Introduction

Subject urgency. It is well-known that a number of problems arising in theoretical physics, hydro and flow dynamics, in the theory of surfaces and shells lead to the necessity of the investigation of mixed type equations.

Many works are devoted to the theory of boundary value problems of mixed type equations:: F. Tricomi [1], S. Gellerstedt [2], F.I. Frankel [3], A.V. Bitsadze [4], K.I. Babenko [5], M.M. Smirnov [6], A.M. Nahushev [7], M.S. Salahitdinov [8], T.D. Dzhuraev [9], T.S. Kalmenov [10], E.I. Moiseev [11], S.M. Ponomarev [12], M.A. Sadybekov [13], M.B. Muratbekov [14]. The following questions have been considered in these works: existence, uniqueness and smoothness of solutions.

However for some classes of mixed type equations these questions are poorly investigated hitherto. Particularly among them is the Dirichlet problem for equations of mixed type, when the factors of the lower order derivatives turn to zero in some points of the considered domain.

The spectral analysis of differential operators studies the spectrum nature depending on the behavior of factors, boundary conditions and domain geometry. Concerning elliptic operators these issues are well studied with typical difficulties found out. The works of G.D. Berkgof, D. Hilbert, R. Courant [15], G. Carleman, E. Titchmarsch [16], L. Hormander [17], A.G. Kostuchenko [18], B.M. Levitan, M. Otelbaev [19], M.G. Gasymov [20], M.Sh. Birman, I.S. Sargsyan [21], R. Oinarov [22] and other authors are devoted to them.

The spectral theory of hyperbolic and mixed type equations has been poorly investigated hitherto comparing to the spectral theory of elliptic type equations.

The spectral questions depending on boundary conditions and domain geometry for operators of mixed type have been studied in the following works: T.S.Kalmenov [23], B.I.Moiseev [24], P.M.Ponomarev [25], M.S.Sadybekov [13], A.S.Berdyshev [26] and others.

The investigation of spectral questions for the operators of mixed type depending on the behavior of coefficients is begun rather recently from works of M.B.Muratbekov [27], T.S.Kalmenov and M.B.Muratbekov [28].

Such questions as the estimate of eigenvalues and singular numbers of differential operators have primary issue in the spectral theory. Concerning operators of mixed type these questions need further investigation.

All of this implies the urgency of the subject of this thesis.

Purpose of the thesis.

1. To investigate spectral questions of the semiperiodical Dirichlet problem for a class of mixed type equations in a rectangle. These questions can be subdivided into the following:

a) discreteness of the spectrum;

b) estimates of singular numbers (Schmidt eigenvalues);

c) estimates of eigenvalues.

2. To find conditions for coefficients of a mixed type equation which provide solvability of the semiperiodical Dirichlet problem in a rectangle.

3. To study the following questions for a class of mixed type equations with growing coefficients in an unbounded domain:

a) separability of a mixed type operator corresponding to the equation considered in an unbounded domain;

b) solvability of the equation or existence of the resolvent $(L + \lambda E)^{-1}$ of an operator corresponding to the equation considered;

c) conditions providing discreteness of the spectrum of a mixed type operator corresponding to the equation considered;

d) asymptotic of $s-$values and their distribution function.

Research methods. The following methods are used in the thesis: the spectral theory of linear operators, the embedding theory of weighted spaces, perturbation theory, the method of compactness, the method of localization, the method of a priori estimate.

Scientific novelty. The following results have been obtained in the thesis:

1. Boundedness and compactness of resolvent of a mixed type operator corresponding to the semiperiodical Dirichlet problem for mixed type equations.

2. Two-sided estimates of Schmidt eigenvalues (s-values)

3. Discreteness of the spectrum of a mixed type operator is proved.

4. Estimates of eigenvalues.

5. All of aforecited results are generalized to the case of coefficients of two variables of the lower order derivatives.

6. Coercitive solvability of the semiperiodical Dirichlet problem for a class of mixed type equations.

7. It is proved, that the condition of boundedness from below for the coefficient of the lowest term is significant for the separability of a mixed type operator in an unbounded domain.

8. Solvability of the semiperiodical Dirichlet problem for a class of mixed type equations with fast-growing coefficients in an unbounded domain and smoothness of solutions.

9. The conditions providing discreteness of the spectrum for a class of mixed type equations considered with fast-growing coefficients.

10. Asymptotic of s–values and their distribution function for a class of mixed type equations.

Theoretical and practical value. The obtained results have theoretical interest and can find an application for the spectral theory of differential operators.

