1 Coercive estimates and estimates of the spectrum of mixed type operators

Before to begin the deduction of results which are obtained in this and the next chapters, we remind on some known definitions and necessary notations.

$\mathbb{R}^n$ is the $n-$dimensional real Euclidean space; in particular, when n=2 we have the two-dimensional Euclidean space of z=(x,y) points, where $-\infty < x < \infty$, $-\infty < y < \infty$.

Ω is a domain in $\mathbb{R}^n$(in particular in $\mathbb{R}^2$). By $\overline{\Omega}$ we denote the closure of the set Ω;

$C^l(\overline{\Omega}), l = 0, 1, ...$ is the set of continuous functions, which have continuous partial derivatives in $\overline{\Omega}$ up to order l; in particular, if Ω is a domain from $\mathbb{R}^2$, then the partial derivatives for a function $u(x, y)$ can be written in the form

$$D^\alpha u = \frac{\partial^\alpha u}{\partial x^{\alpha_1} \partial y^{\alpha_2}}, \quad \text{where} \quad \alpha = \alpha_1 + \alpha_2 \leqslant l$$

α_1 and α_2 are entire non-negative numbers.

$C^\infty(\overline{\Omega})$ is the set of infinitely differentiable functions in $\overline{\Omega}$;

For definiteness we assume that Ω is a domain from $\mathbb{R}^2$, and $\overline{\Omega}$ is its closure.

Definition 1.1 *The closure of the set* $\{(x, y) \in \Omega : u(x, y) \neq 0\}$ *is called the supports of the function* u *determined in the domain* $\overline{\Omega}$ *and denoted by supp* u.

Definition 1.2 *A continuous function* u *in* $\overline{\Omega}$ *with* $\operatorname{supp} u \subseteq \Omega$ *is called a finite function in* $\overline{\Omega}$.

$C_0^\infty(\overline{\Omega})$ is the set of infinitely differentiable and finite functions in $\overline{\Omega}$;

$L_2(\Omega)$ is the Hilbert space consisting of determined and Lebesgue measurable functions in Ω which have the finite norm

$$\|u\|_{2,\Omega} = \left[\int\limits_{\Omega} |u|^2 d\Omega \right]^{1/2};$$

$W_2^k(\Omega)$ is the space of functions from $L_2(\Omega)$, which have generalized Sobolev's derivatives up to order $k \geqslant 1$ which also belongs to $L_2(\Omega)$ with the norm

$$\|f\|_{W_2^k(\Omega)} = \left[\sum_{|\alpha| \leqslant k} |D^\alpha f|^2 d\Omega \right]^{1/2}.$$

Let A be some operator then the definition domain of A is denoted by $D(A)$ and the range of A is denoted by $R(A)$.

Definition 1.2 *If for any x_1 and x_2 belonging to $D(A)$ with $x_1 \neq x_2$ follows that $y_1 = Ax_1 \neq y_2 = Ax_2$, then the operator A is called one-to-one.*

If A maps $D(A)$ on $R(A)$ one-to-one, then there exists the inverse map or inverse operator A^{-1}, which transforms $R(A)$ to $D(A)$.

Definition 1.3. *An operator is called a closed operator if for arbitrary sequence $\{x_n\} \subset D(A)$ with $x_n \to x_0$ and $Ax_n \to y_0$ follows that $x_0 \in D(A)$ and $y_0 = Ax_0$.*

Directly from this definition follows that if the operator A is not closed than it can be extended to a closed operator. This operation is called the closure of the operator A and the operator is called closable operator.

Definition 1.4. *The operator is called completely continuous operator, if it transforms any bounded set into a compact set or as is the same that for every bounded sequence $\{x_n\}$ from $D(A)$ the sequence $\{Ax_n\}$ contains a converging subsequence.*

Let X and Y be normalized spaces and A be a bounded operator from X

to Y. Let us determine a functional φ by the formula

$$\varphi(x) = (x, \varphi) = (Ax, f), x \in X, f \in Y^* \tag{1.1.0}$$

Y^* is the adjoint space to the space Y.

It is easy to check that φ is linear and $D(\varphi) = X$. So, for every $f \in Y^*$ there is an element $\varphi \in X^*$ according to the formula (1.1.0), where X^* is the adjoint space to the space X. Therefore the linear continuou operator $\varphi = A^* f$ is defined. The operator A^* is called adjoint operator to the operator A.

Definition 1.5. *The operator A, applied to the Hilbert space $L_2(\Omega)$ is called self-adjoint, if it is symmetric, i.e. if for any $u, v \in D(A)$ the scalar product relation $\langle Au, v\rangle = \langle u, Av\rangle$ holds and from the identity*

$$\langle Au, v\rangle = \langle u, w\rangle$$

follows that $v \in D(A)$, $w = Av$, where v and w are fixed and u is an arbitrary element from $D(A)$.

Let us give now a very important notion of the spectrum and the resolvent of an operator.

If A is a linear operator in a Hilbert space H, then the complex plane $\mathbb{C}$ can be divided into two parts: a resolvent set (denoted by $\rho(A)$) and a spectrum of the operator A(denoted by $\sigma(A)$), which is divided into a discrete $P_\sigma(A)$ and a continuous spectrum $C_\sigma(A)$.

The resolvent set $\rho(A)$ consists of λ for which the operator $(A - \lambda E)$ has a bounded inverse operator with a dense domain in H, i.e.

$\rho(A) = \{\lambda \in C : (A - \lambda E)^{-1}\text{is defined in whole H}\}$.

If λ belongs to the resolvent set then the operator $(A - \lambda E)^{-1}$ is called a resolvent of the operator A and denoted by $R_\lambda(A)$.

The discrete spectrum is called a set of eigenvalues of an operator A, i.e.

$$P_\sigma(A) = \{\lambda \in C : Au = \lambda u, \text{ for some } u \neq 0 \in \mathrm{H}\}.$$

In other words, when $\lambda \in P_\sigma(A)$ than the operator $(A - \lambda E)^{-1}$ does not exist.

The set of all other points of the spectrum in case of their existence is called the continuous spectrum, i.e.

$$C_\sigma(A) = \big\{\lambda \in C : \text{the operator}(A - \lambda E)^{-1} \text{ exists but is unbounded}\big\}.$$

Definition 1.6. *Let A is a completely continues operator. Then the eigenvalues of the operator $(A^*A)^{1/2}$ are called s-values of the operator A (Schmidt eigenvalues).*

The nonzero s-values we will order according to decreasing magnitude and observing their multiplicities and so

$$s_k(A) = \lambda_k((A^*A)^{1/2}),\ k = 1, 2, ...$$

Let us give another equivalent definition of the s-values. But before we give the definition of the notion of Kolmogorov k-widths and their properties.

Let M be a centrally symmetric subset of H(H is a Hilbert space), i.e. M = –M

The magnitude

$$d_k = \inf_{\{G_k\}} \sup_{u \in M} \inf_{v \in G_k} \|u - v\|,\ \ k = 0, 1, 2, ...$$

is called Kolmogorov k-widths of the set M, where G_k is a k-dimensional subset.

The $k-$widths have the following properties:

$$1) d_0 \geqslant d_1 \geqslant d_2 \geqslant ...;$$

$$2) d_k(\tilde{M}) \leqslant d_k(M), \;\; \tilde{M} \subset M, \;\; k = 1, 2, 3, ...;$$

$$3) d_k(nM) = nd_k(M), \; n > 0, \;\; nM = \{x' = nx, x \in M\}.$$

The assertion of the following theorem allows to give a second equivalent definition of the s-values.

Theorem 1.1. *Let A be a completely continuous operator. Then $s_{k+1}(A)$ (k =1,2,...) coincide with the Kolmogorov k-widths of the set M=AS, the image of the unit ball S =$\{x \in H\colon \|x\| \leqslant 1\}$ under the operator A.*

In many cases this definition proves to be more convenient than the first one.

Other notations and definitions will be given when needed.

1.1 Estimates of the spectrum for a class of mixed type equations

Consider the mixed type operator

$$Lu = -k(y)u_{xx} - u_{yy} + a(y)u_x + c(y)u, \tag{1.1.1}$$

where $k(y)$ is a piecewise continuous function in the segment [-1,1] and $yk(y) > 0$ for $y \neq 0$, $k(0)=0$ (as $y=0$).

Originally we define the operator in the set $C^{\infty}_{0,\pi}(\Omega)$, consisting of infinitely differentiable functions, satisfying the conditions: $u(-\pi, y) = u(\pi, y), u_x(-\pi, y) = u_x(\pi, y)$ and finite as functions of the variable y. Here

$$\Omega = \{(x, y) : -\pi < x < \pi, \ \ -1 < y < 1\}.$$

We note that the operator L admit closure in the metric of $L_2(\Omega)$ and the closure we also denote by L.

The solvability of the semiperiodical Dirichlet problem for the equation, where Lu is defined by the equality (1.1.1), was considered in the work [29].

Let us give necessary notations and definitions for further statements.

Let a function $u(x, y) \in L_2(\Omega)$. Then the following decomposition holds

$$u(x, y) = \sum_{n=-\infty}^{\infty} u_n(y)e^{inx}$$

Definition 1.1.1. *We call the expression [30]*

$$D^{\alpha}_{x}u = e^{\frac{i\pi\alpha}{2}} \sum_{n=0}^{\infty} n^{\alpha}u_n(y)e^{inx} + e^{-\frac{i\pi\alpha}{2}-1} \sum_{n=-\infty}^{-1} |n|^{\alpha}u_n(y)e^{inx}$$

as the fractional derivative $D^{\alpha}_{x}u$ of order $\alpha \geqslant 0$ with respect to x of a function $u(x, y)$. Here the equality is understood in the metric of $L_2(\Omega)$.

The main results

Theorem 1.1.1. *Let the conditions for the functions $a(y)$ and $c(y)$ continuous in the segment [-1,1]*

i) $|a(y)| \geqslant \delta_0 > 0, c(y) \geqslant \delta > 0$ *be fulfilled. Then:*

a) the operator $(L+\lambda E)$ *is continuously invertible for* $\lambda > 0$*;*

b) the operators $r(y)D_x(L+\lambda E)^{-1}, r(y)D_y(L+\lambda E)^{-1}$ *are bounded in* $L_2(\Omega)$*. Here* $D_x = \frac{\partial}{\partial x}, D_y = \frac{\partial}{\partial y}$*;* $r(y)$ *is a continuous function in the segment [-1,1].*

c) the operator $r(y)D_x^{\alpha}(L+\lambda E)^{-1}$ *is completely continuous if* $0 \leqslant \alpha < 1$*.*

Theorem 1.1.2. *Let the conditions of Theorem 1 be fulfilled. Then the following estimate holds for the Schmidt eigenvalues*

$$\frac{1}{k}c^{-1} \leqslant s_k \leqslant c\frac{1}{k^{1/2}}, \quad k = 1, 2, ...,$$

where $c > 0$ *does not depend on* k*.*

Theorem 1.1.3. *Let the condition* $i)$ *be fulfilled. Then:*

a) the spectrum $\sigma(L^{-1})$ *is a discrete set;*

b) for any non-zero $\lambda \in \sigma(L^{-1})$ *the estimate:*

$$|\lambda_k| \leqslant c\frac{1}{k^{1/2}}, \quad k = 1, 2, ...,$$

holds, where $c > 0$ *does not depend on* k*.*

Let us remind that σ_p denotes the set of completely continuous operators such that

$$\|A\|_{\sigma_p}^p = \sum_{k=1}^{\infty} s_k^p(A) < \infty,$$

where $s_k(A)$ are the Schmidt eigenvalues of the completely continuous operator A.

