Chapter 1

Introduction

Mean curvature flow arises when we look at ways of evolving embedded hypersurfaces to minimise the area functional, in fact, mean curvature flow can be defined as the steepest descent flow of the area functional and in some sense, (MCF) evolves hypersurfaces towards minimal surfaces.

Intuitively, (MCF) is an evolutionary process by which each point $\mathbf{x}$ on surface M_t is given a velocity equal to the mean curvature vector, $\mathbf{H}(\mathbf{x}) = -H(\mathbf{x})\boldsymbol{\nu}(\mathbf{x})$, where $H(\mathbf{x}) = \operatorname{div}_{M_t} \boldsymbol{\nu}(\mathbf{x})$ is the mean curvature at $\mathbf{x}$ and $\boldsymbol{\nu}$ the unit normal at $\mathbf{x}$. The process generates a family of surfaces which are said to evolve via mean curvature flow. With this process in mind, we formally define the evolution of a surface under (MCF).

Definition 1.1 (Mean Curvature Flow)**.** A family of manifolds $(M_t)_{t\in[0,T)}$ immersed in $\mathbb{R}^{n+1}$ is said to evolve via mean curvature flow on $[0,T)$ for some $T > 0$ if

$$\frac{d\mathbf{x}}{dt} = \mathbf{H}(\mathbf{x}), \quad \mathbf{x} \in M_t, t \in [0,T) \tag{1.1}$$

An equivalent, and at times more useful, definition of (MCF) is gained by considering the immersions $\mathbf{F}_t = \mathbf{F}(\cdot, t)$

$$\mathbf{F} : M^n \times [0,T) \to \mathbb{R}^{n+1}$$

so that (MCF) can be equivalently cast as

Definition 1.2 (MCF)**.** A family $(\mathbf{F}_t)_{t\in[0,T)}$ of immersions moves by (MCF) if

$$\frac{\partial \mathbf{F}}{\partial t}(\boldsymbol{p}, t) = \mathbf{H}\left(\mathbf{F}(\boldsymbol{p}, t)\right), \quad \boldsymbol{p} \in M^n, t \in [0,T)$$

So the manifolds M_t are now given by $M_t = \mathbf{F}(\cdot, t)(M^n)$. Frequently we will denote $\mathbf{x} = \mathbf{F}(\boldsymbol{p}, t)$ as the position vector on the evolving surface.

Mean curvature flow evolves surfaces in such a way as to minimise the area, indeed (MCF) can be obtained as a result of computing the L^2 gradient flow for variations of the area functional. As a result, under the flow, we have that the local area element $\mu = \mu(\mathbf{x}, t)$ is monotone decreasing (see Appendix B), moreover,

$$\frac{d\mu}{dt} = -H^2\mu$$

so, for compact M_t the total area, not just the area element, decreases monotonically.

Mean curvature flow was first studied by Brakke [6] where the approach was from the point of view of geometric measure theory. Many properties of the solutions to (MCF) have been investigated and it and associated flows remain an area of significant interest. One of the earlier results that stimulated much interest in the flow was Huisken's [14] result that convex, compact hypersurfaces retain their convexity under the flow and in fact become asymptotically round.

Since then, there has been much research on (MCF) and other associated curvature flows such as the Gauß curvature flow or the celebrated Ricci flow, with which Perelman [17], [18] has possibly provided a constructive geometric proof for Thurston's Geometrisation Conjecture.

Much insight into the properties of mean curvature flow is to be gained by studying special classes of solutions. One such class of special solutions which yields many interesting examples is the case of so called 'normal graphs' (see Appendix D). These are surfaces that may be expressed as time-dependent deformations in the normal direction of some fixed base surface M^n in terms of a suitable height ρ.

We can show that evolution by (MCF) of a surface that may be expressed as a normal graph is equivalent, up to tangential diffeomorphisms, to the scalar quasi-linear parabolic partial differential equation on M^n

$$\begin{aligned} \frac{\partial \rho}{\partial t}(\boldsymbol{q}, t) = -|\overline{\nabla}\Phi_\rho(\mathbf{x}, t)| \operatorname{div}_{\mathbb{R}^{n+1}} \left(\frac{\overline{\nabla}\Phi_\rho(\mathbf{x}, t)}{|\overline{\nabla}\Phi_\rho(\mathbf{x}, t)|} \right) \Bigg|_{\mathbf{x}=\tilde{\mathbf{F}}(\boldsymbol{q},t)}, \\ (\boldsymbol{q}, t) \in M^n \times [0, T) \end{aligned} \tag{1.2}$$

