# Appendix A

# Proof of Lemma 4.3

We want to prove that we end up with the same value of $\mathbf{q}_i$ independently of the path we have chosen. It remains to show the following:

**Lemma A.1.** *Any trip around an interior point of the cell decomposition generates a difference vector* $(0,0,0)$.

*Proof.* Let be $\mathbf{p}_0$ be an interior point of the cell decomposition. For improving readability, we do not distinguish between the different vertices and auxiliary vertices converging to $\mathbf{p}_0$ and we denote them indistinctly by 0.

Let $S_0$ be the sequence of all pieces with one end converging to $\mathbf{p}_0$, in counter clockwise order. The elements of $S_0$ are of the form $(s,0)$, with $s, 0 \in V \cup A$. Note that given two different pieces $(s_1, 0_1)$ and $(s_2, 0_2)$, we can have $s_1 = s_2$ or $0_1 = 0_2$. See Figure A.1 for an example. The pairs of pieces of $S_0$ sharing the same supporting bar are exactly those of type $(c)$ (remember that we walk counter clockwise around $\mathbf{p}_0$). The remaining pieces of $S_0$ are those of type $(a)$ or $(b)$ and satisfy that $0 = \bar{0} \in V$ is an end vertex of the supporting bar.

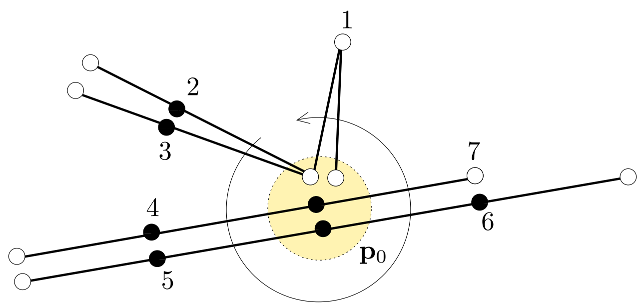


Figure A.1: Crossed pieces when walking counter clockwise around $\mathbf{p}_0$. The pair $\{(0,4),(0,7)\}$ lies on the same supporting bar; both pieces are of type $(c)$. The same for the pair $\{(0,5),(0,6)\}$. The remaining pieces are of type $(a)$ or $(b)$.

The difference vector $\xi$ around $\mathbf{p}_0$ is given by

$$\xi = \sum_{(0,s) \in S_0} \left(\omega_{0s}(\mathbf{p}_0 \times \mathbf{p}_s) + \Delta(0,s)\right) .$$

Grouping terms according to the type of the pieces and by Part 1 of Lemma 4.1 we get

$$\xi = \sum_{(0,s)\in S_0} \omega_{0s}(\mathbf{p}_0 \times \mathbf{p}_s) + \sum_{\substack{(0,s)\in S_0 \\ \text{of type } (b)}} \Delta(0,s) + \sum_{\substack{(0,s)\in S_0 \\ \text{of type } (c)}} \Delta(0,s)\,. \tag{A.1}$$

For the first term of (A.1), we have

$$\begin{aligned}
\sum_{(0,s)\in S_0} \omega_{0s}(\mathbf{p}_0 \times \mathbf{p}_s) &= \sum_{(0,s)\in S_0} \omega_{0s}\left((\mathbf{p}_0 \times \mathbf{p}_s) - (\mathbf{p}_0 \times \mathbf{p}_0)\right) \\
&= \sum_{(0,s)\in S_0} \omega_{0s}(\mathbf{p}_0 \times (\mathbf{p}_s - \mathbf{p}_0)) \\
&= \mathbf{p}_0 \times \sum_{(0,s)\in S_0} \omega_{0s}(\mathbf{p}_s - \mathbf{p}_0) \\
&= \mathbf{p}_0 \times \sum_{(0,s)\in S_0} \omega_{\bar{0}\bar{s}}(\mathbf{p}_{\bar{s}} - \mathbf{p}_{\bar{0}}) \\
&= \mathbf{p}_0 \times \sum_{\substack{(0,s)\in S_0 \\ \bar{0}\in V}} \omega_{0\bar{s}}(\mathbf{p}_{\bar{s}} - \mathbf{p}_0)\,. \qquad (A.2)
\end{aligned}$$

