Chapter 6

Quantitative Analysis of the Realizations of the Combinatorial Types of 3-Polytopes

6.1 Introduction

Characterizing the structure of the set of all d-dimensional polytopes leads to two main lines of study:

(1) to list all possible combinatorial types of polytopes, i.e. to determine which finite lattices correspond to face lattices of polytopes and which do not;

(2) to describe the set of all realizations of a given combinatorial type.

For 3-dimensional polytopes (or 3-polytopes for short), as early as 1922, Steinitz [52, 53] answered the questions about the *realization space*, that is, the space of all polytopes Q that are combinatorially equivalent to a polytope P. The *edge graph* $G(P)$ of a polytope P is the connected graph whose vertex set is the vertex set of the polytope P, and two vertices are adjacent in the graph if they are endpoints of a 1-face of P. Steinitz proved the following theorem.

Theorem 6.1 (Steinitz's Theorem, 1922 [52]). *A graph G is the edge graph of a 3-polytope if and only if G is simple, planar and 3-connected.*

Several proofs for the theorem are given in [48]. Inspecting carefully the proofs, Richter-Gebert concluded that for every 3-polytope P the realization space contains rational points, that is, every 3-polytope can be realized with integral vertex coordinates.

In Part IV of [48] Richter-Gebert presents a proof of Steinitz's Theorem that is based on self-stresses. This approach also proves that the realization space of a 3-polytope contains rational points. Richter-Gebert obtains the following results:

(1) Any 3-polytope P with n vertices can be realized with integral coordinates smaller than 2^{18n^2};

(2) If P contains a triangle facet, then it can be realized with integral coordinates smaller than 43^n.

Previously, in 1994, Onn and Sturmfels [44] proved that a 3-polytope with n vertices can be realized with integral coefficients (vertex coordinates) smaller than n^{169n^3}.

In this chapter we further consider these problems and we prove that

(1′) If P contains a triangle (not necessarily a face), then it can be realized with integral coordinates smaller than 29^n.

(2′) If P contains no triangular facet, but at least a quadrilateral facet, then it can be realized with integral coordinates smaller than 156^n.

(3′) Any 3-polytope P with n vertices can be realized with integral coordinates of absolute value less than $n^{10n}\,2^{10n^2} < 2^{12n^2}$.

For dimension $d \geq 4$, the realization of a d-dimensional polytope is very complicated. d-dimensional polytopes behave very differently from 3-polytopes with respect to realizability. In [48], Richter-Gebert summarizes the counterexamples that are found to contrast with the 3-dimensional case. In the monograph it is also explained that the realizability problem for 4-polytopes is NP-hard, and in order to realize all combinatorial types of integral 4-polytopes with n vertices in the integer grid $\{1, 2, \ldots, f(n)\}^4$, the coordinate size function $f(n)$ has to be at least doubly exponential in n.

This chapter is organized as follows. In Section 6.2 we use the Matrix-Tree Theorem to relate the stress matrix of a graph with its number of spanning forests. In Section 6.3 we work on the size of the smallest integral grid on which all combinatorial types of 3-polytopes can be realized. We study the case when the polytope contains a triangular face, the case when the polytope contains no triangular face but a quadrilateral face, and the case when the polytope contains neither a triangular nor a quadrilateral face, but a pentagonal face.

6.2 A Generalization of the Matrix-Tree Theorem

Let G be a simple, planar and 3-connected graph, and let L be the *Laplacian matrix* of G.

Consider any edge orientation of the graph G. The *incidence matrix* of G (with respect to this orientation) is the $V \times E$ indexed matrix $C = (c_{ie})$ with entries

$$c_{ie} = \begin{cases} 1 & \text{if } i \text{ is at the head of } e \\ -1 & \text{if } i \text{ is at the tail of } e \\ 0 & \text{otherwise} \end{cases}$$

Then we have

Lemma 6.1. *The Laplacian matrix of G can be written in terms of the incidence matrix as*

$$L = C \cdot C^t.$$

The Matrix-Tree Theorem [1] states that the determinant of the matrix obtained from the Laplacian matrix L of a graph G by deleting one row and one column equals the number of spanning trees of G.

In this chapter we consider the stress matrix $\bar{L}$, obtained by deleting the rows and columns corresponding to the boundary points of G. Suppose G is n-vertex graph with k boundary points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$. Let $F^k(G)$ be the set of spanning forests of G with k components, each rooted at one pinned boundary vertex $\mathbf{p}_i$, $i = 1 \ldots k$. We use the following generalization of the Matrix-Tree Theorem:

Theorem 6.2. *Let $\bar{L}_k$ be the stress matrix obtained from L by deleting the rows and columns corresponding to $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$. Then*

$$\det \bar{L}_k = |F^k(G)| \,.$$

Proof. The Binet-Cauchy theorem from linear algebra states that, given an $(r \times s)$-matrix P and an $(s \times r)$-matrix Q, $r \leq s$, $\det(P \cdot Q)$ equals the sum of the products of determinants of corresponding $(r \times r)$-submatrices, where "corresponding" means that we take the same indices for the r columns of P and the r rows of Q.

For $\bar{L}_k$ this means

$$\det \bar{L}_k = \sum_N \det N \cdot \det N^t = \sum_N (\det N)^2,$$

where N runs through all $(n-k) \times (n-k)$ submatrices of $C \setminus \{\text{rows } 1, \ldots, k\}$. The $n-k$ columns of N correspond to a subgraph of G with $n-k$ edges on n vertices, and it remains to show that:

$$\det N = \begin{cases} \pm 1 & \text{if these edges span a forest of } F^k(G) \\ 0 & \text{otherwise} \end{cases}$$

Case 1: Suppose the subgraph contains a cycle. Then since the rows of N corresponding to the vertices of the component containing this cycle sum up to 0, we infer that they are linearly dependent, and hence $\det N = 0$.

