DISSERTATION

The Surgery and Level-Set Approaches to Mean Curvature Flow

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To my mother Bianca

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Erklärung

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John Head

Abstract

Huisken and Sinestrari [HS3] have recently developed a surgery-based approach to extending smooth mean curvature flow beyond the singular time in the two-convex setting. According to their construction one removes, by hand, the regions of large curvature from the hypersurface before a singularity forms. They showed that the procedure can be controlled uniformly across all surgeries by a set of parameters depending just on the initial data.

Within this context we discuss estimates on certain L^p-norms of the mean curvature explicitly in terms of the surgery parameters. Our approach leads to new bounds on the required number of surgeries, and we prove as a corollary that the flow with surgeries converges (in an appropriate limit of the surgery parameters) to the well-known weak solution of the level-set flow introduced in [CGG, ES1].

Zusammenfassung

Huisken und Sinestrari [HS3] entwickelten vor Kurzem einen auf Chirurgie basierenden Ansatz, um den glatten Fluss entlang der mittleren Krümmung, im “zwei-konvexen” Fall, über die singuläre Zeit hinaus fortzusetzen. Gemäß ihrer Konstruktion entfernt man von Hand die Gebiete großer Krümmung aus der Hyperfläche, bevor sich eine Singularität bildet. Sie zeigten, dass sich dieser Vorgang gleichmäßig über alle Chirurgien durch eine Menge von Parametern kontrollieren lässt, welche nur von den Anfangsdaten abhängen.

In diesem Zusammenhang diskutieren wir Abschätzungen für gewisse L^p-Normen der mittleren Krümmung, in expliziter Abhängigkeit von den Chirurgie-Parametern. Unser Ansatz führt zu neuen Abschätzungen über die Anzahl der benötigten Chirurgien. Als ein Corollar beweisen wir, dass der Fluss mit Chirurgien (durch einen geeigneten Grenzübergang in den Chirurgie-Parametern) gegen die bekannte schwache Lösung des Niveauflächenflusses, welcher in [CGG, ES1] eingeführt wurde, konvergiert.

Contents

1 Introduction

We begin with an informal discussion setting up and motivating the subject of this work. The following background material is organised in roughly chronological order with the exception that any mention of weak solutions has been postponed to the end of the section. Our main results are collected in Section 1.2 and we conclude the chapter with a brief description of the layout of the thesis in Section 1.3.

1.1 Background

Mean curvature flow is a geometric deformation process for hypersurfaces of Euclidean space. It arises naturally in differential geometry as the steepest descent flow of the area functional with respect to the L^2-norm on the surface. The problem was introduced more than half a century ago in the context of materials science by Mullins [M] and continues to generate widespread mathematical interest.

Explicitly, given a smooth hypersurface immersion $F_0 : \mathcal{M}^n \to \mathbb{R}^{n+1}$, the solution of mean curvature flow starting from $\mathcal{M}_0 \equiv F_0(\mathcal{M}^n)$ is the one-parameter family $F : \mathcal{M}^n \times [0,T) \to \mathbb{R}^{n+1}$ satisfying

$$\begin{cases} \dfrac{\partial F}{\partial t}(p,t) = -H(p,t)\nu(p,t), \quad p \in \mathcal{M}^n,\ t \geq 0, \\ \quad F(\cdot,0) = F_0, \end{cases} \tag{MCF}$$

where H and ν denote the mean curvature and outward-pointing unit normal respectively. According to our choice of signs the right-hand side is the mean curvature vector $\vec{H}$ and the mean curvature of the round sphere is positive. We write $M_t \equiv F(\cdot,t)(\mathcal{M}^n)$.

(MCF) is a quasi-linear, weakly parabolic system which inherits many properties from and indeed formally resembles the standard heat equation.[1] A conspicuous feature of mean curvature flow is that all relevant curvature quantities associated with $\mathcal{M}_t$ are governed by reaction-diffusion systems of the form

$$\frac{\partial}{\partial t} h^i_j = \Delta_{g(t)} h^i_j + |A|^2 h^i_j.$$

Here the h^i_j are the components of the Weingarten map and $|A|^2$ is the squared norm of the second fundamental form. If, for example, $\mathcal{M}_0$ is closed, the presence of the cubic reaction

[1] A standard calculation reveals that $-H\nu = \Delta_{g(t)} F$, where $\Delta_{g(t)}$ represents the Laplace-Beltrami operator with respect to the induced metric $g(t)$ on the hypersurface at time t.

term on the right-hand side guarantees singularity formation in finite time, motivating a detailed analysis of the geometric structure of the surface in high-curvature regions.

At a qualitative level, mean curvature flow shares many fundamental properties with Hamilton's Ricci flow of metrics on a Riemannian manifold. Ricci flow is likewise a non-linear parabolic problem with corresponding reaction-diffusion systems controlling the evolution of relevant intrinsic curvature quantities and analogous finite-time singularity results. However, it should be pointed that the two flows of course differ in their respective quantitative structures.

A classical PDE approach to the study of mean curvature flow was initiated by Huisken in [H1], where he showed that any convex hypersurface of dimension at least two must contract smoothly to a point in finite time and in an asymptotically round fashion.[2]

Gage and Hamilton [GH] obtained an analogous description of the behaviour of convex curves in the plane. Shortly thereafter, Grayson [G1] proved that any embedded closed curve must become convex before the singular time; that is to say, the global "round point" singularity is the inevitable fate of every embedded closed curve. However, this remarkable classification result is peculiar to one-dimensional curves, compare [G2, A1, E1, Se2].

As we shall now proceed to describe, this behaviour is in fact just one manifestation of a more general diffusion property, to wit: (MCF) induces an asymptotic separation of variables in high curvature regions, where the surface therefore inherits a self-similar structure determined by an elliptic system.

Classification of Mean-Convex Singularities. The structure of singularities can often be probed using rescaling techniques, a now-standard approach in the theory of partial differential equations. In the context of mean curvature flow, these ideas were pioneered by Huisken in [H2, H3] and by Huisken and Sinestrari in [HS1, HS2]. Their work combines to produce a classification theorem in the class of mean-convex surfaces. Roughly speaking, the only singularity profiles which can arise under mean curvature flow and after appropriate parabolic rescaling look asymptotically like

$$\mathbb{S}^{n-k}_t \times \mathbb{R}^k, \quad \Gamma^1_t \times \mathbb{R}^{n-1} \quad \text{or} \quad \Upsilon^{n-k}_t \times \mathbb{R}^k, \quad 0 \le k \le n, \tag{S}$$

where $\mathbb{S}^{n-k}_t$ denotes the $(n-k)$-dimensional shrinking sphere, Γ^1_t is one of the homothetically shrinking immersed curves in $\mathbb{R}^2$ introduced by Mullins [M] (see also Abresch and Langer [AL])[3] and Υ^{n-k}_t represents an $(n-k)$-dimensional convex translating soliton.

Historically one distinguishes between singularities based on the associated rate of curvature blow-up. A singularity is by definition "type I" or "fast" if there exists a finite constant $C > 0$ such that

$$\sup_{\mathcal{M}_t} |A|^2 (T-t) \le C.$$

It is otherwise labelled "type II" or "slow".

[2] This result has an antecedent in Ricci flow theory. Hamilton [Ha1] showed that any metric on a closed three-manifold of positive Ricci curvature must converge after rescaling to one of constant curvature.

[3] This is an infinite family of curves, each of which has self-intersections with the exception of the circle.

It is well-known (see [H1]) that $\sup |A|^2(T-t) \geq 1/2$; the pressing question is therefore whether there exist singularities with a corresponding upper bound $C > 1/2$. The archetypal example of a type I singularity is the shrinking sphere $\mathbb{S}^n_t$, the rescaling limit produced by the evolution of a convex surface. This can be seen as motivation for the preceding definition since on the round sphere we have

$$H(t) = \left(\frac{n}{2(T-t)}\right)^{1/2}.$$

The asymptotic profile of type I singularities has long been understood. In [H2], Huisken introduced the now well-known monotonicity formula for mean curvature flow: the monotone quantity is the area functional weighted with the backward heat kernel in $\mathbb{R}^{n+1}$. This can be thought of as a parabolic analogue of the monotonicity formula for minimal surfaces; similar ideas feature heavily in the theory of semi-linear heat equations and harmonic map heat flow. Huisken used the monotonicity formula to prove that type I singularities approach homothetically shrinking solutions of mean curvature flow.

There is a plethora of known self-similarly shrinking examples, but they remain beyond classification in general. Curvature restrictions can be imposed on the initial data to limit the range of admissible singular profiles: the only compact, mean-convex solution is the shrinking sphere, compare [H2]. Huisken [H3] completed the mean-convex, type I classification by proving that the only non-compact possibilities are the products

$$\mathbb{S}^{n-k}_t \times \mathbb{R}^k \quad \text{and} \quad \Gamma^1_t \times \mathbb{R}^{n-1},$$

where Γ^1_t is one of the homothetically shrinking immersed curves in $\mathbb{R}^2$ introduced by Mullins [M].

Examples of type II singularities are also prevalent, compare [AAG, A2, AV, Wa]. Huisken and Sinestrari [HS1, HS2] showed that, in the mean-convex case, these behave asymptotically like self-similarly translating solutions of mean curvature flow.

The cornerstone in the analysis of type II singularities takes the form of a one-sided pinching estimate on the elementary symmetric polynomials of the principal curvatures.[4] It dictates that the surface must become (weakly) convex near a singularity.[5] White [W2, W3] has developed an analogous theory for weak solutions of mean curvature flow (these are discussed below) using entirely different techniques.

The rescaling procedure can be adapted to the type II setting so as to produce an "eternal limit" (which by definition exists for all $t \in (-\infty, \infty)$). Furthermore the maximum of the curvature is attained on the surface. Hamilton's Harnack inequality for mean curvature flow then implies that the flow moves isometrically via translations, see [Ha2]. That is, any such limit must be a convex, eternal, translating solution Υ^n_t; these are known in the literature as "translating solitons". Moreover, if the rescaled surfaces are not strictly

[4] It is apropos to remark that analogous curvature pinching results, the so-called Hamilton-Ivey estimates, play a decisive role in the analysis of the Ricci flow (see [Ha3]).

[5] Angenent [A1] constructed a self-shrinking torus with mixed curvature which does not become convex.

convex, then as in the type I case they can be decomposed into products

$$\Upsilon_t^{n-k} \times \mathbb{R}^k$$

of flat factors with strictly convex lower-dimensional translating solitons Υ_t^{n-k}.

The classification of translating solitons is another open problem. The so-called "grim reaper" is well-known to be the only one-dimensional example.[6] In addition, there exist unique rotationally symmetric translating solitons in all higher dimensions, compare [AW, CSS].

Wang [Wa] showed that any translating soliton in $\mathbb{R}^3$ must be rotationally symmetric. A perhaps somewhat suprising result is that there exist translating solitons without rotational symmetry for all $n > 2$, see [Wa].

We refer finally to the work of Angenent and Velazquez [AV] for a detailed analysis of the fine asymptotic structure of type II singularities.

Two-Convexity and the Flow with Surgeries. The onset of local singularities precludes even a formal definition of the subsequent evolution in the language of differential geometry. For topological applications, this is of course fatal. Huisken and Sinestrari [HS3] have recently succeeded, however, in extending the classical flow in a topologically controlled way using a surgery-based approach.

The idea is to "manually" remove the regions of large curvature from the surface before the singular time, and continue the smooth evolution using the modified surface. The goal is to repeat the process until a global description of the surface becomes available. Of course, this approach does not give rise to an exact solution of the original initial value problem (MCF) since it introduces a small "error" in the surgery regions at each surgery time.

In [HS3] Huisken and Sinestrari studied *two-convex* surfaces, at each point on which the sum of any two of the principal curvatures is non-negative. Motivation for the curvature condition can be found in the classification described above: the only two-convex solutions in (S) are the shrinking sphere $\mathbb{S}_t^n$, the shrinking cylinder $\mathbb{S}_t^{n-1} \times \mathbb{R}$ and the translating soliton Υ_t^n.[7] Each of these singular profiles is compatible with the surgery algorithm defined in [HS3].[8]

We caution, however, that the foregoing heuristics provide little more than motivation; the results discussed thus far amount to a "zero-order" description of the singular regions and fall short of facilitating a smooth continuation of the evolution. The work of Huisken and Sinestrari [HS3] is in fact independent of the classification of mean-convex singularities and features just one application of rescaling techniques (in the proof of the "neck detection lemma").

They instead used direct *a priori* estimates adapted to the setting of two-convex surfaces to obtain a detailed higher-order description of all possible singularities in dimensions

[6] The grim reaper solution can be written explicitly in closed form: $x = \log \cos y + t$.

[7] Two-convexity is preserved both by the smooth evolution and by the rescaling procedure.

[8] The surgery procedure employed in [HS3] is applicable to singular profiles with precisely one flat direction. For example, the three-convex solution $\mathbb{S}_t^{n-2} \times \mathbb{R}^2$ does not conform to this structure.

$n \geq 3$. Their results make precise the intuitive picture that unless the surface is uniformly convex, any high-curvature region must contain a neck – that is, a piece of the surface which can be represented (up to a homothety) as a graph over a cylinder with small C^k-norm for a suitable $k \geq 1$.

Huisken and Sinestrari then set forth a surgery algorithm for mean curvature flow, analogous to the program introduced by Hamilton in [Ha4] in the context of Ricci flow, according to which the smooth evolution is interrupted shortly before the singular time by a surgery procedure which replaces each neck with two regions diffeomorphic to discs.[9] Any connected components which are known to be diffeomorphic to $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{S}^1$ are also discarded at the surgery time. The smooth flow is then restarted and the whole process is repeated. Huisken and Sinestrari proved that only finitely many surgeries are required, leading to the complete classification of two-convex hypersurfaces in $\mathbb{R}^{n+1}$ $(n \geq 3)$.

Weak Solutions. The study of mean curvature flow from a purely theoretical viewpoint began in earnest with the work of Brakke [B]. He cast the problem in the language of geometric measure theory and studied a weak mean curvature evolution of integral varifolds.

There are now several treatments of weak solutions available in the literature; we note in particular the "phase field" approach developed in [BK, ESS, I1] and based on asymptotics of the scaled Allen-Cahn equation, and the well-known level-set approach [OS, Se2, CGG, ES1, AAG] which we focus on here.[10]

Rather than defining the evolving hypersurfaces as smooth immersions, one may alternatively interpret the surfaces $\mathcal{M}_t$ as level-sets of an appropriate scalar function on $\mathbb{R}^{n+1}$. The idea of representing surfaces as level-sets in the context of differential geometry can be traced back to work of Ohta, Jasnow and Kawasaki [OJK] and was used by Sethian [Se1] and by Osher and Sethian [OS] for numerical analysis of moving interfaces.

We again restrict our attention to the mean-convex setting, allowing us to consider the scalar function $u : \Omega \subset \mathbb{R}^{n+1} \to \mathbb{R}$ which assigns to each point $x \in \Omega$ the time t when $x \in \mathcal{M}_t$. This ansatz reduces (MCF) to the degenerate elliptic boundary value problem

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = -\dfrac{1}{|Du|}, \\ u\Big|_{\partial\Omega} = 0. \end{cases} \tag{$\star$}$$

In order to deal with level-sets which are not regular, it is necessary to define a weak solution of ($\star$). This was accomplished in a more general setting by Evans and Spruck [ES1] and independently by Chen, Giga and Goto [CGG] using viscosity techniques; in the mean-convex case one can alternatively apply variational methods, compare [HI, MS, S].

The weak solution of the level-set flow was reconciled with Brakke's varifold solution by Ilmanen [I2]; see also [ES3, MS]. The regularity theory developed by Brakke has been built

[9] This can thought of as an artificial "fast-forwarding" of the evolution.
[10] See also [A3, AG] for additional proposals in the context of curve shortening flow.

upon in the mean-convex setting by White [W1]. In [ESS], the phase field and level-set approaches were unified (see also [I1]).

Level-set methods have played an important role in the development of various other geometric hypersurface flows. We highlight the fundamental work of Huisken and Ilmanen [HI] on the weak level-set formulation of inverse mean curvature flow, and the theory of weak solutions for non-linear mean curvature flow developed by Schulze [S].

Surgery Parameters. Mean curvature flow with surgeries does not constitute a weak solution in the traditional sense since it relies on a non-canonical modification of the surface at each surgery time. Given a choice of surgery procedure – that is, given a precise choice of "cutting" and "gluing" – it is then controlled by a set of parameters

$$1 << H_0 < H_1 < H_2 < H_3 < \infty$$

which determine when and where surgery is performed.

When the curvature exceeds a certain scale $H_0 = H_0(\mathcal{M}_0)$, the second fundamental form and its derivatives fit one of the profiles described above. The smooth flow is stopped when the curvature reaches $H_3 >> H_0$, and surgery is performed away from the point of maximum curvature at a smaller scale $H_1 = \xi_1 H_3$ ($\xi_1 = \xi_1(\mathcal{M}_0) < 1$) such that the maximum of the curvature after surgery drops by a fixed amount to $H_2 = \xi_2 H_3$ ($\xi_2 = \xi_2(\mathcal{M}_0) < 1$, $\xi_1 < \xi_2$).

