2 Solvability and smoothness of solutions of the semiperiodical Dirichlet problem for mixed type equations

2.1 Separability of a mixed type operator

In this chapter the role of coefficients for the separability of a mixed type operator in an unbounded domain is studied.

Consider the operator

$$Lu = -k(y)u_{xx} - u_{yy} + a(y)u_x + c(y)u \tag{2.1.1}$$

originally defined in $C^{\infty}_{0,\pi}(\Omega)$, where $k(y)$ is a sectionally continuous function in $\mathbb{R}$=(-∞, +∞) and $yk(y) > 0$ for $y \neq$0, $k(0)$=0 (as y=0). Here $C^{\infty}_{0,\pi}(\Omega)$ is a set of infinitely differentiable functions, satisfying the conditions: $u(-\pi, y) = u(\pi, y)$, $u_x(-\pi, y) = u_x(\pi, y)$ and finite as functions of the y variable, where

$$\Omega = \{(x, y) : -\pi < x < \pi, \quad -\infty < y < +\infty\}.$$

It is easy to show that the operator L admits closure in the metric of $L_2(\Omega)$ and the closure is also denoted by L. As usually we denote the domain of the operator by $D(L)$.

Definition. The operator L is called separable if for all functions $u(x, y)\in D(L)$ the estimate

$$\|-k(y)u_{xx} - u_{yy}\|_2 + \|a(y)u_x\|_2 + \|c(y)u\|_2 \leqslant C\left(\|Lu\|_2 + \|u\|_2\right)$$

holds, where $C > 0$ is a constant not depending on $u(x, y)$.

Here and in the following $||\cdot||_2$ denotes the norm of $L_2(\Omega)$.

The basic result of this section is formulated in the following theorem.

Theorem 2.1.1. *Let the conditions :*

i) $a(y)$ *and* $c(y)$ *are piecewise continuous functions in any compact set in* $\mathbb{R}$ *and* $|a(y)| \geqslant \delta_0 > 0$*;*

ii) $c(y) \leqslant c_0 a^2(y)$ *for any* $y \in \mathbb{R}$*, where* $c_0 > 0$ *is a fixed number;*

iii) $\mu_1 = \sup\limits_{|y-t|\leqslant 1} \dfrac{a(y)}{a(t)} < \infty$, $\mu_2 = \sup\limits_{|y-t|\leqslant 1} \dfrac{c(y)}{c(t)} < \infty$

be fulfilled. Then the quantity

$$\gamma = \sup_{u\in D(L)} \frac{||-k(y)_{xx} - u_{yy}||_2}{||Lu||_2 + ||u||_2} \tag{2.1.2}$$

is finite if and only if

$$\inf_{y\in R} c(y) = c > -\infty. \tag{$*$}$$

The necessity will be proved without involving some auxiliary assertions whereas the proof of sufficiency needs a few auxiliary lemmas.

Consider the operator l_n $(n = 0, \pm 1, \pm 2, ...)$ defined by the equality

$$l_n u = -u''(y) + (n^2 k(y) + ina(y) + c(y))u(y) \tag{2.1.3}$$

originally defined in $C_0^\infty(\mathbb{R})$, the set of infinitely differentiable and finite functions in $\mathbb{R}$=(-∞, +∞), where $k(y)$ is a sectionally continuous function in any compact set in $\mathbb{R}$ and a bounded function in $\mathbb{R}$; $yk(y) > 0$ for $y \neq$0 and $k(0)$=0.

The operator (2.1.3) admits closure in $L_2(R)$ and the closure is also denoted by l_n and henceforth, referring to the expression (2.1.3), we will mean this closure.

Let $D(l_n)$ denote the domain of definition of an already closed operator l_n, consisting of finite functions which together with their derivatives up to second order belong to $L_2(\mathbb{R})$.

The following lemmas hold for the operator l_n.

Lemma 2.1.1. *Let the conditions i) and (*) be fulfilled. Then the operator* $l_n + \lambda E$ *is continuously invertible for sufficiently large* $\lambda > 0$.

Lemma 2.1.2. *Let the conditions i)-iii) and the condition (*) be fulfilled. Then the estimate*

$$\left\|\rho(y)|n|^{\alpha}(l_n+\lambda E)^{-1}\right\|_{2\to 2} \leqslant c(\lambda)\sup_{\{j\}}\left\|\rho(y)|n|^{\alpha}\varphi_j(l_n+\lambda E)^{-1}\right\|_{2\to 2}$$

holds for sufficiently great $\lambda > 0$, *where the operator* $l_{n,j}$ *defined by the equality*

$$l_{n,j}u = -u''(y) + (n^2k(y) + ina(y) + c(y))u$$

and the boundary conditions

$$u(\Delta_{\bar{j}}) = u(\Delta_{\stackrel{+}{j}}) = 0\ ,$$

where $\Delta_{\bar{j}}$ *and* $\Delta_{\stackrel{+}{j}}$ *are the left and right ends of the intervals* $\Delta_j = (j-1, j+1)$; $\rho(y)$ *is a continuous function in* $\mathbb{R}$; $c(\lambda)$ *is a constant depending on* λ *and* α *= 0,1.*

Lemma 2.1.3. *Let the conditions of Lemma 2.1.2 be fulfilled. Then*

$$\left\|a(y)|n|(l_n+\lambda E)^{-1}\right\|_{2\to 2} < \infty$$

$$\left\|c(y)(l_n+\lambda E)^{-1}\right\|_{2\to 2} < \infty.$$

The proofs of the lemmas 2.1.1–2.1.3 and the proof of sufficiency of the main theorem will be borrowed from the work [28] and will be cited here in a more compact form for completeness.

Proof of Lemma 2.1.1. Consider the scalar product

$$|\langle l_n u + \lambda u, -inu\rangle| = \left|\int\limits_{-\infty}^{\infty} a(y)|u|^2 dy - in\int\limits_{-\infty}^{\infty}[|u'|^2 + (n^2k(y) + c(y) + \lambda)|u|^2]dy\right|,$$

where $u(y)$ is arbitrary function belongs to $C_0^{\infty}(R)$.

Hence, using the Cauchy-Bunyakovskii inequality, it is not difficult to obtain the following estimate

$$\|(l_n + \lambda E)u\|_2 \geqslant |n|\delta_0\|u\|_2. \tag{2.1.4}$$

And in case n =0 (i.e. $l_0 = -u''(y) + c(y)u(y)$) the following inequality holds for sufficiently large $\lambda > 0$

$$\|(l_0 + \lambda E)u\|_2 \geqslant \delta\|u\|_2, \tag{2.1.5}$$

where $\delta = \inf\limits_{y\in R} c(y) + \lambda > 0$.

Indeed, considering the scalar product

$$|\langle(l_0 + \lambda E)u, u\rangle| = \left|\int\limits_{-\infty}^{\infty} a(y)|u|^2dy - in\int\limits_{-\infty}^{\infty}(c(y) + \lambda)|u|^2dy\right|$$

owing to the Cauchy-Bunyakovskii inequality, we have (2.1.5).

Now, combining (2.1.4) and (2.1.5), we find

$$\|(l_n + \lambda E)u\|_2 \geqslant c_0(|n| + 1)\|u\|_2, \tag{2.1.6}$$

where $c_0 = \min\left\{\frac{\delta_0}{2}, \frac{\delta}{2}\right\}$

Consider the operator $l_{n,j}$ cited in Lemma 2.1.2.

When the conditions i) and (*) are fulfilled the operator $l_{n,j} + \lambda E$ has a continuously inverse operator for sufficiently large $\lambda > 0$ defined in the whole of $L_2(\Delta_j)$ and the inequalities

$$\left\|(l_{n,j} + \lambda E)^{-1}u\right\|_{2\to 2} \geqslant \frac{c}{\lambda^{1/2}}, \tag{2.1.7}$$

$$\left\|\frac{d}{dy}(l_{n,j} + \lambda E)^{-1}\right\|_{2\to 2} \geqslant \frac{c}{\lambda^{1/4}} \tag{2.1.8}$$

hold for the inverse operator, where $c > 0$ is a constant.

Let us prove these assertions.

Integrating by parts it is not difficult to find the estimate

$$\|(l_n + \lambda E)u\|_2 \geqslant c_1\|u\|_2$$

for sufficiently large $\lambda > 0$, for every $u(y) \in D(l_{n,j})$, where $c_1 > 0$ is a constant.

Now if we show that the range set of the operator is everywhere dense in $L_2(\Delta_j)$ then it is clear that the operator $l_{n,j} + \lambda E$ has a continuous inverse for sufficiently large $\lambda > 0$.

Let us suppose the opposite, i.e. the range set $R(l_{n,j}+\lambda E)$ is not everywhere dense in $L_2(\Delta_j)$. Then there exists a element $v \in L_2(v \neq 0)$ such that

$$\left\langle u, l^*_{n,j}v \right\rangle = \langle l_{n,j}, v\rangle = 0 \tag{2.1.9}$$

holds for all $u \in D(l_{n,j}) = D(l_{n,j} + \lambda E)$, where $l^*_{n,j}$ is a conjugate operator to the operator $l_{n,j}$ defined by the equality

$$l^*_n u = -v''(y) + (n^2k(y) + ina(y) + c(y))v(y).$$

From (2.1.9) it follows that $l^*_{n,j}v$=0. Since $k(y)$, $a(y)$ and $c(y)$ are bounded and continuous functions in the segment Δ_j then the function $(n^2k(y)$ - $ina(y)+$ $c(y))$ $v \in L_2(\Delta_j)$ and therefore $v'' \in L_2(\Delta_j)$.

Now, integrating by parts the expression

$$0 =< u, l^*_{n,j}v >= \int\limits_{j-1}^{j+1} u(y)[-\bar{v}''(y) + \overline{(n^2k(y) - \imath n\alpha(y) + c(y))}\bar{v}(y)]\delta y$$

by virtue of the arbitrariness of $u \in D(l_{n,j})$, we have easily made sure that $v(\Delta^-_j) = v(\Delta^+_j)$=0. From here, integrating by parts, we have

$$\left\|l^*_{n,j}v\right\|_2 \geqslant c_1\|v\|_2,$$

where $c_1 > 0$.

From the last inequality, by virtue of $l^*_{n,j}v = 0$, it follows that $v = 0$. Thus we have come to a contradiction that proves the assertion.

It remains to prove the inequalities (2.1.7) and (2.1.8).

Construct the scalar product $< (l_{n,j} + \lambda E)u, u >$, where $u \in D(l_{n,j})$. Integrating by parts and using the boundary conditions we find

$$|< (l_{n,j} + \lambda E)u, u >| = \left| \int\limits_{\Delta j} |u'|^2 dy + \int\limits_{\Delta j} n^2 k(y) + ina(y) + (c(y) + \lambda)|u|^2] dy \right|. \tag{2.1.10}$$

Using the Cauchy-Bunyakovskii inequality and the condition $i)$ we have

$$c_0 \|(l_{n,j} + \lambda E)u\|_2^2 \geqslant c_0 |n|^2 \min_{y \in \Delta_j} |a(y)|^2 \|u\|_2^2. \tag{2.1.11}$$

From (2.1.10), using the Cauchy inequality with $\varepsilon > 0$, we find

$$\frac{1}{2\varepsilon} \|(l_{n,j} + \lambda E)u\|_2^2 \geqslant \int\limits_{\Delta j} |u'|^2 + (c(y) + \lambda - \frac{\varepsilon}{2})|u|^2] dy - n^2 \int\limits_{\Delta j} k(y)|u|^2] dy.$$

Combining this inequality with the inequality (2.1.11) we have

$$\left(\frac{1}{2\varepsilon} + c_0 \right) \|(l_{n,j} + \lambda E)u\|_2^2 \geqslant \int\limits_{\Delta j} |u'|^2 + (-(y) + \lambda - \frac{\varepsilon}{2})|u|^2] dy +$$

$$+ \int\limits_{\Delta j} n^2 (c_0 \min |a(y)|^2 - k(y)|u|^2) dy.$$

From the last inequality, taking the condition (*) into account, we obtain for sufficiently large $\lambda > 0$

$$c(\varepsilon) \|(l_{n,j} + \lambda E)u\|_2^2 \geqslant \lambda \|u\|_2^2, \tag{2.1.12}$$

where $c(\varepsilon) = c_0 + \frac{1}{2\varepsilon}$. The inequality (2.1.12) proves the inequality (2.1.7). From (2.1.10), owing to (2.1.12), the inequality (2.1.8) easily follows.

Consider the operator K defined by the equality

$$Kf = \sum_{\mathrm{j}} \varphi_{\mathrm{j}}(l_{\mathrm{n,j}} + \ \lambda E)\varphi_{\mathrm{j}} f, \ \ f \in L_2(\mathbb{R}),$$

where $\{\varphi_j\}$ is a set of nonnegative functions from $C_0^\infty(\mathbb{R})$ such that $\sum\limits_j \varphi_j \equiv 1$, supp $\varphi_j \subset \Delta_j$ and $\bigcup\limits_j \Delta_j = R$.

