$$F_\alpha : \quad W_p^{k-1/p}(\Gamma, \mathbb{H}(\mathbb{C})) \ \longrightarrow \ W_p^k(\Gamma, \mathbb{H}(\mathbb{C})) \cap ker D_\alpha.$$

Some of the following properties of the operators $P_\alpha,\ Q_\alpha$ and S_α are immediate consequences of Theorem 3.1 and the Cauchy-Pompeiu formula.

THEOREM 3.3. *Let $f \in C^{0,\varepsilon}(\Gamma, \mathbb{H}(\mathbb{C})),\ 0 < \varepsilon \leq 1$. Then the equations*
(i) $(S_\alpha^2 u)(\tau) = u(\tau)$,
(ii) $(F_\alpha P_\alpha u)(\tau) = (F_\alpha u)(\tau)$,
(iii) $(P_\alpha^2 u)(\tau) = (P_\alpha u)(\tau)$,
(iv) $(Q_\alpha^2 u)(\tau) = (Q_\alpha u)(\tau)$,
are valid for any $\tau \in \Gamma$

We refer to [**60**, Chap. 2, Theorem 9] and [**46**, Corollary 4.2.7] for its proofs. The operator S_α is an involution on the space $C^{0,\varepsilon}(\Gamma, \mathbb{H}(\mathbb{C})),\ 0 < \varepsilon \leq 1$, hence, $P_\alpha,\ Q_\alpha$ are mutually complementary projection operators on the same space.

REMARK 3.4. From complex analysis it is well-known that if a function is the boundary value of a *holomorphic function* then the product of this function with a constant is also the boundary value of a *holomorphic function.* We will see throught the following example that this is not true for $\alpha-$*hyperholomorphic functions where* $\alpha \in \mathbb{C}$ in the quaternionic case with the vector quaternion constant.

EXAMPLE 3.5. For the sake of simplicity, we look at the case $\alpha = 0$. If u fullfills $S_0 u = u$ on $\mathbb{R}^2$, using definition (1.24) we get $u \in \mathrm{im} P_0$ then $S_0(e_3 u) = -e_3 u$ on $\mathbb{R}^2$. This is easily seen from

$$S_0(e_3 u)(x) \ = \ -2\frac{1}{4\pi}\int_{\mathbb{R}^2} K_0(x-y)\vec{n}(y)u(y)d\Gamma_y$$

$$
\begin{aligned}
&= -\frac{1}{2\pi}\int\limits_{\mathbb{R}^2} \frac{(x_1-y_1)e_1+(x_2-y_2)e_2}{|x-y|^3}e_3e_3u(y)dy \\
&= -e_3\left[-\frac{1}{2\pi}\int\limits_{\mathbb{R}^2} \frac{(x_1-y_1)e_1+(x_2-y_2)e_2}{|x-y|^3}e_3u(y)dy\right] \\
&= -e_3S_0(u)(x) = -e_3u(x) \in \mathrm{im}Q_0.
\end{aligned}
$$

From these above theorems together with definitions (1.24) we can easily see that any $f \in C^{0,\varepsilon}(\Gamma, \mathbb{H}(\mathbb{C}))$, $0 < \varepsilon \leq 1$, can be represented in a unique way as the sum $f = P_\alpha f + Q_\alpha f$. What then are these "parts" of f, $P_\alpha f$ and $Q_\alpha f$?. The following statement will answer this question.

Theorem 3.6. *Under the assumptions from Theorem 3.1 where Γ is the boundary of a finite domain Ω^+ and an infinite domain Ω^- the following assertions hold.*

(i) *f is the boundary value of a function F from $kerD_\alpha(\Omega^+, \mathbb{H}(\mathbb{C}))$ if and only if*

$$f \in imP_\alpha. \tag{3.3}$$

(ii) *Let f satisfy the condition $(1+\frac{ix}{|x|})f(x) = 0(\frac{1}{|x|})$. In order that f is the boundary value of a function F from $kerD_\alpha(\Omega^-, \mathbb{H}(\mathbb{C}))$, the following condition is necessary and sufficient*

$$f \in imQ_\alpha. \tag{3.4}$$

The readers can find its proof in [**60**, Theorem 10]. Because of Theorem 3.6 the subspace $\mathrm{im}P_\alpha \cap L_2(\Omega, \mathbb{H}(\mathbb{C}))$ is seen to be the set of all $\mathbb{H}(\mathbb{C}) - \alpha-$*regular functions extended* into the domain Ω^+, $\mathrm{im}Q_\alpha \cap L_2(\Omega, \mathbb{H}(\mathbb{C}))$is the set of all $\mathbb{H}(\mathbb{C}) - \alpha-$*regular functions extended* into the domain Ω^- and satistifying the condition $(1+\frac{ix}{|x|})f(x) = 0(\frac{1}{|x|})$. It is easily seen that, in particular, any $f \in C^{0,\varepsilon}(\Gamma, \mathbb{H}(\mathbb{C}))$, $0 < \varepsilon \leq 1$, itself is $\mathbb{H}(\mathbb{C}) - \alpha-$*regular extendable* in this sense into Ω^+, or Ω^- if and only if $Q_\alpha f = 0$ or $P_\alpha f = 0$ on Γ.

Remark 3.7. From the definition of P_α and Q_α, (see (1.24)), the condition (3.3) can be rewritten as

$$f(\tau) = S_\alpha f(\tau) \quad \text{for all } \tau \in \Gamma \tag{3.5}$$

From Theorem 3.6, we can see immediately that the solution of the problem

$$\begin{cases} D_\alpha u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma, \end{cases} \tag{3.6}$$

where $g \in C^{0,\varepsilon}(\Gamma, \mathbb{H}(\mathbb{C}))$ does not alway exist because not all functions g are $\alpha-$*extendable* into Ω^+, i.e., $Q_\alpha g \not\equiv 0$ on Γ. In the case of solvability, according to the Cauchy integral formula, the solution is given by

$$u = F_\alpha g. \tag{3.7}$$

We now come to our first application of the Cauchy-Pompeiu type integral representation of first order as well to solve problem (3.2) as also to give the necessary and sufficient condition for (3.2) to have a solution. These results are contained in Theorem 3.8 and Theorem 3.9.

THEOREM 3.8. *Let* $f \in W_2^k(\Omega, \mathbb{H}(\mathbb{C}))$. *The boundary value problem*

$$\begin{cases} D_\alpha u &= f \qquad \text{in } \Omega, \\ u &= o \qquad \text{on } \Gamma, \end{cases} \tag{3.8}$$

has a solution if and only if $tr_\Gamma Q_\alpha T_{\alpha,1} f = tr_\Gamma T_{\alpha,1} f$.
In the case of solvability the solution is given by $u = T_{\alpha,1} f \in W_2^{k+1}(\Omega, \mathbb{H}(\mathbb{C}))$.