Case 2: Suppose the subgraph contains no cycle but one of its components contains at least two vertices of the boundary of G. Let X be this component. Since X contains more than one boundary vertex, then it has no more than $|E(X)| - 1$ interior vertices of G. The submatrix formed by the columns corresponding to edges in X has $|E(X)|$ columns and at most $|E(X)| - 1$ non-zero rows, hence its rank is at most $|E(X)| - 1$. Consequently, $\det N = 0$.

Case 3: Suppose the subgraph is a forest of $F_k(G)$, with components $X_1, \ldots, X_k$. Consider the component X_1 of the forest. X_1 is a tree rooted at the boundary vertex $\mathbf{p}_1$. Then, in this component, there is a vertex $j_1 \neq \mathbf{p}_1$ of degree 1; let e_1 be the incident edge. Deleting j_1, e_1 we obtain a tree with $E(X_1) - 1$ edges. Again there is a vertex $j_2 \neq \mathbf{p}_1$ of degree 1 with incident edge e_2. Continue in this way until $j_1, \ldots, j_{E(X_1)}$ and $e_1, \ldots, e_{E(X_1)}$ with $j_l \in e_l$ are determined.

We can do this for every component $X_1, \ldots, X_k$ of the subgraph, obtaining a sequence

$$j_1, \ldots, j_{E(X_1)}, j_{E(X_1)+1}, \ldots, j_{E(X_1)+E(X_2)}, \ldots, j_{E(X_1)+\cdots+E(X_{k-1})+1}, \ldots, j_{n-k}$$

of vertices, and a sequence

$$e_1, \ldots, e_{E(X_1)}, e_{E(X_1)+1}, \ldots, e_{E(X_1)+E(X_2)}, \ldots, e_{E(X_1)+\cdots+E(X_{k-1})+1}, \ldots, e_{n-k}$$

of edges.

Now permute the rows and columns of N, to bring j_l into the l-th row and e_l into the l-th column. Since by construction $j_l \notin e_m$ for $l < m$, we see that the new submatrix N' is lower triangular with all elements on the main diagonal equal to ± 1. Thus $\det N = \pm \det N' = \pm 1$, and we are done. □

6.3 Quantitative Analysis of the Size of a Minimal Grid

In this section we consider the size $f(n)$ of a minimal grid $\{1, 2, \ldots, f(n)\}^3$ on which all combinatorial types of 3-polytopes with n vertices can be realized. For upper-bounding the minimal grid size, we are based on Richter-Gebert's construction, which steps from a graph to a concrete polytope using the Maxwell-Cremona Theorem.

The bound we obtain is an exponential function in n if the polytope contains a triangle or a quadrilateral face, and exponential in n^2 for the general case, as in previous results [48], but we get an improvement on the constants.

For the cases when the polytope contains a triangle or a quadrilateral face, we first derive an upper bound for the grid size needed for an equilibrium representation of the edge graph of the polytope.

Assume that a 3-connected planar graph G is given with a choice of a peripheral polygon and an arbitrary assignment of positive stresses ω_{ij} on the interior edges.

Using Tutte's embedding we can easily find an equilibrium stress for the interior vertices, as described in Section 0.2: For a fixed location of the boundary vertices $\mathbf{p}_1, \ldots, \mathbf{p}_k$ in convex position, we solve the following system, yielding equilibrium locations $\mathbf{p}_i = (\mathbf{x}_i, \mathbf{y}_i)$ for the interior vertices:

$$\bar{L}_k \cdot \mathbf{x} = \mathbf{b}_{\mathbf{x}}\,, \qquad \bar{L}_k \cdot \mathbf{y} = \mathbf{b}_{\mathbf{y}}\,,$$

where $\mathbf{x}$ and $\mathbf{y}$ are the obtained coordinates for the interior points, $\bar{L}_k$ is the stress matrix of size $n-k$, and $\mathbf{b}_{\mathbf{x}}$ and $\mathbf{b}_{\mathbf{y}}$ contain the fixed boundary conditions.

Lemma 6.2. *If we choose the boundary points such that the independent vectors $\mathbf{b}_{\mathbf{x}}$ and $\mathbf{b}_{\mathbf{y}}$ are $\det \bar{L}_k$ times an integer vector $\mathbf{u}_{\mathbf{x}}$ and $\mathbf{u}_{\mathbf{y}}$ respectively, we get integer vector solutions in the range $[\,0\,,\, \mathbf{u}^0 \det \bar{L}_k\,]$, being $\mathbf{u}^0$ the absolute value of the largest component of $\mathbf{u}_{\mathbf{x}}$ and $\mathbf{u}_{\mathbf{y}}$.*

Proof. The boundary vertices are vectors with two integer components. We choose $\mathbf{b}_{\mathbf{x}} = \det \bar{L}_k\, \mathbf{u}_{\mathbf{x}}$ and $\mathbf{b}_{\mathbf{y}} = \det \bar{L}_k\, \mathbf{u}_{\mathbf{y}}$. Since all interior vertices are interior to the peripheral convex polygon, their largest coordinate must be in the range $[\,0\,,\, \mathbf{u}^0 \det \bar{L}_k\,]$.

