Appendix D

Normal Graphs

In this chapter to set up the scalar problem which is equivalent to (MCF) we use a similar framework to that seen in [12]. We shall see that a quasi-linear parabolic partial differential equation can be obtained which is equivalent up to tangential diffeomorphisms.

The scalar problem shall be derived in a general setting, with the base surface being any smooth surface.

Suppose that we have a family $(M_t)_{t\in[0,T)}$ of n-dimensional sub-manifolds of $\mathbb{R}^{n+1}$ evolving by (MCF), that is there exists a family of immersions $\mathbf{F} : M_r^n \subset \mathbb{R}^{n+1} \times [0,T) \to \mathbb{R}^{n+1}$ satisfying

$$\frac{d\mathbf{F}}{dt}(\boldsymbol{p},t) = \mathbf{H}(\mathbf{F}(\boldsymbol{p},t)), \quad (\boldsymbol{p},t) \in M_r^n \times [0,T)$$

such that $M_t = \mathbf{F}(\cdot,t)(M_r^n)$.

We are going to investigate a special class of surfaces, the so called normal graphs (over M_r^n). These are immersions of the form

$$\mathbf{F}(\boldsymbol{q},t) = \boldsymbol{q} + \rho(\boldsymbol{q},t)\boldsymbol{\omega}(\boldsymbol{q}), \quad \boldsymbol{q} \in M_r^n, t \in [0,T)$$

for some scalar function ρ, which we call the graph height (above M_r^n). Our aim is to derive the parabolic PDE describing the evolution of the graph height of the evolving surfaces.

D.1 Preliminaries

Consider a foliation of $\mathbb{R}^{n+1}$, given by the smooth level sets of a function $\sigma : \mathbb{R}^{n+1} \to \mathbb{R}$, defining the leaves

$$M_r^n = \left\{\mathbf{x} \in \mathbb{R}^{n+1} : \sigma(\mathbf{x}) = r\right\}, \quad r > 0$$

We may express the outward unit normal at $\boldsymbol{q} \in M_r^n$ by

$$\boldsymbol{\omega}(\boldsymbol{q}) = \frac{\overline{\nabla}\sigma(\boldsymbol{q})}{|\overline{\nabla}\sigma(\boldsymbol{q})|}$$

Upon these level sets, we consider the immersion $\mathbf{X} : M_r^n \times (-\delta, \delta) \to \mathbb{R}^{n+1}, \delta > 0$ defined by

$$\mathbf{X}(\boldsymbol{q}, \varepsilon) = \boldsymbol{q} + \varepsilon \boldsymbol{\omega}(\boldsymbol{q})$$

If we choose δ sufficiently small, then $\mathbf{X}$ is a smooth diffeomorphism onto its image $\mathcal{R}_r$. It is convenient to consider $\mathbf{X}^{-1} = (\mathbf{S}, \Lambda)$, where $\mathbf{S} : \mathcal{R}_r \to M_r^n$ and $\Lambda : \mathcal{R}_r \to (-\delta, \delta)$ where $\mathbf{S}(\mathbf{x})$ is the unique closest point in M_r^n to $\mathbf{x} \in \mathcal{R}_r$ and $\Lambda(\mathbf{x})$ is the signed distance from M_r^n to $\mathbf{x} \in \mathcal{R}_r$. Define the vector $\boldsymbol{\omega}$ by

$$\boldsymbol{\omega}(\mathbf{x}) = \overline{\nabla}\Lambda(\mathbf{x}), \quad \mathbf{x} \in \mathcal{R}_r$$

which is a unit vector since $|\overline{\nabla}\Lambda| = 1$. Furthermore, since $M_r^n = \Lambda^{-1}(\cdot)\{0\}$, this unit vector, ambiently defined on all of $\mathcal{R}_r$, gives the normal vector to M_r^n at $\mathbf{S}(\mathbf{x})$.