PROOF. We begin with considering a new function $\omega := u - T_{\alpha,1} f$.
If u is a solution to the problem (3.8) then ω is a solution to the problem

$$\begin{cases} D_\alpha \omega &= 0 \qquad \text{in } \Omega, \\ \omega &= -tr_\Gamma T_{\alpha,1} f \qquad \text{on } \Gamma. \end{cases} \tag{3.9}$$

By Remark 3.7, the solution of this problem exists if and only if the function ω satisfies the condition

$$-tr_\Gamma T_{\alpha,1} f = S_\alpha(-tr_\Gamma T_{\alpha,1} f) \text{ on } \Gamma. \tag{3.10}$$

If this condition is fulfilled then the solution of the problem (3.9) is

$$\omega = F_\alpha(-tr_\Gamma T_{\alpha,1} f). \tag{3.11}$$

Furthermore, using the Cauchy-Pompeiu integral formula for $T_{\alpha,1} f$ and assertion (i) in Theorem 2.20, we obtain

$$F_\alpha(-tr_\Gamma T_{\alpha,1} f) = -[T_{\alpha,1} f - T_{\alpha,1}(D_\alpha T_{\alpha,1} f)] = 0. \tag{3.12}$$

It implies that

$$0 = \lim_{\Omega^+ \ni x \to \tau \in \Gamma} F_\alpha(-tr_\Gamma T_{\alpha,1} f)(x) = P_\alpha(-tr_\Gamma T_{\alpha,1} f)(\tau).$$

By the definition in (1.24) we have $\frac{1}{2}(I + S_\alpha)(-tr_\Gamma T_{\alpha,1} f) = 0$.
In other words, $-tr_\Gamma T_{\alpha,1} f = -S_\alpha(-tr_\Gamma T_{\alpha,1} f)$. Thus, from equality (3.10) we get

$$tr_\Gamma T_{\alpha,1} f = 0. \tag{3.13}$$

Since the equalities (3.11), (3.12) and (3.13) we get $\omega = 0$ is the solution to the problem (3.9). This means that $u = T_{\alpha,1} f$ is the solution to the problem (3.8).

Now we are looking at the condition (3.10), using the definition of Q_α in (1.24) then $tr_\Gamma Q_\alpha T_{\alpha,1} f = tr_\Gamma T_{\alpha,1} f$. The theorem is completely proved. □

The following theorem solves the boundary value problem (3.2).

THEOREM 3.9. *Let* $f \in W_2^k(\Omega, \mathbb{H}(\mathbb{C}))$, *and* $g \in W_2^{k+1/2}(\Gamma, \mathbb{H}(\mathbb{C}))$. *The boundary value problem (3.2) has a unique solution* $u \in W_2^{k+1}(\Omega, \mathbb{H}(\mathbb{C}))$ *if and only if* $tr_\Gamma Q_\alpha g = tr_\Gamma T_{\alpha,1} f$. *In the case of solvability the solution is given by* $u = F_\alpha g + T_{\alpha,1} f \in W_2^{k+1}(\Omega, \mathbb{H}(\mathbb{C}))$.

PROOF. We use the same idea as in the proof of the above theorem. We also consider the new function $\omega := u - T_{\alpha,1} f$. Then with the aid of Theorem 2.20, if u is a solution to the problem (3.2) then ω is a solution to the problem

$$\begin{cases} D_\alpha \omega &= 0 \qquad \text{in } \Omega, \\ \omega &= g - tr_\Gamma T_{\alpha,1} f \qquad \text{on } \Gamma. \end{cases} \tag{3.14}$$

By equality (3.5), the solution of the problem (3.14) exists if and only if $tr_\Gamma\omega = tr_\Gamma S_\alpha\omega$. It means that

$$u - T_{\alpha,1}f = S_\alpha u - S_\alpha T_{\alpha,1}f \quad \text{on } \Gamma. \tag{3.15}$$

If the condition (3.15) is fulfilled, using Remark 3.7 with the equality (3.7), (see also in [**60**, p. 38]), then the solution of problem (3.14) is $\omega = F_\alpha(g - tr_\Gamma T_{\alpha,1}f)$. Using the Cauchy-Pompeiu formula (1.22) for the function $T_{\alpha,1}f$ we get

$$\omega = F_\alpha g - F_\alpha T_{\alpha,1}f = F_\alpha g - (Id - T_{\alpha,1}D_\alpha)T_{\alpha,1}f = F_\alpha g.$$

As in the above theorem we obtain $F_\alpha T_{\alpha,1}f = 0$, hence, $P_\alpha T_{\alpha,1}f = 0$ on Γ, that is

$$T_{\alpha,1}f = -S_\alpha T_{\alpha,1}f \text{ on } \Gamma.$$

Thus, the condition (3.15) can be rewritten as

$$u - T_{\alpha,1}f = S_\alpha u + T_{\alpha,1}f \text{ on } \Gamma,$$

hence,

$$u - S_\alpha u = 2T_{\alpha,1}f \text{ on } \Gamma \text{ ,i.e., } Q_\alpha g = tr_\Gamma T_{\alpha,1}f.$$

Now, we return to problem (3.2). We obtain its solution in the form

$$u = \omega + T_{\alpha,1}f = F_\alpha g + T_{\alpha,1}f$$

under the necessary and sufficient condition

$$Q_\alpha g = tr_\Gamma T_{\alpha,1}f. \tag{3.16}$$

If a function satisfies $D_\alpha u = f$ and the condition (3.16), due to the Plemelj- Sokhotski formulas we obtain on the boundary the required equality

$$tr_\Gamma u = P_\alpha g + tr_\Gamma T_{\alpha,1}g = P_\alpha g + Q_\alpha g = g.$$

by the aid of the Cauchy- Pompeiu formula (1.22) □

We also refer the readers to [**21, 22, 78, 79, 93**] for other applications of Cauchy-Pompeiu type representations in investigating the classical Riemann problem for the equation $D_\alpha u = f$.

3. Orthogonal decomposition of the space $L_2(\Omega, \mathbb{H}(\mathbb{C}))$

One of the most interesting facts of complex and hypercomplex function theory is the orthogonal decomposition of the space $L_2(\Omega)$,

$$L_2(\Omega) = \ker D \cap L_2(\Omega) \oplus D(\overset{0}{W}{}_2^1(\Omega)) \tag{3.17}$$

where $kerD(\Omega)$ denotes the set of all *holomorphic* i.e of classical *monogenic* functions in Ω. This decomposition has a lot of applications, especially to the theory of partial differential equation, for example in [**31, 33**], to the Stokes system in [**32, 34**]. We also refer to [**57**] for extending the orthogonal decomposition (3.17) to the spaces $L_p(\Omega)$, $1 < p < \infty$.

In this section we investigate a decomposition of the space $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ with respect to $\alpha-$*hyperholomorphic functions.* The proof of this decomposition is based on the existence of an inner product in $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ and the properties of the boundary projections P_α and Q_α. It can be given in using the same idea as in [**46**, Theorem 3.1] (see also [**47, 82**]). However, we change some notations and trivial techniques in accordance with the study of boundary value problems for the Helmholtz equation in the next section. This proof will be presented for sake of preciseness and convenience.