In the following theorem we give an assertion that the resolvent of the operator (1) belongs to the class σ_p.

Theorem 1.1.4. *Let the condition i) be fulfilled. Then the resolvent of the operator L belongs to the class σ_p if $p > 2$.*

For the proofs of the Theorems 1.1.1-1.1.4 we need a few auxiliary assertions and estimates.

Auxiliary lemmas and inequalities

Lemma 1.1.1. *Let the condition i) be fulfilled. Then the operator $L + \lambda E$ is continuously invertible for $\lambda \geqslant 0$ and the equality*

$$(L+\lambda E)^{-1}f = \sum_{n=-\infty}^{\infty} (l_n+\lambda E)^{-1} f_n e^{inx} \qquad (1.1.2)$$

holds in terms of $L_2(\Omega)$, where $(l_n+\lambda E)^{-1}$ is an inverse operator to the closed operator $(l_n+\lambda E)$ originally defined in $C_0^\infty(-1,1)$ by the equality

$$(l_n+\lambda E)u = -u''(y) + (n^2k(y) + ina(y) + c(y) + \lambda)u(y) \qquad (1.1.3)$$

The proof of this lemma can be found in [29].

Lemma 1.1.2. *Let the operator $(l_n+\lambda E)$ be defined by the equality (1.1.3) in the set $C_0^\infty(-1,1)$ $(n = 0, \pm1, \pm2, ...)$ and let the condition i) be fulfilled. Then the estimate*

$$\left\|(l_n+\lambda E)^{-1}\right\|_{2\to 2} \leqslant \frac{c}{\lambda^{1/2}},$$

holds, where $c > 0$ is a constant not depending on n.

Proof: For any $u(y) \in C_0^\infty(-1,1)$ have

$$\langle (l_n+\lambda E)u, u\rangle = \int_{-1}^{1} \left[|u|^2 + (n^2k(y) + ina(y) + c(y) + \lambda)|u|^2\right] dy \qquad (1.1.4)$$

From here and taking the condition i into account we find

$$|\langle (l_n+\lambda E)u, u\rangle| \geqslant \left|\int_{-1}^{1} ina(y)|u|^2 dy\right| \geqslant |n|\delta_0\|u\|^2.$$

Now using the Cauchy-Bunyakovskii inequality we have

$$\|(l_n + \lambda E)u\|_2 \geqslant |n|\delta_0\|u\|_2. \tag{1.1.5}$$

From (1.1.4) and the Cauchy inequality with ε=1 it follows that

$$\frac{1}{2}\|(l_n + \lambda E)u\|_2^2 \geqslant \int\limits_{-1}^{1} \left[|u'|^2 + (c(y) + \lambda)|u|^2\right] dy - \int\limits_{-1}^{1} n^2|k(y)||u|^2 dy.$$

Using the condition $i)$ and that $\lambda > 0$ we find:

$$\frac{1}{2}\|(l_n + \lambda E)u\|_2^2 \geqslant \frac{1}{2}\int\limits_{-1}^{1} \left[|u'|^2 + (c(y) + \lambda)|u|^2\right] dy - \int\limits_{-1}^{1} n^2|k(y)||u|^2 dy \tag{1.1.6}$$

Combining (1.1.5) and (1.1.6) we finally have

$$c^2\|(l_n + \lambda E)u\|_2^2 \geqslant \lambda\|u\|_2^2.$$

The assertion of Lemma 1.1.2 follows from the last inequality.

Lemma 1.1.3. *Let the conditions of Lemma 1.1.2 be fulfilled. Then the estimate*

$$\left\|(l_n + \lambda E)^{-1}\right\|_{2\to 2} \leqslant \frac{1}{|n| \cdot \delta_0}, n = \pm 1, \pm 2, ... \tag{1.1.7}$$

holds.

The proof of Lemma 1.1.3 follows from the inequality (1.1.5).

Lemma 1.1.4. *Let the condition $i)$ be fulfilled. Then the estimate*

$$\left\|\frac{d}{dy}(l_n + \lambda E)^{-1}\right\|_{2\to 2} \leqslant c,$$

holds, where $c > 0$ is a constant.

Proof. From the condition $i)$ and inequalities (1.1.5) and (1.1.6) we have

$$c\|(l_n + \lambda E)u\|_2^2 \geqslant \|u'\|_2^2 + \|u\|_2^2,$$

where $c > 0$ does not depend on u and n.

Hence,

$$\left\| \frac{d}{dy}(l_n + \lambda E)^{-1} \right\|_{2\to 2} = \sup_{f \in \alpha_2(-1,1)} \frac{\left\| \frac{d}{dy}(l_n + \lambda E)^{-1} f \right\|_2}{\|f\|_2} =$$

$$= \sup_{u \in D(l_n + \lambda E)} \frac{\|u'\|_2}{\|(l_n + \lambda E)u\|_2} \leqslant c < \infty.$$

The lemma is proved.

Now we proceed immediately to the proof of the items of the basic theorems.

Proof of Theorem 1.1.1

The proof of item a) of Theorem 1.1.1 immediately follows from Lemma 1.1.1.

Let us prove the item b) of Theorem 1.1.1. By virtue of the item a) and Lemma 1.1.3 we have

$$\left\| r(y) D_x (L + \lambda E)^{-1} f \right\|_2^2 = \left\| r(y) \sum_{n=-\infty}^{\infty} in(l_n + \lambda E)^{-1} f_n e^{inx} \right\|_2^2 =$$

$$= \sum_{n=-\infty}^{\infty} \left\| r(y) in(l_n + \lambda E)^{-1} f_n e^{inx} \right\|_2^2 \leqslant$$

$$\leq \max_{y \in [-1,1]} |r(y)| \sum_{n=-\infty}^{\infty} n^2 \left\| (l_n + \lambda E)^{-1} \right\|_2^2 \|f_n\|_2^2 \leq$$

$$\leqslant c_0 \sup_{\{n\}} |n|^2 \left\| (l_n + \lambda E)^{-1} \right\|_2^2 \sum_{n=-\infty}^{\infty} \|f_n\|_2^2 \leqslant \frac{c_0}{\delta_0^2} \|f\|_2^2.$$

Hence,

$$\left\| r(y) D_x (L + \lambda E)^{-1} \right\|_{2\to 2} \leqslant \frac{c_0}{\delta_0} < \infty.$$

Further we find the norm

$$\left\| r(y) D_y (L + \lambda E)^{-1} f \right\|_2^2 = \sum_{n=-\infty}^{\infty} \left\| r(y) \frac{d}{dy} (l_n + \lambda E)^{-1} f_n(y) \right\|_2^2$$

$$\leqslant \max_{y \in [-1,1]} |r(y)| \sum_{n=-\infty}^{\infty} \left\| \frac{d}{dy}(l_n + \lambda E)^{-1} f_n \right\|_2^2 \leqslant c_0 \sum_{n=-\infty}^{\infty} \left\| \frac{d}{dy}(l_n + \lambda E)^{-1} \right\|_{2\to 2}^2 \|f_n\|_2^2.$$

Hence, by virtue of Lemma 1.1.4, we have

$$\left\|r(y)D_y(L+\lambda E)^{-1}\right\|_{2\to 2}\leqslant c<\infty.$$

The item b) of Theorem 1.1.1 is proved.

Using the operator representation and the definition of the fractional derivative we have

$$r(y)D_x^{\alpha}(L+\lambda E)^{-1}f=r(y)e^{\frac{i\pi\alpha}{2}}\sum_{n=0}^{\infty}n^{\alpha}(l_n+\lambda E)^{-1}f_n(y)e^{inx}+$$

$$+r(y)^{\frac{-i\pi\alpha}{2}}\sum_{n=-\infty}^{1}|n|^{\alpha}(l_n+\lambda E)^{-1}f_n(y)e^{inx}.$$

From Lemma 1.1.1 it follows that $(l_n+\lambda E)$ has the continuous inverse operator $(l_n+\lambda E)^{-1}$ and from Lemma 1.1.4 it is clear that the range of the operator $(l_n+\lambda E)^{-1}$ belongs to $W_2^1(-1,1)$ for any n. Then from well-known theorems of Sobolev spaces (see for example [41]) it follows that the operator $r(y)(l_n+\lambda E)^{-1}$ is completely continuous for every n and the inequality

$$\left\|r(y)|n|^{\alpha}(l_n+\lambda E)^{-1}f\right\|_{2\to 2}\leqslant\frac{|n|^{\alpha}}{|n|\cdot\delta_0},\ \ 0\leqslant\alpha<1 \qquad (1.1.8)$$

holds. The last inequality follows from Lemma 1.1.3.

As for every n the operator $r(y)|n|^{\alpha}(l_n+\lambda E)^{-1}$ is completely continuous from L_2 to L_2 then from well-known theorems for completely continuous operators (see [48]) it follows that the operator $r(y)D_x^{\alpha}(L+\lambda E)^{-1}$ is completely continuous if

$$\mu=\lim_{|n|\to\infty}\mu=\lim_{|n|\to\infty}\left\|r(y)|n|^{\alpha}(l_n+\lambda E)^{-1}\right\|_{2\to 2}=0.$$

From (1.1.8) it is obvious that the number $\mu{\to}0$ as $n\to\infty$. The completely continuity of the operator $r(y)D_x^{\alpha}(L+\lambda E)^{-1}$ is proved. Theorem 1.1.1 is completely proved.

For further statements we need some important estimates and inclusions.

Introduce the sets

$$M = \left\{ u \in L_2(\Omega) : \|Lu\|_2^2 + \|u\|_2^2 \leqslant 1 \right\}$$
$$\tilde{M}_c = \left\{ u \in L_2(\Omega) : \|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2 \leqslant c \right\}$$
$$\tilde{\tilde{M}}_{c^{-1}} = \left\{ u \in L_2(\Omega) : \|u_{xx}\|_2^2 + \|u_{yy}\|_2^2 + \|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2 \leqslant c^{-1} \right\}$$

Then the following lemma holds.

Lemma 1.1.5. *Let the condition i) be fulfilled. Then the inclusions*

$$\tilde{\tilde{M}}_{c^{-1}} \subseteq M \subseteq \tilde{M}_c$$

hold, where $c > 0$ is a constant not depending on $u(x,y)$.