Publications. The basis results of the thesis are printed in the papers [60]-[70] and are also published in materials of conferences and scientific journals "Bulletin of KazNU. Series mathematics, mechanics, informatics", 2000, 2(21), pp. 19-23; "Proceedings of the Minister of Education and the National Academy of Sciences of RK. Series Physics and Mathematics", 2000, 5(213), pp. 24-28., "Differential Equations", 2007, Vol. 43, No. 1, pp. 143–146., Complex Variables and Elliptic Equation. An International Journal, Volume 52 Issue 12, December 2007, 1121-1145, ...

Contents of the thesis. The urgency of the subject has been substantiated, the basic purposes have been formulated, the novelty and theoretical value of the thesis is explained in this introduction.

The first chapter consists of three sections. At the beginning of the first chapter there are cited necessity notations and definitions, the notions of completely continuous, closed, adjoint, self-adjoint operators, the notions of the spectrum and the resolvent of an operator, the definition of s-values. Also the second equivalent definition of s-values is given by means of a well-known theorem concerned with the notation of Kolmogorov widths.

In the section 1.1 the spectral questions of a mixed type operator are studied corresponding to the semiperiodical Dirichlet problem for a class of mixed type equation. The boundedness and compactness of the resolvent of the operator are also proved.

In the rectangle $\Omega = \{(x,y): -\pi < x < \pi, -1 < y < 1\}$ the mixed type operator

$$Lu = -k(y)u_{xx} - u_{yy} + a(y)u_x + c(y)u \tag{1}$$

is considered, originally defined in the set $C^{\infty}_{0,\pi}(\Omega)$, consisting of infinitely differentiable functions, satisfying the conditions

$$u(-\pi, y) = u(\pi, y), \quad u_x(-\pi, y) = u_x(\pi, y) \tag{2}$$

$$u(x, -1) = u(x,\, 1) = 0 \tag{3}$$

where $k(y)$is a piecewise continuous function in the segment [-1,1] and $yk(y) > 0$ for $y \neq 0$, $k(0)$=0 (as y=0).

The spectral properties depending on the behavior of factors are studied for this operator, i.e. the following questions are investigated:

1. Existence of the resolvent of the operator (1)

2. Boundedness and compactness of the operators $r(y)D_x(L + \lambda E)^{-1}$, $r(y)D_y(L + \lambda E)^{-1}$, $r(y)D_x^{\alpha}(L + \lambda E)^{-1}$ ($0 \leqslant \alpha < 1$), where $r(y)$ is a continuous function in the segment [-1, 1];

3. Schmidt eigenvalues (s-values);

4. Eigenvalues of the operator L^{-1}.

For the first time the semiperiodical Dirichlet problem for the equation of mixed type

$$Lu\text{=}f\text{ , for } f(x\text{,}y) \in L_2(\Omega),$$

where Lu is defined by the equality (1) with boundary conditions (2)-(3) has been considered in the work of T.S. Kalmenov [29]. The solvability of this problem has been proved there for arbitrary right hand $f(x,y) \in L_2(\Omega)$ (and the solution $u(x,y) \in W_2^1(\Omega)$) under the following conditions for the coefficients:

$$i) \quad |a(y)| \geqslant \delta_0 > 0, \quad c(y) \geqslant \delta > 0.$$

In particular the question concerning the existence of the resolvent of the operator L has been solved in the work of T.S. Kalmenov, whereas the questions 2)-4) are still open.

The results of the investigations of all these questions are cited in the following theorems.

Theorem 0.1. *Let the continuous functions $a(y)$ and $c(y)$ in the segment [-1,1] fulfill the condition*

i) $|a(y)| \geqslant \delta_0 > 0, c(y) \geqslant \delta > 0$. *Then:*

a) the operator $(L + \lambda E)$ is continuously invertible for $\lambda > 0$;

b) the operators $r(y)D_x(L+\lambda E)^{-1}, r(y)D_y(L+\lambda E)^{-1}$ are bounded in $L_2(\Omega)$. Here $D_x = \frac{\partial}{\partial x}, D_y = \frac{\partial}{\partial y}$; $r(y)$ is a continuous function in the segment [-1,1].

c) the operator $r(y)D_x^\alpha(L + \lambda E)^{-1}$ is completely continuous if $0 \leqslant \alpha < 1$, where D_x^α is understood as a fractional derivative of order α.

The definition of the fractional derivative and also a few other necessary notations and definitions are cited below.