THEOREM 3.10. *Let Ω be a symmetric domain relative to the origin. Then the Hilbert space $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ permits the orthogonal decomposition*

$$L_2(\Omega, \mathbb{H}(\mathbb{C})) = \ker D_\alpha \cap L_2(\Omega, \mathbb{H}(\mathbb{C})) \oplus D_\alpha(\overset{0}{W}{}_2^1(\Omega, \mathbb{H}(\mathbb{C})). \tag{3.18}$$

PROOF. We consider the right-linear subspac $X = L_2(\Omega, \mathbb{H}(\mathbb{C})) \ominus (\ker D_\alpha \cap L_2(\Omega, \mathbb{H}(\mathbb{C}))$.
1) First of all, we prove that

$$D_\alpha(\overset{0}{W}{}_2^1(\Omega, \mathbb{H}(\mathbb{C}))) \subset X. \tag{3.19}$$

Indeed, for all $\varphi \in \ker D_\alpha,\ i.e. D_\alpha\varphi = 0$, and any $\omega \in \overset{0}{W}{}_2^1(\Omega, \mathbb{H}(\mathbb{C}))$, which means that

$$\omega = 0 \text{ on } \Gamma, \text{ and } \omega \in W_2^1(\Omega, \mathbb{H}(\mathbb{C}),$$

not satisfying $\omega \in \ker D_\alpha$ the inner product

$$\begin{aligned}
< D_\alpha\omega, \varphi > &= \int_\Omega \overline{D_\alpha\omega(x)}\varphi(x)dx \\
&= -\int_\Omega D_{r,-\alpha}\overline{\omega(x)}\varphi(x)dx \\
&\overset{D_\alpha\varphi=0}{=} -\int_\Omega \Big(D_{r,-\alpha}\overline{\omega(x)}\varphi(x) + \overline{\omega(x)}D_\alpha\varphi(x)\Big)\,dx \\
&\overset{Stokes}{=} -\int_\Gamma \overline{\omega(x)}\vec{n}(x)\varphi(x)d\Gamma_x \\
&\overset{\omega|_\Gamma=0}{=} 0.
\end{aligned}$$

Thus, $D_\alpha\omega(x) \in X,\ i.e.$, inclusion (3.19) holds.
2) In the second step we will prove that

$$X \subset D_{-\alpha}(\overset{0}{W}{}_2^1(\Omega, \mathbb{H}(\mathbb{C})). \tag{3.20}$$

Actually, let $u \in X$ then we can take for instance $\upsilon = T_{\alpha,1}u$. By assertion (i) of Theorem 2.20 together with Theorem 2.17 we have

$$D_\alpha\upsilon(x) = u(x),\ x \in \Omega \text{ and } \upsilon \in W_2^1(\Omega, \mathbb{H}(\mathbb{C})).$$

For any $\varphi \in \ker D_\alpha \cap L_2(\Omega, \mathbb{H}(\mathbb{C}))$, because $u \in X$ then we obtain

$$\begin{aligned}
0 =< u, \varphi > &= \int_\Omega \overline{u(x)}\varphi(x)dx = \int_\Omega \overline{D_\alpha\upsilon(x)}\varphi(x)dx \\
&= -\int_\Omega D_{r,-\alpha}\overline{\upsilon(x)}\varphi(x)dx \\
&\overset{D_\alpha\varphi=0}{=} -\int_\Omega \Big(D_{r,-\alpha}\overline{\upsilon(x)}\varphi(x) + \overline{\upsilon(x)}D_\alpha\varphi(x)\Big)\,dx
\end{aligned}$$

$$\overset{Stokes}{=} -\int_\Gamma \overline{\upsilon(x)}\vec{n}(x)\varphi(x)d\Gamma_x$$
$$= \overline{\int_\Gamma \overline{\varphi(x)}\vec{n}(x)\upsilon(x)d\Gamma_x}$$

This for the sake of convenience we can rewrite

$$\int_\Gamma \overline{\varphi(y)}\vec{n}(y)\upsilon(y)d\Gamma_y = 0. \tag{3.21}$$

In $\mathbb{R}^3 - \Omega$ we choose an arbitrary x. We get

$$\varphi(y) = K_\alpha^{(1)}(x-y) \text{ then } \overline{\varphi(y)} = K_\alpha^{(1)}(-x-(-y)),$$

and $\vec{n}(-y) = -\vec{n}(y)$, $\upsilon_1 \in W_2^1(\Omega, \mathbb{H}(\mathbb{C}))$ with $\upsilon_1(-y) = \upsilon(y)$, $y \in \Omega$ by using the assumtion that Ω is a domain symmetric with respect to the origin. Therefore, without loss of generality equality (3.21) can be written as

$$\int_\Gamma K_\alpha^{(1)}(-x+y)\vec{n}(-y)\upsilon(-y)d\Gamma_y = 0.$$

If we put $\tilde{\upsilon}(x) = \upsilon(-x)$ and observe $\vec{n}(-y) = -\vec{n}(y)$, then we obtain $F_\alpha\tilde{\upsilon}(x) = 0$, x is an arbitrary point outside Ω. Hence, $F_\alpha\tilde{\upsilon} \equiv 0$ in Ω^-. Using Plemelj- Sokhotski formulas we can easily see that $tr_\Gamma\tilde{\upsilon} = P_\alpha[tr_\Gamma\tilde{\upsilon}]$. Thus, $tr_\Gamma\tilde{\upsilon} \in \text{im } P_\alpha$. Together with Proposition 3.2 we get

$$tr_\Gamma\tilde{\upsilon} \in \text{im } P_\alpha \cap W_2^{1/2}(\Omega, \mathbb{H}(\mathbb{C})).$$

We can see that due to Theorem 3.6, $tr_\Gamma\tilde{\upsilon}$ is the boundary value of a function h from $\ker D_\alpha(\Omega^+, \mathbb{H}(\mathbb{C}))$. It means that

$$\begin{cases} D_\alpha h &= 0 \qquad \text{in } \Omega, \\ tr_\Gamma h &= tr_\Gamma\tilde{\upsilon} \qquad \text{on}\Gamma. \end{cases}$$

Taking the function $\omega = \tilde{\upsilon} - h$ then $\omega \in \overset{0}{W}{}_2^1(\Omega, \mathbb{H}(\mathbb{C}))$ and $D_\alpha\omega = D_\alpha\tilde{\upsilon}$. Moreover, under the assumtion that Ω is a symmetric domain relative to the origin we obtain

$$D_{-\alpha}(-\tilde{\upsilon}(x)) = D_\alpha(-\tilde{\upsilon}(-x)) = D_\alpha(-\upsilon(x)) = -u(x).$$

It concludes $u \in D_{-\alpha}(\overset{0}{W}{}_2^1(\Omega, \mathbb{H}(\mathbb{C})))$. This yields (3.20).