Proof. Let $u(x, y) \in \tilde{\tilde{M}}_{c^{-1}}$. Then

$$\|Lu\|_2^2 + \|u\|_2^2 = \|-k(y)u_{xx} - u_{yy} + a(y)u_x + c(y)u\|_2^2 + \|u\|_2^2 \leqslant$$

$$\leqslant \|-k(y)u_{xx}\|_2^2 + \|u_{yy}\|_2^2 + \|a(y)u_x\|_2^2 + \|c(y)u\|_2^2 + \|u\|_2^2 \leqslant$$

$$\leqslant c(\|u_{xx}\|_2^2 + \|u_{yy}\|_2^2 + \|u_x\|_2^2 + \|u\|_2^2) \leqslant$$

$$\leq c(\|u_{xx}\|_2^2 + \|u_{yy}\|_2^2 + \|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2) \leqslant c \cdot c^{-1} \leqslant 1,$$

where $c = \max_{y \in [-1,1]} \{|k(y)|, |a(y)|, |c(y)|\}$.

From this it follows that

$$\tilde{\tilde{M}}_{c^{-1}} \subseteq M.$$

Let now u∈M . Then by virtue of the item b) of Theorem 1.1.1 we have

$$\|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2) \leqslant c(\|Lu\|_2^2 + \|u\|_2^2) \leqslant c,$$

i.e.

$$M \subseteq \tilde{M}_c.$$

The lemma is proved.

In the Definition 1.6 the notations of s-values and Kolmogorov k-widths have been given. Therefore, referring to them, we give the two following assertions.

Lemma 1.1.6. *Let the condition i) be fulfilled. Then the estimate*

$$c^{-1}\tilde{\tilde{d}}_k \leqslant s_{k+1} \leqslant c\tilde{d}_k, k = 1, 2, ...$$

holds, where $c > 0$ is a constant, s_{k+1} are singular numbers of the operator L^{-1}; $\tilde{d}_k, \tilde{\tilde{d}}_k$ are k-widths of the considered sets $\tilde{M}, \tilde{\tilde{M}}$.

Proof. From Lemma 1.1.5 and from the properties of k-widths it follows that

$$c^{-1}\tilde{\tilde{d}}_k \leqslant d_k \leqslant c\tilde{d}_k.$$

Hence, taking the equality $s_{k+1} = d_k$ (the second definition of s-values) into account, we obtain the proof of Lemma 1.1.6.

Introduce the counting function $N(\lambda) = \sum\limits_{d_k > \lambda} 1$ of those d_k are greater than $\lambda > 0$.

Lemma 1.1.7. *Let the condition of Lemma 1.1.5 be fulfilled. Then the estimate*

$$\tilde{\tilde{N}}(c\lambda) \leqslant N(\lambda) \leqslant \tilde{N}(c^{-1}\lambda) \tag{1.1.9}$$

holds, where $N(\lambda) = \sum\limits_{s_{k+1} > \lambda} 1$, $\tilde{N}(\lambda) = \sum\limits_{\tilde{d}_k > \lambda} 1, \tilde{\tilde{N}}(\lambda) = \sum\limits_{\tilde{\tilde{d}}_k > \lambda} 1.$

Proof. Using Lemma 1.1.6 we find

$$N(\lambda) = \sum_{s_{k+1} > \lambda} 1 \leqslant \sum_{c\tilde{d}_k > \lambda} 1 = \sum_{\tilde{d}_k > c^{-1}\lambda} 1 = \tilde{N}(c^{-1}\lambda)$$

Similarly

$$\tilde{\tilde{N}}(c\lambda) = \sum_{\tilde{\tilde{d}}_k > c\lambda} 1 = \sum_{c^{-1}\tilde{\tilde{d}}_k > \lambda} 1 \leqslant \sum_{s_{k+1} > \lambda} 1 = N(\lambda)$$

From here we finally come to the inequality (1.1.9). The lemma is proved.

Now we proceed to the proofs of Theorems 1.2.2 and 1.2.3.

Proof of Theorem 1.1.2

For the function $\tilde{N}(\lambda)=\sum\limits_{\tilde{d}_k>\lambda}1, \tilde{\tilde{N}}(\lambda)=\sum\limits_{\tilde{\tilde{d}}_k>\lambda}1$ the estimates (proof of this estimates can be found in [49-50])

$$c^{-1}\lambda^{-2}\leqslant \tilde{N}(\lambda)\leqslant c\lambda^{-2} \tag{1.1.10}$$

$$c^{-1}\lambda^{-1}\leqslant \tilde{\tilde{N}}(\lambda)\leqslant c\lambda^{-1} \tag{1.1.11}$$

hold, where c does not depend on $\lambda>0$.

Let $\lambda=\tilde{d}_k$ then $\tilde{N}(\tilde{d}_k)=k$ and from (1.1.10) it follows that

$$c^{-1}d_k^{-2}\leqslant k\leqslant cd_k^{-2}.$$

From here,

$$c^{-1}\frac{1}{k^{1/2}}\leqslant \tilde{d}_k\leqslant c\frac{1}{k^{1/2}}.$$

Just as before we have

$$c^{-1}\frac{1}{k}\leqslant \tilde{\tilde{d}}_k\leqslant c\frac{1}{k}.$$

And now, using Lemma 1.1.6, we find that

$$c^{-1}\frac{1}{k}\leqslant s_k\leqslant c\frac{1}{k^{1/2}}, k=1,2,.... \tag{1.1.12}$$

Theorem 1.1.2 is proved.

Proof of Theorem 1.1.3

For completely continuous operators the Weyl inequality [51]

$$\prod_{j=1}^{k}|\lambda_j(A)|\leqslant\prod_{j=1}^{k}s_j(A),\quad k=1,2,..., \tag{1.1.13}$$

holds, where A is a completely continuous operator, $\lambda_j(A)$ are eigenvalues of this operator arranged as a sequence according to nonincreasing absolute values, the singular numbers $s_j(A)$ are arranged as a nonincreasing sequence.

From (1.1.12) and (1.1.13) we have

$$|\lambda_k|^k \leqslant \prod_{j=1}^{k} |\lambda_j| \leqslant \prod_{j=1}^{k} s_j \leqslant c^k (k!)^{-\frac{1}{2}}$$

Further, using the inequality $e^k k! > k^k (k = 1, 2, ...)$ we find

$$|\lambda_k|^k \leqslant c^k (k!)^{-\frac{1}{2}} \leqslant c^k e^{\frac{k}{2}} k^{-\frac{k}{2}}$$

Hence, we finally have

$$|\lambda_k| \leqslant c k^{-\frac{1}{2}}, \quad k = 1, 2, ...$$

Theorem 1.1.3 is proved.

Proof of Theorem 1.1.4 The assertion of the proof of Theorem 1.1.4 immediately follows from Theorems 1.2.1-1.2.2.

1.2 Estimates of the spectrum of a class of mixed type equations with coefficients of two variables

Consider the differential operator of mixed type

$$Lu = -k(y)u_{xx} - u_{yy} + a(x,y)u_x + c(x,y)u \tag{1.2.1}$$

where $k(y)$ is a sectionally continuous function in [-1,1], $k(0)$=0, $yk(y) > 0$ as $y \neq 0$.

Originally we define the operator in $C^{\infty}_{0,\pi}(\Omega)$, the set of infinitely differentiable functions, satisfying the conditions

$$u(-\pi, y) = u(\pi, y), u_x(-\pi, y) = u_x(\pi, y)$$

and finite as functions of the y variable. Here

$$\Omega = \{(x,y) : -\pi < x < \pi, \ \ -1 < y < 1\}.$$

The closure of the operator L in the metric of $L_2(\Omega)$ is also denoted by L.

Theorem 1.2.1. *Let $a(x,y)$ and $c(x,y)$ be continuous functions in $\overline{\Omega}$, satisfying the condition*

i) $|a(x,y)| \geqslant \delta_0 > 0$, $c(x,y) \geqslant \delta > 0$, δ_0 is a sufficiently large number.

Then the operator ($L+\lambda E$) continuously invertible for a sufficiently great $\lambda > 0$.

Theorem 1.2.2. *Let the conditions of Theorem 1.2.1 be fulfilled. Then the following estimate holds for the Schmidt eigenvalues*

$$c^{-1}\frac{1}{k} \leqslant s_k \leqslant c\frac{1}{k^{1/2}}, k = 1, 2, ...,$$

where $c > 0$ is a constant not depending on k.

Theorem 1.2.3. *Let the conditions of Theorem 1.2.1 be fulfilled. Then:*

a) the spectrum $\sigma((L+\lambda E)^{-1})$ is a discrete set;

b) for any nonzero $\lambda_k \in \sigma((L+\lambda E)^{-1})$

$$|\lambda_k| \leqslant c\frac{1}{k^{1/2}}, k = 1, 2, 3, ...,$$

where $c > 0$ is a constant not depending on k.

The theorem given bellow is similar to Theorem 1.1.4 of the previous section.

Theorem 1.2.4. *Let the condition i) be fulfilled. Then the resolvent of the operator L belongs to the class σ_p if $p > 2$.*

The following assertions are needed below.

Auxiliary assertions and inequalities

Consider the operator

$$(L_j + \lambda E)u = -k(y)u_{xx} - u_{yy} + a(x_j, y)u_x + c(x_j, y)u + \lambda u$$

where $u(x, y) \in C^\infty_{0,\pi}(\Omega)$, $x_j \in (-\pi, \pi)$, $\lambda > 0$.

The operator $L_j + \lambda E$ admits closure and the closure is also denoted by $L_j + \lambda E$.

Lemma 1.2.1. *Let $a(x, y)$ and $c(x, y)$ be continuous functions in $\overline{\Omega}$, satisfying conditions i). Then the operator $L_j + \lambda E$ is continuously invertible for $\lambda > 0$ and the equality*

$$(L_j + \lambda E)^{-1} f = \sum_{n=-\infty}^{\infty} (l_{n,j} + \lambda E)^{-1} f_n e^{inx} \tag{1.2.2}$$

holds in the metric of $L_2(\Omega)$ for it, where $(l_{n,j} + \lambda E)^{-1}$ is an inverse operator to the operator $(l_{n,j} + \lambda E)$ defined by the equality

$$(l_{n,j} + \lambda E)u = -u'' + (n^2 k(y) + ina(x_j, y) + c(x_j, y))u + \lambda u.$$

Proof. Since x_j is a fixed number from the interval $(-\pi, \pi)$, then in this case the factors $a(x_j, y), c(x_j, y)$ depend only on the variable y. Now the proof of the existence of an inverse operator to the operator (1.2.2) reduces to the case of one-dimensional factors $a(y)$ and $c(y)$, i.e. of functions depending only on one variable . Therefore, referring to Lemma 1.1.1 of the previously section, we obtain the proof of Lemma 1.2.1.

Lemma 1.2.2. *Let the conditions of Lemma 1.2.1 be fulfilled. Then the inequalities hold:*

a) $\left\|(L_j+\lambda E)^{-1}\right\|_{2\to 2}\leq \frac{c}{\lambda^{1/2}}$, *where* $c>0$ *is a constant;*

b) $\left\|D_x(L_j+\lambda E)^{-1}\right\|_{2\to 2}\leqslant \frac{1}{\delta_0}$;

c) $\left\|D_y(L_j+\lambda E)^{-1}\right\|_{2\to 2}\leqslant c$, *where* $c>0$ *is constant.*

Proof. From the representation (1.2.2) we have

$$\left\|(L_j+\lambda E)^{-1}f\right\|_2^2=\left\|\sum_{n=-\infty}^{\infty}(l_{n,j}+\lambda E)^{-1}f_n e^{inx}\right\|_2^2=\sum_{n=-\infty}^{\infty}\left\|(l_{n,j}+\lambda E)^{-1}f_n\right\|_2^2,$$

where $(l_{n,j}+\lambda E)u=-u''+(n^2k(y)+ina(x_j,y)+c(x_j,y))u+\lambda u$.

