Fachbereich Mathematik und Informatik
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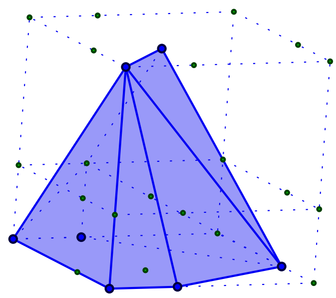
Polytopes with few coordinate values: combinatorial types and diameter bounds



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For Doug.

Contents

Summary

In this thesis we introduce $(0, 1, a)$- and $(0, 1, a_i)$-polytopes and explore their various combinatorial properties. These polytopes are generalizations of $(0, 1)$-polytopes, which arise as fundamental objects of combinatorial optimization and linear programming. While $(0, 1)$-polytopes can be described by having two distinct vertex coordinate values, $(0, 1, a)$- and $(0, 1, a_i)$-polytopes are allowed three. This thesis focuses on the combinatorial structures that result from this constraint relaxation.

In the first chapter we define $(0, 1, a)$- and $(0, 1, a_i)$-polytopes and give interesting examples which motivate the thesis and provide intuition into their geometric and combinatorial structures. In particular, we give examples of combinatorial types of polytopes that distinguish the classes of rational and irrational polytopes, as well as $(0, 1, a)$- and $(0, 1, a_i)$-polytopes.

In the second chapter we fully enumerate the rational $(0, 1, a)$-polytopes in dimension three. In order to do so, we find an upper bound on the number of vertices such a polytope may have, as well as a finite set of a values that suffice to produce all combinatorial types. We then discuss the complexity of a full enumeration of $(0, 1, a)$- and $(0, 1, a_i)$-polytopes in dimension three and four.

The third chapter studies extremal properties of $(0, 1, a)$-polytopes, which lead to the intractability of enumerations in higher dimensions. We find a set of a values from which all combinatorial types of $(0, 1, a)$-polytopes arise; its size grows exponentially with the dimension. This method has also led to a particular nice family of $(0, 1)$-matrices whose set of integral determinants grows exponentially with dimension.

In the fourth chapter we provide a tight upper bound on the diameter of $(0, 1, a_i)$-polytopes. Further, we provide an algorithm that finds a path between any two vertices in linear time.

Chapter 0

Introduction

The aim of this thesis is to study a previously unexplored family of polytopes and present initial results about it. Polytope theory lies within the intersection of combinatorics and geometry, encapsulating the relationship between these two areas of mathematics. Combinatorics extracts interesting properties of discrete objects encoded by their structure. Geometry places these objects into space, allowing us to explore properties such as the size, shape, and relative position of its elements. One of the most natural notions that arises when we concern ourselves with shape is the one of convexity. A geometric object is convex if the shortest path between any two elements of the object remains within the object.

Polytope theory is the study of discrete, convex objects. Polytopes can be intuitively described as the object obtained by "shrink wrapping" a finite collection of points in space. The resulting object is refered to as the convex hull of this point set. Figure 1 gives an example of the convex hull of some point set. Historically, polytopes have initially arisen as geometric objects.

Perhaps the earliest-studied polytopes are known as the Platonic solids, with their beauty and symmetry fascinating geometers since the ancient times. They are constructed with regular polygonal faces so that the same number of congruent faces meet at each vertex. The five Platonic solids are displayed in Figure 2. These polytopes are deeply rooted in their geometry; a slight pertubation of any vertex would destroy its symmetry and thus its status as a Platonic solid.

Since the discovery of the Platonic solids, there has been major developments in the study of polytopes, not just geometrically but also combinatori-

Figure 1: The convex hull of nine points and the resulting polytope.

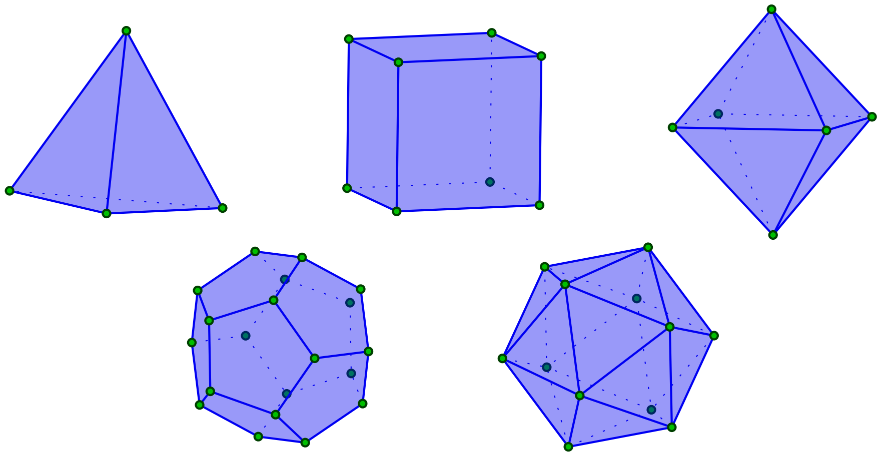


Figure 2: The Platonic solids.

ally. By using combinatorial objects such as the graph and face lattice of a polytope, we can glean information about polytopes that geometric visualization does not necessarily offer.

Perhaps one of the most significant advances in converting polytopes to a combinatorial setting is Steinitz' theorem [47]. This theorem draws a correspondence between 3-connected planar graphs and polytopes in dimension three that allows us to study these polytopes as combinatorial objects independent from their geometric embedding.

In particular, two useful ways we can view polytopes combinatorially is through their face lattices or via their graphs. The face lattice completely encodes the combinatorial structure of the polytope and thus two different geometric realizations of a polytope with isomorphic face lattices are of the same combinatorial type. On the other hand, the graph of a polytope preserves only the vertices, edges, and their respective incidences and adjacencies. While a lot of information is lost when looking at the graph of an arbitrary polytope, the combinatorial structure of a 3-dimensional polytope is completely encapsulated by its graph. Furthermore, Steinitz' theorem gives us a method to produce a geometric realization of the polytope corresponding to a 3-connected planar graph, as demonstrated in Figure 3.

Although no analogous theorem exists in higher dimensions, examining the underlying combinatorial structures of polytopes in dimension four or higher can also be very useful, particularly because here the intuition we gain from visualization fails us.

This thesis exploits this interplay between geometry and combinatorics. In a story akin to that of 3-dimensional polytopes and their graphs, we will take a family of convex polytopes described by relatively rigid geometric conditions and study their associated combinatorial structures.

An important family of polytopes that nicely displays the interaction between geometry and combinatorics are the $(0,1)$-polytopes. If we imagine the

Figure 3: The cube and its graph.

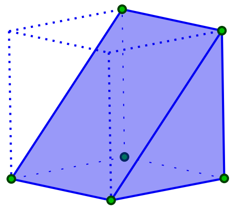


Figure 4: A $(0,1)$-polytope.

n-dimensional cube $C_n = [0,1]^n$, we can form $(0,1)$-polytopes by considering the convex hull of subsets of the vertices of C_n. Figure 4 shows an example of a $(0,1)$-polytope embedded in C_3.

Since the 1970's, the study of $(0,1)$-polytopes has been revisited in various ways. Special cases of $(0,1)$-polytopes have been thoroughly studied through the examination of the polytopes such as the traveling salesman polytopes and the cut polytopes (see [22], [7], [17]). Particular focus has been given in [51] to the family of $(0,1)$-polytopes as a whole. The $(0,1)$-polytopes have particularly nice combinatorial properties. Naddef and Pulleybank have shown in [36] that their graphs admit Hamiltonian paths. Naddef further showed in [35] that these graphs have an especially nice bound on their diameter.

In fact, after a simple affine transformation, it is clear these nice combinatorial features hold for any polytope that satisfies the restriction that each vertex coordinate takes only one of two values. In this thesis, we investigate what happens when we loosen this restriction and allow the vertex coordinates to take one of three distinct values. In particular, we will call these polytopes $(0,1,a)$-polytopes. If we further allow three different values for each coordinate, we create what we will call $(0,1,a_i)$-polytopes. Given these various geometric constraints, we want to examine the resulting combinatorial types as well as other various combinatorial properties.

This thesis is structured as follows: In the remainder of the introduction we will more rigorously define the notions we sketched above as well as other basic but essential concepts that will be used throughout the thesis.

In Chapter 1, we will precisely define the polytope families of $(0,1,a)$- and $(0,1,a_i)$-polytopes, which are the focus of this thesis, as well as provide small examples that should aid us in developing intuition for these families. In particular, we will look at the different classes of combinatorial types $(0,1,a)$- and

$(0,1,a_i)$-polytopes that arise by considering $a \in \mathbb{Z}$, $a \in \mathbb{Q}$, and $a \in \mathbb{R}$. Let $\mathcal{T}(\mathbb{Z})$, $\mathcal{T}(\mathbb{Q})$, and $\mathcal{T}(\mathbb{R})$ respectively denote the collections of combinatorial types of three-dimensional $(0,1,a)$-polytopes for $a \in \mathbb{Z}$, $a \in \mathbb{Q}$, and $a \in \mathbb{R}$. Similarly, we let Let $\mathcal{T}_i(\mathbb{Z})$, $\mathcal{T}_i(\mathbb{Q})$, and $\mathcal{T}_i(\mathbb{R})$ denote the collections of combinatorial types of $(0,1,a_i)$-polytopes. We prove the following strict inclusions:

$$\begin{array}{ccccc} \mathcal{T}_i(\mathbb{Z}) & \subseteq & \mathcal{T}_i(\mathbb{Q}) & \subseteq & \mathcal{T}_i(\mathbb{R}) \\ \cup\!\!\!\!\nmid & & \cup\!\!\!\!\nmid & & \cup\!\!\!\!\nmid \\ \mathcal{T}(\mathbb{Z}) & \subsetneq & \mathcal{T}(\mathbb{Q}) & \subsetneq & \mathcal{T}(\mathbb{R}) \end{array}$$

In Chapter 2 we look into the enumerations of these objects in low dimensions. We prove that this enumeration is possible by finding a finite set of a values from which all possible combinatorial types arise. In particular, the enumeration in dimension three provides a proof of the strict inclusions above.

In Chapter 3 we discuss the complexity of extending the classification to higher dimensions, as well as some extremal properties of these polytopes. We find a finite set of a values that generate all combinatorial types and whose cardinality grows exponentially with respect to the dimension of the polytopes. The technique used to find this set also allows us to find a family of $(0,1)$-matrices with especially nice determinants. In particular, for all $c \in \mathbb{Z}$, such that c is at most the n^{th} Fibonacci number, we can find a $(0,1)$-matrix of size $n \times n$ with determinant c.

Then, in Chapter 4, we discuss a key combinatorial property of the graphs of n-dimensional $(0,1,a_i)$-polytopes: Extending work by Del Pia and Michini, we prove the tight upper bound of $\lfloor \frac{3}{2}n \rfloor$ on their diameter. We also discuss an algorithm for finding a path whose length is linear in n between any two vertices of a $(0,1,a_i)$-polytope.

0.1 Preliminaries on polytope theory

In this section, we define the basic concepts of polytope theory that will be used throughout this thesis. This introduction is intended to be a review as well as a means to establish notation; a reader curious for a more thorough treatment is directed to [52] or [23].

0.1.1 Polytopes as geometric objects

As introduced above, a point set $V \subset \mathbb{R}^n$ is *convex* if for any two points $x, y \in V$, V also contains the line segment

$$[x,y] = \{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\}.$$

Figure 5 gives an example of a non-convex and convex set.

Given some finite point set $V \subset \mathbb{R}^n$, the *convex hull of* V, denoted $\operatorname{conv}(V)$, is the smallest convex set containing V. Let $k \in \mathbb{N}$ and $V = (v_1, \ldots, v_k)$. We

(a) A non-convex set.

(b) A convex set.

Figure 5

have the following description of the convex hull of V:

$$\mathrm{conv}(V) := \{x \in \mathbb{R}^n : x = \sum_{i=1}^{k} \lambda_i v_i \text{ for } \lambda_i \in \mathbb{R},\ 0 \leq \lambda_i \leq 1,\ \sum_{i=1}^{k} \lambda_i = 1\}.$$

While we could consider taking convex hulls of infinite point sets, in this thesis we are only concerned with finite point sets.

Definition 0.1.1. A *polytope* is the convex hull of a finite set of points in $\mathbb{R}^n$. The minimal set $V' \subset V$ such that $P = \mathrm{conv}(V') = \mathrm{conv}(V)$ is the *vertex set of* P, denoted $\mathrm{vert}(P)$.

Similarly to the convex hull, we define the *affine hull of our point set* V by

$$\mathrm{aff}(V) := \{x \in \mathbb{R}^n : x = \sum_{i=1}^{k} \lambda_i v_i \text{ for } \lambda_i \in \mathbb{R},\ \sum_{i=1}^{k} \lambda_i = 1\}.$$

This is the smallest affine subspace containing our point set V.

We say that a polytope is *n-dimensional*, or an *n-polytope*, if the dimension of its affine hull is n. While we can represent a polytope in $\mathbb{R}^m$ for various m, the most natural dimension to view a polytope in is the dimension of its affine hull, and thus, this is what we will assume unless otherwise stated.

Two polytopes P and Q are *affinely isomorphic* if there exists an bijective affine map between P and Q.

Let $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. We say that an inequality $ax \leq b$ is *valid on* P if

$$P \subset \{x \in \mathbb{R}^n : a \cdot x \leq b\}.$$

Further, for a valid inequality $ax \leq b$, $H := \{x \in \mathbb{R}^n : ax = b\}$ is a *face-defining hyperplane of* P. We can further define $H^- := \{x \in \mathbb{R}^n : ax < b\}$ and note that $P \subset H \cup H^-$.

A *face of* P is any set of the form

$$F = P \cap \{x \in \mathbb{R}^n : ax = b\}$$

where $ax \leq b$ is a valid inequality on P. The dimension of F is the dimension of its affine hull. For P an n-polytope, if $\dim(F) = n-1$, then we call F a *facet*

(a) A vertex-defining valid inequality. (b) A facet-defining valid inequality.

Figure 6

(a) Δ_2 (b) Δ_3

Figure 7: Examples of simplices.

of P. The vertex set of P is the set of 0-dimensional faces. An *edge of* P is a 1-dimensional face. Otherwise, we may simply say k*-face* in order to indicate a face of a particular dimension. We also consider the empty space to be the (-1)-dimensional face of P, while P itself is the unique n-face. In Figure 6 we give an example of a vertex-defining inequality as well as a facet-defining inequality. We denote the number of k-faces a polytope P has by $f_k(P)$.

A *polyhedron* P in $\mathbb{R}^n$ is the solution set to a finite collection of linear inequalities. We can write

$$P := \{x \in \mathbb{R}^n : Ax \leq b\}$$

for $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. In particular, P is not necessarily bounded. If P is bounded, it is a polytope. While we defined polytopes by their vertex sets, the fundamental representation theorem for polytopes gives a correspondence between these two representations (see [52, Sect. 2.4]).

Perhaps the simplest of polytopes is called the *simplex*, denoted Δ_n. We can define Δ_n as the convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$. One geometric representation of the simplex, called the *standard simplex*, is $\mathrm{conv}(0, e_1, e_2, \dots, e_n) \subset \mathbb{R}^n$. where the e_i are the standard unit vectors in $\mathbb{R}^n$. The standard simplices in dimension two and three are given in Figure 7.

If each facet of a polytope P is a simplex, then we call P *simplicial*. We say P is *simple* if each vertex is connected to exactly n edges.

Matrix notation

Matrices are an important tool for describing polytopes. A useful object is the *submatrix* of a matrix M, which is given by removing certain rows and columns from the M. We will denote by $M(i, j)$ the submatrix of M given by removing the i^{th} row and j^{th} column of M. Further, $M(i)$ will be used to denote removing the i^{th} row of M.

Let $M(i, j)$ be a submatrix of M. Then the determinant of $M(i, j)$ is the *(i, j) minor of M*. We will also need to consider the *(i, j) cofactor of M*, which is the (i, j) minor multiplied by $(-1)^{i+j}$. The *cofactor matrix of M* is the matrix given by $[m_{i,j}]$, where $m_{i,j}$ is the (i, j) cofactor of M.

0.1.2 Polytopes as combinatorial objects

Now that we have geometrically defined a polytope, we wish to convert many of its properties into a combinatorial setting.

Face lattices of polytopes

We start with a basic combinatorial object.

A *partially ordered set* (or *poset*) is a pair $(S, \preceq)$, where S is a finite set and $\preceq$ is a relation on S such that for all $a, b, c \in S$ we have

(i) $a \preceq a$ (reflexivity),

(ii) if $a \preceq b$ and $b \preceq c$, then $a \preceq c$ (transitivity),

(iii) if $a \preceq b$ and $b \preceq a$, then $a = b$ (anti-symmetry).

By a slight abuse of notation, we may refer simply to a poset S when the partial order is clear. Given $x \preceq y$ in a poset S, the *interval* $[x, y]$ is the poset $\{z \in S : x \leq z \leq y\}$ with the same order as S. For $x < y \in S$, we say *y covers x* if $[x, y] = \{x, y\}$, that is, no element lies strictly between x and y. A poset is *bounded* if it has a unique minimal element denoted $\widehat{0}$, and a unique maximal element denoted $\widehat{1}$. A poset is a *lattice* if it is bounded and every two elements $x, y \in S$ have a unique minimal upper bound in S, called the *join* $x \vee y$, and a unique maximal lower bound in S, called the *meet* $x \wedge y$.

We can visualize face lattices by means of Hasse diagrams. The *Hasse diagram of a partially ordered set P* is the directed graph whose vertices are the elements of P and whose edges are the pairs (x, y) for which y covers x. In this thesis Hasse diagrams will be drawn so that elements are placed higher than the elements they cover.

Example 1. In the poset in Figure 8a, we have that $[x_1, y_1] = \{x_1, w_1, w_2, y_1\}$, y_2 covers w_3, $x_2 \vee w_3 = y_2$, and $x_2 \wedge w_3 = \widehat{0}$. This poset is bounded.

The poset in Figure 8b is an example of a poset that is not a lattice. Here x_1 and x_2 have no unique minimal upper bound $x_1 \vee x_2$ because both x_3 and x_4 are minimal upper bounds on these two elements.

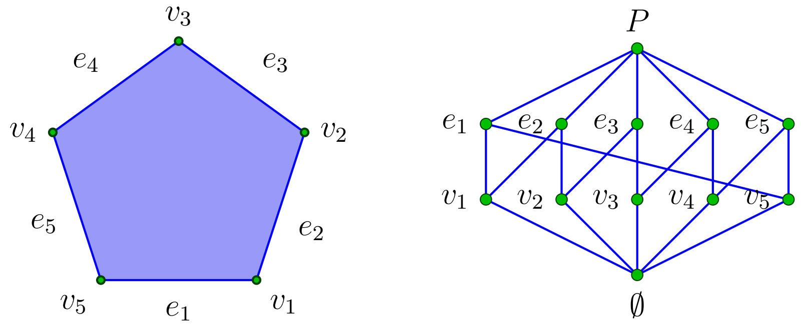


Figure 9: A pentagon P and its face lattice.

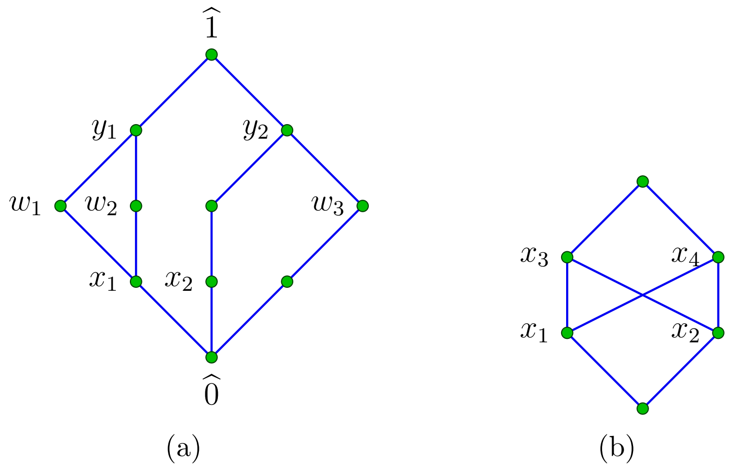


Figure 8: Two examples of posets.

In order to place polytopes in a geometry-free combinatorial setting, we introduce the notion of a face lattice of a polytope. The *face lattice* of a polytope P is the poset $L(P)$ of all the faces of P, partially ordered by inclusion. Figure 9 gives a pentagon and its face lattice.

The face lattices encodes all the combinatorial information about the polytope and are thus the correct setting in which to speak of combinatorial equivalence.

Definition 0.1.2. Two polytopes P and Q are *combinatorially equivalent* if there exists an isomorphism between their face lattices.

However, this notion of combinatorial equivalence is weaker than affine isomorphism. In other words, two polytopes do not have to be affinely equivalent in order to be combinatorially equivalent. If they are combinatorially equivalent, we say they are of the same *combinatorial type*. Figure 10 gives us an example of two different geometric realizations of the same combinatorial type.

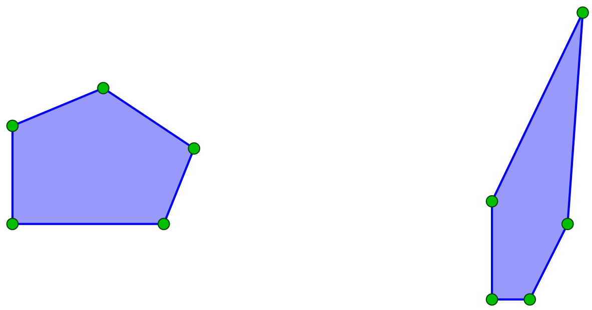


Figure 10: Though not affinely isomorphic, these two pentagons share the face lattice from Figure 9 and are thus combinatorially equivalent.

Graphs of polytopes

Throughout this thesis, we will only be dealing with *simple graphs*: graphs without loops or multiedges. An edge e is *incident* to a vertex u if u is an endpoint of e. If there exists an edge $e = \{u, v\}$ incident to both u and v, then u and v are *adjacent*.