It is not difficult to show that $Kf \in D(l_n)$. Now, applying the operator $(l_n{+}\lambda E)$ to the operator Kf, we have

$$(l_{n,j} + \lambda E)\, Kf = f + \sum_j \varphi_j''\, (l_{n,j} + \lambda E)^{-1}\, \varphi_j f + 2\sum_j \varphi_j \frac{d}{dy}\, (l_{n,j} + \lambda E)^{-1}\, \varphi_j f.$$

Owing to arbitrariness of f we can see that the equality is fulfilled for any function $f \in L_2(R)$. From the inequalities (2.1.7) and (2.1.8) it is not difficult to find the estimate

$$||\, (l_{n,j} + \lambda E)\, \mathrm{K}f||_2 \leqslant ||f||_2 + \left[24c\sum_j \left(\frac{||\varphi_j f||^2}{\sqrt{x}} + \frac{||\varphi_j f||^2}{\sqrt[4]{x}} \right) \right]^{1/2} < \infty.$$

From the aforecited discourses and from the inequality (2.1.6) we conclude that the operator $l_{n,j} + \lambda E$ has a continuous inverse in the space $L_2(\mathbb{R})$ for sufficiently large $\lambda > 0$. Lemma 2.1.1 is proved.

Proof of Lemma 2.1.2. Let the conditions *i*)-*iii*) be fulfilled. Then owing to Lemma 2.1.1 and the fact that $l_{n,j}{+}\lambda E$ is continuously invertible the equality

$$(l_n + \lambda E)^{-1} f = (l_n + \lambda E)^{-1} B_{n,\lambda} f + M_{n,\lambda} f \qquad (2.1.13)$$

holds for any function $f \in C_0^\infty\,(R)$, where

$$B_{n,\lambda} f = \left[E - \sum_j (l_{n,j} + \lambda E)\, \varphi_j\, (l_{n,j} + \lambda E)^{-1}\, \varphi_j \right] f,$$

$$M_{n,\lambda} f = \sum_j \varphi_{jj}\, (l_{n,j} + \lambda E)^{-1}\, \varphi_j f.$$

It is easy to obtain the following estimates for a sufficiently large $\lambda > 0$

$$\left\|(l_{n,j}+\lambda E)^{-1}\right\|_2 \leqslant \frac{\sqrt{2}}{\lambda}, \quad \left\|\frac{d}{dy}(l_{n,j}+\lambda E)^{-1}\right\|_2 \leqslant \frac{1}{\sqrt{\lambda}}. \tag{2.1.14}$$

Now from the representation of the operator $B_{n,\lambda}$ and from the fact that $\sum\limits_j \varphi_{\mathrm{j}}^2 \equiv 1$ and $f - \sum\limits_j \varphi_j^2 f = 0$ we find the following expression for $B_{n,\lambda}$

$$B_{n,\lambda} f = \sum_j 2\varphi_j' \frac{d}{dy}(l_{n,j}+\lambda E)^{-1}\varphi_j^2 f + \sum_j \varphi_j''(l_{n,j}+\lambda E)^{-1}\varphi_j f.$$

Let us estimate the norm of the operator $B_{n,\lambda}$. Due to the fact that only the functions φ_{j-1}, φ_j, φ_{j+1} are not equal to zero in $\Delta_j = [j-1,\ j+1]$ we have

$$\|B_{n,\lambda} f\|_2^2 \leqslant \sum_{j=-\infty}^{\infty} \int_{j-1}^{j+1} \left| \sum_{j-1}^{j+1} \left[\varphi_j''(l_{n,j}+\lambda E)^{-1}\varphi_j f + 2\varphi_j' \frac{d}{dy}(l_{n,j}+\lambda E)^{-1}\varphi_j' f\right]\right|^2 dy.$$

Hence, taking into account that $(a+b+c)^2 \leqslant 3\left(a^2+b^2+c^2\right)$ and $(a+b)^2 \leqslant 2\left(a^2+b^2\right)$, we have

$$\|B_{n,\lambda} f\|_2^2 \leqslant 24c \sum_j \left(\|(l_{n,j}+\lambda E)^{-1}\|_2^2 \|\varphi_j f\|_2^2 + \left\|\frac{d}{dy}(l_{n,j}+\lambda E)^{-1}\right\|_2^2 \|\varphi_j f\|_2^2 \right),$$

where $c = \max\left\{|\varphi_j'|, |\varphi_j''|\right\}$

From the estimates (2.1.14) and the last inequality it is clear that if $\lambda > 0$ is a sufficiently large number then $\|B_{n,\lambda}\| < 1$. Then by a well-known theorem of functional analysis (see for instance [53-54]) the operator $E - B_{n,\lambda}$ has a continuous inverse operator $(l_n+\lambda E)^{-1}$ and due to (2.1.13)

$$(l_n+\lambda E)^{-1} = M_{n,\lambda}(E - B_{n,\lambda})^{-1}$$

holds. This implies that the operator $\rho(y)|n|^\alpha (l_n+\lambda E)^{-1}$ is bounded (or unbounded) in conjunction with the operator $\rho(y)|n|^\alpha M_{n,\lambda}$, i.e. the inequality

$$\left\|\rho(y)|n|^\alpha (l_n+\lambda E)^{-1}\right\|_2 \leqslant c(\lambda)\|\rho(y)|n|^\alpha M_{n,\lambda}\|_2$$

is correct, where $\alpha = 0, 1$; $\rho(y)$ is a continuous function.

From this inequality and from the definition of $M_{n,\lambda}$ we obtain the proof of Lemma 2.1.2.

Proof of Lemma 2.1.3. The estimate

$$\left\|(l_{n,j} + \lambda E)^{-1}\right\|_2 \leqslant \frac{2}{c(\tilde{y}_j)}$$

holds for the operator $(l_{n,j} + \lambda E)^{-1}$, where $c(\tilde{y}_j) = \min\limits_{y \in \bar{\Delta}_j} c(y)$ (the proof of this estimate can be found in the work [28]).

According to the last estimate and Lemma 2.1.2 we have

$$\left\|c\,(y)\,(l_{n,j} + \lambda E)^{-1}\right\|_2 \leqslant c\,(\lambda) \sup_j \left\|c\,(y)\,\varphi_j\,(l_{n,j} + \lambda E)^{-1}\right\|_2 \leqslant c\,(\lambda) \sup_j \frac{c(y)}{c(t)} < \infty.$$

Similarly we find the second inequality. Lemma 2.1.3 is proved.

Let us proceed to the proof of the main theorem, i.e. we prove the necessity and sufficiency of the theorem assertion.

Proof of Theorem 2.1.1.

Necessity. Let $u(x, y) \in D(L)$. Since the system $\left\{e^{inx}\right\}_{n=-\infty}^{\infty}$ is complete in $L_2(\Omega)$ then the decomposition

$$u(x, y) = \sum_{n=-\infty}^{\infty} u_n(y) e^{inx}$$

holds for $u(x, y)$ in the metric of L_2.

It is not difficult to determine that

$$\gamma \geqslant \frac{\|-u''(y)\|^2}{\|-u''(y) + c(y)u(y)\|_2 + \|u(y)\|_2}, \tag{2.1.15}$$

where $u(y) \in D(l_0)$, $l_0 u = -u''(y) + c(y)u(y)$.

Indeed, it is clear from the representation of the function $u(x, y)$ when $n = 0$. In particular this representation implyies that $u(y) \in D(l_0) \subset D(L)$, wherefrom in view of the definition of γ (2.1.15) follows.

Let the conditions i)-iii) be fulfilled and the condition (*) be not satisfied and the quantity γ be finite.

Let us take the sequence of disjoint intervals

$$\Delta_k = \left[-\frac{\pi}{2}+\pi k,\ \frac{\pi}{2}+\pi k\right],\quad \bigcup_{\{k\}}\Delta_k = R,\quad k=0,\ \pm1,\ \pm2,\ \dots .$$

Let $c(y) = -\left(k^2+1\right)$ for $y\in\Delta_k$. It easy to prove that the condition iii) is fulfilled for the function $c(y)$.

Consider the function sequence

$$z_k(y) = \begin{cases} \sin k\left(y-y_k-\frac{\pi}{2}\right), & y\in\Delta_k, \\ 0, \quad y\notin\Delta_k \end{cases}$$

where y_k is the middle point of the interval Δ_k.

Let

$$\alpha(y) = \begin{cases} 1, & |y|\leqslant \frac{\pi}{2}-\delta \\ 0, & |y|>\frac{\pi}{2} \end{cases} \quad \text{and } \alpha(y)\in C_0^\infty\left(-\tfrac{\pi}{2},\ \tfrac{\pi}{2}\right).$$

We replace the function $u(y)\in D(l_0)$ by a sequence of functions $\alpha\left(y-y_k\right)z_k(y)$ (y_k is the middle point) . Then

$$\text{-}\left(\alpha\left(y-y_k\right)z_k\left(y\right)\right)''+c(y)\alpha\left(y-y_k\right)z_k\left(y\right) = -\alpha\left(y-y_k\right)\sin k\left(y-y_k-\frac{\pi}{2}\right)-$$

$$-2k\alpha'\left(y-y_k\right)\cos k\left(y-y_k-\frac{\pi}{2}\right)-\alpha''\left(y-y_k-\right)\sin k\left(y-y_k-\frac{\pi}{2}\right)- \tag{2.1.16}$$

$$-\left(\alpha\left(y-y_k\right)z_k\left(y\right)\right)'' = k^2\alpha\left(y-y_k\right)\sin k\left(y-y_k-\frac{\pi}{2}\right)-$$

$$-2k\alpha'\left(y-y_k\right)\cos k\left(y-y_k-\frac{\pi}{2}\right)-\alpha''\left(y-y_k\right)\sin k\left(y-y_k-\frac{\pi}{2}\right). \tag{2.1.17}$$

Estimate the norm of the expressions (2.1.16) and (2.1.17)

$$\left\|-\left(\alpha\left(y-y_k\right)z_k\left(y\right)\right)''+c(y)\alpha\left(y-y_k\right)z_k\left(y\right)\right\|_2 \leqslant c_1+2kc_2+c_3,$$

$$\left\|-\left(\alpha\left(y-y_k\right)z_k\left(y\right)\right)''\right\|_2 \geqslant k^2c_1-2kc_2-c_3,$$

where

$$c_1 = \left\| \alpha\left(y-y_k\right) \sin k\left(y-y_k-\frac{\pi}{2}\right) \right\|_2,$$
$$c_2 = \left\| \alpha'\left(y-y_k\right) \cos k\left(y-y_k-\frac{\pi}{2}\right) \right\|_2,$$
$$c_3 = \left\| \alpha''\left(y-y_k\right) \sin k\left(y-y_k-\frac{\pi}{2}\right) \right\|_2.$$

All the norms are finite due to the fact that the functions tend to zero outside of Δ_k (since they are finite).

Now, using the inequality (2.1.15) and the last estimates, we find

$$\gamma \geqslant \frac{k^2 c_1 - 2kc_2 - c_3}{c_1 + 2kc_2 + c_3}.$$

Letting k tend to infinity we obtain that γ tend to ∞. It contradicts to the finiteness of γ. The necessity is proved.

Sufficiency. Let the conditions i)-iii) and (*) be fulfilled. Let us prove that γ is finite.

Considering the scalar products $<(L+\lambda E)u,\ u>$ and $<(L+\lambda E)u,\ u_x>$, taking into account the conditions i) and (*), it is not difficult to obtain for the operator L

$$\|(L+\lambda E)\,u\|_2 \geqslant c\|u\|_2$$

for all $u \in D(L)$, where $c > 0$ is a constant.

Due to the last inequality and Lemma 2.1.1 we obtain that the operator $L+\lambda E$ is continuously invertible for sufficiently large $\lambda > 0$ and the representation

$$(L+\lambda E)^{-1} f = \sum_{n=-\infty}^{\infty} \left(l_n + \lambda E\right)^{-1} f_n e^{inx} \qquad (2.1.18)$$

holds for any $f \in L_2\left(\Omega\right)$.

From (2.1.18) and in view of the orthonormal system $\left\{e^{inx}\right\}_{n=-\infty}^{\infty}$ in $L_2\left(-\pi,\ \pi\right)$ it easily follows that

$$\left\| \rho(y) D_x^{\alpha} \left(L+\lambda E\right)^{-1} \right\|_{2\to 2} \leqslant \sup_{\{n\}} \left\| \rho(y) |n|^{\alpha} \left(L+\lambda E\right)^{-1} \right\|_{2\to 2},$$

where $D_x^\alpha = \frac{\partial^\alpha}{\partial x^\alpha}$, $n = 0,\ \pm 1,\ \pm 2,\ \ldots;\quad \rho(y)$ is a continuous function in $\mathbb{R}$.

From here and from Lemma 2.1.2 we find

$$\left\|\rho(y) D_x^\alpha \left(L + \lambda E\right)^{-1}\right\|_{2\to 2} \leqslant c(\lambda) \sup_{\{n\}} \sup_{\{j\}} \left\|\rho(y)|n|^\alpha \varphi_j \left(l_{n,j} + \lambda E\right)^{-1}\right\|_{2\to 2}.$$

Now, if $\rho(y) = a(y)$, $\rho(y) = c(y)$ then according to the last inequality and Lemma 2.1.3 we have

$$\left\|a(y) D_x \left(L + \lambda E\right)^{-1}\right\|_{2\to 2} < \infty, \quad \left\|c(y) \left(L + \lambda E\right)^{-1}\right\|_{2\to 2} < \infty.$$

Using the last inequalities we have

$$\left\|-k(y)u_{xx} - u_{yy}\right\|_2 = \left\|\left(L + \lambda E\right) u - a(y)u_{xx} - c(y)u - \lambda u\right\|_2 \leqslant$$

$$\leqslant \left\|\left(L + \lambda E\right) u\right\|_2 + \left\|a(y)u_{xx}\right\|_2 + \left\|c(y)u\right\|_2 + \left\|\lambda u\right\|_2 \leqslant$$

$$\leqslant \left\|\left(L + \lambda E\right) u\right\|_2 + \left\|a(y) D_x \left(L + \lambda E\right)^{-1} \left(L + \lambda E\right) u\right\|_2 +$$

$$+ \left\|c(y) \left(L + \lambda E\right)^{-1} \left(L + \lambda E\right) u\right\|_2 + \left\|\lambda \left(L + \lambda E\right)^{-1} \left(L + \lambda E\right) u\right\|_2 \leqslant$$

$$\leq c\left\|\left(L + \lambda E\right) u\right\|_2,$$

where $c > 0$ is a constant.