Hence, by virtue of Lemma 1.1.2, we find

$$\left\|(L_j+\lambda E)^{-1}\right\|_{2\to 2}\leqslant \frac{c}{\lambda^{1/2}}.$$

Let us prove the item b). From the representation (1.2.2) we find

$$D_x(L_j+\lambda E)^{-1}f=\sum_{n=-\infty}^{\infty} in(l_{n,j}+\lambda E)^{-1}f_n(y)e^{inx}$$

From here, using Lemma 1.1.3, we compute

$$\left\|D_x(L_j+\lambda E)^{-1}f\right\|_2^2=\left\|\sum_{n=-\infty}^{\infty} in(l_{n,j}+\lambda E)^{-1}f_n(y)e^{inx}\right\|_2^2=$$
$$=\sum_{n=-\infty}^{\infty}\left\|in(l_{n,j}+\lambda E)^{-1}f_n(y)e^{inx}\right\|_2^2\leqslant\sum_{n=-\infty}^{\infty}n^2\left\|(l_{n,j}+\lambda E)^{-1}\right\|_2^2\cdot\|f_n(y)\|_2^2\leqslant$$
$$\leqslant\sup_{\{n\}}\left\{n^2\left\|(l_{n,j}+\lambda E)^{-1}\right\|_2^2\right\}\sum_{n=-\infty}^{\infty}\|f_n(y)\|_2^2\leqslant\sup_{\{n\}}\left\{\frac{n^2}{n^2\delta_0}\right\}\sum_{n=-\infty}^{\infty}\|f_n(y)\|_2^2\leqslant\frac{1}{\delta_0^2}\|f\|_2^2.$$

The last inequality implyies

$$\left\|D_x(L_j+\lambda E)^{-1}\right\|_2^2\leqslant\frac{1}{\delta_0^2}.$$

The item b) of Lemma 1.2.2 is proved.

Let us prove the item c). For this we compute the norm

$$\left\|D_y(L_j+\lambda E)^{-1}f\right\|_2^2=\left\|\sum_{n=-\infty}^{\infty}\tfrac{d}{dy}(l_{n,j}+\lambda E)^{-1}f_n(y)e^{inx}\right\|_2^2\leqslant$$
$$\leqslant\sum_{n=-\infty}^{\infty}\left\|\tfrac{d}{dy}(l_{n,j}+\lambda E)^{-1}f_n(y)\right\|_2^2\leqslant\sum_{n=-\infty}^{\infty}\left\|\tfrac{d}{dy}(l_{n,j}+\lambda E)^{-1}\right\|_2^2\|f_n(y)\|_2^2\leqslant$$
$$\leqslant\sup_{\{n\}}\left\|\tfrac{d}{dy}(l_{n,j}+\lambda E)^{-1}\right\|_2^2\sum_{n=-\infty}^{\infty}\|f_n(y)\|_2^2=\sup_{\{n\}}\left\|\tfrac{d}{dy}(l_{n,j}+\lambda E)^{-1}\right\|_2^2\|f\|_2^2.$$

Hence, using Lemma 1.1.4, we find

$$\left\|D_y(L_j+\lambda E)^{-1}\right\|_2^2\leqslant c^2<\infty$$

Lemma 1.2.2 has been completely proved.

Construct a decomposition of identity corresponding to the covering of the segment [-π,π] with the neighborhoods Δ_j, i.e. we construct N non-negative functions $\varphi_j(x)\in C_0^\infty(-\pi,\pi)$ each of them vanishing outside of Δ_j and such that

$$\sum_{j=1}^{N}\varphi_j^2(x)\equiv 1,\quad x\in[-\pi,\pi].$$

Let K denote the operator defined by the equality

$$Kf=\sum_{j=1}^{N}\varphi_j(L_j+\lambda E)^{-1}\varphi_j f,\quad f\in L_2(\Omega).$$

Consider the action of the operator $(L+\lambda E)$ on Kf:

$$(L+\lambda E)Kf=-k(y)\left(\sum_{j=1}^{N}\varphi_j(L_j+\lambda E)^{-1}\varphi_j f\right)_{xx}-$$
$$-\left(\sum_{j=1}^{N}\varphi_j(L_j+\lambda E)^{-1}\varphi_j f\right)_{yy}+a(x,y)\left(\sum_{j=1}^{N}\varphi_j(L_j+\lambda E)^{-1}\varphi_j f\right)_{x}+$$
$$+c(x,y)\left(\sum_{j=1}^{N}\varphi_j(L_j+\lambda E)^{-1}\varphi_j f\right)+\lambda\left(\sum_{j=1}^{N}\varphi_j(L_j+\lambda E)^{-1}\varphi_j f\right)=$$

$$= -k(y)\left[\sum_{j=1}^{N}(\varphi_j)_{xx}(L_j+\lambda E)^{-1}\varphi_j f + 2\sum_{j=1}^{N}(\varphi_j)_x\left((L_j+\lambda E)^{-1}\varphi_j f\right)_x + \right.$$

$$\left. + \sum_{j=1}^{N}\varphi_j\left((L_j+\lambda E)^{-1}\varphi_j f\right)_{xx}\right] -$$

$$-\left[\sum_{j=1}^{N}(\varphi_j)_{yy}(L_j+\lambda E)^{-1}\varphi_j f + 2\sum_{j=1}^{N}(\varphi_j)_y\left((L_j+\lambda E)^{-1}\varphi_j f\right)_y + \right.$$

$$\left. + \sum_{j=1}^{N}\varphi_j\left((L_j+\lambda E)^{-1}\varphi_j f\right)_{yy}\right] +$$

$$+ a(x,y)\left[\sum_{j=1}^{N}(\varphi_j)_x(L_j+\lambda E)^{-1}\varphi_j f + \sum_{j=1}^{N}\varphi_j\left((L_j+\lambda E)^{-1}\varphi_j f\right)_x\right] +$$

$$+ c(x,y)\sum_{j=1}^{N}\varphi_j(L_j+\lambda E)^{-1}\varphi_j f + \lambda\sum_{j=1}^{N}\varphi_j(L_j+\lambda E)^{-1}\varphi_j f =$$

$$= \sum_{j=1}^{N}\varphi_j\left[-k(y)\left((L_j+\lambda E)^{-1}\varphi_j f\right)_{xx} - \sum_{j=1}^{N}\left((L_j+\lambda E)^{-1}\varphi_j f\right)_{yy} + \right.$$

$$a(x_j,y)\left((L_j+\lambda E)^{-1}\varphi_j f\right)_x + c(x_j,y)(L_j+\lambda E)^{-1}\varphi_j f +$$

$$\left.\lambda(L_j+\lambda E)^{-1}\varphi_j f\right] + a(x,y)\sum_{j=1}^{N}(\varphi_j)_x(L_j+\lambda E)^{-1}\varphi_j f +$$

$$+ \sum_{j=1}^{N}\varphi_j\left(a(x,y)-a(x_j,y)\right)\left((L_j+\lambda E)^{-1}\varphi_j f\right)_x +$$

$$+ \sum_{j=1}^{N}\varphi_j\left(c(x,y)-c(x_j,y)\right)(L_j+\lambda E)^{-1}\varphi_j f -$$

$$-k(y)\left[\sum_{j=1}^{N}(\varphi_j)_{xx}(L_j+\lambda E)^{-1}\varphi_j f + 2\sum_{j=1}^{N}(\varphi_j)_x\left((L_j+\lambda E)^{-1}\varphi_j f\right)_x\right] -$$

$$-\left[\sum_{j=1}^{N}(\varphi_j)_{yy}(L_j+\lambda E)^{-1}\varphi_j f + 2\sum_{j=1}^{N}(\varphi_j)_y\left((L_j+\lambda E)^{-1}\varphi_j f\right)_y\right].$$

From the last equality, taking into account that $(\varphi_j)_y = 0$, $(\varphi_j)_{yy} = 0$, we have

$$(L+\lambda E)Kf=\sum_{j=1}^{N}\varphi_j^2 f+\sum_{j=1}^{N}a(x,y)(\varphi_j)_x(L_j+\lambda E)^{-1}\varphi_j f+$$
$$+\sum_{j=1}^{N}\varphi_j\left(a(x,y)-a(x_j,y)\right)\left((L_j+\lambda E)^{-1}\varphi_j f\right)_x+$$
$$+\sum_{j=1}^{N}\varphi_j\left(c(x,y)-c(x_j,y)\right)(L_j+\lambda E)^{-1}\varphi_j f-$$
$$-k(y)\left[\sum_{j=1}^{N}(\varphi_j)_{xx}(L_j+\lambda E)^{-1}\varphi_j f+2\sum_{j=1}^{N}(\varphi_j)_x\left((L_j+\lambda E)^{-1}\varphi_j f\right)_x\right]$$

Introduce the notations

$$Mf=\sum_{j=1}^{N}\varphi_j\left(a(x,y)-a(x_j,y)\right)\left((L_j+\lambda E)^{-1}\varphi_j f\right)_x,$$

$$Bf=\sum_{j=1}^{N}\varphi_j\left(c(x,y)-c(x_j,y)\right)(L_j+\lambda E)^{-1}\varphi_j f-$$
$$-k(y)\left[\sum_{j=1}^{N}(\varphi_j)_{xx}(L_j+\lambda E)^{-1}\varphi_j f+2\sum_{j=1}^{N}(\varphi_j)_x\left((L_j+\lambda E)^{-1}\varphi_j f\right)_x\right].$$

Lemma 1.2.3. *Let the condition i) be fulfilled. Then there exists $\lambda>0$ such that $\|B\|_{2\to 2}<1$.*

Proof. Estimate the norm of the operator B

$$\|Bf\|_2^2=\Bigg\|\sum_{j=1}^{N}\varphi_j\left(c(x,y)-c(x_j,y)\right)(L_j+\lambda E)^{-1}\varphi_j f-$$
$$-k(y)\left[\sum_{j=1}^{N}(\varphi_j)_{xx}(L_j+\lambda E)^{-1}\varphi_j f+2\sum_{j=1}^{N}(\varphi_j)_x\left((L_j+\lambda E)^{-1}\varphi_j f\right)_x\right]\Bigg\|_2^2=$$

$$=\int_{\Omega}\Bigg|\sum_{j=1}^{N}\varphi_j\left(c(x,y)-c(x_j,y)\right)(L_j+\lambda E)^{-1}\varphi_j f-$$
$$-k(y)\left[\sum_{j=1}^{N}(\varphi_j)_{xx}(L_j+\lambda E)^{-1}\varphi_j f+2\sum_{j=1}^{N}(\varphi_j)_x\left((L_j+\lambda E)^{-1}\varphi_j f\right)_x\right]\Bigg|^2 dxdy\leqslant$$

$$\leqslant\sum_{j=1}^{N}\int_{\Delta_j}\Bigg|\sum_{j-1}^{j+1}\varphi_j\left(c(x,y)-c(x_j,y)\right)(L_j+\lambda E)^{-1}\varphi_j f-$$
$$-k(y)\left[(\varphi_j)_{xx}(L_j+\lambda E)^{-1}\varphi_j f+2(\varphi_j)_x\left((L_j+\lambda E)^{-1}\varphi_j f\right)_x\right]\Bigg|^2 dxdy\leqslant$$

$$\leqslant \sum_{j=1}^{N} \int_{\Delta_j} \left| \sum_{j-1}^{j+1} \varphi_j \left(c(x,y) - c(x_j,y) \right) (L_j + \lambda E)^{-1} \varphi_j f + \right.$$

$$\left. + |k(y)| \left[(\varphi_j)_{xx} (L_j + \lambda E)^{-1} \varphi_j f + 2(\varphi_j)_x \left((L_j + \lambda E)^{-1} \varphi_j f \right)_x \right] \right|^2 dxdy.$$