3)Combining (3.19) and (3.20) and we obtain

$$D_\alpha(\overset{0}{W}{}_2^1(\Omega, \mathbb{H}(\mathbb{C}))) \subset D_{-\alpha}(\overset{0}{W}{}_2^1(\Omega, \mathbb{H}(\mathbb{C}))).$$

If we replace α instead of $-\alpha$ we get

$$D_{-\alpha}(\overset{0}{W}{}_2^1(\Omega, \mathbb{H}(\mathbb{C}))) \subset D_\alpha(\overset{0}{W}{}_2^1(\Omega, \mathbb{H}(\mathbb{C}))).$$

Now we can conclude that

$$D_{-\alpha}(\overset{0}{W}{}_2^1(\Omega, \mathbb{H}(\mathbb{C}))) = D_\alpha(\overset{0}{W}{}_2^1(\Omega, \mathbb{H}(\mathbb{C}))).$$

Together with (3.19), (3.20), we get (3.18). $\square$

Remark 3.11. (i) Due to the validity of this theorem we instantly have the existence of the orthoprojections P_α and Q_α with

$$\begin{aligned} P_\alpha: \ L_2(\Omega,\mathbb{H}(\mathbb{C}))) &\longrightarrow \ \ker D_\alpha \cap L_2(\Omega,\mathbb{H}(\mathbb{C})),\\ Q_\alpha: \ L_2(\Omega,\mathbb{H}(\mathbb{C}))) &\longrightarrow \ D_{-\alpha}(\overset{0}{W}{}_2^1(\Omega,\mathbb{H}(\mathbb{C}))\cap L_2(\Omega,\mathbb{H}(\mathbb{C})). \end{aligned}$$

(ii) Since $\mathrm{im}P_\alpha \subset \ker D_\alpha \subset C^\infty(\Omega,\mathbb{H}(\mathbb{C}))$, we have $D_\alpha Q_\alpha u = D_\alpha(u - P_\alpha u) = D_\alpha u$. This means that, for any $u \in W_2^1(\Omega,\mathbb{H}(\mathbb{C}))$ then the differentiation rule

$$D_\alpha Q_\alpha u = D_\alpha u \tag{3.22}$$

holds.

4. Applications to boundary value problems of the Helmholtz equation

The result of decompositions of the complex quaternion valued Hilbert space in the formerly section is useful. It opens the door for the consideration of boundary value problems of partial differential equations.

This section is concerned about the existence and regularity of the solution to the Dirichlet problem for the Helmholtz equation. This means that the close connection of the decomposition (3.18) with the boundary value problem (3.1) is outlined. Further more, the operator $tr_\Gamma T_{-\alpha} F_\alpha$ is investigated. These results are used later to construct more explicitly the representation formulas of the orthoprojections P_α and Q_α. Finally, by the help of these representations we can get representations for the solution to the boundary valued problem which make use only of the boundary data and of the right-hand side of the differential equation without unknown extension functions.

In the whole section we use the result on the extensions of vector functions defined on the boundary of a bounded domain. We suppose that Ω is a domain with a sufficiently smooth boundary. For each function $g \in W_2^{k+3/2}(\Gamma,\mathbb{H}(\mathbb{C}))$ there exists an extension $h \in W_2^{k+2}(\Omega,\mathbb{H}(\mathbb{C}))$ with $tr_\Gamma h = g$. This is an existence theorem about the existence of the function h defined in $\overline{\Omega}$ with the prescribed regularity and the property that the restriction to the boundary values reproduces again the given function g. However, in general it is not constructive and also says nothing about the extension function being the solution of some differential equation. Of course, the extension is not unique. We refer the readers to [**68**] for more information. We use in this section only an existence result about the extensions. Its proof shows the fact that the regularity of extensions and the smoothness of the boundary are connected.

Now we are looking for the solution of the problem

$$\begin{cases} \Delta u + \alpha^2 u &= f \qquad \text{in } \Omega,\\ u &= 0 \qquad \text{on } \Gamma. \end{cases} \tag{3.23}$$

Theorem 3.12. *Let $f \in W_2^k(\Omega,\mathbb{H}(\mathbb{C}))$. The Dirichlet's problem (3.23) has a solution $u \in W_2^{k+2,loc}(\Omega,\mathbb{H}(\mathbb{C}))$ which may be represented by the formula $u = -T_{-\alpha,1}Q_\alpha T_{\alpha,1} f$.*

Proof. For $f \in W_2^k(\Omega,\mathbb{H}(\mathbb{C}))$, by Remark 2.21 we have $T_{\alpha,1} f \in W_2^{k+1}(\Omega,\mathbb{H}(\mathbb{C}))$. The validity of Theorem 3.10 as well as its proof and Remark 3.11 shows that there exists an $\mathbb{H}(\mathbb{C})-$valued function $u \in \overset{0}{W}{}_2^{k+1}(\Omega,\mathbb{H}(\mathbb{C})$ with $-D_{-\alpha} u = Q_\alpha(T_{\alpha,1} f)$.

On the orther hand, with the aid of the Cauchy-Pompeiu formula (1.23) as well as Theorem 3.8 and noticing that u vanishes on Γ, we get

$$u = T_{-\alpha,1}D_{-\alpha}u = -T_{-\alpha,1}(Q_\alpha(T_{\alpha,1}f)) \text{ and } u \in W_2^{k+2,loc}(\Omega, \mathbb{H}(\mathbb{C})).$$

Hence,

$$(\Delta + \alpha^2)u = D_\alpha(-D_{-\alpha}u) = D_\alpha\{-D_{-\alpha}[-T_{-\alpha,1}(Q_\alpha(T_{\alpha,1}f))]\}. \tag{3.24}$$

Using assertion (i) in Theorem 2.20 we have $D_{-\alpha}T_{-\alpha,1} = Id$. Therefore, by Remark 3.11 $\mathrm{im}P_\alpha \subset \ker D_\alpha$ and the definitions of $P_\alpha,\ Q_\alpha$ in (1.24), the equality (3.24) can be rewritten as

$$\begin{aligned}(\Delta + \alpha^2)u = D_\alpha[Q_\alpha(T_{\alpha,1}f)] \;&=\; D_\alpha[T_{\alpha,1}f - P_\alpha(T_{\alpha,1}f)]\\ &=\; D_\alpha T_{\alpha,1}f \;=\; f.\end{aligned}$$

The theorem is completely proved. □

THEOREM 3.13. *Let $g \in W_2^{k+3/2}(\Gamma, \mathbb{H}(\mathbb{C})),\ k \geq 0$. The first boundary value problem*

$$\begin{cases}\Delta u + \alpha^2 u &= 0 \qquad in\ \Omega,\\ u &= g \qquad on\ \Gamma,\end{cases} \tag{3.25}$$

has a solution $u \in W_2^{k+2,loc}(\Omega, \mathbb{H}(\mathbb{C}))$ of the form

$$u = F_{-\alpha}g + T_{-\alpha,1}P_\alpha D_{-\alpha}h,$$

where h is a $W_2^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))-$extension of g.

