CHAPTER 2

Higher Order Teodorescu Operators in Quaternionic Analysis

In this chapter, using the fundamental solution of the Helmholtz equation, a fundamental solution can be constructed for powers of the factors of the Helmholtz operator. Moreover, we are able to give the explicit forms of the kernel functions for the integral representation formulas of solutions to the higher order D_α equations. This means that we can obtain the representation formulas of solutions to the inhomogenuous equations $D_\alpha^n f = g$. These results will lend assistance aid to investigate some properties of higher order Teodorescu operators.

1. Motivation

Integral representation formulars for differentiable functions are useful, for example they help to determine properties of the functions represented such as smoothness, differentiability, boundary behavious and so on.

It is known that, many chemical and physical processes can be mathematically described via partial differential equations with some conditions which are called boundary value problems (see [**30, 64**]). Integral representations are one of the main tools to solve boundary value problems for partial differential equations. Let us see in the whole this picture in complex analysis. The area integral appearing in the complex Cauchy- Pompeiu representation defines a weakly singular integral operator T. Its properties were studied by I. N. Vekua [**86**]. We refer to Begehr [**18, 19, 89**] or Bojarski [**26**] for solving complex first order partial differential equations based on properties of T as well as of the strongly singular integral operator of Ahlfors-Beurling type Π (see, e.g., [**2, 86**]). Higher order Cauchy- Pompeiu representations were developed in [**9, 10, 11, 14**]. Then, by repeated applications of the T- operator, second order complex equations have been investigated by Begehr (see [**8**]), Dzhuraev [**36, 37**] and a complex fourth order equations is studied by Wen and Kang (see [**90**]). With the idea to generalize the T- operator in order to handle higher order differential equations, the operator $T_{m,n}, (m+n) \geq 0$, as well as a list of its properties is given by Begehr and Hile (see [**16, 17**]). The $T_{m,n}$-operators, in fact, are useful in the study of some boudary value problems for generalized polyanalytic functions of order n in the Sobolev space $W^{1,p}(\Omega)$ (see [**73, 74**]), or for complex elliptic partial differential equations of higher order (see [**3**]).

While these representations are related to powers of the Cauchy-Riemann or the Dirac and the Laplace operators here the Helmholtz operator and its factors are investigated. Iterating the Cauchy- Pompeiu formula and constructing higher order kernel functions lead to higher order Cauchy-Pompeiu representations. In constructing the kernel functions via an integration process the procedure from [**91**], Appendix is used.

In addition, following the techniques of Begehr - Hile [**16**], the integral operators $T_{\alpha,n}, T_{r,\alpha,n}$ are defined (see [**51**]). These operators appear in representation formulas for solutions to the inhomogeneous higher order D_α equation, see [**50**]. The $T_{\alpha,1}$-operator with a kernel given by the fundamental solution of the Helmholtz equation is known as the Teodorescu operator which is introduced in Chapter 1, Section 3. For real α, T_α was investigated in [**41**], for complex α we refer to [**60**], for quaternionic α we again refer to [**64, 66**] where the related formulas are given.

As is shown in [**46, 47**] these operators are useful to study elliptic boundary value problems, and in [**60, 64, 66**] to study Helmholtz equations and the Maxwell system. For this purpose, a modified Teodorescu transform is described precisely in [**6**]. In the case of complex α, by an induction argument beginning with the Cauchy - Pompeiu formula, a higher order representation of functions f in $C^n(\Omega, \mathbb{H}(\mathbb{C}))$, is developed in [**49, 50**]. Therefore, in this chapter we will study some properties of the integral operator $T_{\alpha,n}$, $n \geq 1$, such as existence, mapping properties, differentiability etc.

As the operators T, $T_{m,n}$ have been widely used to study various boundary valued problems for higher order equations in complex analysis, the $T_{\alpha,n}$ should give useful tools in investigating similar problems which can be reduced to such problems and systems to those of Helmholtz equation in quaternionic analysis.

Now, we come to the first of our results. They represent the solution to the inhomogenuous equation $D_\alpha^n f = g$ in Ω, $n \in \mathbb{N}$.

2. Integral representations for higher order D_α equations

The fundamental solution for the operator D_α^n with $n \in \mathbb{N}$ will be constructed by a method as given in [**49**], (see also [**80**], [**91**, Chap. 4]). The advantage of our method, using induction, is that it yields explicit kernel functions. Our main results are contained in Theorem 2.3, Theorem 2.4 and Corollary 2.5. Before doing so, we will present Lemma 2.1. It is used in the case of quaternions to prove Cauchy- Pompeiu type representation formulas in terms of powers of the factors of the Helmholtz operator.

Lemma 2.1. *Let $K_\alpha(x)$ be a fundamental solution for the operator D_α, i.e, a quaternionic function satisfying in distributional sense $D_\alpha K_\alpha(x) = \delta(x)$, $\alpha \neq 0$, and $K_\alpha(x)$ be infinitely often differentiable with respect to α. Then the functions $K_\alpha^{(n)}(x)$, $n \in \mathbb{N}$, determined by the recurrence fomulas*

$$\begin{aligned} K_\alpha^{(1)}(x) &= K_\alpha(x), \\ K_\alpha^{(2)}(x) &= -\frac{\partial}{\partial\alpha} K_\alpha^{(1)}(x), \\ K_\alpha^{(3)}(x) &= \frac{-1}{2}\frac{\partial}{\partial\alpha} K_\alpha^{(2)}(x), \\ K_\alpha^{(k)}(x) &= \frac{-1}{k-1}\frac{\partial}{\partial\alpha} K_\alpha^{(k-1)}(x), \end{aligned}$$

for all $k \in \mathbb{N}^*$, satisfy in distributional sense the equations

$$(D+\alpha)K_\alpha^{(n)}(x) = K_\alpha^{(n-1)}(x), \tag{2.1}$$

$$(D+\alpha)^n K_\alpha^{(n)}(x) = \delta(x). \tag{2.2}$$

Proof. It is easy to see that (2.2) holds for $n = 1$.

First of all, we note that the function space $C_c^\infty(\Omega, \mathbb{H}(\mathbb{C}))$ isdefined as the vector space consisting of functions from Ω to $\mathbb{H}(\mathbb{C})$ with compact support which have continuous derivatives of all orders. Applying (1.20), (1.21)

$$\int_\Omega (D_r+\alpha)K_\alpha^{(n)}(x)\phi(x)dx + \int_\Omega K_\alpha^{(n)}(x)(D-\alpha)\phi(x)dx = \int_{\partial\Omega} K_\alpha^{(n)}(x)\vec{n}(x)\phi(x)d\Gamma_x = 0,$$

$$\int_\Omega (D_r-\alpha)K_\alpha^{(n)}(x)\phi(x)dx + \int_\Omega K_\alpha^{(n)}(x)(D+\alpha)\phi(x)dx = \int_{\partial\Omega} K_\alpha^{(n)}(x)\vec{n}(x)\phi(x)d\Gamma_x = 0,$$

follows.

With the right $\mathbb{H}(\mathbb{C})$-distribution $< f,\phi >:= \int_\Omega f(x)\phi(x)dx$ then

$$< (D_r+\alpha)K_\alpha^{(n)}(x),\ \phi(x) >= -< K_\alpha^{(n)}(x),\ (D-\alpha)\phi(x) >,$$
$$< (D_r-\alpha)K_\alpha^{(n)}(x),\ \phi(x) >= -< K_\alpha^{(n)}(x),\ (D+\alpha)\phi(x) > .$$

Now we can prove Lemma 2.1 for $n=2$. For all $\phi \in C_c^\infty(\Omega,\mathbb{H}(\mathbb{C}))$ and by (1.16) together with the definition of $K_\alpha^{(2)}(x)$ we obtain

$$\begin{aligned}
&< (D_r+\alpha)K_\alpha^{(2)}(x),\phi(x) > \ = \ < (D+\alpha)(-\frac{\partial}{\partial\alpha}K_\alpha^{(1)}(x)),\phi(x) > \\
&=< \frac{\partial}{\partial\alpha}(D+\alpha)K_\alpha^{(1)}(x),\phi(x) > \ + \ < K_\alpha^{(1)}(x),\phi(x) > \\
&= \ -\frac{\partial}{\partial\alpha}\phi(0) + \ < K_\alpha^{(1)}(x),\phi(x) > .
\end{aligned}$$

The δ-distribution, defined as usual by $< \delta(x),\phi(x) >= \phi(0),\ \phi \in C_c^\infty(\Omega,\mathbb{H}(\mathbb{C}))$, can be considered as well a left as a right $\mathbb{H}(\mathbb{C})$-distribution.

Thus, $(D_r+\alpha)K_\alpha^{(2)}(x) = K_\alpha^{(1)}(x)$. By (1.16) and the definitions of $K_\alpha^{(k)}$ we have

$$D_{r,\alpha}K_\alpha^{(k)}(x) = D_\alpha K_\alpha^{(k)}(x) \quad \text{for all } k \in \mathbb{N}^*.$$

Hence, $D_\alpha K_\alpha^{(2)}(x) = (D+\alpha)K_\alpha^{(2)}(x) = K_\alpha^{(1)}(x)$.

Moreover,

$$\begin{aligned}
< (D_r+\alpha)^2K_\alpha^{(2)}(x),\phi(x) > \ &= \ -< (D+\alpha)K_\alpha^{(2)}(x),(D-\alpha)\phi(x) > \\
&= \ -< K_\alpha^{(1)}(x),(D-\alpha)\phi(x) > \\
&= \ < (D+\alpha)K_\alpha^{(1)}(x),\phi(x) > \\
&= \ < \delta(x),\phi(x) > .
\end{aligned}$$

Hence, $(D+\alpha)^2K_\alpha^{(2)}(x) = \delta(x)$.

