Chapter 2

Evolution Equations

2.1 Cylindrical Graphs

Instead of using an equation such as equation (1.2) to analyse the evolution of cylindrical graphs we are instead going to use a more geometric approach and compare the evolving surfaces directly to reference surfaces. For more information on the ideas behind this approach, refer to Chapter C.

We shall be comparing our surfaces to cylinders, and one way to do this is to introduce some way to measure the 'height' of our surface, relative to the cylinder axis. Thus, we consider the function $\sigma : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$\sigma(\mathbf{x}) = |\mathbf{x}_{\perp\boldsymbol{\vartheta}}| \tag{2.1}$$

where $\boldsymbol{\vartheta}$ is a unit vector defining the direction of the axis and $\mathbf{x}_{\perp\boldsymbol{\vartheta}} = \mathbf{x} - \langle \mathbf{x}, \boldsymbol{\vartheta} \rangle \boldsymbol{\vartheta}$, the component of $\mathbf{x}$ not pointing in the direction of $\boldsymbol{\vartheta}$.

Clearly the family of level sets $M^n_\rho = \left\{ \mathbf{x} \in \mathbb{R}^{n+1} : \sigma(\mathbf{x}) = \rho \right\}, \rho > 0$ is the family of n-cylinders of radius ρ with axial direction $\boldsymbol{\vartheta}$ centered at the origin. These will be our reference slices when considering cylindrical graphs.

Now, the outward unit normal $\boldsymbol{\omega}$ to the cylinder M^n_ρ at $\mathbf{x}$ is computed as

$$\begin{aligned} \boldsymbol{\omega}(\mathbf{x}) &= \frac{\overline{\nabla}\sigma(\mathbf{x})}{|\overline{\nabla}\sigma(\mathbf{x})|} \\ &= \frac{\mathbf{x}_{\perp\boldsymbol{\vartheta}}}{|\mathbf{x}_{\perp\boldsymbol{\vartheta}}|} \end{aligned} \tag{2.2}$$

since $|\overline{\nabla}\sigma| \equiv 1$. This is well defined whenever $\sigma(\mathbf{x}) > 0$, i.e. we are not passing through the axis.

If the comparison between the cylindrical reference slices and the surface is to make sense, i.e. to be non-degenerate, we require that the normal of the reference slice at $\mathbf{x}$ is not perpendicular to the normal of the surface for any $\mathbf{x}$. Thus we introduce the gradient function to quantify this relationship. In the following, let M be an n-dimensional hypersurface in $\mathbb{R}^{n+1}$.

Definition 2.1 (Gradient Function)**.** The cylindrical gradient function v on M is defined by

$$v(\mathbf{x}) = \langle \boldsymbol{\nu}, \boldsymbol{\omega} \rangle^{-1}, \quad \mathbf{x} \in M$$

where $\boldsymbol{\nu}$ is the outward pointing unit normal to M at $\mathbf{x}$.

Clearly, this function blows up precisely at points on M at which the unit normal of the reference cylinder M_r^n and M are perpendicular.

We also define the cylindrical height function of M relative to the reference slices.

Definition 2.2 (Height Function)**.** The cylindrical height function u on M is defined by

$$u(\mathbf{x}) = \sigma(\mathbf{x}), \quad \mathbf{x} \in M$$

Worth noting is the fact that the cylindrical height function may be written as

$$u = \langle \mathbf{x}, \boldsymbol{\omega} \rangle$$

Now we may define exactly what we mean when we say a cylindrical graph.

Definition 2.3 (Entire Cylindrical Graph)**.** Let M be an n-dimensional hypersurface in $\mathbb{R}^{n+1}$ such that

$$\begin{cases} u > 0, & \mathbf{x} \in M \\ v < \infty, & \mathbf{x} \in M \end{cases}$$

then we say that M is an entire cylindrical graph.

Note that requiring u to be positive ensures that $\boldsymbol{\omega}$ and hence v is well defined.

Our aim is that, under evolution by (MCF), we may prevent the normal vector of the reference slice and the surface from ever becoming perpendicular, so that our comparison (and analysis) continues to make sense (at least up to a singularity).

The general setup of the problem is presented diagrammatically in Figure 2.1.