Let the function $u(x, y) \in L_2(\Omega)$. Then the following decomposition holds

$$u(x, y) = \sum_{n=-\infty}^{\infty} u_n(y)e^{inx}.$$

Definition 0.1. *We define the fractional derivative $D_x^\alpha u$ of order $\alpha \geqslant 0$ with respect to x of a function $u(x,y)$ as the expression [30]:*

$$D_x^\alpha u = e^{\frac{i\pi\alpha}{2}} \sum_{n=0}^{\infty} n^\alpha u_n(y)e^{inx} + e^{-\frac{i\pi\alpha}{2}} \sum_{n=-\infty}^{-1} |n|^\alpha u_n(y)e^{inx}$$

Here the equality is understood in the metric of $L_2(\Omega)$.

Definition 0.2. *Let A be a completely continuous operator. Then the eigenvalues of the operator $(A^*A)^{1/2}$ are called s-values of the operator A (Schmidt eigenvalues).*

The nonzero s-values be numbered according to decreasing magnitude and observing their multiplicities and so

$$s_k(A) = \lambda_k((A^*A)^{1/2}),\ k = 1, 2, ...$$

Let A be a linear operator in the Hilbert space H then by $\sigma(A)$ we denote the spectrum of the operator .

Theorem 0.2. *Let the conditions of Theorem 0.1 be fulfilled. Then the following estimate holds for the Schmidt eigenvalues*

$$\frac{1}{k}c^{-1} \leqslant s_k \leqslant c\frac{1}{k^{1/2}}, k = 1, 2, ...,$$

where $c > 0$ does not depend on k.

Theorem 0.3. *Let the conditions of Theorem 0.1 be fulfilled. Then*

a) the spectrum $\sigma(L^{-1})$ is a discrete set;

b) the estimate

$$|\lambda_k| \leqslant c\frac{1}{k^{1/2}}, k = 1, 2, ...,$$

holds for any non-zero $\lambda_k \in \sigma(L^{-1})$, where $c > 0$ does not depend on k.

Let us remind that σ_p denotes the set of completely continuous operators such that

$$\|A\|_{\sigma_p}^p = \sum_{k=1}^{\infty} s_k^p(A) < \infty,$$

where $s_k(A)$ are the Schmidt eigenvalues of the completely continuous operator A. It is clear that $s_k \to 0$. The exponent p shows the degree of deviation of the operator from a finite-dimensional operator. The less p, the faster the numbers s_k tend to zero and the better the operator will be approximated using finite-dimensional operators.

Considerable quantity of works are devoted to the question of belonging of the resolvent of various differential operators to the classes σ_p. However, this question for operators of mixed type remains poorly investigated for long time. From theorems 0.1-0.2 the assertions immediately follows that the resolvent of the operator (1) belongs to the class σ_p., i.e. the following theorem holds.

Theorem 0.4. *Let the condition i) be fulfilled. Then the resolvent of the operator Lbelongs to the class σ_p if $p > 2$.*

The methods used for studying spectral questions of the operator (1) lead us to generalize the obtained results to the case of two-dimensional coefficients of the lower terms of a mixed type operator, i. e. for the operator

$$Lu = -k(y)u_{xx} - u_{yy} + a(x, y)u_x + c(x, y)u \tag{4}$$

in the domain $\Omega = \{(x,\, y) :\, -\pi < x < \pi,\, -1 < y < 1\}$. Under the same conditions for the coefficient $k(y)$ like the operator (1) and under the boundary conditions (2)-(3) the following questions are studied:

1) existence of the resolvent of the operator L;

2) estimates of Schmidt eigenvalues (s-values) of the operator L^{-1};

3) estimates of eigenvalues of the operator L^{-1}.

All just listed questions for the operator (4) are considered in section 1.2. The results are formulated in the following theorems.

Theorem 0.5. *Let $a(x, y)$ and $c(x, y)$ be continuous functions in $\overline{\Omega}$, satisfying the condition:*

i) $|a(x, y)| \geqslant \delta_0 > 0$, $c(x, y) \geqslant \delta > 0$, where δ_0 is a sufficiently great number.

Then the operator $(L+\lambda E)$ is continuously invertible for a sufficiently large

$\lambda > 0$.

Theorem 0.6. *Let the conditions of Theorem 0.5 be fulfilled. Then the following estimate holds for Schmidt eigenvalues*

$$c^{-1}\frac{1}{k} \leqslant s_k \leqslant c\frac{1}{k^{1/2}}$$

where $c > 0$ is a constant not depending on k.

Theorem 0.7. *Let the conditions of Theorem 0.5 be fulfilled. Then*

a) the spectrum $\sigma((L + \lambda E)^{-1})$ is a discrete set;

b) for any nonzero $\lambda_k \in \sigma((L + \lambda E)^{-1})$

$$|\lambda_k| \leqslant c\frac{1}{k^{1/2}}, k = 1, 2, 3, ...,$$

where $c > 0$ is a constant not depending on k.