Suppose that for $n \in \mathbb{N}$ the function $K_\alpha^{(n)}$ satisfies

$$(D+\alpha)K_\alpha^{(n)}(x) = K_\alpha^{(n-1)}(x),$$
$$(D+\alpha)^nK_\alpha^{(n)}(x) = \delta(x)$$

in distributional sense for all $\phi \in C_c^\infty(\Omega,\mathbb{H}(\mathbb{C}))$. Then

$$\begin{aligned}
< (D+\alpha)K_\alpha^{(n+1)}(x),\phi(x) > \ &= \ < (D_r+\alpha)[\frac{-1}{n}\frac{\partial}{\partial\alpha}K_\alpha^{(n)}(x)],\phi(x) > \\
&= \ \frac{-1}{n} < \frac{\partial}{\partial\alpha}(D+\alpha)K_\alpha^{(n)}(x) - K_\alpha^{(n)}(x),\phi(x) > \\
&= \ \frac{-1}{n}\frac{\partial}{\partial\alpha} < K_\alpha^{(n-1)}(x),\phi(x) > +\frac{1}{n} < K_\alpha^{(n)}(x),\phi(x) > \\
&= \ \frac{n-1}{n} < \frac{-1}{n-1}\frac{\partial}{\partial\alpha}K_\alpha^{(n-1)}(x),\phi(x) > \\
&\quad +\frac{1}{n} < K_\alpha^{(n)}(x),\phi(x) >
\end{aligned}$$

$$\begin{aligned} &= \frac{n-1}{n} < K_\alpha^{(n)}(x), \phi(x) > + \frac{1}{n} < K_\alpha^{(n)}(x), \phi(x) > \\ &= < K_\alpha^{(n)}(x), \phi(x) > . \end{aligned}$$

Thus, $(D+\alpha)K_\alpha^{(n+1)}(x) = K_\alpha^{(n)}(x)$. In addition

$$\begin{aligned} & < (D+\alpha)^{n+1} K_\alpha^{(n+1)}(x), \phi(x) > \\ = \; & (-1)^n < (D_r+\alpha) K_\alpha^{(n+1)}(x), (D-\alpha)^n \phi(x) > \\ = \; & (-1)^n < K_\alpha^{(n)}(x), (D-\alpha)^n \phi(x) > \\ = \; & < (D+\alpha)^n K_\alpha^{(n)}(x), \phi(x) > \\ = \; & < \delta(x), \phi(x) > . \end{aligned}$$

Then $(D+\alpha)^{n+1} K_\alpha^{(n+1)}(x) = \delta(x)$ in distributional sense. □

Looking at the kernel function $K_\alpha^{(1)}(x)$ we decompose it in the following way:

$$\begin{aligned} K_\alpha^{(1)}(x) &= (\alpha - i\alpha \frac{x}{|x|})(-\frac{e^{i\alpha|x|}}{4\pi|x|}) + \frac{x}{|x|^2}(-\frac{e^{i\alpha|x|}}{4\pi|x|}), \\ K_\alpha^{(n)}(x) : &= \frac{(-1)}{n-1}\frac{\partial}{\partial\alpha} K_\alpha^{(n-1)}(x). \end{aligned}$$

A straightforward calculation by the above lemma and using induction, the following corollary is proved.

Corollary 2.2. *Let α be a complex constant with $\alpha \neq 0$. Then the function*

$$K_\alpha^{(n)}(x) = \frac{(-1)^{n-1}}{(n-1)!}[(n-1) - (n-2)\frac{ix}{|x|} + i\alpha|x| + \alpha x](i|x|)^{n-2}(-\frac{e^{i\alpha|x|}}{4\pi|x|}) \tag{2.3}$$

is a fundamental solution of the operator D_α^n.

Theorem 2.3. *Let Ω be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega =: \Gamma$ and $f \in C^2(\Omega, \mathbb{H}(\mathbb{C})) \cap C^1(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$. Then*

$$\begin{aligned} f(x) = -\int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y) f(y) d\Gamma_y &- \int_\Gamma K_\alpha^{(2)}(x-y)\vec{n}(y) D_{\alpha,y} f(y) d\Gamma_y \\ &+ \int_\Omega K_\alpha^{(2)}(x-y) D_{\alpha,y}^2 f(y) dy. \end{aligned} \tag{2.4}$$

Proof. From the equality (1.22)

$$D_{\alpha,y} f(y) = -\int_\Gamma K_\alpha^{(1)}(y-\tilde{y})\vec{n}(\tilde{y}) D_{\alpha,\tilde{y}} f(\tilde{y}) d\Gamma_{\tilde{y}} + \int_\Omega K_\alpha^{(1)}(y-\tilde{y}) D_{\alpha,\tilde{y}}^2 f(\tilde{y}) d\tilde{y}$$

it follows

$$f(x) = -\int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y) f(y) d\Gamma_y - \int_\Gamma \psi(x,\tilde{y})\vec{n}(\tilde{y}) D_{\alpha,\tilde{y}} f(\tilde{y}) d\Gamma_{\tilde{y}} + \int_\Omega \psi(x,\tilde{y}) D_{\alpha,\tilde{y}}^2 f(\tilde{y}) d\tilde{y},$$

where $\psi(x,\tilde{y}) = \int_\Omega K_\alpha^{(1)}(x-y) K_\alpha^{(1)}(y-\tilde{y}) dy$.

For $x, \tilde{y} \in \Omega$ with $x \neq \tilde{y}$, the quaternionic Cauchy-Pompeiu formula (1.22) and using (2.1) lead to

$$K_\alpha^{(2)}(x-\tilde{y}) = \tilde{\psi}(x,\tilde{y}) + \psi(x,\tilde{y})$$

where $\tilde{\psi}(x,\tilde{y}) = -\int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)K_\alpha^{(2)}(y-\tilde{y})d\Gamma_y$. Then

$$\begin{aligned} f(x) = & - \int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)f(y)d\Gamma_y - \int_\Gamma K_\alpha^{(2)}(x-y)\vec{n}(y)D_{\alpha,y}f(y)d\Gamma_y \\ & + \int_\Omega K_\alpha^{(2)}(x-y)D^2_{\alpha,y}f(y)dy + \int_\Gamma \tilde{\psi}(x,\tilde{y})\vec{n}(\tilde{y})D_{\alpha,\tilde{y}}f(\tilde{y})d\Gamma_{\tilde{y}} \\ & - \int_\Omega \tilde{\psi}(x,\tilde{y})D^2_{\alpha,\tilde{y}}f(\tilde{y})d\tilde{y}. \end{aligned} \tag{2.5}$$

Namely, $K_\alpha^{(1)}(x-y)$ and $K_\alpha^{(2)}(y-\tilde{y})$ are for fixed arbitrary x and $\tilde{y}$, respectively C^1-functions in the whole domain Ω except for the two points x and $\tilde{y}$. Therefore

$$\begin{aligned} \int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)K_\alpha^{(1)}(y-\tilde{y})d\Gamma_y &= \int_{\partial\Omega_\varepsilon} K_\alpha^{(1)}(x-y)\vec{n}(y)K_\alpha^{(1)}(y-\tilde{y})d\Gamma_y + \\ +\int_{|y-x|=\varepsilon} K_\alpha^{(1)}(x-y)\vec{n}(y)K_\alpha^{(1)}(y-\tilde{y})d\Gamma_y &+ \int_{|y-\tilde{y}|=\varepsilon} K_\alpha^{(1)}(x-y)\vec{n}(y)K_\alpha^{(1)}(y-\tilde{y})d\Gamma_y, \end{aligned}$$

where $\Omega_\varepsilon = \Omega - \{y\in\Omega, |y-x|<\varepsilon \text{ or } |y-\tilde{y}|<\varepsilon\}$, $0<\varepsilon$ small enough.

Applying formula (1.20) for Ω_ε to $\int_{\partial\Omega_\varepsilon} K_\alpha^{(1)}(x-y)\vec{n}(y)K_\alpha^{(1)}(y-\tilde{y})d\Gamma_y$, and using that $(D_{r,y}+\alpha)K_\alpha^{(1)}(y-\tilde{y}) = \delta(y-\tilde{y})$ and $(D_{r,y}-\alpha)K_\alpha^{(1)}(x-y) = -\delta(x-y)$, together with

$$\lim_{\varepsilon\to 0}\int_{|y-x|=\varepsilon} K_\alpha^{(1)}(x-y)\vec{n}(y)K_\alpha^{(1)}(y-\tilde{y})d\Gamma_y = -K_\alpha^{(1)}(x-\tilde{y})$$

$$\lim_{\varepsilon\to 0}\int_{|y-\tilde{y}|=\varepsilon} K_\alpha^{(1)}(x-y)\vec{n}(y)K_\alpha^{(1)}(y-\tilde{y})d\Gamma_y = K_\alpha^{(1)}(x-\tilde{y})$$

we obtain $(D_{r,\tilde{y}}-\alpha)\tilde{\psi}(x,\tilde{y}) = 0$. Then $\int_\Omega \tilde{\psi}(x,\tilde{y})D^2_{\alpha,\tilde{y}}f(\tilde{y})d\tilde{y} = \int_\Gamma \tilde{\psi}(x,\tilde{y})\vec{n}(\tilde{y})D_{\alpha,\tilde{y}}f(\tilde{y})d\Gamma_{\tilde{y}}$ by applying (1.20). Inserting this equality into (2.5) we get (2.4). ☐

THEOREM 2.4. *Let $f\in C^n(\Omega,\mathbb{H}(\mathbb{C}))\cap C^{n-1}(\bar{\Omega},\mathbb{H}(\mathbb{C}))$. Then*

$$\begin{aligned} f(x) &= -\sum_{k=1}^n \int_\Gamma K_\alpha^{(k)}(x-y)\vec{n}(y)D^{k-1}_{\alpha,y}f(y)d\Gamma_y \\ &+ \int_\Omega K_\alpha^{(n)}(x-y)D^n_{\alpha,y}f(y)dy. \end{aligned} \tag{2.6}$$

PROOF. For $n=1$ fomula (2.6) coincides with the Cauchy-Pompeiu formula(1.22). We have already shown (2.6) for $n=2$ in Theorem 2.3. In order to prove this formula for any $n>2$ assume it holds for $n-1$. Applying this formula for $D_{\alpha,y}f(y)$ leads to

$$\begin{aligned} D_{\alpha,y}f(y) &= -\sum_{k=1}^{n-1}\int_\Gamma K_\alpha^{(k)}(y-\tilde{y})\vec{n}(\tilde{y})D^k_{\alpha,\tilde{y}}f(\tilde{y})d\Gamma_{\tilde{y}} \\ &+ \int_\Omega K_\alpha^{(n-1)}(y-\tilde{y})D^n_{\alpha,\tilde{y}}f(\tilde{y})d\tilde{y}. \end{aligned}$$