This implyies that $\left\|-k(y)u_{xx} - u_{yy}\right\|_2 \leqslant c\left\|\left(L + \lambda E\right) u\right\|_2$ for any $u \in D(L)$. And it means that $\gamma < \infty$ which is required to be proved. The theorem is completely proved.

2.2 Two-sides estimates of the distribution function of s-values of a class of mixed type differential operators

Introduction and formulation of the basic results

The asymptotic distribution of eigenvalues for elliptic operators in case of an unbounded domain with the coefficients, increasing at infinity, is completely enough investigated in [1]. At the same time only few works are devoted to these questions for operators of hyperbolic and mixed type.

It is obvious that the smoothness degrees of the coefficients in the domain of definition of a mixed type operator, generally, not everywhere correspond to the degree of the operator [2-4].

In this paper the following questions are considered for a class of operators of mixed type in unbounded domains with increasing coefficients:

1) the existence of the resolvent $(L+\lambda E)^{-1}$ for $\lambda > 0$;

2) the compactness of the resolvent;

3) the smoothness of the solution of the equation $Lu = f$;

4) the distribution function of s - values of the operator L^{-1}.

Consider the differential operator of mixed type

$$L_0 u = -k(y)u_{xx} - u_{yy} + a(y)u_x + c(y)u, \tag{2.2.1}$$

in $C^{\infty}_{0,\pi}(\Omega)$, i.e the set, consisting of infinitely differentiable functions, satisfying the conditions: $u(-\pi, y) = u(\pi, y)$, $u_x(-\pi, y) = u_x(\pi, y)$ and being finite as a function of the y variable, $k(y)$ is a sectionally continuous and bounded function in $\mathbb{R}$ and $k(0) = 0$, $yk(y) > 0$, for $y \neq 0$, where

$$\Omega = \{(x, y) : -\pi < x < \pi, -\infty < y < \infty\},$$

It is easy to show, that the operator L_0 allows the closure in the metric of $L_2(\Omega)$ and the closure is also denoted by L.

Further, we assume, that the coefficients $a(y)$, $c(y)$ satisfies the conditions:

i) $|a(y)| \geq \delta_0 > 0$, $c(y) \geq \delta > 0$ being continuous functions in $\mathbb{R}$;

ii) $\mu_1 = \sup\limits_{|y-t|\leq 1} \dfrac{a(y)}{a(t)} < \infty$, $\mu_2 = \sup\limits_{|y-t|\leq 1} \dfrac{c(y)}{c(t)} < \infty$;

iii) $c(y) \leq c_0 a^2(y)$ for $y \in \mathbb{R}$, c_0 is some constant.

The following theorems hold:

Theorem 2.2.1. *Let the conditions i) be fulfilled. Then the operator $L + \lambda E$ is continuously invertible for substantially large $\lambda > 0$.*

Theorem 2.2.2. *Let the conditions i) be fulfilled. Then the resolvent of the operator L is compact if and only if for any $w > 0$*

$$\lim_{|y|\to\infty} \int\limits_{y}^{y+w} (c(t))\, dt = \infty, \tag{2.2.2}$$

Definition 2.2.1. *The operator L is called separable, if for any functions $u \in D(L)$ the estimate*

$$|| - k(y)u_{xx} - u_{yy}||_2 + ||a(y)u_x||_2 + ||c(y)u||_2 \leq C\left(||Lu||_2 + ||u||_2\right)$$

holds, where the constant C does not depend on $u(x,y)$ and $\|\cdot\|_2$ is the norm in $L_2(\Omega)$.

Theorem 2.2.3. *Let the conditions i)-iii) be fulfilled. Then the operator L is separable.*

Definition 2.2.2. *Let A be a completely continuous operator. Then the eigenvalues of the operator $(A^*A)^{1/2}$ are called s - values of the operator A (Schmidt eigenvalues).*

The nonzero s - values of the operator L^{-1} are arranged as a sequence according to decreasing magnitude and observing their multiplicities, so $s_k(L^{-1}) =$

$\lambda_k((L^{-1})^* L^{-1}), \quad k = 1, 2, \ldots.$

We introduce the counting function $N(\lambda) = \sum\limits_{s_k > \lambda} 1$ of those s_k greater than $\lambda > 0$.

T h e o r e m 2.2.4. *Let the conditions i)-iii) be fulfilled. Then the estimate*

$$C^{-1} \sum_{n=-\infty}^{\infty} \lambda^{-1/2} mes \left(y \in R : \left| n^2 + ina(y) + c(y) \right| \leq C^{-1} \lambda^{-1/2} \right) \leq N(\lambda) \leq$$

$$\leq C \sum_{n=-\infty}^{\infty} \lambda^{-1} mes \left(y \in R : |ina(y) + c(y)| \leq C \lambda^{-1} \right),$$

where the constant $C = C(\mu_1, \mu_2)$, $i^2 = -1$ holds.

Existence and compactness of the resolvent of a class of non-semibounded differential operators l_n

Consider the operator

$$(l_{n,j} + \lambda E) = -u'' + (n^2 k(y) + ina(y) + c(y) + \lambda) u, \quad (n = 0, \pm 1, \pm 2, \ldots)$$

determined on a set of functions u, satisfying the requirements

$$u \in C_0^2(\overline{\triangle}_j), \quad u(\triangle_j^-) = u(\triangle_j^+) = 0.$$

Here $\triangle_j^-$ and $\triangle_j^+$ are the right and left ends of the intervals

$$\triangle_j = (j - 1, j + 1).$$

L e m m a 2.2.1. *Let the condition i) be fulfilled. Then there exists the continuous inverse operator $(l_{n,j} + \lambda E)^{-1}$ for $\lambda > 0$ determined in $L_2(\triangle_j)$, where $(l_{n,j} + \lambda E)^{-1}$ is the inverse operator of the closure operator $l_{n,j} + \lambda E$.*

P r o o f. Integrating by parts $< (l_{n,j} + \lambda E)u, u >$ we have for all $u \in D(l_{n,j}) = D(l_{n,j} + \lambda E)$

$$\|(l_{n,j} + \lambda E)u\|_2 \geq c \|u\|_2, \quad c > 0.$$

If we now show, that the set $(l_{n,j}+\lambda E)D(l_{n,j})$ is dense in L_2 hence, it follows that the operator $(l_{n,j}+\lambda E)$ has an inverse operator $(l_{n,j}+\lambda E)^{-1}$. We prove this using the method by contradiction.

Assume, that the set $(l_{n,j}+\lambda E)D(l_{n,j})$ is not dense in $L_2(\triangle_j)$.
Then there exists an element $v \in L_2 (v \neq 0)$ such that $<(l_{n,j}+\lambda E)u, v>=0$ for all $u \in D(l_{n,j})$. This proves that

$$((l_{n,j})^* + \lambda E)\, v = -v'' + (n^2 k(y) - ina(y) + c(y) + \lambda) v = 0$$

in the sense of the theory of distribution.

As the functions $a(y)$, $c(y)$ are bounded and continuous in the segment $\triangle_j$, then the functions $\left(n^2 k(y) - ina(y) + c(y) + \lambda\right) v \in L_2(\triangle_j)$ and hence $v'' \in L_2(\triangle_j)$.

For completing the proof it is enough to be convinced that the element v $(((l_{n,j})^* + \lambda E)\, v = 0)$ belongs to $D(l_{n,j})$, i.e.

$$v(\triangle_j) = v(\triangle_j^+) = 0$$

We can be convinced in it by integrating by parts:

$$0 =< u, ((l_{n,j})^* + \lambda E)\, v >= \int\limits_{\triangle_j} u \left[-\overline{v}'' + \overline{(n^2 k(y) - ina(y) + c(y) + \lambda)\, v} \right] dy =$$

$$= -\int\limits_{\triangle_j} u \overline{v}''\, dy + \int\limits_{\triangle_j} \left(n^2 k(y) + ina(y) + c(y) + \lambda\right) u \overline{v}\, dy =$$

$$= -\int\limits_{\triangle_j} u d\overline{v}' + \int\limits_{\triangle_j} \left(n^2 k(y) + ina(y) + c(y) + \lambda\right) u \overline{v}\, dy =$$

$$= -u\overline{v}'\Big|_{\triangle_j}^{\triangle_j^+} + \int\limits_{\triangle_j} \overline{v}' du + \int\limits_{\triangle_j} \left(n^2 k(y) + ina(y) + c(y) + \lambda\right) u \overline{v}\, dy =$$

$$= \int\limits_{\triangle_j} \overline{v}' du + \int\limits_{\triangle_j} \left(n^2 k(y) + ina(y) + c(y) + \lambda\right) u \overline{v}\, dy =$$

$$= \int\limits_{\triangle_j} u^{'}\, \overline{v}^{\,'} dy + \int\limits_{\triangle_j} \left(n^2k(y) + ina(y) + c(y) + \lambda\right) u\overline{v}\; dy$$

We use here, that $u \in D(l_n)$

Further,

$$0 =< u, ((l_{n,j})^* + \lambda E)\, v >= \int\limits_{\triangle_j} u^{'}\; d\overline{v} + \int\limits_{\triangle_j} \left(n^2k(y) + ina(y) + c(y) + \lambda\right) u\overline{v}\; dy =$$

$$= u^{'}\overline{v}\Big|_{\triangle_j}^{\triangle_j^+} - \int\limits_{\triangle_j} u^{''}\overline{v}\; dy + \int\limits_{\triangle_j} \left(n^2k(y) + ina(y) + c(y) + \lambda\right) u\overline{v}\; dy =$$

$$= u^{'}\overline{v}\Big|_{\triangle_j}^{\triangle_j^+} + \int\limits_{\triangle_j} \left[-u^{''} + \left(n^2k(y) + ina(y) + c(y) + \lambda\right) u\right] \overline{v}\; dy =$$

$$= u^{'}\overline{v}\Big|_{\triangle_j}^{\triangle_j^+} + < (l_n + \lambda)u, v > .$$

By assumption $< (l_{n,j} + \lambda E)u, v >= 0$, and therefore $u^{'}\overline{v}\Big|_{\triangle_j}^{\triangle_j^+} = 0$. Hence, by virtue of arbitrariness of the function u it follows that

$$\overline{v}(\triangle_j) = \overline{v}(\triangle_j^+) = 0$$

Thus, we finally have that

$$v^{''} \in L_2\left(\triangle_j\right), \quad v(\triangle_j) = v(\triangle_j^+) = 0.$$

It remains to prove that the inequality

$$\|\left((l_n)^* + \lambda E\right) v\|_2 \geq |n|\; \delta\; \|v\|_2, \qquad n = \pm 1, \pm 2, \pm 3, \ldots . \tag{2.2.3}$$

holds. Integrating by parts the scalar product and taking into account that the off-integral terms vanish by virtue of the boundary conditions just having been given, we find

$$|< ((l_n)^* + \lambda E)\, v, v >| = \left| \int\limits_{\triangle_j} \left[-v^{''} + \left(n^2k(y) - ina(y) + c(y) + \lambda\right) v\right] \overline{v}\; dy \right| =$$

$$= \left| \int\limits_{\triangle_j} \left[|v^{'}|^2 + \left(n^2 k(y) - ina(y) + c(y) + \lambda \right) |v|^2 \right] dy \right| \geq \left| \int\limits_{\triangle_j} - ina(y) |v|^2 dy \right| .$$

Now, using the Cauchy-Bunyakovskii inequality, we find:

$$\| \left((l_n)^* + \lambda E \right) v \|_2 \geq |n| \; \delta \; \|v\|_2 .$$

By virtue of $\left((l_n)^* + \lambda E \right) v = 0$ it follows from the inequality (2.2.3), that $v = 0$.

Lemma 2.2.1 is completely proved .

Lemma 2.2.2. *Let the condition i) be fulfilled. Then the inequalities*

$$\left\| (l_{n,j} + \lambda E)^{-1} \right\|_{2 \to 2} \leq \frac{c}{\lambda^{1/2}} ; \tag{2.2.4}$$

$$\left\| \frac{d}{dy} (l_{n,j} + \lambda E)^{-1} \right\|_{2 \to 2} \leq \frac{c}{\lambda^{1/4}} ; \tag{2.2.5}$$

hold, where $c > 0$ is a constant, $\lambda > 0$.

Proof. Consider the scalar product $< (l_{n,j} + \lambda E) u, u >$

$$< (l_{n,j} + \lambda E) u, u > = \int\limits_{\triangle_j} \left[-u^{''} + (n^2 k(y) + ina(y) + c(y) + \lambda) u \right] \overline{u} \; dy ,$$

where $\quad u \in D(l_{n,j})$.