The x-coordinates of the interior vertices equal, by Cramer's rule, to

$$x_i = \det \bar{L}^{(i)} / \det \bar{L}_k,$$

where $\bar{L}^{(i)}$ is obtained from $\bar{L}$ by replacing the i-th column by $\mathbf{b}_{\mathbf{x}}$. The number $\det \bar{L}_k$ is then a common factor of all the components of the i-th column of the matrix $\bar{L}^{(i)}$, hence it is a factor of $\det \bar{L}^{(i)}$ and it cancels with the denominator of x_i.

This and the fact that all matrix entries are integers, imply that all the entries of $\mathbf{x}$ are integer numbers.

Analogously, we can see that the entries of $\mathbf{y}$ are integer numbers as well. □

Let $G[\mathbf{p}] = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$ denote the unique equilibrium configuration given by Tutte's Theorem.

6.3.1 Extending the Interior Equilibrium Stress to the Boundary

The resulting equilibrium stress given by Tutte's embedding cannot, in general, be extended to an equilibrium stress on the boundary vertices, except when the outer face is a triangle.

We denote by I and B the interior and boundary vertices of $G[\mathbf{p}]$ respectively.

We want to get equilibrium at the boundary vertices j by offsetting the resulting forces f_j with appropriate negative values on the stresses of the boundary edges:

$$f_j := \sum_{i \in I : \{i,j\} \in E} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j)\,, \text{ for } j \in B \tag{6.1}$$

The following lemma shows how the resulting forces can be expressed in terms of the location of the boundary vertices $\mathbf{p}_j$, $j \in B$:

Lemma 6.3. *1. There are non-negative weights $\tilde{\omega}_{jk} = \tilde{\omega}_{kj} \geq 0$, for $j, k \in B$, independent of $\mathbf{p}$, such that the resulting forces on the boundary vertices are given by*

$$f_j = \sum_{k \in B : k \neq j} \tilde{\omega}_{jk}(\mathbf{p}_k - \mathbf{p}_j)\,, \textit{ for } j \in B\,. \tag{6.2}$$

2. *The weights $\tilde{\omega}$ are rational numbers with denominator the determinant of the reduced Laplacian matrix after removing the rows and columns corresponding to the boundary vertices.*

3. *If we fix vertex $\mathbf{p}_j$ at position 1 and all other boundary vertices at position 0, then $\tilde{\omega}_{jk}$ is the resulting force at vertex k measured with respect to the positive direction.*

In other words, the resulting forces are the same as in a complete graph on the vertex set B with stresses $\tilde{\omega}$.

Proof. Let A be the weighted adjacency matrix with the given weights ω and let D be the diagonal matrix of row sums of A. We subdivide A and D into block matrices indexed by vertex sets I and B. The system can be written as

$$\begin{pmatrix} A_{II} & A_{IB} \\ A_{BI} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p}_I \\ \mathbf{p}_B \end{pmatrix} - \begin{pmatrix} D_I \mathbf{p}_I \\ D_B \mathbf{p}_B \end{pmatrix} = \begin{pmatrix} 0 \\ f_B \end{pmatrix}$$

This is actually an abbreviated representation of two independent systems of equations, for the x-coordinates and for the y-coordinates. The two equations have the same structure of coefficients but different variables and "right-hand sides". Thus $\mathbf{p}_i$ and f_i must be read as representing either the x-coordinate or the y-coordinate of the corresponding vector.

Solving the first equation for $\mathbf{p}_I$ yields

$$\mathbf{p}_I = (D_I - A_{II})^{-1} A_{IB} \mathbf{p}_B,$$

and substituting this into the second equation gives

$$f_B = A_{BI}(D_I - A_{II})^{-1} A_{IB} \mathbf{p}_B - D_B \mathbf{p}_B =: \tilde{A} \mathbf{p}_B\,.$$

Since $A_{BI} = A_{IB}^T$, the coefficient matrix $\tilde{A}$ is symmetric. We can define $\tilde{\omega}$ as the off-diagonal entries of $\tilde{A}$. With this definition, Part 2 is satisfied, since the denominator of $\tilde{A}$ is the determinant of $(D_I - A_{II})$, the reduced Laplacian matrix after removing the rows and columns corresponding to the boundary vertices.

To show that the expression $f_B = \tilde{A}\mathbf{p}_B$ has the form (6.2), we have to check that the row sums of $\tilde{A}$ are 0 and $\tilde{\omega} \geq 0$. Let $\mathbf{1}$ denote the vector of all ones. We have $A_{II}\mathbf{1} + A_{IB}\mathbf{1} = D_I\mathbf{1}$ by definition, and therefore $(D_I - A_{II})^{-1}A_{IB}\mathbf{1} = \mathbf{1}$. If we plug this into the expression of $\tilde{A}\mathbf{1} := A_{BI}(D_I - A_{II})^{-1}A_{IB}\mathbf{1} - D_B\mathbf{1}$, we obtain $A_{BI}\mathbf{1} - D_B\mathbf{1}$, which is again zero, by definition.