Inserting this equality into (1.22) gives

$$
\begin{aligned}
f(x) = -\int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)f(y)d\Gamma_y \quad & - \quad \sum_{k=1}^{n-1}\int_\Gamma \psi_k(x,\tilde{y})\vec{n}(\tilde{y})D^k_{\alpha,\tilde{y}}f(\tilde{y})d\Gamma_{\tilde{y}} \\
& + \quad \int_\Omega \psi_{n-1}(x,\tilde{y})D^n_{\alpha,\tilde{y}}f(\tilde{y})d\tilde{y},
\end{aligned}
$$

where $\psi_k(x,\tilde{y}) = \int_\Omega K_\alpha^{(1)}(x-y)K_\alpha^{(k)}(y-\tilde{y})dy$ for all $k = 1,\ 2, \cdots, (n-1)$. Applying (1.22) with $x \neq \tilde{y}$ and using (2.1) we have

$$
K_\alpha^{(k+1)}(x-\tilde{y}) = -\tilde{\psi}_{k+1}(x,\tilde{y}) + \psi_k(x,\tilde{y}) \quad \text{for all } k = 1,\ 2, \cdots, n-1,
$$

where $\tilde{\psi}_{k+1}(x,\tilde{y}) = \int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)K_\alpha^{(k+1)}(y-\tilde{y}))d\Gamma_y$ for all $k = 1,\ 2,\ \cdots, n-1$. Then

$$
\begin{aligned}
f(x) = -\int_\Gamma K_\alpha(x-y)\vec{n}(y)f(y)d\Gamma_y \quad & - \quad \sum_{k=2}^{n}\int_\Gamma K_\alpha^{(k)}(x-y)\vec{n}(y)D^{k-1}_{\alpha,y}f(y)d\Gamma_y \\
& + \quad \int_\Omega K_\alpha^{(n)}(x-y)D^n_{\alpha,y}f(y)dy \\
& - \quad \sum_{k=1}^{n-1}\int_\Gamma \tilde{\psi}_{k+1}(x,\tilde{y})\vec{n}(\tilde{y})D^k_{\alpha,\tilde{y}}f(\tilde{y})d\Gamma_{\tilde{y}} \\
& + \quad \int_\Omega \tilde{\psi}_n(x,\tilde{y})D^n_{\alpha,\tilde{y}}f(\tilde{y})d\tilde{y}.
\end{aligned}
$$

Note that from (2.1) we have

$$
(D_{r,\tilde{y}} - \alpha)\tilde{\psi}_{k+1}(x,\tilde{y}) = -\tilde{\psi}_k(x,\tilde{y})
$$

and applying Stokes' formula (1.20) again as well as

$$
-(D_{r,\tilde{y}} - \alpha)\tilde{\psi}_1(x,\tilde{y}) = \tilde{\psi}_1(x,\tilde{y}) = 0,
$$

where $\tilde{\psi}_1(x,\tilde{y}) = \int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)K_\alpha^{(1)}(y-\tilde{y})d\Gamma_y$, (see the proof of Theorem 2.3), this proves that

$$
\begin{aligned}
& \sum_{k=1}^{n-1}\int_\Gamma \tilde{\psi}_{k+1}(x,\tilde{y})\vec{n}(\tilde{y})D^k_{\alpha,\tilde{y}}f(\tilde{y})d\Gamma_{\tilde{y}} + \int_\Omega \tilde{\psi}_{n-1}(x,\tilde{y})D^n_{\alpha,\tilde{y}}f(\tilde{y})d\tilde{y} \\
= \quad & -\sum_{k=1}^{n-1}\int_\Omega \tilde{\psi}_k(x,\tilde{y})D^k_{\alpha,\tilde{y}}f(\tilde{y})d\tilde{y} + \sum_{k=1}^{n-1}\int_\Omega \tilde{\psi}_{k+1}(x,\tilde{y})D^{k+1}_{\alpha,\tilde{y}}f(\tilde{y})d\tilde{y} \\
& + \int_\Omega \tilde{\psi}_n(x,\tilde{y})D^n_{\alpha,\tilde{y}}f(\tilde{y})d\tilde{y} = \int_\Omega \tilde{\psi}_1(x,\tilde{y})D_{\alpha,\tilde{y}}f(\tilde{y})d\tilde{y} = 0.
\end{aligned}
$$

This leads to

$$
\begin{aligned}
f(x) \quad = \quad & -\sum_{k=1}^{n}\int_\Gamma K_\alpha^{(k)}(x-y)\vec{n}(y)D^{k-1}_{\alpha,y}f(y)d\Gamma_y \\
& + \quad \int_\Omega K_\alpha^{(n)}(x-y)D^n_{\alpha,y}f(y)dy.
\end{aligned}
$$

$\square$

Because of fomula (2.3) together with the above theorem, the representation of solutions to the inhomogeneous equation $D_\alpha^n f = g$ in Ω can be written in the following form.

COROLLARY 2.5.

$$\begin{aligned} f(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} \int_{\Gamma} & [(k-1)-(k-2)\frac{i(x-y)}{|x-y|} + i\alpha|x-y| + \alpha(x-y)] \\ & \times (i|x-y|)^{k-2}(-\frac{e^{i\alpha|x-y|}}{4\pi|x-y|})\vec{n}(y) D_{\alpha,y}^{k-1} f(y) d\Gamma_y \\ + \frac{(-1)^n}{(n-1)!} \int_{\Omega} & [(n-1)-(n-2)\frac{i(x-y)}{|x-y|} + i\alpha|x-y| + \alpha(x-y)] \\ & \times (i|x-y|)^{n-2}(-\frac{e^{i\alpha|x-y|}}{4\pi|x-y|}) D_{\alpha,y}^{n} f(y) dy. \end{aligned}$$

Now the higher order Teodorescu operators are given. Their properties will be studied in the next sections. The kernel functions $K_\alpha^{(n)}$ are defined by formula (2.3).

DEFINITION 2.6. For a bounded domain Ω in $\mathbb{R}^3$ with piecewise sufficiently smooth boundary Γ, we formally define operators $T_{\alpha,n}, T_{r,\alpha,n}$, where $\alpha \in \mathbb{C}$ acting on $\mathbb{H}(\mathbb{C})$ -valued functions f defined in Ω, according to

$$(T_{\alpha,n} f)(x) := \int_{\Omega} K_\alpha^{(n)}(x-y) f(y) dy,$$

$$(T_{r,\alpha,n} f)(x) := \int_{\Omega} f(y) K_\alpha^{(n)}(x-y) dy.$$

$T_{\alpha,n}$, $T_{r,\alpha,n}$ are called a the *higher order Teodorescu operators.*

REMARK 2.7. If we decompose $K_\alpha^{(1)}$ as

$$K_\alpha^{(1)}(x) = K_{\alpha,1}^{(1)}(x) + K_{\alpha,2}^{(1)}(x) \tag{2.7}$$

where

$$\begin{aligned} K_{\alpha,1}^{(1)}(x) &= (\frac{x}{|x|^2})\,(-\frac{e^{i\alpha|x|}}{4\pi|x|}), \\ K_{\alpha,2}^{(1)}(x) &= (\alpha - i\alpha\frac{x}{|x|})(-\frac{e^{i\alpha|x|}}{4\pi|x|}), \end{aligned}$$

then $K_{\alpha,1}^{(1)}(x)$ coincides up to the factor $(-e^{i\alpha|x|})$ with the kernel function of the operator $T_{0,1}$. This operator was studied in [**47, 81**]. We recall that $T_{0,1}f(x) = \int_\Omega \frac{(x-y)}{4\pi|x-y|^3} f(y)dy$ is the Teodorescu transform and corresponds to the known $T-$ operator from the one variable complex analysis (see [**7, 86**]). It may be expected that $K_{\alpha,1}^{(1)}(x)$ causes analogous properties of $T_{\alpha,1}$ as known for $T_{0,1}$.

Now we begin with the investigations of mapping properties of $T_{\alpha,n}$.

3. Existence and continuity of integrals

In this section, we will prove the existence and continuity of $T_{\alpha,n}$. The operators $T_{r,\alpha,n}$ have analogous properties with respect to this operator acting on the right of the function. We also refer the readers to [**41, 46**] for more details in the discussion of some properties of the integral $T_{\alpha,1}$, with α a real number. $T_{\alpha,1}$ acts on real quaternion - valued functions. The use of complex quaternions as well as α a complex number does not cause changes of the mapping properties of $T_{\alpha,1}$, as was shown in [**45**]. Moreover, the kernel $K_\alpha^{(n)}(x)$ of the operator $T_{\alpha,n}$ has a singularity of order 2 at most and thus it will not affect essentially the properties, induced by $K_\alpha^{(1)}(x)$. Nevertheless, the following properties will be proved more explicitly again for $T_{\alpha,n}, n \geq 1$.

LEMMA 2.8. *Under the same assumptions as in Definition 2.6, for $f \in L_1(\Omega, \mathbb{H}(\mathbb{C}))$, the integral*

$$F(x) = \int\limits_\Omega \frac{1}{|x-y|^{3-\gamma}} f(y)dy \quad \textit{is in} \quad L_1(\Omega, \mathbb{H}(\mathbb{C})) \quad \textit{for all} \quad 0 < \gamma \in \mathbb{R}.$$

PROOF. Notice that here we consider a bounded domain Ω and $\gamma > 0$. Firstly, looking at the integral $\int\limits_\Omega \left(\int\limits_\Omega \frac{1}{|x-y|^{3-\gamma}} dx \right) |f(y)|_{\mathbb{H}(\mathbb{C})} dy$, L. Hedberg has shown that there exists a constant C such that $\int\limits_\Omega \frac{1}{|x-y|^{3-\gamma}} dy \leq C(\text{diam} B)^\gamma$ holds for $0 < \gamma$ and Ω a bounded domain, where B is the smallest cube containing Ω, (see [**53**]).
Hence,

$$\int\limits_\Omega \left(\int\limits_\Omega \frac{1}{|x-y|^{3-\gamma}} dx \right) |f(y)|_{\mathbb{H}(\mathbb{C})} dy \leq M_{(\Omega,\gamma)} \int\limits_\Omega |f(y)|_{\mathbb{H}(\mathbb{C})} dy = M_{(\Omega,\gamma)} ||f||_{L_1},$$

where $M_{(\Omega,\gamma)}$ is a constant depending on Ω and on γ. Using Fubini's Theorem

$$\int\limits_\Omega \left(\int\limits_\Omega \frac{1}{|x-y|^{3-\gamma}} dx \right) |f(y)|_{\mathbb{H}(\mathbb{C})} dy = \int\limits_\Omega \left(\int\limits_\Omega \frac{1}{|x-y|^{3-\gamma}} |f(y)|_{\mathbb{H}(\mathbb{C})} dy \right) dx = \int\limits_\Omega F(x)dx,$$

follows where the involved integrals are finite and hence $F(x)$ is in $L_1(\Omega, \mathbb{H}(\mathbb{C}))$. □

REMARK 2.9. For $x \in \Omega^- := \mathbb{R} - \bar{\Omega}$ we have

$$\left| \int\limits_\Omega \frac{1}{|x-y|^{3-\gamma}} f(y)dy \right| \leq M_{(\Omega,\gamma,x)} ||f||_{L_1}.$$

We now come to our first main result ensuring the existence of the integral $T_{\alpha,n}$ on $L_1(\Omega, \mathbb{H}(\mathbb{C}))$.