Now, we choose a sufficiently smooth function (representing height above the foliation) $\rho : M_r^n \times [0, T) \to (-\delta, \delta)$. and let $\Phi_\rho : \mathcal{R}_r \times [0, T) \to \mathbb{R}$ be the function defined by

$$\Phi_\rho(\mathbf{x}, t) = \Lambda(\mathbf{x}) - \rho(\mathbf{S}(\mathbf{x}), t)$$

Thus we have for each $t \in [0, T)$ the smooth level sets

$$\begin{aligned} \tilde{M}_t &= \Phi_\rho^{-1}(\cdot, t)\{0\} \\ &= \left\{\mathbf{x} \in \mathbb{R}^{n+1} : \Phi_\rho(\mathbf{x}, t) = 0\right\} \end{aligned}$$

Now, consider the immersion $\tilde{\mathbf{F}} : M_r^n \times [0, T) \to \mathbb{R}^{n+1}$ defined by

$$\tilde{\mathbf{F}}(\boldsymbol{q}, t) = \mathbf{X}(\boldsymbol{q}, \rho(\boldsymbol{q}, t))$$

then we have $\tilde{M}_t = \tilde{\mathbf{F}}(\cdot, t)(M_r^n)$. We are now going to force $\tilde{M}_t$ to be a family of surfaces satisfying the evolution equation

$$\left(\frac{d\tilde{\mathbf{F}}}{dt}(\boldsymbol{q}, t)\right)^{\perp} = \mathbf{H}\left(\tilde{\mathbf{F}}(\boldsymbol{q}, t)\right), \quad (\boldsymbol{q}, t) \in M_r^n \times [0, T)$$

The flow defined by this equation is equivalent to (MCF), up to tangential diffeomorphisms, in a way we will now make clear.

Let $\boldsymbol{\varphi} : M_r^n \times [0, T) \to M_r^n$ be a diffeomorphism satisfying

$$\overline{\nabla}_{\boldsymbol{q}} \tilde{\mathbf{F}}(\boldsymbol{\varphi}(\boldsymbol{p}, t), t) \left(\frac{\partial \boldsymbol{\varphi}}{\partial t}(\boldsymbol{p}, t)\right) = -\left(\frac{\partial \tilde{\mathbf{F}}}{\partial t}(\boldsymbol{\varphi}(\boldsymbol{p}, t), t)\right)^{\top} \tag{D.1}$$

Now, let $\mathbf{F}(\boldsymbol{p},t) = \tilde{\mathbf{F}}(\boldsymbol{\varphi}(\boldsymbol{p},t),t)$ and compute

$$\begin{aligned}
\frac{d\mathbf{F}}{dt}(\boldsymbol{p},t) &= \overline{\nabla}_{\boldsymbol{q}}\tilde{\mathbf{F}}(\boldsymbol{\varphi}(\boldsymbol{p},t),t)\left(\frac{\partial\boldsymbol{\varphi}}{\partial t}(\boldsymbol{p},t)\right) + \frac{\partial\tilde{\mathbf{F}}}{\partial t}(\boldsymbol{\varphi}(\boldsymbol{p},t),t) \\
&= -\left(\frac{d\tilde{\mathbf{F}}}{dt}(\boldsymbol{\varphi}(\boldsymbol{p},t),t)\right)^{\top} + \frac{\partial\tilde{\mathbf{F}}}{\partial t}(\boldsymbol{\varphi}(\boldsymbol{p},t),t) \\
&= \left(\frac{\partial\tilde{\mathbf{F}}}{\partial t}(\boldsymbol{\varphi}(\boldsymbol{p},t))\right)^{\perp} \\
&= \mathbf{H}\left(\tilde{\mathbf{F}}(\boldsymbol{\varphi}(\boldsymbol{p},t))\right) \\
&= \mathbf{H}\left(\mathbf{F}(\boldsymbol{p},t)\right)
\end{aligned}$$

and thus, we have shown that this is equivalent to a surface $M_t = \mathbf{F}(\cdot,t)(M_r^n)$ evolving by (MCF), up to these tangential diffeomorphisms.

We may now write the normal vector $\boldsymbol{\nu}$ at $\tilde{\mathbf{F}}(\boldsymbol{q},t)$ as

$$\boldsymbol{\nu}(\tilde{\mathbf{F}}(\boldsymbol{q},t)) = \left.\frac{\overline{\nabla}\Phi_\rho(\mathbf{x},t)}{|\overline{\nabla}\Phi_\rho(\mathbf{x},t)|}\right|_{\mathbf{x}=\tilde{\mathbf{F}}(\boldsymbol{q},t)}$$

and thus the mean curvature H at $\tilde{\mathbf{F}}(\boldsymbol{q},t)$ is

$$H(\tilde{\mathbf{F}}(\boldsymbol{q},t)) = \operatorname{div}_{\mathbb{R}^{n+1}}\left.\left(\frac{\overline{\nabla}\Phi_\rho(\mathbf{x},t)}{|\overline{\nabla}\Phi_\rho(\mathbf{x},t)|}\right)\right|_{\mathbf{x}=\tilde{\mathbf{F}}(\boldsymbol{q},t)}$$