To see that $\tilde{\omega}_{jk} \geq 0$, we argue geometrically. If we position the vertices as in Part 3, by (6.2) $\tilde{\omega}_{jk}$ is the resulting force at vertex k measured with respect to the positive direction. However, it is clear that all vertices $\mathbf{p}_i$ lie in the convex hull between 0 and 1, therefore, by (6.1), $\tilde{\omega}_{jk} \geq 0$. □

Lemma 6.4. *If we set the interior stresses to 1, the weights $\tilde{\omega}$ satisfy*

$$\sum_{k \in B: k \neq j} \tilde{\omega}_{jk} < n - |B|, \quad \textit{for } j \in B\,.$$

Proof. We argue similarly as in Part 3 of Lemma 6.3. Fix the vertex $\mathbf{p}_j$ at the origin 0 and all other boundary vertices at position 1, very close to each other such that they are at distance 1 from $\mathbf{p}_j$. By (6.2), $\sum_{k \in B: k \neq j} \tilde{\omega}_{jk}$ is the resulting force at vertex j. Since all interior vertices $\mathbf{p}_i$, $i \in I$, lie inside the boundary, all interior edges $\{\mathbf{p}_j, \mathbf{p}_i\}$ have length strictly less than 1. If all stresses w_{ij} are set to 1, then all forces at vertex j coming from the interior edges are strictly less than 1. There are $n - |B|$ of these forces, since there are $|I| = n - |B|$ interior vertices, and the lemma follows. □

This inequality holds for any subset $B_s \subset B$ of boundary vertices, since by positioning vertex j and the vertices of $B \setminus B_s$ at position 0, and all vertices $k \in B_s$ different from j at position 1, we have

$$\sum_{k \in B_s: k \neq j} \tilde{\omega}_{jk} < \sum_{k \in B: k \neq j} \tilde{\omega}_{jk} < n - |B|, \quad \text{for } j \in B_s\,.$$

Corollary 6.1. *For any $j, k, l \in B$, we have $\tilde{\omega}_{jk} + \tilde{\omega}_{jl} < n - |B|$.*

Corollary 6.2. *For any $j, k \in B$, we have $\tilde{\omega}_{jk} < n - |B|$.*

Lemma 6.5. *If the outer face is a triangle, then every equilibrium stress on the interior edges and vertices can be extended to an equilibrium stress on the whole framework.*

Proof. Let the boundary vertices be $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$. We can get the desired equilibrium on the full framework by simply setting $\omega_{12} := -\tilde{\omega}_{12}$, $\omega_{23} := -\tilde{\omega}_{23}$, and $\omega_{13} := -\tilde{\omega}_{13}$.

Geometrically, this corresponds to the fact that three lifted vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ lie always in a common plane, and thus there is no difference between the liftings of Part 1 and 2 in Theorem 0.3. □

Lemma 6.6. *If the outer face is a quadrilateral $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4$ with $\mathbf{p}_1 = \binom{0}{0}$, $\mathbf{p}_2 = \binom{1}{0}$, $\mathbf{p}_3 = \binom{x_3}{y_3}$, and $\mathbf{p}_4 = \binom{0}{1}$, then an equilibrium stress on the interior edges and vertices can be extended to an equilibrium stress on the whole framework if and only if*

$$x_3 + y_3 = 1 + \frac{\tilde{\omega}_{13}}{\tilde{\omega}_{24}} \cdot x_3 y_3\,.$$

Proof. The equilibrium equation for $\mathbf{p}_1$ is

$$f_1 + \omega_{12}(\mathbf{p}_2 - \mathbf{p}_1) + \omega_{14}(\mathbf{p}_4 - \mathbf{p}_1) = 0,$$

with $f_1 = \sum_{k=2,3,4} \tilde{\omega}_{1k}(\mathbf{p}_k - \mathbf{p}_1)$, and the equations for $\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$ are similar.

If we solve the system of equilibrium equations for the four boundary vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$ for given values of $\tilde{\omega}$, we obtain the following result:

$$\begin{aligned}
\omega_{12} &= -\tilde{\omega}_{13}x_3 - \tilde{\omega}_{12} \\
\omega_{23} &= -\frac{\tilde{\omega}_{24}}{y_3} - \tilde{\omega}_{23} \\
\omega_{34} &= -\frac{\tilde{\omega}_{24}}{x_3} - \tilde{\omega}_{34} \\
\omega_{14} &= -\tilde{\omega}_{13}y_3 - \tilde{\omega}_{14} \\
x_3 + y_3 &= 1 + \frac{\tilde{\omega}_{13}}{\tilde{\omega}_{24}} \cdot x_3 y_3
\end{aligned}$$

It is clear that the values $\tilde{\omega}_{12}$, $\tilde{\omega}_{23}$, $\tilde{\omega}_{34}$, $\tilde{\omega}_{41}$ do not matter, because their effect can be cancelled individually by modifying ω_{12}, ω_{23}, ω_{34}, ω_{41} appropriately, as in the triangular case.

Note that the assumption about the locations of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_4$ is no loss of generality, since it can always be obtained by an appropriate affine transformation. □

Lemma 6.7. *If the planar framework $G[\mathbf{p}]$ is 3-connected, then $\tilde{\omega}_{24} > 0$ and $\tilde{\omega}_{13} > 0$.*

Proof. Since $G[\mathbf{p}]$ is 3-connected, there exists a path between the boundary vertices $\mathbf{p}_2$ and $\mathbf{p}_4$, otherwise we could disconnect them just by removing $\mathbf{p}_1$ and $\mathbf{p}_3$. Hence, $\tilde{\omega}_{24} > 0$. Analogously, there exists a path between $\mathbf{p}_1$ and $\mathbf{p}_3$, and hence $\tilde{\omega}_{13} > 0$. □

When the outer face is a k-gon, $k \geq 5$, we do not know which conditions must satisfy an interior equilibrium stress for being extended to the boundary. In particular, this question is still open for graphs with a pentagonal outer face: in this case we could not find a solution like in Lemma 6.6. The situation is more complicated because it depends on five values of $\tilde{\omega}_{ij}$, and not just two values $\tilde{\omega}_{13}$ and $\tilde{\omega}_{24}$ as in Lemma 6.6.