THEOREM 2.10. *For $\alpha \in \mathbb{C}$ and $f \in L_1(\Omega, \mathbb{H}(\mathbb{C}))$, the integral $T_{\alpha,n} f(x)$ exists for almost all $x \in \Omega$.*

PROOF. For $n = 1$, with $f \in L_1(\Omega, \mathbb{H}(\mathbb{C}))$, and viewing the formula for $K_\alpha^{(1)}(x-y)$ we observe that

$$\left| K_\alpha^{(1)}(x-y) f(y) \right|_{\mathbb{H}(\mathbb{C})} \leq \frac{\sqrt{2}}{4\pi} |K_\alpha^{(1)}(x-y)|_{\mathbb{H}(\mathbb{C})} |f(y)|_{\mathbb{H}(\mathbb{C})}$$

$$\leq \frac{\sqrt{2}}{4\pi} e^{-\mathrm{Im}\alpha \ \mathrm{diam}\Omega} |f(y)|_{\mathbb{H}(\mathbb{C})} \big(2|\alpha| \frac{1}{|x-y|} + \frac{1}{|x-y|^2}\big).$$

Using Lemma 2.8 leads to the existence of $(T_{\alpha,1}f)(x)$ for almost all $x \in \bar{\Omega}$.

In the case of $n = 2$, let us consider the estimate

$$\begin{aligned} \big|K_\alpha^{(2)}(x-y)f(y)\big|_{\mathbb{H}(\mathbb{C})} &\leq \sqrt{2}|K_\alpha^{(2)}(x-y)|_{\mathbb{H}(\mathbb{C})}|f(y)|_{\mathbb{H}(\mathbb{C})} \\ &\leq \frac{\sqrt{2}}{4\pi} e^{-\mathrm{Im}\alpha \ \mathrm{diam}\Omega} |f(y)|_{\mathbb{H}(\mathbb{C})} \big(2|\alpha| + \frac{1}{|x-y|}\big). \end{aligned}$$

Using Lemma 2.8 again in the case $\gamma \geq 2$, the existence of $(T_{\alpha,2}f)(x)$ for almost all $x \in \bar{\Omega}$ is proved.

If $n \geq 3$ is fixed, we have

$$\begin{aligned} \big|K_\alpha^{(n)}(x-y)f(y)\big|_{\mathbb{H}(\mathbb{C})} &\leq \sqrt{2}|K_\alpha^{(n)}(x-y)|_{\mathbb{H}(\mathbb{C})}|f(y)|_{\mathbb{H}(\mathbb{C})} \\ &\leq \frac{\sqrt{2}}{4\pi} e^{-\mathrm{Im}\alpha \ \mathrm{diam}\Omega} |f(y)|_{\mathbb{H}(\mathbb{C})} \big(2|\alpha||x-y|^{n-2} + |x-y|^{n-3}\big). \end{aligned}$$

Looking at the right - hand side of this inequality, we can easily see that $T_{\alpha,n}f(x)$ for $n \geq 3$ has no singularity. Hence, the existence of the integrals $T_{\alpha,n}f$ follows. □

Remark 2.11. If $x \in \Omega^-$ then $K_\alpha^{(n)}(x-y)$ are bounded continuous functions of y for $n = 1,\ 2$. This means that $T_{\alpha,n}f(x), n = 1,\ 2$, exist for almost all $x \in \mathbb{R}^3$.

For $x \in \Omega^-$ fixed then the integrals $T_{\alpha,n}f(x)$ exist and they tend to infinity for $|x| \to \infty$ for all $n \geq 3$.

Theorem 2.12. *Let the assumptions of Definition 2.6 be satisfied. In addition, let f be a complex - valued function in $L_1(\Omega, \mathbb{H}(\mathbb{C}))$. Then the integral $T_{\alpha,n}f(x)$ converges absolutely for all x in Ω. Moreover, if*

(i) $1 \ \leq q < \ \frac{3}{2}$ *when* $n = 1$,
(ii) $1 \ \leq q < \ 3$ *when* $n = 2$,
(iii) $1 \ \leq q \leq \ +\infty$ *when* $n \geq \ 3$,

then $T_{\alpha,n}f \in L_q(\Omega, \mathbb{H}(\mathbb{C}))$ *with* $||T_{\alpha,n}f||_{L_q} \ \leq \ M_{(\Omega,\alpha,n)}||f||_{L_1}$.

Proof. Firstly, we will define $W_\gamma(x)$ on $\bar{\Omega}$ according to

$$W_\gamma(x) := \int\limits_\Omega \frac{|\omega(y)|}{|x-y|^\gamma} dy,$$

and let ω be an arbitrary function in $L_p(\Omega, \mathbb{H}(\mathbb{C}))$ where $\frac{1}{p} + \frac{1}{q} = 1$.
Again using the estimate of $|K_\alpha^{(n)}(x-y)|$ in Theorem 2.10 shows that the integral $T_{\alpha,n}f(x)$ converges absolutely.

Now, using Hölder's inequality we obtain

$$W_\gamma(x) := \int\limits_\Omega \frac{|\omega(y)|}{|x-y|^\gamma} dy \ \leq \ \left(\int\limits_\Omega \big(\frac{1}{|x-y|^\gamma}\big)^q dy \right)^{\frac{1}{q}} ||\omega||_{L_p} \ \ \leq \ \big(\int\limits_\Omega \frac{1}{|x-y|^\gamma} dy\big)||\omega||_{L_p},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. By Lemma 2.8, hence the middle integral of this inequality exists for $q\gamma \ < \ 3$. Therefore, in the case of $n = 1$ the condition $1 \leq q < \frac{3}{2}$ is sufficient for both values $\gamma = 1$ or $\gamma = 2$ (see the estimates in Theorem 2.10). For $n = 2$, by the estimate of $K^{(2)}_\alpha(x-y)$, we have to consider $\gamma = 1$. Thus the condition is $1 \leq q < 3$. It is easily seen that $1 \leq q \leq +\infty$ is possible for all $n \geq 3$.

Under the assumtions for q together with $W_\gamma(x) \leq \big(\int\limits_\Omega \frac{1}{|x-y|^\gamma} dy\big)||\omega||_{L_p}$, yields $W_\gamma(x) \leq M_{(\Omega,\gamma)}||\omega||_{L_p}$, here $M_{(\Omega,\gamma)}$ is a constant depending on Ω and on γ but not on x, (see the proof of Lemma 2.8). Hence, $W_\gamma(x)$ converges uniformly i.e W_γ is continuous on $\bar{\Omega}$ and for all $\omega \in L_p(\Omega, \mathbb{H}(\mathbb{C}))$ then $W_\gamma(x) \in L_q(\Omega, \mathbb{R})$ for $\frac{1}{p} + \frac{1}{q} = 1$.
Again, by Fubini's theorem we have

$$\int\limits_\Omega \Big(\int\limits_\Omega \frac{|\omega(y)|}{|x-y|^\gamma} dy\Big)|v(x)|_{\mathbb{H}(\mathbb{C})} dx = \int\limits_\Omega \Big(\int\limits_\Omega \frac{|v(x)|}{|x-y|^\gamma} dx\Big)|\omega(y)|_{\mathbb{H}(\mathbb{C})} dy$$

where $\omega \in L_p(\Omega, \mathbb{H}(\mathbb{C}),\ v \in L_1(\Omega, \mathbb{H}(\mathbb{C})$. This due to the fact that the latter integral represents a linear functional on $L_p(\Omega, \mathbb{R})$ gives $\big(\int\limits_\Omega \frac{v(x)}{|x-y|^\gamma} dx\big) \in L_q(\Omega, \mathbb{R})$. Therefore, by the estimates in the above theorem this leads to $T_{\alpha,n} f \in L_q(\Omega, \mathbb{H}(\mathbb{C}))$ for every $f \in L_1(\Omega, \mathbb{H}(\mathbb{C})$.