where $\tilde{\mathbf{F}}(\boldsymbol{q}, t) = \boldsymbol{q} + \rho(\boldsymbol{q}, t)\boldsymbol{\omega}(\boldsymbol{q})$ and $\Phi_\rho(\mathbf{x}, t) = \Lambda(\mathbf{x}) - \rho(\mathbf{S}(\mathbf{x}), t)$ with Λ and $\mathbf{S}$ the signed distance and closest point projection for M^n respectively. While this is a parabolic equation, it is not necessarily uniformly so, especially if the solution is moving towards a focal point of the base surface. Therefore, much care must be taken to ensure that solutions to this equation do not meet with such points.

Perhaps the simplest graphical case is that of an ordinary planar graph, that is, the case where the base surface is an n-dimensional hyperplane in $\mathbb{R}^{n+1}$. In this case, the base surface has no focal points so all that is required for parabolicity is a gradient bound. It has been shown by Ecker and Huisken that for initial data satisfying a Lipschitz growth condition we have long time existence for the flow [10], [11]. Furthermore, the authors show that for initial data with a unique tangent cone at infinity the rescaled flow converges exponentially to a self-similarly expanding solution.

Another interesting choice of base surface is the cylinder. Under the assumption that we have a surface that is a rotationally symmetric graph over a cylinder, we obtain the equation for $\rho : \mathbb{R} \times [0, T) \to \mathbb{R}$

$$\frac{\partial \rho}{\partial t}(z,t) = \frac{\rho''}{1+(\rho')^2} - \frac{n-1}{\rho}, \quad (z,t) \in \mathbb{R} \times [0,T) \tag{1.3}$$

where $\rho'(z,t) = \frac{\partial \rho}{\partial z}(z,t)$.

In this thesis, we shall concern ourselves mainly with the case of graphs over cylinders, and more specifically the case where they have rotational symmetry.

Rotationally symmetric graphs over cylinders has been studied by Altschuler, Angenent and Giga [1] in the non-entire case where the ends are capped perpendicular to the axis. In this paper, the authors show a number of interesting results. Among these are: the number of necks is a non-increasing function of time, the evolution may be extended smoothly through a neck-pinch, and the asymptotic rate of a pinching neck is exactly that of a homothetically shrinking cylinder. In [2], Angenent and Velázquez study rotationally symmetric non-entire solutions, constructing solutions exhibiting type II blowup.

In this thesis, we shall also be investigating rotationally symmetric graphs over cylinders, but here we will be looking at entire solutions, that is, solutions over the entire axis. Also, while Altschuler, et al. [1] worked from a mainly PDE point of view, we shall be deriving estimates and results using a more geometric approach and exploit the maximum principle to maximum effect.

The main result of this thesis is a convergence result for initially cylindrical graphs satisfying an asymptotic steepness and straightness condition. For initial graphs with sufficient asymptotic steepness, we derive an analogous result to that in [10], that is

Theorem 1.3 (Main Theorem)**.** *Suppose that M_0 is a rotationally symmetric entire cylindrical graph, which has at most linear growth, bounded curvature, is bounded below self-similarly (in a sense to be defined in Chapter 5) and suppose also that M_0 satisfies the estimate*

$$\langle \mathbf{x}, \boldsymbol{\nu} \rangle^2 \leqslant c_3 \left(1 + \langle \mathbf{x}, \boldsymbol{\vartheta} \rangle^2\right)^{1-\delta}, \quad \mathbf{x} \in M_0$$

for some $\delta > 0$ and $c_3 < \infty$ then the solution $\tilde{M}_s$ of normalised (MCF) converges as $s \to \infty$ to a limiting surface $\tilde{M}_\infty$ upon which the equation

$$\tilde{\mathbf{x}}^\perp = \tilde{\mathbf{H}}(\tilde{\mathbf{x}}), \quad \tilde{\mathbf{x}} \in \tilde{M}_\infty$$

is satisfied.

Crucial to the proof of the main result will be the:

... evolution equations for basic geometric quantities such as height, gradient and curvature derived in Chapter 2

... uniform geometric height, gradient and curvature bounds derived in Chapter 3,

... existence and extension results of Chapter 4,

... self-similarly expanding barrier surfaces of Chapter 5,

... maximum and comparison principles of Appendix E.