6.3.2 Realizations of Polytopes

We realize a polytope P with edge graph G by first computing an internal equilibrium stress on G using Tutte's Theorem, extending this stress to the boundary, and lifting the configuration using the approach of Richter Gebert [48] described in Section 0.3.

We bound the z-coordinates of the obtained liftings and, in order to embed the vertices on the integer grid, we blow up the obtained coordinates. We need the following lemma.

Lemma 6.8. *If we scale the x- and y-coordinates of the vertices of a 3-polytope P by a factor μ, then the z-coordinates are scaled by a factor μ^2.*

Proof. Let $\mathbf{p} = (\mathbf{p}_x, \mathbf{p}_y, 1)$ be any point of the planar configuration that lifts to the polytope P. As described in Section 0.3, the z-coordinate of $\mathbf{p}$ is obtained from the height function h:

$$h(\mathbf{p}) = \langle \mathbf{p}, \mathbf{q}_i \rangle \qquad \text{if} \quad \mathbf{p} \in c_i\,.$$

The vectors $\mathbf{q}_i$ are computed recursively, choosing a sequence of cells from $\mathbf{q}_0$ to $\mathbf{q}_i$ and walking along this sequence, such that $\mathbf{q}_i = \mathbf{q}_L$ is computed from $\mathbf{q}_{i-1} = \mathbf{q}_R$, when we leave the cell R to enter the cell L crossing the edge (b, t), as

$$\mathbf{q}_L = \omega_{bt}(\mathbf{p}_b \times \mathbf{p}_t) + \mathbf{q}_R \,.$$

By construction, if we scale the x- and y-coordinates of the points $\mathbf{p}_1, \ldots, \mathbf{p}_n$ of the planar configuration embedded at the plane $z = 1$ by a factor μ, the components $(\mathbf{q}_x, \mathbf{q}_y, \mathbf{q}_z)$ of the vector $\mathbf{q}_i$ are scaled by

$$(\mu\,\mathbf{q}_x, \mu\,\mathbf{q}_y, \mu^2\mathbf{q}_z).$$

Then we have

$$h((\mu\,\mathbf{p}_x, \mu\,\mathbf{p}_y, 1)) = \big\langle(\mu\,\mathbf{p}_x, \mu\,\mathbf{p}_y, 1), (\mu\,\mathbf{q}_x, \mu\,\mathbf{q}_y, \mu^2\mathbf{q}_z)\big\rangle = \mu^2 h(\mathbf{p})\,.$$

Therefore the z-coordinate of $\mathbf{p}$ is scaled by μ^2. □

Realizations of Polytopes with a Triangular Facet

Lemma 6.9. *A 3-polytope with n vertices and with a triangular facet can be realized with integral vertex coordinates (x_i, y_i, z_i) in the range*

$$\begin{aligned} 0 \le \quad x_i, y_i \quad &< (n-1)^2\, T(G) \\ 0 \le \quad z_i \quad &< \frac{1}{6}\,(n-3)(n-1)^4\, T(G)^2. \end{aligned}$$

Proof. Assume the outer face of the edge graph of the polytope is a triangle. Fix the outer triangular face to the coordinate points $\mathbf{p}_1 = \binom{0}{0}$, $\mathbf{p}_2 = \binom{1}{0}$, $\mathbf{p}_3 = \binom{0}{1}$, and set the interior stresses to 1. Figure 6.1 shows an equilibrium configuration for a framework with outer triangular face, and a corresponding lifting.

To bound the z-coordinates of the lifting, note that the lifted surface is beneath the triangular pyramid formed by extending the three faces adjacent to the boundary. (They are shaded in the figure.) Their slopes are determined by the stresses ω_{12}, ω_{23}, and ω_{13}, and it is easy to see that the tip of the pyramid, which is the intersection of these three planes, has a height of

$$H = \frac{1}{\frac{1}{|\omega_{12}|} + \frac{1}{|\omega_{23}|} + \frac{1}{|\omega_{13}|}} = \frac{1}{\frac{1}{\tilde\omega_{12}} + \frac{1}{\tilde\omega_{23}} + \frac{1}{\tilde\omega_{13}}},$$

which is strictly bounded by $(n-3)/6$. To see this, use the inequalities $\tilde\omega_{12} + \tilde\omega_{13} < n-3$, $\tilde\omega_{12} + \tilde\omega_{23} < n-3$ and $\tilde\omega_{13} + \tilde\omega_{23} < n-3$, from Corollary 6.1. This implies $1/\tilde\omega_{12} + 1/\tilde\omega_{13} > 4/(n-3)$, $1/\tilde\omega_{12} + 1/\tilde\omega_{23} > 4/(n-3)$ and $1/\tilde\omega_{13} + 1/\tilde\omega_{23} > 4/(n-3)$. Adding these three inequalities, we obtain $1/\tilde\omega_{12} + 1/\tilde\omega_{13} + 1/\tilde\omega_{23} > 6/(n-3)$, and therefore $H < (n-3)/6$.