Finally, using the same ideas gives explicit estimates of $||T_{\alpha,n}||_{L_1 \to L_q}$.
Indeed, firstly notice that as well the function

$$\tilde{T}_{\alpha,n} f(x) = \int\limits_\Omega \big|K^{(n)}_\alpha(x-y)\big|_{\mathbb{H}(\mathbb{C})} |f(y)|_{\mathbb{H}(\mathbb{C})} dy,$$

is defined already where $f \in L_1(\Omega, \mathbb{H}(\mathbb{C}))$, as

$$\begin{aligned} &\int\limits_\Omega \Big(\int\limits_\Omega |K^{(n)}_\alpha(x-y)|_{\mathbb{H}(\mathbb{C})} |f(y)|_{\mathbb{H}(\mathbb{C})} dy\Big)|v(x)|_{\mathbb{H}(\mathbb{C})} dx \\ &= \int\limits_\Omega \Big(\int\limits_\Omega |K^{(n)}_\alpha(x-y)|_{\mathbb{H}(\mathbb{C})} |v(x)|_{\mathbb{H}(\mathbb{C})} dx\Big)|f(y)|_{\mathbb{H}(\mathbb{C})} dy, \end{aligned}$$

where $v \in L_p(\Omega, \mathbb{H}(\mathbb{C}))$.
In the next step we consider $\big(\int\limits_\Omega |K^{(n)}_\alpha(x-y)|_{\mathbb{H}(\mathbb{C})} |v(x)|_{\mathbb{H}(\mathbb{C})} dx\big)$. In the cases listed under conditions (i)-(iii), when $q = +\infty$ we have $p = 1$. We then may apply Lemma 2.8 to a bounded domain large enough to contain $\bar{\Omega}$, and deduce that

$$\int\limits_\Omega |K^{(n)}_\alpha(x-y)|_{\mathbb{H}(\mathbb{C})} |v(x)|_{\mathbb{H}(\mathbb{C})} dx \leq M_{(\alpha,\Omega,n)}||v||_{L_p}.$$

In the cases $1 \leq q < +\infty$ and $1 < p \leq +\infty$, with the list of conditions (i)- (iii) by Lemma 2.8 together with the estimates of $K^{(n)}_\alpha(x-y)$ in Theorem 2.10 we get

$$\int\limits_\Omega |K^{(n)}_\alpha(x-y)|_{\mathbb{H}(\mathbb{C})} |v(x)|_{\mathbb{H}(\mathbb{C})} dx \leq \sup_{x \in \bar{\Omega}} \left(\int\limits_\Omega |K^{(n)}_\alpha(x-y)|^q dx\right)^{\frac{1}{q}} ||v||_{L_p}.$$

This leads to $\int\limits_\Omega \big(\int\limits_\Omega |K^{(n)}_\alpha(x-y)|_{\mathbb{H}(\mathbb{C})} |f(y)|_{\mathbb{H}(\mathbb{C})} dy\big)|v(x)|_{\mathbb{H}(\mathbb{C})} dx \leq M_{(\Omega,\alpha,n)}||f||_{L_1}||v||_{L_p}$. This inequality completes the proof of the theorem. □

4. Differentiability of integrals

If we look for applications of the $T_{\alpha,n}-$ operators then we need their mapping properties within Sobolev spaces. For this purpose, in this section we will investigate differentiability of the higher Teodorescu transforms. We refer to S. G. Mikhlin, S. Prössdorf [**69**] for an excellent book about singular integral operators but all of whose kernels are related the volume potentials. These are the operators of Calderon-Zygmund type, hence, the Teodorescu transform in the case $\alpha = 0$ do not cause much problems (see [**16, 42**], [**47**, Chap. 4] and references therein). However, in the cases $\alpha \neq 0$, the situation becomes more complicated. We can not immediately apply the theory of Calderon and Zygmund. This means that we have to give the estimate of the kernels of the higher Teodorescu operators in order to be able to use these theories. We start with exchanging differentiation and integration of higher order Teodorescu transforms.

Lemma 2.13. *Let $f \in C^1_c(\Omega, \mathbb{H}(\mathbb{C}))$ then*

(i) *for $k = 1, 2, 3$,*

$$\partial_k (T_{\alpha,1} f)(x) = \int_\Omega [\partial_{k,x} K^{(1)}_\alpha (x-y)] f(y) dy \;+\; \bar{e}_k \frac{f(x)}{3},$$

$$\partial_k (T_{r,\alpha,1} f)(x) = \int_\Omega f(y) [\partial_{k,x} K^{(1)}_\alpha (x-y)] dy \;+\; \frac{f(x)}{3} \bar{e}_k,$$

$$\partial_{r,k} (T_{r,\alpha,1} f)(x) = \int_\Omega f(y) [\partial_{r,k,x} K^{(1)}_\alpha (x-y)] dy \;+\; \frac{f(x)}{3} \bar{e}_k,$$

$$\partial_{r,k} (T_{\alpha,1} f)(x) = \int_\Omega [\partial_{r,k,x} K^{(1)}_\alpha (x-y)] f(y) dy \;+\; \bar{e}_k \frac{f(x)}{3},$$

(ii) *with $k = 1, 2, 3$,*

$$\partial_k (T_{\alpha,n} f)(x) = \int_\Omega [\partial_{k,x} K^{(n)}_\alpha (x-y)] f(y) dy,$$

$$\partial_k (T_{r,\alpha,n} f)(x) = \int_\Omega f(y) [\partial_{k,x} K^{(n)}_\alpha (x-y)] dy,$$

$$\partial_{r,k} (T_{r,\alpha,n} f)(x) = \int_\Omega f(y) [\partial_{r,k,x} K^{(n)}_\alpha (x-y)] dy,$$

$$\partial_{r,k} (T_{\alpha,n} f)(x) = \int_\Omega [\partial_{r,k,x} K^{(n)}_\alpha (x-y)] f(y) dy,$$

for all $n \geq 2$.

Proof. (i) In order to prove (i), we need to prove the first formula, the other formulas are shown in the same way. Let us consider the decomposition in (2.7). Firstly, notice that $K^{(1)}_{\alpha,1}(x-y)$ can be written as

$$K^{(1)}_{\alpha,1}(x-y) = -\frac{1}{4\pi}\left(\frac{x-y}{|x-y|^3} + \frac{e^{i\alpha|x-y|}-1}{|x-y|}\frac{x-y}{|x-y|^2}\right)$$

which has singularities of order less than or equal to two only. Therefore, we have

$$\partial_k(T_{\alpha,1}f)(x) = \partial_k \int\limits_{\Omega} (-\frac{1}{4\pi})\frac{x-y}{|x-y|^3}f(y)dy$$
$$+\partial_k \int\limits_{\Omega} (-\frac{1}{4\pi})\big[\frac{e^{i\alpha|x-y|}-1}{|x-y|}\frac{x-y}{|x-y|^2} + \big(\frac{e^{i\alpha|x-y|}-1}{|x-y|} + \frac{1}{|x-y|}\big)\big(\alpha - i\alpha\frac{x-y}{|x-y|}\big)\big]f(y)dy.$$

For the latter integral, it is allowed to exchange differentiation and integration because the kernel has a singularity of order 1.

On the orther hand, the first integral has a singularity of order 2, this integral is exactly $\partial_k(T_{0,1}f)(x)$ and is investigated very carefully in [**69**, chapter IX, §7], (see also [**46, 69**]). For the reader's convennience, we will present this proof.

Under the assumption $f \in C^1_c(\Omega, \mathbb{H}(\mathbb{C}))$, i.e., it has compact support, then the function $T_{0,1}f$ can be written as

$$(T_{0,1}f)(x) = -\frac{1}{4\pi}\int\limits_{\Omega} \frac{x-y}{|x-y|^3}f(y)dy = \frac{1}{4\pi}\int\limits_{\mathbb{R}^3} \frac{y}{|y|^3}f(x-y)dy$$

Notice that its kernel is weakly singular. Using the Stokes' formula we obtain

$$\begin{aligned}
\partial_k(T_{0,1}f)(x) &= \frac{1}{4\pi}\partial_{k,x}\int\limits_{\mathbb{R}^3} \frac{y}{|y|^3}f(x-y)dy = \frac{1}{4\pi}\lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^3-B(0,\varepsilon)} \frac{y}{|y|^3}\partial_{k,x}f(x-y)dy \\
&= -\frac{1}{4\pi}\lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^3-B(0,\varepsilon)} \left[\partial_{k,y}\left(\frac{y}{|y|^3}f(x-y)\right) - \partial_{k,y}\frac{y}{|y|^3}f(x-y)\right]dy \\
&= -\frac{1}{4\pi}\lim_{\varepsilon\to 0}\int\limits_{|y|=\varepsilon} \frac{y}{|y|^3}\frac{y_k}{|y|}f(x-y)dS_\varepsilon + \frac{1}{4\pi}\lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^3-B(0,\varepsilon)} \partial_{k,y}\frac{y}{|y|^3}f(x-y)dy \\
&= \frac{1}{4\pi}\lim_{\varepsilon\to 0}\int\limits_{|y-x|=\varepsilon} \frac{x-y}{|x-y|^3}\frac{y_k-x_k}{|x-y|}f(y)dS_\varepsilon \\
&\quad -\frac{1}{4\pi}\lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^3-B(x,\varepsilon)} \partial_{k,x}\frac{x-y}{|x-y|^3}f(y)dy \\
&= \frac{1}{4\pi}\lim_{\varepsilon\to 0}\int\limits_{\partial B(x,\varepsilon)} \frac{1}{\varepsilon^2}\sum_{i=1}^{3}\frac{(x_i-y_i)e_i}{|x-y|}\cos(r,y_k)f(y)dS_\varepsilon \\
&\quad -\frac{1}{4\pi}\lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon} \partial_{k,x}\frac{x-y}{|x-y|^3}f(y)dy
\end{aligned}$$

$$= \frac{1}{4\pi}\lim_{\varepsilon\to 0}\int\limits_{\partial B(x,1)}\sum_{i=1}^{3}(x_i-y_i)e_i\cos(r,y_k)f(x+\varepsilon\omega)dS_1$$
$$-\frac{1}{4\pi}\lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon}\partial_{k,x}\frac{x-y}{|x-y|^3}f(y)dy$$

where $\omega = \frac{y-x}{|y-x|}$ and $n_k = \cos(r,y_k) = \frac{y_k-x_k}{|y-x|}$, hence $\vec{n} = (n_1,n_2,n_3)$ is the outward pointing normal vector on $\partial B(x,\varepsilon)$ at the point y. If $\varepsilon\to 0$ then the first integral converges uniformly to $\int\limits_{\partial B(x,1)}\sum\limits_{i=1}^{3}(x_i-y_i)e_i\cos(r,y_k)dS_1 f(x)$.

Since

$$\begin{aligned}\int\limits_{\partial B(x,1)}\sum_{i=1}^{3}(x_i-y_i)e_i\cos(r,y_k)dS_1 &= \sum_{i=1}^{3}e_i\int\limits_{B(x,1)}\frac{\partial(x_i-y_i)}{\partial y_k}dy\\ &= \bar{e}_k\int\limits_{\partial B(x,1)}\left(\int\limits_0^1 r^2dr\right)dS_1 = 4\pi\frac{\bar{e}_k}{3},\end{aligned}$$

this leads to $\frac{1}{4\pi}\int\limits_{\partial B(x,1)}\sum\limits_{i=1}^{3}(x_i-y_i)e_i\cos(r,y_k)dS_1 f(x) = \frac{\bar{e}_k}{3}f(x)$.

