Appendix C

Geometric Graphs

We will be dealing primarily with cylindrical graphs, but first we must define precisely what we mean by that. There are at least two ways to deal look at this concept, each with their advantages. An alternate method is covered in Appendix D. The method we shall be employing takes a more geometric point of view, where we shall be defining a graph in terms of 'reference slices'. A similar derivation to ours may be found in [5].

C.1 Local Graphs

First, suppose that we have a function $\sigma : \Omega \to \mathbb{R}$ such that $\overline{\nabla}\sigma \neq 0, \forall \mathbf{x} \in \Omega$, where $\overline{\nabla}$ is the gradient on $\mathbb{R}^{n+1}$ and $\Omega \subset \mathbb{R}^{n+1}$ is some open set. Now, let M_t be a surface evolving by mean curvature flow, that is

$$\frac{d\mathbf{x}}{dt} = \mathbf{H}(\mathbf{x})$$

where $\mathbf{x}$ is the position vector on M_t and $\mathbf{H}$ is the mean curvature vector at $\mathbf{x}$.

Let us define the reference slices, with which we shall be comparing M_t.

Definition C.1 (Reference Slices)**.** Let σ be a function with non-vanishing gradient in an open set $\Omega \subset \mathbb{R}^{n+1}$ and let $\boldsymbol{\xi}$ be some fixed vector in $\mathbb{R}^{n+1}$, then denoting $\sigma_{\boldsymbol{\xi}}(\mathbf{x}) = \sigma(\mathbf{x} - \boldsymbol{\xi})$, we define the reference slice $M^n_\rho(\boldsymbol{\xi})$ centered at $\boldsymbol{\xi}$ by

$$M^n_\rho(\boldsymbol{\xi}) = \left\{\mathbf{x} \in \mathbb{R}^{n+1} : \sigma_{\boldsymbol{\xi}}(\mathbf{x}) = \rho\right\}, \quad \rho > 0 \tag{C.1}$$

Now, on Ω, since σ has non-vanishing gradient, we may define the unit normal $\boldsymbol{\omega}_{\boldsymbol{\xi}}$ on $M^n_\rho(\boldsymbol{\xi})$ by

$$\boldsymbol{\omega}_{\boldsymbol{\xi}}(\mathbf{x}) = \beta_{\boldsymbol{\xi}}(\mathbf{x})\overline{\nabla}\sigma_{\boldsymbol{\xi}}(\mathbf{x}) \tag{C.2}$$

where $\beta_{\boldsymbol{\xi}} = |\overline{\nabla}\sigma_{\boldsymbol{\xi}}(\mathbf{x})|^{-1}$.

We shall also define the gradient function $v_{\boldsymbol{\xi}}$ (relative to $\boldsymbol{\xi}$) on $M_t \cap \Omega$ by

Definition C.2 (Gradient Function)**.**

$$v_{\boldsymbol{\xi}}(\mathbf{x},t) = \langle \boldsymbol{\nu}, \boldsymbol{\omega}_{\boldsymbol{\xi}} \rangle^{-1}, \quad \mathbf{x} \in M_t \cap \Omega \tag{C.3}$$

The gradient function holds the essence of what to us will signify a graph. We shall say that $M_t \cap \Omega$ is a local graph over $M^n_\rho(\boldsymbol{\xi})$ if $v_{\boldsymbol{\xi}}(\mathbf{x},t) < \infty$ for all $\mathbf{x} \in M_t \cap \Omega$. In other words, the normal vectors of $M^n_\rho(\boldsymbol{\xi})$ and M_t are nowhere perpendicular.