Integrating by parts the last term and using the finiteness of $u(y)$, we find that

$$|< (l_{n,j} + \lambda E) u, u >| = \left| \int\limits_{\triangle_j} |u^{'}|^2 dy + \int\limits_{\triangle_j} \left(n^2 k(y) + ina(y) + c(y) + \lambda \right) |u|^2 \; dy \right| . \tag{2.2.6}$$

Hence, taking into account that $a(y)$ does not changes the sign, we have

$$\| (l_{n,j} + \lambda E) u \|_2^2 \geq n^2 \left[\min_{y \in \triangle_j} |a(y)| \right]^2 ||u||_2^2 \tag{2.2.7}$$

Using the Cauchy inequality we find from (2.2.6)

$$\frac{1}{2\varepsilon} \| (l_{n,j} + \lambda E) u \|_2^2 + \frac{\varepsilon}{2} ||u||_2^2 \geq \int\limits_{\triangle_j} \left[|u^{'}|^2 + (c(y) + \lambda) |u|^2 \right] dy - \int\limits_{\triangle_j} n^2 |k(y)| |u|^2 dy$$

or

$$\frac{1}{2\varepsilon}\left\|(l_{n,j}+\lambda E)u\right\|_2^2 \geq \int\limits_{\triangle_j}\left[|u^{'}|^2+\left(c(y)+\lambda-\frac{\varepsilon}{2}\right)|u|^2\right]dy-\int\limits_{\triangle_j}n^2|k(y)||u|^2dy.$$

Taking $i)$ into account, from the last inequality we find

$$\frac{1}{2\varepsilon}\left\|(l_{n,j}+\lambda E)u\right\|_2^2 \geq \frac{1}{2}\int\limits_{\triangle_j}\left[|u^{'}|^2+\left(c(y)+\lambda\right)|u|^2\right]dy-\int\limits_{\triangle_j}n^2|k(y)||u|^2dy. \tag{2.2.8}$$

Combining the inequalities (2.2.7) and (2.2.8), we find

$$c(\varepsilon)\left\|(l_{n,j}+\lambda E)u\right\|_2^2 \geq \lambda\left\|u\right\|_2^2. \tag{2.2.9}$$

The inequality (2.2.4) is proved.

By virtue of (2.2.9) it follows from (2.2.6) that

$$\frac{c}{\sqrt{\lambda}}\left\|(l_{n,j}+\lambda E)u\right\|_2^2 \geq \int\limits_{\triangle_j}\left[|u^{'}|^2+\left(c(y)+\lambda\right)|u|^2\right]dy-\int\limits_{\triangle_j}n^2|k(y)||u|^2dy. \tag{2.2.10}$$

Further, multiplying both parts of the inequality (2.2.7) by $\frac{c}{\sqrt{\lambda}}$ and then combining it with (2.2.10), we find

$$\frac{c}{\sqrt{\lambda}}\left\|(l_{n,j}+\lambda E)u\right\|_2^2 \geq ||u^{'}||_2^2,$$

where $c>0$ is a constant.

The last inequality proves Lemma 2.2.2.

Lemma 2.2.3. *Let the condition i) be fulfilled. Then*

a) $\left\|l_nu\right\|_2 \geq |n|\delta_0\left\|u\right\|_2$, $u\in D(l_n)$, $n=\pm1,\pm2,\pm3,\ldots$, $\left\|l_nu\right\|_2\geq\delta\left\|u\right\|_2$ *for* $n=0$;

b) $c\left\|l_nu\right\|_2 \geq \left(\|u^{'}\|_2+\|\sqrt{c(y)}u\|_2+\|\sqrt{|n||a(y)|}u\|_2\right)$, $u\in D(l_n)$, $n=\pm1,\pm2,\pm3,\ldots$,

where $c>0$ *is a constant not depending on* u *and* n.

P r o o f . We have for any $u \in C_0^\infty(R)$

$$|< l_n u, inu >| = \left| \int\limits_{-\infty}^{\infty} l_n u \, \overline{inu} \, dy \right| =$$

$$= \left| \int\limits_{-\infty}^{\infty} \left(-u^{''} + (n^2 k(y) + ina(y) + c(y))u \right) \, \overline{inu} \, dy \right| =$$

$$= \left| \int\limits_{-\infty}^{\infty} \left(-u^{''}(-i)n \, \overline{u} \; + \left(n^2 k(y)(-i)n + i(-i)n^2 a(y) + (-i)nc(y) \right) |u|^2 \right) \; dy \right| =$$

$$= \left| -in \int\limits_{-\infty}^{\infty} \left[|u^{'}|^2 + (n^2 k(y) + c(y))|u|^2 \right] \; dy + \int\limits_{-\infty}^{\infty} n^2 a(y)|u|^2 \; dy \right| .$$

Hence, using the Cauchy-Bunyakovskii inequality, we find:

$$\|l_n u\| \, \|inu\| \geq |< l_n u, inu >| \geq \int\limits_{-\infty}^{\infty} n^2 |a(y)| \; |u|^2 \; dy$$

$$\|l_n u\| \, n \, \|u\| \geq n^2 \int\limits_{-\infty}^{\infty} \delta_0 |u|^2 \; dy \geq n^2 \delta_0 ||u||_2^2.$$

From the last inequality, we have

$$\|l_n u\|_2^2 \geq n^2 \delta_0^2 \, \|u\|_2^2$$

The item a) of Lemma 2.2.3 is proved.

Further, consider the quadratic form

$$< l_n u, u >= \int\limits_{-\infty}^{\infty} \left[(-u^{''} + \left(n^2 k(y) + ina(y) + c(y) \right) u \right] \; \overline{u} \, dy,$$

where $u \in C_0^\infty(R)$.

Integrating the last equality, we find

$$|< l_n u, u >| = \left| \int\limits_{-\infty}^{\infty} \left[|u^{'}|^2 + (n^2 k(y) + ina(y) + c(y)) \right] |u|^2 \; dy \right| .$$

Using a property of complex numbers we have

$$\|l_n u\| \cdot \|u\| \geq \left| \int\limits_{-\infty}^{\infty} \left[|u^{'}|^2 + (n^2 k(y) + c(y)) \right] |u|^2 \, dy \right|.$$

From this, using the Cauchy inequality with $\varepsilon > 0$, we find that

$$\frac{1}{2\varepsilon} \|l_n u\|_2^2 + \frac{\varepsilon}{2} \|u\|_2^2 \geq \int\limits_{-\infty}^{\infty} \left[|u^{'}|^2 + c(y)|u|^2 \right] \, dy - \int\limits_{-\infty}^{\infty} n^2 |k(y)||u|^2 dy. \quad (2.2.11)$$

On the basis of the condition i), from (2.2.11) we will have

$$\frac{1}{2\varepsilon} \|l_n u\|_2^2 \geq \frac{1}{2} \int\limits_{-\infty}^{\infty} \left[|u^{'}|^2 + c(y)|u|^2 \right] \, dy - n^2 \int\limits_{-\infty}^{\infty} |k(y)||u|^2 dy. \quad (2.2.12)$$

Combining the inequalities a) and (2.2.12), we find the inequality

$$c \, \|l_n u\|_2 \geq \left(\|u^{'}\|_2 + \|\sqrt{c(y)} u\|_2 \right). \quad (2.2.13)$$

Further, consider the scalar multiplicity for any $u \in C_0^\infty(R)$

$$< l_n u, u >= \int\limits_{-\infty}^{\infty} \left[(-u^{''} + \left(n^2 k(y) + ina(y) + c(y) \right) u \right] \, \overline{u} dy.$$

Integrating by parts the last equality and taking into account that $u \in C_0^\infty(R)$, we find

$$|< l_n u, u >| \geq \left| \int\limits_{-\infty}^{\infty} ina(y)|u|^2 \, dy \right|.$$

From this and taking condition i) into account, we find

$$\frac{1}{2\varepsilon} \|l_n u\|_2^2 + \frac{\varepsilon}{2} \|u\|_2^2 \geq \frac{1}{2} \int\limits_{-\infty}^{\infty} |n||a(y)||u|^2 \, dy + \frac{1}{2} \int\limits_{-\infty}^{\infty} |n||a(y)||u|^2 \, dy$$

or

$$c_1(\varepsilon) \, \|l_n u\|_2 \geq \|\sqrt{|n||a(y)|} u\|_2.$$

This gives us along with (2.2.13)

$$c(\varepsilon) \, \|l_n u\|_2 \geq \left(\|u^{'}\|_2 + \|\sqrt{c(y)} \, u\|_2 + \|\sqrt{|n||a(y)|} \, u\|_2 \right).$$

Lemma 2.2.3 is completely proved.

We take the set $\{\varphi_j\}$ of non-negative functions from $C_0^\infty(R)$ such that

$$\sum_j \varphi_j^2 \equiv 1, \quad supp\ \ \varphi_j \in \triangle_j, \quad \underset{j}{\cup} \triangle_j \equiv \mathbb{R}$$

(sums are taken for all the integers j without indicating the limits).

Let K denote the operator, defined by the equality

$$Kf = \sum_j \varphi_j \left(l_{n,j} + \lambda E\right)^{-1} \varphi_j f, \quad f \in L_2(\mathbb{R}).$$

L e m m a 2.2.4. *Let the condition i) be fulfilled. Then the equality*

$$(l_n + \lambda E)Kf = f - B\lambda f \tag{2.2.14}$$

holds for any $f \in C_0^\infty(R)$, *where* $B_\lambda f = Kf = \sum_j \varphi_j^{''} \left(l_{n,j} + \lambda E\right)^{-1} \varphi_j f + 2\sum_j \varphi_j^{'} \frac{d}{dy} \left(l_{n,j} + \lambda E\right)^{-1} \varphi_j f$

P r o o f. Let $f \in C_0^\infty(\mathbb{R})$ and consider the influence of the operator K on f:

$$Kf = \sum_j \varphi_j \left(l_{n,j} + \lambda E\right)^{-1} \varphi_j f, \quad f \in L_2(\mathbb{R}). \tag{2.2.15}$$

Because $f \in C_0^\infty(\mathbb{R})$ the sum (2.2.15) is finite. Therefore, the following calculations are valid:

$$(l_n + \lambda E)\, Kf = (l_n + \lambda E) \sum_j \varphi_j \left(l_{n,j} + \lambda E\right)^{-1} \varphi_j f =$$

$$= \left(-\frac{d^2}{dy^2} + \left(n^2 k(y) + ina(y) + c(y)\right)\right) \sum_j \varphi_j \left(l_{n,j} + \lambda E\right)^{-1} \varphi_j f =$$

$$= -\frac{d^2}{dy^2} \sum_j \varphi_j \left(l_{n,j} + \lambda E\right)^{-1} \varphi_j f + \left(n^2 k(y) + ina(y) + c(y)\right) \times$$

$$\times \sum_j \varphi_j \left(l_{n,j} + \lambda E\right)^{-1} \varphi_j f = -\sum_j \varphi_j^{''} \left(l_{n,j} + \lambda E\right)^{-1} \varphi_j f -$$

$$-2\sum_j \varphi_j' \left[(l_{n,j}+\lambda E)^{-1}\varphi_j f\right]' - \sum_j \varphi_j \frac{d^2}{dy^2}(l_{n,j}+\lambda E)^{-1}\varphi_j f +$$

$$+(n^2k(y))\sum_j \varphi_j (l_{n,j}+\lambda E)^{-1}\varphi_j f + ina(y)\sum_j \varphi_j (l_{n,j}+\lambda E)^{-1}\varphi_j f +$$

$$+c(y)\sum_j \varphi_j (l_{n,j}+\lambda E)^{-1}\varphi_j f = -\sum_j \varphi_j \frac{d^2}{dy^2}(l_{n,j}+\lambda E)^{-1}\varphi_j f +$$

$$+\sum_j \varphi_j (n^2k(y))(l_{n,j}+\lambda E)^{-1}\varphi_j f + \sum_j ina(y)\varphi_j (l_{n,j}+\lambda E)^{-1}\varphi_j f +$$

$$+\sum_j c(y)\varphi_j (l_{n,j}+\lambda E)^{-1}\varphi_j f - \sum_j \varphi_j'' (l_{n,j}+\lambda E)^{-1}\varphi_j f -$$

$$-2\sum_j \varphi_j' \left[(l_{n,j}+\lambda E)^{-1}\varphi_j f\right]' = \sum_j \varphi_j \left(-\frac{d^2}{dy^2} + \left(n^2k(y)+ina(y)+c(y)\right)\right) \times$$

$$\times (l_{n,j}+\lambda E)^{-1}\varphi_j f - \sum_j \varphi_j'' (l_{n,j}+\lambda E)^{-1}\varphi_j f - 2\sum_j \varphi_j' \left[(l_{n,j}+\lambda E)^{-1}\varphi_j f\right]' =$$

$$= f - \sum_j \varphi_j'' (l_{n,j}+\lambda E)^{-1}\varphi_j f - 2\sum_j \varphi_j' \frac{d}{dy}(l_{n,j}+\lambda E)^{-1}\varphi_j f.$$

Here, we take into account that $\sum\limits_j \varphi_j^2 \equiv 1$. Lemma 2.2.4 is proved.