2.2 The Evolution Equations

In the next section, we shall derive evolution equations for the quantities governing the evolution of cylindrical graphs such as height and gradient functions. In general, these equations will be of the form $\left(\frac{d}{dt} - \Delta\right) f = Q$, where Q is some function of f and other geometric quantities on M_t. Using these equations along with delicate test- and cutoff-function arguments,

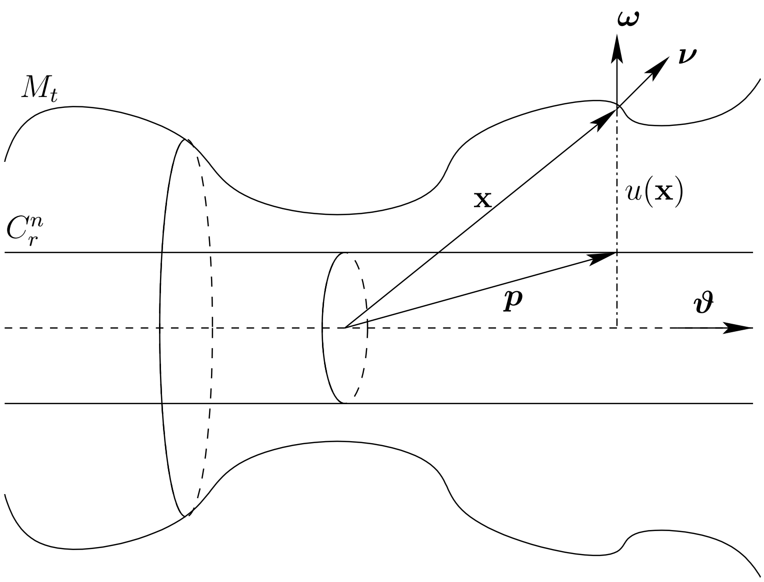

Figure 2.1: Cylindrical Graph

we will be able to apply maximum principles to derive estimates on these quantities.

We have this useful identity for the derivative of the normal vector on the cylinder.

Lemma 2.4. *For any vector* $\mathbf{Z}$ *we have*

$$\overline{\nabla}_{\mathbf{Z}}\boldsymbol{\omega} = \frac{1}{u}\mathbf{Z}_\tau \tag{2.3}$$

where $\mathbf{Z}_\tau = \mathbf{Z} - \langle \mathbf{Z}, \boldsymbol{\vartheta}\rangle \boldsymbol{\vartheta} - \langle \mathbf{Z}, \boldsymbol{\omega}\rangle \boldsymbol{\omega}$ *is the twisting component of* $\mathbf{Z}$.

Proof. We calculate

$$\begin{aligned}\overline{\nabla}_{\mathbf{Z}}\boldsymbol{\omega} &= \overline{\nabla}_{\mathbf{Z}}\left(\frac{\mathbf{x}_{\perp\boldsymbol{\vartheta}}}{|\mathbf{x}_{\perp\boldsymbol{\vartheta}}|}\right)\\ &= \frac{1}{|\mathbf{x}_{\perp\boldsymbol{\vartheta}}|}\left[\mathbf{Z} - \langle \mathbf{Z}, \boldsymbol{\vartheta}\rangle \boldsymbol{\vartheta}\right] - \frac{1}{|\mathbf{x}_{\perp\boldsymbol{\vartheta}}|}\mathbf{Z}|\mathbf{x}_{\perp\boldsymbol{\vartheta}}|\boldsymbol{\omega}\\ &= \frac{1}{u}\left[\mathbf{Z} - \langle \mathbf{Z}, \boldsymbol{\vartheta}\rangle \boldsymbol{\vartheta} - \langle \mathbf{Z}, \boldsymbol{\omega}\rangle \boldsymbol{\omega}\right]\end{aligned}$$

□

Here, by twisting, we mean the component of the vector which is directed around the axis. In other words, it is the projection of the vector onto the tangent space of the cylinder, minus the axial part.