Since all interior vertices are interior to the peripheral convex polygon, their largest coordinate must be in the range $[0, 1]$. Now, let us blow up all x- and y-coordinates by the common denominator, which is the determinant of the reduced Laplacian matrix $\bar L_3$ obtained from L by removing the rows and columns corresponding to the boundary vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$. By Lemma 6.2, this yields integer coordinates. And by Lemma 6.8, this yields integral z-coordinates as well, which have been multiplied by $(\det \bar L_3)^2$. Hence,

$$0 \le x_i, y_i \le \det \bar L_3, \quad 0 \le z_i < (n-3)(\det \bar L_3)^2/6.$$

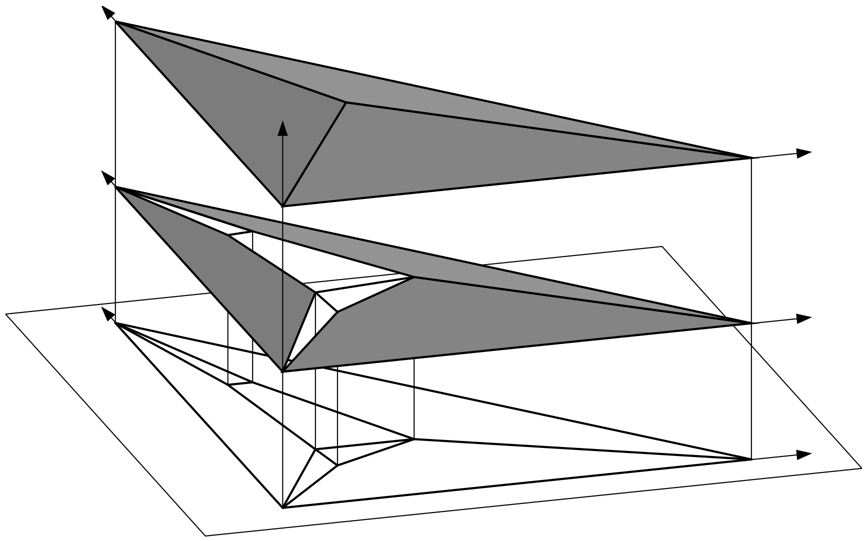

Figure 6.1: A graph with a triangular outer face, its lifting, and on top, the containing pyramid formed by the planes of the faces adjacent to the outer face.

We write $\det \bar{L}_3$ in terms of the number of spanning trees of G using Theorem 6.2 and Lemma 5.12, and the lemma follows. □

The method can also be applied if there is a triangle which is not a face: We can cut the polytope at this triangle, realize the two parts separately and glue them together, one on the top and one on the bottom:

Lemma 6.10. *A 3-polytope with n vertices whose graph contains a triangle can be realized with integral vertex coordinates (x_i, y_i, z_i) in the range*

$$\begin{aligned} 0 \leq \quad x_i, y_i \quad &< (n-1)^2\, T(G) \\ 0 \leq \quad z_i \quad &< \frac{1}{6}\,(n-3)(n-1)^4\, T(G)^2. \end{aligned}$$

Proof. We use Part 3 of Theorem 0.3. For estimating the combined z-range from the two halves, let n_1 and n_2 be the number of vertices of each of them respectively. Then, using Lemma 6.9,

$$z_i < (n_1 - 3)(\det \bar{L}_3)^2/6 + (n_2 - 3)(\det \bar{L}_3)^2/6 = (n-3)(\det \bar{L}_3)^2/6.$$

and by Theorem 6.2 and Lemma 5.12 we obtain again the bounds. □

Theorem 6.3. *A 3-polytope with n vertices whose graph G contains a triangle can be realized with integral vertex coordinates (x_i, y_i, z_i) in the range*

$$\begin{aligned} 0 \leq \quad x_i, y_i \quad &< n^2 \left(\frac{16}{3}\right)^n = n^2\, 5.\bar{3}^n \\ 0 \leq \quad z_i \quad &< \frac{1}{6}\, n^5 \left(\frac{16}{3}\right)^{2n} = \frac{1}{6}\, n^5\, 28.\bar{4}^n. \end{aligned}$$

Proof. Substituting in Lemma 6.10 the bounds on the number of spanning trees given in Theorem 5.7, we obtain

$$0 \leq x_i, y_i \leq (n-1)^2 \left(\frac{16}{3}\right)^n < n^2 \left(\frac{16}{3}\right)^n \tag{6.3}$$

and

$$0 \leq z_i < \frac{1}{6}(n-3)(n-1)^4 \left(\frac{16}{3}\right)^{2n} < \frac{1}{6} n^5 \left(\frac{16}{3}\right)^{2n}. \tag{6.4}$$

□

Realizations of Polytopes with a Quadrilateral Facet

Lemma 6.11. *A 3-polytope with n vertices whose graph G contains no triangle but a quadrilateral face can be realized with integral vertex coordinates (x_i, y_i, z_i) in the range*

$$\begin{aligned} |x_i| &\leq 2n(n-1)^6\, T(G)^2, \\ |y_i| &\leq n(n-1)^6\, T(G)^2, \\ |z_i| &< \tfrac{1}{3} n^2 (n-4)(n-1)^{12}\, T(G)^4. \end{aligned}$$

Proof. Assume the outer face of the edge graph of the polytope is a quadrilateral $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4$, with fixed coordinate points $\mathbf{p}_1 = \binom{0}{0}$, $\mathbf{p}_2 = \binom{1}{0}$, $\mathbf{p}_3 = \binom{x_3}{y_3}$, and $\mathbf{p}_4 = \binom{0}{1}$, x_3, y_3 being such that $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4$ is convex.

We apply Lemma 6.6 as follows: By relabeling the points, we may assume that $\tilde{\omega}_{13} \geq \tilde{\omega}_{24}$. Then we set $x_3 := 2$ and

$$y_3 := \frac{\tilde{\omega}_{24}}{2\tilde{\omega}_{13} - \tilde{\omega}_{24}}. \tag{6.5}$$

We have $0 < y_3 \leq 1$. Note that the first inequality holds since the outer face is convex, and it is true if and only if $\tilde{\omega}_{24} > 0$, which fits with Lemma 6.7.