Lemma 2.2.5. *Let the condition i) be fulfilled. Then some $\lambda > 0$ is found such that $\|B_\lambda| < 1$.*

Proof. We estimate the norm of the operator B_λ:

$$\|B_\lambda f\|_2^2 = \|\sum_j \varphi_j'' (l_{n,j}+\lambda E)^{-1}\varphi_j f + 2\sum_j \varphi_j' \frac{d}{dy}(l_{n,j}+\lambda E)^{-1}\varphi_j f\|_2^2 =$$

$$= \int_{-\infty}^{\infty} \left|\sum_j \varphi_j'' (l_{n,j}+\lambda E)^{-1}\varphi_j f + 2\sum_j \varphi_j' \frac{d}{dy}(l_{n,j}+\lambda E)^{-1}\varphi_j f\right|^2 dy =$$

$$= \sum_j \int_{j-1}^{j+1} \left|\sum_{j-1}^{j+1} \varphi_j'' (l_{n,j}+\lambda E)^{-1}\varphi_j f + 2\varphi_j' \frac{d}{dy}(l_{n,j}+\lambda E)^{-1}\varphi_j f\right|^2 dy.$$

Here, we have used that in $\overline{\triangle}_j = [j-1, j+1]$ only φ_{j-1}, φ_j, $\varphi_{j+1} \neq 0$. From this and by virtue of the Holder inequality we find that

$$\|B_\lambda f\|_2^2 \leq 24c_0 \sum_j \left(\| \left(l_{n,j} + \lambda E\right)^{-1} \varphi_j f\|_2^2 + \| \frac{d}{dy} \left(l_{n,j} + \lambda E\right)^{-1} \varphi_j f\|_2^2 \right),$$

where $c_0 = max\{|\varphi_j''|, \varphi_j'\}$

From the last inequality, using Lemma 2.2.2, we have

$$\|B_\lambda f\|_2^2 \leq 24c_0 \sum_j \left(\| \left(l_{n,j} + \lambda E\right)^{-1} \|_2^2 \|\varphi_j f\|_2^2 + \| \frac{d}{dy} \left(l_{n,j} + \lambda E\right)^{-1} \|_2^2 \|\varphi_j f\|_2^2 \right) \leq$$

$$\leq 24c_0 \left(\sum_j \left(\frac{c}{\lambda^{1/2}} \|\varphi_j f\|_2^2 + \frac{c}{\lambda^{1/4}} \|\varphi_j f\|_2^2 \right) \right) \leq$$

$$\leq 24c_0 \left(\frac{c}{\lambda^{1/2}} \sum_j \int_{-\infty}^{\infty} |\varphi_j f|^2 dy + \frac{c}{\lambda^{1/4}} \sum_j \int_{-\infty}^{\infty} |\varphi_j f|^2 dy \right) \leq$$

$$\leq 24c_0 c \left(\frac{1}{\lambda^{1/2}} \int_{-\infty}^{\infty} \sum_j |\varphi_j f|^2 dy + \frac{1}{\lambda^{1/4}} \int_{-\infty}^{\infty} \sum_j |\varphi_j f|^2 dy \right) \leq$$

$$\leq 24c_0 c \left(\frac{1}{\lambda^{1/2}} \int_{-\infty}^{\infty} \left(\sum_j \varphi_j^2 \right) |f|^2 dy + \frac{1}{\lambda^{1/4}} \int_{-\infty}^{\infty} \left(\sum_j \varphi_j^2 \right) |f|^2 dy \right) \leq$$

$$\leq 24c_0 c \left(\frac{1}{\lambda^{1/2}} \|f\|_2^2 + \frac{1}{\lambda^{1/4}} \|f\|_2^2 \right) \|B_\lambda f\|_2^2 \leq$$

$$\leq 24c_0 c \left(\frac{1}{\lambda^{1/2}} + \frac{1}{\lambda^{1/4}} \right) \|f\|_2^2. \tag{2.2.16}$$

The last inequality proves the lemma for substantially large positive λ.

Theorem 2.2.5. *Let the condition i) be fulfilled. Then the operator $(l_n + \lambda E)$ is continuously invertible for substantially large $\lambda > 0$ and the equality*

$$(l_n + \lambda E)^{-1} = K(E - B\lambda)^{-1} \tag{2.2.17}$$

holds.

P r o o f . The operator $(E-B\lambda)$ is bounded along with its inverse operator. Therefore the set $M = \{\varphi = (E - B\lambda)f \,:\, f \in C_0^\infty(R)\}$ is dense in $L_2(R)$. From the inequality (2.2.13) we find for $\varphi = (E - B\lambda)f,\ f \in C_0^\infty(R)$ that $K(E - B\lambda)^{-1}\varphi \in D(l_n)$ and $(l_n + \lambda E)K(E - B\lambda)^{-1}\varphi = \varphi$. Hence, we have that $y = K(E - B\lambda)^{-1}f$ is a solution of the equation $(l_n + \lambda E)y = f$. The uniqueness follows from Lemma 2.2.3 Thus Theorem 2.2.5 is proved.

T h e o r e m 2.2.6. *Let the condition i) be fulfilled. Then the operator $(l_n)^{-1}$ is completely continuous if and only if for any $w > 0$*

$$\lim_{|y|\to\infty} \int\limits_{y}^{y+w} c(t)dt = \infty, \tag{2.2.18}$$

P r o o f . *N e c e s s i t y.* Let the condition of the lemma do not hold. Then there exists a sequence of intervals $Q_d(y_j) \subset \mathbb{R}$ such that

$$\sup \int\limits_{Q_d(y_j)} c(t)dt < 0, \qquad \text{where} \quad d > 0 \tag{2.2.19}$$

i.e. when the intervals $Q_d(y_j)$, preserving the length, diverge to infinity.

Let $w(x) \in C_0^\infty(Q(0))$ and consider the set of functions such that $u_j(y) = w(y - y_j)$. It is not difficult to establish the inequality

$$\| - u_j'' + (n^2k(y) + ina(y) + c(y))u_j\|_2^2 \leq c$$

by virtue of (2.2.19), where c does not depend on j.

Assume

$$F_j(y) = -u_j'' + (n^2k(y) + ina(y) + c(y))u_j,$$

$$\sup F_j(y) \subseteq Q_d(y_j)$$

Now we show that $F_j(y)$ weakly converge to zero:

$$|< F_j(y), v(y) >| = \left|\int\limits_{-\infty}^{\infty} F_j(y)v(y)dy\right| = \left|\int\limits_{Q_d(y_j)} F_j(y)v(y)dy\right| \leq$$

$$\leq \left(\int_{Q_d(y_j)} F_j^2(y)dy \right)^{1/2} \left(\int_{Q_d(y_j)} v^2(y)dy \right)^{1/2} \tag{2.2.20}$$

So $v \in L_2(\mathbb{R})$, it is obvious that $\int\limits_{Q_d(y_j)} v^2(y)dy \to 0$ as $j \to \infty$. Hence and from (2.2.20) it follows that the sequence $\{F_j\} \to 0$ weakly.

It is directly seen that

$$\|u_j(y)\|_2 = c > 0. \tag{2.2.21}$$

For this reason, if the operator $(l_n)^{-1}$ is compact, then $\{u_j\}$ must converge to zero in the L_2 norm. But this is impossible by virtue of (2.2.21). The necessity is proved.

Sufficiency. Let $L_2^1(\mathbb{R}, c(y))$ denote the space obtained by supplementing the norm

$$\|u : L_2^1(\mathbb{R}, c(y))\| = \left(\int_R \left[|u^{'}|^2 + c(y)|u|^2 \right] dy \right)^{1/2}.$$

From Lemma 2.2.3 it follows that

$$R(l_n^{-1}) \subset L_2^1(\mathbb{R}, c(y)).$$

By virtue of results of [1], any bounded set in $L_2^1(\mathbb{R}, c(y), \alpha a(y))$ is compact in $L_2(\mathbb{R})$ if and only if the condition

$$Q^*(y) \to \infty \text{ as } |y| \to \infty, \tag{2.2.22}$$

is fulfilled, where $Q^*(y) = \inf\limits_{d \to 0} \left\{ d^{-1} \geq \int\limits_{y-\frac{d}{2}}^{y+\frac{d}{2}} c(t)dt \right\}.$

From this it follows that it is enough to show the equivalency of the conditions (2.2.22) and (2.2.18).

Let (2.2.22) be not fulfilled. Then there exists a sequence of points y_n, n =0,1,2,..., and a constant $c > 0$ such that $Q^*(y) \leq c_1$. By virtue of the equality

$$d_n^{-1} = \int\limits_{x_n - \frac{d_n}{2}}^{x_n + \frac{d_n}{2}} c(t)dt,$$

which follows from the definition of $Q^*(y)$, we obtain that there exists intervals $\triangle_n$, diverging to infinity, preserving the length such that

$$\int\limits_{\triangle_n} c(t)dt \leq c_1.$$

The last inequality shows that the condition (2.2.18) is not fulfilled.

On the other hand if condition (2.2.18) is not fulfilled, then there exist pairwise disjoint intervals $\triangle_n$ of same length, diverging to infinity. From the definition of $Q^*(y)$, we obtain, that $Q^*(y_n) \leq c_1$, where y_n is the center of $\triangle_n$. This means that (2.2.22) is not fulfilled. From this it follows that (2.2.22) and (2.2.18) are equivalent. The sufficiency of Theorem 2.2.6 is proved.

Weighted estimates for the non-semibounded operator l_n

Lemma 2.2.6. *Let the conditions $i) - iii)$ be fulfilled. Then the inequalities*

$$\|(l_{n,j} + \lambda E)^{-1}\|_{2\to 2} \leq \frac{1}{|n||a(\widetilde{y}_j)|}, \quad n = \pm 1, \pm 2, ... \tag{2.2.23}$$

$$\|(l_{n,j} + \lambda E)^{-1}\|_{2\to 2} \leq \frac{2}{c(y_j) + \lambda}, \tag{2.2.24}$$

hold for any $j \in \mathbb{N}$, where $c(y_j) = \min\limits_{y\in\overline{\triangle}_j} c(y)$, $|a(\widetilde{y}_j)| = \min\limits_{y\in\overline{\triangle}_j} |a(y_j)|$.

Proof. For any $u \in C_0^\infty(\triangle_j)$ we have

$$< (l_{n,j}+\lambda E)u, u >= \left| \int\limits_{\triangle_j} \left[|u'|^2 + (n^2k(y) + c(y) + \lambda)|u|^2 \right] dy + in \int\limits_{\triangle_j} a(y)|u|^2 dy \right|. \tag{2.2.25}$$

Hence, taking the condition $i)$ into account and using the Cauchy-Bunyakovskii inequality, we find

$$\|(l_{n,j}+\lambda E)u\|_2\|u\|_2 \geq \left| in\int\limits_{\triangle_j} a(y)|u|^2dy \right| \geq |in| \min_{y\in\overline{\triangle}_j} |a(y)|\|u\|_2^2 \geq |n||a(\widetilde{y}_j)|\|u\|_2^2.$$

Therefore

$$\|(l_{n,j}+\lambda E)u\|_2^2 \geq \left[|n|\cdot|a(\widetilde{y}_j)|\right]^2 \|u\|_2^2, \qquad n=\pm1,\pm2,,\dots. \tag{2.2.26}$$

By the definition of the operator norm the following computations hold:

$$\|(l_{n,j}+\lambda E)^{-1}\|_{2\to2} = \sup_{f\in L_2}\frac{\|(l_{n,j}+\lambda E)^{-1}f\|_2}{\|f\|_2} = \sup_{u\in D(l_{n,j})}\frac{\|u\|_2}{\|(l_{n,j}+\lambda E)u\|_2}.$$

Now, using the inequality (2.2.26), we find that

$$\|(l_{n,j}+\lambda E)^{-1}\|_{2\to2} \leq \frac{1}{|n||a(\widetilde{y}_j)|}, \qquad n=\pm1,\pm2,,\dots.$$

The inequality (2.2.23) is proved.

From the inequality (2.2.25), by virtue of the Cauchy inequality with $\varepsilon>0$, we find that

$$\frac{1}{2(c(y_j)+\lambda)}\|(l_{n,j}+\lambda E)u\|_2^2 + \frac{c(y_j)+\lambda}{2}\|u\|_2^2 \geq$$
$$\geq \int\limits_{\triangle_j}\left[|u^{'}|^2+(c(y_j)+\lambda)|u|^2\right]dy - \int\limits_{\triangle_j} n^2|k(y)||u|^2dy,$$

where $\varepsilon = c(y_j)+\lambda$.

$$\frac{1}{2(c(y_j)+\lambda)}\|(l_{n,j}+\lambda E)u\|_2^2 \geq \int\limits_{\triangle_j}|u^{'}|^2dy + \frac{c(y_j)+\lambda}{2}\int\limits_{\triangle_j}|u|^2dy - \int\limits_{\triangle_j} n^2|k(y)||u|^2dy \tag{2.2.27}$$

Dividing both parts of the inequality (2.2.26) by $2(c(y_j)+\lambda)$ gives

$$\frac{1}{2(c(y_j)+\lambda)}\|(l_{n,j}+\lambda E)u\|_2^2 \geq \frac{\left[|n|\cdot|a(\widetilde{y}_j)|\right]^2}{2(c(y_j)+\lambda}\|u\|_2^2 \tag{2.2.28}$$

As a result, combining (2.2.27) and (2.2.28), we get the inequality

$$\frac{1}{c(y_j)+\lambda}\|(l_{n,j}+\lambda E)u\|_2^2 \geq$$

$$\geq \|u^{'}\|_2^2 + \frac{c(y_j)+\lambda}{2}\|u\|_2^2 + n^2\int\limits_{\triangle_j}\left[\frac{a(\widetilde{y}_j)^2}{2(c(y_j)+\lambda)} - |k(y)|\right]|u|^2dy \qquad (2.2.29)$$

If one takes the conditions *ii*)-*iii*) into account, then from the last inequality it is easy to check

$$\frac{1}{c(y_j)+\lambda}\|(l_{n,j}+\lambda E)u\|_2^2 \geq \|u^{'}\|_2^2 + \frac{c(y_j)+\lambda}{2}\|u\|_2^2.$$

Hence,

$$2\|(l_{n,j}+\lambda E)u\|_2^2 \geq (c(y_j)+\lambda^2)\|u\|_2^2 \qquad (2.2.30)$$

By virtue of (2.2.30), the estimate

$$\|(l_{n,j}+\lambda E)^{-1}\|_{2\to 2} \leq \frac{2}{c(y_j)+\lambda}$$

is obvious. The lemma 2.2.6 is proved.