Now, using (D.1), we obtain

$$\left\langle\frac{d\tilde{\mathbf{F}}}{dt}(\boldsymbol{q},t),\boldsymbol{\nu}(\tilde{\mathbf{F}}(\boldsymbol{q},t))\right\rangle = -H(\tilde{\mathbf{F}}(\boldsymbol{q},t))$$

however, we also have

$$\left\langle\frac{d\tilde{\mathbf{F}}}{dt}(\boldsymbol{q},t),\boldsymbol{\nu}(\tilde{\mathbf{F}}(\boldsymbol{q},t))\right\rangle = \frac{\partial\rho}{\partial t}(\boldsymbol{p},t)\left\langle\boldsymbol{\nu}(\tilde{\mathbf{F}}(\boldsymbol{q},t)),\boldsymbol{\omega}(\boldsymbol{q})\right\rangle$$

which combined with the fact that

$$\left\langle\boldsymbol{\nu}(\tilde{\mathbf{F}}(\boldsymbol{q},t)),\boldsymbol{\omega}(\boldsymbol{q})\right\rangle = |\overline{\nabla}\Phi_\rho(\tilde{\mathbf{F}}(\boldsymbol{q},t),t)|^{-1}$$

we obtain the PDE for ρ

Theorem D.1 (Mean Curvature Flow of graph height)**.** *The graph height ρ over a fixed base surface $M_r^n = \sigma^{-1}(\cdot)\{r\}$ of a surface $\tilde{M}_t = \tilde{\mathbf{F}}(\cdot,t)(M_r^n)$ satisfying*

$$\left(\frac{d\tilde{\mathbf{F}}}{dt}(\boldsymbol{q},t)\right)^{\perp} = \mathbf{H}\left(\tilde{\mathbf{F}}(\boldsymbol{q},t)\right) \tag{D.2}$$

where

$$\tilde{\mathbf{F}}(\boldsymbol{q},t) = \boldsymbol{q} + \rho(\boldsymbol{q},t)\boldsymbol{\omega}(\boldsymbol{q})$$

evolves according to the equation

$$\frac{\partial \rho}{\partial t}(\boldsymbol{q},t) = -|\overline{\nabla}\Phi_\rho(\mathbf{x},t)|\,\mathrm{div}_{\mathbb{R}^{n+1}}\left(\frac{\overline{\nabla}\Phi_\rho(\mathbf{x},t)}{|\overline{\nabla}\Phi_\rho(\mathbf{x},t)|}\right)\Bigg|_{\mathbf{x}=\tilde{\mathbf{F}}(\boldsymbol{q},t)} \tag{D.3}$$

for $(\boldsymbol{q},t) \in M_r^n \times [0,T)$*, so long as* $\tilde{\mathbf{F}}(\boldsymbol{q},t) \notin \ker \overline{\nabla}\sigma$*.*

D.2 Uniform Parabolicity

We shall see that under the right circumstances, D.3 is a quasilinear parabolic partial differential equation, in fact with a uniform gradient bound, it is uniformly parabolic. To do this, let us begin by expanding the derivatives. Setting $\mathbf{x} = \tilde{\mathbf{F}}(\boldsymbol{q},t)$ and $\boldsymbol{q} = \mathbf{S}(\mathbf{x})$, using the chain rule, we obtain

$$\boldsymbol{\nu}(\mathbf{x}) = \frac{\boldsymbol{\omega}(\mathbf{x}) - \tilde{\nabla}\rho}{\sqrt{1+|\tilde{\nabla}\rho|^2}}$$

where $\tilde{\nabla}\rho = \left\langle \overline{\nabla}_{\boldsymbol{q}}\rho, \overline{\nabla}_{\mathbf{e}_k}\mathbf{S}\right\rangle \mathbf{e}_k$ and $\overline{\nabla}_{\boldsymbol{q}}\rho = \overline{\nabla}\rho(\boldsymbol{q},t)$.

Lemma D.2. *For each* $\mathbf{x} \in \mathcal{R}_r$*, there exists an orthonormal basis* $\{\mathbf{e}_i\}_{i=1}^{n+1}$ *of* $\mathbb{R}^{n+1}$ *such that*

$$\left\langle \overline{\nabla}_{\mathbf{e}_i}\mathbf{S}(\mathbf{x}), \mathbf{e}_j\right\rangle = \left\langle \overline{\nabla}_{\mathbf{e}_j}\mathbf{S}(\mathbf{x}), \mathbf{e}_i\right\rangle$$

for all $1 \leqslant i,j \leqslant n+1$*.*

Proof. Arrange $\{\mathbf{e}_i\}_{i=1}^{n+1}$ such that

$$\mathbf{e}_i = \begin{cases} \gamma_i, & i \leqslant n \\ \boldsymbol{\omega}, & i = n+1 \end{cases}$$

where $\{\boldsymbol{\gamma}_i\}_{i=1}^{n}$ is an orthonormal basis for $T_{\boldsymbol{q}}M_r^n$.