We define a *path* to be a sequence of vertices such that any two consecutive vertices are adjacent. A graph G is *connected* if for any two vertices $v, w \in G$, there exists a path from v to w. We say a graph is *k-connected* if it has at least k vertices and remains connected with the removal of any $k - 1$ vertices.

A path comprised of m edges is said to have length m. For vertices v and w in a graph G, we denote by $\delta(v, w)$ the length of the shortest path between v and w. The *diameter* of G, $\delta(G)$, is the maximal $\delta(v, w)$ for all pairs of vertices v, w.

A *cycle on G* is a path in G with the same start and end vertex. A *Hamiltonian cycle* is a cycle that visits every vertex in G exactly once.

Example 2. The graph G in Figure 11 is a connected, graph. A Hamiltonian path of G of length five is given in blue.

For a polytope P, the vertices and edges define a graph, called the *graph of P*, denoted $G(P)$. On one hand, a graph loses a lot of combinatorial information about the polytope. For example, it may be very difficult to answer whether a graph is the graph of a polytope. In general, it is impossible to reconstruct a polytope from its graph; see [52, Ch. 3]. On the other hand, graphs provide a simpler format in which we can study certain combinatorial properties of polytopes, in particular, the diameter. The *diameter of a polytope* P is defined to be the same as the diameter of the graph of P.

0.1.3 Polytopal complexes and spheres

A *polytopal complex* $\mathcal{C}$ is a finite collection of polytopes in $\mathbb{R}^n$ such that

(i) the empty polytope is in $\mathcal{C}$,

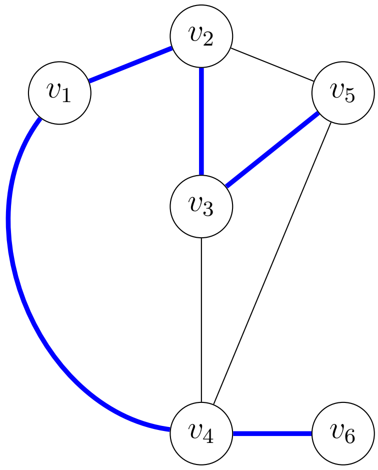


Figure 11: A graph with a Hamiltonian path.

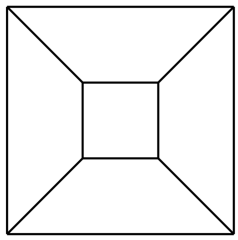


Figure 12: A Schlegel diagram of a cube.

(ii) if $P \in \mathcal{C}$, then all the faces of P are also in $\mathcal{C}$,

(iii) the intersection $P \cap Q$ of two polytopes $P, Q \in \mathcal{C}$ is a face of both P and Q.

The dimension of $\mathcal{C}$ is the largest dimension of a polyhedron in $\mathcal{C}$.

A useful construction that allows us to study polytopes in one dimension lower is a Schlegel diagram. Let P be a polytope in $\mathbb{R}^n$ and F be a facet of P. A *Schlegel diagram of P based at F* is the projection of the faces of P that do not contain F from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$ through a point beyond F. The result is a polytopal subdivision of F in $\mathbb{R}^{n-1}$, which is combinatorially equivalent to the complex $\mathcal{C}((\partial P) \setminus \{F\})$ of all proper faces of P other than F. We can see a Schlegel diagram of C_3 in Figure 12.

Schlegel diagrams are particularly useful in lower dimensions. They provide us with a way to view 4-polytopes in $\mathbb{R}^3$, as well as view 3-polytopes as planar graphs.

Another useful object in the study of polytopes is a (combinatorial) n-sphere. An *n-sphere* is a regular CW complex that is homeomorphic to the topological n-sphere such that any two faces intersect at a common face. For example, the boundary of an n-polytope is an $(n-1)$-sphere. However, not every sphere is the boundary of a polytope. In [12], Bokowski and Garms give an example of a 3-sphere with ten vertices that is nonpolytopal.

Chapter 1

First examples

The goal of this chapter is to precisely define the $(0,1,a)$-polytopes introduced in the previous chapter, as well as to provide examples illuminating distinctions between the various generalizations. We will begin with a closer examination of some well-known families of polytopes.

1.1 Introduction to $(0,1)$-polytopes

In this section we formally introduce $(0,1)$-polytopes. A $(0,1)$-polytope P is a polytope with vertex set $\text{vert}(P) \subseteq \{0,1\}^n$. In other words, these are polytopes with all vertex coordinate values either 0 or 1. The vertex set of a $(0,1)$-polytope in dimension n is a subset of the vertex set of the n-cube, $C_n = [0,1]^n$. But while the n-cube is an extremely well-understood polytope, $(0,1)$-polytopes prove to be much more complicated.

On the one hand, $(0,1)$-polytopes retain many nice combinatorial properties from the n-cube. For instance, for an n-dimensional $(0,1)$-polytope P, since $\text{vert}(P) \subset \text{vert}(C_n)$, we know P has at most 2^n vertices. In [35], Naddef provides a very simple proof to bound the diameter of $(0,1)$-polytopes by their dimension, which is also the diameter of the cube of that dimension. Additionally, it is demonstrated in [36] that all $(0,1)$-polytopes have a Hamiltonian cycle.

On the other hand, there are many aspects of n-cubes that are well known, but very difficult to understand for $(0,1)$-polytopes. For example, the n-cube has exactly $2n$ facets and the facet description is straightforward. Thus, it is natural to ask how many facets a $(0,1)$-polytope in dimension n may have, and how complicated a description of these facets might be. However, it has yet to be determined what the maximum number of facets could be. The best-known bounds grows factorially with respect to n [20]. Furthermore, these facets do not necessarily have a simple description, meaning the coefficients of the inequalities in the facet description can be exponentially large. We will discuss this further in Chapter 3.

Figure 1.1: The two $(0,1)$-polytopes in dimension two, up to combinatorial types.

In fact, we can extend these properties to all polytopes that can be realized using just two coordinate vertex values.

Lemma 1.1.1. *Let P be an n-dimensional polytope such that*

$$\mathit{vert}(P) \subseteq \{a_1, b_1\} \times \cdots \times \{a_n, b_n\}.$$

Then P is affinely isomorphic to a $(0,1)$-polytope P'.

Proof. Let P be an n-dimensional polytope such that

$$\mathrm{vert}(P) \subseteq \{a_1, b_1\} \times \cdots \times \{a_n, b_n\}.$$

Then for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define

$$T(x) := \begin{pmatrix} \frac{1}{b_1 - a_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{b_2 - a_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \frac{1}{b_n - a_n} \end{pmatrix} \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ \vdots \\ x_n - a_n \end{pmatrix}.$$

Then $T(x)$ is an affine transformation and $T(P) = P'$ such that $\mathrm{vert}(P') \subseteq \{0,1\}^n$. $\square$

1.1.1 Enumeration of $(0,1)$-polytopes

There have been some attempts to classify $(0,1)$-polytopes according to combinatorial types. In [36] it is shown that we may assume without loss of generality that we are working with full-dimensional $(0,1)$-polytopes. In dimension two, the $(0,1)$-polytopes are very straight forward to enumerate. They are simply the triangle and the square, as shown in Figure 1.1
In dimension three, there are only eight distinct combinatorial types, which are given in Figure 1.2.

However, this seemingly simple list of polytopes is deceiving. The enumeration in dimension four is fairly simple; there are 172 distinct combinatorial types. The combinatorial types of $(0,1)$-polytopes in dimension five were enumerated by Aichholzer in [2]. However, $(0,1)$-polytopes get so complicated that, by dimension 6, the search and storage of all combinatorial types is, at the very least, impractical (see [51]).

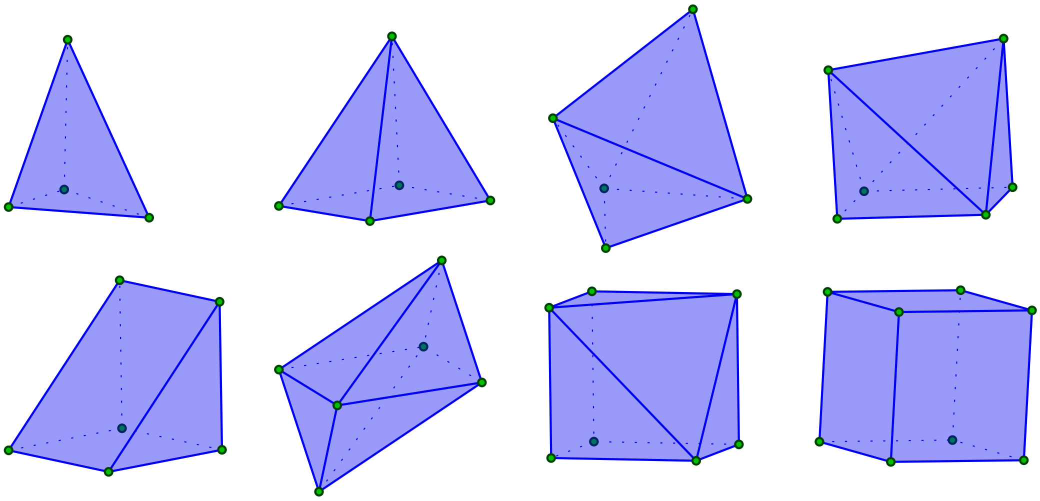


Figure 1.2: The eight distinct combinatorial types of 3-dimensional $(0,1)$-polytopes.

1.2 $(0,1,2)$-polytopes

The *half-integral polytopes* are a natural extension of $(0,1)$-polytopes. These polytopes arise in the study of inside-out polytopes, fractional matching polytopes, fractional stable set polytopes, and many more; see [11], [9], [1], [40], [45]. We define half-integral polytopes by their vertex set: each vertex coordinate must take its value in $\{0, \frac{1}{2}, 1\}$. It is not hard to see that these polytopes are affinely isomorphic to what we will call $(0,1,2)$-*polytopes*, polytopes whose vertex coordinates must take values in $\{0,1,2\}$.

While every $(0,1)$-polytope is clearly also a $(0,1,2)$-polytope, already in dimension two we have polytopes distinguishing the class of $(0,1,2)$-polytopes from $(0,1)$-polytopes. See, for instance, the four distinct combinatorial types of $(0,1,2)$-polytopes in Figure 1.3.

$(0,1,2)$-polytopes are the jumping-off point for the families of generalizations that is the focus of this thesis. For both $(0,1)$-polytopes and $(0,1,2)$-polytopes, we explicitly specify the allowable coordinate values for their vertices; i.e. they must assume values in $\{0,1\}$ and $\{0,1,2\}$, respectively. While Lemma 1.1.1 shows that all polytopes whose vertex coordinates are restricted to two values are affinely isomorphic to $(0,1)$-polytopes, $(0,1,2)$-polytope do not extend this concept to three values. A $(0,1,2)$-polytope is affinely isomorphic to a polytope P with set

$$\mathrm{vert}(P) \subseteq \{k_1, \frac{k_1+l_1}{2}, l_1\} \times \cdots \times \{k_n, \frac{k_n+l_n}{2}, l_n\}.$$

Note, up to a translation, we can always assume 0 is an allowed value. We now consider what happens when we loosen this ratio restriction. In Section

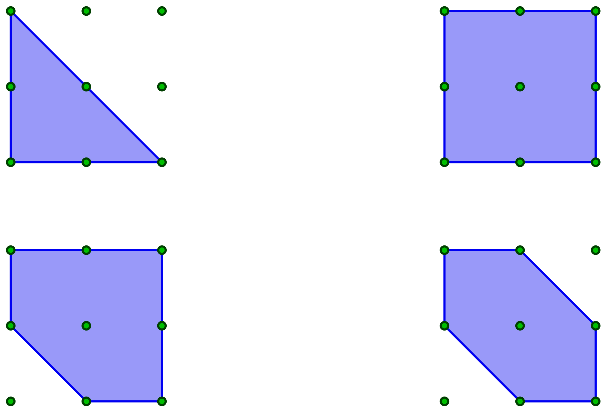


Figure 1.3: The four distinct combinatorial types of 2-dimensional $(0, 1, 2)$-polytopes.

1.3 and 1.4, we will look at the set of polytopes whose vertex coordinates are restricted to three values, and discuss the combinatorial distinctions that arise when a is integral, rational, or irrational. Then, in Section 1.5, we will explore the polytopes whose vertex coordinates can take one of three values chosen independently for each coordinate direction.

1.3 $(0, 1, a)$-polytopes

In the previous section we saw $(0, 1, 2)$-polytopes, which are required to have very specific coordinate values. Now we will investigate what happens when we loosen this restriction.

Definition 1.3.1. We say that a polytope $P \subset \mathbb{R}^n$ is *a* $(0, 1, a)$*-polytope* for $a \in \mathbb{R}$, if for every $v \in \text{vert}(P)$, $v \in \{0, 1, a\}^n$.

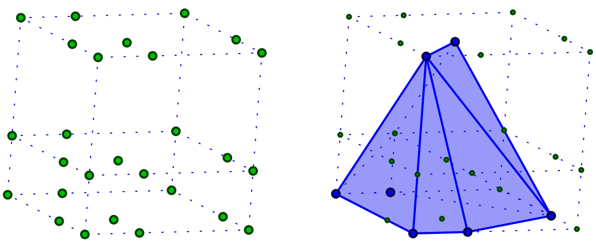
However, we see that any polytope whose vertex coordinates are restricted to three values is affinely isomorphic to a $(0, 1, a)$-polytope.

Lemma 1.3.2. *Let* $k, l, m \in \mathbb{R}$. *There exists an affine map* $f : \mathbb{R} \to \mathbb{R}$ *such that* $f(\{m, l, k\}) = \{0, 1, a\}$ *for* $a \in \mathbb{R}_{\geq 2}$.

Proof. We may assume that $k < l < m$.
Case 1: $l \leq \frac{1}{2}(m + k)$. Consider

$$f(x) = \frac{x - k}{l - k}.$$

Then, $f(k) = 0$, $f(l) = 1$, and $f(m) = a$ for some $a \in \mathbb{R}$ where, since $l \leq \frac{1}{2}(m + k)$, $a \geq 2$.



$$\begin{pmatrix} 0 & 1 & 0 & 1 & 3 & 1 & 0 & 1 \\ 0 & 0 & 3 & 3 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 3 \end{pmatrix}$$

Figure 1.4: Example of a $(0,1,3)$-grid, $(0,1,3)$-polytope embedding in a grid, and a $(0,1,3)$-matrix.

Case 2: $l > \frac{1}{2}(m+k)$. Consider

$$f(x) = \frac{x-m}{l-m}.$$

Then, $f(m) = 0$, $f(l) = 1$, and $f(k) = a$ for some $a \in \mathbb{R}$ where, since $l > \frac{1}{2}(m+k)$, $a \geq 2$, completing the proof. $\square$

Proposition 1.3.3. *Let P be an n-polytope such that vert$(P) \subset \{k,l,m\}^n$. Then P is affinely isomorphic to a $(0,1,a)$-polytope for $a \in \mathbb{R}_{\geq 2}$.*

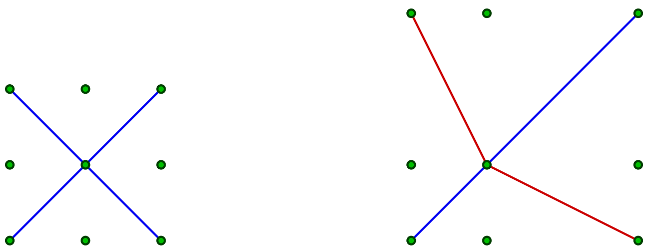
Proof. We see for each $v \in \text{vert}(P)$, $v_i \in \{k,l,m\}$. Thus we apply the affine transformation from Lemma 1.3.2 coordinate-wise to each vertex of P. Since $v_i \in \{k,l,m\}$ for each $v \in \text{vert}(P)$, for each i we get the same a value. $\square$

Thus, when considering $(0,1,a)$-polytopes, we can restrict our attention to $a \in \mathbb{R}_{\geq 2}$.

In connection to n-dimensional $(0,1,a)$-polytopes, we will also refer to $(0,1,a)$*-grids in dimension* n as well as $(0,1,a)$*-matrices of size* $n \times n$. By $(0,1,a)$-grids, we mean the grid $\{0,1,a\}^n$, of which the vertices of an n-dimension $(0,1,a)$-polytope will be a subset. When we talk about an n-dimensional $(0,1,a)$-polytope, we assume it is embedded in the n-dimensional $(0,1,a)$-grid. We will also refer to the convex hull of the nodes of the $(0,1,a)$-grid as the $(0,a)$-cube. A matrix $M = [m_{ij}]$ such that $m_{ij} \in \{0,1,a\}$ is what we will call a $(0,1,a)$-matrix. These are formed by putting a $(0,1,a)$-polytope's vertices in the columns of the matrix.

Example 3. Let $a = 3$, Figure 1.4 gives an example of the 3-dimensional $(0,1,3)$-grid, a $(0,1,3)$-polytope embedded in this grid, and a corresponding $(0,1,3)$-matrix.

Before we study $(0,1,a)$-polytopes in any more depth, we first want to convince ourselves that this really is a different family from the family of $(0,1,2)$-polytopes. In other words, we would be interested in a combinatorial type of a $(0,1,a)$-polytope whose combinatorial type cannot be realized when the vertex coordinates are forced to take values in $\{0,1,2\}$.



(a) $(0,1,2)$-grid in dimension 2. (b) $(0,1,a)$-grid in dimension 2.

Figure 1.5

(a) The convex hull of $(a,0)$, $(0,a)$, $(1,1)$, (a,a) for $a=2$. (b) The convex hull of $(a,0)$, $(0,a)$, $(1,1)$, (a,a) for $a=3$.

Figure 1.6

Although the set of combinatorial types for $(0,1,2)$-polytopes and $(0,1,a)$-polytopes in dimension two are the same, there are differences in their geometric realizations. Considering the $(0,1,a)$-grid in dimension two, we see that the point $(1,1)$ only lies on one of the hyperplanes defined by the diagonals, as shown in Figure 1.5.

We can see a noticeable distinction between the $(0,1,a)$- and $(0,1,2)$-polytopes by varying a in $(0,1,a)$-grids of dimension $n>2$. As a changes, hyperplanes which contain at least $n+1$ points may form as well as “break”. This break of the hyperplanes induces different combinatorial types of $(0,1,a)$-polytopes to the $(0,1,2)$-polytopes for $n>2$.

With Figure 1.5, we can get a bit of intuition that helps us piece together a $(0,1,a)$-polytope whose combinatorial type cannot be realized as a $(0,1,2)$-polytope.

While both combinatorial types in Figure 1.6 arise for $a=2$ as well as for any $a \in \mathbb{R}_{>2}$, we can use this geometric difference to obtain a combinatorial type of a 3-dimensional $(0,1,a)$-polytope that is not realizable as a $(0,1,2)$-polytope.

Proposition 1.3.4. *There exists a combinatorial type of a $(0,1,a)$-polytope that cannot be realized as a $(0,1,2)$-polytope.*

Proof. Let P be the 3-dimensional $(0,1,3)$-polytope whose vertices can be read

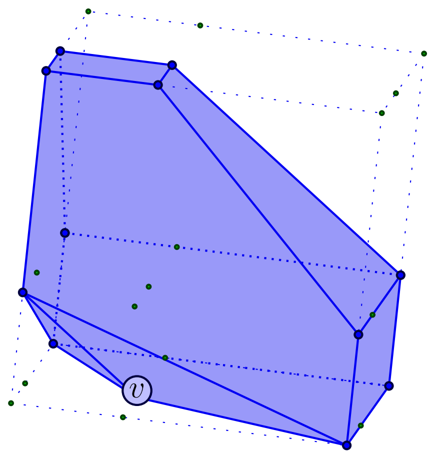


Figure 1.7: A $(0,1,3)$-polytope that cannot be realized as a $(0,1,2)$-polytope.

as the columns of the following $(0,1,3)$-matrix:

$$\begin{pmatrix} 1 & 3 & 0 & 3 & 0 & 3 & 0 & 3 & 0 & 1 & 0 & 1 \\ 1 & 0 & 3 & 3 & 0 & 0 & 3 & 3 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 \end{pmatrix}.$$

P cannot be realized as a $(0,1,2)$-polytope, as it does not appear in the full enumeration of $(0,1,2)$-polytopes in Section 2.2. $\square$

We can still gain intuition by considering Figure 1.7, which is a realization of P. P takes advantage of the geometry from Figure 1.6. We notice that the vertex labeled v would not be a vertex if the threes in the matrix above were replaced by twos. This vertex lies on a line which is broken by the requirements of a $(0,1,3)$-polytope. Then, by adding vertices on top of this facet, we create a combinatorial type that does not arise for $a = 2$.

Therefore, we see that the family of $(0,1,a)$-polytopes strictly contains the family of $(0,1,2)$-polytopes, even in dimension three. So it makes sense to study these polytopes more closely.

We will start this closer examination by discussing different classes of $(0,1,a)$-polytopes. In particular, we will discuss the implications of restricting a such that $a \in \mathbb{Z}$, $a \in \mathbb{Q}$, and $a \in \mathbb{R}$.

1.4 Classes of combinatorial types of $(0,1,a)$-polytopes

In this section we will discuss different classes of combinatorial types of $(0,1,a)$-polytopes, namely those created by allowing $a \in \mathbb{Z}$, $a \in \mathbb{Q}$, and $a \in \mathbb{R}$.

Let $\mathcal{T}(\mathbb{Z})$, $\mathcal{T}(\mathbb{Q})$, and $\mathcal{T}(\mathbb{R})$ denote the collections of combinatorial types of $(0,1,a)$-polytopes for $a \in \mathbb{Z}$, $a \in \mathbb{Q}$, $a \in \mathbb{R}$, respectively. It is clear that

$\mathcal{T}(\mathbb{Z}) \subseteq \mathcal{T}(\mathbb{Q}) \subseteq \mathcal{T}(\mathbb{R})$. However, in this section we show that these inclusions are actually strict.