Thus, we are ready to define a local graph:

Definition C.3 (Local Graph)**.** If for every $\mathbf{x}_0 \in M_t$ there exists a vector $\boldsymbol{\xi} \in \mathbb{R}^{n+1}$ and an open neighborhood $\Omega \subset \mathbb{R}^{n+1}$ of $\mathbf{x}_0$ such that

$$\begin{cases} \beta_{\boldsymbol{\xi}} < \infty, & \mathbf{x} \in \Omega \\ v_{\boldsymbol{\xi}} < \infty, & \mathbf{x} \in \Omega \cap M_t \end{cases}$$

then we say that M_t is a local graph (with respect to the reference slicing $M^n_\rho(\boldsymbol{\xi})$). Furthermore, if we may choose $\boldsymbol{\xi}$ independent of $\mathbf{x}_0$, then we call M_t an entire graph (in which case we may extend Ω to cover all of M_t).

It is not immediately clear whether an entire graph will, under evolution by (MCF), remain an entire graph for all future times. In fact, it is not even clear whether a local graph will remain a local graph. It could degenerate to a local entire graph, or worse.

Another quantity that will be interesting for us will be the 'height' of M_t. We define the height function $u_{\boldsymbol{\xi}}$ (relative to $\boldsymbol{\xi}$) on $M_t \cap \Omega$ by the restriction of σ to $M_t \cap \Omega$, that is

Definition C.4 (Height Function)**.**

$$u_{\boldsymbol{\xi}}(\mathbf{x},t) = \sigma_{\boldsymbol{\xi}}(\mathbf{x}), \quad \mathbf{x} \in M_t \cap \Omega \tag{C.4}$$

We return now to our question of whether we may express M_t as a local graph over $M^n_\rho(\boldsymbol{\xi})$ if it was initially one. To answer this, we are going to need to derive the evolution equations of the quantities $u_{\boldsymbol{\xi}}$ and $v_{\boldsymbol{\xi}}$.

C.2 Evolution Equations

Let us now compute these equations, noticing that we may for simplicity without any loss of generality assume that on the neighbourhood Ω that we are working on, set $\boldsymbol{\xi} = 0$. Thus, we denote $\sigma_0 = \sigma$, $\beta_0 = \beta$, $u_0 = u$ and $v_0 = v$.

We define the tangential gradient ∇ on M_t as the projection of the gradient on $\mathbb{R}^{n+1}$ onto the tangent space of M_t. Thus we compute for u the identity

$$\begin{aligned}\nabla u &= \overline{\nabla} u - \left\langle \overline{\nabla} u, \boldsymbol{\nu} \right\rangle \boldsymbol{\nu} \\ &= \overline{\nabla} \sigma - \left\langle \overline{\nabla} \sigma, \boldsymbol{\nu} \right\rangle \boldsymbol{\nu} \\ &= \frac{\boldsymbol{\omega}}{\beta} - \left\langle \frac{\boldsymbol{\omega}}{\beta}, \boldsymbol{\nu} \right\rangle \boldsymbol{\nu} \\ &= \frac{1}{\beta}\left[\boldsymbol{\omega} - \frac{1}{v}\boldsymbol{\nu}\right]\end{aligned} \tag{C.5}$$

and thus the Laplacian of u is given by

$$\begin{aligned}\Delta u &= \operatorname{div}_{M_t} \nabla u \\ &= -\frac{1}{v\beta} \operatorname{div}_{M_t} \boldsymbol{\nu} + \operatorname{div}_{M_t} \overline{\nabla} \sigma \\ &= \frac{1}{\beta} \langle \mathbf{H}, \boldsymbol{\omega} \rangle + \operatorname{div}_{M_t} \overline{\nabla} \sigma\end{aligned} \tag{C.6}$$