We use here that only $\varphi_{j-1}(x)$, $\varphi_j(x)$, $\varphi_{j+1}(x)$ are not equal to zero in Δ_j. Hence, taking into account that $\left||a| + |b| + |c|\right|^2 \leqslant 3(a^2 + b^2 + c^2)$ and by virtue of Lemma 1.2.2, we have

$$\|Bf\|_2^2 \leqslant 36 \sum_{j=1}^{N} \left[\left\| \varphi_j \left(c(x,y) - c(x_j,y) \right) (L_j + \lambda E)^{-1} \varphi_j f \right\|_2^2 + \right.$$

$$+ |k(y)|^2 \left\| (\varphi_j)_{xx} (L_j + \lambda E)^{-1} \varphi_j f \right\|_2^2 +$$

$$\left. + |k(y)|^2 \left\| (\varphi_j)_x \left((L_j + \lambda E)^{-1} \varphi_j f \right)_x \right\|_2^2 \right] \leqslant$$

$$\leqslant 36 \sum_{j=1}^{N} \left[\max_{x \in \Delta_j} |(c(x,y) - c(x_j,y))|^2 \left\| (L_j + \lambda E)^{-1} \right\|_{2\to 2}^2 \|\varphi_j f\|_2^2 + \right.$$

$$+ \max_{y \in [-1,1]} |k(y)|^2 \max_{x \in \bar{\Delta}_j} |(\varphi_j)_{xx}|^2 \left\| (L_j + \lambda E)^{-1} \right\|_{2\to 2}^2 \|\varphi_j f\|_2^2$$

$$\left. + \max_{y \in [-1,1]} |k(y)|^2 \max_{x \in \bar{\Delta}_j} |(\varphi_j)_x|^2 \left\| D_x (L_j + \lambda E)^{-1} \right\|_{2\to 2}^2 \|\varphi_j f\|_2^2 \right].$$

From here, by virtue of Lemma 1.2.2 and taking the boundedness and continuity of the functions $k(y)$, $c(x, y)$, $\varphi_j(x)$, $\varphi_j'(x)$, $\varphi_j''(x)$ into account, we find

$$\|Bf\|_2^2 \leqslant 36c \sum_{j=1}^{N} \left[\frac{\max\limits_{x \in \Delta_j} |(c(x,y) - c(x_j,y))|^2}{\lambda} + \frac{c_0}{\lambda} \right] \|\varphi_j f\|_2^2 + 36 \sum_{j=1}^{N} \frac{c_1}{\delta_0} \|\varphi_j f\|_2^2,$$

where $c_0 = \max\limits_{y \in [-1,1]} |k(y)|^2 \max\limits_{x \in \bar{\Delta}_j} |(\varphi_j)_{xx}|^2$, $c_1 = \max\limits_{y \in [-1,1]} |k(y)|^2 \max\limits_{x \in \bar{\Delta}_j} |(\varphi_j)_x|^2$.

By the choice of a sufficiently small domain $\bar{\Delta}_j$ we can estimate from above $\max\limits_{x \in \bar{\Delta}_j} |c(x,y) - c(x_j,y)|$ by an arbitrarily preassigned small number $\varepsilon > 0$. Therefore we can get the inequality

$$\|Bf\|_2^2 \leqslant 36 \left[\frac{c}{\lambda} (\varepsilon + c_0) + \frac{c_1}{\delta_0} \right] \sum_{j=1}^{N} \|\varphi_j f\|_2^2 \leqslant 36 \left[\frac{c}{\lambda} (\varepsilon + c_0) + \frac{c_1}{\delta_0} \right] \|f\|_2^2$$

The last inequalities proves the lemma for sufficiently large δ_0 and λ.

Lemma 1.2.4. *Let the condition i) be fulfilled. Then* $\|M\|_{2\to 2} < \frac{1}{2}$.

Proof. Estimate the norm of the operator M

$$\|Mf\|_2^2 = \left\| \sum_{j=1}^{N} \varphi_j \left(a(x,y) - a(x_j,y)\right) \left((L_j + \lambda E)^{-1} \varphi_j f\right)_x \right\|_2^2 =$$

$$= \int_{\Omega} \left| \sum_{j=1}^{N} \varphi_j \left(a(x,y) - a(x_j,y)\right) \left((L_j + \lambda E)^{-1} \varphi_j f\right)_x \right|^2 dxdy \leqslant$$

$$\leqslant \sum_{j=1}^{N} \int_{\Delta_j} \left| \sum_{j-1}^{j+1} \varphi_j \left(a(x,y) - a(x_j,y)\right) \left((L_j + \lambda E)^{-1} \varphi_j f\right)_x \right|^2 dxdy \leqslant$$

$$\leqslant 9 \sum_{j=1}^{N} \left\| \varphi_j \left(a(x,y) - a(x_j,y)\right) \left((L_j + \lambda E)^{-1} \varphi_j f\right)_x \right\|_2^2 \leqslant$$

$$\leqslant 9 \sum_{j=1}^{N} \max_{x \in \bar{\Delta}_j} |a(x,y) - a(x_j,y)| \left\| D_x (L_j + \lambda E)^{-1} \right\|_{2\to 2}^2 \|\varphi_j f\|_2^2.$$

Similarly to the previous lemma we estimate from above $\max\limits_{x \in \bar{\Delta}_j} |a(x,y) - a(x_j,y)|$ by an arbitrarily preassigned number $\varepsilon > 0$ by the choice of a sufficiently small domain $\bar{\Delta}_j$. By virtue of Lemma 1.2.2 the norm $\left\| D_x (L_j + \lambda E)^{-1} \right\|_{2\to 2}^2$ is estimated by the number $\frac{1}{\delta_0}$. We come to the following inequality

$$\|Mf\|_2^2 \leqslant 9 \frac{1}{\delta_0} \varepsilon \sum_{j=1}^{N} \|\varphi_j f\|_2^2 \leqslant \frac{9\varepsilon}{\delta_0} \|f\|_2^2.$$

In the last inequality we choose $\varepsilon > 0$ so small (moreover, by the condition of the lemma, δ_0 is a sufficiently large number) that $\|M\|_{2\to 2}^2 \leqslant \frac{9 \cdot \varepsilon}{\delta_0} \leqslant \frac{1}{2}$. The lemma is proved.

Proof of Theorem 1.2.1.

By virtue of the Lemmas 3.3 and 3.4, the operator $(E + B + M)$ is bounded together with its inverse. Therefore the set $R = \{\varphi : \ \varphi = (E + B + M) f, \ f \in C_{0,\pi}^{\infty}(\bar{\Omega})$ is dense in $L_2(\Omega)$. From the Lemmas 3.3 and 3.4 for $\varphi = (E + B + M) f$, $f \in C_{0,\pi}^{\infty}(\bar{\Omega})$ we obtain, that $K (E + B + M)^{-1} \varphi \in D(L)$ and $(L + \lambda E) K (E + B + M)^{-1} \varphi$ φ. From the last equality it follows that $u = K (E + B + M)^{-1} f$ is a solution of the equation (1.2.1).

Now it remains to show that $(L+\lambda E)^{-1}$ is a one-to-one inverse operator to the operator $(L+\lambda E)$ (the uniqueness of the solution). It is enough for this to obtain the inequality

$$c\|(L+\lambda E)\|_2 \geqslant \|u\|_2, \tag{1.2.3}$$

where $c>0$ is a constant.

Owing to the fact that the representation

$$u=(L+\lambda E)^{-1} f = K\,(E+B+M)^{-1} f, \quad f\in L_2(\Omega),$$

holds, we have

$$\|u\|_2^2 = \left\|K\,(E+B+M)^{-1} f\right\|_2^2 = \left\|\sum_{j=1}^{N}\varphi_j(L_j+\lambda E)^{-1}\varphi_j(E+B+M)^{-1}f\right\|_2^2 \leqslant$$

$$\leqslant 9\sum_{j=1}^{N}\left\|\varphi_j(L_j+\lambda E)^{-1}\varphi_j(E+B+M)^{-1}f\right\|_2^2 \leqslant$$

$$\leqslant 9\sum_{j=1}^{N}\left\|(L_j+\lambda E)^{-1}\right\|_{2\to 2}^2\left\|\varphi_j(E+B+M)^{-1}f\right\|_2^2.$$

Hence, by virtue of Lemma 1.2.2, we have

$$\|u\|_2^2 \leqslant \frac{9c_0}{\lambda}\sum_{j=1}^{N}\left\|\varphi_j(E+B+M)^{-1}f\right\|_2^2 \leqslant \frac{9c_0}{\lambda}\left\|(E+B+M)^{-1}f\right\|_2^2$$

Since the operator $(E+B+M)^{-1}$ is a bounded operator from $L_2(\Omega)$ to $L_2(\Omega)$ than the last inequality implyies

$$\|u\|_2^2 \leqslant c\|f\|_2^2,$$

where $c=\frac{9c_0c_1}{\lambda}$. From the last inequality we immediately come to the inequality (1.2.3). Theorem 1.2.1 is proved.

Proofs of Theorems 1.2.2 and 1.2.3

By virtue of Theorem 1.2.1 the representation

$$u = (L+\lambda E)^{-1} f = K(E+B+M)^{-1} f$$

holds for arbitrary $f \in L_2(\Omega)$.