By Part 2 of Lemma 6.3, the weights $\tilde{\omega}$ are rational numbers with denominator the determinant of the reduced Laplacian matrix $\bar{L}_4$, after removing the four rows and columns corresponding to the boundary vertices. In particular, $\tilde{\omega}_{13}$ and $\tilde{\omega}_{24}$ are rational numbers with a common denominator $\det \bar{L}_4$.

Let D_y be the denominator of the rational number y_3, after cancelling the common denominator $\det \bar{L}_4$. If we multiply all coordinates by D_y and then by $\det \bar{L}_4$, we get integral coordinates for all $\mathbf{p}_i$ in the range $0 \leq x_i \leq 2D_y \det \bar{L}_4$ and $0 \leq y_i \leq D_y \det \bar{L}_4$.

Since $\tilde{\omega}_{13}$ and $\tilde{\omega}_{24}$ have common denominator $\det \bar{L}_4$, D_y is bounded by twice the numerator of $\tilde{\omega}_{13}$, which is at most $(n-4)\det \bar{L}_4$, by Corollary 6.2:

$$D_y \leq 2(n-4)\det \bar{L}_4\,.$$

We want to bound now the height of the tip of the 4-face polytope formed by extending the four faces adjacent to the boundary. Suppose we rotate the edge $\{3,4\}$ counterclockwise about $\mathbf{p}_4$ until $\mathbf{p}_3\mathbf{p}_4$ is parallel to $\mathbf{p}_1\mathbf{p}_2$, keeping the slope of the incident face fixed, i. e., $y_3 = 1$ and $\mathbf{p}_3 = (2, 1)$. Since $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{p}_3\mathbf{p}_4$ are parallel, the intersection of their incident faces is a line

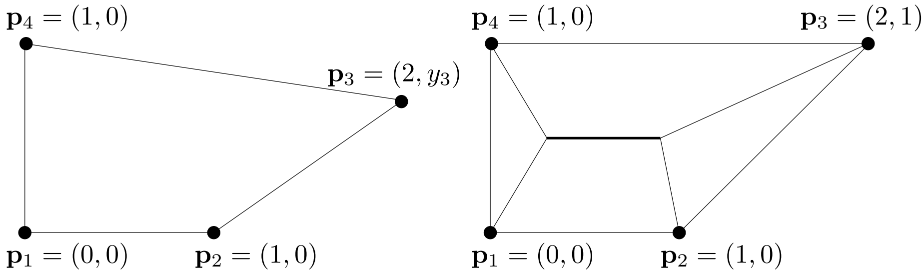

Figure 6.2: Left: the original position of the boundary points. Right: Position of the boundary points after the rotation, together with the projection of the 4-face polytope formed by extending the four faces adjacent to the boundary. The intersection of the faces incident to $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{p}_3\mathbf{p}_4$ is drawn thick.

ℓ parallel to the plane defined by $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4$. See Figure 6.2. Now, the face incident to $\{3,4\}$ lies above the old face before the rotation, and therefore the height of ℓ is an upper bound for the tip of the original 4-face polytope.

The height of ℓ is given by

$$H = \frac{1}{\frac{1}{|\omega_{34}|} + \frac{2}{|\omega_{12}|}} \leq \frac{1}{3} \max\{|\omega_{34}|, |\omega_{12}|\} \leq \frac{1}{3} \max\{|\tilde{\omega}_{34}|, |\tilde{\omega}_{12}|\},$$

which, by Corollary 6.2, is strictly bounded by $(n-4)/3$.

Therefore, after scaling the z-coordinate using Lemma 6.8, we have

$$|x_i| \leq 2n\,(\det \bar{L}_4)^2$$
$$|y_i| \leq n\,(\det \bar{L}_4)^2$$
$$|z_i| < \left(n(\det \bar{L}_4)^2\right)^2 \frac{n-4}{3}$$

By Theorem 6.2 and Lemma 5.13, we write $\det \bar{L}_4$ in terms of the number of spanning trees of G, obtaining the result. □

If there is a 4-gon which is not a face, we can try to apply the same trick as in the case where the polytope contains a triangle: to cut the polytope at this 4-gon, realize the two parts separately and glue them together, one on the top and one on the bottom. But the bound on the z-coordinate is not as easy to compute as before, since the quadrilateral face that we want to glue is not necessarily horizontal, hence the stresses of the two parts must be computed jointly.

Theorem 6.4. *A 3-polytope with n vertices with no triangular facet but a quadrilateral facet can be realized with integral vertex coordinates (x_i, y_i, z_i) in the range*

$$|x_i| < 2n^7 \left[\exp\left(\ln 4 - \frac{1}{8}\right)\right]^{2n} < 2n^7\, 12.460815^n,$$
$$|y_i| < n^7 \left[\exp\left(\ln 4 - \frac{1}{8}\right)\right]^{2n} < n^7\, 12.460815^n,$$
$$|z_i| < \frac{1}{3} n^{15} \left[\exp\left(\ln 4 - \frac{1}{8}\right)\right]^{4n} < \frac{1}{3} n^{15}\, 155.271910^n.$$

Proof. If we substitute in Lemma 6.11 the bounds on the number of spanning trees given in Theorem 5.10, we obtain

$$\begin{aligned}|x_i| &< 2n(n-1)^6 \exp\left(\ln 4 - \frac{1}{8}\right)^{2n}\\ |y_i| &< n(n-1)^6 \exp\left(\ln 4 - \frac{1}{8}\right)^{2n}\\ |z_i| &< \frac{1}{3}\, n^2(n-4)(n-1)^{12} \exp\left(\ln 4 - \frac{1}{8}\right)^{4n}\end{aligned}$$

and the theorem follows. □

Realizations of Polytopes with no Triangular Facet and no Quadrilateral Facet

In this case we could not find a solution like in Theorem 6.4, because we do not know how to extend an interior equilibrium stress to the boundary vertices. The smallest face cycle of the edge graph G of the polytope has length 5, because G is a planar graph without 3-faces and 4-faces with a planar dual graph of average degree less than 6, thus the dual graph has minimum degree 5, and hence G must contain a pentagonal face. The situation is more complicated because it depends on five values of $\tilde{\omega}_{ij}$, and not just two values $\tilde{\omega}_{13}$ and $\tilde{\omega}_{24}$ as in Lemma 6.6.