The starting point for our work in this thesis is the observation made by Huisken and Sinestrari in [HS3] that, for a fixed choice of H_0, ξ_1, ξ_2, the remaining parameters H_1, H_2, H_3 are not uniquely determined; they can in fact be made arbitrarily large.[11]

This prompts us to consider an increasing sequence of parameters

$$\{H_k^i\}_{i\geq 1} = \{H_1^i, H_2^i, H_3^i\}_{i\geq 1},$$

corresponding to a whole sequence $\{\mathcal{M}_t^i\}_{i\geq 1}$ of mean curvature flows with surgeries, along which the surgery times grow and the necks cut out during surgery become increasingly narrow. Huisken and Sinestrari proved that, for each set of finite parameters, only finitely many surgeries are required. In the limit $H_k^i \to \infty$ (that is, $H_1^i, H_2^i, H_3^i \to \infty$ with fixed ratios ξ_1, ξ_2) infinitely many surgeries may be necessary.

This begs the question: how does the limiting object relate to the weak solution of mean curvature flow?

1.2 Main Results

In this section we collect rough statements of our main results. We assume throughout that the initial surface $\mathcal{M}_0$ is smooth, closed, two-convex and of dimension $n \geq 3$.

[11] See [ACK] for a discussion of related ideas in the context of rotationally symmetric Ricci flows.

Smooth Mean Curvature Flow. For a smooth, closed, two-convex surface evolving according to classical mean curvature flow we establish an upper bound – which behaves like $t^{-1/2}$ for small t – on the L^p-norms of the mean curvature for all $p < n-1$ (see Theorem 2.5).

Theorem 1.1 (Smooth L^p Estimate) *Let $\mathcal{M}_t$ be the smooth solution of mean curvature flow starting from a given smooth, closed, two-convex initial hypersurface $\mathcal{M}_0$ in $\mathbb{R}^{n+1}$. In addition set $p = n-1-\varepsilon$. Then there exist constants C_1 and C_2 depending just on $n, \varepsilon, \mathcal{M}_0$ such that*

$$\|H\|_{L^p(\mathcal{M}_t)} \le \frac{C_1}{\varepsilon^{1/2}} \exp\left(C_2 t\right)\left(\exp\left(2C_2 t\right) - 1\right)^{-\frac{1}{2}}$$

for all $\varepsilon > 0$ and for all $t > 0$ as long as the solution remains smooth.

This should be compared with the estimates obtained by Ecker and Huisken in [EH]. Our result relies heavily on the two-convex estimates in [HS3] which give rise to the critical exponent $p = n-1$ (see Remark 2.7).

Remark 1.2 (First Singular Set) Our methods have applications to the regularity theory developed in [E3], which deals with special case in which a smooth solution of mean curvature flow develops a singularity for the first time. Let $\mathcal{M}_t$, $0 \le t < t_0$, be the smooth solution of mean curvature flow starting from a given smooth, closed, two-convex initial hypersurface $\mathcal{M}_0$ in $\mathbb{R}^{n+1}$. Our L^p estimates combine with Ecker's theorem in [E3] to give

$$\dim(\operatorname{sing}(\mathcal{M}_{t_0})) \le 1,$$

where $\dim(\operatorname{sing}(\mathcal{M}_{t_0}))$ denotes the Hausdorff dimension of the singular set $\operatorname{sing}(\mathcal{M}_{t_0})$ at time t_0 (see Section 4.2 for precise definitions).

Mean Curvature Flow with Surgeries. Consider the solution of mean curvature flow with surgeries constructed by Huisken and Sinestrari. We show that the L^p-norms of the mean curvature are non-increasing under the surgery procedure defined in [HS3] in order to arrive at the following result.

Theorem 1.3 (L^p Estimate for Flow with Surgeries) *Let $\mathcal{M}_t$, $0 \le t \le t_0$, be the solution of mean curvature flow with surgeries starting from a given smooth, closed, two-convex initial hypersurface $\mathcal{M}_0$ in $\mathbb{R}^{n+1}$. Again let $p = n-1-\varepsilon$. Then there exists a constant $C = C(\varepsilon, t_0, \mathcal{M}_0)$ such that*

$$\begin{aligned} C\int_{\mathcal{M}_0} H^p\,d\mu \ge \int_{\mathcal{M}_{t_0}} H^p\,d\mu + p(p-1)\int_0^{t_0}\int_{\mathcal{M}_t} |\nabla H|^2 H^{p-2}\,d\mu\,dt \\ + \frac{\varepsilon}{2(n-1)}\int_0^{t_0}\int_{\mathcal{M}_t} H^{p+2}\,d\mu\,dt \end{aligned}$$

for all $\varepsilon > 0$.

These quantities are of course geometric and can therefore be used to bound the required number of surgeries using ideas developed by Huisken and Sinestrari in [HS3].

Corollary 1.4 (Number of Surgeries) *For any sufficiently small $\varepsilon > 0$, the number of surgeries N satisfies*

$$N \leq C(H_1^i)^{1+\varepsilon}, \tag{1.1}$$

where $C = C(\varepsilon, n, \mathcal{M}_0)$.

Corollary 1.4 is crucial for the following application.

Approximating Weak Solutions. As in Section 1.1 above, we consider an increasing sequence of surgery parameters $\{H_k^i\}_{i\geq 1} = \{H_1^i, H_2^i, H_3^i\}_{i\geq 1}$ and the corresponding solutions $\{\mathcal{M}_t^i\}_{i\geq 1}$ of mean curvature flow with surgeries. Let u_i be the level-set function determined by $\mathcal{M}_t^i$ (see Chapter 3 for precise definitions). We prove that the flow with surgeries in fact agrees with the unique weak solution u_L of the level-set flow in the limit $H_k^i \to \infty$ in the following sense.

Theorem 1.5 (Convergence to Weak Solution) *Let $\Omega \subset \mathbb{R}^{n+1}$ such that $\mathcal{M}_0 = \partial\Omega$ is a smooth, closed, two-convex initial hypersurface. Let u_L be the weak solution of the level-set flow on Ω, and denote by u_i the level-set functions representing the solutions $\mathcal{M}_t^i$ of mean curvature flow with surgeries starting from $\mathcal{M}_0$. Then*

$$u_i \longrightarrow u_L$$

uniformly on $\bar{\Omega}$ as $i \to \infty$.

A different version of this result has been independently obtained by Lauer [L]. Our proof relies on a combination of global barrier arguments and quantitative local techniques which are well-suited to the study of necks. We obtain estimates on the rate of convergence explicitly in terms of the surgery parameters, see Theorem 3.7.

Regularity Estimates for Weak Solution. Theorem 1.5 should be interpreted not only as a “consistency check” for the artificial surgery construction, but in addition as a new approximation scheme for the weak solution. In order to demonstrate the utility of our result, let us denote by Γ_t the level-sets of the weak solution and by $\mathbf{H}$ the generalised mean curvature of Γ_t. We show that at almost every time, $|\mathbf{H}|$ is bounded in L^p for all $p < n-1$. Moreover, we obtain convergence as $H_k^i \to \infty$ (see Section 4.1):

Corollary 1.6 (Higher Convergence) *We have*

$$\int_{\mathcal{M}_t^i} H^p \, d\mu \to \int_{\Gamma_t} |\mathbf{H}|^p \, d\mu$$

for all $p < n-1$ and for all $t \geq 0$.

1.3 Outline

The work in this thesis is laid out as follows.

Chapter 2 begins by setting up notation adapted to the context of two-convex surfaces; this will be maintained throughout. Our first object of study is classical mean curvature flow. We perform the central computation leading to Theorem 1.1, and point out various features and consequences. In particular we record an appropriate statement of the result which can be applied to the main theorem in [E3].

The chapter continues with a discussion of mean curvature flow with surgeries. While the definitions are often lengthy, the essential ingredients from [HS3] have been included in order to keep the presentation self-contained. The reader familiar with the surgery construction can omit the first half of Section 2.2 and proceed immediately to the estimates for necks. The ultimate objective of this analysis is Corollary 1.4, a new bound on the required number of surgeries.

Chapter 3 deals with an application of the integral estimates to the "surgery limit". We review the relevant theory from the weak solution literature and provide an appropriate formulation of the flow with surgeries in the language of level-sets. We then carefully explain our geometric barrier construction. The tools employed here are the familiar smooth avoidance principle, Brakke's "clearing out lemma" and Corollary 1.4. This approach leads to quantitative estimates on the rate of convergence.

Chapter 4 records some preliminary consequences of our main results. In Section 4.1 we refine our global approximation theorem and extract a convergence statement for the individual level-sets. The integral estimates in Chapter 2 can then be passed to limits. These results make use of the regularity theory developed by White [W1].

Finally, in Section 4.2, we return to the classical mean curvature evolution and discuss the situation in which a smooth solution develops a singularity for the first time. We describe the application of the "smooth" estimates in Chapter 2 to the work by Ecker [E3] on the size of the singular set at the so-called "first singular time".

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2 Integral Estimates for Classical MCF and the Number of Surgeries

In this chapter we bound certain L^p-norms of the mean curvature ($p < n-1$) for both smooth mean curvature flow and mean curvature flow with surgeries in the two-convex setting. We then show how these ideas can be applied to obtain a new bound on the number of surgeries required for the flow constructed by Huisken and Sinestrari in [HS3].

Notation. We denote by $g = \{g_{ij}\}$ the induced metric on the hypersurface $\mathcal{M}_t = F(\cdot, t)(\mathcal{M}^n)$. We then denote by $d\mu$ the surface measure, by $A = \{h_{ij}\}$ the second fundamental form and by $\lambda_1 \leq \cdots \leq \lambda_n$ the ordered principal curvatures at the space-time point $(p, t) \in \mathcal{M}^n \times [0, T]$.

We assume throughout that the dimension n of the hypersurface is at least 3 and that the initial surface is two-convex, i.e. $\lambda_1 + \lambda_2 \geq 0$ everywhere on $\mathcal{M}_0$. Following [HS3, Def. 2.5], we introduce some notation which will clarify our exposition. Recall that if $\lambda_1 + \lambda_2 \geq 0$ on $\mathcal{M}_0$, then by the strong maximum principle $\lambda_1 + \lambda_2 > 0$ on $\mathcal{M}_t$ for all $t > 0$.

Definition 2.1 (Class of Two-Convex Surfaces, [HS3]) We denote by $\mathcal{C}(R, \alpha)$, $\alpha = (\alpha_0, \alpha_1, \alpha_2)$, the class of smooth, closed hypersurface immersions satisfying

$$\lambda_1 + \lambda_2 \geq \alpha_0 H, \quad H \geq \alpha_1 R^{-1}, \quad |\mathcal{M}| \leq \alpha_2 R^n, \tag{2.1}$$

for some positive constants $R, \alpha_0, \alpha_1, \alpha_2$.

We choose the parameter R such that $|A|^2 \leq R^{-2}$ on $\mathcal{M}_0$ by setting

$$\sup_{\mathcal{M}_0} |A|^2 = R^{-2}.$$

Huisken and Sinestrari [HS3, Prop. 2.6] showed that any smooth, closed, strictly two-convex surface belongs to $\mathcal{C}(R, \alpha)$ and satisfies $|A|^2 \leq R^{-2}$ for some R, α. Furthermore, $\mathcal{C}(R, \alpha)$ is invariant under both smooth mean curvature flow and standard surgery.

Since R represents the scale of the surface it will feature explicitly in the estimates below. For later reference we point out that the surgery parameters H_k ($k = 1, 2, 3$) can be written in the form $\tilde{H}_k R^{-1}$ where each $\tilde{H}_k$ depends only on scale-free properties of $\mathcal{M}_0$ (see Section 2.2). We stress that the inequality $|A|^2 \leq R^{-2}$ is not invariant under mean curvature flow and pertains only to the initial data. This assumption is made throughout but will not be repeated.

2.1 L^p Estimates for H under Smooth MCF

Motivated by the search for an estimate on the required number of surgeries in the context of mean curvature flow with surgery, we begin with a discussion of classical mean curvature flow. Our starting point is the following result from [H1].

Lemma 2.2 (Evolution Equations, [H1]) *The surface measure and mean curvature satisfy the evolution equations*

$$\frac{\partial}{\partial t} d\mu = -H^2 d\mu, \tag{2.2}$$

$$\frac{\partial}{\partial t} H = \Delta H + |A|^2 H, \tag{2.3}$$

as long as the solution of mean curvature flow remains smooth.

Smooth Calculation. Suppose for now that $\mathcal{M}_t$ is a smooth solution of mean curvature flow starting from an initial surface $\mathcal{M}_0 \in \mathcal{C}(R, \alpha)$ for some R, α. Then we may use Lemma 2.2 to compute

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{M}_t} H^p \, d\mu &= \int_{\mathcal{M}_t} pH^{p-1} \left(\Delta H + |A|^2 H\right) - H^{p+2} \, d\mu & (2.4) \\ &= - p(p-1) \int_{\mathcal{M}_t} |\nabla H|^2 H^{p-2} \, d\mu + \int_{\mathcal{M}_t} H^p \left(p|A|^2 - H^2\right) d\mu & (2.5) \end{aligned}$$

for all $p \in \mathbb{R}$. We will be interested in $p > 0$ (an estimate for $p = 0$ is immediate from (2.2)). In order to deal with the mixed term on the right-hand side we appeal to the roundness estimate [HS3, Thm. 5.3], a result which is adapted to the two-convex geometry (see Remark 2.7).[1] While the solution currently under consideration is smooth, we state the full result including surgeries (see Remark 2.6 below).

Theorem 2.3 (Roundness Estimate, [HS3]) *i) Let $\mathcal{M}_0 \in \mathcal{C}(R, \alpha)$. Then for any $\eta > 0$ there exists a constant $C_\eta = C_\eta(\mathcal{M}_0) > 0$ such that the smooth solution $\mathcal{M}_t$ of mean curvature flow satisfies*

$$|A|^2 - \frac{H^2}{n-1} \leq \eta H^2 + C_\eta R^{-2}. \tag{2.6}$$

ii) There exists $\tilde{\eta} > 0$ such that the following holds. The parameters controlling the surgery procedure can be chosen such that (2.6) holds also for a solution of mean curvature flow with surgeries for all $0 < \eta < \tilde{\eta}$.

[1] See also [HS1] for $n = 2$. Note that in this case two-convexity and mean-convexity coincide.

Applying Theorem 2.3 to our expression (2.5) we arrive at

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{M}_t} H^p\,d\mu \leq &-p(p-1)\int_{\mathcal{M}_t}|\nabla H|^2 H^{p-2}\,d\mu \\ &+\left(p\eta - \frac{n-1-p}{n-1}\right)\int_{\mathcal{M}_t} H^{p+2}\,d\mu + pC_\eta R^{-2}\int_{\mathcal{M}_t} H^p\,d\mu.\end{aligned}$$

We therefore restrict our attention to $p < n-1$. To make this concrete, let $\varepsilon > 0$ and fix $p = n-1-\varepsilon$. We then make an appropriate choice of η (adapted to p) in the roundness estimate above, say

$$\eta_\varepsilon = \frac{\varepsilon}{2(n-1)(n-1-\varepsilon)}.$$

We henceforth suppress the subscript ε on η and write C_ε in place of C_{η_ε} for ease of notation. We will be most interested in small ε, but we point out that if this choice does not satisfy $\eta < \tilde{\eta}$ then we may instead employ

$$\eta = \min\left\{\frac{\varepsilon}{2(n-1)(n-1-\varepsilon)}, \frac{\tilde{\eta}}{2}\right\}$$

in the roundness estimate without having to modify any of the following calculations. We hereby obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{M}_t} H^p\,d\mu \leq &-p(p-1)\int_{\mathcal{M}_t}|\nabla H|^2 H^{p-2}\,d\mu \\ &-\frac{\varepsilon}{2(n-1)}\int_{\mathcal{M}_t} H^{p+2}\,d\mu + pC_\varepsilon R^{-2}\int_{\mathcal{M}_t} H^p\,d\mu.\end{aligned}$$

We emphasize that this holds for all $\varepsilon > 0$.[2] Applying Hölder's inequality we find

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{M}_t} H^p\,d\mu \leq &-p(p-1)\int_{\mathcal{M}_t}|\nabla H|^2 H^{p-2}\,d\mu \\ &-\frac{\varepsilon}{2(n-1)}\left(\int_{\mathcal{M}_t} H^p\,d\mu\right)^{\frac{p+2}{p}}\left|\mathcal{M}_t\right|^{-\frac{2}{p}} + pC_\varepsilon R^{-2}\int_{\mathcal{M}_t} H^p\,d\mu.\end{aligned}$$

Finally, in view of (2.2) and Definition 2.1 we conclude

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{M}_t} H^p\,d\mu \leq &-p(p-1)\int_{\mathcal{M}_t}|\nabla H|^2 H^{p-2}\,d\mu \\ &-\frac{\varepsilon}{2(n-1)}\alpha_2^{-\frac{2}{p}}R^{\frac{-2n}{p}}\left(\int_{\mathcal{M}_t} H^p\,d\mu\right)^{\frac{p+2}{p}} + pC_\varepsilon R^{-2}\int_{\mathcal{M}_t} H^p\,d\mu.\end{aligned}$$

[2] When $\varepsilon = 0$ (that is, when $\eta = 0$) the constant C_ε from the roundness estimate blows up and we lose control on the right hand side; see Remark 2.7.