Note that as $\cos(r,y_j) = \frac{y_j-x_j}{|y-x|}$, we can write

$$\begin{aligned}-\partial_{k,x}\frac{x-y}{|x-y|^3} &= \partial_{k,x}\sum_{i=1}^{3}\frac{(x_i-y_i)\bar{e}_i}{|x-y|^3}\\ &= \sum_{k\neq j=1}^{3}-3\frac{(x_k-y_k)(x_j-y_j)\bar{e}_i}{|x-y|^5}-\frac{3(x_k-y_k)^2\bar{e}_k}{|x-y|^5}+\frac{\bar{e}_k}{|x-y|^3}\\ &= \frac{1}{|x-y|^3}\left(-3\sum_{i=1}^{3}\frac{(x_k-y_k)(x_i-y_i)}{|x-y|^2}\bar{e}_i+\sum_{j=1}^{3}\frac{(x_j-y_j)^2}{|x-y|^2}\bar{e}_k\right),\end{aligned}$$

and because of

$$\begin{aligned}&\int\limits_{\partial B(x,1)}\left(-3\sum_{i=1}^{3}\frac{(x_k-y_k)(x_i-y_i)}{|x-y|^2}\bar{e}_i+\sum_{j=1}^{3}\frac{(x_j-y_j)^2}{|x-y|^2}\bar{e}_k\right)dS_1\\ =&\int\limits_{\partial B(x,1)}\left(-3\sum_{i=1}^{3}\frac{(x_i-y_i)}{|x-y|}\bar{e}_i\cos(r,y_k)+\sum_{j=1}^{3}\frac{(x_j-y_j)}{|x-y|}\bar{e}_k\cos(r,y_j)\right)dS_1\\ =&\left(-3\frac{4\pi}{3}\bar{e}_k+\sum_{j=1}^{3}\frac{4\pi}{3}\bar{e}_k\right) = 0,\end{aligned}$$

the integral $\int\limits_{\Omega_\varepsilon}\partial_{k,x}\frac{x-y}{|x-y|^3}f(y)dy$ converges uniformly in x to the singular integral $\int\limits_{\Omega}\partial_{k,x}\frac{x-y}{|x-y|^3}f(y)dy$, for $\varepsilon\to 0$. Summing up we have (i).

(ii) In the case $n = 2$, we have

$$T_{\alpha,2}f(x) = \int\limits_{\Omega} (1 + i\alpha|x-y| + \alpha(x-y))\left(\frac{e^{i\alpha|x-y|}}{4\pi|x-y|}\right)f(y)dy.$$

The kernel of this integral has a singularity of order 1, hence it can be allowed to exchange the differentiation and the integration. Therefore, (ii) holds for $n = 2$.
For $n \geq 3$, the kernels of these integrals have no singularities, thus (ii) is easily seen. $\square$

Now we will study the properties of $\partial_k T_{\alpha,n}$. To do this, firstly we need the following lemma to hold for $0 \leq \gamma \leq 2$. This lemma can be proved by using the same idea as in [**47**, Chap. 3] for $\gamma = 0$. However, we will present this proof for the reader's convenience.

Lemma 2.14. *Let Ω be a bounded domain in $\mathbb{R}^3$, $\omega \in L_p(\Omega, \mathbb{H}(\mathbb{C}))$. We define the operator*

$$W_{2-\gamma}\omega(x) = \int\limits_{\Omega} \frac{1}{|x-y|^{2-\gamma}}\omega(y)dy, \qquad 0 \leq \gamma \leq 2,$$

being well-define in $L_p(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ and

$$||W_{2-\gamma}\omega||_{L_p} \leq M_{(\Omega,\gamma)}||\omega||_{L_p}, \qquad \textit{for } 1 < p < +\infty.$$

Proof. Firstly, we will estimate the integral for $x \in \Omega$

$$\begin{aligned}
\int\limits_{\Omega} \left(\frac{1}{|x-y|^{2-\gamma}}\right)^{\alpha} dy = \int\limits_{\Omega} \frac{1}{|x-y|^{(2-\gamma)\alpha}}dy \quad &\leq \int\limits_{|y|<(\mathrm{diam}\Omega)} \frac{1}{|y|^{(2-\gamma)\alpha}}dy \\
&\leq 4\pi \frac{(\mathrm{diam}\Omega)^{3-(2-\gamma)\alpha}}{3-(2-\gamma)\alpha}
\end{aligned} \tag{2.8}$$

with the condition $(2-\gamma)\alpha < 3$.

By using Hölder's inequality for $1 < p < +\infty$ and $x \in \Omega$ then applying inequality (2.8) in the case $\alpha = 1$ we have the following estimate

$$\begin{aligned}
\left|\int\limits_{\Omega} \frac{1}{|x-y|^{2-\gamma}}\omega(y)dy\right| &\leq \int\limits_{\Omega} \frac{1}{|x-y|^{2-\gamma}}|\omega(y)|dy \quad = \\
= \quad & \int\limits_{\Omega} \left(\frac{1}{|x-y|^{2-\gamma}}\right)^{1-\frac{1}{p}} \left(\frac{1}{|x-y|^{2-\gamma}}\right)^{\frac{1}{p}} |\omega(y)|dy \\
\leq \quad & \int\limits_{\Omega} \left[\left(\left(\frac{1}{|x-y|^{2-\gamma}}\right)^{\frac{1}{q}}\right)^{q} dy\right]^{\frac{1}{q}} \left(\int\limits_{\Omega} \left[\left(\frac{1}{|x-y|^{2-\gamma}}\right)^{\frac{1}{p}}|\omega(y)|\right]^{p} dy\right)^{\frac{1}{p}} \\
\leq \quad & \int\limits_{\Omega} \left(\frac{1}{|x-y|^{2-\gamma}}dy\right)^{\frac{1}{q}} \int\limits_{\Omega} \left(\frac{1}{|x-y|^{2-\gamma}}|\omega(y)|^{p}dy\right)^{\frac{1}{p}} \\
\leq \quad & \left(\frac{4\pi}{1+\gamma}(\mathrm{diam}\Omega)^{1+\gamma}\right)^{\frac{1}{q}} \int\limits_{\Omega} \left(\frac{1}{|x-y|^{2-\gamma}}|\omega(y)|^{p}dy\right)^{\frac{1}{p}},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Thus,

$$
\begin{aligned}
&\int\limits_{\Omega}\left|\int\limits_{\Omega}\frac{1}{|x-y|^{2-\gamma}}\omega(y)dy\right|^{p}dx\\
\leq\;& \int\limits_{\Omega}\left[\left(\frac{4\pi}{1+\gamma}(\mathrm{diam}\Omega)^{1+\gamma}\right)^{\frac{1}{q}}\int\limits_{\Omega}\left(\frac{1}{|x-y|^{2-\gamma}}|\omega(y)|^{p}dy\right)^{\frac{1}{p}}\right]^{p}dx\\
\leq\;& (\frac{4\pi}{1+\gamma})^{\frac{p}{q}}(\mathrm{diam}\Omega)^{(1+\gamma)\frac{p}{q}}\int\limits_{\Omega}\left(\int\limits_{\Omega}\frac{1}{|x-y|^{2-\gamma}}|\omega(y)|^{p}dy\right)dx\\
\leq\;& (\frac{4\pi}{1+\gamma})^{\frac{p}{q}}(\mathrm{diam}\Omega)^{(1+\gamma)\frac{p}{q}}\int\limits_{\Omega}\left(\int\limits_{\Omega}\frac{1}{|x-y|^{2-\gamma}}dx\right)|\omega(y)|^{p}dy\\
\leq\;& (\frac{4\pi}{1+\gamma})^{\frac{p}{q}+1}(\mathrm{diam}\Omega)^{(1+\gamma)(\frac{p}{q}+1)}\int\limits_{\Omega}|\omega(y)|^{p}dy,
\end{aligned}
$$

i.e $\left(\int\limits_{\Omega}|W_{2-\gamma}(x)|^{p}dx\right)^{\frac{1}{p}} \leq \quad M_{(\Omega,\gamma)}||\omega||_{L_p}$, where $M_{\Omega,\gamma}=\frac{4\pi}{1+\gamma}(\mathrm{diam}\Omega)^{1+\gamma}$.
$W_{2-\gamma}$ is a linear bounded operator, hence it is a continuous operator from $L_p(\Omega,\mathbb{H}(\mathbb{C}))$ to $L_p(\Omega,\mathbb{H}(\mathbb{C}))$. The lemma is proved. □

We note that in the case $\alpha=0$ the following theorem is taken from [**47**, Chap. 3]. Its proof is essentially due to K. Gürlebeck and W. Sprössig by using a theorem of Calderon - Zygmund ([**69**, § 3, Chap. 11], for $\Omega=\mathbb{R}^3$.

THEOREM 2.15. *The operator*

$$\partial_k T_{0,1}: L_p(\mathbb{R}^3,\mathbb{H}(\mathbb{C})) \longrightarrow L_p(\mathbb{R}^3,\mathbb{H}(\mathbb{C})), \qquad 1 < p < +\infty,$$

is well-defined, continuous and

$$||\partial_k T_{0,1}||_{\mathcal{L}(L_p(\mathbb{R}^3,\mathbb{H}(\mathbb{C})))} \leq C(4\pi)^{\frac{-1}{p}}+\frac{1}{3}, \qquad 1 < p < +\infty.$$

Analogously to the above theorem for a bounded domain we have the following same result for $T_{\alpha,n}$. We refer to [**45**] for proving this property in the case $n=1$. For sake of completeness, here we also give its proof.

