Appendix D

Expected value of the impedance phase

Estimation of the expected phase value by determining the probability function distribution. By using the polar transformation the tensor element Z has the expression:
$x = |Z|, \quad \Re(Z) = x\cos(\phi), \;\; \Im(Z) = x\sin(\phi)$
$Re(Z)$ and $Im(Z)$ will each be considered as normal distributed random variables (r.v.'s); then the variables x and ϕ are r.v.'s too.
We define $\quad a = \Re(\hat{Z}) \quad$ and $\quad b = \Im(\hat{Z})$
as the expected values of Z, i.e., the mean values assumed to be the measured data following a Gaussian distribution (a and b). The real and imaginary parts a and b have equal and uncorrelated Gaussian errors $s = \Delta Z$.
The two r.v. x, ϕ density function (df) will take the form :

$$g(x,\phi) = \frac{x}{2\pi s^2}\exp(-\frac{1}{2s^2}\left((x\cos(\phi)-a)^2 + (x\sin(\phi)-b)^2\right)$$

where the phase ϕ density function (df) is defined by :

$$g_\phi(\phi) = \int\limits_0^\infty g(x,\phi)dx$$

Solving this integration results in:

$$g_\phi(\phi) = \frac{e^{-\frac{a^2+b^2}{2s^2}}}{2\pi} + \frac{e^{-\frac{(b\cos\phi - a\sin\phi)^2}{2s^2}} Erf \;\; \frac{a\cos\phi+b\sin\phi}{\sqrt{2}s} \;\; (a\cos\phi+b\sin\phi)}{2\sqrt{(2\pi)}s} + \frac{e^{-\frac{(b\cos\phi - a\sin\phi)^2}{2s^2}}(a\cos\phi+b\sin\phi)}{2\sqrt{(2\pi)}s}$$

The final expression for the probability function $F(\bar{\phi})$ of the phase is:
$\int\limits_0^{\bar{\phi}} g_\phi(\phi)\, d\phi =$

$$= \frac{\bar{\phi}\; e^{-\frac{a^2+b^2}{2s^2}}}{2\pi} + \frac{Erf\left(\frac{b}{\sqrt{2}s}\right) - Erf\left(\frac{b\cos\bar{\phi}-a\sin\bar{\phi}}{\sqrt{2}s}\right)}{4}$$

$$+\int_0^{\bar{\phi}} \left[\frac{Erf\left(\frac{a\cos\phi + b\sin\phi}{\sqrt{2}s}\right)(a\cos\phi + b\sin\phi)\ e^{-\frac{(b\cos\phi - a\sin\phi)^2}{2s^2}}}{2\sqrt{(2\pi)}s} \right] d\phi$$

$for\ 0 \leq \bar{\phi} < 2\pi$.

From the former expression, the expected value $E(\phi) = \int_0^{2\pi} (\phi\ g_\phi(\phi))\ d\phi$ can be obtained:

$$E(\phi) = \int_0^{2\pi} \phi \frac{e^{-\frac{a^2+b^2}{2s^2}}}{2\pi} d\phi + \int_0^{2\pi} \phi \frac{(a\cos\phi + b\sin\phi)}{2s\sqrt{(2\pi)}} e^{\frac{-(b\cos\phi - a\sin\phi)^2}{2s^2}} d\phi$$

$$+ \int_0^{2\pi} \phi \frac{Erf\left(\frac{a\cos\phi + b\sin\phi}{\sqrt{2}s}\right)(a\cos\phi + b\sin\phi)}{2s\sqrt{(2\pi)}} e^{\frac{-(b\cos\phi - a\sin\phi)^2}{2s^2}} d\phi$$

By using a trapezoidal numerical integration, the expected value $E(\phi)$ can be solved with high accuracy.

For significant errors, the expected value of the phase shifts down with respect to the measured data, which is in accordance with the up-shifting of the expected value of the apparent resistivity proportional to the impedance element error (from the χ^2 statistical analysis on MT apparent resistivities, appendix C).