Consequently,

$$\begin{aligned} u_x &= D_x (L+\lambda E)^{-1} f = D_x K (E+B+M)^{-1} f = \\ &= D_x \sum_{j=1}^{N} \varphi_j (L+\lambda E)^{-1} \varphi_j (E+B+M)^{-1} f = \\ &= \sum_{j=1}^{N} (\varphi_j)_x (L+\lambda E)^{-1} \varphi_j (E+B+M)^{-1} f + \\ &+ \sum_{j=1}^{N} \varphi_j D_x (L+\lambda E)^{-1} \varphi_j (E+B+M)^{-1} f, \end{aligned}$$

$$\begin{aligned} \|u_x\|_2^2 = &\left\| \sum_{j=1}^{N} (\varphi_j)_x (L+\lambda E)^{-1} \varphi_j (E+B+M)^{-1} f + \right. \\ &\left. + \sum_{j=1}^{N} \varphi_j D_x (L+\lambda E)^{-1} \varphi_j (E+B+M)^{-1} f \right\|_2^2 \leqslant \\ &\leqslant 18 \sum_{j=1}^{N} \left\| (\varphi_j)_x (L+\lambda E)^{-1} \varphi_j (E+B+M)^{-1} f \right\|_2^2 + \\ &+ 18 \sum_{j=1}^{N} \left\| \varphi_j D_x (L+\lambda E)^{-1} \varphi_j (E+B+M)^{-1} f \right\|_2^2 \leqslant \\ &\leqslant 18 \sum_{j=1}^{N} c_0 \left\| (L+\lambda E)^{-1} \right\|_{2\to 2}^2 \left\| \varphi_j (E+B+M)^{-1} f \right\|_2^2 + \\ &+ 18 \sum_{j=1}^{N} \left\| D_x (L+\lambda E)^{-1} \right\|_{2\to 2}^2 \left\| \varphi_j (E+B+M)^{-1} f \right\|_2^2. \end{aligned}$$

Further, by virtue of Lemma 1.2.2, we have

$$\|u_x\|_2^2 \leqslant 18 \sum_{j=1}^{N} \left(\frac{c_0}{\lambda} + \frac{1}{\delta_0} \right) \left\| \varphi_j (E+B+M)^{-1} f \right\|_2^2 \leqslant 18 c_1 \left\| (E+B+M)^{-1} f \right\|_2^2,$$

where $c_1 = \left(\frac{c_0}{\lambda} + \frac{1}{\delta_0} \right)$.

Owing to the fact that the operator $(E+B+M)^{-1}$ is a bounded operator from $L_2(\Omega)$ to $L_2(\Omega)$, we have

$$\|u_x\|_2^2 \leqslant c^2\|f\|_2^2$$

or

$$\|u_x\|_2 \leqslant c\|f\|_2,$$

where c=18c_1.

Just as before we obtain the estimate for u_y

$$\|u_y\|_2 \leqslant c\|f\|_2$$

and the estimate for u

$$\|u\|_2 \leqslant c\|f\|_2.$$

Its proof is given in Theorem 1.2.1.

Thanks to these inequalities we have

$$\|u_x\|_2 + \|u_y\|_2 + \|u\|_2 \leqslant c\|Lu\|_2.$$

Further, reproducing the computations and argument used in proving of Theorems 1.1.2 and 1.1.3, we obtain the proof of Theorems 1.2.2 and 1.2.3.

1.3 Properties of the resolvent and estimates of eigenvalues for a mixed type operator

In the rectangle

$$\Omega = \{(x,\ y):\quad -\pi < x < \pi, \quad -1 < y < 1\ \}$$

consider the operator

$$Lu = -k(y)u_{xx} - u_{yy} + a(y)u_x + c(y)u \tag{1.3.1}$$

with sectionally continuous and finite coefficients, originally defined in $C^{\infty}_{0,\,\pi}(\Omega)$, is the set consisting of infinitely differentiable functions, finite as functions of the y variable and satisfying the conditions

$$u(-\pi, y) = u(\pi, y), \qquad u_x(-\pi, y) = u_x(\pi, y) \tag{1.3.2}$$

$$u(x, -1) = u(x, 1) = 0. \tag{1.3.3}$$

The operator L admits closure in the metric of $L_2(\Omega)$ and the closure is also denoted by L.

Let the coefficients of the operator satisfy the conditions

a) $a(y)$, $c(y)$, $k(y)$ are piecewise continuous functions in [-1,1];

$c(y) \geqslant \delta > 0$, $a(y)$ does not change its sign ($a(y) \geqslant 0$ or $a(y) \leqslant 0$);

b) the condition

$$\lim_{|t|\to\infty}\ \sup_{y\in[-1,\,1]} \frac{t^2}{[K^*_t(y)]^2} < c,$$

is fulfilled for some $m > 0$, where $c > 0$ is the fixed number and $K^*_t(y)$ is a averaging function defined by the equality (introduced by M. Otelbaev (see [52]))

$$K^*_t(y) = \inf_{d>0}\{d^{-1}; d^{-1} \geqslant \int\limits_{y-\frac{d}{2}}^{y+\frac{d}{2}} K_t(\tau)d\tau\}$$

where $K_t(\tau) = t^2(m|a(\tau)| \,-\, |k(\tau)|) + c(\tau) > 0, \forall \tau \in [-1, 1], -\infty < t < \infty$.

The basic results of this chapter is formulated in the following theorems.

Theorem 1.3.1. *Let the conditions a)–b) be fulfilled. Then the operator $L + \lambda E$ is continuously invertible for a sufficiently large $\lambda > 0$.*

Theorem 1.3.2. *Let the conditions a) – b) be fulfilled. Then there exists a sequence of positive eigenvalues of the operator (1.3.1) and the estimates*

$$c^{-1}k^2 \leqslant \lambda_k \leqslant ck^2, \quad k = 1,\ 2, ...,$$

hold for them, where c is a constant.

For the proofs of these theorem we need some auxiliary lemmas.

Consider the operator defined by the equality

$$l_t u = -u''(y) + (t^2 k(y) + ita(y) + c(y))u(y)$$

or omitting the variable y of u we just write

$$l_t u = -u'' + (t^2 k(y) + ita(y) + c(y))u$$

in the set $C_0^\infty(-1,\ 1), \quad -\infty < t < \infty$.

Lemma 1.3.1. *Let the condition a) be fulfilled. Then the inequality*

$$(m+1)(t^2+1)||(l_t+\lambda E)u||_2^2 \geqslant \frac{1}{2}\int_{-1}^{1} [|u'|^2 + (t^2(m|a(y)| - |k(y) + c(y+1))|u|^2]dy$$

holds for arbitrary $u \in C_0^\infty(-1,\ 1)$ and for a sufficiently large $\lambda > 0$, where $m > 0$ is a constant, $-\infty < t < \infty$.

Proof. Consider the scalar product

$$| < (l_t + \lambda E)u,\ -itu > | = \left| \int_{-1}^{1} t^2 a(y)|u|^2 dy - \right.$$
$$\left. it\int_{-1}^{1} (t^2 k(y) + c(y) + \lambda|u|^2 dy + it\int_{-1}^{1} u''(y)\bar{u}(y)dy \right|.$$

Integrating by parts the last component and using the boundedness of the function $u(y)$, we have

$$|< (l_t + \lambda E)u, \, - itu >| = \left| \int_{-1}^{1} t^2 a(y)|u|^2 dy - it \int_{-1}^{1} [|u'|^2 + (t^2 k(y) + c(y) + \lambda|u|^2]dy \right|.$$

From here

$$| < (l_t + \lambda E)u, \, - itu > | \geqslant \left| \int_{-1}^{1} t^2 a(y)|u|^2 dy \right|.$$

Since $a(y)$ does not change its sign then

$$| < (l_t + \lambda E)u, \; -itu > | \geqslant \int_{-1}^{1} t^2 |a(y)||u|^2 dy$$

By virtue of the Cauchy-Bunyakovskii inequality from this it follows that

$$||(l_t + \lambda E)u||_2 \; || - itu||_2 \geqslant \int_{-1}^{1} t^2 |a(y)| \; |u|^2 dy$$

On the basis of this inequality and using the Cauchy inequality with $\varepsilon > 0$ we find for $\varepsilon = 1/(t^2+1)$

$$\frac{1}{2}(t^2 + 1)||(l_t + \lambda E)u||_2^2 + \frac{1}{2}\frac{1}{t^2 + 1}||itu||_2^2 \geqslant \int_{-1}^{1} t^2 |a(y)||u|^2 dy. \qquad (1.3.4)$$

Now we consider the scalar product

$$| < l_t u + \lambda u, u > | = \left| - \int_{-1}^{1} u'' \, \bar{u} dy + \int_{-1}^{1} (t^2 k(y) + c(y + \lambda))|u|^2 dy + it \int_{-1}^{1} a(y)|u|^2 dy \right|.$$

Integrating by parts the first component and using the boundedness of the function $u(y)$, we have

$$|< l_t u + \lambda u, \; u >| = \left| \int_{-1}^{1} [|u|^2 + (t^2 k(y) + c(y) + \lambda))|u|^2] dy + it \int_{-1}^{1} a(y)|u|^2 dy \right|.$$

Hence

$$|<l_tu+\lambda u,u>|\geqslant\left|\int\limits_{-1}^{1}[|u'|^2+(t^2k(y)+c(y)+\lambda)|u|^2]dy\right|\geqslant$$
$$\geqslant\int\limits_{-1}^{1}[|u'|^2+(c(y)+\lambda)|u|^2]dy-\int\limits_{-1}^{1}t^2|k(y)||u|^2dy$$

Similarly, using the Cauchy-Bunyakovsky inequality and afterwards the Cauchy inequality with $\varepsilon>0$ (ε=1/(t^2+1)), we have

$$\frac{(t^2+1)}{2}||l_tu+\lambda u||_2^2+\frac{1}{2}\left(\frac{1}{t^2+1}\right)||u||_2^2\geqslant$$
$$\geqslant\int\limits_{-1}^{1}[|u'|^2+(c(y)+\lambda)|u|^2]dy-\int\limits_{-1}^{1}t^2|k(y)|u|^2dy$$

Taking the condition a) into account, from the last inequality and the inequality (1.3.4) we finally find

$$(m+1)(t^2+1)||(l_t+\lambda E)u||_2^2\geqslant$$
$$\geqslant\frac{1}{2}\int\limits_{-1}^{1}[|u'|^2+(t^2(m|a(y)|-|k(y)|)+c(y))|u|^2]dy$$

for some $m>0$ and a sufficiently large $\lambda>0$. The lemma is proved.

Next some notations are introduced which will be useful hereinafter.

Let $d(y)=[K_t{}^*(y)]^{-1}$, then $\Delta_{d(y)}(y)$ is the interval $(y-\frac{d}{2},\quad y+\frac{d}{2})$, $\Delta^{(k)}=\Delta_{d_k}(y_k)=(y_k-\frac{d}{2},\quad y_k+\frac{d}{2})$

From the definition of the function $K_t{}^*(y)$ given above it is clear that it is positive. But, moreover, the function $K_t{}^*(y)$ is continuous. The proof of this assertion can be found in the work [53]. We just note that for proving of the continuity the Lipschitz condition is used which is sufficient for the continuity of the function. In particular, the estimate

$$|[K_t^*(y_0)]^{-1}-[K_t^*(y)]^{-1}|\leqslant 2|y_0-y|,\qquad \text{for all } y\in\Delta_{\frac{d_0}{2}}(y_0)$$

is obtained in the work [53], where $\Delta_{\frac{d_0}{2}}(y_0)=(y_0-\frac{d_0}{4},\ y_0+\frac{d_0}{4})\subset(-1,\ 1)$

Lemma 1.3.2. *For the interval (-1, 1) there exist no more than a countable cover* $\{\Delta^{(k)}\}$ *of disjoint intervals* $\Delta^{(k)}$ *contained in the interval (-1, 1) accurate within countable set.*

The proof of this lemma will be borrowed from the work [53] and will be cited here for completeness of the statement.