Remark. Let

$$(l_{n,\lambda_0}+\lambda E)u = u^{''} + \left(n^2k(y) + i(a(y)+\lambda_0)n + (c(y)+\lambda)\right)u$$

be defined in $L_2(\mathbb{R})$, where the sign of the real number λ_0 coincides with the sign of $a(y)$.

Consider the equation

$$(l_n+\lambda E)u = u^{''} + (n^2k(y) + ina(y) + (c(y)+\lambda)u = f \in L_2(\mathbb{R})$$

or

$$u^{''} + (n^2k(y) + i(a(y)+\lambda_0)n + (c(y)+\lambda)u - i\lambda_0 nu(y) = f. \qquad (2.2.31)$$

According to Theorem 2.1 the operator $(l_{n,\lambda_0}+\lambda E)$ generates a one-to-one transformation and by Lemma 2.2.3 the estimate

$$\|(l_{n,\lambda_0}+\lambda E)^{-1}\|_{2\to 2} \leq \frac{1}{|n|(\delta+\lambda_0)}$$

holds. From this it follows that the above equation is equivalent to the equation

$$v - i\lambda_0 n(l_{n,\lambda_0} + \lambda E)^{-1} v = f, \tag{2.2.32}$$

where $(l_{n,\lambda_0} + \lambda E)u = v,\ u = (l_{n,\lambda_0} + \lambda E)^{-1} v$

From the last inequality it is clear that

$$\|i\lambda_0 n(l_{n,\lambda_0} + \lambda E)^{-1}\|_{2\to 2} \le \frac{\lambda_0 |n|}{|n|(\delta + \lambda_0)} < 1.$$

According to the conditions *ii)*-*iii)* this inequality shows, that we can guarantee the correctness of the inequality

$$n^2 \left[\frac{c_0 \left(a(\tilde{y}_j) + \lambda_0 \right)^2}{2(c(y_j) + \lambda)} - k(y) \right] \ge 0$$

selecting corresponding $c_0,\ \lambda_0$.

Lemma 2.2.7. *Let the conditions of Lemma 2.2.6. be fulfilled and let $\lambda > 0$ be such that $\|B_\lambda\| < 1$. Then the estimate*

$$\|\rho(y)|n|^\alpha (l_n + \lambda E)^{-1}\|_2^2 \le c(\lambda) \sup_j \|\rho(y)|n|^\alpha \varphi_j (l_{n,j} + \lambda E)^{-1}\|_2^2 \tag{2.2.33}$$

holds, where α=0,1 and $\rho(y)$ is a continuous function in $\mathbb{R}$.

Proof: It is seen from representation (2.2.16), that the operator $\rho(y)|n|^\alpha (l_n + \lambda E)^{-1}$ is bounded (or unbounded) along with the operator $\rho(y)|n|^\alpha K(E - B\lambda)^{-1}$. For this reason the following aim will be a norm estimate of the last operator $\rho(y)|n|^\alpha K(E - B\lambda)^{-1}$. For any $f \in L_2(\mathbb{R})$ we have

$$\|\rho(y)|n|^\alpha (l_n + \lambda E)^{-1} f\|_2^2 = \|\rho(y)|n|^\alpha K(E - B\lambda)^{-1}\|_2^2 =$$

$$= \|\rho(y)|n|^\alpha \sum_j \varphi_j (l_{n,j} + \lambda E)^{-1} \varphi_j (E - B\lambda)^{-1} f\|_2^2 =$$

$$= \int_{-\infty}^{\infty} \left| \sum_j \rho(y)|n|^\alpha \varphi_j (l_{n,j} + \lambda E)^{-1} \varphi_j (E - B\lambda)^{-1} f \right|^2 dy \le$$

$$\leq \sum_j \int\limits_{j-1}^{j+1} \left| \sum_j \rho(y)|n|^{\alpha} \varphi_j (l_{n,j} + \lambda E)^{-1} \varphi_j (E - B\lambda)^{-1} f \right|^2 dy.$$

It is easy to verify that in $\triangle_j = [j-1, j+1]$ only φ_{j-1}, φ_{j+1}, $\varphi_j \neq 0$. Taking this into account, by virtue of the Holder inequality, we have

$$\|\rho(y)|n|^{\alpha}(l_n + \lambda E)^{-1} f\|_2^2 \leq$$

$$\leq \sum_j \int\limits_{j-1}^{j+1} \left| \sum_{j-1}^{j+1} \rho(y)|n|^{\alpha} \varphi_j (l_{n,j} + \lambda E)^{-1} \varphi_j (E - B\lambda)^{-1} f \right|^2 dy \leq$$

$$\leq 3 \sum_j \int\limits_{j-1}^{j+1} \sum_{j-1}^{j+1} \left| \rho(y)|n|^{\alpha} \varphi_j (l_{n,j} + \lambda E)^{-1} \varphi_j (E - B\lambda)^{-1} f \right|^2 dy \leq$$

$$\leq 12c \sum_j \|\rho(y)|n|^{\alpha} \varphi_j (l_{n,j} + \lambda E)^{-1} \varphi_j (E - B\lambda)^{-1} f\|_2^2 \leq$$

$$\leq 12c \sum_j \|\rho(y)|n|^{\alpha} \varphi_j (l_{n,j} + \lambda E)^{-1}\|_2^2 \|\varphi_j (E - B\lambda)^{-1} f\|_2^2 \leq$$

$$\leq 12c \sup_j \|\rho(y)|n|^{\alpha} \varphi_j (l_{n,j} + \lambda E)^{-1}\|_2^2 \sum_j \|\varphi_j (E - B\lambda)^{-1} f\|_2^2 \leq$$

$$\leq 12c \sup_j \|\rho(y)|n|^{\alpha} \varphi_j (l_{n,j} + \lambda E)^{-1}\|_2^2 \sum_j \int\limits_{-\infty}^{\infty} |\varphi_j (E - B\lambda)^{-1} f|^2 dy =$$

$$= 12c \sup_j \|\rho(y)|n|^{\alpha} \varphi_j (l_{n,j} + \lambda E)^{-1}\|_2^2 \int\limits_{-\infty}^{\infty} (\sum_j \varphi_j^2) \left((E - B\lambda)^{-1} f\right)^2 \, dy =$$

$$= 12c \sup_j \|\rho(y)|n|^{\alpha} \varphi_j (l_{n,j} + \lambda E)^{-1}\|_2^2 \|(E - B\lambda)^{-1} f\|_2^2 \leq$$

$$\leq 12c \sup_j \|\rho(y)|n|^{\alpha} \varphi_j (l_{n,j} + \lambda E)^{-1}\|_2^2 \|(E - B\lambda)^{-1}\|_{2\to 2} \|f\|_2^2 \leq$$

$$\leq 12c(\lambda) \sup_j \|\rho(y)|n|^{\alpha} \varphi_j (l_{n,j} + \lambda E)^{-1}\|_2^2 \|f\|_2^2.$$

From here we have

$$\|\rho(y)|n|^{\alpha}(l_n + \lambda E)^{-1}\|_{2\to 2}^2 \leq 12 \cdot c(\lambda) \sup_j \|\rho(y)|n|^{\alpha} \varphi_j (l_{n,j} + \lambda E)^{-1}\|_2^2.$$

Lemma 2.2.7 is proved.

L e m m a 2.2.8. *Let the conditions of Lemma 2.2.7 be fulfilled. Then the following estimates hold:*

a) $\left\|c(y)(l_n+\lambda E)^{-1}\right\|_{2\to 2} \le c_1 < \infty$;

b) $\left\|ina(y)(l_n+\lambda E)^{-1}\right\|_{2\to 2} \le c_2 < \infty$;

c) $\left\|\dfrac{d}{dy}(l_n+\lambda E)^{-1}\right\|_{2\to 2} \le c_3 < \infty$.

P r o o f. By Lemma 2.2.7

$$\left\|c(y)(l_n+\lambda E)^{-1}\right\|_{2\to 2} \le c(\lambda)\sup_j \left\|c(y)\varphi_j(l_n+\lambda E)^{-1}\right\|_{2\to 2}.$$

From here and from lemma 2.2.6 we find that

$$\left\|c(y)(l_n+\lambda E)^{-1}\right\|_{2\to 2} \le c(\lambda)\sup_j \left\|c(y)\varphi_j(l_n+\lambda E)^{-1}\right\|_{2\to 2} \le$$

$$\le c(\lambda)\sup_j \frac{\max\limits_{y\in\overline{\triangle}_j}|c(y)\varphi_j|}{c(y_j)+\lambda} \le c(\lambda)\sup_j \frac{\max\limits_{y\in\overline{\triangle}_j} c(y)}{\min\limits_{y\in\overline{\triangle}_j} c(y)} \le c(\lambda)\sup_{|y-t|\le 1}\frac{c(y)}{c(t)} < c_1 < \infty.$$

Further,

$$\left\|ina(y)(l_n+\lambda E)^{-1}\right\|_{2\to 2} \le c(\lambda)\sup_j \left\|ina(y)\varphi_j(l_{n,j}+\lambda E)^{-1}\right\|_{2\to 2} \le$$

$$\le c(\lambda)\sup_j |n||a(y)\varphi_j| \left\|(l_{n,j}+\lambda E)^{-1}\right\|_{2\to 2} \le c(\lambda)\sup_j \frac{|n|\max\limits_{y\in\overline{\triangle}_j}|a(y)|}{|n||a(y_j)|} \le$$

$$\le c(\lambda)\sup_j \frac{\max\limits_{y\in\overline{\triangle}_j}|a(y)|}{\min\limits_{y\in\overline{\triangle}_j}|a(y)|} \le c(\lambda)\sup_{|y-t|\le 1}\frac{a(y)}{a(t)} \le c_2 < \infty.$$

Similarly, we obtain that

$$\left\|\frac{d}{dy}(l_n+\lambda E)^{-1}\right\|_{2\to 2} \le c_3 < \infty$$

Lemma 2.2.8. is proved.

L e m m a 2.2.9. *Let the conditions of Lemma 2.2.7. be fulfilled. Then the estimate*

$$\|u^{'}\|_2 + \|ina(y)u\|_2 + \|c(y)u\|_2 \le c(\|l_n u\|_2 + \|u\|_2) \tag{2.2.34}$$

holds, where $c > 0$ is a constant.

P r o o f . Using Lemma 2.2.8., we have

$$\|c(y)u\|_2 = \|c(y)(l_n+\lambda E)^{-1}(l_n+\lambda E)u\|_2 \le \|c(y)(l_n+\lambda E)^{-1}\|_{2\to 2}\cdot\|(l_n+\lambda E)u\|_2 \le$$

$$\le c_1(\|(l_n+\lambda E)u\|_2) \le c_1\left(\|(l_n u\|_2 + \lambda\|u\|_2\right) \le c_1\cdot c_2(\lambda)\left(\|(l_n u\|_2 + \|u\|_2\right) \le$$

$$\le c\left(\|(l_n u\|_2 + \|u\|_2\right) \tag{2.2.35}$$

for any $u \in D(l_n)$.

Just as before

$$\|ina(y)u\|_2 = \|ina(y)(l_n+\lambda E)^{-1}(l_n+\lambda E)u\|_2 \le$$

$$\le \|ina(y)(l_n+\lambda E)^{-1}\|_{2\to 2}\|(l_n+\lambda E)u\|_2 \le c_1(\|(l_n+\lambda E)u\|_2) \le$$

$$\le c_1\left(\|(l_n u\|_2 + \lambda\|u\|_2\right) \le c_1 c_2(\lambda)\left(\|(l_n u\|_2 + \|u\|_2\right) \le$$

$$\le c\left(\|(l_n u\|_2 + \|u\|_2\right) \tag{2.2.36}$$

$$\|\frac{d}{dy}u\|_2 = \|\frac{d}{dy}(l_n+\lambda E)^{-1}(l_n+\lambda E)u\|_2 \le$$

$$\le c_3(\|(l_n+\lambda E)u\|_2) \le c\left(\|(l_n u\|_2 + \|u\|_2\right). \tag{2.2.37}$$

Now, combining (2.2.35)-(2.2.36), we arrive at the inequality (2.2.34). Lemma 2.2.9. is proved.

Separability of the operator l_n^+

Consider the operator

$$l_n^+ u = -u'' + \left(n^2 + ina(y) + c(y)\right)u \tag{2.2.38}$$

in $L_2(\mathbb{R})$ $(n = 0, \pm 1, \pm 2, \pm 3, ...)$

L e m m a 2.2.10. *Let the conditions i)-ii) be fulfilled. Then*

a) the operator $l_n^+ + \lambda E$ is continuously invertible for $\lambda > 0$;

b) for any $u \in D\,(l_n)$ the estimate

$$\| -u^{''}\|_2^2 + \|n^2 u\|_2^2 + \|ina(y)u\|_2^2 + \|c(y)u\|_2^2 \leq c\left(\|l_n u\|_2^2 + \|u\|_2^2\right) \qquad (2.2.39)$$

holds, where $c = c(\mu_1, \mu_2, \lambda)$.