We note that $\mathbf{S}$ may be expressed as $\mathbf{S}(\mathbf{x}) = \mathbf{x} - \Lambda(x)\boldsymbol{\omega}(x)$ and compute

$$\begin{aligned} \overline{\nabla}_{\mathbf{e}_i}\mathbf{S} &= \mathbf{e}_i - \mathbf{e}_i(\Lambda)\boldsymbol{\omega} - \Lambda\overline{\nabla}_{\mathbf{e}_i}\boldsymbol{\omega} \\ &= \mathbf{e}_i - \mathbf{e}_i(\Lambda)\boldsymbol{\omega} - \Lambda b_{ik}\boldsymbol{\gamma}_k \end{aligned}$$

where b_{ij} is the second fundamental form of M_r^n at $\boldsymbol{q}$.

Now, we observe that $\overline{\nabla}_{\boldsymbol{\omega}}\mathbf{S} = 0$ and that $\overline{\nabla}_{\gamma_i}\mathbf{S} \in T_{\boldsymbol{q}}M_r^n$ (since $\mathbf{S}(\mathcal{R}_r) = M_r^n$) and compute

$$\left\langle \overline{\nabla}_{\mathbf{e}_i}\mathbf{S}, \mathbf{e}_j\right\rangle = \delta_{ij} - \Lambda b_{ij}$$

which is clearly symmetric, and thus we have the result. □

Expanding the second order derivatives, we obtain

$$
\begin{aligned}
\frac{\partial \rho}{\partial t}(\boldsymbol{q},t) &= -\Delta_{\mathbb{R}^{n+1}}\Phi_\rho + |\overline{\nabla}\Phi_\rho|^{-2}\left\langle \overline{\nabla}_{\overline{\nabla}\Phi_\rho}\overline{\nabla}\Phi_\rho, \overline{\nabla}\Phi_\rho \right\rangle \\
&= \left(\delta_{kl} - |\overline{\nabla}\Phi_\rho|^{-2}\left\langle \overline{\nabla}\Phi_\rho, \mathbf{e}_k\right\rangle \left\langle \overline{\nabla}\Phi_\rho, \mathbf{e}_l\right\rangle\right)\Big[\left\langle \overline{\nabla}_{\mathbf{e}_k}\mathbf{S}, \mathbf{e}_i\right\rangle \left\langle \overline{\nabla}_{\mathbf{e}_l}\mathbf{S}, \mathbf{e}_j\right\rangle \rho_{ij} \\
&\qquad + \left\langle \overline{\nabla}_{\mathbf{e}_k}\overline{\nabla}_{\mathbf{e}_l}\mathbf{S}, \mathbf{e}_i\right\rangle \rho_i\Big] \\
&\quad + |\overline{\nabla}\Phi_\rho|^{-2}\left\langle \overline{\nabla}_{\overline{\nabla}\Phi_\rho}\boldsymbol{\omega}, \overline{\nabla}\Phi_\rho\right\rangle - \operatorname{div}_{\mathbb{R}^{n+1}}\boldsymbol{\omega}
\end{aligned}
$$

remembering of course, that all expressions on the right-hand side are evaluated at $\mathbf{x} = \tilde{\mathbf{F}}(\boldsymbol{q},t)$.

Thus, we obtain an equation for ρ.

Proposition D.3. *If $(M_t)_{t\in[0,T)}, T > 0$ satisfies (D.2) then the graph height $\rho : M_r^n \times [0,T) \to \mathbb{R}$ satisfies the scalar evolution equation*

$$
\frac{\partial \rho}{\partial t}(\boldsymbol{q},t) = a^{ij}(\rho, \overline{\nabla}_{\boldsymbol{q}}\rho)\rho_{ij} + b^i(\rho, \overline{\nabla}_{\boldsymbol{q}}\rho)\rho_i + f(\rho) \tag{D.4}
$$