Therefore, in this case we have to resort to realizing the polar polytope first, as suggested by Richter-Gebert [48] for the case when G has no triangle. However, we obtain an improved bound even in this case, by applying Theorem 5.11.

Theorem 6.5. *Let P be a 3-polytope with n vertices that has no triangular facet and no quadrilateral facet. Then there is a realization of P with integral vertex coordinates of absolute values less than $n^{10n}\, 2^{10n^2} < 2^{12n^2}$.*

Proof. Let G the edge graph of P and let G^Δ be its dual graph. The graph G^Δ contains a triangular face. Hence the coordinate bound for the polar polytope P^Δ is achieved by Lemma 6.9. The height of the vertices of P^Δ lies in the range

$$0 \le z_i < \frac{1}{6}\,(n(G^\Delta)-3)(n(G^\Delta)-1)^4\, T(G^\Delta)^2.$$

where $n(G^\Delta)$ is the number of vertices of the dual graph G^Δ. By Lemma 5.3, $T(G^\Delta) = T(G)$. Since G has smallest face cycle 5, by Theorem 5.11 we have

$$T(G) < \exp\left(\frac{2}{3}\ln 3 + \frac{1}{3}\ln 4 - \frac{4}{27}\right)^n < 2.847263^n.$$

Hence,

$$\begin{aligned}0 \le z_i &< \frac{1}{6}\, n(G^\Delta)^5\, T(G)^2\\ &< \frac{1}{6}\, n^{\frac{10}{3}} \exp\left(\frac{2}{3}\ln 3 + \frac{1}{3}\ln 4 - \frac{4}{27}\right)^{2n}\\ &< \frac{1}{6}\, n^{\frac{10}{3}} 8.106907^n.\end{aligned}$$

We have used that, since G has smallest face cycle 5, $n(G^{\Delta}) \leq 2/3n - 4/3$.

It can be shown that there is a realization $\mathbf{P}^{\Delta}$ of P^{Δ} with integral vertex coordinates of absolute values less than $\frac{1}{6}\,n^{10/3}8.106907^n$ that contains the origin in its interior. The proof is analogous to the proof of Lemma 13.2.5 in [48]. It consists in showing that the generated polytope contains a grid point g in its interior, and then apply a translation that moves g to the origin.

Now we proceed similarly as in [48, Lemma 13.2.6]. The hyperplanes that support the faces $f_1, \ldots, f_n$ of this polytope are of the form

$$H_i := \left\{(x, y, z) \in \mathbb{R}^3 \,|\, a_i x + b_i y + c_i z + 1 = 0\right\}.$$

(a_i, b_i, c_i) for $i = 1, \ldots, n$ are the vertices of the desired realization $\mathbf{P}$ of P. They can be calculated by solving the system of equations $a_i x_{i_j} + b_i y_{i_j} + c_i h_{i_j} + 1 = 0$. Here $(x_{i_j}, y_{i_j}, h_{i_j})$ for $j = 1, \ldots, 3$ are the coordinates of points in $\mathbf{P}^{\Delta}$ that lie on f_i. Solving these equations by Cramer's rule leads to coefficients a_i, b_i, c_i of the form

$$\det \begin{pmatrix} * & * & * \\ * & * & * \\ 1 & 1 & 1 \end{pmatrix} \Big/ \det \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix},$$

where $*$ are entries of absolute value less than $\frac{1}{6}\,n^{10/3}8.106907^n$. The divisor is always identical for a fixed index i. The absolute value of the dividend and the divisor is strictly less than $6 \cdot \left(1/6\,n^{10/3}8.106907^n\right)^3 = 1/36\,n^{10}\,8.106907^{3n}$. Multiplying all the points (a_i, b_i, c_i) by the divisors of the remaining points, we can transform the rational coordinates (the above quotient) into integral coordinates of absolute value less than

$$\begin{aligned} \left(\frac{1}{36}\,n^{10}\,8.106907^{3n}\right)^n &< n^{10n}\left(\left(\frac{1}{36}\right)^{\frac{1}{n}} 8.106907^3\right)^{n^2} \\ &< n^{10n}\,533^{n^2} \\ &< (4.5 \cdot 533)^{n^2} \\ &< 2^{12n^2}. \end{aligned}$$

The third inequality holds since $n^{10} < 4.5^n$, for each $n \geq 20$. Note that $n \geq 20$, since the smallest polytope with neither triangular nor quadrilateral facets is the dodecahedron, with 20 vertices. From the second inequality we could also get a bound of $n^{10n}\,2^{10n^2}$, which is also less than 2^{12n^2}. □

This theorem improves the result by Richter Gebert [48] which states that any 3-polytope with a triangular facet can be realized with positive integral vertex coordinates of absolute value less than 2^{18n^2}. Actually, Richter-Gebert mentions that with a more careful analysis he can also prove a bound of 2^{13n^2}.