Theorem 1.4.1. *Let $\mathcal{T}(\mathbb{Z})$, $\mathcal{T}(\mathbb{Q})$, and $\mathcal{T}(\mathbb{R})$ be defined as above. Then*

$$\mathcal{T}(\mathbb{Z}) \subsetneq \mathcal{T}(\mathbb{Q}) \subsetneq \mathcal{T}(\mathbb{R}).$$

We will begin by examining the case $a \in \mathbb{Q}$. In other words, we are insisting that the polytope has a realization that only allows the coordinates to take one of three rational values.

Lemma 1.4.2. *There is a combinatorial type of a $(0,1,a)$-polytope that cannot be realized as an integral $(0,1,a)$-polytope.*

Proof. Let P be the 3-dimensional $(0,1,\frac{5}{2})$-polytope whose vertices can be read as the columns of the following $(0,1,\frac{5}{2})$-matrix:

$$\begin{pmatrix} \frac{5}{2} & \frac{5}{2} & 1 & 1 & 0 & \frac{5}{2} & 0 & 1 \\ 0 & \frac{5}{2} & \frac{5}{2} & 1 & \frac{5}{2} & \frac{5}{2} & 1 & \frac{5}{2} \\ 0 & 0 & 0 & 1 & 1 & 1 & \frac{5}{2} & \frac{5}{2} \end{pmatrix}.$$

P cannot be realized as a $(0,1,a)$-polytope for $a \in \mathbb{Z}$.

The full enumeration of $(0,1,a)$-polytopes for $a \in \mathbb{Z}$ in Section 2.2 confirms that P cannot be realized as a $(0,1,a)$-polytope for $a \in \mathbb{Z}$, and thus the class of $(0,1,a)$-polytopes for $a \in \mathbb{Q}$ is distinct from the one for $a \in \mathbb{Z}$. $\square$

In the general case of $a \in \mathbb{R}$, we are considering the case of all polytopes, whose coordinates are allowed to take one of three values in $\mathbb{R}$.

This distinction from the case where $a \in \mathbb{Q}$ is particularly interesting when we consider 3-dimensional polytopes. Rationality of vertex coordinates is a desirable property of polytopes when doing computations with the polytopes. A fundamental theorem of 3-dimensional polytopes by Steinitz states that every simple three-connected planar graph is the graph of a three-dimensional polytope [47]. Another useful result of the proof of Steinitz' theorem is that the steps used to realize the polytope from a 3-connected planar graph may also be done in rational 3-space. In fact, an even stronger corollary may be achieved:

Corollary 1.4.3 (see [23, Sect. 13.2])**.** *For every 3-polytope $P \subset \mathbb{R}^3$ and for each $\epsilon > 0$ there exists a 3-polytope P' combinatorially equivalent to P such that, in any preassigned cartesian system of coordinates in $\mathbb{R}^3$, all vertices of P' have rational coordinates, with the distance between corresponding vertices of P and P' being less than ϵ.*

This corollary tells us that by only slightly perturbing the vertices of a 3-polytope P, we can achieve a realization of P with rational coordinates. Of course, once we have these rational coordinates we can easily get integer

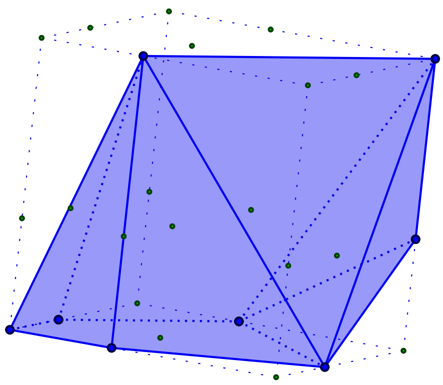


Figure 1.8: A realization of a combinatorial type of a $(0, 1, a)$-polytope in dimension three that requires a to be irrational.

coordinates, but we may have to scale the polytope by a large factor in order to do so.

This ability to realize any 3-polytope with rational or integral coordinates allows us to control the computational complexity of working with 3-polytopes.

While the original proof required an exponential number of bits to write down these vertices, an algorithm to obtain vertex coordinates has been found to only require linearly many with respect to the number of vertices. However, if we place the restriction on the number of different values the coordinates may take, we lose this rational realizability.

Lemma 1.4.4. *There is a combinatorial type of a $(0, 1, a)$-polytope that cannot be realized as a rational $(0, 1, a)$-polytope.*

Proof. Let P be the 3-dimensional $(0, 1, a)$-polytope for $a = \frac{1}{2}\sqrt{5} + \frac{3}{2}$ whose vertices can be read as the columns of the following $(0, 1, \frac{5}{2})$-matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 1 & a & a & 1 & a \\ 0 & 0 & 1 & a & 1 & a & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 1 & a & a \end{pmatrix}.$$

P cannot be realized as a $(0, 1, a)$-polytope for $a \in \mathbb{Q}$. The full enumeration of the $(0, 1, a)$-polytopes for $a \in \mathbb{Q}$ in Section 2.2 confirms the fact P cannot be realized as a $(0, 1, a)$-polytope for $a \in \mathbb{Q}$. $\square$

Figure 1.8 is a realization of P for $a = \frac{1}{2}\sqrt{5} + \frac{3}{2}$. This polytope, however, can be easily realized with rational vertices. In fact, the polytope may be realized with four integer coordinates by the vertex set

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 3 & 3 & 1 & 3 \\ 0 & 0 & 1 & 3 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 4 \end{pmatrix}.$$

However, by restricting this polytope to three vertex coordinate values, we force an irrational realization.

Lemma 1.4.4 shows that the requirement to take rational or irrational vertex coordinates once we fix a number of coordinates gives us a new notion of what it means for a polytope to be rational.

Proof of Theorem 1.4.1. Lemma 1.4.2 and Lemma 1.4.4 show the strict inclusions

$$\mathcal{T}(\mathbb{Z}) \subsetneq \mathcal{T}(\mathbb{Q}) \subsetneq \mathcal{T}(\mathbb{R}).$$

□

1.5 $(0,1,a_i)$-polytopes

The second level of generalization of $(0,1,2)$-polytopes will be referred to as $(0,1,a_i)$-polytopes. As stated earlier, polytopes affinely isomorphic to $(0,1,a)$-polytope may take one of three values. So the i^{th} coordinate may take a value in $\{0, k_i, ak_i\}$. The $(0,1,a_i)$-polytopes no longer have this conforming ratio requirement. Instead the i^{th} coordinate may now take values in $\{0, k_i, a_i k_i\}$ for $a_i \in \mathbb{R}$.

Definition 1.5.1. An n-dimensional polytope $P \subset \mathbb{R}^n$ is *a* $(0,1,a_i)$*-polytope* for some vector $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ if for every $v \in \text{vert}(P)$ and $i \in [n]$, the value of the i^{th} coordinate v_i is in the set $\{0,1,a_i\}$. Note that for $(0,1,a)$-polytopes, $\mathbf{a} = (a, a, \ldots, a)$.

Similarly as before, we can view any polytope whose vertex coordinates are restricted to three values per coordinate direction as affinely isomorphic to a $(0,1,a_i)$-polytope.

Proposition 1.5.2. *Let P be an n-polytope such that vert$(P) \subset \prod_{i=1}^{n} \{k_i, l_i, m_i\}$. Then P is affinely isomorphic to a $(0,1,a_i)$-polytope for $\boldsymbol{a} \in \mathbb{R}^n_{\geq 2}$.*

Proof. We see for each $v \in \text{vert}(P)$, $v_i \in \{k_i, l_i, m_i\}$. Thus we apply the affine transformation from Lemma 1.3.2 coordinate-wise to each vertex of P. Since $\{k_i, l_i, m_i\}$ may be distinct for each i, we may have a distinct a_i for each i. □

Thus, when considering $(0,1,a)$-polytopes, we can restrict our attention to $a \in \mathbb{R}_{\geq 2}$.

The $(0,1,a_i)$ grid also extends this notion of "breaking" a hyperplane, as we saw earlier with the $(0,1,a)$-polytopes. Even in dimension two, we see the distinction from $(0,1,2)$- and $(0,1,a)$-grids as shown in Figure 1.9.

Similarly as for $(0,1,a)$-polytopes, we would like to look at the sets $\mathcal{T}_i(\mathbb{Z})$, $\mathcal{T}_i(\mathbb{Q})$, and $\mathcal{T}_i(\mathbb{R})$: the sets of combinatorial types of $(0,1,a_i)$-polytopes for $\mathbf{a} \in \mathbb{Z}$, $\mathbf{a} \in \mathbb{Q}$, and $\mathbf{a} \in \mathbb{R}$, respectively. In particular, the goal of the remainder of the section is show the following:

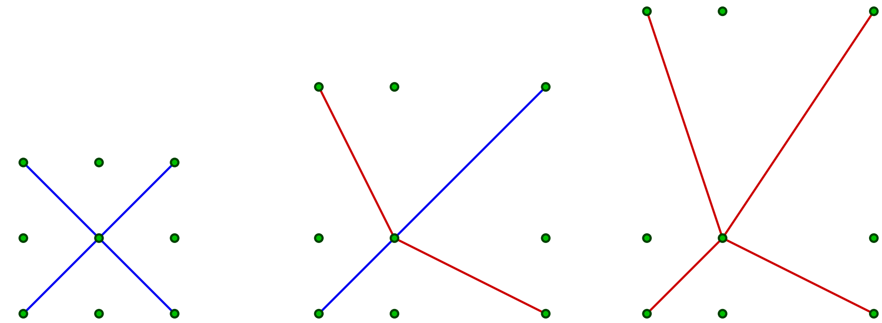


Figure 1.9: A $(0, 1, 2)$-, $(0, 1, a)$- and $(0, 1, a_i)$-grid.

Theorem 1.5.3. *Let $\mathcal{T}_i(\mathbb{Z})$, $\mathcal{T}_i(\mathbb{Q})$, and $\mathcal{T}_i(\mathbb{R})$ and $\mathcal{T}(\mathbb{Z})$, $\mathcal{T}(\mathbb{Q})$, and $\mathcal{T}(\mathbb{R})$ be defined as above. Then*

$$\begin{array}{ccccc} \mathcal{T}_i(\mathbb{Z}) & \subseteq & \mathcal{T}_i(\mathbb{Q}) & \subseteq & \mathcal{T}_i(\mathbb{R}) \\ \cup\!\!\!\!\nmid & & \cup\!\!\!\!\nmid & & \cup\!\!\!\!\nmid \\ \mathcal{T}(\mathbb{Z}) & \subsetneq & \mathcal{T}(\mathbb{Q}) & \subsetneq & \mathcal{T}(\mathbb{R}). \end{array}$$

In particular, we are examing the containments of collections of combinatorial types from Theorem 1.4.1 in the setting of $(0, 1, a_i)$-polytopes.

Lemma 1.5.4. *There exists an integral $(0, 1, a_i)$-polytope that cannot be realized as a rational $(0, 1, a)$-polytope.*

Proof. Let P be the 3-dimensional $(2, 3, 4)$-polytope whose vertices can be read as the columns of the following matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 2 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 4 \end{pmatrix}.$$

As P is combinatorially equivalent to the irrational $(0, 1, a)$-polytope in Lemma 1.4.4, we see that P cannot be realized as a $(0, 1, a)$-polytope for $a \in \mathbb{Q}$. $\square$

Thus, P is a polytope whose combinatorial type is contained in $\mathcal{T}_i(\mathbb{Z})$ but not $\mathcal{T}(\mathbb{Q})$. We see that $\mathcal{T}(\mathbb{Z}) \subsetneq \mathcal{T}_i(\mathbb{Z})$ and $\mathcal{T}(\mathbb{Q}) \subsetneq \mathcal{T}_i(\mathbb{Q})$.

We would also like to show that there exists a polytope that can be realized as a $(0, 1, a_i)$-polytope but not a $(0, 1, a)$-polytope. To find a candidate, we explore a combinatorial structure of $(0, 1, a)$-polytopes. For this, the following result is helpful.

Theorem 1.5.5 (see [23, Sect. 5.3])**.** *Let P be a polytope in $\mathbb{R}^n$. Then there exists an $\epsilon = \epsilon(P) > 0$ such that every polytope $P' \subset \mathbb{R}^n$ for which the Hausdorff metric $\rho(P, P')$ is less than ϵ satisfies $f_k(P) \geq f_k(P')$ for all k.*

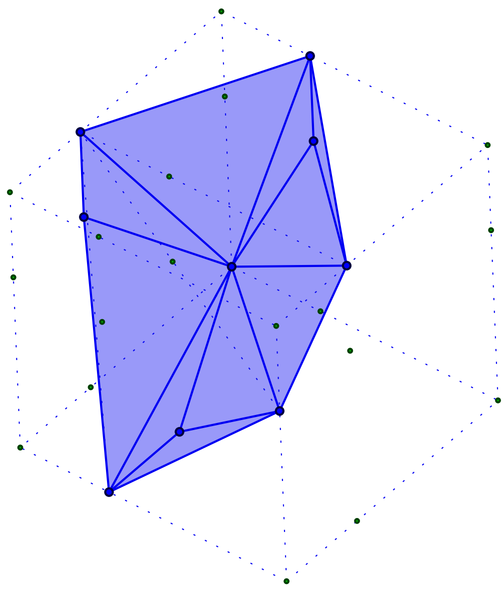


Figure 1.10: A $(0, 1, 3)$-polytope with vertex degree eight.

From Theorem 1.5.5 one can derive that by slightly perturbing the vertices of a polytope P, the number of vertices will not decrease, nor will the maximal vertex degree.

Lemma 1.5.6. *A $(0, 1, a)$-polytope in dimension three can have vertex degree at most eight. Further, this bound is tight.*

Proof. We have checked the full database of rational $(0, 1, a)$-polytopes in dimension three as provided by the enumeration in Section 2.2.1. While there are 39 combinatorial types of these polytopes with a vertex of degree eight, there are no rational $(0, 1, a)$-polytopes in dimension three with maximal vertex degree greater than eight.

This also implies that there cannot exist an irrational $(0, 1, a)$-polytope with degree greater than eight: Otherwise, a slight perturbation of a would result in a rational polytope with smaller maximal vertex degree, contradicting Theorem 1.5.5.

Thus, no vertex of a 3-dimensional $(0, 1, a)$-polytope can have vertex degree more than eight.

The $(0, 1, 3)$-polytope whose vertices can be read as the columns of the following matrix confirms the tightness of this bound.

$$\begin{pmatrix} 1 & 1 & 3 & 3 & 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 0 & 3 & 3 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 3 & 3 \end{pmatrix}$$

This polytope can be seen in Figure 1.10.

□

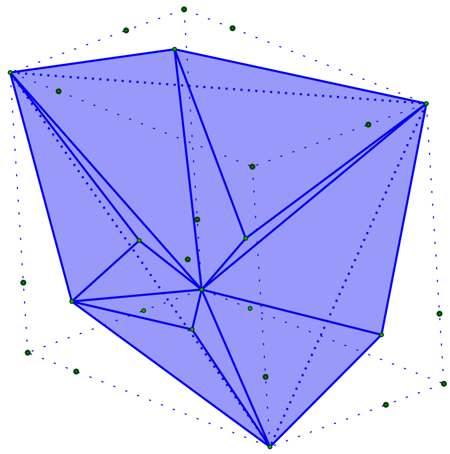


Figure 1.11: A $(0, 1, a_i)$-polytope that cannot be realized as a $(0, 1, a)$-polytope.

Lemma 1.5.7. *There exists a $(0, 1, a_i)$-polytope that cannot be realized as a $(0, 1, a)$-polytope.*

Proof. Consider the $(0, 1, a_i)$-polytope for $\mathbf{a} = (5, 3, 4)$ whose vertices can be read as the columns of the following matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 5 & 1 & 0 & 1 & 5 & 5 \\ 0 & 1 & 1 & 0 & 3 & 3 & 3 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 4 & 4 & 4 & 1 \end{pmatrix}$$

The vertex $(0, 0, 0)$ has degree nine, as shown in Figure 1.11. Thus, by Lemma 1.5.6, P cannot be realized as a $(0, 1, a)$-polytope for $a \in \mathbb{R}$. $\square$

Proof of Theorem 1.5.3. By Theorem 1.4.1 as well as by Lemma 1.5.4 and Lemma 1.5.7 we obtain the following overview of the different collections of combinatorial types of $(0, 1, a)$- and $(0, 1, a_i)$-polytopes:

$$\begin{array}{ccccc} \mathcal{T}_i(\mathbb{Z}) & \subseteq & \mathcal{T}_i(\mathbb{Q}) & \subseteq & \mathcal{T}_i(\mathbb{R}) \\ \cup\!\!\!\!\!\not\;\, & & \cup\!\!\!\!\!\not\;\, & & \cup\!\!\!\!\!\not\;\, \\ \mathcal{T}(\mathbb{Z}) & \subsetneq & \mathcal{T}(\mathbb{Q}) & \subsetneq & \mathcal{T}(\mathbb{R}). \end{array}$$ $\square$

The strict containments $\mathcal{T}_i(\mathbb{Z}) \subsetneq \mathcal{T}_i(\mathbb{Q}) \subsetneq \mathcal{T}_i(\mathbb{R})$ are missing from Theorem 1.5.3.

Conjecture 1. The containments implied by Theorem 1.5.3 are the only containments amongst the collections of combinatorial types $\mathcal{T}_i(\mathbb{Z})$, $\mathcal{T}_i(\mathbb{Q})$, and $\mathcal{T}_i(\mathbb{R})$.

In this chapter we have explored two methods of proving these strict containments: by completely enumerating various families of polytopes, and by finding a combinatorial structure that excludes polytopes from a class. In the next chapter we will discuss the intractability of a complete enumeration of $(0,1,a_i)$-polytopes. Thus, the more propitious approach for proving these strict containments would be to find combinatorial structures exclusive to particular classes of $(0,1,a_i)$-polytopes.

Many polytopes in general cannot be realized as $(0,1,a_i)$-polytopes. The following proposition helps illuminate this fact.

Proposition 1.5.8. *Let $P \subset \mathbb{R}^{n+1}$ be a n-dimensional $(0,1,a_i)$-polytope for $\boldsymbol{a} \in \mathbb{R}^{n+1}$. Then P is affinely isomorphic to an n-dimensional $(0,1,a_i')$-polytope $P' \subset \mathbb{R}^n$ where $\boldsymbol{a}' \in \mathbb{R}^n$ is a sub-vector of $\boldsymbol{a}$.*

Proof. Because P is not full dimensional, P satisfies an equation of the form

$$c_1x_1 + \cdots + c_nx_n + c_{n+1}x_{n+1} = c_0.$$

We can permute the coordinates to get $c_{n+1} \neq 0$. Consider

$$\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$$

where $\pi(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$. Then $\pi(P) = P'$ is a injective projection of P that is a $(0,1,a_i)$-polytope in $\mathbb{R}^n$ where $\mathbf{a}' \in \mathbb{R}^n$ is a sub-multiset of $\mathbf{a}$. $\square$

Thus, by applying Proposition 1.5.8 repeatedly, we can view any face of a $(0,1,a_i)$-polytope as a full-dimensional polytope. Thus, we can easily build polytopes that cannot be realized as $(0,1,a_i)$-polytopes. For instance, the largest $(0,1,a_i)$-polytope in dimension two is a hexagon. Thus, any polytope that has a 2-face with more than six vertices cannot be realized as a $(0,1,a_i)$-polytope.

Chapter 2

Enumerations and complexity discussions for $n \leq 4$.

The aim of this chapter is to discuss the enumeration of low-dimensional $(0, 1, a)$-polytopes. In order to motivate this classification as well as to establish some useful results, we will first look into the progress made in enumerating combinatorial types of general polytopes with a low dimension.

2.1 Classifying general polytopes

In dimension two, there is a precise description of all combinatorial types. There is exactly one distinct combinatorial type with k vertices for $k \geq 3$. Thus, we may refer to the polygons with this combinatorial type as k-gons. However, these polytopes are relatively straightforward and we will now move on to dimension three, where things get more interesting.

2.1.1 For dimension three

Polytopes in dimension three are one of the most thoroughly-studied family of polytopes. The reason for this runs even deeper than their experimental and visual accessibility. Referred to by himself as "the fundamental theorem of convex types", Steinitz' theorem has laid the foundation for the extent to which we understand 3-polytopes today.

Theorem 2.1.1 ([46], see [23]). *A graph G is the graph of a 3-polytope if and only if G is planar and 3-connected.*

It is not too hard to convince oneself that the graph of a 3-polytope is planar and 3-connected: Balinski's theorem tells us that the graph of an n-polytope is n-connected ([8]), while a Schlegel diagram of 3-polytopes produce a planar projection of the polytope's graph. The fact that any graph that is planar and 3-connected can be realized as the graph of a 3-polytope is a bit more subtle and therein lies the difficulties of Steinitz' proof. The proof argues

number of vertices	number of comb. types
4	1
5	2
6	7
7	34
8	257
9	2606
10	$32,300$
11	$440,564$
12	$6,384,634$

Table 2.1: Number of combinatorial types of 3-polytopes with at most 12 vertices.

that any 3-connected planar graph may be built up from the complete graph on four vertices while preserving realizability of the corresponding polytope. For more details about this proof, the reader is directed to [23].

The true beauty of this theorem is how our extensive knowledge of 3-connected planar graphs can now be applied to the combinatorial types of 3-polytopes. In particular, we can now enumerate 3-polytopes with k vertices by simply enumerating the 3-connected planar graphs with k vertices. Of course, we are still restricted by computational limitations. An enumeration done in [18] is shown in Table 2.1.

2.1.2 In higher dimensions

It is not nearly so straightforward to classify polytopes in dimension four as there does not exist an analogous theorem to Steinitz' for dimension greater than three. Despite the lack of such a precise characterization of 4-polytopes, there still exists partial enumerations.

Due to results by [32] and [29], we have for $f_0 \leq n+3$, that each $(n-1)$-sphere is isomorphic to some n-polytope. Thus, in dimension four, the polytopes with up to seven vertices can be classified by classifying the 3-spheres with up to seven vertices.