Proof. Let us take an arbitrary point $y_1 \in$(-1, 1) and assign $\Delta^{(1)} = \Delta_{d(y_1)}$, $d(y_1)$=$[K_t * (y_1)]^{-1}$. Suppose that $\Delta^{(1)}$ does not cover (y_1, b). Among the intervals $\{\Delta_{d(y)}(y): \quad y_1 \leqslant y < d, \quad d(y) = [K_t^*(y)]^{-1}\}$ there are both overlapping (for example, the interval $\Delta^{(1)}$ is itself) and disjoint intervals with the interval $\Delta^{(1)}$. The last statement follows from the following property of $K_t^*(y)$: from the definition of $K_t^*(y)$ easily follows that $[K_t^*(y)]^{-1} \to 0$ as $y \to 1$, therefore $\Delta^{(1)}$ and $\Delta_{d(y)}$ do not intersect if y is close to 1.

Consider the function $\psi(y) = y - \dfrac{[K_t^*(y)]^{-1}}{2}$ in the segment $[y_1, 1]$. The range of values of this function is $y_1 - \varepsilon \leqslant \psi(y) \leqslant 1$. Therefore $\quad y_1 + \dfrac{d_1}{2} \in (y_1 - \varepsilon, 1)$. From here, by virtue of the continuity of $\psi(y)$ there exists a $y \in (y_1, 1)$ such that $\psi(y) = y_1 + \frac{d_1}{2}$, i.e. the left end of $\Delta_{d(y)}$ coincide with the right end of $\Delta^{(1)}$. We denote by y_2 the least among such (it exists), $\Delta^{(2)} = \Delta_{d(y)}(y_2)$. If on some step (y_1, y) is contained in $\{\Delta^{(n)}\}_{n \leqslant k}$ (accurate within the ends of intervals) then the construction process of intervals to the righthand side from $\Delta^{(1)}$ is completed. Otherwise it can be proceeded indefinitely.

The constructed intervals must cover $(y_1, 1)$ since otherwise their centers will converge to a point $y < 1$ in which $[K_t^*(y)]^{-1} = \lim\limits_{k \to \infty} [K_t^*(y_n)]^{-1} = 0$. It is contrary to the continuity and the positiveness of this function in the interval (-1, 1). In exactly the same way intervals be constructed to the left side from $\Delta^{(1)}$. Lemma 1.3.2 is proved.

Lemma 1.3.3. *Let the conditions* a) *and* b) *be fulfilled. Then there exist a* *constant* $c_0 > 0$ *such that the inequality*

$$\int\limits_{\Delta_{d(y)}} [|u'|^2 + (t^2(m|a(\tau)| - |k(\tau)|) + c(\tau)|u|^2]d(\tau) \geq$$

$$\geq c_0^{-1}\left(\int\limits_{\Delta_{d(y)}} |u'|^2 d\tau + d^{-2}(y)\int\limits_{\Delta_{d(y)}} |u|^2 d\tau\right)$$

holds for any $u \in C_0^\infty(\Delta_{d(y)}(y))$, *where* $\Delta_{d(y)}(y) \subset (-1,\ 1)$ *and* $m > 0$ *is a constant.*

Reproducing the computations and arguments used in the work [53] we obtain the proof of the next lemma.

Lemma 1.3.4. *Let the conditions a) – b) be fulfilled. Then the estimate*

$$||(l_t + \lambda E)u||_2 \geqslant c||u||_2 \tag{1.3.5}$$

holds for $\lambda > 0$ *for all* $u \in D(l_t)$, *where* $c > 0$ *is a constant.*

Proof. Based on the Lemmas 1.3.1–1.3.3 we have

$$(t^2+1)(m+1)||(l_t + \lambda E)u||_2^2 \geqslant \frac{1}{2}\int\limits_{-1}^{1} [|u'|^2 + (t^2(m|a(y)| - |k(y)|) +$$

$$+c(y) + \lambda)|u|^2]dy \geqslant \frac{1}{2}\sum_{\{k\}}\left(\int\limits_{\Delta_{d_k}(y_k)} |u'|^2 dy + d_k^{-2}(y_k)\int\limits_{\Delta_{dk}(y_k)} |u|^2 dy\right),$$

where the constant c_0 from Lemma 1.3.3 without loss of generality is taken as 1 in every interval $\Delta_d(y_k)$.

From here

$$m(t^2+1)||(l_t + \lambda E)u||_2^2 \geqslant \frac{1}{2}\sum_{\{k\}} d_k^{-2}(y_k)\int\limits_{\Delta_{d_k}(y_k)} |u|^2 dy \geqslant$$

$$\geqslant \inf_{y\in(-1,\,1)} [K_t^*(y)]^2 \sum_{\{k\}}\int\limits_{\Delta_{d_k}(y_k)} |u|^2 dy = \inf_{y\in(-1,\,1)} [K_t^*(y)]^2 \int\limits_{-1}^{1} |u|^2 dy =$$

$$= \inf_{y\in(-1,\,1)} [K_t^*(y)]^2||u||_2^2.$$

From the last inequality, taking into account the condition b), we find

$$(m+1)(t^2+1)||(l_t+\lambda E)u||_2^2 \geqslant t^2 c||u||_2^2$$

From this we finally obtain the inequality (1.3.5). The lemma is proved.

Lemma 1.3.5. *Let the conditions $a) - b)$ be satisfied. Then the operator $l_t+\lambda E$ is continuously invertible for $\lambda > 0$.*

Proof. Lemma 1.3.4 implyies that there exists a bounded inverse operator for the operator $l_t+\lambda E$ in the range $R(l_t+\lambda E)$. Now if we show that the range of the operator $l_t+\lambda E$ is everywhere dense in L_2(-1, 1) then the inverse operator $(l_t+\lambda E)^{-1}$ will be continuous in the whole of L_2(-1, 1).

Assume by contradiction that the range is not dense in L_2(-1, 1). Then there exists an element $\upsilon\in L_2$ ($\upsilon\neq 0$) such as $\upsilon\perp R(l_t+\lambda E)$, i.e.

$$< l_t u + \lambda u, \upsilon >= 0 \text{ for all } u \in D(l_t).$$

Then it is clear by the Riesz theorem that $\upsilon\in D(l_t{}^*)$ and this means $l_t{}^*\upsilon\in L_2$(-1,1), where $l_t{}^*$is a conjugate operator to l_t, i.e. the equality

$$< (l_t + \lambda E)u, \upsilon >=< u, (l_t^* + \lambda E)\upsilon >= 0$$

holds and

$$(l_t^* + \lambda E)\upsilon = -\upsilon'' + (t^2 k(y) - ita(y) + c(y) + \lambda)\upsilon = 0.$$

As $a(y)$, $c(y)$, $k(y)$ are bounded functions then $(t^2k(y)$ – $ita(y) + c(y)+$ $+\lambda)\upsilon\in L_2$ (-1,1). From here $\upsilon''\in L_2$(-1,1).

Now if we show that υ=0 this will be to contradiction and the lemma will be proved.

Indeed, consider the scalar product

$$\langle u,\ l_t^* \upsilon + \lambda \upsilon \rangle = 0.$$

The following computations are correct for it

$$0 = \langle u,\ l_t^* \upsilon + \lambda \upsilon \rangle = \int_{-1}^{1} u\Big[-\bar{\upsilon}'' + \overline{(t^2 k(y) - ita(y) + c(y) + \lambda)\upsilon}\Big]dy =$$
$$= -\int_{-1}^{1} u\bar{\upsilon}'' dy + \int_{-1}^{1} (t^2 k(y) + ita(y) + c(y) + \lambda)u\bar{\upsilon}dy = -\int_{-1}^{1} ud(\bar{\upsilon}')+$$
$$\int_{-1}^{1} (t^2 k(y) + ita(y) + c(y) + \lambda)u\bar{\upsilon}dy =$$

$$= -u\bar{\upsilon}'|_{-1}^{1} + \int_{-1}^{1} \bar{\upsilon}' du + \int_{-1}^{1} (t^2 k(y) + ita(y) + c(y) + \lambda)u\bar{\upsilon}dy = \int_{-1}^{1} u' d\bar{\upsilon}+$$
$$+\int_{-1}^{1} (t^2 k(y) + ita(y) + c(y) + \lambda)u\bar{\upsilon}dy = u'\bar{\upsilon}|_{-1}^{1} - \int_{-1}^{1} u''\bar{\upsilon}\, dy+$$
$$+\int_{-1}^{1} (t^2 k(y) + ita(y) + c(y) + \lambda)u\bar{\upsilon}dy = u'\bar{\upsilon}|_{-1}^{1} + \int_{-1}^{1} (l_t u + \lambda u)\cdot \bar{\upsilon}dy =$$
$$= \langle (l_t + \lambda E)u, \upsilon \rangle + u'\bar{\upsilon}|_{-1}^{1} = 0.$$

By assumption $< (l_t + \lambda E)u, \upsilon >= 0$ and it implies $u'\bar{\upsilon}|_{-1}^{1} = 0$ or $u'(1)\bar{\upsilon}(1) = u'(-1)\bar{\upsilon}(-1)$.

Let $\bar{\upsilon}(1) = \alpha$, $\bar{\upsilon}(-1) = \beta$. Assume that $\alpha \neq 0$ or $\beta \neq 0$ and by virtue of arbitrariness of the function u we take

$$u(y) = (y+1)^2 (y-1),\ (\ u(-1) = u(1) = 0\ ).$$

Then

$$u'(y) = 2(y^2 - 1) + (y+1)^2 \text{ and } u'(-1) = 0,\ u'(1) = 4.$$

From this it follows that $\bar{\upsilon}(-1) = \bar{\upsilon}(1) = 0$ or that is the same $\upsilon(-1) = \upsilon(1) = 0$. Now it easy to prove that the inequality

$$\|(l_t^* + \lambda E)\upsilon\|_2 \geqslant \|\upsilon\|_2$$

holds for the function $\upsilon\in L_2(-1,1)$, $\upsilon(-1)=\upsilon(1)=0$ which follows from the similar computations and reasoning obtained for the operator l_t.

Owing to $l_t{}^*\upsilon+\lambda\upsilon=0$ from the last inequality it follows that $\upsilon=0$. Lemma 1.3.5 is completely proved.

Consider the operator

$$L'u=-k(y)u_{xx}-u_{yy}-a(y)u_x+c(y)u$$

originally defined in $C^{\infty}_{0,\pi}(\Omega)$.

It is easy to prove that the L' admits closure and the closure is also denoted by L'.

Lemma 1.3.6. *Let the conditions a) – c) be fulfilled. Then*

$$D(L)\subseteq D(L'^*),$$

where $D(L)$ and $D(L'^*)$ are the domains of definition of the operator L and the conjugate operator to the operator L'.

Proof. According to the definition of a conjugate operator the equality

$$\langle L'u,\upsilon\rangle=\langle u,L'^*\upsilon\rangle$$

holds for any $u(x,y)\in D(L')$, $\upsilon(x,y)\in D(L'^*)$. If we prove that the last equality also holds for any $\upsilon\in D(L)$ then the assertion of the lemma will be proved.