Let
$$\varphi = \exp\left(-\frac{pC_\varepsilon}{R^2}t\right)\int_{\mathcal{M}_t} H^p\,d\mu.$$
We have proved that φ is non-increasing under the smooth evolution in the two-convex setting for all $p < n-1$.

Lemma 2.4 (Smooth Monotonicity) *Let $\mathcal{M}_t$ be a smooth solution of mean curvature flow starting from $\mathcal{M}_0 \in \mathcal{C}(R,\alpha)$ and fix $p = n-1-\varepsilon$. Then there exists a constant $C_\varepsilon = C_\varepsilon(\mathcal{M}_0)$ such that*
$$\begin{aligned}\frac{d}{dt}\varphi \le &-p(p-1)\exp\left(-\frac{pC_\varepsilon}{R^2}t\right)\int_{\mathcal{M}_t}|\nabla H|^2 H^{p-2}\,d\mu\\ &-\frac{\varepsilon}{2(n-1)}\alpha_2^{-\frac{2}{p}}R^{\frac{-2n}{p}}\exp\left(-\frac{pC_\varepsilon}{R^2}t\right)\left(\int_{\mathcal{M}_t}H^p\,d\mu\right)^{\frac{p+2}{p}}\end{aligned}$$
for all $\varepsilon > 0$ as long as the solution remains smooth.

Hence any L^p-norm of the mean curvature is bounded under smooth mean curvature flow on any finite time interval for all $p < n-1$. In fact, solving the ODE
$$\frac{d}{dt}\varphi \le -\frac{\varepsilon}{2(n-1)}\alpha_2^{-\frac{2}{p}}R^{\frac{-2n}{p}}\exp\left(\frac{2C_\varepsilon}{R^2}t\right)\varphi^{\frac{p+2}{p}}$$
we conclude that
$$\varphi \le \alpha_2 R^{n-p}\left(\frac{\varepsilon}{2(n-1)pC_\varepsilon}\left(\exp\left(\frac{2C_\varepsilon}{R^2}t\right)-1\right)\right)^{-\frac{p}{2}}.$$
This corresponds to an L^p-estimate for the mean curvature under smooth mean curvature flow which behaves like $t^{-1/2}$ for small t.[3] The following result is a precise statement of Theorem 1.1.

Theorem 2.5 (Smooth L^p Estimate) *Let $\mathcal{M}_t$ be a smooth solution of mean curvature flow starting from $\mathcal{M}_0 \in \mathcal{C}(R,\alpha)$ and set $p = n-1-\varepsilon$. Then there exists a constant $C_\varepsilon = C_\varepsilon(\mathcal{M}_0)$ such that*
$$\|H\|_{L^p(\mathcal{M}_t)} \le \alpha_2^{1/p}R^{(n-p)/p}\exp\left(\frac{C_\varepsilon}{R^2}t\right)\left(\frac{\varepsilon}{2(n-1)pC_\varepsilon}\left(\exp\left(\frac{2C_\varepsilon}{R^2}t\right)-1\right)\right)^{-\frac{1}{2}}$$
for all $\varepsilon > 0$ and for all $t > 0$ as long as the solution remains smooth.

[3] This agrees with the scaling of the interior estimates obtained by Ecker and Huisken in [EH].

Remark 2.6 (Surgery) The additional surgery estimate in the second part of Theorem 2.3 has not yet been used. In the next section, however, we will apply Lemma 2.4 to a solution of mean curvature flow with surgeries, and in this setting it will be necessary to call upon the roundness estimate (with the same constants) before and after surgeries.

We now make an informal remark on the critical norm $p = n - 1$.

Remark 2.7 ($p = n - 1$) It is in fact transparent in the calculation above that the coefficient of H in the roundness estimate (2.6) determines the values of p which are susceptible to our approach. The critical exponent $p = n - 1$ (that is, $\varepsilon = 0$) therefore arises naturally from the geometric observation that on the round cylinder

$$|A|^2 - \frac{1}{n-1}H^2 \equiv 0.$$

It is in this way that the two-convex geometry declares itself.

Remark 2.8 (Double Integral Estimate) As above, let $p = n - 1 - \varepsilon$. Another consequence of Lemma 2.4 is an estimate of the form

$$\begin{aligned}\exp\left(pC_\varepsilon R^{-2}t_0\right)\int_{\mathcal{M}_0} H^p\,d\mu \geq \int_{\mathcal{M}_{t_0}} H^p\,d\mu &+ \frac{4}{p}(p-1)\int_0^{t_0}\int_{\mathcal{M}_t}\left|\nabla(H^{\frac{p}{2}})\right|^2 d\mu\,dt \\ &+ \frac{\varepsilon}{2(n-1)}\int_0^{t_0}\int_{\mathcal{M}_t} H^{p+2}\,d\mu\,dt\end{aligned}$$

for a smooth solution $\mathcal{M}_t$ of mean curvature flow on some time interval $0 \leq t \leq t_0$ and for all $\varepsilon > 0$.

Before proceeding to apply these results to a solution of mean curvature flow with surgeries we point out that, in the two-convex setting, the mean curvature controls the full norm of the second fundamental form. As was shown in [HS3, Prop. 2.7], the two-convex inequality $-\lambda_1 \leq \lambda_2$ implies

$$-\lambda_1 \leq \lambda_3 \leq \lambda_1 + \lambda_2 + \lambda_3 \leq H \quad \text{and} \quad \lambda_n \leq \lambda_1 + \lambda_2 + \lambda_n \leq H.$$

This gives rise to the estimates $-H \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq H$ and the resultant inequality

$$|A|^2 \leq n\left(\max_{1\leq i\leq n}|\lambda_i|\right)^2 \leq nH^2, \tag{2.7}$$

which is of course false in the more general mean-convex setting.

Theorem 2.5 is therefore equivalent in our context to an estimate on the L^p-norms of $|A|$: setting $p = n - 1 - \varepsilon$ as above, we obtain

$$\||A|\|_{L^p(\mathcal{M}_t)} \leq n^{1/2} \alpha_2^{1/p} R^{(n-p)/p} \exp\left(\frac{C_\varepsilon}{R^2} t\right) \left(\frac{\varepsilon}{2(n-1)pC_\varepsilon}\left(\exp\left(\frac{2C_\varepsilon}{R^2} t\right) - 1\right)\right)^{-\frac{1}{2}}$$

for all $\varepsilon > 0$ and $t > 0$. This will not be required for our later applications.

Applying (2.7) this time to Remark 2.8 we arrive at the following result, which has interesting applications to recent work by Ecker [E3] on the size of the singular set at the first singular time.

Remark 2.9 (Integrability Condition) Let $\mathcal{M}_t$, $0 \leq t < t_0$, be a smooth solution of mean curvature flow starting from $\mathcal{M}_0 \in \mathcal{C}(R, \alpha)$. Then we have shown that there exists a constant $C_\varepsilon = C_\varepsilon(\mathcal{M}_0)$ such that

$$\int_0^{t_0} \int_{\mathcal{M}_t} |A|^{n+1-\varepsilon} \, d\mu \, dt \leq \frac{C(n)}{\varepsilon} \exp\left(\frac{pC_\varepsilon}{R^2} t_0\right) \int_{\mathcal{M}_0} H^{n-1-\varepsilon} \, d\mu$$

for all $\varepsilon > 0$ where $C(n) = 2(n-1)n^{p/2+1}$ and $p = n - 1 - \varepsilon$.

For a description of the application to the size of the singular set at the first singular time, the reader can immediately skip ahead to Section 4.2. In the next section, we begin our analysis of the surgery construction developed by Huisken and Sinestrari.

2.2 L^p Estimates for H Across Surgery.

Our goal is to bound a "higher-order" geometric quantity under mean curvature flow with surgeries and use it to improve the estimate in [HS3] on the required number of surgeries. This section is devoted to the first task; the second is dealt with in Section 2.3. We define and discuss the flow with surgeries as constructed by Huisken and Sinestrari [HS3], before turning to integral estimates across surgery regions and revisiting the calculation from Lemma 2.4 in this modified context.

We henceforth assume that the reader is familiar with the theory developed by Huisken and Sinestrari. For the sake of completeness we have included the necessary definitions from [HS3], but our summary is not intended to be comprehensive. The following material is taken directly from [HS3].

Mean Curvature Flow with Surgeries. [HS3, Sect. 2] defines the solution of mean curvature flow with surgeries starting from a smooth hypersurface immersion $F_0 : \mathcal{M}_1 \to \mathbb{R}^{n+1}$ in some class $\mathcal{C}(R, \alpha)$. It consists of a sequence of

i) intervals $[0,T_1],[T_1,T_2],\dots,[T_{N-1},T_N]$,

ii) manifolds $\mathcal{M}_1,\mathcal{M}_2,\dots,\mathcal{M}_N$, and

iii) smooth mean curvature flows $F_t^i:\mathcal{M}_i\to\mathbb{R}^{n+1}$, $t\in[T_{i-1},T_i]$

such that

a) the initial hypersurface for the family F^1 is given by $F_0:\mathcal{M}_1\to\mathbb{R}^{n+1}$, and

b) the initial hypersurface for the flow $F_t^i:\mathcal{M}_i\to\mathbb{R}^{n+1}$ on $[T_{i-1},T_i]$ for $2\le i\le N$ is obtained from $F^{i-1}_{T_{i-1}}$ by:

   - performing standard surgery (as defined below) on finitely many disjoint necks, replacing each of them with two spherical caps; and
   - removing finitely many disconnected components (these are diffeomorphic either to $\mathbb{S}^n$ or to $\mathbb{S}^{n-1}\times\mathbb{S}^1$).

Abusing notation slightly we again write $\mathcal{M}_t$ for the solution of mean curvature flow with surgeries. The surgery time T_N is the "extinction" time of $\mathcal{M}_t$ if all connected components of $\mathcal{M}_{T_N}$ can be identified as copies of $\mathbb{S}^n$ or $\mathbb{S}^{n-1}\times\mathbb{S}^1$, or alternatively if this can be achieved after performing finitely many surgeries on $\mathcal{M}_{T_N}$.

In order to control the L^p-norms of the mean curvature under mean curvature flow with surgeries, we therefore require estimates on the curvature of necks before and after surgery. The following definitions from [HS3] can be omitted by the reader familiar with the surgery theory.

Necks. A neck is a region which is geometrically close to a piece of the standard cylinder in a precise quantitative way [HS3, Def. 3.7, 3.9]. This concept is of course independent of mean curvature flow:

Definition 2.10 (Hypersurface Neck, [HS3]) Let $F:\mathcal{M}^n\to\mathbb{R}^{n+1}$ be a smooth hypersurface with induced metric g and Weingarten map W, and let $\mathcal{N}:\mathbb{S}^{n-1}\times[a,b]\to(\mathcal{M}^n,g)\subset\mathbb{R}^{n+1}$ be a local diffeomorphism. Then $\mathcal{N}$ is an (ϵ,k)-hypersurface neck if

$$|r^{-2}(z)g-\bar g|_{\bar g}\le\epsilon,\quad |\bar D^j(r^{-2}(z)g)|_{\bar g}\le\epsilon\quad\text{and}\quad\left|\left(\frac{d}{dz}\right)^j\log r(z)\right|\le\epsilon$$

uniformly for $1\le j\le k$, and if in addition

$$|W(q)-r^{-1}(z)\bar W|\le\epsilon r^{-1}(z)\quad\text{and}\quad|\nabla^l W(q)|\le\epsilon r^{-l-1}(z)$$

for $1\le l\le k$ and for all $q\in\mathbb{S}^{n-1}\times\{z\}$ and all $z\in[a,b]$. Here $\bar g$ is the standard metric on the cylinder and $r:[a,b]\to\mathbb{R}$ is the average radius of the cross-section $\mathcal{N}(\mathbb{S}^{n-1}\times\{z\})$ with respect to the pullback of g on $\mathcal{M}^n$.

In order to deal with overlapping necks Huisken and Sinestrari introduced the concept of a maximal normal hypersurface neck. The following conditions [HS3, Def. 3.8, 3.11] are sufficient to guarantee uniqueness of $\mathcal{N}$ up to isometries (see also [Ha4]).

Definition 2.11 (Maximal Normal Hypersurface Neck, [HS3]) Let $\mathcal{N}$ be an (ϵ, k)-hypersurface neck. Then $\mathcal{N}$ is a maximal normal (ϵ, k)-hypersurface neck if

i) each cross-section $\Sigma_z = \mathcal{N}(\mathbb{S}^{n-1} \times \{z\}) \subset (\mathcal{M}, g)$ has constant mean curvature;

ii) the restriction of $\mathcal{N}$ to each $\mathbb{S}^{n-1} \times \{z\}$ equipped with the standard metric is a harmonic map to Σ_z equipped with the metric induced by g (for $n = 3$ we require the additional condition that the center of mass of the pull-back of g on $\mathbb{S}^2 \times \{z\}$ considered as a subset of $\mathbb{R}^3 \times \{z\}$ lies at the origin $\{0\} \times \{z\}$);

iii) the volume of any subcylinder with respect to the pullback of g is given by

$$Vol(\mathbb{S}^{n-1} \times [v, w], g) = \int_v^w r(z)^n \, dz;$$

iv) for any Killing vector field $\bar{V}$ on $\mathbb{S}^{n-1} \times \{z\}$ we have that

$$\int_{\mathbb{S}^{n-1} \times \{z\}} \bar{g}(\bar{V}, U) \, d\mu = 0$$

where U is the unit normal vector field to Σ_z in $(\mathcal{M}, g)$ and $d\mu$ is the measure of the metric $\bar{g}$ on the standard cylinder;

v) whenever $\mathcal{N}^*$ is another neck satisfying i)-iv) with $\mathcal{N} = \mathcal{N}^* \circ G$ for some diffeomorphism G, then the map G is onto.

Surgery is performed on hypersurface necks in normal form (see [HS3, Thm. 8.1] or Theorem 2.15 below).

Surgery. Huisken and Sinestrari [HS3, Sect. 3] defined the following "standard surgery" procedure for hypersurface necks. Let $\mathcal{N} : \mathbb{S}^{n-1} \times [a, b] \to \mathcal{M}$ be a maximal normal (ϵ, k)-hypersurface neck with appropriate parameters (ϵ, k). Let $z_0 \in [a, b]$ such that $[z_0 - 4\Lambda, z_0 + 4\Lambda] \subset [a, b]$ for an appropriate constant $\Lambda > 0$. For the pair $(\mathcal{N}, z_0)$ and given parameters $0 < \tau < 1$ and $B > 10\Lambda$, surgery with parameters τ, B at the cross section $\mathcal{N}_{z_0} = \mathcal{N}(\mathbb{S}^{n-1} \times \{z_0\})$ replaces the neck of length L by two spherical caps as follows.

We denote by $\bar{C}_{z_0} : \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n+1}$ the straight cylinder best approximating the surface at the cross section $\mathcal{N}_{z_0}$: the radius of $\bar{C}$ is chosen as the mean radius $r(z_0) = r_0$, a point on its axis if given by the center of mass of $\mathcal{N}_{z_0}$ with its induced metric, and its axis is parallel to the average of the unit normal field to $\mathcal{N}_{z_0} \subset (\mathcal{M}, g)$, taken again with respect to the induced metric.