PROOF. For $g \in W_2^{k+3/2}(\Gamma, \mathbb{H}(\mathbb{C}))$ there exists a $W_2^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))-$extension with $tr_\Gamma h = g$. With $u = \upsilon + h$ the boundary value problem (3.25) will be transformed into

$$\begin{cases}(\Delta + \alpha^2)\upsilon &= -(\Delta + \alpha^2)h \qquad \text{in } \Omega,\\ \upsilon &= 0 \qquad \text{on } \Gamma.\end{cases}$$

Applying Theorem 3.12 to the above boundary value problem we see that its solution can be represented as

$$\upsilon = -T_{-\alpha,1}Q_\alpha T_{\alpha,1}[-(\Delta + \alpha^2)h] \tag{3.26}$$

Noticing that $(\Delta + \alpha^2) = -D_\alpha D_{-\alpha}$, using the Cauchy-Pompeiu formula for $(-D_{-\alpha}h)$, we have

$$-D_{-\alpha}h = F_\alpha(-D_{-\alpha}h) + T_{\alpha,1}D_\alpha(-D_{-\alpha}h).$$

Inserting this equality into (3.26) yields

$$\upsilon = T_{-\alpha,1}Q_\alpha F_\alpha D_{-\alpha}h - T_{-\alpha,1}Q_\alpha D_{-\alpha}h = -T_{-\alpha,1}Q_\alpha D_{-\alpha}h,$$

as $Q_\alpha[F_\alpha D_{-\alpha}h] = 0$ because of the properties of orthoprojections $Q_\alpha P_\alpha = 0$, *i.e.*, $\mathrm{im}P_\alpha \subset \ker Q_\alpha$ and F_α maps into $\mathrm{im}P_\alpha$.

Using the definition (1.24), $Q_\alpha = Id - P_\alpha$ and the Cauchy-Pompeiu (1.23) for $D_{-\alpha}h$ i.e., $h = F_{-\alpha}h + T_{-\alpha,1}D_{-\alpha}h$ we have

$$\begin{aligned}\upsilon = -T_{-\alpha,1}Q_\alpha D_{-\alpha}h \;&=\; -T_{-\alpha,1}D_{-\alpha}h + T_{-\alpha,1}P_\alpha D_{-\alpha}h\\ &=\; -h + F_{-\alpha}h + T_{-\alpha,1}P_\alpha D_{-\alpha}h.\end{aligned}$$

Puting $u = \upsilon + h$ and noticing that $tr_\Gamma h = g$ then

$$u = F_{-\alpha}g + T_{-\alpha,1}P_\alpha D_{-\alpha}h \in W_2^{k+2,loc}(\Omega, \mathbb{H}(\mathbb{C})).$$

□

Now we can conclude the existence and uniqueness of the solution to problem (3.1) by the following theorem.

THEOREM 3.14. *Let* $f \in W_2^k(\Omega, \mathbb{H}(\mathbb{C}))$, $g \in W_2^{k+3/2}(\Gamma, \mathbb{H}(\mathbb{C}))$, *and suppose* α^2 *is not an eigenvalue of* $\{-\delta, tr\}$. *Then the Dirichlet problem (3.1) has the unique solution*

$$u = F_{-\alpha} g + T_{-\alpha,1} P_\alpha D_{-\alpha} h - T_{-\alpha,1} Q_\alpha T_{\alpha,1} f$$

belonging to $W_2^{k+2,loc}(\Omega, \mathbb{H}(\mathbb{C}))$ *where* h *denotes a* $W_2^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))-$*extension of* g.

PROOF. 1) The existence of the solutions.

By Theorem 3.12 and 3.13 the sum of the solutions of the boundary value problems (3.23) and (3.25) solves problem (3.1). Its solution can be represented as

$$u = F_{-\alpha} g + T_{-\alpha,1} P_\alpha D_{-\alpha} h - T_{-\alpha,1} Q_\alpha T_{\alpha,1} f \in W_2^{k+2,loc}(\Omega, \mathbb{H}(\mathbb{C})).$$

2) The uniqueness of the solution.

To show this, we only have to prove the uniqueness of the solution to the problem

$$\begin{cases} (\Delta + \alpha^2) u &= 0 \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \Gamma. \end{cases}$$

Indeed, if u is a solution to the above problem then using Theorem 3.8 we see that its solution can be represented as $u = T_{-\alpha,1} D_{-\alpha} u$.
By Remark 3.11 and $tr_\Gamma u = 0$ then $D_{-\alpha} u \in \text{ im} Q_\alpha$. It means that there exits a function ω such that $Q_\alpha \omega = D_{-\alpha} u$. From the properties of Q_α in Theorem 3.3 we have

$$Q_\alpha (D_{-\alpha} u) = Q_\alpha^2 \omega = Q_\alpha \omega = D_{-\alpha} u, \ i.e., \ D_{-\alpha} u = Q_\alpha (D_{-\alpha} u).$$

This together with $tr_\Gamma u = 0$, and then using Theorem 3.8 again we can easily see that $u = T_{-\alpha,1} Q_\alpha (D_{-\alpha} u)$. Hence, $u = T_{-\alpha,1} D_{-\alpha} u = T_{-\alpha,1} Q_\alpha (D_{-\alpha} u)$.
On the orther hand, $D_{-\alpha} u \in \ker D_\alpha$, which means that $D_{-\alpha} u \in \text{im} P_\alpha$. Repeating the above process again for P_α instead of Q_α with using assertion (iii) of Theorem 3.3 we also get

$$D_{-\alpha} u = P_\alpha (D_{-\alpha} u) \text{ and } u = T_{-\alpha,1} P_\alpha (D_{-\alpha} u).$$

Now, we can conclude

$$u = T_{-\alpha,1} D_{-\alpha} u = T_{-\alpha,1} Q_\alpha (D_{-\alpha} u) = T_{-\alpha,1} P_\alpha (D_{-\alpha} u).$$

This equality, together with $P_\alpha + Q_\alpha = Id$ leads to $T_{-\alpha,1} D_{-\alpha} u = 0$. Using the Cauchy-Pompeiu formula (1.22) and noticing that $tr_\Gamma u = 0$ we get $u = 0$. Thus, the uniqueness of the solution follows. □

We now intend to investigate some properties of $tr_\Gamma T_{-\alpha,1} F_\alpha$. There exist spaces where it becomes an isomorphism. These results will lend assistance aid to construct more explicitly the representation formulas of the orthoprojections P_α and Q_α. For this purpose, let us now analyze in more detail the subspace $\text{im} Q_\alpha$.

PROPOSITION 3.15. *We have* $tr_\Gamma T_{-\alpha,1} u = 0$ *if and only if* $u \in imQ_\alpha$.