However, this is not true for a higher number of vertices. In [12], Bokowski and Garms give a simple proof method for showing that a 3-sphere with ten vertices discovered by Altshuler (see [4]) is nonpolytopal.

This complicates the classification of 4-polytopes with the additional step that once all the 3-spheres are identified, they must then be checked for polytopality. However, this is not entirely impossible. The simplicial polytopes with eight vertices were completely classified by Grünbaum and Sreedharan in [24]. Altshuler and Steinberg later provided a complete enumeration of all 3-spheres with eight vertices and determined their polytopality in [6]. Currently,

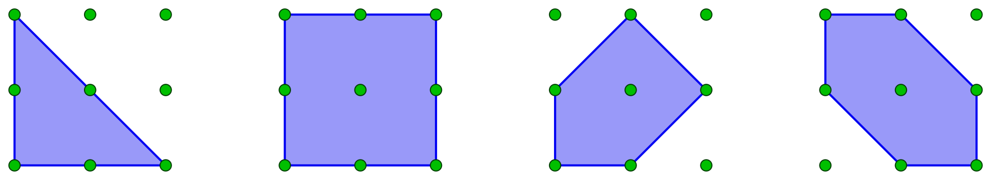


Figure 2.1: The combinatorial types of $(0, 1, a)$-polytopes in dimension 2.

an enumeration for polytopes with more vertices exists only for simplicial polytopes (see [19]).

Indeed, nonpolytopal 3-spheres are not rare (see [41], [37]), as Altshuler demonstrates in [5] by classifying the simplicial 3-spheres with nine vertices and determining whether they are polytopal or not. Of the 1296 simplicial 3-spheres, 154 of them are nonpolytopal.

Another important consequence of the lack of a Steinitz-like theorem is we now lose the ability to realize all 4-polytopes using rational coordinates. In fact, Perles was able to use Gale diagrams to show that there is an 8-polytope with 12 vertices whose combinatorial type can only be realized with irrational coordinates. For more information on Gale diagrams and the construction of this particular polytope, the reader is directed to [52].

2.2 Classifying $(0, 1, a)$-polytopes in low dimension

This section is dedicated to enumerating $(0, 1, a)$-polytopes to the fullest of our computational capabilities. As we saw in the previous section, a low dimension and low number of vertices vastly improve the viability of the task at hand.

As in the case of an arbitrary polytope in low dimensions, we can first look at enumerating the combinatorial types of 2-dimensional $(0, 1, a)$-polytopes. These turn out to be even more straightforward than the case of arbitrary polytopes; the most vertices a $(0, 1, a)$-polytope in dimension two can have is six. Figure 2.1 gives one realization of each combinatorial type in dimension two.

2.2.1 For $n = 3$

As we have seen above for 3-polytopes with few enough vertices, we would like to classify $(0, 1, a)$-polytopes in dimension three. In this context, Steinitz' theorem would cause us to lose too much information to ensure each coordinate takes one of only three values.

However, in contrast to general polytopes, we have a natural upper bound of 3^n on the maximal number of vertices a $(0,1,a)$-polytope could have, which greatly facilitates our enumeration. This limit of possible vertices allows a fairly straightforward enumeration technique. We can take the convex hull of all (ordered) subsets of the $(0,1,a)$-grid and check for distinct combinatorial types. However, in order to make this idea feasible, it is useful to find a tighter upper bound on the number of vertices a 3-dimensional $(0,1,a)$-polytope may have. Additionally, we would need to establish a finite set of a values to evaluate in order to make make this an achievable goal.

Range of number of vertices

Because the $(0,1,a)$-polytopes have vertex sets which are subsets of the $(0,1,a)$-grid, we have an obvious limit of 3^n on the number of vertices an n-dimensional $(0,1,a)$-polytope may have. However, this bound is clearly not tight, and for computation purposes it would be beneficial to have a tighter upper bound, at least in dimension three.

Let $v_2(n)$, $v_a(n)$, and $v_i(n)$ be the maximum number of vertices an n-dimensional $(0,1,2)$-, $(0,1,a)$-, and $(0,1,a_i)$-polytope can have, respectively.

Proposition 2.2.1. *Let $v_2(3)$, $v_a(3)$, and $v_i(3)$ be as defined above. Then,*

$$v_2(3) = v_a(3) = v_i(3) = 16.$$

Proof. Clearly, $v_2(n) \leq v_a(n) \leq v_i(n)$. It is left to be shown that $16 \leq v_2(3)$ and $v_i(3) \leq 16$.

We first show that $v_i(3) \leq 16$. We can imagine the $(0,1,a_i)$-grid in dimension three as having three levels of a 2-dimensional $(0,1,a_i)$-grid, with levels 0, 1, and a_3, corresponding to the last coordinate. Notice that a cross-section of each level will also be a polygon. In order to maximize the number of vertices of a $(0,1,a_i)$-polytope, we need to jointly maximize the number of grid vertices of these cross-sections.

First we will show that a polytope cannot have more than 17 vertices. One level can have at most six vertices. However, it cannot be that each level has six vertices. Figure 2.2 demonstrates the two possible alignments for the hexagon. Regardless of the alignment, there will be four instances where three possible vertices lie on a vertical line. Thus, there is no arrangement with 18 vertices.

There also cannot exist a $(0,1,a_i)$-polytope in dimension three with 17 vertices. This polytope would have to have two levels with six vertices, and one level with 5 vertices. The key observation here is that any $(0,1,a_i)$-pentagon must contain at least one of these red vertices as a vertex, as shown in Figure 2.3. Any pentagon must have at least four boundary vertices. If the pentagon has the inner node as a vertex, it cannot also have $(0,0)$, $(a_1,0)$, $(0,a_2)$, and (a_1,a_2) as vertices.

Figure 2.2: The two possible realizations of hexagons in a 2-dimensional $(0, 1, a)$-grid.

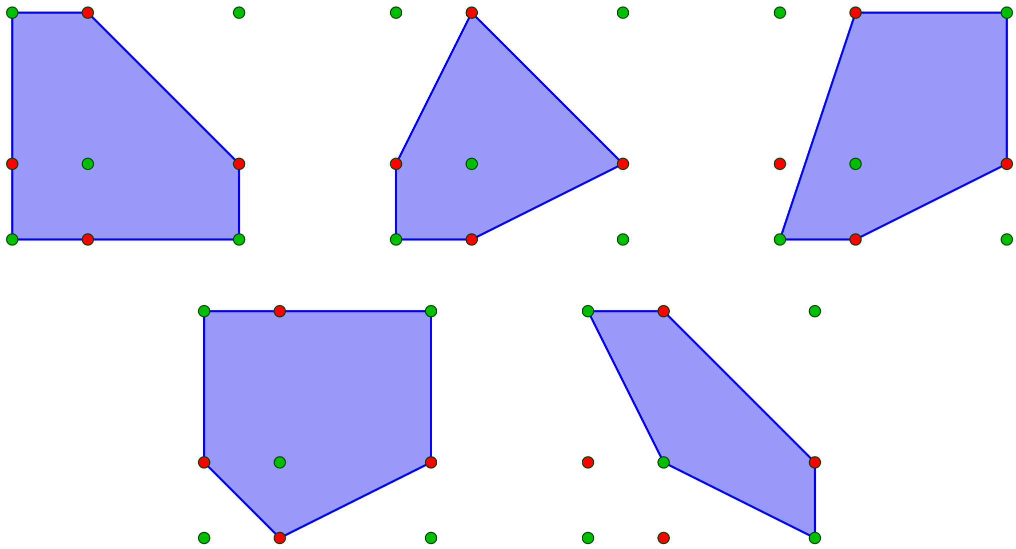


Figure 2.3: Possible configurations of pentagons.

Thus, three points in any such arrangement would lie on a vertical line, resulting in less than 17 vertices. Therefore, a $(0, 1, a_i)$-polytope in dimension three can have at most 16 vertices.

To see that $16 \leq v_2(3)$, consider the polytope defined by vertices given as columns in the following matrix:

$$\begin{pmatrix} 1 & 2 & 2 & 0 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

This is a $(0, 1, 2)$-polytope with 16 vertices, thus completing the proof. $\square$

The polytope demonstrating the tightness of the bound in Proposition 2.2.1 is given in Figure 2.4. This polytope can be easily imagined as a hexagon on level 0, the whole $(0, 2)$-square on level 1, and a rotated hexagon on level 2. However, as the enumeration in Section 2.2 reveals, this is not the only $(0, 1, a)$-polytope with 16 vertices.

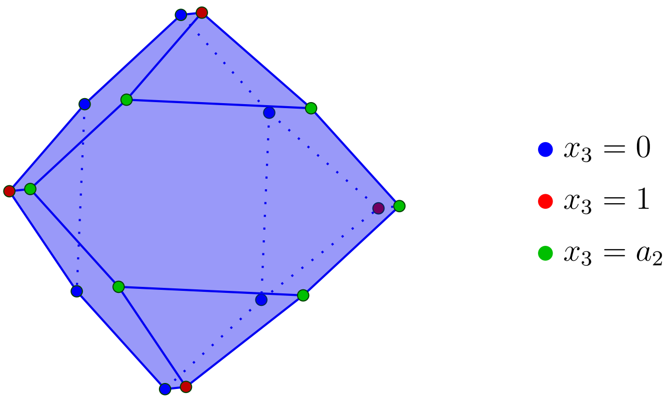


Figure 2.4: A $(0,1,2)$-polytope with 16 vertices.

Due to Proposition 2.2.1, we just need to enumerate the combinatorial types of $(0,1,a)$-polytopes with up to 16 vertices.

Additionally, comparing the enumeration of the ten 3-polytopes with four, five, or six vertices with that of $(0,1)$-polytopes, we see that all 3-polytopes with six or less vertices can be realized as $(0,1)$-polytopes. Thus, we can begin our enumeration with seven vertices.

For the enumeration, we need to check if the convex hulls of subsets of the $(0,1,a)$-grid of size seven through sixteen return 3-polytopes with a combinatorial type that has not previously been found. This is certainly a doable computation, with $117,487,335$ subsets to check, although nicely parallelizable in the number of vertices a polytope has. The number of subsets necessary to check also decreases after we disregard the ones that lose dimension or vertices.

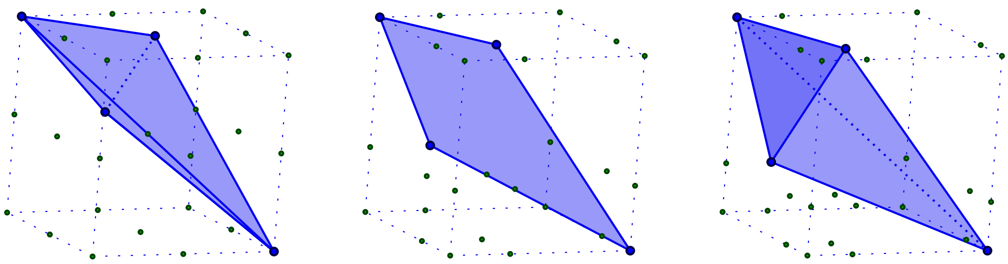
Reducing to a finite set of a values

While we have found a reasonable restriction in the number of vertices we need to consider, the second step to making an enumeration of $(0,1,a)$-polytopes computationally feasible is to find a finite set of a values for which we need to check for combinatorial types.

The general idea behind the enumerations is that we fix some points in the $(0,1,a)$-grid, starting with $a=2$ and look at the convex hull. We can then see what happens to the combinatorial structures as we increase a. There could be occurences where four or more points of the $(0,1,a)$-grid lie on a common hyperplane. Some of these hyperplanes, ie. $x_1 = 0$, will continue to contain these points and thus never "break" as a changes. Other sets of four points will lay on a plane for a particular a value, but form a tetrahedron otherwise.

Example 4. The hyperplane in the $(0,1,a)$-grid defined by

$$2x_1 + x_2 + x_3 = 6$$



(a) $(0,1,2)$-grid (b) $(0,1,3)$-grid (c) $(0,1,4)$-grid

Figure 2.5: As a is pushed out, the tetrahedron in 2.5a is flattened out, until all four vertices lie on a plane in 2.5b, and then the vertices are further pulled out to once more form a tetrahedron 2.5c.

contains the points $(3,0,0)$, $(1,3,1)$, $(1,1,3)$, and $(0,3,3)$ when $a = 3$. However, the points $(a,0,0)$, $(1,a,1)$, $(1,1,a)$, and $(0,a,a)$ will not lie on a hyperplane for any other $a \neq 3$. Figure 2.5 shows these transitions.

Thus, we need to find all instances of a where four points lie on a hyperplane. It could be that these four points do not share a facet or that one of the four points lies in the convex hull of the other three. However, it could also correspond to a change in combinatorial type.

In order to do, we can consider four points v_1, v_2, v_2, v_4 of the $(0,1,a)$-grid. If at least these four points lie on a plane, then this could correspond to a polytope having facet that is not a simplex, which could in turn result in a new combinatorial type.

Four vertices will lie on a plane exactly when the determinant of

$$M = \begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

is zero. These determinants will be polynomials in a of degree at most three. In particular, we want to evaluate for which values of a the determinants are zero but not constantly zero. Let

$$M_3^a := \{M \in \{0,1,a\}^{4\times 4} : m_{4j} = 1 \text{ for all } j \in [4]\}.$$

where m_{4j} is the $(4,j)$ entry of M, and we can define the set of determinants that are not constantly zero

$$\mathcal{M}_3^a := \{\det(M) : M \in M_3^a, \det(M) \not\equiv 0\}.$$

Then, we are interested in the set of a values

$$\mathcal{A}_3 := \{a \in \mathbb{R}_{\geq 2} : d|_a = 0 \text{ for some } d \in \mathcal{M}_3^a\}.$$

Because there are only finitely-many determinants, we also know $\mathcal{A}_3$ is a finite set.

Theorem 2.2.2. *Let $\mathcal{A}_3$ be defined as above and $x, y \in \mathbb{R}$ such that $[x,y] \cap \mathcal{A}_3 = \emptyset$. Then all combinatorial types of polytopes that arise as $(0,1,x)$-polytopes can also be realized as $(0,1,y)$-polytopes.*

Proof. Let $\mathcal{A}_3$ be defined as above and $x, y \in \mathbb{R}$ such that $[x,y] \cap \mathcal{A}_3 = \emptyset$. Let P_x be a full-dimensional $(0,1,x)$-polytope. Let $V(a)$ be a subset of the nodes of the 3-dimensional $(0,1,a)$-grid such that $V(x)$ is the vertex set of P_x. Let $P_y = \mathrm{conv}(V(y))$. We claim that P_y is combinatorially equivalent to P_x. We can completely describe a polytope by its vertices in facets. Thus, we need to check that a facet-defining hyperplane of P_x corresponds to a facet-defining hyperplane and both contain corresponding vertices.

Let $H(a)$ be a hyperplane such that $H(x)$ is a facet-defining hyperplane of P_x for the facet F. If H is also a facet-defining hyperplane of the $(0,2)$-cube, then clearly $H(y)$ is a facet-defining hyperplane of P_y containing the corresponding vertices. Otherwise, F contains exactly three vertices, and the corresponding vertices of P_y are contained in $H(y)$. Further, $H(y)$ is a facet-defining hyperplane of P_y. Because $H(x)$ is a facet-defining hyperplane of P_x, $P_x \subset H(x) \cup H(x)^-$. But since for all $a \in R$ such that $x \le a \le y$, $a \notin \mathcal{A}_n$, then $P_y \subset H(y) \cup H(y)^-$, completing the proof. □

Therefore, we only need to check a values that are the roots of the determinants as well as one representative per interval between the roots in order to obtain all possible distinct combinatorial types. This gives us the means to enumerate all combinatorial types for a finite number of a values.

In order to do so, we will first enumerate all determinants, and in particular their roots. Then, as we saw in Theorem 1.4.1, it is sensible to make a distinction in our classification of integer, rational, or real $(0,1,a)$-polytopes. Therefore, we will break up the investigation of 3-dimensional $(0,1,a)$-polytopes into three cases, $a \in \mathbb{Z}$, $a \in \mathbb{Q}$, and $a \in \mathbb{R}$. Then we will discuss what roles these restrictions play in the enumerations of $(0,1,a)$-polytopes.

Computations and hardware

The computations for the results presented in the remainder of Section 2.2.1 were set up with the computer algebra system Sage [43]. Each case was either handled on a desktop PC with 8GB of RAM, or on a cluster of about 300 Xeon CPUs with about 3GB RAM each. The time needed to complete each computation varied on the size of the set checked to be a vertex set, as well as on the number of distinct face lattices produced. The quickest computations took minutes, while the lengthiest took months. One bottleneck identified in the process is the memory leak of the Sage face lattice function. Since the implementation of the program, however, new methods have been developed for identifying polytopal combinatorial equivalence with Sage. Thus, the efficiency of the program could potentially be improved.

Proposition 2.2.3. *Consider the set $\mathcal{A}_3$ as defined above. $\mathcal{A}_3$ contains exactly the following 7 elements:*

$$
\begin{aligned}
&2\\
\delta + \frac{1}{3\delta} + 1 &\approx 2.3247\\
\sqrt{2} + 1 &\approx 2.4142\\
\frac{\sqrt{5}}{2} + \frac{3}{2} &\approx 2.618\\
&3\\
\gamma + \frac{7}{9\gamma} + \frac{4}{3} &\approx 3.1479\\
\sqrt{2} + 2 &\approx 3.4142
\end{aligned}
$$

for

$$\delta = \sqrt[3]{\frac{\sqrt{69}}{18} + \frac{1}{2}} \text{ and } \gamma = \sqrt[3]{\frac{\sqrt{93}}{18} + \frac{47}{54}}.$$

Proof. Sage was used to find the determinants for all $M_3^a \in \mathcal{M}_3^a$. There were determinants in the form 0, ± 1, ± 2, as well as polynomials in a with degree at most 3. Solving for the roots of the polynomials and finding the ones greater or equal to two, we obtain the result. ☐

Thus, Table 2.2 displays all intervals where distinct combinatorial types of 3-dimensional $(0, 1, a)$-polytopes could arise.

Case $a \in \mathbb{Z}$

By limiting our a values such that $a \in \mathbb{Z}$, we can focus on integral roots of the determents of matrix M_3^a.

From above we have that there are exactly two of them greater or equal to two, namely 2 and 3. Additionally, since no root is greater than 4, we know that the combinatorial structure of a $(0, 1, a)$-polytope does not change once a is at least 4. Thus, we have the following proposition:

Proposition 2.2.4. *There are three classes that can result in distinct combinatorial types of $(0, 1, a)$-polytopes in dimension 3 for $a \in \mathbb{Z}$. In particular, the combinatorial types of $(0, 1, a)$-polytopes may be distinct for:*

(1) $a = 2$ (2) $a = 3$ (3) $a \geq 4$

In order to complete the enumeration, we utilize Sage, finding all possible ordered combinations of $7 \leq k \leq 16$ points in the $(0, 1, a)$-grid for $a = 2$, $a = 3$, and $a = 4$, and check whether the combinatorial type has already been found.

In particular, Table 2.3 gives the number of distinct combinatorial types.

Case no.	a values
(1)	$a = 2$
(2)	$2 < a < \delta + \frac{1}{3\delta} + 1$
(3)	$a = \delta + \frac{1}{3\delta} + 1$
(4)	$\delta + \frac{1}{3\delta} + 1 < a < \sqrt{2} + 1$
(5)	$a = \sqrt{2} + 1$
(6)	$\sqrt{2} + 1 < a < \frac{\sqrt{5}}{2} + \frac{3}{2}$
(7)	$a = \frac{\sqrt{5}}{2} + \frac{3}{2}$
(8)	$\frac{\sqrt{5}}{2} + \frac{3}{2} < a < 3$
(9)	$a = 3$
(10)	$3 < a < \gamma + \frac{7}{9\gamma} + \frac{4}{3}$
(11)	$a = \gamma + \frac{7}{9\gamma} + \frac{4}{3}$
(12)	$\gamma + \frac{7}{9\gamma} + \frac{4}{3} < a < \sqrt{2} + 2$
(13)	$a = \sqrt{2} + 2$
(14)	$a > \sqrt{2} + 2$

Table 2.2: The 14 different classes of a values needed to be checked for $\delta = \sqrt[3]{\frac{\sqrt{69}}{18} + \frac{1}{2}}$ and $\gamma = \sqrt[3]{\frac{\sqrt{93}}{18} + \frac{47}{54}}$.

no. of vertices	$a = 2$	$a = 3$	$a \geq 4$	total
6	7	7	7	7
7	34	34	34	34
8	193	249	249	249
9	680	1415	1406	1444
10	1758	4401	4419	4561
11	2049	6150	6196	6392
12	955	3134	3153	3379
13	207	700	704	739
14	29	91	92	97
15	3	10	10	10
16	1	2	2	2
total	5916	16193	16272	16914

Table 2.3: The number of distinct combinatorial types for $a \in \mathbb{Z}$.

This is a complete enumeration of $(0, 1, a)$-polytopes for $a \in \mathbb{Z}$ in dimension three. It is worth observing that while all 3-polytopes with seven vertices cannot be realized with coordinates 0 or 1, it is possible to realize them with coordinates 0,1, or 2.

Case $a \in \mathbb{Q}$

Classifying all $(0, 1, a)$-polytopes for $a \in \mathbb{Q}$ is equivalent to enumerating all polytopes in dimension three whose vertex coordinates are restricted to three distinct integers. In order to do so, we have a similar proposition to Proposition 2.2.4 above.