Let $u_n(x,y)\in C^{\infty}_{0,\pi}(\Omega)$ and $u_n\to u\in D(L')$, $\upsilon_n(x,y)\in C^{\infty}_{0,\pi}(\Omega)$ and $\upsilon_n\to\upsilon\in D(L)$. Then the following equality holds for any u_n, $\upsilon_n\in C^{\infty}_{0,\pi}(\Omega)$

$$\langle L'u_n,\upsilon_n\rangle=\langle u_n,L\upsilon_n\rangle. \tag{1.3.7}$$

Let us prove this equality. We have for any u, $\upsilon\in C^{\infty}_{0,\pi}(\Omega)$ (index n for u_n,

$v_n$is omitted here in order to avoid complications in the notations):

$$\langle L'u, v\rangle = \int\limits_{\Omega} (-k(y)u_{xx} - u_{yy} - a(y)u_x + c(y)u)\bar{v}dxdy = \int\limits_{\Omega} -k(y)u_{xx}\bar{v}dxdy -$$
$$-\int\limits_{\Omega} u_{yy}\bar{v}dxdy - \int\limits_{\Omega} a(y)u_x\bar{v}dxdy + \int\limits_{\Omega} c(y)u\bar{v}dxdy = -\int\limits_{-1}^{1} k(y)\left[\int\limits_{-\pi}^{\pi} \bar{v}du_x\right]dy -$$
$$-\int\limits_{-\pi}^{\pi} dx\int\limits_{-1}^{1} \bar{v}du_y - \int\limits_{-1}^{1} a(y)\left[\int\limits_{-\pi}^{\pi} \bar{v}du\right]dy + \int\limits_{\Omega} c(y)u\bar{v}dxdy = -\int\limits_{-1}^{1} k(y)\left[u_x\bar{v}|_{-\pi}^{\pi} -\right.$$
$$\left.-\int\limits_{-\pi}^{\pi} u_x d\bar{v}\right] dy - \int\limits_{-\pi}^{\pi} \left[u_y\bar{v}|_{-1}^{1} - \int\limits_{-1}^{1} u_y d\bar{v}\right]dx - \int\limits_{-1}^{1} a(y)\left[u\bar{v}|_{-\pi}^{\pi} - \int\limits_{-\pi}^{\pi} ud\bar{v}\right]dy +$$
$$+\int\limits_{\Omega} c(y)u\bar{v}dxdy.$$

By virtue of the boundary conditions (1.3.2)-(1.3.3) we have for functions from $C_{0,\pi}^{\infty}(\Omega)$

$$\langle L'u, v\rangle = -\int\limits_{-1}^{1} k(y)\left[\int\limits_{-\pi}^{\pi} \bar{v}_x du\right]dy - \int\limits_{-\pi}^{\pi} \left[\int\limits_{-1}^{1} \bar{v}_y du\right]dx - \int\limits_{-1}^{1} a(y)\left[\int\limits_{-\pi}^{\pi} \bar{v}_x udx\right]dy +$$
$$+\int\limits_{\Omega} c(y)u\bar{v}dxdy = -\int\limits_{-1}^{1} k(y)\left[u\bar{v}_x|_{-\pi}^{\pi} - \int\limits_{-\pi}^{\pi} u\bar{v}_{xx}dx\right]dy - \int\limits_{-\pi}^{\pi} \left[u\bar{v}_y|_{-1}^{1} - \int\limits_{-1}^{1} u\bar{v}_{yy}dy\right]dx$$
$$-\int\limits_{-1}^{1}\int\limits_{-\pi}^{\pi} a(y)u\bar{v}_x dxdy + \int\limits_{\Omega} c(y)u\bar{v}dxdy.$$

And owing to the boundary conditions (1.3.2)-(1.3.3) we finally have

$$\langle L'u, v\rangle = \int\limits_{\Omega} u(-k(y)\bar{v}_{xx} - \bar{v}_{yy} + a(y)\bar{v}_x + c(y)\bar{v})dxdy = \langle u, Lv\rangle.$$

It proves the inequality (1.3.7).

Now proceeding to the limit in the equality (1.3.7) we have

$$\langle L'u, v\rangle = \langle u, Lv\rangle.$$

The lemma is proved.

Proof of Theorem 1.3.1

Let $u(x, y) \in C_{0,\pi}^{\infty}(\Omega)$ and Lu=f. Then the representation

$$(L + \lambda E)u_k = (L + \lambda E)\sum_{n=-k}^{k} u_n(y)e^{inx} = \sum_{n=-k}^{k} (l_n + \lambda E)u_n e^{inx}$$

holds for the operator $L+\lambda E$, where $u_k = \sum\limits_{n=-k}^{k} u_n(y)e^{inx}$ and $u_k(x,y) \to u(x,y)$, as $k \to \infty$.

By virtue of Lemma 1.3.5 we have for $\lambda > 0$ (replace the index t by n in the operator l_t)

$$||(L+\lambda E)u_k||_2^2 = \sum_{n=-k}^{k} ||(l_n+\lambda E)u_n||_2^2 \geqslant \sum_{n=-k}^{k} c^2||u_n||_2^2 = c^2||u_k||_2^2.$$

Proceeding to the limit in this inequality we finally have

$$\|(L+\lambda E)u\|_2 \geqslant c\|u\|_2$$

and make sure that the last estimate holds for any $u(x,y) \in C_{0,\pi}^{\infty}(\Omega)$.

Since the operator allows the closure then by virtue of the continuity of the norm the last estimate holds for every $u(x,y) \in D(L)$.

Now we show that the kernel of the operator L contains only the null element, i.e. $N(L)$=Ker(L)=$\{0\}$.

Reproducing the computations and arguments used for proving of invertibility of the operator $l_n+\lambda E$, we obtain that the operator $l_n'+\lambda E$ has a continuous inverse operator $(l_n'+\lambda E)^{-1}$, where

$$l_n' + \lambda E = -u'' + (n^2 k(y) - ina(y) + c(y) + \lambda)u.$$

It is known that the equality

$$f(x,y) = \sum_{n=-\infty}^{\infty} f_n(y)e^{inx}$$

holds for any $f \in L_2(\Omega)$. It is easy to show from here that

$$u_k(x,y) = \sum_{n=-k}^{k} (l_n'+\lambda E)^{-1} f_n(y)e^{inx}$$

is a solution of the problem

$$(L'+\lambda E)u = f_k, \tag{1.3.1'}$$

$$u|_{-\pi} = u|_{\pi}, \ u_x|_{-\pi} = u_x|_{\pi}, \tag{1.3.2'}$$

$$u(x,-1) = u(x,1) = 0, \tag{1.3.3'}$$

where

$$f_k(x,y) = \sum_{n=-k}^{k} f_n(y)e^{inx}.$$

By virtue of Lemma 1.3.5 we have for $\lambda > 0$

$$\|f_k(x,y)\|_2^2 = \|(L+\lambda E)u_k\|_2^2 = \sum_{n=-k}^{k} \|(l'+\lambda E)u_n\|_2^2 \geqslant$$
$$\geqslant c^2 \sum_{n=-k}^{k} \|u_n\|_2^2 = c^2\|u_k\|_2^2$$

or

$$\|f_k(x,y)\|_2 \ \geqslant c\|u_k\|_2. \tag{1.3.8}$$

From this and from $f_k(x,y) \to f(x,y)$ it follows that the sequence $\{u_k(x,y)\}_{k=1}^{\infty}$ is fundamental.

By virtue of the completeness of the space $L_2(\Omega)$ we have

$$u_k(x,y) \to u(x,y) \in L_2. \tag{1.3.9}$$

Therefore, there exists a strong solution of the problem $(1.3.1') - (1.3.3')$ for every $f(x,y) \in L_2$.

One can immediately prove that the definition of a strong solution is equivalent to the closure of the operator $L' + \lambda E$ originally defined in $\overset{\circ}{W}{}^{2}_{2,\pi}(\Omega)$.

From aforesaid it follows that the range set of the operator $L'+\lambda E$ coincides with the entire $L_2(\Omega)$, i.e.

$$R(L' + \lambda E) = L_2(\Omega). \tag{1.3.10}$$

From the general theory of linear operators it is well known that

$$L_2(\Omega) = R(L' + \lambda E) \perp N((L' + \lambda E)^*)$$

From this and from (1.3.10) we find that $N((L' + \lambda E)^*) = \{0\}$.

Consequently, using Lemma 1.3.6, we have

$$N(L + \lambda E) = Ker(L + \lambda E) \subseteq N((L' + \lambda E)^*) = Ker((L' + \lambda E)^*) = \{0\}$$

i.e. $N(L+\lambda E)=Ker(L+\lambda E)=\{0\}$.

Reproducing the computations and arguments used for proving of the inequality (1.3.8) and using the completeness of the space $L_2(\Omega)$, we obtain that there exists the unique strong solution of the problem (1.3.1)-(1.3.3) for any f such that

$$c\|u\|_2 \leqslant \|f\|_2$$

and the representation

$$u = (L + \lambda E)^{-1} f = \sum_{n=-\infty}^{\infty} (l_n + \lambda E)^{-1} f_n \, e^{inx}$$

holds for it, where $c > 0$ is a constant. Here we use the fact that $Ker(L+\lambda E)=\{0\}$. The theorem is completely proved.

Proof of Theorem 1.3.2. We will find the eigenfunctions of the problems (1.3.1)-(1.3.3) in the form

$$u(x, y) = \sum_{n=-\infty}^{\infty} u_n(y) e^{inx}.$$

Then we have the Sturm-Liouville spectral problem for the functions $u(y)$

$$-u''(y) + (n^2 k(y) + ina(y) + c(y))u = \lambda u, \qquad (1.3.11)$$

$$u(-1) = u(1) = 0. \qquad (1.3.12)$$

When the condition $a)$ is fulfilled and in case n=0 the problem (1.3.11)-(1.3.12) will take the form

$$l_0 u = -u''(y) + c(y)u = \lambda u,$$

$$u(-1) = u(1) = 0.$$

It easy to prove that the domain of the operator l_0 coincide with the space $W_2^2(-1,1)$ owing to the boundedness of $c(y)$.

Let M_0 denote the unit disk in $W_2^2(-1,1)$. The two-sided estimates

$$c^{-1}\lambda^{-\frac{1}{2}} \leqslant N_0(\lambda) \leqslant c\lambda^{-\frac{1}{2}}$$

holds for the $k-$widths of the set M_0, where $N_0(\lambda)$ is a quantity of the $k-$widths $\overset{0}{d}_k$ of the set M_0 greater than $\lambda > 0$ and $c > 0$ is a constant. From the last inequality and from the properties of $N_0(\lambda)$ we have

$$c^{-1}\frac{1}{k^2} \leqslant \overset{0}{d}_k \leqslant c\frac{1}{k^2}$$

Moreover, taking the self-adjointness of the operator l_0 and the correctness of the equality $\lambda_{k+1}(l_0^{-1}) = \overset{0}{d}_k$ into account, we finally have for the eigenvalues of the operator l_0

$$c^{-1}k^2 \leqslant \lambda_k(l_0) \leqslant ck^2\,, \quad k = 1,\, 2,\, ...$$

The theorem is proved.