To the left of z_0, standard surgery leaves the collar $\mathbb{S}^{n-1} \times [a, z_0 - 3\Lambda]$ untouched, and replaces the cylinder $\mathcal{N}(\mathbb{S}^{n-1} \times [z_0 - 3\Lambda, z_0])$ by a ball attached smoothly to $\mathcal{N}_{z_0 - 3\Lambda}$. The other side is similarly defined. For convenience we set $z_0 - 4\Lambda = 0$ and consider a normal parametrisation $\mathcal{N} : \mathbb{S}^{n-1} \times [0, 4\Lambda] \to \mathcal{M}$.

i) (*Bending In*) Let

$$u(z) \equiv r_0 \exp\left(-\frac{B}{z - \Lambda}\right)$$

on $[\Lambda, 3\Lambda]$ for $B > 10\Lambda$ in Gaussian normal coordinates. For a parameter $0 < \tau < 1$, define:

$$\tilde{\mathcal{N}}(\omega, z) := \mathcal{N}(\omega, z) - \tau u(z)\nu(\omega, z).$$

ii) (*Symmetrising*) Denote by $\varphi : [0, 4\Lambda] \to \mathbb{R}^+$ a fixed smooth transition function with $\varphi = 1$ on $[0, 2\Lambda]$, $\varphi = 0$ on $[3\Lambda, 4\Lambda]$ and $\varphi' \leq 0$, and by $\tilde{C}_{z_0} : \mathbb{S}^{n-1} \times [0, 4\Lambda] \to \mathbb{R}^{n+1}$ the bent cylinder defined by

$$\tilde{C}_{z_0}(\omega, z) := \bar{C}_{z_0}(\omega, z) - \tau u(z)\nu_{\bar{C}}(\omega, z).$$

We then interpolate to obtain an axially symmetric surface

$$\hat{\mathcal{N}}(\omega, z) := \varphi(z)\tilde{\mathcal{N}}(\omega, z) + (1 - \varphi(z))\tilde{C}_{z_0}(\omega, z).$$

The function φ depends only on Λ, and it can be defined in such a way that all of its derivatives are smaller if Λ is larger. In particular, if we assume $\Lambda \geq 10$, each derivative of φ is bounded by some fixed constant.

iii) (*Capping Off*) We change u on $[3\Lambda, 4\Lambda]$ to a function $\hat{u}$ to ensure that $\tau\hat{u}(z) \to r(z_0) = r_0$ as z approaches some $z_1 \in (3\Lambda, 4\Lambda]$, such that $\tilde{C}_{z_0}([3\Lambda, 4\Lambda])$ is a smoothly attached axially symmetric and uniformly convex cap. In this region there is a fixed upper bound on the curvature and each of its derivatives, independent of $\Lambda \geq 10$ and the surgery parameters τ, B.

We are now ready to control the L^p-norms of the mean curvature across surgery. In particular, we show that each of these quantities drops by a controlled amount as a result of the surgery construction.

Curvature Estimates for Necks. Since the proof of Lemma 2.4 is direct and doesn't rely on any auxiliary quantities, the L^p-norms of the mean curvature can be estimated directly and will succumb to coarse techniques. These results are independent of mean curvature flow and will be combined with the smooth calculation from the previous section in Theorem 2.16.

We consider an (ϵ, k)-hypersurface neck

$$\mathcal{N} : \mathbb{S}^{n-1} \times [a, b] \to \mathcal{M}$$

contained in $\mathcal{M} \subset \mathbb{R}^{n+1}$, and we discuss the surgery corresponding to the pair $(\mathcal{N}, z_0)$ where $z_0 \in [a, b]$. We denote by r_0 the mean radius of the cross-section $\mathcal{N}_{z_0} = \mathcal{N}(\mathbb{S}^{n-1} \times \{z_0\})$, that is

$$|\mathcal{N}_{z_0}| = \omega_{n-1}(r_0)^{n-1}$$

where ω_{n-1} is the area of the standard unit $(n-1)$-sphere. We refer to r_0 as the scale of the neck. We also write $\mathcal{N}^+$ $(\mathcal{N}^-)$ for the neck before (after) surgery. Similarly, $\mathcal{M}^+$ $(\mathcal{M}^-)$ will denote the surface before (after) surgery.

Definition 2.12 (Length Parameter, [HS3]) We define the length of a hypersurface neck $\mathcal{N} : \mathbb{S}^{n-1} \times [a, b] \to \mathbb{R}^{n+1}$ to be $b - a$.

According to this definition, length is a scale-invariant quantity. The length of the neck plays a crucial role in the sequel and will depend solely on the dimension n. The distance (with respect to the metric) between the two ends of the standard embedded cylinder of length L and radius r_0 is $r_0 L$.

The scale-invariant surgery procedure replaces a neck $\mathcal{N}(\mathbb{S}^{n-1} \times [0, L])$ of length L with two spherical caps diffeomorphic to discs. The parameters ϵ, k describe the quality of the neck and the parameters τ, B control the surgery procedure itself. Λ is a length parameter which must be sufficiently large but can otherwise be chosen freely in terms of n (we require $L \geq 20 + 8\Lambda$). Surgery leaves the collars $[0, \Lambda]$ and $[L - \Lambda, L]$ unchanged and replaces the cylinder $\mathcal{N}(\mathbb{S}^{n-1} \times [\Lambda, L - \Lambda])$ with two caps attached smoothly to $\mathcal{N}_\Lambda$ and to $\mathcal{N}_{L-\Lambda}$.

In the next lemma we suppose that $\mathcal{M}^+$ is obtained from $\mathcal{M}^-$ by performing standard surgery on finitely many disjoint hypersurface necks with scale r_0.

Lemma 2.13 (L^p Estimate Across Surgery) *For each $p \geq 0$ the following property holds. We can choose $L = L(n)$ sufficiently large such that*

$$\int_{\mathcal{M}^-} H^p \, d\mu - \int_{\mathcal{M}^+} H^p \, d\mu \geq C(r_0)^{n-p} \tag{2.8}$$

where $C = C(n, L)$.

Proof. Let $\mathcal{N}^- : \mathbb{S}^{n-1} \times [0, L] \to \mathcal{M}^- \to \mathbb{R}^{n+1}$ be an (ϵ, k)-hypersurface neck with scale r_0 in normal form. Choosing ϵ sufficiently small we can arrange that

$$H(p) \geq \frac{9}{10}\left(\frac{n-1}{r_0}\right) \quad \text{for all } p = (w, z) \in \mathcal{N}^- \text{ such that } z \in [0, L].$$

For ease of notation we write $\mathcal{U}^- \subset \mathcal{M}^-$ for the subset of $\mathcal{M}^-$ altered by the given surgery and $\mathcal{U}^+$ for the subset of $\mathcal{M}^+$ replacing $\mathcal{U}^-$. We then estimate $|\mathcal{U}^-| \geq (9/10)L\omega_{n-1}(r_0)^n$ so that

$$\int_{\mathcal{U}^-} H^p \, d\mu \geq C_1(n) L r_0^{n-p}.$$

Let us restrict our attention for the moment to the left side of the neck. During surgery we pinch the neck in on $[\Lambda, 3\Lambda]$ and attach an axially symmetric convex cap on $[3\Lambda, 4\Lambda]$. We choose the parameter τ in the surgery construction sufficiently small so as to ensure that the curvature remains close to that of the cylinder on $[\Lambda, 3\Lambda]$, for example

$$\frac{9}{10}\left(\frac{n-1}{r_0}\right) \leq H(p) \leq \frac{11}{10}\left(\frac{n-1}{r_0}\right) \quad \text{for all } p = (w,z) \in \mathcal{N}^+ \text{ such that } z \in [\Lambda, 3\Lambda].$$

In addition, the curvature of the convex cap attached to $\mathcal{N}_{3\Lambda}$ can be made as close as we like to that of a standard sphere:

$$\frac{9}{10}\left(\frac{n}{r_0}\right) \leq H(p) \leq \frac{11}{10}\left(\frac{n}{r_0}\right) \quad \text{for all } p = (w,z) \in \mathcal{N}^+ \text{ such that } z \in [3\Lambda, 4\Lambda].$$

We apply the same analysis to the other end of the neck. $\mathcal{U}^+$ consists of two copies of $\mathcal{U}_1^+ \cup \mathcal{U}_2^+$ where $\mathcal{U}_2^+$ denotes the convex cap attached to $\mathcal{N}_{3\Lambda}$ as described in step iii) of the surgery procedure and $\mathcal{U}_1^+$ denotes the bent cylinder in between. We can arrange that $(9/5)\Lambda\omega_{n-1}(r_0)^n \leq |\mathcal{U}_1^+| \leq (11/5)\Lambda\omega_{n-1}(r_0)^n$ and

$$C_2(n)\Lambda r_0^{n-p} \leq \int_{\mathcal{U}_1^+} H^p \, d\mu \leq C_3(n)\Lambda r_0^{n-p}.$$

Finally, step iii) of the surgery construction can be adapted such that $(9/10)\omega_n(r_0)^n \leq |\mathcal{U}_2^+| \leq (11/10)\omega_n(r_0)^n$ and

$$C_4(n)(r_0)^{n-p} \leq \int_{\mathcal{U}_2^+} H^p \, d\mu \leq C_5(n) r_0^{n-p}.$$

According to the construction in [HS3], we can first choose Λ (sufficiently large and just depending on n), and we can then choose

$$L = C + 8\Lambda \geq 20 + 8\Lambda$$

sufficiently large in terms of n such that for each $p \geq 0$ we have

$$\int_{\mathcal{U}^-} H^p \, d\mu \geq 2 \int_{\mathcal{U}^+} H^p \, d\mu.$$

This completes the proof. $\square$

Remark 2.14 ($p = 0$) The $p = 0$ case of Lemma 2.13 is from [HS3] – see Remark 2.17. Notice that, in view of Lemma 2.2, it implies

$$\left|\mathcal{M}_0\right| \geq \left|\mathcal{M}_{t_0}\right| + \int_0^{t_0} \int_{\mathcal{M}_t} H^2 \, d\mu \, dt, \tag{2.9}$$

where $\mathcal{M}_t$ denotes the solution of mean curvature flow with surgeries on the time interval $0 \leq t \leq t_0$. The estimate (2.9) is the foundation of Brakke's definition of weak mean curvature flow. In the special case of smooth surfaces, Brakke's definition requires that

$$\int_{\mathcal{M}_0} \phi \, d\mu \geq \int_{\mathcal{M}_{t_0}} \phi \, d\mu + \int_0^{t_0} \int_{\mathcal{M}_t} \phi H^2 + H \langle \nabla \phi, \nu \rangle \, d\mu \, dt$$

for all non-negative $\phi = \phi(x) \in C_c^1(\mathbb{R}^{n+1})$. The error terms introduced at each surgery time can be estimated as follows,

$$\int_{\mathcal{U}^+} \phi \, d\mu - \int_{\mathcal{U}^-} \phi \, d\mu \leq C(n, L)(r_0)^n \sup_{\mathcal{U}^+} \phi.$$

This error disappears as $r_0 \to 0$ (a property which is expected in light of the results in the next chapter).

L^p **Estimate for Mean Curvature Flow with Surgeries.** We now arrive at our main result for a two-convex solution of mean curvature flow with surgeries ($n \geq 3$). In what follows:

- $\mathcal{M}_t$ denotes the solution of mean curvature flow with surgeries;
- T_j ($j = 1, 2, ..., N$) are the surgery times;
- $\mathcal{M}_{T_j^-}$ ($\mathcal{M}_{T_j^+}$), denotes the surface at time T_j before (after) surgery has been performed.

The solution $\mathcal{M}_t$ is determined by a set of parameters H_1, H_2, H_3 which control the choice of surgery times and locations. We recall the main result from [HS3].

Theorem 2.15 (Existence & Finite Extinction, [HS3]) *Let $\mathcal{M}_0 \in \mathcal{C}(R, \alpha)$ with $n \geq 3$. Then there exist constants $\omega_1, \omega_2, \omega_3 > 1$ depending only on α such that the following holds. If we set $H_2 = \omega_2 H_1$ and $H_3 = \omega_3 H_2$, then for any $H_1 \geq \omega_1 R^{-1}$ there exists an associated mean curvature flow with surgeries on $0 \leq t \leq T_N$ starting from $\mathcal{M}_0$ and satisfying the properties:*

i) each surgery is performed at the earliest time T_j such that $\max H(\cdot, T_j^-) = H_3$;

ii) after the two-step surgery procedure, $\max H(\cdot, T_j^+) \leq H_2$;

iii) all surgeries start from a cross-section of a normal hypersurface neck with mean radius $r_0 = (n-1)/H_1$;

iv) $N < \infty$.[4]

[4]In fact, $N \leq C(n)(H_1)^n$ - see Section 2.3.

The surgery times T_j therefore depend on the choice of surgery parameters. Furthermore, the number of surgeries N may depend on the surgery parameters; this is discussed in the next section. We first combine the smooth calculation from the previous section with the curvature estimates for necks to prove:

Theorem 2.16 (L^p Estimate for Flow with Surgeries) *We can choose $L = L(n)$ sufficiently large such that the following property holds. Let $\mathcal{M}_0 \in \mathcal{C}(R, \alpha)$ with $n \geq 3$ and fix $p = n - 1 - \varepsilon$. Then the solution $\mathcal{M}_t$ of mean curvature flow with surgeries starting from $\mathcal{M}_0$ satisfies*

$$C(T) \int_{\mathcal{M}_0} H^p \, d\mu \geq \int_{\mathcal{M}_T} H^p \, d\mu + p(p-1) \int_0^T \int_{\mathcal{M}_t} |\nabla H|^2 H^{p-2} \, d\mu \, dt + \frac{\varepsilon}{2(n-1)} \int_0^T \int_{\mathcal{M}_t} H^{p+2} \, d\mu \, dt$$

for all $\varepsilon > 0$ and for all $0 < T \leq T_N < \infty$. The constant $C(T)$ depends also on ε and $\mathcal{M}_0$.

Proof. Note that the proof of Lemma 2.4 relies just on Theorem 2.3. Since part ii) of that result guarantees that the estimate survives surgery without any modifications to the constants, we conclude that Lemma 2.4 holds on each smooth time interval $[0, T_1]$, $[T_1, T_2]$,...,$[T_m, T]$. We can therefore integrate on each interval $[T_j, T_{j+1}]$ (where $T_0 := 0$) to obtain

$$\begin{aligned}\exp\left(-pC_\varepsilon R^{-2} T_j\right) \int_{\mathcal{M}_{T_j}} H^p \, d\mu \geq{} & \exp\left(-pC_\varepsilon R^{-2} T_{j+1}\right) \int_{\mathcal{M}_{T_{j+1}}} H^p \, d\mu \\ & + p(p-1) \int_{T_j}^{T_{j+1}} \left(\exp\left(-pC_\varepsilon R^{-2} t\right) \int_{\mathcal{M}_t} |\nabla H|^2 H^{p-2} \, d\mu \right) dt \\ & + \frac{\varepsilon}{2(n-1)} \int_{T_j}^{T_{j+1}} \left(\exp\left(-pC_\varepsilon R^{-2} t\right) \int_{\mathcal{M}_t} H^{p+2} \, d\mu \right) dt.\end{aligned}$$

Furthermore, by Lemma 2.13,

$$\int_{\mathcal{M}_{T_{j+1}^-}} H^p \, d\mu \geq \int_{\mathcal{M}_{T_{j+1}^+}} H^p \, d\mu.$$

Note that we simply disregard any contribution made by the components discarded at the surgery time. Since $\exp\left(-pC_\varepsilon R^{-2} t\right)$ is continuous in t, we can combine the results from the finitely many time intervals to obtain the desired result. $\square$

In the final section of this chapter, we extract a bound on the required number of surgeries from Lemma 2.13 and Theorem 2.16.

2.3 Number of Surgeries

The central result obtained by Huisken and Sinestrari – see [HS3, Thm. 8.1] or Theorem 2.15 – says that, for any choice of parameters (sufficiently large), there exists a corresponding mean curvature flow with surgeries which terminates after finitely many surgery times. More precisely, Huisken and Sinestrari used an area-based argument to arrive at a bound on N which depends explicitly on the surgery parameters H_1, H_2, H_3. In this section we briefly review their result and combine it with the higher order L^p-estimates from the previous section to derive a new quantitative estimate on N which will be essential for the applications in the next chapter.

Remark 2.17 (Huisken-Sinestrari Estimate) Let $\mathcal{M}_t$ be the solution of mean curvature flow with surgeries starting from some $\mathcal{M}_0 \in \mathcal{C}(R, \alpha)$. Observe that from Lemma 2.2, Lemma 2.13 and Definition 2.1 we have the area bound

$$|\mathcal{M}_t| \leq \alpha_2 R^n.$$

Setting $p = 0$ in Lemma 2.13 we observe moreover that each surgery reduces the area of the surface by at least some fixed multiple of $(H_k)^{-n}$ $(k = 1, 2, 3)$, where the H_k denote the surgery parameters associated with $\mathcal{M}_t$ (see Theorem 2.15 above). Huisken and Sinestrari [HS3] proved using this argument that N satisfies

$$N \leq C(n)(H_k)^n. \tag{2.10}$$

The estimate (2.10) is sufficient to establish that mean curvature flow with surgeries must terminate after a finite number of surgery times for each finite choice of H_k.

We will see in the proof of Theorem 3.7 below, however, that (2.10) is not strong enough for our later applications (see Remark 3.17). We therefore exploit the property that each surgery consumes a fixed multiple of $(H_k)^{n-p}$ for $p > 0$ to extract a sharper bound on N.

Corollary 2.18 (Number of Surgeries) *We can choose $L = L(n)$ sufficiently large such that the following property holds. Let $\mathcal{M}_0 \in \mathcal{C}(R, \alpha)$ with $n \geq 3$ and consider the solution $\mathcal{M}_t$, $t \in [0, T_N]$, of mean curvature flow with surgeries starting from $\mathcal{M}_0$ with parameters H_k $(k = 1, 2, 3)$. For any sufficiently small $\varepsilon > 0$ there exists a constant $C = C(\varepsilon, n, L, \mathcal{M}_0)$ such that*

$$N \leq C(H_1)^{1+\varepsilon}. \tag{2.11}$$

Here, as above, N denotes the number of surgeries.