The first equality holds since $\mathbf{p}_0 \times \mathbf{p}_0$ is always zero. The second and third equations hold by the linearity of the cross product. The fourth equality holds since, given a piece $(b,t)$ of the bar $(\bar{b},\bar{t})$,

$$\omega_{bt}(\mathbf{p}_t - \mathbf{p}_b) = \omega_{\bar{b}\bar{t}}(\mathbf{p}_{\bar{t}} - \mathbf{p}_{\bar{b}})\,.$$

The fifth equality holds from the following: since we surround a vertex $\mathbf{p}_0$, the pairs of oriented pieces supported by the same bar, contribute to the sum with two terms of equal value but different sign, thus they cancel. Hence only the remaining pieces, those of type $(a)$ or $(b)$, contribute to the sum.

By Part 2 of Lemma 4.1 and by the linearity of the cross product, we have for the second term of (A.1):

$$\sum_{\substack{(0,s)\in S_0 \\ \text{of type } (b)}} \Delta(0,s) = \sum_{\substack{(0,s)\in S_0 \\ \text{of type } (b)}} F_0^{[s,\bar{s}]}\mathbf{p}_0 \times (\mathbf{p}_{\bar{s}} - \mathbf{p}_0)^{\perp}\,. \tag{A.3}$$

We analyze now the third term of (A.1). We pair the pieces of $S_0$ of type $(c)$ according to their supporting bar. It is easy to see that they can be paired in such a way. Let $P_0$ be the set of all the oriented pairs. An element of $P_0$ is of the form $\{(0,s_1),(0,s_2)\}$, and its supporting bar is $(\bar{s_2},\bar{s_1})$ (see Figure A.2). We have

$$\sum_{\substack{(0,s)\in S_0 \\ \text{of type } (c)}} \Delta(0,s) = \sum_{\{(0,s_1),(0,s_2)\}\in P_0} \left(\Delta(0,s_1) + \Delta(0,s_2)\right)\,. \tag{A.4}$$

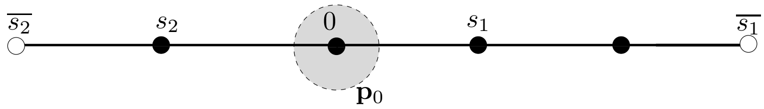


Figure A.2: A pair of $P_0$

Given a pair $\{(0,s_1),(0,s_2)\}$ of $P_0$, we obtain

$$\begin{aligned}
&\Delta(0,s_1)+\Delta(0,s_2)\\
&\quad = F_{\bar{s_2}}^{[s_1,\bar{s_1}]}\mathbf{p}_{\bar{s_2}} \times (\mathbf{p}_{\bar{s_1}}-\mathbf{p}_{\bar{s_2}})^{\perp} - F_{\bar{s_1}}^{[0,\bar{s_2}]}\mathbf{p}_{\bar{s_1}} \times (\mathbf{p}_{\bar{s_1}}-\mathbf{p}_{\bar{s_2}})^{\perp}\\
&\qquad\qquad + F_{\bar{s_1}}^{[s_2,\bar{s_2}]}\mathbf{p}_{\bar{s_1}} \times (\mathbf{p}_{\bar{s_2}}-\mathbf{p}_{\bar{s_1}})^{\perp} - F_{\bar{s_2}}^{[0,\bar{s_1}]}\mathbf{p}_{\bar{s_2}} \times (\mathbf{p}_{\bar{s_2}}-\mathbf{p}_{\bar{s_1}})^{\perp}\\
&\quad = F_{\bar{s_2}}^{[0,0]}\mathbf{p}_{\bar{s_2}} \times (\mathbf{p}_{\bar{s_1}}-\mathbf{p}_{\bar{s_2}})^{\perp} - F_{\bar{s_1}}^{[0,0]}\mathbf{p}_{\bar{s_1}} \times (\mathbf{p}_{\bar{s_1}}-\mathbf{p}_{\overline{\bar{s_2}}})^{\perp}\\
&\quad = \Bigg(\sum_{\substack{k:\{k;\bar{s_2},\bar{s_1}\}\in\mathcal{L}\\ \mathbf{p}_k=\mathbf{p}_0}} \omega_{k\bar{s_2}\bar{s_1}} - \sum_{\substack{k:\{k;\bar{s_1},\bar{s_2}\}\in\mathcal{L}\\ \mathbf{p}_k=\mathbf{p}_0}} \omega_{k\bar{s_1}\bar{s_2}}\Bigg)\mathbf{p}_0 \times (\mathbf{p}_{\bar{s_1}}-\mathbf{p}_{\bar{s_2}})^{\perp}\,. \qquad \text{(A.5)}
\end{aligned}$$