PROOF. 1) Let $u \in \mathrm{im}Q_\alpha$ then there exists a function $\omega \in L_2(\Omega, \mathbb{H}(\mathbb{C}))$ such that $Q_\alpha \omega = u,\ u \in W_2^1(\Omega, \mathbb{H}(\mathbb{C}))$. Since assertion (iv) of Theorem 3.3 we get $Q_\alpha u = Q_\alpha^2 \omega = Q_\alpha \omega = u$. Hence, $tr_\Gamma T_{-\alpha,1} u = tr_\Gamma T_{-\alpha,1} Q_\alpha u$.

On the orther hand, $u \in \mathrm{im}Q_\alpha \subset D_{-\alpha}(\overset{0}{W}{}_2^1(\Omega, \mathbb{H}(\mathbb{C})))$. Therefore there exists a function υ in $W_2^1(\Omega, \mathbb{H}(\mathbb{C}))$ such that

$$\begin{cases} D_{-\alpha}\upsilon &= Q_\alpha u \qquad \text{in } \Omega, \\ \upsilon &= 0 \qquad\quad\ \ \text{on } \Gamma. \end{cases}$$

By Theorem 3.8 we get its solution $\upsilon = T_{-\alpha,1} Q_\alpha u$ satistifying $tr_\Gamma \upsilon = tr_\Gamma T_{-\alpha,1} Q_\alpha u$. Therefore, $tr_\Gamma T_{-\alpha,1} u = 0$.

2) Suppose that $tr_\Gamma T_{-\alpha,1} u = 0$. By the definitions of P_α and Q_α as well as $\mathrm{im}P_\alpha \subset \ker D_\alpha$ then the function u can be decomposed into the sum $u = u_1 + u_2$, where $u_1 \in \ker D_\alpha$ and $u_2 \in \mathrm{im}Q_\alpha$. Under the assumtion we get

$$tr_\Gamma T_{-\alpha,1} u_1 + tr_\Gamma T_{-\alpha,1} u_2 = 0.$$

It is clear that the first term $T_{-\alpha,1} u_1$ belongs to $\ker(\Delta + \alpha^2)$ and the second term, by using the above discussions with $u_2 \in \mathrm{im}Q_\alpha$ then $tr_\Gamma T_{-\alpha,1} Q_\alpha u_2 = tr_\Gamma T_{-\alpha,1} u_2 = 0$.
Thus, we have $tr_\Gamma T_{-\alpha,1} u_1 = 0$. Therefore, $T_{-\alpha,1} u_1$ is a solution of the boundary value problem

$$\begin{cases} (\Delta + \alpha^2) T_{-\alpha,1} u_1 &= 0 \qquad \text{in } \Omega, \\ T_{-\alpha,1} u_1 &= 0 \qquad \text{on } \Gamma. \end{cases}$$

From Theorem 3.14 we can conclude that the only solution of this problem is $T_{-\alpha,1} u_1 \equiv 0$, whence $u_1 \equiv 0$. Hence, we get $u = u_2 \in \mathrm{im}Q_\alpha$. □

Next the operator $T_{-\alpha,1} F_\alpha$ will be investigated.

PROPOSITION 3.16. *The formula*

$$\ker\ tr_\Gamma T_{-\alpha,1} F_\alpha \cap (W_2^{1/2}(\Gamma, \mathbb{H}(\mathbb{C})) \cap imP_\alpha) = \{0\}$$

is valid.

PROOF. Let be $u \in \ker\ tr_\Gamma T_{-\alpha,1} F_\alpha$, *i.e.*, $tr_\Gamma T_{-\alpha,1} F_\alpha u = 0$. Now, we have to show that if $u \in (W_2^{1/2}(\Gamma, \mathbb{H}(\mathbb{C})) \cap imP_\alpha)$ then $u \equiv 0$.

Indeed, from the assumtion $tr_\Gamma T_{-\alpha,1} F_\alpha u = 0$, using Proposition 3.15 we get $F_\alpha u \in \mathrm{im}Q_\alpha$. Since Remark 3.11, $\mathrm{im}Q_\alpha \subset D_{-\alpha}(\overset{0}{W}{}_2^1(\Omega, \mathbb{H}(\mathbb{C})))$, leads to the existence of the functions $\omega \in \overset{0}{W}{}_2^1(\Omega, \mathbb{H}(\mathbb{C}))$ satistifying $D_{-\alpha}\omega = F_\alpha u$.
Apparently, $F_\alpha u \in \ker\ D_\alpha$. By Proposition 3.2, we get

$$\begin{cases} (\Delta + \alpha^2)\omega &= D_\alpha F_\alpha u = 0 \quad \text{in } \Omega, \\ \omega &= 0 \qquad\qquad\quad \text{on } \Gamma. \end{cases}$$

Using the uniqueness result of Theorem 3.14 now yields that $\omega = 0$. Hence, we may conclude that also $D_{-\alpha}\omega = 0$, whence $F_\alpha u = 0$. Moreover, with $tr_\Gamma T_{-\alpha,1} F_\alpha u = 0$ and $u \in \mathrm{im}P_\alpha, i.e., u \in \ker D_\alpha$ then $F_\alpha u = 0$, whence $u = 0$. The Proposition 3.16 is completely proved. □

THEOREM 3.17. *The operator*

$$tr_\Gamma T_{-\alpha,1}F_\alpha \ : \ imP_\alpha \cap W_2^{k+1/2}(\Gamma,\mathbb{H}(\mathbb{C})) \ \longrightarrow \ im\ Q_{-\alpha}\cap W_2^{k+3/2}(\Gamma,\mathbb{H}(\mathbb{C}))$$

is an isomorphism.

PROOF. 1) $tr_\Gamma T_{-\alpha,1}F_\alpha$ is injective.
By the aid of the properties of $T_{-\alpha,1}$ (see Remark 2.21) and the Proposition 3.2 together with the trace theorem we obtain

$$tr_\Gamma T_{-\alpha,1}F_\alpha(W_2^{k+1/2}(\Gamma,\mathbb{H}(\mathbb{C})) \ \subset \ W_2^{k+3/2}(\Gamma,\mathbb{H}(\mathbb{C})).$$

Now, let $\omega \in \mathrm{im}P_\alpha \cap W_2^{k+1/2}(\Gamma,\mathbb{H}(\mathbb{C}))$ such that $tr_\Gamma T_{-\alpha,1}F_\alpha(\omega)=0$. By Proposition 3.16 we get $\omega=0$. Thus, $tr_\Gamma T_{-\alpha,1}F_\alpha$ is injective.

2) $tr_\Gamma T_{-\alpha,1}F_\alpha$ is a surjective mapping.
Let ω be an arbitrary element from im $Q_{-\alpha}\cap W_2^{k+3/2}(\Gamma,\mathbb{H}(\mathbb{C}))$. This function can be written as $\omega = Q_{-\alpha}g$ for some function g defined on Γ, also $g\in W_2^{k+3/2}(\Gamma,\mathbb{H}(\mathbb{C}))$. Now, we are looking for some function $\upsilon \in W_2^{k+2}(\Gamma,\mathbb{H}(\mathbb{C}))$ satisfying $\upsilon=\omega$ on Γ.