Proof. It follows from the proof of Theorem 2.15 in [HS3] that, given any choice of length parameter L (sufficiently large), we can find a set of finite constants $\{\omega_1, \omega_2, \omega_3\}$ depending only on α such that for all $H_1 \geq \omega_1 R^{-1}$ there exists a mean curvature flow with surgeries satisfying the desired properties.

We then apply the argument in Remark 2.17 to the higher L^p-norms of the mean curvature. It follows from Theorem 2.16 that for any $\varepsilon > 0$ we have

$$\int_{\mathcal{M}_T} H^{n-1-\varepsilon}\, d\mu \leq C(\varepsilon, \mathcal{M}_0, T)$$

at any finite time T. In addition, for any $\varepsilon > 0$, Lemma 2.13 guarantees that at each surgery time

$$\int_{\mathcal{M}_{T_j^-}} H^{n-1-\varepsilon}\, d\mu - \int_{\mathcal{M}_{T_j^+}} H^{n-1-\varepsilon}\, d\mu \geq C(n)(H_k)^{-(1+\varepsilon)}.$$

This is again independent of any contribution made by the components discarded at the surgery time. This completes the proof. □

In the next chapter we consider the object produced by letting $H_k \to \infty$. In fact, we use Corollary 2.18 to reconcile the solution provided by the non-canonical surgery construction with the unique weak solution of mean curvature flow studied in [CGG, ES1].

3 Approximating Weak Solutions using MCF with Surgeries

Mean curvature flow with surgeries and the theory of weak solutions are independent attempts at a geometrically reasonable model of mean curvature flow which exists for all time. The primary goal of the present chapter is to effect a reconciliation between the two approaches. The central result, Theorem 3.7 below, is that the solution of mean curvature flow with surgeries converges to the weak solution in an appropriate limit of the surgery parameters. Moreover, we obtain quantitative estimates on the rate of convergence using Corollary 2.18 and Brakke's "clearing out lemma". A different version of this result was independently obtained by Lauer [L].

3.1 Convergence to the Weak Solution

In order to set up a precise statement of our main theorem, we rapidly recall some basic definitions and results from the well-developed theory of weak solutions. For further details we refer to [B, CGG, ES1, I2, W2]. The main result appears in Theorem 3.7.

Remark 3.1 (Curvature Assumptions) We will assume throughout this chapter that $\mathcal{M}_0$ is the mean-convex, connected boundary of a bounded, open subset of $\mathbb{R}^{n+1}$. The curvature assumption is unnecessary from the point of view of existence results for weak solutions, but our main intention is to discuss the corresponding flow with surgeries, which restricts us further to domains with two-convex boundaries of dimension at least three.

Level-Set Ansatz. Let $\Omega \subset \mathbb{R}^{n+1}$ be open and bounded such that $\mathcal{M}_0 = \partial\Omega$ has non-negative mean curvature. By the strong maximum principle, H is strictly positive on the smooth solution $\mathcal{M}_t$ of mean curvature flow for all $t > 0$ as long as it exists. The hypersurfaces $\mathcal{M}_t$ can therefore be represented as the level-sets

$$\mathcal{M}_t = \{x \in \Omega \mid u(x) = t\} \tag{3.1}$$

of a continuous scalar "time" function $u : \bar{\Omega} \to \mathbb{R}$ satisfying the degenerate elliptic boundary value problem

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = -\dfrac{1}{|Du|}, \\ \qquad u\Big|_{\partial\Omega} \; = \; 0. \end{cases} \tag{$\star$}$$

If u is smooth at a point $x \in \Omega$ with $Du(x) \neq 0$, then $(\star)$ simply states that the level-sets of u near x solve (MCF) in the classical sense.

Weak Solutions. We briefly recall one way [HI, MS, S] to define a weak concept of solutions to $(\star)$ in the mean-convex case using the energy functional

$$J_u(v) := \int_\Omega |Dv| - \frac{v}{|Du|}\, dx.$$

For more general formulations we refer to [CGG, ES1] and to [B, I2] (see also Remark 3.4).

Definition 3.2 (Weak Solution) Given $u \in C^{0,1}(\bar{\Omega})$ such that $|Du|^{-1} \in L^1(\Omega)$, we say that u is a weak solution of $(\star)$ on Ω if

$$J_u(u) \leq J_u(v) \tag{3.2}$$

for any Lipschitz continuous function v on Ω such that $\{u \neq v\} \subset\subset \Omega$, and if u satisfies $u > 0$ on Ω and $\{u = 0\} = \partial\Omega$.

Notation. We hereafter write u_L for the weak solution of the level-set flow and we define

$$\Gamma_t := \begin{cases} \partial \left\{ x \in \Omega \;\middle|\; u_L(x) > t \right\} & \text{for all } t \leq T \\ \emptyset & \text{for all } t > T \end{cases}$$

to be the t-slices of u_L, where the extinction time T is given by

$$T := \sup_\Omega u_L.$$

We will often have occasion to consider the regions $\Omega_t := \{u_L > t\}$ enclosed by the level-sets $\Gamma_t = \partial\Omega_t$ of the weak solution.

Properties. With the preceding definitions in hand, we turn now to the geometric properties the weak solution. The next result is well-known and can be found for example in [CGG, ES1].

Theorem 3.3 (Properties of Weak Solution, [ES1]) *Let $\Omega \subset \mathbb{R}^{n+1}$ be open and bounded such that $\partial\Omega$ has non-negative mean curvature. Then there exists a unique weak solution u_L of $(\star)$ on Ω such that*

i) Γ_t agrees with the smooth solution $\mathcal{M}_t$ of mean curvature flow starting from $\mathcal{M}_0 = \partial\Omega$ if and so long as the latter exists, and

ii) if $\mathcal{M}_t$, $t_1 \leq t \leq t_2$, is any smooth, compact mean curvature flow with positive mean curvature then

$$\mathcal{M}_{t_1} \cap \Gamma_{t_1} = \emptyset \implies \mathcal{M}_t \cap \Gamma_t = \emptyset$$

for all $t_1 \leq t \leq t_2$.

It is straightforward to verify that property ii) is equivalent to the statement

$$\frac{d}{dt}\mathrm{dist}(\mathcal{M}_t, \Gamma_t) \geq 0.$$

Remark 3.4 (Weak Solutions via Smooth Avoidance, [I2]) The weak solution of the level-set flow can in fact be defined using Theorem 3.3 ii) as the unique maximal set $\mathrm{graph}(u_L) \subset \mathbb{R}^{n+1} \times \mathbb{R}^+$ such that the slices

$$\partial K_t := \{x \in \mathbb{R}^{n+1} \bigm| (x,t) \in \mathrm{graph}(u_L)\}$$

satisfy the avoidance principle for $t \geq 0$. Here $K_t = \{x \in \Omega \,|\, u_L(x) \geq t\}$.

Associated with the variational structure of Definition 3.2 is a fundamental one-sided area minimisation property. We write ∂^* for the reduced boundary of a set.

Definition 3.5 (Outward Minimising) Let $U \subset \mathbb{R}^{n+1}$ be an open set. We say that the set $E \subset \mathbb{R}^{n+1}$ is outward minimising in U if

$$|\partial^* E \cap K| \leq |\partial^* F \cap K| \tag{3.3}$$

for any $F \supset E$ such that $F \setminus E \subset\subset U$ and any compact set $K \supset (F \setminus E)$.

This concept also plays a decisive role in the theory of weak solutions for various other hypersurface flows, compare [HI, S].

Proposition 3.6 (Outward Minimising, [W1]) *Let $\Omega \subset \mathbb{R}^{n+1}$ be open and bounded and suppose $\partial\Omega$ has non-negative mean curvature. Then the sets $\Omega_t = \{u_L > t\}$ enclosed by the level-sets of the weak solution of mean curvature flow generated by Ω are outward minimising in Ω.*

It is well-known [ES1] that u_L can be approximated uniformly in C^0 by smooth, non-compact solutions of mean curvature flow. Indeed, one establishes existence of a weak solution satisfying (3.2) using functions u^ε solving the regularised boundary value problem

$$\begin{cases} \mathrm{div}\left(\dfrac{Du^\varepsilon}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}}\right) = -\dfrac{1}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}}, \\ \left. u^\varepsilon \right|_{\partial\Omega} = 0, \end{cases}$$

compare [ES1, MS]. From a geometric point of view, these are (after rescaling) higher-dimensional, translating, graphical solutions of mean curvature flow. In this chapter we set forth an alternate approximation scheme based on closed solutions of mean curvature flow with surgeries.

Mean Curvature Flow with Surgeries. We henceforth restrict our attention to domains Ω in $\mathbb{R}^{n+1}$, $n \geq 3$, such that $\partial\Omega$ is two-convex. Our first task is to interpret the solution of mean curvature flow with surgeries in the language of level-sets.

Given $\partial\Omega = \mathcal{M}_0 \in \mathcal{C}(R, \alpha)$, we have the solution $\mathcal{M}_t$, $t \in [0, T_N]$, of the flow with surgeries corresponding to a choice of parameters $\{H_1, H_2, H_3\}$. Consider the level-set function u which assigns to each $x \in \Omega$ the time t such that $x \in \mathcal{M}_t$. When $t \notin \{T_1, \dots, T_N\}$, we have

$$\mathcal{M}_t = \{x \in \Omega \mid u = t\}$$

and the smoothness of the classical solution implies smoothness of the time function.

Now consider any surgery time T_j; let $E_{T_j^-}$ be the closed domain in $\mathbb{R}^{n+1}$ bounded by $\mathcal{M}_{T_j^-}$ and let $F_{T_j^+}$ be the open set in $\mathbb{R}^{n+1}$ enclosed by $\mathcal{M}_{T_j^+}$. Note that since we are in the mean-convex setting the surgery procedure gives rise to points $x \in \Omega$ such that $x \notin \mathcal{M}_t$ for any t. In order to produce a continuous function on $\bar{\Omega}$ we therefore define

$$u(x) := \begin{cases} t & \text{for all } x \in \mathcal{M}_t \\ T_j & \text{for all } x \in E_{T_j^-} \setminus F_{T_j^+} \end{cases}$$

for each region $E_{T_j^-} \setminus F_{T_j^+}$ overlooked due to surgery. These regions are therefore by definition plateaus in graph(u) and the level-sets $\{u = T_j\}$ $(j = 1, \dots, N)$ may not be smooth hypersurfaces. Clearly $u \in C^{0,1}(\bar{\Omega})$ and we have

$$\mathcal{M}_{T_j^-} = \partial\left(\operatorname{int}\{x \in \Omega \mid u \geq T_j\}\right) \quad \text{and} \quad \mathcal{M}_{T_j^+} = \partial\left\{x \in \Omega \mid u > T_j\right\}.$$

For convenience we define the sets $\Sigma_t := \{u > t\}$ and $\tilde{\Sigma}_t := \operatorname{int}\{u \geq t\}$ so that $\mathcal{M}_{T_j^+} = \partial\Sigma_{T_j}$ and $\mathcal{M}_{T_j^-} = \partial\tilde{\Sigma}_{T_j}$. Away from the surgery times, $\mathcal{M}_t = \partial\Sigma_t = \partial\tilde{\Sigma}_t$. Finally, $\mathcal{M}_t := \emptyset$ for all $t > T_N$.

Approximating Sequence. Theorem 2.15 produces a set of parameters $\omega_1, \omega_2, \omega_3$ which depend only on α and which produce a flow with surgeries starting from $\partial\Omega$ and satisfying properties i)-iv) for any choice $H_1 \geq \omega_1 R^{-1}$ with $H_2 = \omega_2 H_1$ and $H_3 = \omega_3 H_2$.

We therefore consider any increasing sequence of parameters $\{H_k^i\}_{i\geq 1} = \{H_1^i, H_2^i, H_3^i\}_{i\geq 1}$, corresponding to a sequence $\{\mathcal{M}_t^i\}_{i\geq 1}$ of mean curvature flows with surgeries, along which the surgery times grow and the necks removed at these times become increasingly thin.[1] The ratios ω_2, ω_3 are fixed along the sequence – that is, $H_2^i = \omega_2 H_1^i$ and $H_3^i = \omega_3 H_2^i$ for each i. The following is a precise statement of Theorem 1.5.

[1]Recall that each $\mathcal{M}_t^i$ exists on a time interval $0 \leq t \leq T_N$ where N and T_N depend on i.

Theorem 3.7 (Convergence to Weak Solution) *Let $\mathcal{M}_0 \in \mathcal{C}(R,\alpha)$ with $n \geq 3$ such that $\mathcal{M}_0 = \partial\Omega$ for some open, bounded $\Omega \subset \mathbb{R}^{n+1}$. Let u_L be the weak solution of the level-set flow on Ω, and denote by u_i the level-set functions representing the solutions $\mathcal{M}_t^i$ of mean curvature flow with surgeries starting from $\mathcal{M}_0$ with parameters H_1^i, H_2^i, H_3^i. For all sufficiently small $\varepsilon > 0$ we have*

$$\sup_{\bar{\Omega}} |u_i - u_L| \leq C(H_1^i)^{-1+\varepsilon}$$

where $C = C(n, \varepsilon, \mathcal{M}_0)$.

The primary tools from the weak theory will be the clearing out lemma (Theorem 3.12 below) and the smooth avoidance principle; the necessary estimates will come from Proposition 3.6. The theorem will follow from Corollary 2.18, Lemma 3.10 and Proposition 3.11.

We first use Theorem 3.3 ii) to show that u_i is bounded above by u_L for each i. Then, in Section 3.3, we use the clearing out lemma to show that the weak solution can be used in addition as a lower barrier after an appropriate time-translation which depends explicitly on the parameters H_k^i and converges to zero as $i \to \infty$.

3.2 A Barrier Result

This section is concerned with a global barrier result for the level-set functions u_i and u_L. We use the avoidance principle to show that, for each set of parameters H_k^i, the resultant mean curvature flow with surgeries respects the weak solution as an "external" barrier. This is a natural consequence of the fact that the surgery procedure does not interfere with surfaces outside the neck on which it is performed.

We recall the concept of the "solid tube" [HS3, Prop. 3.25] enclosed by a hypersurface neck:

Proposition 3.8 (Solid Tube, [HS3]) *Given a normal (ϵ, k)-hypersurface neck $\mathcal{N} : \mathbb{S}^{n-1} \times [0, L] \to \mathcal{M}^n \subset \mathbb{R}^{n+1}$ with parameters $L \geq 20 + 8\Lambda \geq 100$, $0 < \epsilon \leq \epsilon_0$ and $k \geq k_0$ depending on n, there exists a unique local diffeomorphism*

$$G : \bar{B}_1^n \times [0, L] \to \mathbb{R}^{n+1}$$

such that

i) G (restricted to the cylinder) agrees with $\mathcal{N}$;

ii) each cross-section $G(\bar{B}_1^n \times \{z_0\}) \subset \mathbb{R}^{n+1}$ is an embedded area minimising hypersurface;

iii) G restricted to each slice $\bar{B}_1^n \times \{z_0\}$ is a harmonic diffeomorphism; and

iv) G is ϵ-close in C^{k+1}-norm to the standard isometric embedding of a solid cylinder in $\mathbb{R}^{n+1}$.

We want to compare the flow with surgeries to the weak evolution. Before the first surgery time, $\mathcal{M}_t^i$ and Γ_t agree – indeed they correspond to the smooth solution for $0 \leq t < T_1$, compare Theorem 3.3. At the first surgery time, $\mathcal{M}^i_{T_1^-} = \Gamma_{T_1}$ and, by construction (see Section 2.2 and proof of Lemma 3.10),

$$\mathcal{M}^i_{T_1^+} \subset E_{T_1^-} = \bar{\Omega}_{T_1}.$$

The pressing question is therefore: what happens for $t > T_1$? The following is general result which is quintessentially parabolic in nature. It dictates that two surfaces which agree except on some subset must separate instantaneously under the smooth evolution.

Lemma 3.9 (Tearing Apart, [ES2]) *Let $W \subset \mathbb{R}^{n+1}$ be open and bounded and consider a subset $\hat{W} \subset W$. Suppose that $\mathcal{M}_0 = \partial W$ and $\hat{\mathcal{M}}_0 = \partial \hat{W}$ are smooth and mean-convex with $\hat{\mathcal{M}}_0 \subset \bar{W}$ and $\hat{\mathcal{M}}_0 \neq \mathcal{M}_0$. Then the corresponding solutions $\mathcal{M}_t, \hat{\mathcal{M}}_t$ of mean curvature flow satisfy*

$$\hat{\mathcal{M}}_t \cap \mathcal{M}_t = \emptyset$$

for $t > 0$ as long as they remain smooth.

We conclude therefore that $\mathcal{M}_t^i$ is trapped inside Ω_t for all $t > T_1$. This corresponds to the following global barrier result; we present a simply proof using only the smooth avoidance principle.

Lemma 3.10 (Upper Barrier) *Let Ω, u_i and u_L be as in Theorem 3.7. Then for each i we have*

$$u_i(x) \leq u_L(x) \tag{3.4}$$

for all $x \in \bar{\Omega}$.