Now, since $\operatorname{div}_{\mathbb{R}^{n+1}} \mathbf{X} = \operatorname{div}_{M^n} \mathbf{X} + \left\langle \overline{\nabla}_{\boldsymbol{\omega}} \mathbf{X}, \boldsymbol{\omega} \right\rangle$, and also $\operatorname{div}_{\mathbb{R}^{n+1}} \mathbf{X} = \operatorname{div}_{M_t} \mathbf{X} + \left\langle \overline{\nabla}_{\boldsymbol{\nu}} \mathbf{X}, \boldsymbol{\nu} \right\rangle$ for $\mathbf{X}$ a vector-field defined ambiently on $\mathbb{R}^{n+1}$, we have

$$\operatorname{div}_{M_t} \mathbf{X} = \operatorname{div}_{M^n} \mathbf{X} + \left\langle \overline{\nabla}_{\boldsymbol{\omega}} \mathbf{X}, \boldsymbol{\omega} \right\rangle - \left\langle \overline{\nabla}_{\boldsymbol{\nu}} \mathbf{X}, \boldsymbol{\nu} \right\rangle \tag{C.7}$$

and so, we compute

$$\begin{aligned}\operatorname{div}_{M_t} \overline{\nabla} \sigma &= \operatorname{div}_{M^n} \overline{\nabla} \sigma + \left\langle \overline{\nabla}_{\boldsymbol{\omega}} \overline{\nabla} \sigma, \boldsymbol{\omega} \right\rangle - \left\langle \overline{\nabla}_{\boldsymbol{\nu}} \overline{\nabla} \sigma, \boldsymbol{\nu} \right\rangle \\ &= \frac{1}{\beta} \operatorname{div}_{M^n} \boldsymbol{\omega} - \frac{1}{\beta^2} \boldsymbol{\omega}(\beta) + \frac{1}{v\beta^2} \boldsymbol{\nu}(\beta) - \frac{1}{\beta} \left\langle \overline{\nabla}_{\boldsymbol{\nu}} \boldsymbol{\omega}, \boldsymbol{\nu} \right\rangle \\ &= \frac{1}{\beta}\left[H_{M^n_\rho} - \nabla u(\beta) - \chi\right]\end{aligned} \tag{C.8}$$

where $\chi = \left\langle \overline{\nabla}_{\boldsymbol{\nu}} \boldsymbol{\omega}, \boldsymbol{\nu} \right\rangle$ and $H_{M^n_\rho}$ is the mean curvature at $\mathbf{x}$ of M^n_ρ.

Finally, we compute the time derivative of u, finding where $\mathbf{X} = \mathbf{H}$

$$\begin{aligned}\frac{du}{dt} =& X(\sigma) \\ &= \left\langle \overline{\nabla} \sigma, \mathbf{X} \right\rangle \\ &= \frac{1}{\beta} \langle \mathbf{H}, \boldsymbol{\omega} \rangle\end{aligned} \tag{C.9}$$

and thus we have

Proposition C.5. *The evolution equation of the height function on $M_t \cap \Omega$ is*

$$\left(\frac{d}{dt} - \Delta\right) u = -\frac{1}{\beta}\left[H_{M^n} - \nabla u(\beta) - \chi\right] \tag{C.10}$$

Height estimates are going to be very important to us in this study, giving decay and growth bounds, and many will come from maximum principle type arguments on this equation. We will also require estimates on the gradient function, since it is one of our aims to show that the surfaces under (MCF) actually do remain graphs over the cylinder.

Proposition C.6. *The gradient function v on $M_t \cap \Omega$ satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right) v = -|A|^2 v - 2v^{-1}|\nabla v|^2 + v^2(H\chi - \boldsymbol{\omega}(H)) \tag{C.11}$$

Proof. Again, we proceed similarly to [3], and calculate where $\mathbf{X} = \mathbf{H}$

$$\begin{aligned} \frac{dv}{dt} &= \mathbf{X} \langle \boldsymbol{\nu}, \boldsymbol{\omega} \rangle^{-1} \\ &= -v^2 \left[\langle \overline{\nabla}_{\mathbf{X}} \boldsymbol{\nu}, \boldsymbol{\omega} \rangle + \langle \boldsymbol{\nu}, \overline{\nabla}_{\mathbf{X}} \boldsymbol{\omega} \rangle \right] \\ &= -v^2 \langle \nabla H, \boldsymbol{\omega} \rangle + H v^2 \chi \end{aligned} \tag{C.12}$$