The proof of item a) is similar to the proof of Theorem 2.1.

For the proof of item b), we need a few lemmas.

Consider the operator

$$l_{n,j}^{+} = -u^{''} + \left(n^2 + ina(y) + c(y)\right)u,$$

$$u(\triangle_j^{-}) = u(\triangle_j^{+}) = 0,$$

where $\triangle_j = (j-1, j+1)$, $\triangle_j^{-}$ and $\triangle_j^{+}$ are the left and right end points of the intervals $\triangle_j$, $j = \pm 1, \pm 2, \pm 3, \ldots$.

Lemma 2.2.11. *Let the conditions i) be fulfilled. Then there exists the continuous inverse operator $(l_{n,j}^{+} + \lambda E)^{-1}$ defined in $L_2(\triangle_j)$.*

This lemma can be proved just as Lemma 2.2.1.

Lemma 2.2.12. *Let the conditions i) be fulfilled. Then the inequalities*

$$\left\|(l_{n,j}^{+} + \lambda E)^{-1}\right\|_{2\to 2} \leq \frac{1}{\lambda}; \qquad (2.2.40)$$

$$\left\|\frac{d}{dy}(l_{n,j}^{+} + \lambda E)^{-1}\right\|_{2\to 2} \leq \frac{1}{\lambda^{1/2}}; \qquad (2.2.41)$$

$$\left\|(l_{n,j}^{+} + \lambda E)^{-1}\right\|_{2\to 2} \leq \frac{1}{c(y_j)}; \qquad (2.2.42)$$

$$\left\|(l_{n,j}^{+} + \lambda E)^{-1}\right\|_{2\to 2} \leq \frac{1}{(|n|+1)\,|a(y_j)|}, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots; \qquad (2.2.43)$$

$$\left\|(l_{n,j}^{+} + \lambda E)^{-1}\right\|_{2\to 2} \leq \frac{c}{|n|^2}, \quad n = \pm 1, \pm 2, \pm 3, \ldots \qquad (2.2.44)$$

hold, where $c(y_j) = \min\limits_{y\in\overline{\triangle}_j} c(y)$, $|a(y_j)| = \min\limits_{y\in\overline{\triangle}_j} |a(y)|$.

P r o o f . Consider the scalar product $< (l^+_{n,j} + \lambda E)u, u >$, where $u \in C_0^\infty(\mathbb{R})$. Integrating by parts and taking into account that the integral free terms will vanish, we find

$$< (l^+_{n,j} + \lambda E)u, u >= \int\limits_{\triangle_j} |u^{'}|^2 + \left[n^2 + ina(y) + c(y) + \lambda\right] |u|^2 dy \qquad (2.2.45)$$

From this, using the Cauchy-Bunyakovskii inequality, we find

$$\left\|(l^+_{n,j} + \lambda E)u\right\|_2 \|u\|_2 \geq \int\limits_{\triangle_j} \left[n^2 + \lambda\right] |u|^2 dy. \qquad (2.2.46)$$

Now, from (2.2.46)

$$\left\|(l^+_{n,j} + \lambda E)u\right\|_2 \geq \lambda \|u\|_2 \,, \qquad (2.2.47)$$

$$\left\|(l^+_{n,j} + \lambda E)u\right\|_2 \|u\|_2 \geq \int\limits_{\triangle_j} n^2 |u|^2 dy \qquad (2.2.48)$$

follows. From the attained estimates follows (2.2.40) and (2.2.44).

Using the Cauchy-Bunyakovskii inequality and taking (2.2.47) into account, we have from inequality (2.2.45)

$$\frac{1}{\lambda} \left\|(l^+_{n,j} + \lambda E)u\right\|_2^2 \geq \left\|u^{'}\right\|_2^2 .$$

The inequality (2.2.41) is proved. From inequality (2.2.45) it follows that

$$\left|< (l^+_{n,j} + \lambda E)u, u >\right| \geq \left|\int\limits_{\triangle_j} na(y)|u|^2 dy\right|,$$

$$\left|< (l^+_{n,j} + \lambda E)u, u >\right| \geq \int\limits_{\triangle_j} c(y)|u|^2 dy.$$

From this, taking conditions i) into account, we have

$$\left\|(l^+_{n,j} + \lambda E)u\right\|_2 \geq |n||a(y_j)| \|u\|_2 \,, \quad n = \pm 1, \pm 2, \pm 3, ...,$$

$$\left\|(l^+_{n,j} + \lambda E)u\right\|_2 \geq c(y_j) \|u\|_2 \,.$$

The inequalities (2.2.42) and (2.2.43) are proved. Lemma 2.2.12 is completely proved.

P r o o f of the item b) of Lemma 2.2.10.

Introduce an operator defined by the equality

$$Kf=\sum_j \varphi_j\left(l_{n,j}^{+}+\lambda E\right)^{-1}\varphi_j f,\quad f\in L_2(\mathbb{R}).$$

It is easy to find the representation

$$\left(l_n^{+}+\lambda E\right)^{-1}=K\left(E+B_\lambda\right)^{-1}, \tag{2.2.49}$$

where

$$B_\lambda f=\sum_j \varphi_j^{''}\left(l_{n,j}^{+}+\lambda E\right)^{-1}\varphi_j f+2\sum_j \varphi_j^{'}\frac{d}{dy}\left(l_{n,j}^{+}+\lambda E\right)^{-1}\varphi_j f.$$

The details of the proofs are just as in the conclusion of (2.2.16) and we will not state them here. The proof can be finished now using Lemma 2.2.12.

From the equality (2.2.49) we have

$$\left\|n^2(l_n^{+}+\lambda E)^{-1}\right\|_{2\to 2}=\left\|n^2K(E+B_\lambda)^{-1}\right\|_{2\to 2},$$

$$\left\|ina(y)(l_n^{+}+\lambda E)^{-1}\right\|_{2\to 2}=\left\|ina(y)K(E+B_\lambda)^{-1}\right\|_{2\to 2},$$

$$\left\|c(y)(l_n^{+}+\lambda E)^{-1}\right\|_{2\to 2}=\left\|c(y)K(E+B_\lambda)^{-1}\right\|_{2\to 2}.$$

Hence, since the operator $(E+B_\lambda)^{-1}$ is bounded, we find that

$$\left\|n^2(l_n^{+}+\lambda E)^{-1}\right\|_{2\to 2}\le c(\lambda)\left\|n^2K\right\|_{2\to 2},$$

$$\left\|ina(y)(l_n^{+}+\lambda E)^{-1}\right\|_{2\to 2}\le c(\lambda)\left\|ina(y)K\right\|_{2\to 2},$$

$$\left\|c(y)(l_n^{+}+\lambda E)^{-1}\right\|_{2\to 2}\le c(\lambda)\left\|c(y)K\right\|_{2\to 2}.$$

From these inequalities and from the definition of K, we find the inequalities

$$\left\|n^2(l_n^{+}+\lambda E)^{-1}\right\|_{2\to 2}\le c(\lambda)\sup_{\{j\}}\left\|n^2\varphi_j(l_{n,j}^{+}+\lambda E)^{-1}\right\|_{2\to 2},$$

$$\left\| ina(y)(l_n^+ + \lambda E)^{-1} \right\|_{2\to 2} \le c(\lambda) \sup_{\{j\}} \left\| ina(y)\varphi_j (l_{n,j}^+ + \lambda E)^{-1} \right\|_{2\to 2},$$
$$\left\| c(y)(l_n^+ + \lambda E)^{-1} \right\|_{2\to 2} \le c(\lambda) \sup_{\{j\}} \left\| c(y)\varphi_j (l_{n,j}^+ + \lambda E)^{-1} \right\|_{2\to 2}.$$

By virtue of inequality (2.2.44) we have

$$\left\| n^2(l_n^+ + \lambda E)^{-1} \right\|_{2\to 2} \le c(\lambda) \sup_{\{j\}} \left\| n^2\varphi_j (l_{n,j}^+ + \lambda E)^{-1} \right\|_{2\to 2} \le$$
$$\le c(\lambda) n^2 \sup_{\{j\}} |\max|\varphi_j|| \left\| (l_{n,j}^+ + \lambda E)^{-1} \right\|_{2\to 2} \le c(\lambda)\frac{n^2}{n^2} < \infty. \tag{2.2.50}$$

In exactly the same way, taking the condition $ii)$ into account and using the estimates (2.2.42)-(2.2.44), we find

$$\left\| ina(y)(l_n^+ + \lambda E)^{-1} \right\|_{2\to 2} \le c(\lambda) \sup_{\{j\}} \left\| ina(y)\varphi_j (l_{n,j}^+ + \lambda E)^{-1} \right\|_{2\to 2} \le$$
$$\le c(\lambda)|n| \sup_{\{j\}} \left| \max_{y\in\overline{\triangle}_j} |a(y)\varphi_j| \; a(y) \right| \left\| (l_{n,j}^+ + \lambda E)^{-1} \right\|_{2\to 2} \le$$
$$\le c(\lambda)|n| \sup_{\{j\}} \frac{\max\limits_{y\in\overline{\triangle}_j} |a(y)|}{(|n|+1)|a(y_j)|} \le c(\lambda)\frac{|n|}{(|n|+1)} \cdot \frac{\max\limits_{y\in\overline{\triangle}_j} |a(y)|}{\min\limits_{y\in\overline{\triangle}_j} |a(y)|} \le$$
$$\le c(\lambda) \sup_{|y-t|\le 2} \frac{a(y)}{a(t)} < c. \tag{2.2.51}$$
$$\left\| c(y)(l_n^+ + \lambda E)^{-1} \right\|_{2\to 2} \le c(\lambda) \sup_{\{j\}} \left\| c(y)\varphi_j (l_{n,j}^+ + \lambda E)^{-1} \right\|_{2\to 2} \le$$
$$\le c(\lambda) \sup_{\{j\}} |c(y)\varphi_j| \left\| (l_{n,j}^+ + \lambda E)^{-1} \right\|_{2\to 2} \le c(\lambda) \sup_{\{j\}} \frac{\max\limits_{y\in\overline{\triangle}_j} c(y)}{c(y_j)} \le$$
$$\le c(\lambda) \sup_{\{j\}} \frac{\max\limits_{y\in\overline{\triangle}_j} c(y)}{\min\limits_{y\in\overline{\triangle}_j} c(y)} \le c(\lambda) \sup_{|y-t|\le 2} \frac{c(y)}{c(t)} < c. \tag{2.2.52}$$

Using inequalities (2.2.50)-(2.2.52), we have

$$\| -u''\|_2^2 = \| \left(l_n^+ + \lambda E\right) u - \left(n^2 + ina(y) + c(y) + \lambda\right) u\|_2^2 \le$$
$$\le \| \left(l_n^+ + \lambda E\right) u\|_2^2 + \|n^2 u\|_2^2 + \|ina(y)u\|_2^2 + \|c(y)u\|_2^2 + \|\lambda u\|_2^2 \le$$

$$\leq \| \left(l_n^+ + \lambda E\right) u\|_2^2 + \|n^2 \left(l_n^+ + \lambda E\right)^{-1} \left(l_n^+ + \lambda E\right) u\|_2^2 +$$

$$+\|ina(y) \left(l_n^+ + \lambda E\right)^{-1} \left(l_n^+ + \lambda E\right) u\|_2^2 +$$

$$+\|c(y) \left(l_n^+ + \lambda E\right)^{-1} \left(l_n^+ + \lambda E\right) u\|_2^2 \leq c\| \left(l_n^+ + \lambda E\right) u\|_2^2 \leq$$

$$\leq c(\lambda) \left(\|l_n^+ u\|_2^2 + \|u\|_2^2\right).$$

From this it follows that

$$\|u''\|_2^2 + \|n^2 u\|_2^2 + \|ina(y)u\|_2^2 + \|c(y)u\|_2^2 \leq c \left(\|l_n^+ u\|_2^2 + \|u\|_2^2\right)$$

for all $u \in D(l_n^+)$, where $c = c(\mu_1, \mu_2, \lambda)$.

Estimates of the s - values of the operator $(l_n)^{-1}$

Introduce the sets

$$M = \left\{u \in L_2(\mathbb{R}) : \left\|-u'' + \left(n^2 k(y) + ina(y) + c(y)\right) u\right\|_2^2 \leq 1\right\},$$

$$\widetilde{M}_c = \left\{u \in L_2(\mathbb{R}) : \|u'\|_2^2 + \|ina(y)u\|_2^2 + \|c(y)u\|_2^2 \leq c\right\},$$

$$\approx{M}_{c_0^{-1}} = \left\{u \in L_2(\mathbb{R}) : \left\|-u'' + \left(n^2 + ina(y) + c(y)\right) u\right\|_2^2 \leq c_0^{-1}\right\},$$

where $c = c(\mu_1, \mu_2)$, $\quad c_0 = c(\mu_1, \mu_2, \lambda)$.

L e m m a 2.2.13. *Let the conditions i)-iii) be fulfilled. Then the inclusions*

$$\overset{\approx}{M}_{c_0^{-1}} \subseteq M \subseteq \widetilde{M}_c$$

hold, where $c = c(\mu, \mu_1)$, $\quad c_0 = c(\mu_1, \mu_2, \lambda)$.