THEOREM 2.16. *The operators $\partial_k T_{\alpha,n}: L_p(\Omega,\mathbb{H}(\mathbb{C})) \longrightarrow L_p(\Omega,\mathbb{H}(\mathbb{C}))$ are well-defined and continuous for all $n \geq 1$ and $1\leq k\leq 3$.*

PROOF. From Lemma 2.13 we have

$$\partial_k(T_{\alpha,1}f)(x)=\partial_k(T_{0,1}f)(x) + \frac{1}{4\pi}\int\limits_{\Omega}\tilde{K}_{\alpha,11}(x-y)f(y)dy,$$

$$\partial_k(T_{\alpha,2}f)(x)=\frac{1}{4\pi}\int\limits_{\Omega}\tilde{K}_{\alpha,21}(x-y)f(y)dy,$$

where

$$\begin{aligned}
\tilde{K}_{\alpha,11}(x-y) \quad &= \frac{x_k-y_k}{|x-y|}\frac{x-y}{|x-y|}\frac{e^{i\alpha|x-y|}-1}{|x-y|}\left(\frac{i\alpha}{|x-y|}-\frac{3}{|x-y|^2}\right)\\
&+\frac{e^{i\alpha|x-y|}-1}{|x-y|}\frac{1}{|x-y|^2}e_k\\
&+\left(\frac{e^{i\alpha|x-y|}-1}{|x-y|}+\frac{1}{|x-y|}\right)\left(i\alpha\frac{e_k}{|x-y|}-i\alpha\frac{x_k-y_k}{|x-y|}\frac{x-y}{|x-y|}\frac{1}{|x-y|}\right)\\
&-\frac{x_k-y_k}{|x-y|}(\alpha-i\alpha\frac{x-y}{|x-y|})\left[\frac{e^{i\alpha|x-y|}-1}{|x-y|}\left(i\alpha-\frac{3}{|x-y|}\right)+\left(i\alpha-\frac{3}{|x-y|}\right)\right],
\end{aligned}$$

$$\begin{aligned}
\tilde{K}_{\alpha,21}(x-y) = \quad & \frac{x_k-y_k}{|x-y|}\left(\frac{e^{i\alpha|x-y|}-1}{|x-y|}+\frac{1}{|x-y|}\right)\left(i\alpha-\frac{1}{|x-y|}\right)\\
&+\alpha^2\frac{x_k-y_k}{|x-y|}e^{i\alpha|x-y|}\ +\ \alpha e_k\left(\frac{e^{i\alpha|x-y|}-1}{|x-y|}+\frac{1}{|x-y|}\right)\\
&+\alpha\left[\frac{x-y}{|x-y|}\frac{x_k-y_k}{|x-y|}\left((i\alpha|x-y|-1)-(\frac{e^{i\alpha|x-y|}-1}{|x-y|}+\frac{1}{|x-y|})\right)\right].
\end{aligned}$$

We can use analogous estimations as in Theorem 2.10 for the kernel functions $\tilde{K}_{\alpha,11}(x-y)$, $\tilde{K}_{\alpha,21}(x-y)$. These estimations together with Theorem 2.15 reduce these problems to the real integral

$$W_{2-\gamma}\omega(x) = \int\limits_{\Omega}\frac{1}{|x-y|^{2-\gamma}}\omega(y)dy, \qquad 0\le\gamma\le 2.$$

It is well-defined in $L_p(\Omega,\mathbb{H}(\mathbb{C}))$ and continuous. By Lemma 2.14 and noticing that $C_c^\infty(\Omega,\mathbb{H}(\mathbb{C}))$ is dense in $L_p(\Omega,\mathbb{H}(\mathbb{C}))$ the theorem follows for $n=1$ and $n=2$.

For $n=3$, we need only consider $W_{2-\gamma}\omega(x)$ for $0\le\gamma\le 1$. In the case $n>3$, then $\partial_k T_{\alpha,n}f(x)$ has no singularity. By the assumption that Ω is a bounded domain we get

$$||\partial_k T_{\alpha,n}f(x)||_{L_p}\ \le\ M_{(\Omega,\alpha,n)}||f||_{L_p}.$$

Hence, the assertion of the theorem holds for all $1\le n\in\mathbb{N}$. □

We now are able to come to our main result in the following section continuing the mapping properties of $T_{\alpha,n}$ between spaces of continuous functions.

5. Mapping properties of $T_{\alpha,n}$

In this section we will give the most important properties of $T_{\alpha,n}$ in Theorem 2.20 and Remark 2.21. They are one of main tools for dealing some boundary value problems for the Helmholtz equation which are discussed in the next chapter.

THEOREM 2.17. *The operator* $T_{\alpha,n}:\ L_p(\Omega,\mathbb{H}(\mathbb{C}))\longrightarrow W_p^1(\Omega,\mathbb{H}(\mathbb{C}))$ *is well-defined and continuous for all* $n\ \ge\ 1$.

PROOF. From Theorem 2.16 we can see that $\partial_k T_{\alpha,n}: L_p(\Omega,\mathbb{H}(\mathbb{C}))\longrightarrow L_p(\Omega,\mathbb{H}(\mathbb{C}))$ is well-defined and continuous for all $n\ \ge\ 1$. In order to show $T_{\alpha,n}f\ \in\ W_p^1(\Omega,\mathbb{H}(\mathbb{C}))$ for every $f\in L_p(\Omega,\mathbb{H}(\mathbb{C}))$, we at first verify that the operator $T_{\alpha,n}$ acts invariantly on $L_p(\Omega,\mathbb{H}(\mathbb{C}))$.

Indeed, for $n = 1$ by Theorem 2.12 if $1 \le q \le \frac{3}{2}$, the linear operator $T_{\alpha,1}$ is bounded, i.e., $T_{\alpha,1}$ also belongs to $\mathcal{L}(L_1(\Omega, \mathbb{H}(\mathbb{C})),\ L_q(\Omega, \mathbb{H}(\mathbb{C})))$. This implies that

$$T_{\alpha,1} \ \in \ \mathcal{L}(L_q(\Omega, \mathbb{H}(\mathbb{C})),\ L_q(\Omega, \mathbb{H}(\mathbb{C}))) \text{ for } 1 \le q \le \ \frac{3}{2}.$$

Consequently, by Theorem 2.16 we get

$$T_{\alpha,1} \ \in \ \mathcal{L}(L_q(\Omega, \mathbb{H}(\mathbb{C})),\ W_q^1(\Omega, \mathbb{H}(\mathbb{C}))) \text{ for } 1 \le q \le \ \frac{3}{2}.$$

Sobolev's imbedding theorems, the algebraic expression (1.7), implies that

$$W_q^1(\Omega, \mathbb{H}(\mathbb{C}))) \ \subset \ L_{\frac{3q}{3-q}}(\Omega, \mathbb{H}(\mathbb{C})) \text{ for } 1 \le q \le \ \frac{3}{2}.$$

Hence,

$$T_{\alpha,1} \ \in \ \mathcal{L}(L_q(\Omega, \mathbb{H}(\mathbb{C})),\ L_r(\Omega, \mathbb{H}(\mathbb{C}))) \text{ for } 1 \le q \le \ \frac{3}{2},\ 1 \ \le r \le \ 3.$$

It is clear that $T_{\alpha,1} \ \in \ \mathcal{L}(L_q(\Omega, \mathbb{H}(\mathbb{C})),\ L_r(\Omega, \mathbb{H}(\mathbb{C})))$ for $1 \ \le r \le \ 3$. Using Theorem 2.16 again we get $T_{\alpha,1}f \ \in \ W_r^1(\Omega, \mathbb{H}(\mathbb{C}))$ for all $1 \ \le r \le \ 3$ and $f \in L_q(\Omega, \mathbb{H}(\mathbb{C}))$. Sobolev's imbedding theorems, the algebraic expression (1.7), guarantees that

$$W_r^1(\Omega, \mathbb{H}(\mathbb{C}))) \ \subset \ L_{\frac{3r}{3-r}}(\Omega, \mathbb{H}(\mathbb{C})) \text{ for } r \ < \ 3.$$

This means that

$$T_{\alpha,1} \ \in \ \mathcal{L}(L_s(\Omega, \mathbb{H}(\mathbb{C})),\ L_s(\Omega, \mathbb{H}(\mathbb{C}))) \text{ for } 1 \le s \le \ +\infty.$$

In the case $n = 2$, if $1 \le q \le \ 3$ we have

$$T_{\alpha,2} \ \in \ \mathcal{L}(L_1(\Omega, \mathbb{H}(\mathbb{C})),\ W_q^1(\Omega, \mathbb{H}(\mathbb{C}))).$$

Consequently, we can say that

$$T_{\alpha,2} \ \in \ \mathcal{L}(L_q(\Omega, \mathbb{H}(\mathbb{C})),\ W_q^1(\Omega, \mathbb{H}(\mathbb{C}))) \text{ for } 1 \le q \le \ 3.$$

Using Sobolev's imbedding theorems again leads to

$$T_{\alpha,2} \ \in \ \mathcal{L}(L_s(\Omega, \mathbb{H}(\mathbb{C})),\ L_s(\Omega, \mathbb{H}(\mathbb{C}))) \text{ for } 1 \le s \le \ +\infty.$$

The assertion of this theorem follows immediately from Theorem 2.12 for every $n \ge 3$. $\square$

REMARK 2.18. In the case $p > 3$, by Sobolev's imbedding theorems, the algebraic expression (1.8),

$$W_q^1(\Omega, \mathbb{H}(\mathbb{C}))) \ \subset \ C_b(\Omega, \mathbb{H}(\mathbb{C}))), i.e., T_{\alpha,n} \ \in \ \mathcal{L}(L_p(\Omega, \mathbb{H}(\mathbb{C})), C_b(\Omega, \mathbb{H}(\mathbb{C}))).$$

Analogously to Corollary 3.10 in [**47**] we have $T_{\alpha,n}: \ L_p(\Omega, \mathbb{H}(\mathbb{C})) \ \longrightarrow \ L_p(\Omega, \mathbb{H}(\mathbb{C}))$ is compact for every $1 \le n \in \mathbb{N},\ p \in [1, +\infty)$. This may be proved by the same discussion as in [**76**, Theorem 7.83].