Proof. Observe that $\mathcal{M}_0^i = \Gamma_0 = \partial\Omega$ implies $\mathcal{M}_\delta^i \subset\subset \Omega$ for all $\delta > 0$. By Theorem 3.3,

$$\frac{d}{dt}\mathrm{dist}(\mathcal{M}^i_{t+\delta}, \Gamma_t) \geq 0$$

as long as $\mathcal{M}^i_{t+\delta}$ remains smooth. However, it is straightforward to see that this property is preserved by the surgery construction.

Each standard surgery is performed on an (ϵ, k)-hypersurface neck $\mathcal{N}_0$ of length L which encloses a solid tube $G_0 : \bar{B}_1^n \times [0, L] \to \mathbb{R}^{n+1}$. As in Chapter 2, we denote by $\mathcal{U}^+$ the two regions (diffeomorphic to discs) introduced by each standard surgery (see Section 2.2 above). By construction, $\mathcal{U}^+ \subset G_0(\bar{B}_1^n \times [0, L])$ and therefore

$$\mathcal{M}^i_{T_j^+} \subset E_{T_j^-}$$

(this is property is clearly respected by step two of the surgery procedure in which finitely many components are discarded). It follows that

$$\mathcal{M}^i_{T_j^- + \delta} \subset\subset \Omega_{T_j} \quad \Longrightarrow \quad \mathcal{M}^i_{T_j^+ + \delta} \subset\subset \Omega_{T_j}.$$

In fact, $\operatorname{dist}(\mathcal{M}^i_{t+\delta}, \Gamma_t)$ is non-decreasing across each surgery time T_j $(j = 1, \ldots, N)$ and

$$\mathcal{M}^i_{t+\delta} \cap \Gamma_t = \emptyset \quad \text{and} \quad \mathcal{M}^i_{t+\delta} \subset\subset \Omega_t$$

for all $t \geq 0$.

Since there are only finitely many surgery times, and since u_i and u_L are continuous, this yields the desired result. □

3.3 Time-Shifting the Weak Solution

In this section we complete the proof of Theorem 3.7. Having established that $u_i \leq u_L$ on $\bar{\Omega}$ for each i, our goal now is translate u_L "vertically" in time until it sits below u_i. We use the "clearing out lemma" to analyse the behaviour of the weak solution locally within hypersurface necks and control the construction in a quantitative way.

"Time-Shifting". The heuristic idea can be described as follows. As before we have $\partial\Omega = \mathcal{M}_0 \in \mathcal{C}(R, \alpha)$ and the flow with surgeries $\mathcal{M}^i_t$ for a given choice of parameters. At the first surgery time T_1, more precisely at T_1^-, $\mathcal{M}^i_t$ agrees with the classical and weak solutions. We therefore freeze $\mathcal{M}^i_{T_1^-}$ and run the weak solution a little longer, until

$$\Gamma_{T_1+t_w} \subset\subset \Sigma_{T_1}.$$

That is, we give the weak solution enough time to vacate the regions modified by surgery. This must happen for some constant t_w in light of the two-convex assumption on the initial data, and we will show that t_w can be controlled explicitly in terms of the surgery parameters with the expected parabolic scaling. We then perform surgery on $\mathcal{M}^i_{T_1^-}$, after which

$$\mathcal{M}^i_{T_1^+} \cap \Gamma_{T_1+t_w} = \emptyset.$$

We can then restart the two evolutions. Suppose that at any surgery time T_j we have

$$\mathcal{M}^i_{T_j^+} \cap \Gamma_{T_j+jt_w} = \emptyset \quad \text{and} \quad \Gamma_{T_j+jt_w} \subset\subset \Sigma_{T_j}.$$

It follows from the avoidance principle that

$$\mathcal{M}^i_t \cap \Gamma_{t+jt_w} = \emptyset$$

on the interval $[T_j^+, T_{j+1}^-]$ until the next surgery time. Note that we need not keep track of the information on the precise distance between the two solutions. At each subsequent surgery time T_{j+1}, we again freeze $\mathcal{M}^i_{T_{j+1}^-}$ and apply an additional translation t_w to the weak solution. Since only finitely many surgeries are required, this process need be repeated only finitely many times. The fundamental quantity to control is therefore the combined scaling of the estimates on t_w and N. The length parameter L again plays a decisive role.

This is the geometric foundation of the following barrier principle.

Proposition 3.11 (Lower Barrier) *Let Ω, u_i and u_L be as in Theorem 3.7. We can choose $L = L(n)$ sufficiently large such that for each i we have*

$$u_L(x) - Nt_w \leq u_i(x)$$

for all $x \in \bar{\Omega}$, where t_w satisfies

$$t_w \leq C(n)L^2(H_1^i)^{-2}. \tag{3.5}$$

The proof of Proposition 3.11 requires techniques which take into account the local geometry of necks.

Clearing Out. The "clearing out lemma" is due originally to Brakke [B, Sect. 6.3].[2] It dictates that if the surface has small area ratio with respect to a ball of given radius, then the solution of mean curvature flow must clear out of a smaller concentric ball in a precise quantitative way – that is, it must vacate the ball of half the radius, for example, after a waiting time proportional to the square of that radius:

Theorem 3.12 (Clearing Out Lemma, [B]) *There exist constants $\theta, C > 0$ depending only on n such that, for any $x_0 \in \mathbb{R}^{n+1}$ and $\rho > 0$, the estimate*

$$|\Gamma_{t_0} \cap B_\rho(x_0)| \leq \theta \rho^n$$

implies

$$\Gamma_t \cap B_{\rho/2}(x_0) = \emptyset$$

where

$$t - t_0 \leq C\rho^2. \tag{3.6}$$

The parabolic scaling of the estimate on the waiting time is crucial; it leads to a refined upper bound on the time t_w required by the weak solution to vacate the regions modified by surgery.

Proposition 3.11 will follow from the next two results, which we state and prove separately, in combination with Theorem 3.12. We require a detailed description of the structure of the regions altered by surgery.

Surgery Regions. In [HS3, Pf. of Thm. 8.1] Huisken and Sinestrari established that at each surgery time, all points with mean curvature exceeding H_2^i are contained in one of finitely many disjoint regions $\mathcal{A}_l$, each of which either covers an entire connected component of known topology or has a boundary $\partial\mathcal{A}_l$ consisting of either one or two cross-sections with mean radius

$$r_\partial^i = \frac{2(n-1)}{H_1^i}.$$

More precisely, each $\mathcal{A}_l$ must assume one of five possible structures:

[2] See also [ES2, Thm. 7.3], as well as [E1, Prop. 4] for a streamlined proof of the "smooth" clearing out lemma.

a) $\partial\mathcal{A}_l = \emptyset$

- $\mathcal{A}_l$ is uniformly convex and diffeomorphic to $\mathbb{S}^n$;
- $\mathcal{A}_l$ is the union of a hypersurface neck $\mathcal{N}_0$ with two regions diffeomorphic to discs and forms a connected component diffeomorphic to $\mathbb{S}^n$;
- $\mathcal{A}_l$ is a maximal hypersurface neck $\mathcal{N}_0$ which covers an entire connected component of $\mathcal{M}^i_{T_j^-}$ and is diffeomorphic to $\mathbb{S}^{n-1}\times\mathbb{S}^1$;

b) $\partial\mathcal{A}_l \neq \emptyset$

- $\mathcal{A}_l$ is the union of a hypersurface neck $\mathcal{N}_0$ with a region diffeomorphic to a disc, and has one boundary component with mean radius r^i_∂;
- $\mathcal{A}_l$ is a hypersurface neck $\mathcal{N}_0$ with two boundary components (each of which has mean radius r^i_∂) and is therefore diffeomorphic to $\mathbb{S}^{n-1}\times[0,1]$.

We denote by $\mathcal{G}_l$ the region enclosed by $\mathcal{A}_l$. According to the construction in [HS3] (see Section 2.2 above), all connected components $\mathcal{A}_l$ of known topology are discarded at the surgery time. In addition, one standard surgery is performed at the nearest cross-section to each boundary component (whenever they arise) with mean radius $r^i_0 = (n-1)/H^i_1$, forming a connected component diffeomorphic to $\mathbb{S}^n$ which is also discarded.

The points modified at a surgery time T_j therefore belong either to hypersurface necks – these include in particular the pieces of the surface on which surgery is actually performed – or to components of known topology which are disconnected from the rest of the surface.

The spheres and tori discarded at the surgery time are referred to in [HS3] as "components removed afterwards". These may contain pieces which are not cylindrical, but any such region is also precisely controlled and will be dealt with in Lemma 3.16.

Area Estimates. We first consider the regions in which the surgery procedure takes place. Let T_j be a surgery time and consider a hypersurface neck $\mathcal{N}_0 \subset \mathcal{M}^i_{T_j^-}$ on which surgery is performed. Let G_0 be the associated solid tube. In the next lemma, we show that the hypothesis (3.12) in the clearing out lemma is satisfied if we choose ρ proportional to the surgery scale r^i_0. Note that $\Gamma_t = \partial^*\Omega_t$ for a.e. $0 \le t \le T$ – see Proposition 4.3 below.

Lemma 3.13 (Area Bound) *We can choose $L = L(n)$ sufficiently large such that the following property holds. Let t_0 be such that $\Gamma_{t_0} = \partial^*\Omega_{t_0}$ and $\Gamma_{t_0} \subset \tilde{\Sigma}^i_{T_j}$. Then there exists $C = C(n)$ such that, setting*

$$\rho_0 = CL(H^i_k)^{-1},$$

we have the estimate

$$|\Gamma_{t_0} \cap B_{\rho_0}(x)| \le \theta\rho_0^n \tag{3.7}$$

at each $x \in G_0$ which is modified by the surgery procedure. Here $\theta = \theta(n)$ is the constant from Theorem 3.12.

Proof. We are given a hypersurface neck $\mathcal{N}_0 \subset \mathcal{A}_l$ on which surgery is to be performed. Let $G_0 : \bar{B}_1^n \times [0, L] \to \mathbb{R}^{n+1}$ and let x be a point in G_0 which is altered by surgery; by assumption, $x \in G_0(\bar{B}_1^n \times [\Lambda, L - \Lambda])$. Since G_0 can be made as close as we wish to the standard isometric embedding of a piece of the solid cylinder in $\mathbb{R}^{n+1}$, we can therefore arrange that at each such x we have

$$|\mathcal{N}_0 \cap B_{(\Lambda r_0^i)}(x)| \leq 4\Lambda\omega_{n-1}(r_0^i)^n.$$

We choose Λ (and therefore L) sufficiently large such that

$$\frac{\omega_{n-1}}{\Lambda^{n-1}} \leq \frac{\theta}{4}$$

which in turn implies that

$$|\mathcal{N}_0 \cap B_{(\Lambda r_0^i)}(x)| \leq \theta(\Lambda r_0^i)^n. \tag{3.8}$$

Given this choice of Λ we set

$$\rho_0 = \frac{(n-1)\Lambda}{H_1^i}.$$

We now verify that a weak solution trapped inside $\mathcal{N}_0$ satisfies (3.7) with this choice of ρ_0.

In order to show that

$$|(\Gamma_{t_0} \cap G_0) \cap B_{\rho_0}(x)| \leq \theta\rho_0^n,$$

we use Proposition 3.6, the area minimisation property of the weak solution. In fact, direct comparison of the set $\Omega_{t_0} \cap G_0$ with the perturbation $\Omega_{t_0} \cup G_0$ yields the estimate

$$|(\Gamma_{t_0} \cap G_0) \cap B_{\rho_0}(x)| \leq |\mathcal{N}_0 \cap B_{\rho_0}(x)|$$

courtesy of the outward minimising property (3.3).

To complete the proof, it is necessary to confirm that no other part of the surface can interfere with $B_{\rho_0}(x)$ - that is,

$$B_{\rho_0}(x) \cap (\bar{\Omega}_{t_0} \setminus G_0) = \emptyset.$$

To this end, let $\mathcal{B}_{g(t)}(p, r) \subset \mathcal{M}^n$ be the closed ball of radius $r > 0$ around $p \in \mathcal{M}^n$ with respect to the metric $g(t)$:

$$\mathcal{B}_{g(t)}(p, r) = \{q \in \mathcal{M}^n \mid d_{g(t)}(p, q) \leq r\}.$$

Consider a space-time point (p_0, T_j) such that p_0 lies at the center of the neck $\mathcal{N}_0 \subset \mathcal{M}_{T_j}^i$ and set

$$R_0 = \frac{n-1}{H(p_0, T_j)} \quad \text{and} \quad \mathcal{B}_0 = \mathcal{B}_{g(T_j)}(p_0, R_0 L).$$

Following [HS3], we define

$$\mathcal{P}(p,t,r,\omega) := \big\{(q,s) \;\big|\; q \in \mathcal{B}_{g(t)}(p,r),\; s \in [t-\omega, t]\big\} \tag{3.9}$$

to be the "backward parabolic neighbourhood" of (p,t). In the context of smooth mean curvature this definition is unambiguous, but in the setting of mean curvature flow with surgeries further explanation is required: recall that the solution of the flow with surgeries is a family of smooth flows

$$F^j : \mathcal{M}_j \times [T_{j-1}, T_j] \to \mathbb{R}^{n+1}.$$

Away from the surgery times, the ball $\mathcal{B}_{g(t)}(p,r)$ belongs to the manifold $\mathcal{M}_j$ corresponding to the interval $[T_{j-1}, T_j]$ containing t. When t corresponds to a surgery time, it is necessary to distinguish between the manifolds before and after surgery.

Assume for the moment that no point in $\mathcal{B}_{g(T_j)}(p_0, R_0L)$ belongs to a region which has been modified by surgery between $T_j - r_0^2\omega$ and T_j. That is, any surgery which has occurred during this time interval only interferes with parts of the surface disjoint from $\mathcal{B}_{g(T_j)}(p_0, R_0L)$.

Huisken and Sinestrari [HS3, Lem. 7.4] showed that at any point in $\mathcal{P}(p_0, T_j, R_0L, R_0^2\omega)$ the Weingarten operator of the surface and its spacial derivatives (up to order k and after appropriate rescaling) are ϵ-close to the corresponding quantities associated with the standard shrinking cylinder. Furthermore, for any $t \in [T_j - \omega R_0^2, T_j]$, they showed [HS3, Lem. 7.9] that the point (p_0, t) lies at the center of an (ϵ, k_0-1)-hypersurface neck $\mathcal{N}_t \subset \mathcal{B}_0$ of length at least $L-2$ (here $k_0 \geq 2$). Let

$$\varrho(r,s) = \big(r^2 - 2(n-1)s\big)^{1/2}$$

for $s \leq 0$. Note that $\varrho(r,s)$ is the radius at time s of a standard n-dimensional cylinder (with initial radius r) evolving (in s) by mean curvature flow. Then in addition the mean radius $r(z)$ of every cross-section of $\mathcal{N}_t$ is given by

$$\varrho(R_0, t - T_j)(1 + O(\epsilon))$$

and there exists a unit vector $\chi \in \mathbb{R}^{n+1}$ such that

$$|\,\langle \nu(p,t), \chi\rangle\,| \leq \epsilon$$

for all $p \in \mathcal{N}_t$.

It is clear that we can therefore choose $\omega = C(n)L^2$ sufficiently large (see Remark 3.14 below) to ensure that $B_{\rho_0}(x)$ is completely contained within the solid tube enclosed by the hypersurface neck at the earlier time $T_j - \omega r_0^2$. Notice that no surgery can interfere with $\mathcal{N}_t$ on the time interval $[T_j - \omega R_0^2, T_j]$ since the curvature there is below the surgery scale H_1^i. By the curvature assumption on the initial data, each point $x \in \mathbb{R}^{n+1}$ satisfies $x \in \Gamma_t$ for at most one t. This ensures that the ball does not touch any part of the weak solution outside the neck $\mathcal{N}_0$ and therefore completes the proof. $\square$

Remark 3.14 (The Parameter ω) In [HS3, Lem. 7.9] there is an additional requirement that $\omega \leq \omega_0$ for some $\omega_0 = \omega_0(n)$. The restriction ω_0 arises from an application of the pointwise gradient estimate [HS3, Thm. 6.1]

$$|\nabla A|^2 \leq C_1|A|^4 + C_2$$

where $C_1 = C_1(n)$ and $C_2 = C_2(\mathcal{M}_0) > 0$. However, careful examination of the proofs of the gradient estimate in Chapter 6 and the neck detection results in Chapter 7 reveals that ω_0 can be ignored. This is pointed out also in [HS3, Rem. 6.2], where it is explained that [HS3, Pf. of Thm. 6.1] is sufficient to establish the stronger estimate

$$|\nabla A|^2 \leq C_1\delta|A|^4 + C_\delta$$

for any $\delta > 0$ where now $C_\delta = C_\delta(\delta, \mathcal{M}_0)$. Choosing δ smaller corresponds to increasing ω_0, and since δ can be made as small as we wish, this justifies our claim in the proof of Lemma 3.13 that ω can be chosen freely.