Lemma 2.5. *For cylindrical graphs, the function* $\chi = \left\langle \overline{\nabla}_{\boldsymbol{\nu}} \boldsymbol{\omega}, \boldsymbol{\nu} \right\rangle$ *is given by*

$$\chi = \frac{\kappa}{u} \tag{2.4}$$

where $\kappa = \langle \boldsymbol{\nu}_\mathcal{T}, \boldsymbol{\nu}_\mathcal{T} \rangle$ *is the twist function.*

Proof. By Lemma 2.4 we have

$$\begin{aligned}
\chi &= \left\langle \overline{\nabla}_{\boldsymbol{\nu}} \boldsymbol{\omega}, \boldsymbol{\nu} \right\rangle \\
&= \left\langle \frac{1}{u} \left[\boldsymbol{\nu} - \langle \boldsymbol{\nu}, \boldsymbol{\vartheta} \rangle \boldsymbol{\vartheta} - \langle \boldsymbol{\nu}, \boldsymbol{\omega} \rangle \boldsymbol{\omega} \right], \boldsymbol{\nu} \right\rangle \\
&= \frac{1}{u} \langle \boldsymbol{\nu}_\mathcal{T}, \boldsymbol{\nu}_\mathcal{T} \rangle
\end{aligned}$$

□

We see from its definition that κ measures at a point on M_t how much the normal vector is directed or twisting around the axis of the cylinder.

For a derivation of graph evolution equations in a more general setting, please refer to Appendix C. However, here we present the evolution equations for a specific graph setting; the cylindrical graph.

Proposition 2.6. *The height function* u *of a cylindrical graph satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta \right) u = - \frac{n - 1 - \kappa}{u} \tag{2.5}$$

Proof. This follows from a simple application of Proposition C.5, however in the cylindrical case it is far easier to just compute (2.5) directly, using Lemma 2.5 and the fact that $u = \langle \mathbf{x}, \boldsymbol{\omega} \rangle$.

First, compute the time derivative of u, finding where $\mathbf{X} = \mathbf{H}$

$$\begin{aligned}
\frac{du}{dt} &= \mathbf{X}(u) \\
&= \langle \mathbf{H}, \boldsymbol{\omega} \rangle + \left\langle \mathbf{x}, \frac{1}{u} \mathbf{H}_\mathcal{T} \right\rangle \\
&= \langle \mathbf{H}, \boldsymbol{\omega} \rangle
\end{aligned}$$

since $\mathbf{x}_\mathcal{T} = 0$.

Now, compute the Laplacian of u and we obtain

$$\begin{aligned}
\Delta u &= \boldsymbol{\tau}_i \boldsymbol{\tau}_i \langle \mathbf{x}, \boldsymbol{\omega} \rangle \\
&= \boldsymbol{\tau}_i \left[\langle \boldsymbol{\tau}_i, \boldsymbol{\omega} \rangle + \left\langle \mathbf{x}, \frac{1}{u} \boldsymbol{\tau}_{i\mathcal{T}} \right\rangle \right] \\
&= \langle \mathbf{H}, \boldsymbol{\omega} \rangle + \left\langle \boldsymbol{\tau}_i, \frac{1}{u} \boldsymbol{\tau}_i \right\rangle \\
&= \langle \mathbf{H}, \boldsymbol{\omega} \rangle + \frac{n - 1 - \kappa}{u}
\end{aligned}$$

since $\langle \boldsymbol{\tau}_i, \boldsymbol{\tau}_i \rangle = n - 1 - \kappa$. The result follows from these two equations. □

If the height function u is initially uniformly bounded, the evolution equation (2.5) immediately implies an *a priori* estimate for the height, since the right hand side of (2.5) is negative.

Proposition 2.7. *The gradient function v of a cylindrical graph satisfies the evolution equation*

$$\left(\frac{d}{dt}-\Delta\right)v=-|A|^2v-2v^{-1}|\nabla v|^2+\left(\frac{n-1-\kappa}{u^2}\right)v+\frac{2H_\tau v^2}{u} \tag{2.6}$$

where $H_\tau=\operatorname{div}_{M_t}\boldsymbol{\nu}_\tau$ is the twist mean curvature.

Proof. Firstly,

$$\begin{aligned}\boldsymbol{\omega}(H)&=\boldsymbol{\omega}(g^{ij}h_{ij})\\&=g^{ij}\boldsymbol{\omega}(h_{ij})+\boldsymbol{\omega}(g^{ij})h_{ij}\end{aligned}$$