Proposition 2.2.5. *There are nine cases we need to check for distinct combinatorial types of* $(0, 1, a)$*-polytopes in dimension 3 for* $a \in \mathbb{Q}$*. Using the numbering from Table 2.2, the enumerations of* $(0, 1, a)$*-polytopes may be distinct for:*

(1) $a = 2$

(2) $2 < a < \delta + \frac{1}{3\delta} + 1$

(4) $\delta + \frac{1}{3\delta} + 1 < a < \sqrt{2} + 1$

(6) $\sqrt{2} + 1 < a < \frac{\sqrt{5}}{2} + \frac{3}{2}$

(8) $\frac{\sqrt{5}}{2} + \frac{3}{2} < a < 3$

(9) $a = 3$

(10) $3 < a < \gamma + \frac{7}{9\gamma} + \frac{4}{3}$

(12) $\gamma + \frac{7}{9\gamma} + \frac{4}{3} < a < \sqrt{2} + 2$

(14) $a > \sqrt{2} + 2$

for

$$\delta = \sqrt[3]{\frac{\sqrt{69}}{18} + \frac{1}{2}} \text{ and } \gamma = \sqrt[3]{\frac{\sqrt{93}}{18} + \frac{47}{54}}.$$

Proof. Similarly to above, we need to check for new combinatorial types for the rational roots as well as in the intervals between the irrational roots. Thus, we have the nine classes where new combinatorial types may arise. □

Using the labeling for the different cases from Table 2.2, Table 2.4 gives the number of distinct combinatorial types of $(0, 1, a)$-polytopes with k vertices for $a \in \mathbb{Q}$.

This is a complete enumeration for $(0, 1, a)$-polytopes for $a \in \mathbb{Q}$ in dimension three. We notice that six combinatorial types are missing for $(0, 1, a)$-polytopes with 8 vertices. This means that these types require either irrational coordinates when restricted to only three coordinate values, or that they have no realization with only three coordinate values.

k	(1)	(2)	(4)	(6)	(8)	(9)	(10)	(12)	(14)	total
6	7	7	7	7	7	7	7	7	7	7
7	34	34	34	34	34	34	34	34	34	34
8	193	243	246	247	249	249	249	249	249	252
9	680	1140	1200	1215	1384	1415	1400	1402	1406	1467
10	1758	3476	3619	3665	4356	4401	4393	4398	4419	4649
11	2049	5153	5217	5243	6139	6150	6176	6177	6196	6491
12	955	2802	2809	2814	3134	3134	3148	3148	3153	3420
13	207	663	663	663	700	700	704	704	704	745
14	29	90	90	90	91	91	92	92	92	97
15	3	10	10	10	10	10	10	10	10	10
16	1	2	2	2	2	2	2	2	2	2
total	5916	13620	13897	13990	16106	16193	16215	16223	16272	17173

Table 2.4: The number of distinct combinatorial types of $(0,1,a)$-polytopes with k vertices for $a \in \mathbb{Q}$.

Case $a \in \mathbb{R} \setminus \mathbb{Q}$

In this section we will look at the classification of the most general case of $(0,1,a)$-polytopes in dimension three. By allowing $a \in \mathbb{R}$, we are enumerating all 3-polytopes with their vertex coordinates restricted to three distinct real numbers. After our enumeration of for $a \in \mathbb{Z}$ and $a \in \mathbb{Q}$, it only remains to enumerate the $(0,1,a)$-polytopes such that a is one of the five irrational roots of M_3^a:

$$\begin{array}{cccc}
\text{Case} & a \text{ value} & & \\
(3) & \delta + \frac{1}{3\delta} + 1 & \approx & 2.3247 \\
(5) & \sqrt{2}+1 & \approx & 2.4142 \\
(7) & \frac{\sqrt{5}}{2} + \frac{3}{2} & \approx & 2.618 \\
(11) & \gamma + \frac{7}{9\gamma} + \frac{4}{3} & \approx & 3.1479 \\
(13) & \sqrt{2}+2 & \approx & 3.4142
\end{array}$$

for

$$\delta = \sqrt[3]{\frac{\sqrt{69}}{18} + \frac{1}{2}} \text{ and } \gamma = \sqrt[3]{\frac{\sqrt{93}}{18} + \frac{47}{54}}.$$

However, we may first further reduce the amount of work required by examining a bit more closely these planes containing at least four points of a $(0,1,a)$-grid for irrational a. In other words, one can examine the convex hull of $\{v_1, v_2, v_3, v_4\}$ such that

$$\det \begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \\ 1 & 1 & 1 & 1 \end{pmatrix} = 0,$$

for $a \in \{\delta + \frac{1}{3\delta} + 1, \sqrt{2}+1, \frac{\sqrt{5}}{2} + \frac{3}{2}, \beta + \frac{7}{9\delta} + \frac{4}{3}, \sqrt{2}+2\}$. Because these four points will lie on a plane, we have two possibilities: They either form a quadrilateral or a triangle whose convex hull contains the fourth point. Of these five irrational roots, all but $a = \sqrt{2}+1$ admit this first possibility.

Proposition 2.2.6. *Any $(0,1,a)$-polytope for $a = \sqrt{2}+1$ can be realized as a $(0,1,a')$-polytope for $\sqrt{2}+1 < a' < \frac{\sqrt{5}}{2} + \frac{3}{2}$.*

Proof. Let P_a be a $(0,1,a)$-polytope for $a = \sqrt{2}+1$. Let $a' \in \mathbb{R}$ such that $\sqrt{2}+1 < a' < \frac{\sqrt{5}}{2} + \frac{3}{2}$. Let $V(a)$ be a subset of the nodes of the 3-dimensional $(0,1,a)$-grid such that $V(a)$ is the vertex set of P_a. Let $P_{a'} = \mathrm{conv}(V(a'))$ where we obtain $V(a')$ by increasing a to a'. We claim that $P_{a'}$ is combinatorially equivalent to P_a. We can completely describe a polytope by its vertices in facets. Thus, we need to check that a facet-defining hyperplane of P_a corresponds to a facet-defining hyperplane and both contain corresponding vertices.

Let $H(a)$ be a facet-defining hyperplane of P_a for the facet F. If H is also a facet-defining hyperplane of the $(0,2)$-cube, then clearly $H(a')$ is a facet-defining hyperplane of $P_{a'}$ containing the corresponding vertices. Otherwise, for $a = \sqrt{2}+1$, any time four points lie on a hyperplane that is not a facet-defining hyperplane of the $(0,2)$-cube, one point lies in the convex hull of a triangle formed by the three other points. Thus F contains exactly three vertices, and the corresponding vertices of $P_{a'}$ are contained in $H(a')$. $H(a')$ doesn't contain any extra vertices because Further, $H(a')$ is a facet-defining hyperplane of $P_{a'}$. Because $H(a)$ is a facet-defining hyperplane of P_a, $P_a \subset H(a) \cup H(a)^-$. But since for all $x \in R$ such that $a < x < a'$, $x \notin \mathcal{A}_n$, then $P_{a'} \subset H(a') \cup H(a')^-$, completing the proof.

□

Thus, for $a = \sqrt{2}+1$, no new combinatorial types will arise which are not already in the enumeration for cases (4) and (6). Therefore, we only need to enumerate the combinatorial types arising for the four other irrational roots of our $(0,1,a)$-matrices given above.

Unfortunately, this reduction is not enough in order to completely classify all $(0,1,a)$-polytopes. With the current computational limitations, only the $(0,1,a)$-polytopes for $a \in \mathbb{R}$ with eight and nine vertices could be classified within a reasonable time frame. The number of distinct combinatorial types corresponding to the cases listed above are given in Table 2.5. In total, there are 252 and 1590 distinct combinatorial types of $(0,1,a)$-polytopes with 8 and 9 vertices, respectively.

We would like to note that there are only five combinatorial types of 3-polytopes with eight vertices that cannot be realized as a $(0,1,a)$-polytope. One is the pyramid over a 7-gon. The Schlegel diagrams for the other four are given in Figure 2.6.

no. of vertices	(3)	(7)	(11)	(13)
8	245	246	250	249
9	1209	1339	1403	1422

Table 2.5: The number of distinct combinatorial types for $a \in \mathbb{Z} \setminus \mathbb{Q}$.

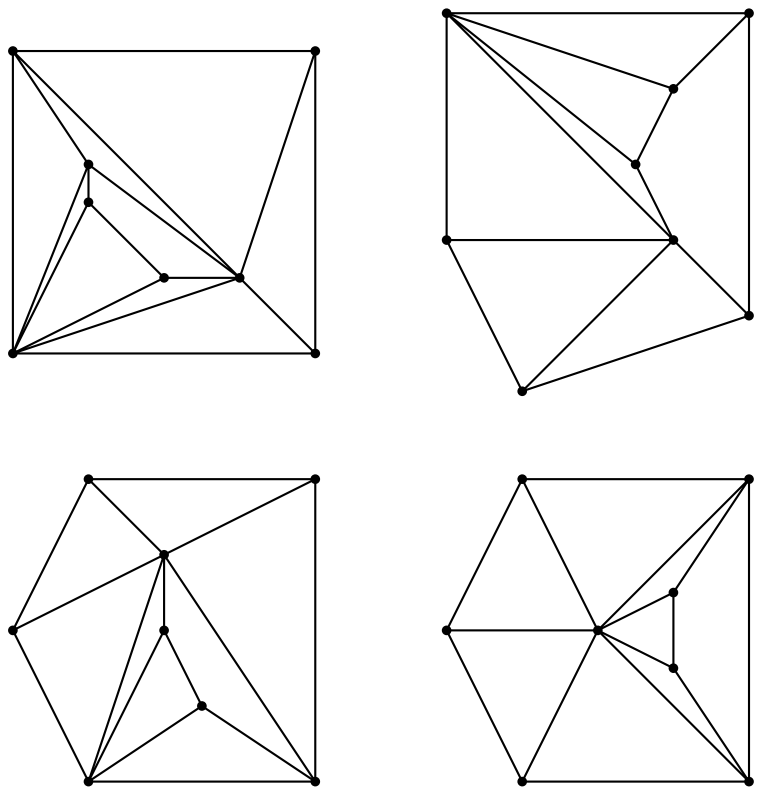


Figure 2.6: Schlegel diagrams of the 3-polytopes with eight vertices that cannot be realized as $(0, 1, a)$-polytopes, excluding the pyramid over the 7-gon.

2.2.2 Classifying $(0, 1, a)$-polytopes in dimension four

In this section, we would like to explore what happens when we attempt to apply the techniques used above to the case $n = 4$. While an enumeration in dimension four turns out to be computationally infeasible, we are able to get some concrete results which help lay a foundation for the complexity discussions in the next chapter.

Bounding the number of vertices

As in the dimension three case, the first step in classifying the $(0, 1, a)$-polytopes is to determine the maximal possible number of vertices. Recall the polytope from Proposition 2.2.1, created by embedding a hexagon in the plane $x_3 = 0$, a $(0, 2)$-square in the plane $x_3 = 1$, and a hexagon rotated by 90 degrees in the plane $x_3 = 2$. This polytope P^3_m resulted in a maximal number of vertices for $(0, 1, a)$-polytopes. We can use a similar construction for P^3_m as in Proposition 2.2.1 in order to find a $(0, 1, a)$-polytope in dimension four with many vertices. We embed P^3_m in the hyperplane $x_4 = 0$, embed a 3-dimensional $(0, 2)$-cube in the hyperplane $x_4 = 1$, and a reflected P^3_m in the hyperplane $x_4 = 2$.

Example 5. Using the construction described above, the resulting polytope has 40 vertices and a Schegel diagram as given in Figure 2.7.

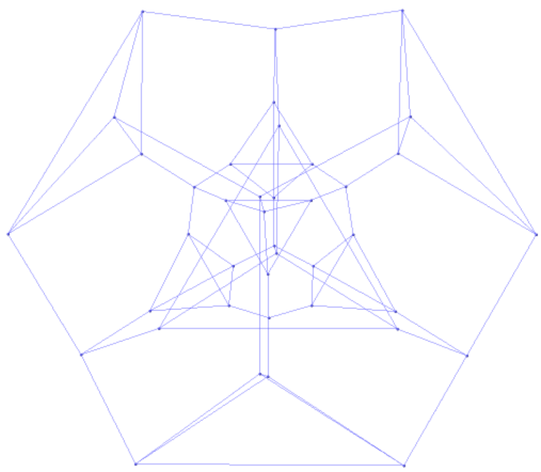


Figure 2.7: Schlegel diagram of a 4-dimensional $(0, 1, a)$-polytope with 40 vertices.

By the polytope in Example 5, we obtain a lower bound for the maximal number of vertices a $(0, 1, a)$-polytope in dimension four can have. However, proving that this bound is tight is more difficult than in the example for $n = 3$. In the case $n = 3$, there is simply one combinatorial type for a polygon with a specific number of vertices, and thus, all the different realizations of the polygon on the $(0, 1, a)$-grid are easily obtained.

We use the lower bound of 40 vertices, and realize quickly enough that this already renders a complete classification as infeasible.

Because all 4-dimensional polytopes with six vertices can also be realized as $(0,1)$-polytopes, we may begin our search with seven vertices. While there are 14 distinct combinatorial types of $(0,1)$-polytopes with seven vertices, there are 31 4-polytopes altogether with seven vertices. Thus, we would need to see how many of these 31 can be realized as $(0,1,a)$-polytopes.

Therefore, by using this lower bound of the maximal number of vertices, this still gives us $1,208,925,819,614,628,822,791,982$ convex hulls of ordered subsets of $\{0,1,a\}^4$ to check, varying from size 7 to 40.

We could investigate further, supposing that this computation is feasible, or if we want to limit the classification to $(0,1,a)$-polytopes with a particular (small) number of vertices. The next step would be to look at the different a values from which distinct combinatorial types could arise.

Evaluating the a values

Similar to the dimension three case, we would look at all possible roots of the determinants of matrices in the form

$$M = \begin{pmatrix} | & | & | & | & | \\ v_1 & v_2 & v_3 & v_4 & v_5 \\ | & | & | & | & | \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \text{ for } v_i \in \{0,1,a\}^4$$

is zero. Let

$$M_4^a := \{M \in \{0,1,a\}^{5\times 5} : m_{5j} = 1 \text{ for all } j \in [5]\}.$$

We can define the set of determinants that aren't constantly zero

$$\mathcal{M}_4^a := \{\det(M) : M \in M_4^a, \det(M) \not\equiv 0\}.$$

The first step to evaluate how difficult a classification could become is to examine the different roots. Sage was once again utilized for these computations [43]. There are altogether 165 distinct roots greater than or equal to two. Four of them are integral: 2, 3, 4, and 5; one of them is rational: $\frac{5}{2}$; and 160 of them are irrational. Additionally, the largest root is slightly larger than 6.

If we limit $a \in \mathbb{Z}$, then an enumeration is somewhat attainable. The four integral roots must be checked for combinatorial types, as well as $a = 7$ because the combinatorial type may change after the largest root. However, these six cases could possibly be enumerated for a limited number of vertices.

If we allow $a \in \mathbb{Q}$, then we only have one additional root, namely the rational root $a = \frac{5}{2}$. However, we would also have to consider the interval between the irrational roots of the determinants. Here the computations become less feasible. This gives us an a value to check for every interval between the roots, as well as one for each root. Thus we would have to run programs checking 170 different values of a for distinct combinatorial types.

In order to completely enumerate the $(0, 1, a)$-polytopes in dimension four, we would have to check the 170 equivalence classes from the $a \in \mathbb{Q}$ case, as well as the 160 instances where a is an irrational root of one of the $(0, 1, a)$-matrices.

Therefore, it seems fairly unreasonable to expect an enumeration of $(0, 1, a)$-polytopes in dimension four unless we limit ourselves to a small number of vertices and restrict a to be integral.

2.3 Complexity of enumerating $(0, 1, a_i)$-polytopes

In this section we would like to show the computational limitations of enumerating $(0, 1, a_i)$-polytopes. Because $(0, 1, a_i)$-polytopes are generalizations of $(0, 1, a)$-polytopes, we can restrict our attention to dimension three.

In Section 2.2.1 we computed the set $\mathcal{A}_3$ to obtain a finite set of a values to check for distinct combinatorial types for $(0, 1, a)$-polytopes. This gave us the parameter space for $(0, 1, a)$-polytopes as $\mathbb{R}_{\geq 2}$ subdivided by elements of $\mathcal{A}_3$. We did similar computations with Sage [43] to find all the determinants of matrices $M = [m_{ij}]$ for $1 \leq i, j \leq 4$, where $m_{ij} \in \{0, 1, a_i\}$ and $m_{4j} = 1$. Let $\mathcal{M}_3^{(a_1,a_2,a_3)}$ be the set of all such matrices M. Then the parameter space of $(0, 1, a_i)$-polytopes is $\{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2 \leq a_1 \leq a_2 \leq a_3\}$ subdivided by the zero sets of the determinants of the matrices of $\mathcal{M}_3^{(a_1,a_2,a_3)}$. These are polynomials in variables (a_1, a_2, a_3).

While there are constant determinants with values $0, \pm 1, \pm 2$ and linear determinants, the majority of the determinants are nonlinear. Of these determinants, 334 have zeros for $a_1, a_2, a_3 \geq 2$. In order to estimate how many of these have distinct zeros, we can randomly choose different points in the cone $\{x \in \mathbb{R}^3 : 2 \leq x_1 \leq x_2 \leq x_3\}$. By comparing whether or not these random points lie on, above, or below these hyperplanes or curves, we can see how many different combinations of a_i's could potentially result in distinct combinatorial types.

Similar to using signed vectors on hyperplane arrangements to describe different regions, we can create signed vectors with respect to the determinants with solutions for $a_1, a_2, a_3 \geq 2$. However, in contrast to doing this with hyperplane arrangements, the following example shows that two equivalent signed vectors can correspond to different regions when dealing with nonlinear curves.

Example 6. In the diagram given in Figure 2.8, the two purple regions each have a corresponding signed vector of $(-, -, +, +)$. The illustrates an example of distinct regions not necessarily corresponding to distinct signed vectors.

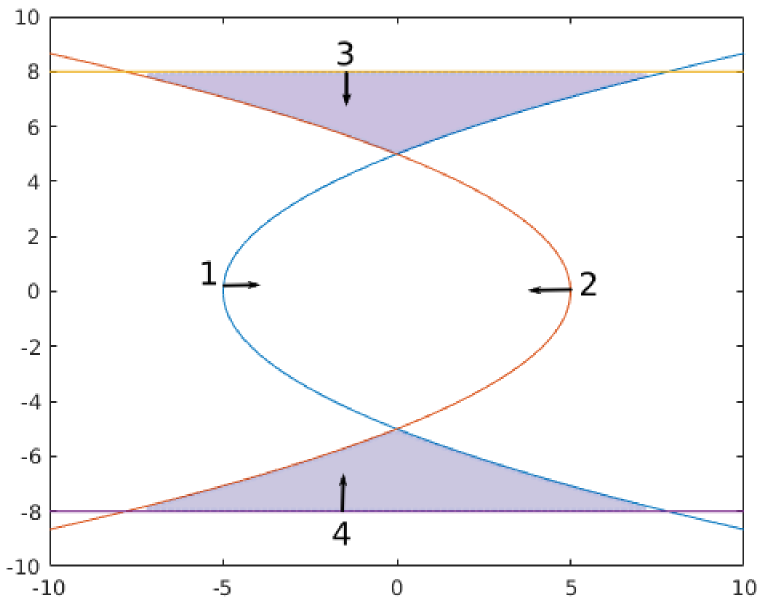


Figure 2.8: A curve arrangement with two sections with identical signed vectors.

Computing the signed vectors for about 475,000 random rational points, we found approximately 540 distinct signed vectors. By the example above, we can consider this at best a lower bound of the number of vectors of a values we would need to check to produce distint combinatorial types of polytopes. Even in the case where the a_i's are restricted to integral values is infeasible. There are 188 different $\mathbf{a} \in \mathbb{Z}^3_{\geq 2}$ for which we would need to check for distinct combinatorial types.

These results demonstrate that even limited to a small number of vertices and integral values, an enumeration of $(0, 1, a_i)$-polytopes is computationally infeasible.

Chapter 3

Enumerations and complexity discussions for higher n.

In this chapter we will discuss the classification of $(0,1,a)$-polytopes in higher dimensions as well as some extremal properties. To motivate this chapter and to develop a bit of insight as to how expensive we can expect the enumeration to be, we first look at related results of $(0,1)$-polytopes.

3.1 Motivation from $(0,1)$-polytopes

In this section, we want to expand on the discussion of $(0,1)$-polytopes given in Section 1.1, with more of a focus on higher dimension. We first examine some extremal properties, which will illuminate the complexities of $(0,1)$-polytopes in higher dimensions. These complications manifest in the enumeration of their combinatorial types.

3.1.1 Extremal properties

We saw in Chapter 1 that the enumeration of $(0,1)$-polytopes is too large for $n \geq 6$. This may seem slightly counter-intuitive, as we have a clear construction for a $(0,1)$-polytopes in dimension n achieving the maximal number of vertices, namely the n-cube with 2^n vertices.

To highlight why this enumeration is impractical, we first discuss the number of facets a $(0,1)$-polytope may have, and then we look into their connection with $(0,1)$-matrices and how this can enlighten our understanding.

Many facets

Our intuition from $(0,1)$-polytopes with low dimension may mislead us into thinking that such a polytope may have at most 2^n facets. (This is in fact true for $n \leq 4$.) However, this is not at all the case in higher dimensions. Let

$f_{n-1}(n)$ be the maximal number of facets of an n-dimensional $(0,1)$-polytope. The following theorem establishes a lower bound on $f_{n-1}(n)$.

Theorem 3.1.1 (Gatzouras, Giannopoulos, Markoulakis [21]). *For $f_{n-1}(n)$ as defined above,*

$$f_{n-1}(n) \geq \left(\frac{cn}{\log^2 n}\right)^{n/2}$$

for some constant c.

However, many experts believe this bound is far from being tight, and that the actual maximal number of facets is closer to the following best-known upper bound:

Theorem 3.1.2 (Fleiner, Kaibel, Rote [20]). *For large enough n, an n-dimensional $(0,1)$-polytope has no more than*

$$f_{n-1}(n) \leq 30(n-2)!$$

facets.

Through these bounds, one begins to realize that the facet descriptions of $(0,1)$-polytopes are much more complicated than initally thought. In the remainder of this section we will develop results that demonstrate the complexity of not only bounding the number of facets of $(0,1)$-polytopes, but also of computing the facet-defining inequalities.