where

$$
a^{ij} = \left(\delta_{kl} - |\overline{\nabla}\Phi_\rho|^{-2}\left\langle \overline{\nabla}\Phi_\rho, \mathbf{e}_k\right\rangle \left\langle \overline{\nabla}\Phi_\rho, \mathbf{e}_l\right\rangle\right)\left\langle \overline{\nabla}_{\mathbf{e}_k}\mathbf{S}, \mathbf{e}_i\right\rangle \left\langle \overline{\nabla}_{\mathbf{e}_l}\mathbf{S}, \mathbf{e}_j\right\rangle
$$

and

$$
\begin{aligned}
b^i &= \left(\delta_{kl} - |\overline{\nabla}\Phi_\rho|^{-2}\left\langle \overline{\nabla}\Phi_\rho, \mathbf{e}_k\right\rangle \left\langle \overline{\nabla}\Phi_\rho, \mathbf{e}_l\right\rangle\right)\left\langle \overline{\nabla}_{\mathbf{e}_k}\overline{\nabla}_{\mathbf{e}_l}\mathbf{S}, \mathbf{e}_i\right\rangle \\
&\quad - |\overline{\nabla}\Phi_\rho|^{-2}\left\langle \overline{\nabla}_{\overline{\nabla}\Phi_\rho}\boldsymbol{\omega}, \overline{\nabla}_{\mathbf{e}_i}\mathbf{S}\right\rangle
\end{aligned}
$$

and

$$
f = -H_{M^n_{r+\rho(\boldsymbol{q},t)}}
$$

with $H_{M^n_{r+\rho(\boldsymbol{q},t)}} = \operatorname{div}_{\mathbb{R}^{n+1}}\overline{\nabla}\Lambda(\mathbf{x})|_{\mathbf{x}=\tilde{\mathbf{F}}(\boldsymbol{q},t)}$. Note that these derivatives are evaluated at $\mathbf{x} = \tilde{\mathbf{F}}(\boldsymbol{q},t)$, and thus (may) depend implicitly upon ρ.

It is quite easy to see that this equation is parabolic.

Proposition D.4. *The matrix a^{ij} is positive definite on any compact region of $\mathcal{R}_r$ upon which $|\overline{\nabla}_{\boldsymbol{q}}\rho|$ is bounded above.*

Proof. Recall that the gradient function v defined as usual by

$$
v(\mathbf{x},t) = \langle \boldsymbol{\nu}(\mathbf{x}), \boldsymbol{\omega}(\mathbf{x})\rangle^{-1}, \quad \mathbf{x} \in M_t, t \in [0,T)
$$

measures the angle between the normal vector (or equivalently tangent plane) of the base surface M_r^n at $\boldsymbol{q} = \mathbf{S}(\mathbf{x})$ and the normal vector (or tangent plane) of the evolving surface M_t at $\mathbf{x}$. If this function is bounded above, then the projection of tangent vectors between these planes is non-zero.

On a surface expressed as a normal graph we have

$$v(\mathbf{x}, t) = \sqrt{1 + |\overline{\nabla}\rho(\mathbf{S}(\mathbf{x}), t)|^2}$$

thus, for $\mathbf{x} \in \mathcal{R}_r$, bounding $|\overline{\nabla}_{\boldsymbol{q}}\rho|$ is equivalent to obtaining a bound on v.

Now, since $a^{ij} = \left\langle (\overline{\nabla}_{\gamma_i}\mathbf{S})^\top, (\overline{\nabla}_{\gamma_j}\mathbf{S})^\top \right\rangle$ we can easily see that it composed only from the contraction of the metric on M_t at $\mathbf{x}$ and the metric of M_r^n at $\boldsymbol{q}$. Furthermore, the assumption of bounded gradient implies that these metrics are non-orthogonal. Thus, we have that a^{ij}, the composition of two non-orthogonal positive definite metrics, is itself positive definite. □

Theorem D.5. *There exists a neighborhood $\mathcal{R}_r$ of M_r^n on which the graph height ρ of a surface $(M_t)_{t\in[0,T)}$ evolving via Equation (D.2) satisfies a quasi-linear parabolic partial differential equation.*

Remark D.6. The region $\mathcal{R}_r$ depends on the principle curvatures of the base surface M_r^n since ([13], [7]) in terms the normal coordinate basis

$$\mathbf{e}_i = \begin{cases} \boldsymbol{\gamma}_i, & i \leqslant n \\ \boldsymbol{\omega}, & i = n+1 \end{cases}$$

we have

$$D^2\Lambda(\mathbf{x}) = \operatorname{diag}\left(\frac{\kappa_1}{1-\Lambda\kappa_1}, \ldots, \frac{\kappa_n}{1-\Lambda\kappa_n}, 0\right)$$

where $\{\kappa_i\}_{i=1}^n$ are the principle curvatures of M_r^n at $\boldsymbol{q}$. Thus it is clear that if $\rho < \min(\kappa_i^{-1})$ that the setup is valid.

