Introduction

The importance of complex analysis for mathematical physics is that the Cauchy-Riemann differential operator and its complex conjugate provide a factorization of the two-dimensional Laplace operator. For $W(z) = u(x, y) - iv(x, y)$ being a solution of the Cauchy-Riemann system which can be written in complex form as

$$\frac{\partial W}{\partial \bar{z}} = 0, \tag{0.1}$$

where $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. Similarly, $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ is used. The Laplace operator $\Delta = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$ arises from various applications in many partial differential equations in mathematical physics e.g., from the Poinsson, the wave and the heat equation. For the equation (0.1), two linearly independent fundamental solutions of these equation

$$W_1 = 2\frac{\partial \ln|z|}{\partial z} = \frac{1}{z},$$

$$W_2 = 2i\frac{\partial \ln|z|}{\partial z} = \frac{i}{z},$$

are connected with the kernel of the Cauchy-type integral. In case of the inhomogeneous Cauchy-Riemann equation

$$\frac{\partial W}{\partial \bar{z}} = f, \tag{0.2}$$

they lead to the Cauchy-Pompeiu integral representation formulas. Because of the Cauchy integral and the Cauchy-Pompeiu formulas complex analytic methods are very powerful. We refer to Begehr [**7**], Dzhuraev [**36**] and Vekua [**86**] for numerous applications in dealing partial differential equations in complex analysis.

A most simple and sufficiently natural generalization of the Lapalace is the Helmholtz equation. The Helmholtz operator,$(\Delta + \alpha^2)$, $\alpha \in \mathbb{C}$, plays an important role and it offen arises in applications of physics. It can also be factorized using the Cauchy- Riemann operator in case of two dimensions. However, a lot of physical problems are not only particular circumstances at two dimensions. For example, an overwhelming majority of physically meaningful problems can not be reduced to two-dimensional models. Here for physics the particular case of quaternions are of special interest as it is related to four dimensions. Recently, in the book [**60**], the Maxwell equations were investigated and boundary value problems for electromagnetic fields solved via quaternionic analysis. Here the wave equation is of importance which is formally related with the Helmholtz equation of three variables and can be treated by complex analytic methods.

Moreover, the Dirac operator D generalizing the Cauchy-Riemann operator in higher dimensional spaces provides a factorization of the Laplace operator. Likewise the Helmholtz operator, $(\Delta + \alpha^2)$, $\alpha \in \mathbb{C}$, can be factorized in quaternionic analysis by a certain first order partial differential operator $D_\alpha := D + \alpha$ and $D_{-\alpha} := D - \alpha$ where $D := \sum_{k=1}^{3} e_k \frac{\partial}{\partial x_k}$.

Powers of this operator and of the Helmholtz operator lead to model equations of higher order. Their counterparts are studied extensively and a basis is founded for investigating more general higher order equations by singular integral equations for certain potentials related to the leading term of the equation. The potentials are provided by higher order Pompeiu integral operators. They appear as *"iteration"* of the first order Pompeiu operator and their kernels are fundamental solutions to the related differential operator. At this point we would like to note that the idea of *"iteration"* to obtain a higher order Cauchy-Pompeiu representation and higher order Cauchy kernels was introduced by Begehr [**9, 10, 11, 14**] and Dzhuraev [**36**] where the representations are related to powers of the Cauchy-Riemann or the Dirac operator and the Laplace operator. It has been then used, by Mshimba [**73, 74**], to treat some boundary value problems for generalized polyanalytic functions of order n in the Sobolev space $W^{1,p}(\Omega)$ or by Akal [**3**], Dzhuraev [**36, 37**] for complex elliptic partial differential equations of higher order. Subject of our work is the Helmholtz operator in quaternionic analysis and its factors under these ideas. The thesis may be summarized as follows.

From the quaternionic form of the Stokes theorem Cauchy-Pompeiu representation formulas are provided related to both factors of the Helmholtz operator. They lead by iteration to a second order Cauchy-Pompeiu formula related to the Helmholtz equation. Further iterations lead to higher order Cauchy-Pompeiu representations related to powers of the factors of the Helmholtz operator and of the Helmholtz operator itself. These iterations result as well a fundamental solutions to the higher order operators as define higher order Pompeiu integral operators. These integral operators in quaternionic analysis are also called Teodorescu operators. Their properties being important for treating boundary valued problems for the related model equation and for solving more general partial differential equations are studied in detail. Then the Dirichlet boundary value problem is treated for the inhomogeneous D_α, the inhomogeneous Helmholtz equation and the inhomogeneous higher order Helmholtz equation by using the properties of $T_{\alpha,n}$ as well as the projections defined by the type of singular Cauchy operator. As applications some orthogonal decompositions of the space of square integralble quaternionic functions on bounded regular domains are provided. Finally, as generalization of powers of the D_α and of the Helmholtz operator certain polynomial operators are investigated consisting of products of powers of D_α operators for different α's. A fundamental solution is constructed and general Cauchy-Pompeiu representations are proved.

The thesis is divided into five chapters. In Chapter 1, we briefly recall some basic concepts of the algebraic structure properties of complex quaternions. We also include in this chapter some important results such as the quaternionic Stokes formula and the Cauchy-Pompeiu integral representation formulas of first order.

In Chapter 2, using the fundamental solution of the Helmholtz equation we can construct the explicit forms of the fundamental solution for powers of the factors of the Helmholtz operator. Thus, the Cauchy-Pompeiu type representation formulas in terms of powers of the factors of the Helmholtz operator are proved. These results lend assistance aid to investigate some properties of higher order Teodorescu operators. We refer to [**16, 47**] for the Teodorescu transform in the case $\alpha = 0$. They are operators of Calderon-Zygmund type and do not cause problems. However, in the general cases $\alpha \neq 0$, the situation becomes more complicated. How to overcome these difficulties in investigating their mapping properties are very careful shown in this chapter.

In Chapter 3, using the fundamental solution of the Helmholtz equation in order to build a systematical theory for the metaharmonic functions introduced and developed in Chapter 1 and Chapter 2 the Dirichlet problem for the Helmholtz equation and the higher order Helmholtz equation is studied.

In Chapter 4, we use the same ideas as in Chapter 2 to obtain the representations of solutions to inhomogeneous power Helmholtz equations. More interesting, the orthogonal decompositions in complex quaternion-valued Hilbert spaces with respect to *left(right)* $\alpha-$*hyperholomorphic functions* as well as *poly-left(right)* $\alpha-$*hyperholomorphic functions* and *polymetaharmonic functions* are provided. Using these results the Dirichlet problem for the inhomogeneous bimetaharmonic equation

$$\begin{cases} (\Delta+\alpha^2)^2 u &= 0 \qquad \text{in } \Omega, \\ tr_\Gamma u &= g_1 \qquad \text{on } \Gamma, \\ tr_\Gamma D_\alpha u &= g_2 \qquad \text{on } \Gamma, \end{cases}$$

is considered.

In Chapter 5, we use the same approach and ideas as in Chapter 2 and Chapter 4 to obtain similar results for a more general polynomial differential equation. This operator $\prod_{\nu=1}^{j}(D+\alpha_\nu)^{k_\nu}$ is composed as a product of powers of D_α's with different α's. The Cauchy-Pompeiu integral representation formulas for its solution are proved.

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