3.1.2 Determinants of $(0,1)$-matrices

Another tool we have to assist in the study of $(0,1)$-polytopes is $(0,1)$-matrices. As we saw in the previous chapter, the different properties of matrices created by the vertices of $(0,1,a)$-polytopes played a large role in determining the complexity of their enumeration. Therefore, we would like to examine properties of $(0,1)$-matrices, and in particular their determinants, in order to gain insight on $(0,1)$-polytopes. We would like to find an upper bound for the largest possible determinant value, as well describe the set of values the determinants may take.

Keeping these goals in mind, in the remainder of the section we will discuss some known results concerning determinants of $(0,1)$-matrices. Many of the following results were obtained in the setting of $(-1,1)$-matrices, ie. matrices with entries in $\{-1,1\}$. In order to apply this work to $(0,1)$-matrices, the following proposition gives us a bijection between these two families that induces a nice bijection between their determinants.

Proposition 3.1.3 (Williamson [49], see Ziegler [51]). *Let A be an $n \times n$ $(0,1)$-matrix. Let $\mathbf{1}$ be the column vector of all ones in $\mathbb{R}^n$ and $\mathbf{1}\mathbf{1}^t$ be the $n \times n$ matrix of ones. The map*

n	η_n
1	1
2	1
3	2
4	3
5	5
6	9
7	32
8	56
9	144
10	320
11	1458
12	3645
13	9477
14	25515
15	131072
16	1114112

Table 3.1: The maximal determinant obtained by an $n \times n$ matrix.

$$\varphi : A \longmapsto \begin{pmatrix} 1 & \boldsymbol{1} \\ \boldsymbol{1} & \boldsymbol{1}\boldsymbol{1}^t - 2A \end{pmatrix} =: \widehat{A}.$$

establishes a bijection between the $(0,1)$*-matrices of size* $n \times n$ *and the* $(-1,1)$*-matrices of size* $(n+1) \times (n+1)$ *for which all entries in the first row and column are* 1.
Furthermore, the bijection φ *satisfies* $\det(\widehat{A}) = (-2)^n \det(A)$.

Hadamard maximal determinant problem

One property of $(0,1)$-matrices we would like to know more about is the maximal size of the determinant. This problem is close related to the Hadamard maximal determinant problem. Denote by η_n the maximal determinant of a $(0,1)$-matrix of size $n \times n$. Table 3.1 gives some known results. For the complete list of results and further details, the reader is directed to [38].

The empirical evidence above suggests that the size of η_n grows at an exponential rate with n. Additionally, we have the following lemma, which places an exponential upper bound on the size of the determinant. Because of its simplicity, we also include the proof as given in [51].

Lemma 3.1.4 (The Hadamard bound)**.** *The maximal determinant of a* $(0,1)$*-matrix of size* $n \times n$ *is bounded by*

$$\eta_n \le 2 \left(\frac{\sqrt{n+1}}{2} \right)^{n+1}.$$

Proof. The Hadamard inequality states that the determinant of a square matrix is at most the product of the norms of its columns, with equality if and only if all columns are othogonal to each other. Applied to the case of a $(-1,1)$-matrix $\widehat{A}$ of size $(n+1)\times(n+1)$, this yields

$$\det(\widehat{A}) \le \sqrt{n+1}^{\,n+1}.$$

By Proposition 3.1.3, we see that for a $(0,1)$-matrix A of size $n \times n$

$$\det(A) \le \frac{\sqrt{n+1}^{\,n+1}}{2^n}.$$

□

With some more work, this upper bound of the maximal size of the determinant of a $(0,1)$-matrix can also give us some geometric intuition for $(0,1)$-polytopes.

Let us assume that A is an invertible $(0,1)$-matrix of size $n \times n$. Then we denote the largest entry of $B = A^{-1}$ with $\chi(A)$. But we know by Cramer's rule that all entries b_{ij} of B are of the form

$$b_{ij} = (-1)^{i+j}\frac{\det(A(i,j))}{\det(A)}.$$

We let $\chi(n)$ denote the largest entry of any invertible $(0,1)$-matrix of size $n \times n$.

Theorem 3.1.5 (Alon-Vũ [3], see [51])**.** *The maximal absolute value of an entry in the inverse of an invertible* $(0,1)$*-matrix of size* $n\times n$ *over all invertible* $(0,1)$*-matrix of size* $n \times n$, $\chi(n)$, *can be bounded by*

$$\frac{n^{n/2}}{2^{2n+o(n)}} \le \chi(n) \le \eta_{n-1} \le \frac{n^{n/2}}{2^{n-1}}.$$

Furthermore, $(0,1)$*-matrices that realize the lower bound can be effectively constructed.*

This theorem has an immediate geometric application to $(0,1)$-polytopes.

Corollary 3.1.6 ([3], see [51])**.** *The largest integer coefficient* $\mathrm{coeff}(n)$ *in any facet description of a full-dimensional* $(0,1)$*-polytope in* $\mathbb{R}^n$ *satisfies*

$$\frac{(n-1)^{n-1/2}}{2^{2n+o(n)}} \le \chi(n-1) \le \mathrm{coeff}(n) \le \eta_{n-1} \le \frac{n^{n/2}}{2^{n-1}}.$$

The geometric implications of the work done by Alon and Vũ tell us that the facet descriptions of $(0,1)$-polytopes are sufficiently more complicated than our intuition for the cases $n = 2,3$ may lead us to believe. Furthermore, Corollary 3.1.6 tells us that even generating geometric data for $(0,1)$-polytopes in higher dimensions will be computationally very expensive. This further hints at the limitations of enumerating combinatorial types of $(0,1,a)$-polytopes in high dimensions.

n	σ_n
1	1
2	1
3	2
4	3
5	5
6	9
7	18
8	40
9	102
10	268
11	738 (conjectured)
12	2172

Table 3.2

Spectrum of the determinant function problem

Another question related to determinants of $(0, 1)$-matrices, a generalization of the Hadamard maximal determinant problem, is referred to as the "spectrum of the determinant function problem". The spectrum of the determinant functionis the set of values taken by determinants of $(0, 1)$-matrices of size $n \times n$. We define σ_n to be the largest integer such that there exists a $(0, 1)$-matrix with determinant equal to σ_n, and additionally such that for any $k \in \mathbb{Z}$ where $0 \leq k \leq \sigma_n$, there also exists some $(0, 1)$-matrix with determinant equal to k. Table 3.2 gives the known results for σ_n. Once more, for the complete list of results and further details, the reader is directed to [39].

The data from Table 3.2 suggests that σ_n grows exponentially in n. In fact, there is a conjecture of Gil Kalai in the polymath blog [26] that claims this is indeed the case.

Conjecture 2. There is a $c \in \mathbb{R}$ such that for every $m \in \mathbb{Z}$ with $|m| \leq c(\sqrt{n}/2)^n$, there is an $n \times n$ $(0, 1)$-matrix whose determinant is m.

This conjecture would give a description of a particularly nice subset of exponential size within the set of determinants of $(0, 1)$-matrices. While this conjecture still remains open, we can still ask ourselves more general questions involving the determinants of $(0, 1)$-matrices.

Lower Hessenberg $(0, 1)$-Matrices

We would like to construct a family of $(0, 1)$-matrices with distinct determinants in order to create a large subset for the spectrum of the determinant function problem. One family that is particularly useful is a special type of lower Hessenberg matrices: matrices that are nearly triangular, with the exception of nonzero entries on the super diagonal.

In particular, we define the $(n \times n)$-matrix $H_n = [h_{i,j}]$ by

$$h_{i,i-k} = \left\{ \begin{array}{ll} 1, & k \in \{-1, 0, 2, 4, \dots | i - k > 0\}, \\ 0, & \text{otherwise}, \end{array} \right\}$$

which we will refer to as the *lower Hessenberg* $(0,1)$*-matrix.*

Example 7. We can also describe these lower Hessenberg $(0,1)$-matrices as having ones on the main and super diagonal, zeros above the superdiagonal, and alternating zeros and ones below the main diagonal. For example, we have

$$H_7 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The Fibonacci sequence

We will now introduce a famous sequence of numbers that will play a large role in the remainder of this chapter. First given its name in 1876 by François Edouard Anatole Lucas, the Fibonacci sequence has fascinated mathematicians since before 1000 CE. The Fibonacci sequence can be defined recursively. We let

$$F_1 = 1, \quad F_2 = 1$$

and define

$$F_n = F_{n-1} + F_{n-2}.$$

The sequence is possibly most famously used to describe the growth of a hypothetical rabbit population, but it continues to prove its ubiquity in nature and mathematics. There are many beautiful identities and properties of the Fibonacci sequence, a few of which we will utilize in this chapter. The curious reader is directed to [31] for a more complete treatment of this sequence.

And indeed, we see the first appearance of the Fibonacci sequence in the following proposition:

Proposition 3.1.7 (Ching [14])**.** *Let H_n be defined as above. Then* $\det(H_n) = F_n$*, where F_n is the n^{th} Fibonacci number.*

Due to nice results of Fibonacci numbers, we also achieve desirable properties of matrices which involve these matrices, H_n. The following theorem follows from a result by Carmichael in 1913:

n	F_n		factorization
1	1		
2	1		
3	2		prime
4	3		prime
5	5		prime
6	8	=	2^3
7	13		prime
8	21	=	3×7
9	34	=	2×17
10	55	=	5×11
11	89		prime
12	144	=	$2^4 \times 3^2$
13	233		prime
14	377	=	13×29
15	610	=	$2 \times 5 \times 61$
16	987	=	$3 \times 7 \times 47$
17	1597		prime
18	2584	=	$2^3 \times 17 \times 19$
19	4181	=	37×113
20	6765	=	$3 \times 5 \times 11 \times 41$
21	10946	=	$2 \times 13 \times 421$
22	17711	=	89×199
23	28657		prime
24	46368	=	$2^5 \times 3^2 \times 7 \times 23$
25	75025	=	$5^2 \times 3001$

Table 3.3: The factorization of Fibonacci numbers.

Theorem 3.1.8 ([50])**.** *If $n \neq 1, 2, 6, 12$, then the n^{th} term of the Fibonacci sequence contains at least one primitive divisor.*

This theorem tells us that for $n \notin \{1, 2, 6, 12\}$ and for all $k < n$, F_n has some prime divisor that does not divide F_k. The factorizations are shown in Table 3.3 for the first 25 Fibonacci terms, with the primitive divisors in red.

Before we are able to utilize this intriguing property of Fibonacci numbers, we need some terminology for partitions. A *partition of an integer* n is a multiset of integers $S = \{p_1, p_2, \ldots, p_k\}$ such that

$$\sum_{i=1}^{k} p_k = n.$$

We call S' the *underlying set of* S if for all $s \in S'$, $s \in S$.

We will denote by $\rho(n)$ the number of partitions of n. In [25], G.H. Hardy and S. Ramanujan mostly completed an exact formula for $\rho(n)$. This formula was later completed and perfected by H. Rademacher in [42]. The asymptotic of this formula is given by

$$\rho(n) \approx \frac{1}{4n\sqrt{3}} \exp\left(\pi\left(\frac{2n}{3}\right)^{\frac{1}{2}}\right).$$

However, work done by Maróti in [33] gives us a simpler working lower bound that holds for all integers n:

$$\rho(n) > \frac{e^{2\sqrt{n}}}{14}.$$

We can now use partitions to obtain a result of Fibonacci numbers.

Corollary 3.1.9. *Let $I = \{i_1, \ldots i_q\}$ and $J = \{j_1, \ldots, j_p\}$ be partitions of some integer with maximal elements at least 3. Let I' and J' be the underlying sets of I and J respectively. Suppose either $I', J' \subseteq \{1, 2, 3, 4\}$, or $I'\Delta J' \nsubseteq \{1, 2, 3, 4, 6, 12\}$, where $I'\Delta J'$ is the symmetric difference of I' and J'. Then*

$$F_{i_1} \cdots F_{i_q} = F_{j_1} \cdots F_{j_p}$$

implies $I = J$.

Before we get to the proof, we will examine this corollary a bit closer. As long as a partition either contains only 1, 2, 3 and 4 as parts, or as long as a the symmetric difference contains at least one integer not equal to 1, 2, 3, 4, 6 or 12 as a part, then the product of the Fibonacci numbers indexed by the elements of the partitions will be unique to this partition. It is also not difficult to find examples necessitating the hypotheses.

Example 8. To see that we cannot depend on F_6 or F_{12} for uniqueness of Fibonacci poducts, consider the following partitions. Let $I = \{1, 2, 6\}$ and $J = \{3, 3, 3\}$, both partitions of 9. It follows that

$$F_1 \cdot F_2 \cdot F_6 = 1 \cdot 1 \cdot 8 = 8 = 2 \cdot 2 \cdot 2 = F_3 \cdot F_3 \cdot F_3.$$

Furthermore, consider $I = \{2, 2, 2, 2, 12\}$ and $J = \{3, 3, 3, 3, 4, 4\}$, both partitions of 20. We have

$$F_2 \cdot F_2 \cdot F_2 \cdot F_2 \cdot F_{12} = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 144 = 144 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = F_3 \cdot F_3 \cdot F_3 \cdot F_3 \cdot F_4 \cdot F_4.$$

Thus we have two examples of two distinct partitions of an integer with the same corresponding Fibonacci product.

We will now show that by avoiding these particular situations, we are ensured distinct products.

Proof. Let $I = \{i_1, \ldots i_q\}$ and $J = \{j_1, \ldots, j_p\}$ be distinct partitions of some integer with maximal element at least 3. We further assume for I' and J', the underlying sets of I and J, that either $I', J' \subseteq \{1, 2, 3, 4\}$ or $I' \Delta J' \not\subseteq \{1, 2, 3, 4, 6, 12\}$. Then, let K be a multiset such that $K = I \cap J$.

If K is empty, then we let m be the largest element of I or J not contained in $\{6, 12\}$. If $m \geq 5$, then F_m has a factor that no other Fibonacci number indexed by an element of either multiset has, and thus $F_{i_1} \cdots F_{i_q} \neq F_{j_1} \cdots F_{j_p}$. Otherwise, we are in the case that $I' \subseteq \{1, 2, 3, 4\}$ and $J' \subseteq \{1, 2, 3, 4\}$. If there exists an element of the partition equal to 3 or 4, then once again, the Fibonacci number indexed by this element contains a factor that no other Fibonacci number indexed by an element in either multiset has.

If K is nonempty. We consider $\widehat{I} = I \setminus K$ and $\widehat{J} = J \setminus K$. Then $\widehat{I} \cap \widehat{J} = \emptyset$, and thus by the case above we see

$$\begin{aligned}\prod_{i \in I} F_i &= \prod_{\hat{i} \in \widehat{I}} F_{\hat{i}} \cdot \prod_{k \in K} F_k \\ &\neq \prod_{\hat{j} \in \widehat{J}} F_{\hat{j}} \cdot \prod_{k \in K} F_k \\ &= \prod_{j \in J} F_j.\end{aligned}$$

□

In order to put a lower bound on the number of partitions of $n \in \mathbb{Z}$ that satisfy the hypotheses of Corollary 3.1.9, we create an injection from the partitions of $n - 2$ to partitions of n, which will now all satisfy the conditions.

Lemma 3.1.10. *The are at least $\rho(n-2)$ partitions of n that satisfy the hypotheses of Corollary 3.1.9.*

Proof. We define an injective function τ from the partitions of $n - 2$ to partitions of n that satisfy the hypotheses of Corollary 3.1.9. Let $I = \{i_1, \ldots i_q\}$ be a partition of $n - 2$ such that $i_1 \leq i_2 \leq \cdots \leq i_q$. Let r be the largest index such that $i_r = 1$.

If $I = \{1, 1, \ldots, 1\}$, then $\tau(I) = \{n\}$.

If I already satisfies the conditions of Corollary 3.1.9, we send

$$I = \{i_1, \ldots i_q\} \mapsto \{1, 1, i_1, \ldots, i_q\}.$$

If I does not satify the conditions of Corollary 3.1.9 and $i_q = 12$, we send

$$I = \{i_1, \ldots i_q\} \mapsto \{i_{r+1}, i_{r+2}, \ldots, i_{q-1}, i_q + r + 2\}.$$

Otherwise, it must be that I does not satify the conditions of Corollary 3.1.9 and $i_q = 6$.

If $r \neq 5$, then we send

$$I = \{i_1, \dots i_q\} \mapsto \{1, i_{r+1}, i_{r+2}, \dots, i_{q-1}, i_q + r + 1\}.$$

If $r = 5$, then we send

$$I = \{i_1, \dots i_q\} \mapsto \{i_6, i_7, \dots, i_{q-1}, i_q + 7\}.$$

Thus, we have at least $\rho(n-2)$ partitions of n that satisfy the hypotheses of Corollary 3.1.9. $\square$

Another nice property of Fibonacci numbers is known as Zeckendorf's theorem:

Theorem 3.1.11 (Zeckendorf's theorem, see [13])**.** *For any positive integer N, there exists positive integers $c_i \geq 2$, with $c_{i+1} > c_i + 1$, such that*

$$N = \sum_{i=0}^{k} F_{c_i}.$$

We can use these properties of the Fibonacci numbers to develop a lower bound on the spectrum of the determinant function problem.

Theorem 3.1.12. *A lower bound on the number of distinct determinants of $(0,1)$-matrices of size $n \times n$ is the number of partitions of $n-2$, for $n > 2$.*

Proof. We consider the following $(n \times n)$ matrix:

$$M = \begin{pmatrix} H_{i_1} & & 0 \\ & H_{i_2} & \\ & & \ddots & \\ 0 & & & H_{i_k} \end{pmatrix}$$

The $\det(M)$ is given by $\det(H_{i_1}) \cdots \det(H_{i_k})$. Then, by Lemma 3.1.10, for each distinct partition of $n-2$, we have a distinct partition of n that satisfies the conditions of Corollary 3.1.9. Thus we obtain $\rho(n-2)$ distinct products of the Fibonacci numbers indexed by the partition, giving us $\rho(n-2)$ distinct determinants. $\square$

Thus, we see that $(0,1)$-matrices of size $n \times n$ have at least as many distinct determinants as there are partitions of $n-2$.

In particular, these matrices give us a half-exponential lower bound to the spectrum problem for $(0,1)$-matrices.

3.2 $(0, 1, a)$-polytopes in higher dimension

The goal of this section is to examine the feasibility of enumerating $(0, 1, a)$-polytopes in higher dimension. We will follow a similar strategy as presented for $(0, 1)$-polytopes in Section 3.1.

3.2.1 Lower bound on maximum number of vertices

We first establish a lower bound for the maximal number of vertices we can expect from a $(0, 1, a)$-polytope in dimension n. By doing this, we can obtain a lower bound for the complexity of a complete enumeration of $(0, 1, a)$-polytopes in dimension n.

Proposition 3.2.1. *There exists an n-dimensional $(0, 1, 2)$-polytope P_m^n with*

$$\binom{n}{\lfloor \frac{n}{3} \rfloor} 2^{\lceil \frac{2n}{3} \rceil}$$

vertices.

Proof. Consider the set

$$S := \left\{ x \in \{-1, 0, 1\}^n : |x|^2 = \left\lceil \frac{2n}{3} \right\rceil \right\}$$

where $|\cdot|^2$ is the standard Euclidean norm. Then there are $\binom{n}{\lfloor \frac{n}{3} \rfloor}$ different ways to choose where to place zeros and $2^{\lceil \frac{2n}{3} \rceil}$ ways to pick plus or minus one, and thus

$$|S| = \binom{n}{\lfloor \frac{n}{3} \rfloor} 2^{\lceil \frac{2n}{3} \rceil}.$$

Let $\widehat{P}_m^n = \mathrm{conv}(S)$. Since each $x \in S$ lies on the sphere of radius $\lceil \frac{2n}{3} \rceil^{\frac{1}{2}}$, each X is a vertex of $\widehat{P}_m^n$. $\widehat{P}_m^n$ is clearly affinely isomorphic to a $(0, 1, 2)$-polytope P_m^n with

$$\binom{n}{\lfloor \frac{n}{3} \rfloor} 2^{\lceil \frac{2n}{3} \rceil}$$

vertices. $\square$

Using Stirling's formula, we can obtain the asymptotic approximation of this lower bound of the maximal number of vertices a $(0, 1, a)$-polytope may have of

$$\frac{3^{n+1}}{2\sqrt{\pi n}}.$$

This lower bound is not far from the trivial upper bound 3^n. This indicates that the number of distinct combinatorial types should also increase dramatically with n. Thus, an enumeration could only be tractable for a $(0, 1, a)$-polytope with a small number of vertices.

3.2.2 Enumeration complexity with various a values

In order to fully determine the complexity of enumerating $(0,1,a)$-polytopes in higher dimensions, it is important to determine some subset of the set of different a values we would need to evaluate.

In particular, we want to examine the following setting: Let

$$M_n^a \in \{0,1,a\}^{(n+1)\times(n+1)}$$

be a matrix such that $m_{n+1,j} = 1$ for $j \in [n+1]$. We can view M_n^a as a matrix whose columns are formed by the coordinates of $n+1$ vertices of a $(0,1,a)$-polytope P in dimension n and then extended by a row of ones. This matrix will have determinant equal to zero when the $n+1$ vertices all lie on a hyperplane, and thus could potentially lie on a non-simplex facet of P. However, we may also disregard the determinants that are constantly zero, as these correspond to points that will always lie on a facet, regardless of our value a. We define

$$\mathcal{M}_n^a := \{M_n^a : \det(M_n^a) \not\equiv 0\}.$$

We are interested in the following set:

$$\mathcal{A}_n := \{a \in \mathbb{R}_{\geq 2} : \exists M_n^a \in \mathcal{M}_n^a \text{ s.t.} \det(M_n^a) = 0\}.$$

We have the following generalization of Theorem 2.2.2.