D.3 Cylindrical Case

In the case of cylindrical normal graphs, the signed distance function to the cylinder C_r^n is given by

$$\Lambda(\mathbf{x}) = |\mathbf{x}_{\perp\boldsymbol{\vartheta}}| - r$$

and the closest point projection is

$$\mathbf{S}(\mathbf{x}) = r\left(\frac{\mathbf{x}_{\perp\boldsymbol{\vartheta}}}{|\mathbf{x}_{\perp\boldsymbol{\vartheta}}|}\right) + \langle \mathbf{x}, \boldsymbol{\vartheta}\rangle\, \boldsymbol{\vartheta}$$

Let $\{\boldsymbol{\gamma}_i\}_{i=1}^n$ be an orthonormal basis for $T_{\boldsymbol{q}}M_r^n$ with $\boldsymbol{\gamma}_n = \boldsymbol{\vartheta}$ (making the basis in fact normal coordinates) and arrange

$$\mathbf{e}_i = \begin{cases} \boldsymbol{\gamma}_i, & i < n \\ \boldsymbol{\vartheta}, & i = n \\ \boldsymbol{\omega}, & i = n+1 \end{cases}$$

Now, we compute the first derivatives of the closest point projection

$$\overline{\nabla}_{\mathbf{e}_i}\mathbf{S} = \left(\frac{r}{|\mathbf{x}_{\perp\boldsymbol{\vartheta}}|}\right)\mathbf{e}_{i\tau} + \langle \mathbf{e}_i, \boldsymbol{\vartheta}\rangle\,\boldsymbol{\vartheta}$$

and the second derivatives

$$\overline{\nabla}_{\mathbf{e}_i}\overline{\nabla}_{\mathbf{e}_j}\mathbf{S} = -\left(\frac{r}{|\mathbf{x}_{\perp\boldsymbol{\vartheta}}|^2}\right)\left[\left\langle \mathbf{e}_{i\tau}, \mathbf{e}_{j_\tau}\right\rangle \boldsymbol{\omega} + \langle \mathbf{e}_i, \boldsymbol{\omega}\rangle\,\mathbf{e}_{j_\tau} + \langle \mathbf{e}_j, \boldsymbol{\omega}\rangle\,\mathbf{e}_{i\tau}\right]$$

Also, we we have that

$$H_{M^n_{r+\rho}} = \frac{n-1}{\rho + r}$$

all of which were evaluated at $\mathbf{x} = \tilde{\mathbf{F}}(\boldsymbol{q}, t)$.

Set $b_{ij} = \left\langle \overline{\nabla}_{\mathbf{e}_i}\mathbf{S}, \mathbf{e}_j\right\rangle$, then

$$b_{ij} = \begin{cases} \left(\frac{r}{r+\rho}\right), & i = j < n \\ 1, & i = j = n \\ 0, & \text{otherwise} \end{cases}$$

We also define $\tilde{\nabla}\rho$ by

$$\tilde{\nabla}\rho = \rho_k b_{kl}\mathbf{e}_l$$

and without loss of generality, we may set

$$\left\langle \overline{\nabla}_{\boldsymbol{q}}\rho(\boldsymbol{q}, t), \boldsymbol{\omega}(\boldsymbol{q})\right\rangle = 0, \quad \forall \boldsymbol{q} \in M^n_r$$

then we have the equation

$$\frac{\partial \rho}{\partial t}(\boldsymbol{q}, t) = a^{ij}\rho_{ij} - \frac{n-1}{\rho + r} - \left(\frac{1}{\rho + r}\right)\left(\frac{|(\tilde{\nabla}\rho)_\tau|^2}{1 + |\tilde{\nabla}\rho|^2}\right)$$
$$\boldsymbol{q} \in M^n_r, t \in [0, T) \quad \text{(D.5)}$$

where

$$a^{ij} = b_{ik}b_{jl}\left(\delta_{kl} - \frac{\tilde{\nabla}_k\rho\tilde{\nabla}_l\rho}{1 + |\tilde{\nabla}\rho|^2}\right)$$

It is easily seen that (D.5) is a quasilinear parabolic partial differential equation.