P r o o f . Let $u \in \overset{\approx}{M}_{c_0^{-1}}$. Then, by virtue of Lemma 2.2.10, we find

$$\|l_n u\|_2^2 = \left\|-u'' + \left(n^2 k(y) + ina(y) + c(y)\right) u\right\|_2^2 \leq$$

$$\leq \|-u''\|_2^2 + \left\|n^2 k(y) u\right\|_2^2 + \|ina(y)u\|_2^2 + \|c(y)u\|_2^2 \leq$$

$$\leq \|-u''\|_2^2 + \left\|n^2 u\right\|_2^2 + \|ina(y)u\|_2^2 + \|c(y)u\|_2^2 \leq$$

$$\leq c_0 \left\| -u'' + \left(n^2 + ina(y) + c(y) \right) u \right\|_2^2 \leq c_0 c_0^{-1} \leq 1,$$

where $c_0 = c(\mu_1, \mu_2, \lambda)$.

Therefore $u \in M$, i.e. $\overset{\approx}{M}_{c_0^{-1}} \subseteq M$. Let now $u \in M$. Then, by virtue of Lemma 2.2.9, we have

$$\|-u'\|_2^2 + \|ina(y)u\|_2^2 + \|c(y)u\|_2^2 \leq$$

$$\leq c \|l_n u\|_2^2 = \left\| -u'' + \left(n^2 k(y) + ina(y) + c(y) \right) u \right\|_2^2 \leq c.$$

From this it follows that $M \subseteq \tilde{M}_c$.

L e m m a 2.2.14. *Let the conditions of Lemma 2.2.13 be fulfilled. Then the estimates*

$$c_0^{-1} \overset{\approx}{d}_k \leq s_{k+1} \leq c \tilde{d}_k, \quad k = 1, 2, ...,$$

hold with constants $c = c(\mu_1, \mu_2)$, $c_0 = c(\mu_1, \mu_2, \lambda)$, where s_{k+1} are the singular numbers of the operator $(l_n)^{-1}$, d_k, $\tilde{d}_k$, $\overset{\approx}{d}_k$ are the widths of the respective sets M, $\tilde{M}$, $\overset{\approx}{M}$.

P r o o f . From Lemma 2.2.13 and from the relations of the widths we have

$$c_0^{-1} \overset{\approx}{d}_k \leq d_k \leq c \tilde{d}_k.$$

Hence, taking the equality $s_{k+1} = d_k$ into account, we obtain the proof of Lemma 2.2.14.

L e m m a 2.2.15. *Let the conditions of Lemma 2.2.14 be fulfilled. Then the estimates*

$$\overset{\approx}{N}(c_0 \lambda) \leq N(\lambda) \leq \tilde{N}(c^{-1} \lambda),$$

$$N(\lambda) = \sum_{s_{k+1} > \lambda} 1, \quad \tilde{N}(\lambda) = \sum_{\tilde{d}_k > \lambda} 1, \quad \overset{\approx}{N}(\lambda) = \sum_{\overset{\approx}{d}_k > \lambda} 1,$$

hold, where $c = c(\mu_1, \mu_2)$, $c_0 = c(\mu_1, \mu_2, \lambda)$.

P r o o f . By the definition of the function $N(\lambda)$ and by virtue of Lemma 2.2.14, we have

$$N(\lambda) = \sum_{s_{k+1}>\lambda} 1 \leq \sum_{c\widetilde{d}_k>\lambda} 1 = \sum_{\widetilde{d}_k>c^{-1}\lambda} 1 = \widetilde{N}(c^{-1}\lambda).$$

Similarly

$$\widetilde{\widetilde{N}}(c_0\lambda) = \sum_{\widetilde{\widetilde{d}}_k>c_0\lambda} 1 = \sum_{c_0^{-1}\widetilde{d}_k>\lambda} 1 \leq \sum_{s_{k+1}>\lambda} 1 = N(\lambda).$$

Lemma 2.2.15 is proved.

L e m m a 2.2.16. *Let the conditions i)-iii) be fulfilled. Then the estimates*

$$c^{-1}\lambda^{-1/2} mes\left(y \in \mathbb{R} : \left|n^2 + ina(y) + c(y)\right| \leq c^{-1}\lambda^{-1/2}\right) \leq N(\lambda) \leq$$

$$\leq c\lambda^{-1} mes\left(y \in \mathbb{R} : |ina(y) + c(y)| \leq c\lambda^{-1}\right),$$

hold, where $c = c(\mu_1, \mu_2)$, $i^2 = -1$, $N(\lambda) = \sum\limits_{s_k>\lambda} 1$ *are the number of* s_k *is greater than* $\lambda > 0$*;* s_k *are the singular numbers of the operator* $(l_n)^{-1}$.

P r o o f . By $L^2_{2,a(y),c(y)}$ and $L^1_{2,a(y),c(y)}$ we denote the space obtained by replenishment of $C_0^\infty(\mathbb{R})$ concerning the norms

$$\left|u, L^2_{2,a(y),c(y)}\right| = \left(\int\limits_{-\infty}^{\infty} \left[|u''|^2 + |n^2 + ina(y) + c(y)|^2 |u|^2\right] dy\right)^{1/2},$$

$$\left|u, L^1_{2,a(y),c(y)}\right| = \left(\int\limits_{-\infty}^{\infty} \left[|u'|^2 + |ina(y) + c(y)|^2 |u|^2\right] dy\right)^{1/2}.$$

It is clear that $\widetilde{\widetilde{M}} \subset L^2_{2,a(y),c(y)}$, $\widetilde{M} \subset L^1_{2,a(y),c(y)}$. And now the proof of Lemma 2.2.16 follows from Lemmas 2.2.14 and 2.2.15 and from the results of [1].

Proofs of Theorems 2.2.1-2.2.4

P r o o f o f T h e o r e m 2.2.1. From Theorem 2.2.1 we find, that

$$u_k = \sum_{n=-k}^{k} (l_n + \lambda E)^{-1} f_n(y) e^{inx} \tag{2.2.53}$$

is a solution of the equation

$$(l_n + \lambda E)u_k = f_k \in L_2(\Omega), \tag{2.2.54}$$

where $\qquad f_k \xrightarrow{L_2} f, \qquad f_k = \sum_{n=-k}^{k} f_n(y)e^{inx},$

$$(l_n + \lambda E)u = -u'' + \left(n^2k(y) + ina(y) + c(y) + \lambda\right) u$$

By virtue of an inequality of Lemma 2.2.3 we have

$$\|u_k\| \le c\|f_k\|, \tag{2.2.55}$$

where c is a constant not depending on k.

As $f_k \xrightarrow{L_2} f$, then from (2.2.55) we find that

$$\|u_k - u_m\|_2 \le c\|f_k - f_m\|_2 \to 0 \quad \text{for} \quad k, m \to \infty.$$

Hence, by virtue of completeness of the space L_2, it follows that there exists a unique function $u \in L_2(\Omega)$, such that

$$u_k \to u \quad \text{for} \quad k \to \infty. \qquad (2.2.56).$$

From (2.2.54), (2.2.55) it follows, that

$$\|u_k - u\|_2 \to 0, \quad \|f_k - f\|_2 \to 0 \quad \text{for} \quad k \to \infty.$$

The last inequality shows that $u \in L_2(\Omega)$ is a solution of the equation $Lu = f$ and from (2.2.53) we have that

$$u = (L + \lambda E)^{-1}f = \sum_{n=-\infty}^{\infty} (l_n + \lambda E)^{-1} f_n(y)e^{inx} \qquad (2.2.57).$$

Theorem 2.2.1 is proved.

Proof of Theorem 2.2.2. Using Lemma 2.2.3, it is easy to see that

$$\lim_{|n|\to\infty} \|(l_n + \lambda E)^{-1}\|_{2\to 2} = 0.$$

Seeing this, from (2.2.57) it follows that the operator $(l_n + \lambda E)^{-1}$ is compact if and only if $(l_n + \lambda E)^{-1}$ is completely continuous. And now the proof of the theorem follows from Theorem 2.2.2.

Proof of Theorem 2.2.3. From (2.2.57) and from the fact that the system $\{e_{inx}\}_{n=-\infty}^{\infty}$ in $L_2(-\pi, \pi)$ is orthonormal we make sure that

$$\|\rho(y) D_x^\alpha (L + \lambda E)^{-1}\|_{2\to 2} = \sup_{\{n\}} \|\rho(y)|n|^\alpha (l_n + \lambda E)^{-1}\|_{2\to 2}, \tag{2.2.58}$$

where $D_x^\alpha = \dfrac{\partial^\alpha}{\partial x^\alpha}$ and α=0,1, $\rho(y)$ is a continuous function in $\mathbb{R}$.

Indeed, for any $f(x, y) \in L_2(\Omega)$

$$\|\rho(y) D_x^\alpha (L + \lambda E)^{-1} f\|_2^2 = \sum_{n=-\infty}^{\infty} \|\rho(y)|n|^\alpha (l_n + \lambda E)^{-1} f_n\|_2^2, \tag{2.2.59}$$

where $f(x, y) =< f_n(y), e^{inx} >$, $n = 0, \pm 1, \pm 2, ...$.

Seeing this, we find, that

$$\|\rho(y)|n|^\alpha (l_n + \lambda E)^{-1} f_n\|_2 \leq \|\rho(y) D_x^\alpha (L + \lambda E)^{-1} f\|_2,$$

from which we have

$$\|\rho(y)|n|^\alpha (l_n + \lambda E)^{-1}\|_{2\to 2} \leq \|\rho(y) D_x^\alpha (L + \lambda E)^{-1}\|_{2\to 2}. \tag{2.2.60}$$

On the other hand, from (2.2.59) we have

$$\|\rho(y) D_x^\alpha (L + \lambda E)^{-1} f\|_2^2 \leq \sup_{\{n\}} \|\rho(y)|n|^\alpha (l_n + \lambda E)^{-1}\|_{2\to 2} \sum_{n=-\infty}^{\infty} \|f_n\|^2 =$$

$$= \sup_{\{n\}} \|\rho(y)|n|^\alpha (l_n + \lambda E)^{-1}\|_{2\to 2} \|f_n\|_2^2.$$

As a result we obtain the inequality

$$\|\rho(y) D_x^\alpha (L + \lambda E)^{-1} f\|_2 \leq \sup_{\{n\}} \|\rho(y)|n|^\alpha (l_n + \lambda E)^{-1}\|_2. \tag{2.2.61}$$

Owing to the inequalities (2.2.60), (2.2.61) we obtain the proof of inequality (2.2.58).

From inequality (2.2.61), as $\rho(y) = c(y)$, by virtue of Lemma 2.2.8 we find

$$\|c(y)(l_n + \lambda E)^{-1}\| \le c_1 < \infty.$$

In exactly the same way

$$\|a(y)D_x(l_n + \lambda E)^{-1}f\|_{2\to 2} \le c_2 < \infty$$

follows. Or for any $u \in D(L)$

$$\|c(y)u\|_2 \le \|c(y)(L+\lambda E)^{-1}(L+\lambda E)u\|_2 \le \|c(y)(L+\lambda E)^{-1}\|_{2\to 2}\|(L+\lambda E)u\|_2 \le$$

$$\le c_1 \left(\|(Lu\|_2 + \lambda\|u\|_2\right) \le c_1c_1(\lambda) \left(\|(Lu\|_2 + \|u\|_2\right) \le c(\lambda) \left(\|(Lu\|_2 + \|u\|_2\right),$$

where $c(\lambda) = c_1c_1(\lambda)$;

$$\|a(y)u_x\|_2 \le \|a(y)D_x(L + \lambda E)^{-1}(L + \lambda E)u\|_2 \le$$

$$\le \|a(y)D_x(L + \lambda E)^{-1}\|_{2\to 2}\|L + \lambda E)u\|_2 \le$$

$$\le c_2(\|(L + \lambda E)u\|_2) \le c_2c_2(\lambda) \left(\|(Lu\|_2 + \|u\|_2\right) \le c(\lambda) \left(\|(Lu\|_2 + \|u\|_2\right).$$

With the help of these inequalities we deduce the inequalities

$$\| - k(y)u_{xx} - u_{yy}\|_2 = \|(L + \lambda E)u - a(y)u_x - c(y)u - \lambda u\|_2 \le$$

$$\le \|(L + \lambda E)u\|_2 + \|a(y)u_x\|_2 + \|c(y)u\|_2 + \|\lambda u\|_2 \le c(\lambda) \left(\|(Lu\|_2 + \|u\|_2\right).$$

So, we have proved that

$$\| - k(y)u_{xx} - u_{yy}\|_2 + \|a(y)u_x\|_2 + \|c(y)u\|_2 \le c(\lambda) \left(\|(Lu\|_2 + \|u\|_2\right).$$

Theorem 2.2.3 is proved.

P r o o f o f T h e o r e m 2.2.4. From Theorem 2.2.1 it follows that

$$L^{-1}f = \sum_{n=-\infty}^{\infty} l_n^{-1} f_n(y)e^{inx}, \tag{2.2.62}$$

where $f =< f_n(y), e^{inx} >$.

From (2.2.62) it follows that if s is a singular point of the operator L^{-1}, then s is a singular number of one of the operators l_n^{-1} and vice versa, if s is a singular number of one of the operators l_n^{-1}, then s is a singular point of the operator L^{-1}.

From this and from Lemma 2.2.16 the proof of Theorem 2.2.4 easily follows.