Now, we compute

$$\begin{aligned}\boldsymbol{\omega}(h_{ij})&=-\boldsymbol{\omega}\left\langle\boldsymbol{\nu},\overline{\nabla}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_j\right\rangle\\&=-\left\langle\boldsymbol{\nu},\overline{\nabla}_{\boldsymbol{\omega}}\overline{\nabla}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_j\right\rangle-\left\langle\overline{\nabla}_{\boldsymbol{\omega}}\boldsymbol{\nu},\overline{\nabla}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_j\right\rangle\\&=-\left\langle\boldsymbol{\nu},\overline{\nabla}_{\boldsymbol{\tau}_i}\overline{\nabla}_{\boldsymbol{\tau}_j}\boldsymbol{\omega}\right\rangle\\&=-\left\langle\boldsymbol{\nu},\overline{\nabla}_{\boldsymbol{\tau}_i}\left(\frac{1}{u}\boldsymbol{\tau}_{j_\tau}\right)\right\rangle\qquad\text{by Lemma 2.4}\\&=\frac{1}{u^2}\left\langle\boldsymbol{\nu}_\tau,\boldsymbol{\tau}_i(u)\boldsymbol{\tau}_j\right\rangle\\&\quad-\frac{1}{u}\left\langle\boldsymbol{\nu},-h_{ij}\boldsymbol{\nu}_\tau-\left\langle\boldsymbol{\tau}_j,\overline{\nabla}_{\boldsymbol{\tau}_i}\boldsymbol{\omega}\right\rangle\boldsymbol{\omega}-\left\langle\boldsymbol{\tau}_j,\boldsymbol{\omega}\right\rangle\overline{\nabla}_{\boldsymbol{\tau}_i}\boldsymbol{\omega}\right\rangle\\&=\frac{1}{u^2}\left\langle\boldsymbol{\nu}_\tau,\boldsymbol{\tau}_i(u)\boldsymbol{\tau}_j+\boldsymbol{\tau}_j(u)\boldsymbol{\tau}_i\right\rangle\\&\quad+\frac{\kappa h_{ij}}{u}+\frac{v^{-1}}{u}\left\langle\overline{\nabla}_{\boldsymbol{\tau}_i}\boldsymbol{\omega},\boldsymbol{\tau}_j\right\rangle\end{aligned}$$

which leads to

$$\begin{aligned}g^{ij}\boldsymbol{\omega}(h_{ij})&=\frac{2}{u^2}\left\langle\boldsymbol{\nu}_\tau,\nabla u\right\rangle+\frac{H\kappa}{u}+\frac{v^{-1}}{u}\operatorname{div}_{M_t}\boldsymbol{\omega}\\&=-\frac{2\kappa v^{-1}}{u^2}+\frac{H\kappa}{u}+\left(\frac{n-1-\kappa}{u^2}\right)v^{-1}\end{aligned}\tag{2.7}$$

Similarly, we compute

$$\begin{aligned}\boldsymbol{\omega}(g^{ij})&=-\boldsymbol{\omega}(g_{ij})\\&=-2\left\langle\overline{\nabla}_{\boldsymbol{\tau}_i}\boldsymbol{\omega},\boldsymbol{\tau}_j\right\rangle\\&=-\frac{2}{u}\left\langle\boldsymbol{\tau}_{i_\tau},\boldsymbol{\tau}_{j_\tau}\right\rangle\end{aligned}$$

so we obtain

$$h_{ij}\boldsymbol{\omega}(g^{ij}) = -\frac{2}{u}h_{ij}\left\langle \boldsymbol{\tau}_{i\tau}, \boldsymbol{\tau}_{j\tau}\right\rangle \tag{2.8}$$

With H_τ as defined above, we compute

$$\begin{aligned} H_\tau &= \operatorname{div}_{M_t}\left(\boldsymbol{\nu} - \langle \boldsymbol{\nu}, \boldsymbol{\vartheta}\rangle \boldsymbol{\vartheta} - \langle \boldsymbol{\nu}, \boldsymbol{\omega}\rangle \boldsymbol{\omega}\right) \\ &= H - h_{ij}\langle \boldsymbol{\tau}_i, \boldsymbol{\vartheta}\rangle \langle \boldsymbol{\tau}_j, \boldsymbol{\vartheta}\rangle - h_{ij}\langle \boldsymbol{\tau}_i, \boldsymbol{\omega}\rangle \langle \boldsymbol{\tau}_j, \boldsymbol{\omega}\rangle \\ &\quad - \left\langle \boldsymbol{\nu}, \overline{\nabla}_{\boldsymbol{\tau}_i}\boldsymbol{\omega}\right\rangle \langle \boldsymbol{\omega}, \boldsymbol{\tau}_i\rangle - \langle \boldsymbol{\nu}, \boldsymbol{\omega}\rangle \left(\frac{n-1-\kappa}{u}\right) \\ &= h_{ij}\left\langle \boldsymbol{\tau}_{i\tau}, \boldsymbol{\tau}_{j\tau}\right\rangle + \frac{\kappa v^{-1}}{u} - \left(\frac{n-1-\kappa}{u}\right) v^{-1} \end{aligned}$$

