# Appendix B

# Formal Definition of Recursively Constructible Family of Graphs

We say that a sequence of graphs $\{G_n\}_{n\geq 0}$ is a *recursively constructible* family of graphs if it can can be built from a given initial graph by means of a repeated fixed sequence of elementary operations involving addition of vertices and edges, and deletion of edges.

The concept of recursively constructible family of graphs appears for the first time in a joint work with Marc Noy [43]. There, these families are formally defined as follows.

Given a graph $G$ and $U \subseteq V(G)$, let $G[U]$ be the subgraph induced by $U$, and let $N_G(U)$ be the set of vertices in $V(G)$ adjacent to some vertex in $U$. The symbol $\cup$ denotes disjoint union.

Then we require the existence of a positive integer $r$, and a (labeled) graph $M$, such that:

(a) $V(G_0) = W_0$, $E(G_0) = E_0$.

(b) $V(G_n) = V(G_{n-1}) \cup W_n$.

(c) $E(G_n) = (E(G_{n-1}) \setminus S) \cup E_n$, where $S \subseteq \bigcup_{i=1}^{r} E_{n-i}$ is a set of edges with one end in $W_0$ and the other end in $\bigcup_{i=1}^{r} W_{n-i}$.

(d) All edges of $E_n$ go between $W_n$ and $W_0 \cup (\bigcup_{i=0}^{r} W_{n-i})$, for $n > r$.

(e) The graph $G_n\left[W_0 \cup (\bigcup_{i=0}^{r} W_{n-i})\right]$ is equal to $M$ for $n > r$. In particular, $G_n[W_n]$ is always the same graph.

When we say 'equal' in condition (e) we mean the following. The vertices in $G_n$ are labeled by consecutive integers. The first labels correspond to the initial vertices $W_0$, next labels to $W_1$, and so on. And within a given $W_n$, the labels are also ordered, so that $G_n[W_n]$ is always the same labeled graph. Then the mapping that assigns the vertices of $M$ to the vertices of $W_0 \cup W_n \cup W_{n-1} \cup \cdots \cup W_{n-r}$ in the order of increasing labels is a graph isomorphism. Also, the set $S$ of edges that can be removed in condition (c) is always the same, a fact captured again by condition (e).

The elementary operations which constitute the step from $G_{n-1}$ to $G_n$ are then of the following three kinds:

1. Addition of a new set $W_n$ of vertices adjacent to vertices only in $W_0 \cup (\bigcup_{i=0}^{r} W_{n-i})$.

2. Deletion of a fixed set $S$ of edges with one end in $W_0$ and the other end in $\bigcup_{i=1}^{r} W_{n-i}$.

3. Addition of a fixed new set $E_n$ of edges incident only to vertices in $W_0 \cup (\bigcup_{i=0}^{r} W_{n-i})$.

Operation 1 is the first step for enlarging the graph. The edges between $W_n$ and $W_0$ are crucial in order to capture families with 'cyclic boundary conditions', like for example the family of prisms $\{P_{W\times k}\}_{k\geq 0}$ in Figure 5.3. Then we also need to remove the corresponding edges added in the previous step, and this is where Operation 2 comes into play. Operation 3 specifies the edges between the new and the old vertices, and also within the new vertices.

Our definition allows to connect vertices in $W_n$ not only to $W_{n-1}$ but also to $W_{n-2}$ up to $W_{n-r}$. The reason is to be able to capture powers of graphs and, more generally, circulant graphs.