Remark 3.15 (Alternative Proofs[3]) We outline an alternative approach. Let $x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$. Ecker [E1] observed that

$$\left(\frac{d}{dt} - \Delta\right)(|x|^2 - (n-\beta)x_{n+1}^2 + 2\beta t) \leq 0$$

for $0 \leq \beta \leq n$ (see Dierkes [D] for the elliptic case). He therefore obtained the following hyperboloidal barrier courtesy of the maximum principle: if

$$\mathcal{M}_{t_0} \subset \left\{(n-1-\beta)x_{n+1}^2 \geq x_1^2 + \cdots + x_n^2 - \iota^2\right\}$$

then

$$\mathcal{M}_{t+t_0} \subset \left\{(n-1-\beta)x_{n+1}^2 \geq x_1^2 + \cdots + x_n^2 - \iota^2 + 2\beta(t-t_0)\right\}$$

for $t - t_0 \leq \iota^2/(2\beta)$ and where $0 \leq \beta < n-1$. Returning to the setting of necks, we observe that ι must be made proportional to r_0^i and therefore that the scaling of this estimate agrees with the scaling of the estimate on the waiting time provided by the clearing out lemma. Note that the barriers here could be constructed locally within the neck, allowing one to circumvent the argument on the length of the backward parabolic neighbourhood.

We are now able to deal with points $x \in \mathcal{G}_l$ which do not sit inside a hypersurface neck.

Components Removed Afterwards. Let T_j be a surgery time. Consider any $\mathcal{A}_l \subset \mathcal{M}_{T_j}^i$ and the corresponding domain $\mathcal{G}_l \subset \mathbb{R}^{n+1}$. Let $\mathcal{S} \subset \mathcal{G}_l$ be the open set in $\mathbb{R}^{n+1}$ enveloped by a component removed at the surgery time T_j. We state and prove a general result which gives an upper bound on the extinction time

$$T^* = \sup\left\{\, t \geq 0 \;\middle|\; \Gamma_t \neq \emptyset \,\right\}$$

of the weak solution generated by any component discarded at the surgery time.

[3] The self-similarly shrinking torus constructed by Angenent [A1] is another well-known barrier which can be used to estimate the time required for a neck to pinch off under mean curvature flow.

Lemma 3.16 (Extinction of Discarded Components) *Let Ω and $\mathcal{M}_t^i$ be as in Theorem 3.7, and let H_1^i, H_2^i, H_3^i be the corresponding parameters. Consider a discarded component $\mathcal{S}$ produced by $\mathcal{M}_t^i$ at a surgery time (as described above) and let u_L^* be the weak solution of the level-set flow generated by $\mathcal{S}$. There exists a constant $C = C(n)$ such that*

$$T^* \leq C(H_1^i)^{-2} \tag{3.10}$$

where T^* denotes the extinction time of u_L^* on $\mathcal{S}$.

Proof. Components removed afterwards must be diffeomorphic either to $\mathbb{S}^n$ or to $\mathbb{S}^{n-1} \times \mathbb{S}^1$. The only way a copy of $\mathbb{S}^{n-1} \times \mathbb{S}^1$ can occur is in the form of a maximal normal (ϵ, k)-hypersurface neck without boundary. An argument similar to the proof of Lemma 3.13 in combination with the clearing out lemma implies that $T^* \leq C_1(n)(H_k^i)^{-2}$. Indeed the same argument can be applied to any neck which arises as a subset of a discarded component.

In the remaining cases $\partial\mathcal{S}$ is diffeomorphic to $\mathbb{S}^n$. Hence $\partial\mathcal{S}$ can arise (1) as a uniformly convex component, (2) as the union of a hypersurface neck with two regions diffeomorphic to discs or (3) as a component which becomes disconnected from the rest of the surface as a result of surgery.

If two surgeries are performed on a region $\mathcal{A}_l$ with two boundary components then the resultant connected component satisfies $T^* \leq C_1(n)(H_k^i)^{-2}$ by the argument above. The only other regions are therefore enclosed by uniformly convex connected components or by pieces of the surface which are diffeomorphic to discs and which border a hypersurface neck on one side.

In case (1) we can use the curvature bound on any uniformly convex connected component [HS3, Thm. 7.14] in combination with Myer's theorem and an appropriate spherical barrier to obtain $T^* \leq C_2(n)(H_k^i)^{-2}$. To complete the proof, we must deal with the remaining convex regions which are diffeomorphic to discs.

Huisken and Sinestrari [HS3, Thm. 8.2] showed that a neck can either close up and end with a convex cap or border a disc which was inserted by a previous surgery [HS3, Lem. 7.12]. In either situation we can apply the avoidance principle, this time using a straight cylinder as a smooth barrier.

By [HS3, Pf. of Thm. 8.2] the curvature of this cylinder is bounded below by H_1^i up to a constant. We have already established that after a time bounded above by $C_1(n)(H_k^i)^{-2}$ the weak solution must clear out of the bordering neck. By the curvature assumption, it cannot re-enter the collar of the neck. Then by comparison with the smooth evolution of a standard cylinder the weak solution (if not already empty) disappears completely after an additional time bounded above by $C_3(n)(H_k^i)^{-2}$. Choosing $C = \max\{C_1, C_2\} + C_3$ completes the proof. □

Proof of Proposition 3.11. Starting off as in the proof of Lemma 3.10 above, we have $\mathcal{M}_0^i = \Gamma_0 = \partial\Omega$ and therefore $\Omega_\delta \subset\subset \Omega$ for all $\delta > 0$. Theorem 3.3 gives

$$\Omega_{T_1+\delta} \subset\subset \tilde{\Sigma}_{T_1}^i$$

for all $\delta > 0$. Applying Proposition 4.3, Lemma 3.13, the clearing out lemma and Lemma 3.16 we obtain

$$\Omega_{T_1+\delta+t_w} \subset\subset \Sigma^i_{T_1}$$

for all $\delta > 0$ and for some t_w satisfying (3.5). Repeating the argument on each smooth time interval we conclude

$$\Omega_{t+\delta+Nt_w} \subset\subset \Sigma^i_t$$

for all $\delta > 0$ and for all $t \geq 0$, and the result follows from the continuity of the level-set functions. □

Proof of Theorem 3.7. Combine Lemma 3.10, Proposition 3.11 and Corollary 2.18. □

Remark 3.17 (Number of Surgeries) We emphasize the need for the estimates in Chapter 2. If we replace the bound from Corollary 2.18 with the estimate

$$N \leq C(H^i_1)^n,$$

the size of the time-translation Nt_w blows up as $H^i_1 \to \infty$.

4 Remarks on Regularity Estimates for MCF

Here we assemble some preliminary consequences of our undertaking in the previous chapters. We begin by investigating the finer properties of our approximation result: we borrow heavily from [S] to make precise the sense in which the surfaces $\mathcal{M}_t^i$ approximate the level-sets Γ_t, and pass the estimates from Chapter 2 to limits. Our approach makes use of the regularity theory developed by White [W1]. Finally, we point out an application of Remark 2.9 to recent work by Ecker [E3] on the size of the singular set at the first singular time.

4.1 Regularity Estimates for Weak Solutions

As a first consequence of Theorem 3.7 we show that the convergence of the level-set functions implies convergence, in the sense of Radon measures, of the individual level-sets at almost every time. The L^p estimates from Chapter 2 can then be passed to limits.

Preliminaries. As in Chapter 3, we have an increasing sequence of parameters $\{H_k^i\}_{i\geq 1} = \{H_1^i, H_2^i, H_3^i\}_{i\geq 1}$ and corresponding sequence $\mathcal{M}_t^i$ of mean curvature flows with surgeries.

Definition 4.1 (Radon Measures) We define the Radon measures

i) $\mu_t^i := \mathcal{H}^n \llcorner \partial\Sigma_t^i \equiv \mathcal{H}^n \llcorner \partial\{u_i > t\}$,

ii) $\tilde{\mu}_t^i := \mathcal{H}^n \llcorner \partial\tilde{\Sigma}_t^i \equiv \mathcal{H}^n \llcorner \partial(\mathrm{int}\{u_i \geq t\})$,

iii) $\mu_t := \mathcal{H}^n \llcorner \partial^*\Omega_t \equiv \mathcal{H}^n \llcorner \partial^*\{u_L > t\}$.

It is well-known that the family of Radon measures μ_t is a Brakke flow [B].

Theorem 4.2 (No Mass Drop, [MS]) *Let $\Omega \subset \mathbb{R}^{n+1}$ be open and bounded such that $\partial\Omega$ is and smooth and has non-negative mean curvature. Then μ_t is continuous in time for all $0 \leq t \leq T$.*

In fact, [MS, Cor. 1.2] assumes only that $\partial\Omega$ is C^1 and mean-convex with respect to the generalised mean curvature. We recall the following properties of the level-set flow.

Proposition 4.3 (Further Properties, [ES3, MS]) *We have:*

i) $\mathcal{H}^{n+1}(\{Du_L = 0\}) = 0$ *and in particular* $\mathcal{H}^{n+1}(\{u_L = t\}) = 0$ *for all* $t \geq 0$*;*

ii) for a.e. $t \geq 0$, $\Gamma_t = \partial^*\Omega_t$ $\mathcal{H}^n$*-a.e.;*

iii) for a.e. $t \geq 0$, Γ_t *is a unit density, n-rectifiable varifold with generalised mean curvature* $\mathbf{H}$.

Property i) is commonly referred to as "non-fattening" and implies that

$$\partial^*\{u_L > t\} = \partial^* K_t$$

for all $t \geq 0$ (recall from Remark 3.4 that $K_t = \{u_L \geq t\}$). Since $u \in C^{0,1}(\bar{\Omega}) \subset BV(\Omega)$, comparison of the respective co-area formulas yields

$$\int_0^T \big|\, \{u_L = t\} \setminus \partial^*\{u_L > t\} \,\big|\, dt = 0.$$

We therefore have

$$\partial^*\{u_L > t\} = \{u_L = t\} \quad \mathcal{H}^n\text{-a.e.} \tag{4.1}$$

for a.e. $t \geq 0$. Following [S, MS], this leads us to a natural definition.

Definition 4.4 (Set of Good Times) We denote by $I \subset [0, T]$ of the set of times such that (4.1) holds.

That is, $\mu_t = \mathcal{H}^n \llcorner \Gamma_t$ for all $t \in I$. The set I has full $\mathcal{L}^1$ measure and we begin by establishing convergence for each $t \in I$.

In order to refine this result, we will require an estimate on the size of the singular set for mean curvature flow.

Definition 4.5 (Singular Points, [W1]) A point $(x, t) \in \mathrm{graph}(u_L)$ is called a regular point if

i) there exists a neighbourhood around (x, t) in $\mathbb{R}^{n+1} \times \mathbb{R}$ in which the set enclosed by $\mathrm{graph}(u_L)$ is a smooth manifold with boundary, and

ii) the tangent plane to $\mathrm{graph}(u_L)$ is not $\mathbb{R}^{n+1} \times \{0\}$.

If (x, t) is not a regular point then it is by definition a singular point.

The set of singular points is then called the singular set $\mathrm{sing}(u_L)$ of the weak solution. The following result is White's famous theorem controlling the size of the singular set, see [W1].

Theorem 4.6 (Singular Set, [W1]) *The singular set*

$$\mathrm{sing}(u_L) \subset \mathrm{graph}(u_L)$$

has parabolic Hausdorff dimension at most $n-1$. In particular, $\mathrm{sing}(\Gamma_t)$ *is closed and satisfies*

$$\mathcal{H}^n(\mathrm{sing}(\Gamma_t)) = 0$$

for all $t > 0$. Here $\mathrm{sing}(\Gamma_t) = \{x \in \Omega \mid (x,t) \in \mathrm{sing}(u_L)\} \subset \mathbb{R}^{n+1}$ *denotes the t-level-set of* $\mathrm{sing}(u_L)$.

Remark 4.7 (Projection onto $\mathbb{R}^{n+1}$) If we denote by Ω_S the projection of $\mathrm{sing}(u_L)$ onto Ω, then Ω_S is closed and satisfies

$$\mathcal{H}^n(\Omega_S) = 0.$$

Finally, for convenient reference, we include a statement of Brakke's local regularity theorem [B, 6.11] (see also [I2, Thm. 12.1]). In the following form it applies only to smooth solutions $\mathcal{M}_t$ of mean curvature flow and can be found in [E2, Sect. 3].

Theorem 4.8 (Brakke's Regularity Theorem, [B]) *There exist constants $\theta, C > 0$ depending only on n such that if for some $\rho > 0$ and for all $t \in [T - C\rho^2, T]$ we have*

i) $|\mathcal{M}_t \cap B_\rho(x_0)| \leq \frac{3}{2}\omega_n \rho^n$,

ii) $\frac{\omega_n}{2}\left(\frac{\rho}{2}\right)^n \leq |\mathcal{M}_t \cap B_{\rho/2}(x_0)| \leq \frac{3\omega_n}{2}\left(\frac{\rho}{2}\right)^n$, *and*

iii) $\int_A (x - x_0)^2_{n+1}\, d\mu \leq \theta \rho^{n+2}$,

where $A = \mathcal{M}_{T-C\rho^2} \cap B_\rho(x_0)$, then x_0 is a regular point.

Mass Bounds and Weak Compactness. From Remark 2.17 we have the uniform area bound

$$|\mathcal{M}_t^i| \leq |\mathcal{M}_0| \leq \alpha_2 R^n \tag{4.2}$$

for all $0 \leq t \leq T$. By the weak compactness theorem for Radon measures there exist subsequences $\{\mu_t^{i_j}\}_{j\geq 1}$, $\{\tilde{\mu}_t^{i_j}\}_{j \geq 1}$ and Radon measures μ, $\tilde{\mu}$ such that

$$\mu_t^{i_j} \to \mu \quad \text{as measures,} \tag{4.3}$$
$$\tilde{\mu}_t^{i_j} \to \tilde{\mu} \quad \text{as measures,} \tag{4.4}$$

for all $0 \leq t \leq T$. Theorem 3.7 then implies that

$$\mathrm{spt}(\mu) \subset \{u_L = t\} \quad \text{and} \quad \mathrm{spt}(\tilde{\mu}) \subset \{u_L = t\}. \tag{4.5}$$

We now want to establish that $\mu = \tilde{\mu} = \mu_t$ for all $t \in I$ and that this property is independent of the choice of subsequence. We will follow Schulze [S, Pf. of Prop. 5.10] – this requires us to prove that the solution of mean curvature flow with surgeries remains outward minimising after each surgery time.

Outward Minimising. Recall from Definition 3.5 that a set $E \subset U$ is outward minimising in some open set U if

$$|\partial^* E \cap K| \leq |\partial^* F \cap K|$$

for any $F \supset E$ such that $F \setminus E \subset\subset U$ and any compact set $K \supset (F \setminus E)$. The proof is not quite straightforward, however, since the condition is global in nature and the solution of mean curvature flow with surgeries will in general become disconnected after the first surgery time. Our approach makes use of well-known non-existence results for minimal surfaces and the length parameter L again plays a crucial role.

Proposition 4.9 (Outward Minimising) *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open bounded set such that $\partial\Omega \in \mathcal{C}(R, \alpha)$ is connected and consider the solution $\mathcal{M}_t^i$ of mean curvature flow with surgeries starting from $\mathcal{M}_0 = \partial\Omega$. We can choose $L = L(n)$ sufficiently large such that the sets*

$$\tilde{\Sigma}_t^i = \operatorname{int}\{u_i \geq t\} \quad \textit{and} \quad \Sigma_t^i = \{u_i > t\}$$

are outward minimising in Ω for all $0 \leq t \leq T_N$.

Proof. It is well known that the solution of classical mean curvature flow starting from any mean-convex initial data $\partial\Omega$ preserves the outward minimising property for as long as it exists (this follows from smoothness, $(\star)$, the curvature assumption and the divergence theorem).

Hence $\mathcal{M}_{T_1^-}^i$ is outward minimising and we now want to show that this property is preserved by surgery and again holds true for $\mathcal{M}_{T_1^+}^i$. Since $\mathcal{M}_{T_1^-}^i$ is a valid comparison set, we must first check that the area of the surface $\mathcal{M}_{T_1^+}^i$ after surgery is strictly less than that of $\mathcal{M}_{T_1^-}^i$. To this end we recall from [HS3] (see $p = 0$ case of Lemma 2.13 above) that

$$\left|\mathcal{M}_{T_1^-}^i\right| - \left|\mathcal{M}_{T_1^+}^i\right| \geq C(H_k^i)^{-n}.$$

Now $\mathcal{M}_{T_1^+}^i$ is obtained from $\mathcal{M}_{T_1^-}^i$ by performing surgery on finitely many independent hypersurface necks and by subsequently discarding finitely many connected components of known topology. Since $\mathcal{M}_{T_1^-}^i$ is outward minimising, any $F \supset \Sigma_{T_1}^i$ with $|F| \leq |\mathcal{M}_{T_1^+}^i|$ must in turn satisfy

$$F \subset (\tilde{\Sigma}_{T_1}^i \cup \mathcal{M}_{T_1^-}^i)$$

and $\partial^* F$ must agree with $\mathcal{M}_{T_1^-}^i$ outside the regions altered at the surgery time.