Thus combining this identity with (2.7) and (2.8), we have

$$\boldsymbol{\omega}(H) = \frac{\kappa H - 2H_\tau}{u} - \left(\frac{n-1-\kappa}{u^2}\right) v^{-1} \tag{2.9}$$

and then the proposition follows from (C.6). □

Note that we cannot immediately (or easily) obtain an *a priori* estimate on v using Equation 2.6 and the maximum principle since it is not clear what the sign of the right hand side of (2.6) is. To obtain this estimate we will require a test function argument, as developed in Chapter 3.

Remark 2.8 (Rotational Symmetry)**.** In the case where M_0 (and hence M_t, since (MCF) is invariant to isometries of $\mathbb{R}^{n+1}$, including rotations) is a surface of revolution (of a graph) about the $\boldsymbol{\vartheta}$ axis, we have both $\kappa = 0$ and $H_\tau = 0$ everywhere. This is clear, since for a rotationally symmetric surface we may express the normal as $\boldsymbol{\nu} = \langle \boldsymbol{\nu}, \boldsymbol{\vartheta}\rangle \boldsymbol{\vartheta} + \langle \boldsymbol{\nu}, \boldsymbol{\omega}\rangle \boldsymbol{\omega}$ and thus $\boldsymbol{\nu}_\tau = 0$ on M_t.

2.3 Normalised Equations

The evolution equations that we shall use to analyse the flow throughout Chapter 3 are

(i) $\left(\frac{d}{dt} - \Delta\right) u = -\frac{n-1-\kappa}{u}$

(ii) $\left(\frac{d}{dt} - \Delta\right) v = -|A|^2 v - 2v^{-1}|\nabla v|^2 + \left(\frac{n-1-\kappa}{u^2}\right) v + \frac{2H_\tau v^2}{u}$

(iii) $\left(\frac{d}{dt} - \Delta\right) H = H|A|^2$

(iv) $\left(\frac{d}{dt} - \Delta\right) |A|^2 = -2|\nabla A|^2 + 2|A|^4$

For derivations of the evolution equations for the curvature quantities H and $|A|^2$, equations (iii) and (iv) respectively, refer to Appendix B.

The derivation of the last two equations, which we don't include in this chapter since they don't depend at all on the particular graph setting, can be found in Appendix B.

All of our estimates will use these equations or combinations thereof. The ultimate objective of taking combinations (multiples, sums, functions) of these quantities is to obtain evolution equations suitable for application of the maximum principle. Chapter 3 is concerned with finding such combinations, or test-functions.

A further set of equations can be obtained if we rescale the flow so as to preserve certain rates or quantities. Suppose we have a solution $(M_t)_{t\in[t_0,T)}$, $T > 0$ of (MCF) which is developing a singularity ($T < \infty$), or perhaps expanding out to infinity ($T = \infty$). If we suspect the rate at which the surface is asymptotically expanding about a point $\mathbf{x}_0$ is, say $\lambda(t)$, i.e. that there exists some limiting surface M such that

$$\lim_{t\to T}\left(\frac{M_t - \mathbf{x}_0}{\lambda(t)}\right) = \tilde{M}$$

then it is prudent to 'look at M_t on the scale of $\lambda(t)$', in other words, if we want to show properties of the surface as time runs to the singular time (or infinity), then something that we might do is look at it on a different scale (in both time and space) such that the contraction (or be it expansion) is normalised out.

We make the ansatz that (at least in the case of cylindrical graphs) the asymptotic rate of contraction (or expansion) is given by

$$\lambda(t) = \sqrt{1 + \alpha(t - t_0)}, \quad t \in [t_0, T)$$

for some $\alpha \neq 0$, and $T > 0$ is given by

$$T = \begin{cases} t_0 + \frac{\xi}{\alpha}, & \alpha < 0 \\ \infty, & \alpha > 0 \end{cases}$$

where $\xi = \operatorname{sign}\alpha$.

This choice is motivated by the hypothesis that singularities (and expansion) of (MCF) are modelled by self-similar solutions. For more details, see Chapter 5. In the case that $\alpha < 0$ we are interested in singularities, and in the case that $\alpha > 0$ we are interested in expansion.