The first equation follows from the definition of $\Delta$, with the supporting bar $(\bar{s_2},\bar{s_1})$ oriented as $(0,s_1)$, or $(\bar{s_1},\bar{s_2})$ oriented as $(0,s_2)$. For the second equation we used $-\mathbf{x}^{\perp}=(-\mathbf{x})^{\perp}$ and the fact that

$$[0,\bar{s_i}] = 0 \cup [s_i,\bar{s_i}]\,, \qquad \text{for } i=1,2$$

by construction. The third equation follows from the definition of $F_{\bar{s_2}}^{[0,0]}$ and $F_{\bar{s_1}}^{[0,0]}$, the linearity of the cross product and the representation of $\mathbf{p}_k$ as a convex combination of $\mathbf{p}_{\bar{s_2}}$, $\mathbf{p}_{\bar{s_1}}$, and we write $\mathbf{p}_0$ instead of $\mathbf{p}_k$.

From (A.2), (A.3), (A.4) and (A.5), the equation (A.1) can be rewritten as

$$\begin{aligned}
\xi =&\mathbf{p}_0 \times \Bigg(\sum_{\substack{(0,s)\in S_0\\ 0\in V}} \omega_{0\bar{s}}(\mathbf{p}_{\bar{s}}-\mathbf{p}_0)\\
&\quad + \sum_{\substack{(0,s)\in S_0\\ \text{of type } (b)}} \Bigg(\sum_{k:\{k;0,\bar{s}\}\in\mathcal{L}} -\eta_0^{k0\bar{s}} + \sum_{k:\{k;\bar{s},0\}\in\mathcal{L}} \eta_0^{k\bar{s}0}\Bigg)(\mathbf{p}_{\bar{s}}-\mathbf{p}_0)^{\perp}\\
&\quad + \sum_{\{(0,s_1),(0,s_2)\}\in P_0} \Bigg(\sum_{\substack{k:\{k;\bar{s_2},\bar{s_1}\}\in\mathcal{L}\\ \mathbf{p}_k=\mathbf{p}_0}} \omega_{k\bar{s_2}\bar{s_1}} - \sum_{\substack{k:\{k;\bar{s_1},\bar{s_2}\}\in\mathcal{L}\\ \mathbf{p}_k=\mathbf{p}_0}} \omega_{k\bar{s_1}\bar{s_2}}\Bigg)(\mathbf{p}_{\bar{s_1}}-\mathbf{p}_{\bar{s_2}})^{\perp}\Bigg)\\
=&\mathbf{p}_0 \times \Bigg(\sum_{\substack{(0,s)\in S_0\\ 0\in V}} \omega_{0\bar{s}}(\mathbf{p}_{\bar{s}}-\mathbf{p}_0) + \sum_{0\in V}\mathbf{F}_{ST}(0)\Bigg)\\
=&\mathbf{p}_0 \times (0,0,0)\\
=&(0,0,0)\,.
\end{aligned}$$

The vector $\mathbf{p}_0$ is a common factor of the three terms of the sum, thus the first equation follows. For the second and third equations, one can check that the second term of the cross product is nothing else than the resulting force at $\mathbf{p}_0$, which is the sum of the classic and the self-touching stress over all the vertices converging to $\mathbf{p}_0$, and we had equilibrium at every vertex. $\square$