Consider any hypersurface neck $\mathcal{N}_0 \subset \mathcal{M}_{T_1^-}^i$ in normal form enclosing a solid tube G_0, and suppose that F minimises area among all surfaces outside $\mathcal{M}_{T_1^+}^i$. Then $F \cap G_0$ is a properly embedded minimal surface in $G_0 \setminus \bar{\mathcal{U}}^+$ where $\bar{\mathcal{U}}^+$ represents the closed domain bounded by the surgery caps $\mathcal{U}^+$ introduced during the surgery procedure.

We claim that there exists no connected minimal surface joining the two surgery caps. This follows from well-known non-existence results for minimal surfaces joining two sufficiently separated boundary components. We again must choose our length parameter L sufficiently large (but depending only on n), and the claim then follows for example from the elliptic result of Dierkes [D], or from Ecker's parabolic generalisation as described in Remark 3.15.

Finally, to see that no minimal surface can extend from $\mathcal{U}^+$ into $G_0 \setminus \bar{\mathcal{U}}^+$ without joining the two ends, we note that G_0 can be foliated by surgery caps and that this would produce a contradiction with the maximum principle.

Since, by assumption, $\mathcal{M}_0$ is connected, the only way that a component can become disconnected from the rest of the surface is as a result of surgery (unless the entire surface is discarded at the first surgery time). Removing any such component therefore preserves the outward minimising property. We can then repeat our argument a finite number of times on each smooth time interval and at each subsequent surgery time. □

Remark 4.10 (Limits of Outward Minimising Sets) Schulze [S, Lem. 5.6] showed that the outward minimising property is preserved by L^1_{loc} convergence. Note that the uniform convergence $u_i \to u_L$ implies L^1 convergence $\Sigma^i_t, \tilde{\Sigma}^i_t \to \Omega_t$ courtesy of Proposition 4.3.

Weak Convergence. We now have the necessary estimates – namely Theorem 3.7 and Proposition 4.9 – to follow [S, Pf. of Prop. 5.10] and establish that $\mu^i_t, \tilde{\mu}^i_t \to \mu_t$. We briefly review the argument for μ^i_t (the same approach of course works for $\tilde{\mu}^i_t$).

It follows from (4.3) and Proposition 4.9 that μ is absolutely continuous with respect to $\mathcal{H}^n$-measure. Then in view of (4.5) and (4.1), the differentiation theorem for Radon measures gives

$$\mu = \mu_t \llcorner f$$

where

$$f(x) = \lim_{\rho \to 0} \frac{\mu(B_\rho(x))}{\mu_t(B_\rho(x))} \quad \mathcal{H}^n\text{-a.e.}$$

Lower semi-continuity implies $f \geq 1$ $\mathcal{H}^n$-a.e. $x \in \Gamma_t$. Futhermore, a rescaling argument can be combined with the outward minimising property to show that $f \leq 1$ for $\mathcal{H}^n$-a.e. $x \in \partial^*\Omega_t$. These estimates are independent of the subsequence and therefore

$$\mu^i_t \to \mu_t \quad \text{and} \quad \tilde{\mu}^i_t \to \mu_t \tag{4.6}$$

for all $t \in I$ in the sense of measures.

We obtain the desired convergence for all $t \in [0, T]$ courtesy of Theorem 4.2 .

Allard's Compactness Theorem. We denote by δV the first variation of a varifold V. Note that if V is the multiplicity-one varifold associated with a smooth, embedded manifold M (possibly with boundary) and if $U \subset \mathbb{R}^{n+1}$ is an open, bounded set then

$$|\delta V|(U) = \mathcal{H}^{n-1}(\partial M \cap U) + \int_{M \cap U} |H| \, d\mu.$$

We recall Allard's compactness theorem for integral varifolds (see [I2, 1.9]).

Theorem 4.11 (Allard's Compactness Theorem) *Let $\{\mu_i\}_{i \geq 1}$, where $\mu_i = \mu_{V_i} = |V_i|$, be a sequence of integer n-rectifiable Radon measures with corresponding integer n-rectifiable varifolds $V_i = V_{\mu_i}$ satisfying*

$$\sup_{i \geq 1} (\mu_i(U) + |\delta V_i|(U)) < \infty$$

for each $U \subset\subset \mathbb{R}^{n+1}$. Then there exists an integer n-rectifiable Radon measure $\mu = \mu_V$ and associated integer n-rectifiable varifold $V = V_\mu$ such that

i) $\mu_{i_j} \to \mu$ as Radon measures,

ii) $V_{i_j} \to V$ as varifolds,

iii) $\delta V_{i_j} \to \delta V$ as vector-valued Radon measures,

iv) $|\delta V| \leq \liminf_{i_j \to \infty} |\delta V_{i_j}|$ as Radon measures,

for some subsequence i_j.

We want to apply Theorem 4.11 to the special case in which each V_i is the multiplicity-one varifold corresponding to a smooth, closed manifold. From Theorem 2.16 we have

$$\int_{\mathcal{M}_t^i} H^p \, d\mu \leq C(T)$$

for all $t \geq 0$ and all $p < n-1$. Applying Hölder's inequality and (4.2) we obtain

$$\begin{aligned} \int_{\mathcal{M}_t^i} H \, d\mu &\leq (\alpha_2 R^n)^{\frac{p-1}{p}} \left(\int_{\mathcal{M}_t^i} H^p \, d\mu \right)^{\frac{1}{p}} && (4.7) \\ &\leq C(T), && (4.8) \end{aligned}$$

for all $t \geq 0$. Using Allard's compactness theorem and (4.6) we can find a subsequence $\{\partial \Sigma_t^{i_j}\}_{j \geq 1}$, $\{\partial \tilde{\Sigma}_t^{i_j}\}_{j \geq 1}$ such that

$$\begin{aligned} \partial \Sigma_t^{i_j} &\to \Gamma_t \quad \text{as varifolds,} \\ \partial \tilde{\Sigma}_t^{i_j} &\to \Gamma_t \quad \text{as varifolds.} \end{aligned}$$

This is again independent of the choice of subsequence.

Corollary 4.12 (Weak Convergence) *For all* $0 \leq t \leq T$ *we have*

$$\begin{aligned} \mu_t^i &\to \mu_t \quad \textit{as Radon measures,} \\ \tilde{\mu}_t^i &\to \mu_t \quad \textit{as Radon measures,} \\ \partial\Sigma_t^i &\to \Gamma_t \quad \textit{as varifolds,} \\ \partial\tilde{\Sigma}_t^i &\to \Gamma_t \quad \textit{as varifolds.} \end{aligned}$$

This will allow us in particular to pass the regularity estimates from Chapter 2 to limits.

Regularity Estimates. Recall that if the total first variation $|\delta V|$ is a Radon measure and is absolutely continuous with respect to μ, then the generalised mean curvature vector $\mathbf{H} = \mathbf{H}_\mu$ is defined by

$$\delta V(X) = -\int \langle \mathbf{H}, X \rangle \, d\mu$$

for all $X \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$. Theorem 2.16, Hölder's inequality and Theorem 4.11 imply that Γ_t carries a generalised mean curvature vector $\mathbf{H}$ for all $0 \leq t \leq T$.

We therefore have from Allard's compactness theorem and Corollary 4.12 that

$$\mu_t^i \llcorner \vec{H}_i \to \mu_t \llcorner \mathbf{H} \quad \text{as vector-valued Radon measures,} \tag{4.9}$$

where $\vec{H}_i(p,t) = -H_i(p,t)\nu_i(p,t)$ denotes the mean curvature vector associated with $\mathcal{M}_t^i$. We hereafter suppress the subscript i.

Again following [S], we can therefore quote Hutchinson [Hu, Thm. 4.4.2] to obtain the lower semi-continuity property

$$\int_{\Gamma_t} |\mathbf{H}|^p \, d\mu \leq \liminf_{i\to\infty} \int_{\mathcal{M}_t^i} H^p \, d\mu, \quad 0 \leq t \leq T, \tag{4.10}$$

for all $p < n-1$. We have arrived at the following result:

Proposition 4.13 (L^p Estimate for Weak Solution) *We have*

$$\int_{\Gamma_t} |\mathbf{H}|^p \, d\mu \leq C(T), \quad 0 \leq t \leq T, \tag{4.11}$$

for all $p < n-1$.

Remark 4.14 ($p > n$ and Allard's Regularity Theorem) Also from Theorem 2.16 we have the uniform double integral estimate

$$\int_0^T \int_{\mathcal{M}_t^i} H^{p+2}\, d\mu\, dt \le C(T)$$

for all $p < n - 1$. Fatou's lemma yields

$$\liminf_{i\to\infty} \int_{\mathcal{M}_t^i} H^{p+2}\, d\mu < \infty, \qquad \text{a.e. } 0 \le t \le T.$$

Then from (4.9) and [Hu, Thm. 4.4.2] we find

$$\int_{\Gamma_t} |\mathbf{H}|^{p+2}\, d\mu \le \liminf_{i\to\infty} \int_{\mathcal{M}_t^i} H^{p+2}\, d\mu, \qquad \text{a.e. } 0 \le t \le T.$$

Applying Fatou's lemma once more gives

$$\begin{aligned} \int_0^T \int_{\Gamma_t} |\mathbf{H}|^{p+2}\, d\mu\, dt &\le \liminf_{i\to\infty} \int_0^T \int_{\mathcal{M}_t^i} H^{p+2}\, d\mu\, dt \\ &\le C(T). \end{aligned}$$

Since this holds for all $p < n - 1$ we can invoke Allard's regularity theorem.

The estimate (4.10) can in fact be turned into an equality using Theorems 4.6 and 4.8 – or alternatively using the additional estimates available in Theorem 2.16 (see Remark 4.17).

Higher Convergence. In a first step we show that, away from the singular set, the surfaces $\mathcal{M}_t^i$ converge smoothly to Γ_t.

Lemma 4.15 (Smooth Convergence) *We have*

$$\mathcal{M}_t^i \to \Gamma_t \quad \textit{smoothly}$$

away from Ω_S.

Proof. We follow [MS, Pf. of Lem. 3.3]. Definition 4.5 says that for each $x_0 \in \Omega \setminus \Omega_S$ there exists a ball $B_\rho(x_0)$ such that the level-sets $\{u_L = t\}$ evolve smoothly inside $B_\rho(x_0)$.

We can therefore apply Corollary 4.12 and Brakke's local regularity theorem in combination with a rescaling argument similar to [E2, Pf. of Prop. 3.3] to obtain the lemma. □

Corollary 4.16 (L^p Convergence) *We have*

$$\int_{\mathcal{M}_t^i} H^p \, d\mu \to \int_{\Gamma_t} |\mathbf{H}|^p \, d\mu, \quad 0 \leq t \leq T,$$

for all $p < n-1$.

Proof. By Hölder's inequality,

$$\int_{\mathcal{M}_t^i \cap A} H^p \, d\mu \leq \left|\mathcal{M}_t^i \cap A\right|^{1-\frac{p}{q}} \left(\int_{\mathcal{M}_t^i \cap A} H^q \, d\mu \right)^{\frac{p}{q}}$$

for any $p < q < n-1$ and for all $0 \leq t \leq T$. Given any $\delta > 0$, we can therefore use Theorem 4.6 and Theorem 2.16 to find a neighbourhood B of $\operatorname{sing}(\Gamma_t)$ such that

$$\lim_{i \to \infty} \int_{\mathcal{M}_t^i \cap B} H^p \, d\mu \leq C\delta \tag{4.12}$$

for all $p < n-1$ and for all $0 \leq t \leq T$. Lemma 4.15 then gives the desired result. □

Note that the same approach in combination with Remark 4.14 gives

$$\int_{\mathcal{M}_t^{i_j}} H^{p+2} \, d\mu \to \int_{\Gamma_t} |\mathbf{H}|^{p+2} \, d\mu, \qquad \text{a.e. } 0 \leq t \leq T,$$

for all $p < n-1$ with respect to a subsequence i_j.

Remark 4.17 (Rellich's Theorem) Recall from Theorem 2.16 that

$$\int_0^T \int_{\mathcal{M}_t^i} \left|\nabla \left(H^{\frac{p}{2}}\right)\right|^2 d\mu \, dt \leq C(T).$$

Fatou's lemma yields

$$\liminf_{i \to \infty} \int_{\mathcal{M}_t^i} \left|\nabla \left(H^{\frac{p}{2}}\right)\right|^2 d\mu < \infty \qquad \text{a.e. } 0 \leq t \leq T.$$

For each such t, there is a subsequence i_j such that

$$\sup_{j \geq 1} \int_{\mathcal{M}_t^{i_j}} \left|\nabla \left(H^{\frac{p}{2}}\right)\right|^2 d\mu < \infty.$$

We can therefore combine Rellich's theorem with Remark 4.14, Allard's regularity theorem and (4.12) as in [S] to obtain the desired convergence with respect to a subsequence for a.e. t.

4.2 The Size of the Singular Set at the First Singular Time

This final section is concerned with the special circumstance in which a smooth solution of mean curvature flow develops a singularity for the first time. We describe an application of Remark 2.9 to recent work by Ecker [E3] on the size of the singular set at the first singular time.

The First Singular Time. Let $\mathcal{M}_t$ be a smooth solution of mean curvature flow on some time interval $0 \le t < t_0$. We adapt Definition 4.5 to this setting:

Definition 4.18 (First Singular Set, [E3]) The space-time point (x_0, t_0) is singular if

i) there is a sequence of times $t_j \nearrow t_0$ and points $x_j \in \mathcal{M}_{t_j}$ such that $x_j \to x_0$, and

ii) there is no smooth extension of $\mathcal{M}_t$ beyond t_0 in any neighbourhood of x_0.

We denote by $\operatorname{sing}(\mathcal{M}_{t_0}) \subset \mathbb{R}^{n+1}$ the set of points $x \in \mathbb{R}^{n+1}$ such that (x, t_0) is a singular point. If $\operatorname{sing}(\mathcal{M}_{t_0}) \neq \emptyset$, we call t_0 the first singular time.

In [E3, Thm. 1.1], Ecker showed that the integrability condition

$$\int_0^{t_0} \int_{\mathcal{M}_t} |A|^p \, d\mu \, dt < \infty$$

implies

$$\mathcal{H}^{n+2-p}(\operatorname{sing}(\mathcal{M}_{t_0})) = 0$$

for all $2 \le p \le n+2$ and

$$\operatorname{sing}(\mathcal{M}_{t_0}) = \emptyset$$

for $p \ge n+2$.

Now let $\mathcal{M}_0 \in \mathcal{C}(R, \alpha)$ and consider the smooth solution $\mathcal{M}_t$ of mean curvature flow on $0 \le t < t_0$. From Remark 2.9 we have

$$\int_0^{t_0} \int_{\mathcal{M}_t} |A|^{n+1-\varepsilon} \, d\mu \, dt \le \frac{C(n)}{\varepsilon} \exp\left(\frac{pC_\varepsilon}{R^2} t_0\right) \int_{\mathcal{M}_0} H^{n-1-\varepsilon} \, d\mu$$

for all $\varepsilon > 0$ where $C(n) = 2(n-1)n^{p/2+1}$ and $C_\varepsilon = C_\varepsilon(\mathcal{M}_0)$. Applying Ecker's theorem we find in particular that

$$\mathcal{H}^{1+\varepsilon}(\operatorname{sing}(\mathcal{M}_{t_0})) = 0$$

for all $\varepsilon > 0$. This gives rise to the following corollary [E3, Cor. 1.7].

Corollary 4.19 (First Singular Set, [E3]) *Let $\mathcal{M}_0 \in \mathcal{C}(R,\alpha)$ and consider the smooth solution $\mathcal{M}_t$, $0 \le t < t_0$, of mean curvature flow starting from $\mathcal{M}_0$. Then we have*

$$\dim(\operatorname{sing}(\mathcal{M}_{t_0})) \le 1$$

where $\dim(\operatorname{sing}(\mathcal{M}_{t_0}))$ *denotes the Hausdorff dimension of the set* $\operatorname{sing}(\mathcal{M}_{t_0})$.

Remark 4.20 (Examples) It is useful to review relevant known examples. In dimension two, the symmetric torus of positive mean curvature collapses onto a circle at a finite extinction time T. In this case,

$$\mathcal{H}^1(\operatorname{sing}(\mathcal{M}_T)) < \infty.$$

Turning briefly to the non-compact setting, it is of course well-known that the round cylinder $\mathbb{S}^{n-1}_{r_0} \times \mathbb{R}$ with initial radius r_0 contracts to a line at time

$$T = \frac{r_0^2}{2(n-1)}.$$

In the closed case, however, the precise description of the structure of singular regions provided by Huisken and Sinestrari [HS3] leads one to speculate that the $\mathcal{H}^1$-measure of the singular set is in fact finite.

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