Using Theorem 3.9, this problem has a solution $\upsilon = F_{-\alpha}\omega + T_{-\alpha,1}D_{-\alpha}\upsilon$ under the condition $Q_{-\alpha}\omega = tr_\Gamma T_{-\alpha,1}D_{-\alpha}\upsilon$. By properties of $Q_{-\alpha}$ we get

$$Q_{-\alpha}\omega = Q^2_{-\alpha}g = Q_{-\alpha}g=\omega.$$

On the orther hand, since the Plemelj-Sokhotzkij's formula and assertion (ii) of Theorem 3.3 we have $F_{-\alpha}\omega=0$. Thus, $tr_\Gamma T_{-\alpha,1}D_{-\alpha}\upsilon=0$.
Now, one can find $u=D_{-\alpha}\upsilon$ such that $u\in\ker D_\alpha$, and satistifying $tr_\Gamma u = tr_\Gamma D_{-\alpha}\upsilon$. Its solution is $u=F_\alpha(tr_\Gamma D_{-\alpha}\upsilon)$ if and only if $tr_\Gamma D_{-\alpha}\upsilon \in$ im P_α, *i.e.*, $tr_\Gamma u\in$ im P_α.
This means that, the continuous function υ is always choosable in such a way that

$$\begin{cases}(\Delta+\alpha^2)\upsilon &= 0 \qquad \text{in } \Omega,\\ \upsilon &= \omega \qquad \text{on } \Gamma,\end{cases}$$

and with $u = tr_\Gamma D_{-\alpha}\upsilon \in$ im $P_\alpha \cap W_2^{k+1/2}(\Gamma,\mathbb{H}(\mathbb{C}))$. □

Now, we come to clarify the structure of the orthogonal projections P_α and Q_α.

THEOREM 3.18. *Let $k\geq 1$. Then for $u\in W_2^k(\Omega,\mathbb{H}(\mathbb{C}))$ we have $P_\alpha u$, $Q_\alpha u \in W_2^k(\Omega,\mathbb{H}(\mathbb{C}))$. Furthermore, the orthoprojections P_α and Q_α allow the representations*

$$\begin{aligned} P_\alpha &= F_\alpha(tr_\Gamma T_{-\alpha,1}F_\alpha)^{-1}tr_\Gamma T_{-\alpha,1}\\ Q_\alpha &= Id - F_\alpha(tr_\Gamma T_{-\alpha,1}F_\alpha)^{-1}tr_\Gamma T_{-\alpha,1}. \end{aligned} \tag{3.27}$$

PROOF. Under the assumtion $u\in W_2^k(\Omega,\mathbb{H}(\mathbb{C}))$, Remark 2.21 immediately follows that $T_{-\alpha,1}u\in W_2^{k+1}(\Omega,\mathbb{H}(\mathbb{C}))$. By the trace theorem we get $tr_\Gamma T_{-\alpha,1}u \in W_2^{k+1/2}(\Gamma,\mathbb{H}(\mathbb{C}))$. Theorem 3.17 leads to

$$(tr_\Gamma T_{-\alpha,1}F_\alpha)^{-1}tr_\Gamma T_{-\alpha,1}u\in W_2^{k-1/2}(\Gamma,\mathbb{H}(\mathbb{C})).$$

Since Proposition 3.2 we get

$$F_\alpha(tr_\Gamma T_{-\alpha,1}F_\alpha)^{-1}tr_\Gamma T_{-\alpha,1}u \in W_2^k(\Omega,\mathbb{H}(\mathbb{C}))\cap \ker D_\alpha.$$

For the sake of brevity we put

$$\tilde{P}_\alpha = F_\alpha(tr_\Gamma T_{-\alpha,1}F_\alpha)^{-1}tr_\Gamma T_{-\alpha,1},\ \tilde{Q}_\alpha = Id-\tilde{P}_\alpha.$$

A straightforward calculation delivers $\tilde{P}_\alpha^2 = \tilde{P}_\alpha$. Obviously, $(Id - \tilde{P}_\alpha)^2 = (Id - \tilde{P}_\alpha)$ and $(Id - \tilde{P}_\alpha)\tilde{P}_\alpha = \tilde{P}_\alpha(Id - \tilde{P}_\alpha) = 0$. Furthermore, we consider

$$\begin{aligned}\tilde{Q}_\alpha u = (Id - \tilde{P}_\alpha)u &= u - F_\alpha(tr_\Gamma T_{-\alpha,1}F_\alpha)^{-1}tr_\Gamma T_{-\alpha,1}u \\ &\overset{Theorem 2.20}{=} D_{-\alpha}T_{-\alpha,1}u - D_{-\alpha}T_{-\alpha,1}F_\alpha(tr_\Gamma T_{-\alpha,1}F_\alpha)^{-1}tr_\Gamma T_{-\alpha,1}u \\ &= D_{-\alpha}\omega\end{aligned}$$

with $\omega = T_{-\alpha,1}u - T_{-\alpha,1}F_\alpha(tr_\Gamma T_{-\alpha,1}F_\alpha)^{-1}tr_\Gamma T_{-\alpha,1}u$. Obviously, $tr_\Gamma\omega = 0$, and $\omega \in W_2^{k+1}(\Omega, \mathbb{H}(\mathbb{C}))$.

Hence, $\tilde{Q}_\alpha u \in D_{-\alpha}(\overset{0}{W}{}_2^{k+1}(\Omega, \mathbb{H}(\mathbb{C})))$. For the uniqueness of the orthoprojections we obtain $\tilde{P}_\alpha = P_\alpha$, and $\tilde{Q}_\alpha = Q_\alpha$. □

By the special structure of P_α in formula (3.27) the following Corollary holds.

COROLLARY 3.19. *Each solution u of the problem (3.1) belongs to $W_2^k(\Omega, \mathbb{H}(\mathbb{C}))$.*

The result of this Corollary can be generalized to $W_p^k(\Omega, \mathbb{H}(\mathbb{C}))-$spaces. Therefore, we conclude this section by the result about the regularity of the solution of problem (3.1).

THEOREM 3.20. *Suppose $f \in W_p^k(\Omega, \mathbb{H}(\mathbb{C}))$, $g \in W_p^{k+2-1/p}(\Gamma, \mathbb{H}(\mathbb{C}))$, with $k \geq 0$, $1 < p < \infty$, then the boundary value problem (3.1) has the unique solution*

$$u = F_{-\alpha}g + T_{-\alpha,1}P_\alpha D_{-\alpha}h - T_{-\alpha,1}Q_\alpha T_{\alpha,1}f$$

belonging to $W_p^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))$ where h denotes a $W_2^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))-$extension of g.