THEOREM 2.19. *When Ω is a bounded domain in $\mathbb{R}^3$ then*

$$T_{\alpha,n}: \ C(\bar{\Omega}, \mathbb{H}(\mathbb{C})) \ \longrightarrow \ C(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$$

is bounded and

(i) $||T_{\alpha,1}||_{\mathcal{L}(C(\bar{\Omega}), C(\bar{\Omega}))} \ \le \ \frac{\sqrt{2}}{4\pi} e^{-\mathrm{Im}\alpha \mathrm{diam}\Omega} \max\limits_{x \in \Omega} \left\{ \int\limits_\Omega \big(2|\alpha| \frac{1}{|x-y|} + \frac{1}{|x-y|^2}\big) dy \right\},$

(ii) $||T_{\alpha,2}||_{\mathcal{L}(C(\bar{\Omega}), C(\bar{\Omega}))} \ \le \ \frac{\sqrt{2}}{4\pi} e^{-\mathrm{Im}\alpha \mathrm{diam}\Omega} \max\limits_{x \in \Omega} \left\{ \int\limits_\Omega \big(2|\alpha| + \frac{1}{|x-y|}\big) dy \right\},$

(iii) $||T_{\alpha,n}||_{\mathcal{L}(C(\bar{\Omega}),C(\bar{\Omega}))} \leq \frac{\sqrt{2}}{4\pi} e^{-\mathrm{Im}\alpha \mathrm{diam}\Omega} \max_{x\in\Omega} \left\{ \int_\Omega \left(2|\alpha||x-y|^{n-2} + |x-y|^{n-3}\right) dy \right\}$, *for every* $n \geq 3$.

Proof. Let $f \in C(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$, and we note that $C(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$ is dense in $L_1(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$. For $p > 3$, by the remarks preceding the theorem, together with Theorem 2.16 we have $f \in L_p(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$ and $T_\alpha f \in C_b(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$. With an arbitrarily fixed $x \in \bar{\Omega}$ we get the estimates

$$\begin{aligned} ||T_{\alpha,1} f(x)||_{\mathbb{H}(\mathbb{C})} &\leq \sqrt{2} \int_\Omega |K_\alpha^{(1)}(x-y)||f(y)|dy \\ &\leq \frac{\sqrt{2}}{4\pi} ||f||_{C(\bar{\Omega})} e^{-\mathrm{Im}\alpha \max_y \{|x-y|\}} \left\{ \int_\Omega \left(2|\alpha| \frac{1}{|x-y|} + \frac{1}{|x-y|^2}\right) dy \right\}, \\ ||T_{\alpha,2} f(x)||_{\mathbb{H}(\mathbb{C})} &\leq \sqrt{2} \int_\Omega |K_\alpha^{(2)}(x-y)||f(y)|dy \\ &\leq \frac{\sqrt{2}}{4\pi} ||f||_{C(\bar{\Omega})} e^{-\mathrm{Im}\alpha \max_y \{|x-y|\}} \left\{ \int_\Omega \left(2|\alpha| + \frac{1}{|x-y|}\right) dy \right\}, \end{aligned}$$

and for $n \geq 3$,

$$\begin{aligned} ||T_{\alpha,n} f(x)||_{\mathbb{H}(\mathbb{C})} &\leq \sqrt{2} \int_\Omega |K_\alpha^{(2)}(x-y)||f(y)|dy \\ &\leq \frac{\sqrt{2}}{4\pi} ||f||_{C(\bar{\Omega})} e^{-\mathrm{Im}\alpha \max_y \{|x-y|\}} \left\{ \int_\Omega \left(2|\alpha||x-y|^{n-2} + |x-y|^{n-3}\right) dy \right\}. \end{aligned}$$

Taking the maximum with respect to $x \in \overline{\Omega}$, the norm inequalities (i)-(iii) hold. □

Theorem 2.20. *The following assertions hold.*

(i) *The operator* $T_{\alpha,1}$ *is the algebraic right - inverse to the operator* D_α, *i.e., for any* $f \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \cap C(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$, *we have*

$$D_\alpha T_{\alpha,1} f(x) = f(x), \quad \textit{for every } x \in \Omega,$$

(ii) $D_\alpha T_{\alpha,n} f(x) = T_{\alpha,n-1} f(x)$, *for every* $x \in \Omega,\ n \geq 2$,

(iii) $D_\alpha^n T_{\alpha,n} f(x) = f(x)$, *for every* $x \in \Omega,\ n \geq 1$.

Proof. (i)In Lemma 2.13 it has been shown that $\partial T_{\alpha,1}$ is a strongly singular integral which a singularity of order 3. Thus in order to prove (i) we need some steps based on two techniques which can be found in [**27**, Chapter III, Theorem 3], see also [**46, 93**].

Firstly, we prove that (i) holds for $f \in C_c^1(\Omega, \mathbb{H}(\mathbb{C}))$. Indeed, for every $f \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$ having its compact support supp $f \subset\subset \Omega$, the function $T_{\alpha,1} f$ can be written as

$$\begin{aligned} T_{\alpha,1} f(x) &= \int_\Omega K_\alpha^{(1)}(x-y) f(y) dy = \int_{\mathbb{R}^3} K_\alpha^{(1)}(x-y) f(y) dy \\ &= -\int_{\mathbb{R}^3} K_\alpha^{(1)}(y) f(x-y) dy. \end{aligned}$$

Because $f(x)$ has compact support this permits us to exchange differentiation and integration. Thus we have

$$\begin{aligned}
D_\alpha T_{\alpha,1} f(x) &= -D_{\alpha,x} \int\limits_{\mathbb{R}^3} K_\alpha^{(1)}(y) f(x-y) dy \\
&= -\lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^3 - B(0,\varepsilon)} D_{\alpha,x} K_\alpha^{(1)}(y) f(x-y) dy \\
&= -\lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^3 - B(0,\varepsilon)} \left(\sum_{k=1}^3 e_k K_\alpha^{(1)}(y) \frac{\partial}{\partial x_k} f(x-y) + \alpha K_\alpha^{(1)}(y) f(x-y) \right) dy \\
&= -\lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^3 - B(0,\varepsilon)} \left(-\sum_{k=1}^3 e_k K_\alpha^{(1)}(y) \frac{\partial}{\partial y_k} f(x-y) + \alpha K_\alpha^{(1)}(y) f(x-y) \right) dy \\
&= -\lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^3 - B(0,\varepsilon)} \left(-D_y (K_\alpha^{(1)}(y) f(x-y)) + (D_{\alpha,y} K_\alpha^{(1)}(y)) f(x-y) \right) dy
\end{aligned}$$

where $B(0,\varepsilon) = \{y \in \mathbb{R}^3, |y| \ < \ \varepsilon\}$.
We note that $D_{\alpha,y} K_\alpha^{(1)}(y) = 0$ for $y \neq 0$, and f has compact support. Using the Stokes formula, hence, this equality can be rewritten as

$$\begin{aligned}
D_\alpha T_{\alpha,1} f(x) &= -\lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^3 - B(0,\varepsilon)} -D_y (K_\alpha^{(1)}(y) f(x-y)) dy \\
&= -\lim_{\varepsilon\to 0} \int\limits_{\partial B(0,\varepsilon)} K_\alpha^{(1)}(y) \vec{n}(y) f(x-y)) d\Gamma_y \\
&= -\lim_{\varepsilon\to 0} \int\limits_{\partial B(0,\varepsilon)} K_\alpha^{(1)}(y) \vec{n}(y) f(x)) d\Gamma_y \\
&\quad + \lim_{\varepsilon\to 0} \int\limits_{\partial B(0,\varepsilon)} K_\alpha^{(1)}(y) \vec{n}(y) [f(x) - f(x-y))] d\Gamma_y .
\end{aligned}$$

By $\lim\limits_{\varepsilon\to 0} \int\limits_{\partial B(0,\varepsilon)} K_\alpha^{(1)}(y) \vec{n}(y) f(x)) d\Gamma_y = -f(x)$ as well as $f \in C_c^1(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$, we obtain

$$D_\alpha T_{\alpha,1} f(x) = f(x).$$

We prove next that (i) holds for any $f \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$. To that purpose, let $x \in \Omega$ be arbitrary. Thus, we can select a neighborhood $\mathcal{U}_x$ of x such that $\mathcal{U}_x \subset\subset \Omega$.
Take a $C^1(\Omega, \mathbb{H}(\mathbb{C}))-$ function $\psi|_{\mathcal{U}_x} = 1$ and supp$\psi \subset\subset \Omega$. Then we have $f = \psi f + (1-\psi) f$ and

$$\psi f \in C_c^1(\Omega, \mathbb{H}(\mathbb{C})), \qquad (\psi f)|_{\mathcal{U}_x} = f,$$

$$(1-\psi) f \in C^1(\Omega, \mathbb{H}(\mathbb{C})), \qquad ((1-\psi) f)|_{\mathcal{U}_x} = 0,$$

From the first step we get $D_\alpha T_\alpha \psi f = \psi f = f$ in $\mathcal{U}_x$. As $(1-\psi) f$ equals zero in $\mathcal{U}_x$ we obtain

$$\begin{aligned} D_\alpha T_{\alpha,1}(1-\psi)f(x) &= D_\alpha\left(\int_\Omega K_\alpha^{(1)}(x-y)(1-\psi)f(y)dy\right) \\ &= D_\alpha\left(\int_{\Omega-\mathcal{U}_x} K_\alpha^{(1)}(x-y)(1-\psi)f(y)dy\right) \\ &= \int_{\Omega-\mathcal{U}_x} [D_{\alpha,x}K_\alpha^{(1)}(x-y)](1-\psi)f(y)dy = 0. \end{aligned}$$

Hence, $D_\alpha T_{\alpha,1}f(x) = D_\alpha T_{\alpha,1}\psi f(x) + D_\alpha T_{\alpha,1}(1-\psi)f = f$. Because x is taken arbitrarily in Ω then assertion (i) of the theorem follows.

(ii) By Lemma 2.13, the operator $D_{\alpha,x}$ acting on $T_{\alpha,n}f(x)$ can be interchanged with integration for any $n \geq 2$ as in these cases the singularity at $y = x$ of the kernels $D_{\alpha,x}K_\alpha^{(n)}(x-y), n \geq 2$ is not worse than $O(\frac{1}{|x-y|^2})$, allowing differentiation under the integral of $T_{\alpha,n}f$. By using Lemma 2.1, (also in [**49**, Lemma 2.5]) the identify (ii) holds.

(iii) By induction, together with (i), (ii) we obtain (iii). □

Remark 2.21. For every $n \geq 1$ the operator $T_{\alpha,n} : \ L_p(\Omega, \mathbb{H}(\mathbb{C})) \longrightarrow W_p^n(\Omega, \mathbb{H}(\mathbb{C}))$ is well-defined and continuous.

Consequently, we also obtain

$$T_{\alpha,n} : \ W_p^k(\Omega, \mathbb{H}(\mathbb{C})) \longrightarrow W_p^{k+n}(\Omega, \mathbb{H}(\mathbb{C})).$$