The theorem given bellow is similar to the aforecited Theorem 0.4.

Theorem 0.8. *Let the condition i) be fulfilled. Then the resolvent of the operator L belongs to the class σ_p if $p > 2$.*

In the section 1.3 of the first chapter the coercitive solvability of the Dirichlet problem for a class of mixed type equations is studied.

The solvability of the Dirichlet semiperiodical problem for mixed type equations, determined in a rectangle or on an infinity bar, when the factors of the lower order derivatives are bounded away zero, is obtained in the works [29] and [31].

However, we note that in all of these investigations the case, when the coefficients of the lower order derivatives turn to zero in some points of the considered domain, practically remains unaddressed. The results of studying this question are cited in the following theorems.

The operator L defined by the equality (1) and the boundary conditions (2)–(3) are considered in the rectangle $\Omega = \{(x, y) : -\pi < x < \pi, \ \ -1 < y < 1\}$

with piecewise continuous and bounded coefficients in the segment [-1,1].

Theorem 0.9. *Let the conditions be fulfilled:*

a) $a(y)$, $c(y)$, $k(y)$ *are piecewise continuous functions in [-1,1];*

$c(y) \geqslant \delta > 0$, $a(y)$ *does not change its sign* ($a(y) \geqslant 0$ *or* $a(y) \leqslant 0$)*;*

b) the following conditions are fulfilled for some $m > 0$*:*

$$\lim_{|t|\to\infty} \sup_{y\in[-1,\,1]} \frac{t^2}{[K_t^*(y)]^2} < c,$$

where $c > 0$ *is a fixed number and* $K_t^*(y)$ *is a averaging function defined by the equality (introduced by M. Otelbaev (see [52]))*

$$K_t^*(y) = \inf_{d>0}\{d^{-1}; d^{-1} \geqslant \int\limits_{y-\frac{d}{2}}^{y+\frac{d}{2}} K_t(\tau)d\tau\}$$

where $K_t(\tau) = t^2(m|a(\tau)| \, - \, |k(\tau)|) + c(\tau) > 0, \ \tau \in [-1, \quad 1], \ -\infty < t < \infty.$

Then the operator $L + \lambda E$ *is continuously invertible for a sufficiently large* $\lambda > 0$.

Theorem 0.10. *Let the conditions* $a) - b)$ *be fulfilled. Then there exist a positive sequence of eigenvalues of the operator (1) and the estimates*

$$c^{-1}k^2 \leqslant \lambda_k \leqslant ck^2, \qquad k = 1, 2, ...,$$

hold for them, where c is a constant.

We note that the conditions $a) - b)$ in the theorems 0.9-0.10 cover the conditions announced in [29]. Let us give an example of coefficients which do not satisfy those conditions.

Example. $a(y) = |sign(sin(1000\pi y)|$, $k(y)$=sign(sin(1000πy). The function $c(y)$ can be taken as a positive constant. It is clear that similar functions do not satisfy the condition i). Besides, one can see that the conditions for the coefficient $k(y)$ can be mitigated(softened).

In the second chapter coercitive solvability, smoothness (separability of the corresponding operator) and approximate qualities of solutions of a class of mixed type equation in unbounded domains are studied.

The first section is devoted to the question of separability of a mixed type operator in an unbounded domain.

The questions concerned with the smoothness of solutions of differential equations inseparably linked with the questions concerned with the separability of differential operators have been attracted relentless practical and theoretical interest of mathematicians for many years.

In case of bounded domains and smooth factors the questions concerned with smoothness have been well studied with the help of methods which became standard and are announced in the monographs: J.-L.Lions and E.Madjenes [32], O.A.Ladyjenskaya and N.N.Uraltseva [33] and others. These questions can be found also in the works: P.I. Lizorkin and S.M. Nikolskii [34], M.I. Vishik V.V. Grushin [35], V.P. Glushko [36], S.N. Krujkov [37] and other authors.

But apparently, V. Everitt and M. Girtz /[38-40]/ were the first who begun to study systematically the question concerned with smoothness (separability) of solutions and for all that in unbounded domains. Particularly, the definition of the fundamental problem of differential operators' separability belongs to both.

Later on their results were improved in the works of M.Otelbaev /[41-43]/ and K.H. Boimatov /[44-45]/. Moreover they suggest another method for investigation of the separability problem.

All aforesaid results have been obtained for equations (operators) of elliptic type. The studying of the problem of separability has been recently begun from the works: M.B. Muratbekov [46], T.S. Kalmenov M.B. Muratbekov [28].