Theorem 3.2.2. *Let $\mathcal{A}_n$ be defined as above and $x, y \in \mathbb{R}$ such that $[x,y] \cap \mathcal{A}_n = \emptyset$. Then all combinatorial types of polytopes that arise as $(0,1,x)$-polytopes can also be realized as $(0,1,y)$-polytopes.*

Proof. Let $\mathcal{A}_n$ be defined as above and $x, y \in \mathbb{R}$ such that $[x,y] \cap \mathcal{A}_n = \emptyset$. Let P_x be a full-dimensional $(0,1,x)$-polytope. Let $V(a)$ be a subset of the nodes of the n-dimensional $(0,1,a)$-grid such that $V(x)$ is the vertex set of P_x. Let $P_y = \operatorname{conv}(V(y))$. We claim that P_y is combinatorially equivalent to P_x. We can completely describe a polytope by its vertices in facets. Thus, we need to check that a facet-defining hyperplane of P_x corresponds to a facet-defining hyperplane and both contain corresponding vertices.

Let $H(a)$ be a hyperplane such that $H(x)$ is a facet-defining hyperplane of P_x for the facet F. If H is also a facet-defining hyperplane of the 3-cube, then clearly $H(y)$ is a facet-defining hyperplane of P_y containing the corresponding vertices. Otherwise, because $x \notin \mathcal{A}_n$, F contains exactly n vertices. The corresponding vertices of P_y are contained in $H(y)$. Further, $H(y)$ is a facet-defining hyperplane of P_y. Because $H(x)$ is a facet-defining hyperplane of P_x, $P_x \subset H(x) \cup H(x)^-$. But since for all $a \in R$ such that $x \leq a \leq y$, $a \notin \mathcal{A}_n$, then $P_y \subset H(y) \cup H(y)^-$, completing the proof. $\square$

The goal of this section is to describe the set $\mathcal{A}_n$. We will try to answer the same questions posed in Section 3.1. In particular, we can look at upper and lower bounds of the size of $\mathcal{A}_n$ as well as the distance between elements and the maximal size of an element.

Maximal element size

The first observation one may make is that $\mathcal{A}_n$ is a finite set. This follows immediately from the fact that there are a finite number of matrices, and thus a finite number of determinants. We would like to investigate non-trivial upper and lower bounds on the size of $\mathcal{A}_n$. Our determinants will be polynomial of degree at most n, with integer coefficients. Let $c_n a^n + c_{n-1}a^{n-1} + \cdots + c_0$ be a determinant of a $(0, 1, a)$-matrix of size $n \times n$. We observe that since we have a finite number of determinants, there also exists an $M \in \mathbb{Z}$ such that $c_i \leq M$ for $i \in [n]$. This leads us to a simple well-known upper bound on the maximal value of an element of $\mathcal{A}_n$.

Proposition 3.2.3. *Let $P(a) = c_n a^n + c_{n-1}a^{n-1} + \cdots + c_0$ be a polynomial of degree n such that $c_i \in \mathbb{Z}$, $|c_i| \leq M$, and $c_n \neq 0$. Then, any root of $P(a)$ is bounded above by nM.*

Proof. Let $c_n a^n + c_{n-1}a^{n-1} + \cdots + c_0 = 0$. Then,

$$-c_n = \frac{c_{n-1}}{a} + \cdots + \frac{c_0}{a^n}.$$

Now we can assume $a \geq 1$, or else the bound is obvious, and thus by the integrality of c_n we have

$$1 \leq |c_n| \leq \left|\frac{c_{n-1}}{a}\right| + \cdots + \left|\frac{c_0}{a^n}\right| \leq \frac{nM}{a},$$

and thus

$$a \leq nM.$$

□

Thus, with our finite number of determinants and polynomials, we can set a global bound on the maximal root of a determinant for each n.

Separation of roots and an upper bound on size

With the upper bound on the size of a root of the determinant of a $(0, 1, a)$-matrix, if we could also put a lower bound on the distance between two roots, this would give us an upper bound on the number of distinct roots.

Let $P(a)$ be a polynomial of degree n with integer coefficients c_i and complex roots α_i such that

$$P(a) = c_n a^n + c_{n-1}a^{n-1} + \cdots + c_0 = c_n \prod_{i=1}^{n} (a - \alpha_i).$$

We define $\mathrm{sep}(P)$, the *minimum root separation of P*, by

$$\mathrm{sep}(P) = \min_{\alpha_j \neq \alpha_k} |\alpha_j - \alpha_k| .$$

Theorem 3.2.4 (Horowitz, 1973, see [15])**.** *Let $P = c_n a^n + c_{n-1}a^{n-1} + \cdots + c_0 = 0$ be a polynomial with integer coefficients of degree $n \geq 2$ such that $d = \sum_{i=0}^{n} |c_i|$. Then,*

$$sep(P) \geq (nd)^{-4n+5}.$$

By combining Theorem 3.2.4 and Proposition 3.2.3, we see that there are at most $nM(nd)^{4n+5}$ elements in our set $\mathcal{A}_n$.

While this gives us a nice upper bound on the number of elements in $\mathcal{A}_n$, a lower bound would be nice to further understand the complexity of an enumeration.

3.2.3 Determinants of $(0,1,a)$-matrices

Similarly to how we have developed a family of $(0,1)$-matrices of size $n \times n$ with at least $\frac{e^{2\sqrt{n-2}}}{14}$ distinct positive determinants, our goal in this section is to develop an exponential lower bound for the number of distinct determinants of $(0,1,a)$-matrices.

Recall from Section 3.1.2 the lower Hessenberg $(0,1)$-matrices, which we define by

$$H_n = h_{i,i-k} = \left\{ \begin{array}{ll} 1, & k \in \{-1,0,2,4,\ldots | i-k > 0\}, \\ 0, & \text{otherwise}, \end{array} \right\}.$$

In the remainder of the section we will investigate a handful of slightly altered versions of H_n and the nice properties that ensue, namely, their determinants. These matrices will then be pieced together to form a family of matrices with Fibonacci-many distinct determinants, giving us the desired exponential lower bound on the size of $\mathcal{A}_n$. In particular, this bound also gives a lower bound of values of $a \in \mathcal{A}_n$.

Fibonacci-many determinants

We are now equipped to establish a lower bound for the number of determinants of $(0,1,a)$-matrices. To do so, we first define a family of matrices C_n, where we can define $C_n := H_{n+1}(1, n+1)$.

Example 9. We can also describe C_n as having all zeros on the main diagonal, one on the super diagonal, and on the diagonal above the super. Below the main diagonal we have alternating diagonals of ones and zeros.

$$C_6 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Lemma 3.2.5. *The matrix C_n, as defined above, has* $\det(C_n) = (-1)^{n+1}$.

Proof. We induct on n. For $n = 2$ we see

$$\det(C_2) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 = (-1)^3.$$

For a bit more clarity, we also look at the case $n = 3$.

$$\begin{aligned} \det(C_3) &= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \\ &= 0 \cdot (0 - 1) - 1 \cdot (0 - 0) + 1 \cdot (1 - 0) \\ &= 1 = (-1)^4. \end{aligned}$$

We assume $\det(C_k) = (-1)^{k+1}$ holds for all $k < n$.

First, we make a few observations about the nature of C_n for $n \geq 3$. First of all, the n and $n-2$ row are equal except for the last column. And secondly, the $n-1$ and $n-3$ row are equal except for the last two columns.

In order to compute the determinant of C_n, we notice that there are ones only in the $n-1$ and $n-2$ row of the last column. Thus we can use Laplace's formula to see

$$\det(C_n) = -\det(C(n-1, n)) + \det(C(n-2, n)).$$

Note that $C(n-1, n)$ and $C(n-2, n)$ are matrices of size $(n-1) \times (n-1)$.

We see that the last two rows of $C(n-1, n)$ are equal giving us

$$\det(C(n-1, n)) = 0.$$

Thus we just need to examine $C(n-2, n)$. Here, the $(n-1)-1$ row and the $(n-1)-2$ row vary only in the last column, where the $(n-1)-1$ row ends in a zero, and the $(n-1)-2$ row ends in a one. Thus we can subtract the $(n-1)-1$ row from the $(n-1)-2$ row so the $(n-1)-2$ row is all zeros except for a one in the last column. Thus, $\det(C(n-2, n))$ is equal to the determinant of its $((n-1)-2, n-1)$ submatrix. This is exactly the matrix C_n, with the last two columns deleted, and its $n-2$ row and $n-3$ row deleted. However, without the last two columns, these rows are identical to the $n-1$

and n row. Thus, the $((n-1)-2, n-1)$ submatrix of $C(n-2,n)$ is actually just equal to C_{n-2}. By induction, we thus have

$$\det(C_n) = \det(C_{n-2}) = (-1)^{n-1} = (-1)^{n+1},$$

completing the proof. $\square$

We define a further family of matrices, $\widehat{C}_n$, which we obtain by changing the last row of C_n to a row of all ones.

Example 10. For example, we have

$$\widehat{C}_6 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Lemma 3.2.6. *The matrix defined above, $\widehat{C}_n$, has $\det(\widehat{C}_n) = (-1)^{n+1}$.*

Proof. Because $\widehat{C}_n$ can be obtained by adding the second-to-last row of C_n to the last row of C_n, we have

$$\det(\widehat{C}_n) = \det(C_n) = (-1)^{n+1}.$$

$\square$

Similarly to above, we can also define a family of $n \times n$ matrices, $\widehat{H}_n$ by exchanging the last row of H_n with a row of ones.

Lemma 3.2.7. *Consider $\widehat{H}_n$ defined above for $n \geq 3$. Then*

$$\det\left(\widehat{H}_n\right) = F_{n-2}.$$

Proof. We induct on n. For $n = 3$:

$$\begin{aligned} \det(\widehat{H}_3) &= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \\ &= 1 \cdot (1-1) - 1 \cdot (0-1) + 0 \cdot (0-1) \\ &= 1 = F_1. \end{aligned}$$

In the last column there are only ones in the last two rows. So we can use Laplace expansion on the last column. We notice that the (n,n) submatrix of

$\widehat{H}_n$ is equivalent to H_{n-1}. Additionally, we note that the $(n-1, n)$ submatrix of $\widehat{H}_n$ is equivalent to $\widehat{H}_{n-1}$. Thus,

$$\begin{aligned}
\det\left(\widehat{H}_n\right) &= \det(H_{n-1}) - \det\left(\widehat{H}_{n-1}\right) \\
&= F_{n-1} - F_{n-3} \\
&= F_{n-2}.
\end{aligned}$$

□

Lemma 3.2.8. *Let Neg_n be the matrix that we define by taking H_n and replacing all zeros in the last row of H_n with -1's. Then*

$$\det(Neg_n) = F_{n+1}.$$

Proof. We induct on n. For $n = 3$ we see

$$\begin{aligned}
\det(Neg_n) &= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} \\
&= 1 \cdot (1+1) - 1 \cdot (0-1) + 0 \\
&= 3 = F_4.
\end{aligned}$$

We assume this holds for $k < n$. We now consider Neg_n. Because the last column of Neg_n has all zeros except for ones in the last two rows, we use Laplace expansion on the last column.

However, we see that the (n, n) submatrix of Neg_n is equal to H_{n-1} and that the $(n-1, n)$ minor of Neg_n is equal to Neg_{n-1} where the last row was multiplied by -1. Thus we get

$$\begin{aligned}
\det(Neg_n) &= \det(H_{n-1}) + \det(Neg_{n-1}) \\
&= F_{n-1} + F_n \\
&= F_{n+1}
\end{aligned}$$

which completes the proof. □

We define one last matrix family before we are ready to piece them together to achieve our goal of Fibonacci many distinct determinants. We define $\overline{M}_n$ to be a matrix of size $(n+1) \times n$, formed by extending H_n by a row of ones. We will then look at the submatrix of $\overline{M}_n$, $\overline{M}_n(j)$, given by removing the j^{th} row. Note that $\overline{M}_n(j)$ is an $n \times n$ matrix.

Theorem 3.2.9. *Let $\overline{M}_n(j)$ be defined as above for $n \geq 3$ and $1 \leq j \leq n-1$. Then,*

$$\det\left(\overline{M}_n(j)\right) = (-1)^{n+j} F_j.$$

We prove the special case of $j = n-1$ separately as a lemma before we are finally ready to proceed with the full proof of Theorem 3.2.9.

Lemma 3.2.10. *Let $M := \overline{M}_n(n-1)$ be the matrix defined above. Then,*

$$\det(M) = -F_{n-1}.$$

Proof. Since there are only ones in the last column in the last two rows, we use Laplace expansion on the last column. We notice that the $(n-1)^{\text{st}}$ and $(n-2)^{\text{nd}}$ row are the same up to the last two columns. Additionally, the $(n-1, n)$ minor of M is $\widehat{H}_{n-1}$ Thus, we can use Laplace's expansion along the last column as well as Lemma 3.2.7 to see

$$\begin{aligned}\det(M) &= -\det(H_{n-2}) - \det(\widehat{H}_{n-1}) \\ &= -(F_{n-2} + F_{n-3}) \\ &= -F_{n-1}.\end{aligned}$$

$\square$

We are now ready to prove Theorem 3.2.9.

Proof. We induct. For $n = 3$ and $j = 1$, we have

$$\det\left(\overline{M}_3(1)\right) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1 = (-1)^{3+1}F_1.$$

The case of $j = 2$ is handled in Lemma 3.2.10.

We assume the theorem holds for all $k < n$. We want to examine $\overline{M}_n(j)$ as a block matrix:

$$\left(\begin{array}{c|c} H_{j-1} & B \\ \hline A & \widehat{C}_{n-j+1} \end{array}\right)$$

where for simplicity of notation, we let $n - j + 1 = m$. B is a $(j-1) \times m$ matrix of all zeros except for a one as the $(j-1, 1)$ entry, and A is a matrix of alternating zeros and ones, starting with a one for even j, and a zero for odd j, and the last row is all ones. We use the Schur determinant identity, which tells us that

$$\det\left(\overline{M}_n(j)\right) = \det(\widehat{C}_m)\det(H_{j-1} - B\widehat{C}_m^{-1}A).$$

Lemma 3.2.6 prescribes the determinant of $\det(\widehat{C}_m)$, so it remains only to find $\det(H_{j-1} - B\widehat{C}_m^{-1}A)$. Within our investigation, we will first look at what the matrix $B\widehat{C}_m^{-1}$ looks like, then $B\widehat{C}_m^{-1}A$, and finally $H_{j-1} - B\widehat{C}_m^{-1}A$.

Claim 1: We have

$$B\widehat{C}_m^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -F_1 & -F_2 & \cdots & -F_{m-3} & -F_{m-2} & -F_{m-3} & F_{m-1} \end{pmatrix}.$$

Because B is a matrix of all zeros except a one in the $(j-1)$-entry, this corresponds to the last row of $B\widehat{C}_m^{-1}$ being the first row of $\widehat{C}_m^{-1}$, with the remaining rows having zeroes for each entry. Thus, Claim 1 really boils down to showing that the first row of $\widehat{C}_m^{-1}$ is equal to

$$(-F_1, -F_2, \cdots, -F_{m-3}, -F_{m-2}, -F_{m-3}, F_{m-1}).$$

The $(1,i)$-entry of $\widehat{C}_m^{-1}$ is equal to

$$\frac{1}{\det(\widehat{C}_m)} \operatorname{adj}(\widehat{C}_m)_{1,i} = (-1)^{m+1} \operatorname{adj}(\widehat{C}_m)_{1,i}.$$

The first row of the $\operatorname{adj}(\widehat{C}_m)$ is given by the first column of the cofactor matrix of $\widehat{C}_m$. However, the $(i,1)$ cofactor matrix is exactly $M_{m-1}(i)$. Thus, by our inductive hypothesis, we have for $1 \leq i < m-2$ that

$$\begin{aligned} \operatorname{inv}(\widehat{C}_m(1,i)) &= (-1)^{m+1} \operatorname{adj}(\widehat{C}_m)_{1,i} \\ &= (-1)^{m+1} (-1)^{1+i} (-1)^{m-1+i} F_i \\ &= (-1)^{2m+2i+1} F_i \\ &= -F_i. \end{aligned}$$

Thus we just need to look at the cases $i = m-2$, $i = m-1$ and $i = m$ individually.

For the case $i = m-2$, we have that the $(m-2,1)$ cofactor matrix of $\widehat{C}_m$ is $\overline{M}_{m-1}(m-2)$ and so, by Lemma 3.2.10, we have

$$\begin{aligned} \operatorname{inv}(\widehat{C}_m(1,m-2)) &= (-1)^{m+1} (-1)^{m-2+1} (-1) F_{(m-1)-1} \\ &= (-1)^{2m} (-1) F_{m-2} \\ &= -F_{m-2}. \end{aligned}$$

For the case $i = m-1$, we have that the $(m-1,1)$ cofactor matrix of $\widehat{C}_m$ is $\widehat{H}_{m-1}$ and so, by Lemma 3.2.7, we have

$$\begin{aligned}
\operatorname{inv}(\widehat{C}_m(1,i)) &= (-1)^{n-j}\operatorname{adj}(\widehat{C}_m)_{1,i} \\
&= (-1)^{n-j}(-1)^{1+n-j}F_{m-3} \\
&= (-1)^{2n-2j+1}F_{m-3} \\
&= -F_{m-3}.
\end{aligned}$$

For the case $i = m$, we have that the $(m,1)$ cofactor matrix of $\widehat{C}_m)$ is H_{m-1}. By Proposition 3.1.7, we have $\det(H_{m-1}) = F_{m-1}$. Thus,

$$\begin{aligned}
\operatorname{inv}(\widehat{C}_m(1,m)) &= (-1)^{m+1}(-1)^{m+1}F_{m-1} \\
&= F_{m-1}.
\end{aligned}$$

Therefore, we see that Claim 1 holds.

We now will look into what $B\widehat{C}_m^{-1}A$ should look like. We make the following claim:

Claim 2: $H_{j-1} - B\widehat{C}_m^{-1}A$ is identical to H_{j-1}, with the exception that instead of the last row consisting of alternating zeros and ones, we have alternating ones and -1's.

This claim will result from the fact that $B\widehat{C}_m^{-1}A$ is a matrix of all zeros, except for the last row, which is a row of alternating zeros and ones.

To show this claim, we first notice that A is a $m \times (j-1)$ matrix of alternating zeros and ones, with the exception that the last row is all ones. This property of A will have us taking sums of even or odd Fibonacci numbers.

It is also useful to note that the first row of A and the last row of H_{j-1} are identical.

We have the following column types: If m is odd, we have either

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Then we have

$$\begin{aligned}
&\begin{pmatrix} -F_1, -F_2, \cdots, -F_{m-3}, -F_{m-2}, -F_{m-3}, F_{m-1} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \\
&= -(F_2 + F_4 + \cdots + F_{m-3} + F_{m-3}) + F_{m-1} \\
&= -(F_{m-3} + F_{m-2} - 1) + F_{m-1} \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
&\begin{pmatrix} -F_1, -F_2, \cdots, -F_{m-3}, -F_{m-2}, -F_{m-3}, F_{m-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \\
&= -(F_1 + F_3 + \cdots + F_{m-4} + F_{m-2}) + F_{m-1} \\
&= -F_{m-1} + F_{m-1} \\
&= 0.
\end{aligned}$$

Thus, for m odd, the last row of $B\widehat{C}_m^{-1}A$ will be $(1, 0, 1, \cdots)$ if the last row of H_{j-1} begins with a zero, or $(0, 1, 0, \cdots)$ if the last row of H_{j-1} begins with a one. Either way, after the subtraction, the last row of H_{j-1} will consist of ones and -1's.

For m even, we have either

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

$$\begin{aligned}
&\begin{pmatrix} -F_1, -F_2, \cdots, -F_{m-3}, -F_{m-2}, -F_{m-3}, F_{m-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \\
&= -(F_1 + F_3 + \cdots + F_{m-3} + F_{m-3}) + F_{m-1} \\
&= -(F_{m-2} + F_{m-3}) + F_{m-1} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
&\begin{pmatrix} -F_1, -F_2, \cdots, -F_{m-3}, -F_{m-2}, -F_{m-3}, F_{m-1} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \\
&= -(F_2 + F_4 + \cdots + F_{m-2}) + F_{m-1} \\
&= -(F_{m-1} - 1) + F_{m-1} \\
&= 1.
\end{aligned}$$

Thus, for m even, the last row of $B\widehat{C}_m^{-1}A$ will be $(1,0,1,\cdots)$ if the last row of H_{j-1} begins with a zero, or $(0,1,0,\cdots)$ if the last row of H_{j-1} begins with a one. This is exactly the situation from above with m odd. Thus, regardless of the parity of m, in $H_{j-1} - B\widehat{C}_m^{-1}A$, we subtract only from the last row of H_{j-1}, and then we only subtract one from the entries that are zeros. Thus, $H_{j-1} - B\widehat{C}_m^{-1}A$ is identical to H_{j-1} with the exception that the last row has -1's in place of zeros.

Therefore, we use Lemma 3.2.8 to get $\det(H_{j-1} - B\widehat{C}_m^{-1}A) = F_j$. And thus,

$$\begin{aligned}
\det(M_n(j)) &= \det(\widehat{C}_m)\det(H_{j-1} - B\widehat{C}_m^{-1}A) \\
&= (-1)^{m-1}F_j \\
&(-1)^{n-j}F_j
\end{aligned}$$

giving us our desired result. □

Corollary 3.2.11. *There are at least Fibonacci-many distinct polynomial determinants of $(0,1,a)$-matrices. Furthermore, each of these determinants have a root at a distinct a value.*

Proof. We consider the following matrix:

$$M_n^a = \left(\begin{array}{cccc} & & & * \\ & H_n & & * \\ & & & \vdots \\ & & & * \\ & & & * \\ & & & 0 \\ 1 & 1 & \dots & 1 \end{array} \right),$$

where $*$ may take a value of 0 or a. We can get our family of desired $(0, 1, a)$-matrices by choosing $k \leq n-1$ entries where there is a $*$ to insert an a, and all the others are zero. Suppose we choose the $j_1, \dots, j_k$ rows to insert an a. Then, because the $(k, n+1)$ submatrix of M_n^a is equal to $\overline{M}_n(k)$, we can use Theorem 3.2.9 coupled with Laplace expansion along the last column to see that

$$\begin{aligned} \det(M_n^a) = F_n &+ (-1)^{n+1+j_1} a(-1)^{n+j_1} F_{j_1} + (-1)^{n+1+j_2} a(-1)^{n+j_2} F_{j_2} \\ &+ \dots + (-1)^{n+1+j_k} a(-1)^{n+j_k} F_{j_k} \\ &= F_n - a(F_{j_1} + F_{j_2} + \dots + F_{j_k}). \end{aligned}$$

We then use Zeckendorf's theorem to see that every integer between 1 and $F_n - 1$ can be achieved with a sum of distinct Fibonacci numbers preceding F_n.