Now, computing the Laplacian,

$$\Delta v = -v^2 \Delta \langle \boldsymbol{\nu}, \boldsymbol{\omega} \rangle + 2v^{-1}|\nabla v|^2 \tag{C.13}$$

Focusing on the first term, using normal coordinates and the fact that $[\boldsymbol{\omega}, \boldsymbol{\tau}_i] = 0$ (see Appendix B) we calculate

$$\begin{aligned} \Delta \langle \boldsymbol{\nu}, \boldsymbol{\omega} \rangle &= \boldsymbol{\tau}_i \boldsymbol{\tau}_i \langle \boldsymbol{\nu}, \boldsymbol{\omega} \rangle \\ &= \boldsymbol{\tau}_i \left(\langle \overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\nu}, \boldsymbol{\omega} \rangle + \langle \boldsymbol{\nu}, \overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\omega} \rangle \right) \\ &= \langle \overline{\nabla}_{\boldsymbol{\tau}_i} \overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\nu}, \boldsymbol{\omega} \rangle + 2 \langle \overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\nu}, \overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\omega} \rangle + \langle \boldsymbol{\nu}, \overline{\nabla}_{\boldsymbol{\tau}_i} \overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\omega} \rangle \\ &= \langle \Delta \boldsymbol{\nu}, \boldsymbol{\omega} \rangle + 2 \langle h_{ik} \boldsymbol{\tau}_k, \overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\omega} \rangle + \langle \boldsymbol{\nu}, \overline{\nabla}_{\boldsymbol{\omega}} \overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\tau}_i \rangle \quad \text{by GW(i)} \\ &= \langle \nabla H - |A|^2 \boldsymbol{\nu}, \boldsymbol{\omega} \rangle + 2h_{ik} \langle \boldsymbol{\tau}_k, \overline{\nabla}_{\boldsymbol{\omega}} \boldsymbol{\tau}_i \rangle \quad \text{by Proposition A.7} \\ &\quad + \left\langle \boldsymbol{\nu}, \overline{\nabla}_{\boldsymbol{\omega}} \left(-h_{ii} \boldsymbol{\nu} + \left(\overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\tau}_i \right)^{\top} \right) \right\rangle \quad \text{and GW(ii)} \\ &= \langle \nabla H, \boldsymbol{\omega} \rangle - |A|^2 v^{-1} + h_{ik} \boldsymbol{\omega}(g_{ik}) \\ &\quad - h_{ii} \langle \boldsymbol{\nu}, \overline{\nabla}_{\boldsymbol{\omega}} \boldsymbol{\nu} \rangle - \boldsymbol{\omega}(h_{ii}) \langle \boldsymbol{\nu}, \boldsymbol{\nu} \rangle + \left\langle \boldsymbol{\nu}, \overline{\nabla}_{\boldsymbol{\omega}} \left(\overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\tau}_i \right)^{\top} \right\rangle \\ &= \langle \nabla H, \boldsymbol{\omega} \rangle - |A|^2 v^{-1} - h_{ik} \boldsymbol{\omega}(g^{ik}) - g^{ik} \boldsymbol{\omega}(h_{ik}) \\ &= \langle \nabla H, \boldsymbol{\omega} \rangle - |A|^2 v^{-1} - \boldsymbol{\omega}(H) \end{aligned} \tag{C.14}$$

since in normal coordinates $\left\langle \boldsymbol{\nu}, \overline{\nabla}_{\boldsymbol{\omega}} \left(\overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\tau}_i \right)^{\top} \right\rangle$ vanishes.

We note here that $\boldsymbol{\omega}(H)$ measures the rate of change of mean curvature as we deform M_t in the direction of $\boldsymbol{\omega}$. Later, in the case of cylindrical graphs, we shall calculate $\boldsymbol{\omega}(H)$ explicitly. □