The operator

$$Lu = -k(y)u_{xx} - u_{yy} + a(y)u_x + c(y)u \tag{5}$$

is studied in this thesis originally defined in $C^{\infty}_{0,\pi}(\Omega)$, where $k(y)$ is a sectionally continuous function in R=(-∞, +∞) and $yk(y) > 0$ for $y \neq 0$, $k(0)$=0 (as y=0). Here $C^{\infty}_{0,\pi}(\Omega)$ is the set of infinitely differentiable functions, satisfying the conditions

$$u(-\pi, y) = u(\pi, y), \;\; u_x(-\pi, y) = u_x(\pi, y)$$

and finite as functions of the y variable, where

$$\Omega = \{(x, y) : -\pi < x < \pi, \;\; -\infty < y < \infty\}$$

Let us introduce the definition of separability of the operator L which is needed for the statement of the basic theorem of this chapter. But we note before that $D(L)$ denotes the domain of definition of the closed operator (5), which will be again denoted by L.

Definition 0.3. *The operator L is called separable if for all functions $u(x, y) \in D(L)$ the estimate*

$$\|-k(y)u_{xx} - u_{yy}\|_2 + \|a(y)u_x\|_2 + \|c(y)u\|_2 \leqslant C\left(\|Lu\|_2 + \|u\|_2\right)$$

holds, where $C > 0$ is a constant not depending on $u(x, y)$.

Theorem 0.11. *Let the conditions be fulfilled:*

i) $a(y)$ and $c(y)$ are piecewise continuous functions in any compact set in $\mathbb{R}$ and $|a(y)| \geq \delta_0 > 0$;

ii) $c(y) \leqslant c_0 a^2(y)$ for any $y \in \mathbb{R}$, where $c_0 > 0$ is a fixed number;

iii) $\mu_1 = \sup\limits_{|y-t|\leqslant 1} \frac{a(y)}{a(t)} < \infty$, $\mu_2 = \sup\limits_{|y-t|\leqslant 1} \frac{c(y)}{c(t)} < \infty$

Then the quantity

$$\gamma = \sup_{u\in D(L)} \frac{||-k(y)u_{xx} - u_{yy}||_2}{||Lu||_2 + ||u||_2}. \tag{6}$$

is finite if and only if

$$\inf_{y\in R} c(y) = c > -\infty. \tag{$*$}$$

Therefore, it is proved that the condition of boundedness from bellow of the coefficient $c(y)$ is significant for the separability of the mixed type operator.

In the second section the coercitive solvability and the approximate qualities of solutions of the semiperiodical Dirichlet problem for a class of mixed type equations are investigated in an unbounded domain. The separability (smoothness of solutions) is also studied for a mixed type operator corresponding to the considered equation.

Suppose that the conditions $i) - iii)$ are fulfilled for the operator (5). Then the following theorems hold.

Theorem 0.12. *Let the conditions $i)$ be fulfilled. Then the operator $L+\lambda E$ is continuously invertible for substantially large $\lambda > 0$.*

Theorem 0.13. *Let the conditions $i)$ be fulfilled. Then the resolvent of the operator L is compact if and only if for any $w > 0$*

$$\lim_{|y|\to\infty} \int\limits_{y}^{y+w} (c(t))\, dt = \infty. \tag{7}$$

Theorem 0.14. *Let the conditions $i) - iii)$ be fulfilled. Then the operator L is separable.*

Definition 0.5. *Let A be a completely continuous operator. Then the eigenvalues of the operator $(A^*A)^{1/2}$ are called s - values of the operator A (Schmidt eigenvalues).*

The nonzero s - values of the operator L^{-1} are arranged as a sequence according to decreasing magnitude and observing their multiplicities, so $s_k(L^{-1}) = \lambda_k((L^{-1})^* L^{-1}), \quad k = 1, 2, ...$

We introduce the counting function $N(\lambda) = \sum\limits_{s_k > \lambda} 1$ of those s_k greater than $\lambda > 0$.

Theorem 0.15. *Let the conditions $i) - iii)$ be fulfilled. Then the following estimate holds:*

$$C^{-1} \sum_{n=-\infty}^{\infty} \lambda^{-1/2} mes \left(y \in R : \left| n^2 + ina(y) + c(y) \right| \leq C^{-1} \lambda^{-1/2} \right) \leq N(\lambda) \leq$$

$$\leq C \sum_{n=-\infty}^{\infty} \lambda^{-1} mes \left(y \in R : |ina(y) + c(y)| \leq C \lambda^{-1} \right),$$

where the constant $C = C(\mu_1, \mu_2), \quad i^2 = -1.$