Further, a is a root of this determinant when

$$a = \frac{F_n}{F_{j_1} + F_{j_2} + \dots + F_{j_k}}.$$

Because at least $F_n - 1$ of these terms are distinct, we have Fibonacci many distinct roots.

□

Corollary 3.2.12. *For all $m \in \mathbb{N}$ such that $m \leq F_n$, there exists a $(0, 1)$-matrix with determinant equal to m.*

Proof. Similar to Corollary 3.2.11, we consider the following matrix:

$$M_n^1 = \left(\begin{array}{cccc} & & & * \\ & H_n & & * \\ & & & \vdots \\ & & & * \\ & & & * \\ & & & 0 \\ 1 & 1 & \dots & 1 \end{array} \right),$$

where $*$ may take a value of 0 or 1. Suppose we choose the $j_1, \ldots, j_k$ rows to insert an 1.Then,

$$\begin{aligned}\det(M_n^1) &= F_n + (-1)^{n+1+j_1}(-1)^{n+j_1}F_{j_1} + (-1)^{n+1+j_2}(-1)^{n+j_2}F_{j_2} \\ &\quad + \cdots + (-1)^{n+1+j_k}(-1)^{n+j_k}F_{j_k} \\ &= F_n - (F_{j_1} + F_{j_2} + \cdots + F_{j_k}),\end{aligned}$$

which again by Zeckendorf's theorem (Theorem 3.1.11), yields all positive integers up to F_n.

□

This corollary gives a nice partial result to Conjecture 2. Although the conjecture is stronger, Corollary 3.2.12 provides at least exponentially-many positive integral determinants of $(0, 1)$-matrices.

Summary of complexity of $\mathcal{A}_n$

In the work above, we have seen that with an exponential number of vertices in the dimension n, a complete enumeration of $(0, 1, a)$-polytopes is extremely infeasible. Even limiting the number of vertices, we have also shown that there are exponentially-many distinct, positive a values we would have to check for combinatorial types.

Chapter 4

Bounds on the diameter

In this chapter, we discuss a key combinatorial property of the graphs of $(0, 1, a_i)$-polytopes, namely a quite small diameter.

4.1 Background and motivation

The ability to bound the diameter of a general polytope has been of particular interest since the creation of the simplex algorithm by Dantzig in 1947 (see [34]). The simplex algorithm gives a method for optimizing a linear functional with respect to a system of linear equations and inequalities. The solution set of this system can be viewed as a polyhedron. It is not hard to see that if a maximum exists, it will be achieved at a vertex. Thus, the idea is to travel from vertex to vertex along edges that continue to increase the functional until a local maximum is found. By convexity, this is also a global maximum.

The idea of travelling from vertex to vertex along edges is strongly connected to the notion of diameter. Recall that for two vertices u and v of a polytope P, we will denote the shortest path from u to v by $\delta(u, v)$. The *diameter of* P, denoted $\delta(P)$, is the largest $\delta(u, v)$, for all $u, v \in P$. The diameter is essentially telling us how far apart two vertices can be in the graph-theoretical way.

Bounds on the diameter of polytopes and polyhedra with respect to their dimension and number of facets would provide a lower bound for the computational time of the simplex algorithm. However, it is important to note that it is not enough to know the diameter of a polytope is small, we also need a way to choose the most direct path. But a bound on the diameter at least tells us of the existence of such a path.

In particular, the Hirsch conjecture was an attempt to provide this bound. It stated that the diameter of an n-polytope with f facets should be no more than $f - n$ (see [28]). This conjecture, however, was disproved by Francisco Santos in 2011 [44]. But there are still many families that satisfy the Hirsch conjecture, including our motivating family of polytopes.

Theorem 4.1.1 ([35])**.** *The Hirsch conjecture is true for all* $(0,1)$*-polytopes.*

The proof of Theorem 4.1.1 is remarkably simple. The idea is to first bound the diameter of a $(0,1)$-polytope by its dimension, and then it quickly follows that this is enough to ensure the diameter is no more than $f-n$.

Establishing the Hirsch bound for the diameter of polytopes has been of particular focus amongst experts in the field (see [10], [27], [48]). Santos and Kim give a nice survey of the research and results relating to the conjecture in [28]. However, it also suffices to find any polynomial upper bound on the diameter in order to establish the existence of a path that needs only polynomial time. Thus, we consider the following conjecture from [28]:

Conjecture 3 (Polynomial Hirsch conjecture)**.** There is a polynomial function $p(f,n)$ such that, for any polyhedron P of dimension n with f facets, $\delta(P) \leq p(f,n)$.

The goal of this chapter is to provide a linear bound in n for $(0,1,a_i)$-polytopes. This bound was proven by Del Pia and Michini in [16] for the special case of $(0,1,2)$-polytopes.

Theorem 4.1.2. *The maximal diameter of a* $(0,1,a_i)$*-polytope of dimension* n *is* $\lfloor \frac{3}{2}n \rfloor$.

Before we begin the proof, we will give some notation and other results that will assist us in the proof.

4.2 Notation and preliminaries

Let P be an n-polytope with vertex v. For $c \in \mathbb{R}^n$ and $c_0 \in \mathbb{R}$, let $cx \leq c_0$ be a valid inequality such that

$$\{v\} = P \cap \{x : cx = c_0\}.$$

We choose some $c_1 < c_0$ with $cv' < c_1$ for all $v' \in \text{vert}(P)/v$. Then, the *vertex figure of* P *at* v is defined as the polytope

$$P/v := P \cap \{x : cx = c_1\}.$$

Notice that for any two vertices u and v of P, there exists a $0 < \lambda < 1$, such that

$$\lambda u + (1-\lambda)v \in P/u.$$

Proposition 4.2.1. *Let* u *and* v *be vertices of a polytope* P *in* $\mathbb{R}^n$ *such that* $u_i < v_i$*. Then,* u *has some neighbor* u' *such that* $u_i < u'_i$*.*

Proof. Suppose by way of contradiction that for any neighbor u' of u, $u_i \geq u'_i$.

We consider the vertex figure of u, P/u. In particular for u and any other $w \in \text{vert}(P)$, there exists some $0 < \lambda < 1$ such that $\lambda u + (1-\lambda)w \in P/u$. However for any $x \in P/u$, $x_i \leq u_i$, while $\lambda u_i + (1-\lambda)v_i > u_i$ for $\lambda > 0$. $\square$

Recall Proposition 1.5.8, which tells us for $P \subset \mathbb{R}^{n+1}$ an n-dimensional $(0,1,a_i)$-polytope for $\mathbf{a} \in \mathbb{R}^{n+1}$, that P is affinely isomorphic to an n-dimensional $(0,1,a'_i)$-polytope $P' \subset \mathbb{R}^n$ where $\mathbf{a'} \in \mathbb{R}^n$ is a sub-multiset of $\mathbf{a}$.

By repeatedly applying Proposition 1.5.8, we can view any $(0,1,a_i)$-polytope as full dimensional. In particular, the faces of $(0,1,a_i)$-polytopes can be viewed as full-dimensional $(0,1,a_i)$-polytopes. Therefore, we can denote the largest diameter achieved by an n-dimensional $(0,1,a_i)$-polytope by δ^n, and a k-face of a polytope will have diameter at most δ^k.

Further, to simplify notation, we define a partition of the vertices of a $(0,1,a_i)$-polytope P. For each $i \in [n]$, we define

$$\begin{aligned} V_i^0 &:= \{v \in \text{vert}(P) : v_i = 0\} \\ V_i^1 &:= \{v \in \text{vert}(P) : v_i = 1\} \\ V_i^2 &:= \{v \in \text{vert}(P) : v_i = a_i\}. \end{aligned}$$

Notice that V_i^0 and V_i^2 are either empty, or faces of P.

4.3 Proof of diameter bound

Our aim for this section is to prove that the tight upper bound of the diameter of an n-dimensional $(0,1,a_i)$-polytope is $\lfloor \frac{3}{2} n \rfloor$.

Lemma 4.3.1 (Del Pia, Michini [16])**.** *There exists a $(0,1,2)$-polytope P in dimension n with diameter $\delta(P) = \lfloor \frac{3}{2} n \rfloor$.*

Proof. We can recursively construct a polytope H_n achieving this bound even by using coordinates in $\{0,1,2\}$. We define H_1 to be the line segment with diameter $1 = \lfloor \frac{3}{2} \rfloor$. For $n = 2$, we consider a hexagon, which has diameter $3 = \frac{3}{2} \cdot 2$. In order to form higher dimensional polytopes that achieve the bound, we consider taking products of H_2. It is well known that for polytopes P, Q, $\delta(P \times Q) = \delta(P) + \delta(Q)$. For H_n where n is even, we take the product of $\frac{n}{2}$ hexagons. Thus

$$\begin{aligned} \delta(H_n) &= \sum_{i=1}^{n/2} 3 \\ &= \frac{3}{2} n. \end{aligned}$$

Similarly, for n odd, we take the product of H_{n-1} with a line segment. Thus

$$\begin{aligned}\delta(H_n) &= \frac{3}{2}(n-1)+1\\ &= \left\lfloor \frac{3}{2}n \right\rfloor.\end{aligned}$$

□

Thus, it only remains to that show for P a $(0,1,a_i)$-polytope, $\delta(P) \leq \lfloor \frac{3}{2}n \rfloor$.

Proof of Theorem 4.1.2. We will break the proof down into cases. In each case we will show that from any two vertices we can reach a common facet in one step, or a common $(n-2)$-face in three steps. By induction this implies that $\delta^n \leq \lfloor \frac{3}{2}n \rfloor$.
We induct on the dimension n. The cases $n=0$ and $n=1$ are clearly true. We consider an $(0,1,a_i)$-polytope P and suppose the hypothesis is true for all $(0,1,a_i)$-polytopes for $k < n$. In particular, we want to show for $u, v \in P$, that one of the following two inequalities holds:

$$\delta(u,v) \leq \delta^{n-1}+1 \tag{4.1}$$

$$\delta(u,v) \leq \delta^{n-2}+3 \tag{4.2}$$

We proceed with a case-wise analysis.

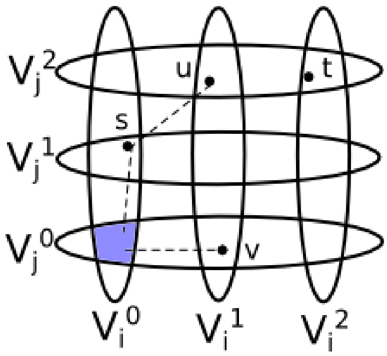
Case 1: Assume $u_i = v_i = 0$ or $u_i = v_i = a_i$ for some $i \in [n]$. Then u and v are contained in a common face. Thus $\delta(u,v) \leq \delta^{n-1}$.

Case 2: Suppose there exists an $i \in [n]$ such that for $u \in V_i^p$ and $v \in V_i^q$, $|p-q| = 1$. Then we would need one step to get from u to a common face sharing v, and thus (4.1) is satisfied.

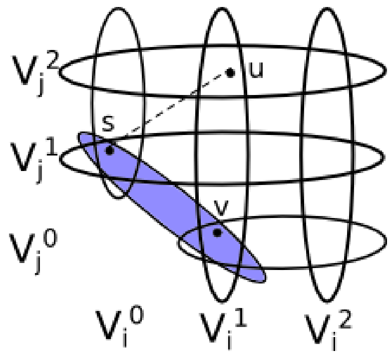
Case 3: Assume neither Case 1 nor Case 2 hold and there exists an $i \in [n]$ such that $u_i = v_i = 1$.

We note that if $u_k = v_k = 1$ for all $k \in [n]$ then we are done. By Proposition 4.2.1, u must have a neighbor $s \in V_i^0$ and a neighbor $t \in V_i^2$. Then there must exist a j such that either $s_j \neq u_j$ or $t_j \neq u_j$. If this is not true, then by taking $\lambda = \frac{t_i-u_i}{t_i-s_i}$ we have $\lambda s + (1-\lambda)t = u$, contradicting the fact that u is a vertex. Thus we can assume that $s_j < u_j$.

Suppose $u_j \neq v_j$. If $s \in V_i^0 \cap V_j^0$ then we are done because, by applying Proposition 4.2.1 to the facet defined by V_j^0, v must have some neighbor in $V_i^0 \cap V_j^0$. Thus Equation (4.2) is satisfied, as illustrated in Figure 4.1a.



(a) $V_i^0 \cap V_j^0 \neq \emptyset$.



(b) $V_i^0 \cap V_j^0 = \emptyset$.

Figure 4.1: Case 3: $u_i \neq v_i$.

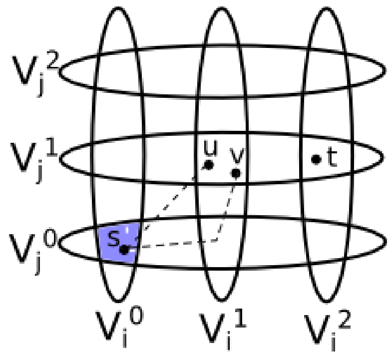


Figure 4.2: Case 3: $u_i = v_i = 1$.

Thus we may assume $s \in V_i^0 \cap V_j^1$. If $V_i^0 \cap V_j^0 \neq \emptyset$, then both s and v have a neighbor in $V_i^0 \cap V_j^0$, satisfying (4.2). If $V_i^0 \cap V_j^0 = \emptyset$, then $V_i^1 \cap V_j^1$ actually defines a facet, giving us that (4.1) holds, as shown in Figure 4.1b.

Suppose $u_j = v_j = 1$. Then by applying Proposition 4.2.1 twice, v needs two steps to enter a common facet with s, satisfying (4.1) as shown in Figure 4.2. $\diamond$

Case 4: Assume for all $i \in [n]$, $|u_i - v_i| = a_i$.

For all $i \in [n]$, let $u \in V_i^2$ and $v \in V_i^0$. Suppose there exists a vertex $u' \in V_i^0$ such that u and u' are adjacent vertices in P. Since u' and v share a common facet V_i^0, we have that (4.1) is satisfied, as illustrated in Figure 4.3.

Now suppose no such u' exists. By a similar argument from Case 3, there exists some $j \in [n]$ such that $w \notin V_j^2$. Thus, w is at most one step away from the at most $(n-2)$-dimension face defined by $V_i^0 \cap V_j^0$ which contains v, as shown in Figure 4.4. Thus, (4.2) is satisfied. $\diamond$

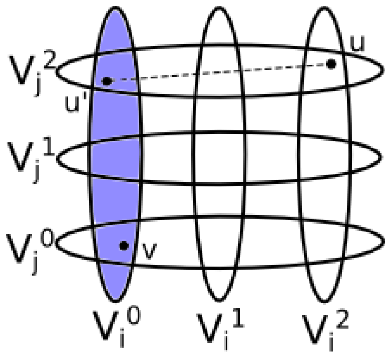


Figure 4.3: A neighbor of u is contained in V_i^0.

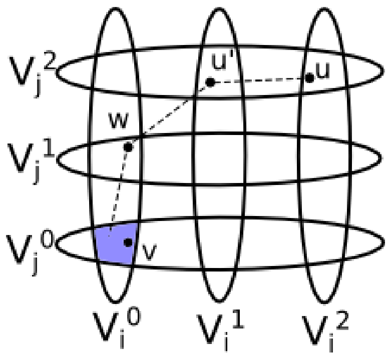


Figure 4.4: No neighbor of u is contained in V_i^0.

Thus, we have that in any case, one of the following two inequalities hold:

$$\delta(u, v) \leq \delta^{n-1} + 1$$
$$\delta(u, v) \leq \delta^{n-2} + 3.$$

Therefore, if we are in a case where (4.1) holds, then

$$\begin{aligned}
\delta(u, v) &\leq \delta^{n-1} + 1 \\
&\leq \lfloor \frac{3}{2}(n-1) \rfloor + 1 \\
&\leq \lfloor \frac{3}{2}n \rfloor.
\end{aligned}$$

And similarly, if we are in a case where (4.2) holds, then

$$\begin{aligned}
\delta(u, v) &\leq \delta^{n-2} + 3 \\
&\leq \lfloor \frac{3}{2}(n-2) \rfloor + 3 \\
&\leq \lfloor \frac{3}{2}n \rfloor.
\end{aligned}$$

Thus, by induction, we have that for all d, the diameter of P is at most $\lfloor \frac{3}{2}n \rfloor$, completing the proof. $\square$

4.4 Additional remarks

Finding a polynomial path

This bound tells us that there exists a path between any two vertices of a $(0, 1, a_i)$-polytope whose length is at most $\lfloor \frac{3}{2}n \rfloor$. Akin to the work done in [30], the proof also lends itself to finding an algorithm which gives this path of length at most $2n$.

Let u and v be vertices of a $(0, 1, a_i)$-polytope P. We can procede with each coordinate one at a time, needing at most two steps for both u and v to reach V_i^0 for $i \in [n]$. By viewing V_i^0 as a $(0, 1, a_i)$-polytope in a lower dimension, we can continue the process.

We may assume that $u_i = 0$ (and $v = 1$ or 2) or $u_i = 1$ (and $v = 1$). Otherwise we can use an affine transformation so that we are in one of these two cases.

If $u_i = 0$, then we know v is at most two steps away to some vertex $v' \in V_i^0$. Thus, u and v' are contained in the $(0, 1, a_i)$-grid of one dimension $n - 1$. Thus we procede with a $(0, 1, a_i)$-polytope in one dimension less.

If $u_i = 1$ and, without loss of generality, $v_i = 1$, then both vertices have neighbors u' and v' respectively in V_i^0. Thus two steps are needed to enter the common facet V_i^0, so we again can limit our view of our $(0, 1, a_i)$-polytope to V_i^0 in one dimension lower.

We can do this until a path is found from both u and v to the origin. Note, this does not mean the origin is necessarily a vertex of our original polytope P, but it will be after the appropriate affine transformations and dimension reductions. Thus by using affine transformations and for each $i \in [n]$ minimizing both u_i and v_i, we will find a connecting path between u and v of length at most $2n$.

Returning to the Hirsch conjecture

It is also worth noting that Theorem 4.1.2 implies that at least the $(0, 1, a_i)$-polytopes with more than $\lfloor \frac{5}{2}n \rfloor$ facets satisfy the Hirsch conjecture.

Conjecture 4. All $(0, 1, a_i)$-polytopes satisfy the Hirsch conjecture.

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Zusammenfassung

In der vorliegenden Arbeit stellen wir $(0,1,a)$- und $(0,1,a_i)$-Polytope vor und erforschen ihre verschiedenen kombinatorischen Eigenschaften. Diese Polytope sind Verallgemeinerungen von $(0,1)$-Polytopen, die fundamentale Objekte der kombinatorischen Optimierung und linearen Programmierung darstellen. Während $(0,1)$-Polytope durch Ecken mit zwei verschiedenen Koordinatenwerten beschrieben werden, sind $(0,1,a)$- und $(0,1,a_i)$-Polytope durch Ecken mit drei verschiedenen Koordinatenwerten characterisiert. Diese Dissertation konzentriert sich auf die durch diese Verallgemeinerung entstehenden neuen kombinatorischen Strukturen.

Im ersten Kapitel werden diese Polytope definiert und interessante Beispiele präsentiert, die sowohl die vorliegende Arbeit motivieren sollen, als auch Intuition für die neuen geometrischen und kombinatorischen Strukturen vermitteln sollen. Insbesondere geben wir Beispiele an, die zeigen, dass sich die kombinatorischen Typen von $(0,1,a)$- und $(0,1,a_i)$-Polytopen mit ganzzahligen, rationalen und irrationalen Koordinaten jeweils unterscheiden können.

Im zweiten Kapitel werden dreidimensionale $(0,1,a)$-Polytope mit rationalen Koordinaten vollständig enumeriert. Hierfür wird in einem ersten Schritt die maximale Anzahl an Ecken, die ein solches Polytop haben kann, nach oben beschränkt. In einem zweiten Schritt wird gezeigt, dass nur endlich viele Werte von a zu verschiedenen kombinatorischen Typen führen. Im Anschluss daran diskutieren wir die Komplexität der Enumeration der kombinatorischen Typen aller $(0,1,a)$- und $(0,1,a_i)$-Polytope in den Dimensionen drei und vier.

Das dritte Kapitel behandelt extremale Eigenschaften von $(0,1,a)$-Polytopen, die dazu führen, dass Enumerationen in höheren Dimensionen nicht mehr möglich sind. In diesem Zusammenhang definieren wir eine Menge von Werten a mit exponentiell wachsender Kardinalität, sodass verschiedene Werte a zu Polytopen mit verschiedenen kombinatorischen Typen führen können. Diese Methode führt ebenfalls zu einer besonders schönen exponentiell wachsenden Klasse von $(0,1)$-Matritzen, deren Determinanten alle ganzzahlig sind.

Im vierten Kapitel diskutieren wir den Durchmesser von $(0,1,a_i)$-Polytopen und geben insbesondere eine lineare obere Schranke an. Wir präsentieren einen Algorithmus, der in linearer Zeit zwischen je zwei Ecken einen kurzen Weg findet.

Selbstständigkeitserklärung

Gemäß §7 (4) der Promotionsordnung des Fachbereichs Mathematik und Informatik der Freien Universität Berlin versichere ich hiermit, dass ich alle Hilfsmittel und Hilfen angegeben und auf dieser Grundlage die Arbeit selbstständig verfasst habe. Des Weiteren versichere ich, dass ich diese Arbeit nicht schon einmal zu einem früheren Promotionsverfahren eingereicht habe.

Berlin, den

Katy Beeler