To that end, let us normalise our solution, defining new space and time variables,

$$\tilde{\mathbf{x}}(s) = \psi(t)\mathbf{x}(t), \quad t \in [t_0, T)$$

and

$$s = \xi \log \lambda(t), \quad t \in [t_0, T)$$

where $\psi(t) = \sqrt{\frac{\xi\alpha}{2}}\lambda^{-1}(t)$,

Denote the normalised surface by $\tilde{M}_s$. The normalisation has the effect of scaling out homothety while extending the (finite, in the case $\alpha < 0$) time interval to be defined for all positive times s and we obtain the normalised flow $(\tilde{M}_s)_{s\in[0,\infty)}$. The following results are from [15].

Proposition 2.9 (Normalised Mean Curvature Flow)**.**

$$\frac{d\tilde{\mathbf{x}}}{ds} = \tilde{\mathbf{H}} - \xi\tilde{\mathbf{x}}, \quad \tilde{\mathbf{x}} \in \tilde{M}_s, s \in [0,\infty) \tag{2.10}$$

Proof. Since $\frac{d\psi}{dt} = -\xi\psi^3$ and $\frac{ds}{dt} = \psi^2$, we compute

$$\begin{aligned}\mathbf{H} &= \frac{d\mathbf{x}}{dt} \\ &= \frac{d}{dt}\left(\psi^{-1}\tilde{\mathbf{x}}\right) \\ &= -\frac{1}{\psi^2}\frac{d\psi}{dt}\tilde{\mathbf{x}} + \frac{1}{\psi}\frac{d\tilde{\mathbf{x}}}{ds}\frac{ds}{dt} \\ &= \psi\left(\frac{d\tilde{\mathbf{x}}}{ds} + \xi\tilde{\mathbf{x}}\right)\end{aligned}$$

which yields the result, since $\tilde{\mathbf{H}} = \psi^{-1}\mathbf{H}$. □

Lemma 2.10. *Suppose P and Q are quantities formed by contractions of g_{ij} and h_{ij}, P and satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right)P = Q$$

and $\tilde{P}$ has degree γ (i.e. $\tilde{P} = \psi^\gamma P$), then $\tilde{Q}$ has degree $\gamma - 2$ and

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)\tilde{P} = \tilde{Q} - \xi\gamma\tilde{P}$$

Proof. Much the same as above, we calculate

$$\begin{aligned}Q &= \left(\frac{d}{dt} - \Delta\right)P \\ &= \left(\frac{d}{dt} - \Delta\right)\psi^{-\gamma}\tilde{P} \\ &= \psi^{-\gamma}\left(\frac{d}{dt} - \Delta\right)\tilde{P} - \gamma\psi^{-\gamma-1}\frac{d\psi}{dt}\tilde{P} \\ &= \psi^{-(\gamma-2)}\left(\left(\frac{d}{ds} - \tilde{\Delta}\right)\tilde{P} + \xi\gamma\tilde{P}\right)\end{aligned}$$

and defining $\tilde{Q} = \psi^{\xi-2}Q$ the conclusion follows. □

Applying this lemma, we obtain the rescaled equations on $\tilde{M}_s$

(i) $\left(\frac{d}{ds} - \tilde{\Delta}\right)\tilde{u} = -\frac{n-1-\tilde{\kappa}}{\tilde{u}} - \xi\tilde{u}$

(ii) $\left(\frac{d}{ds} - \tilde{\Delta}\right)\tilde{v} = -|\tilde{A}|^2\tilde{v} - 2\tilde{v}^{-1}|\nabla\tilde{v}|^2 + \left(\frac{n-1-\tilde{\kappa}}{\tilde{u}^2}\right)\tilde{v} + \frac{2\tilde{H}_\tau\tilde{v}^2}{\tilde{u}}$

(iii) $\left(\frac{d}{ds} - \tilde{\Delta}\right)\tilde{H} = \tilde{H}|\tilde{A}|^2 + \xi\tilde{H}$

(iv) $\left(\frac{d}{ds} - \tilde{\Delta}\right)|\tilde{A}|^2 = -2|\nabla\tilde{A}|^2 + 2|\tilde{A}|^4 + 2\xi|\tilde{A}|^2$

Note that the rescaled equation for $\tilde{v}$ is identical to that of v.