We now come to our last result of this chapter and obtain the representation of the unique of solution of a boundary value problem of the higher order Helmholtz equations.

5. Boundary value problem of the higher order Helmholtz equations

In this section, we investigate more general boundary value problems of the higher order Helmholtz equations.

$$\begin{cases}(\Delta + \alpha^2)^n u &= f \quad \text{in } \Omega, \\ u &= g_0 \quad \text{on } \Gamma, \\ (\Delta + \alpha^2)u &= g_1 \quad \text{on } \Gamma, \ \ldots, \\ \ldots & \ldots \\ (\Delta + \alpha^2)^{n-1}u &= g_{n-1} \quad \text{on } \Gamma.\end{cases} \tag{3.28}$$

This type of equations can be to reduced to the first order Helmholtz equation. Firstly, combining Theorem 3.14, Theorem 3.18 and Theorem 3.20 we compose them into the following theorem.

THEOREM 3.21. *Under the assumtion of the Theorem 3.20, the boundary value problem (3.1) has the unique solution*

$$u = F_{-\alpha}g + T_{-\alpha,1}F_\alpha(tr_\Gamma T_{-\alpha,1}F_\alpha)^{-1}Q_{-\alpha}g - T_{-\alpha,1}Q_\alpha T_{\alpha,1}f.$$

belonging to $W_p^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))$

PROOF. The uniqueness and the existence of the solution were shown in Theorem 3.14. Therefore, it is necessary to show that u permits the above mentioned representation. For this reason we have only to prove that the represention of u satisfies the patial differential equation (3.1). First of all, we now look at the following problem where h denotes a $W_2^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))-$extension of g

$$\begin{cases} D_{-\alpha}h & = D_{-\alpha}h & \text{in } \Omega, \\ h & = g & \text{on } \Gamma. \end{cases}$$

By Theorem 3.9 we get

$$Q_{-\alpha}g = tr_\Gamma T_{-\alpha,1} D_{-\alpha}h.$$

Hence, by Therem 3.14 the solution to problem (3.1) can be rewitten as

$$u = F_{-\alpha}g + T_{-\alpha,1}F_\alpha(tr_\Gamma T_{-\alpha,1}F_\alpha)^{-1}Q_{-\alpha}g - T_{-\alpha,1}Q_\alpha T_{\alpha,1}f.$$

On the contrary, using assertion (i) of Theorem 2.20 and Proposition 3.2 we have

$$D_{-\alpha}u = F_\alpha(tr_\Gamma T_{-\alpha,1}F_\alpha)^{-1}Q_{-\alpha}g - Q_\alpha T_{\alpha,1}f.$$

Using Proposition 3.2 and assertion (i) of Theorem 2.20 again together with equality (3.22) yield $(\Delta + \alpha^2)u = -D_\alpha D_{-\alpha}u = f$.
For the boundary condition, as in the discussions in Theorem 3.12 and Theorem 3.13 we compute

$$tr_\Gamma T_{-\alpha,1}Q_\alpha T_{\alpha,1}f = 0,$$

and

$$tr_\Gamma F_{-\alpha}g + tr_\Gamma T_{-\alpha,1}F_\alpha(tr_\Gamma T_{-\alpha,1}F_\alpha)^{-1}Q_{-\alpha}g = P_\alpha g + Q_\alpha g = g.$$

□

We now look at the following problem

$$\begin{cases} (\Delta+\alpha^2)^2 u & = f & \text{in } \Omega, \\ u & = g_0 & \text{on } \Gamma, \\ (\Delta+\alpha^2)u & = g_1 & \text{on } \Gamma, \end{cases} \tag{3.29}$$

where $f \in L_2(\Omega, \mathbb{H}(\mathbb{C}))$, $g_0 \in W_2^{7/2}(\Gamma, \mathbb{H}(\mathbb{C}))$, $g_1 \in W_2^{3/2}(\Gamma, \mathbb{H}(\mathbb{C}))$.

Firstly, $\upsilon = (\Delta + \alpha^2)u$ as the unique solution of the problem

$$\begin{cases} (\Delta+\alpha^2)\upsilon & = f & \text{in } \Omega, \\ \upsilon & = g_1 & \text{on } \Gamma, \end{cases}$$

is

$$\upsilon = F_{-\alpha}g_1 + T_{-\alpha,1}F_\alpha(tr_\Gamma T_{-\alpha,1}F_\alpha)^{-1}Q_{-\alpha}g_1 - T_{-\alpha,1}Q_\alpha T_{\alpha,1}f.$$

Next, the unique solution of the problem

$$\begin{cases} (\Delta+\alpha^2)u & = \upsilon & \text{in } \Omega, \\ u & = g_0 & \text{on } \Gamma, \end{cases}$$

is

$$u = F_{-\alpha}g_0 + T_{-\alpha,1}F_\alpha(tr_\Gamma T_{-\alpha,1}F_\alpha)^{-1}Q_{-\alpha}g_0 - T_{-\alpha,1}Q_\alpha T_{\alpha,1}\upsilon.$$

Therefore, the unique solution of the problem (3.28) is represented as

$$\begin{aligned} u \quad = \quad & F_{-\alpha}g_0 + T_{-\alpha,1}F_\alpha(tr_\Gamma T_{-\alpha,1}F_\alpha)^{-1}Q_{-\alpha}g_0 \\ & -T_{-\alpha,1}Q_\alpha T_{\alpha,1}\left(F_{-\alpha}g_1 + T_{-\alpha,1}F_\alpha(tr_\Gamma T_{-\alpha,1}F_\alpha)^{-1}Q_{-\alpha}g_1\right) \\ & + \left(T_{-\alpha,1}Q_\alpha T_{\alpha,1}\right)^2 f. \end{aligned}$$

Consequently, by induction the above theorem allows a generalization in the following manner. However, this type of boundary value problem for higher order Laplacian as well for generalized Vekua type problem are investigated in [**47**, Chap. 4], see also [**82, 83**]. Here, we present our similar result with respect to the higher order Helmholtz equation for sake of completeness.

THEOREM 3.22. *Let* $f \in L_2(\Omega, \mathbb{H}(\mathbb{C}))$, $g_k \in W_2^{2n-\frac{4k+1}{2}}(\Gamma, \mathbb{H}(\mathbb{C}))$. *Then the unique solution of the boundary value problem (3.28) admits the reprentation*

$$\begin{aligned} u \quad = \quad & F_{-\alpha}g_0 + \sum_{\nu=1}^{n}\left(T_{-\alpha,1}Q_\alpha T_{\alpha,1}\right)^{\nu-1}\left(F_{-\alpha}g_{\nu-1} + T_{-\alpha,1}F_\alpha(tr_\Gamma T_{-\alpha,1}F_\alpha)^{-1}Q_{-\alpha}g_{\nu-1}\right) \\ & +(-1)^n\left(T_{-\alpha,1}Q_\alpha T_{\alpha,1}\right)^n f. \end{aligned}$$