Chapter 4

A Generalization of the Maxwell-Cremona Theorem for Self-Touching Configurations

4.1 Introduction

The classic Maxwell-Cremona Theorem [20, 21, 22, 48, 56] is a powerful tool that establishes a bijection between the set of classical equilibrium stresses of a configuration in $\mathbb{R}^2$ and the set of three-dimensional polyhedral terrains in $\mathbb{R}^3$ that project onto it.

In this chapter we study how this theorem translates to the case of self-touching configurations. We present a generalization of the Maxwell-Cremona Theorem and establish a correspondence between the set of stresses of a planar self-touching configuration and the set of generalized three-dimensional polyhedral terrains, that is, three-dimensional polyhedral terrains with jump discontinuities, that project onto it.

The *lifting*, that is, the direction from the self-touching stress to the generalized three-dimensional polyhedral terrain, is unique up to the addition of a linear function, but the *projection*, which is the direction from the generalized polyhedral terrain to the self-touching stress, is in general not unique.

In [16], the authors use the Maxwell-Cremona Theorem for solving the Carpenter's Rule Problem. Inspired in their idea, we developed this generalization of the Maxwell-Cremona Theorem as a tool for proving Theorem 3.1, but we did not succeed. Maybe in the future will be found some nice application of our generalization.

4.2 Basics

4.2.1 The Cell Decomposition

Suppose we are given a self-touching configuration, which we can assume to be connected. We distinguish between *vertices* and *points*, the converging positions of the vertices. We denote by $\mathbf{p}_v$ the coordinates of a vertex v. We assume that the configuration is embedded in $\mathbb{R}^3$ in the plane $z = 1$, so that each point $\mathbf{p}_v$ has coordinates $(x_v, y_v, 1)$.

The points of the self-touching configuration induce a cell decomposition: if it is connected, the peripheral cell is realized as a polygon and the interior cells form a proper cell decomposition of it by polygons. See Figure 4.1 for an example. Let c_i, $i = 1, \ldots, m$, be the interior cells, and let c_0 be the exterior cell.

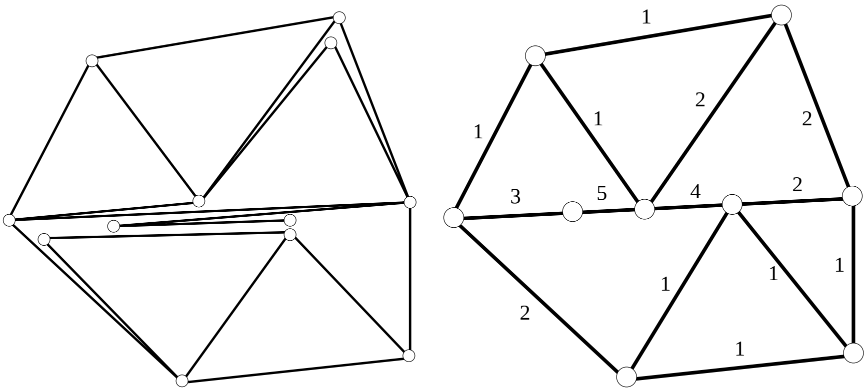


Figure 4.1: Left: Vertices and bars of a self-touching configuration. Right: points and edges of the corresponding cell decomposition. Numbers denote edge multiplicities.

Several bars of the linkage may converge to the same overlapping position. We distinguish between *edges* of the cell decomposition, *bars* of the self-touching configuration, and *pieces* of bars. The cell decomposition induces a subdivision of the bars of the self-touching configuration into pieces. See an example in Figure 4.2. The edges are the segments in the cell decomposition. In general, an edge of the cell decomposition represents several overlapping pieces of bars. Given an edge $e \in E$, let S_e be the set of overlapping pieces lying along e, and let $\lambda_e = |S_e|$ be the *multiplicity* of e. Note that given an oriented edge $e = (\beta, \tau)$, for each oriented piece $(b, t) \in S_{\beta\tau}$ we have $\mathbf{p}_\beta = \mathbf{p}_b$ and $\mathbf{p}_\tau = \mathbf{p}_t$. Let E, B, S be the sets of edges, bars and pieces respectively. Along the chapter, edges, bars and pieces are assumed to be oriented in some arbitrary way, to be able to distinguish their different sides, left and right.

We introduce *auxiliary vertices* on the interior of the bars, delimiting the pieces. Let A be the set of auxiliary vertices. Obviously, two auxiliary vertices on the same bar have different coordinates. Note that a point of the cell decomposition can represent several vertices of V and several auxiliary vertices of A.

Let (b, t) be an oriented piece. We denote by $(\bar{b}, \bar{t})$ its *supporting bar*, i. e., the oriented bar containing the piece. See Figure 4.2.

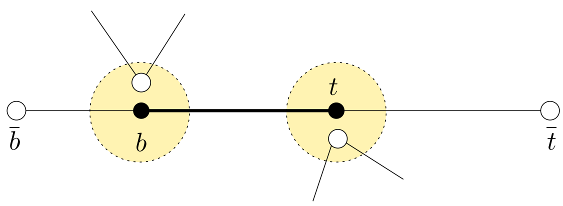


Figure 4.2: A piece (b, t) and its supporting bar $(\bar{b}, \bar{t})$. Vertices are drawn in white and auxiliary vertices in black (we use this convention in the whole chapter).

4.2.2 The Self-Touching Forces

Given a self-touching configuration, we say that the triplet $\{k; i, j\}$ belongs to $\mathcal{L}$ if $\mathbf{p}_k$ is a vertex pushing against the directed bar $(\mathbf{p}_i, \mathbf{p}_j)$ and $\mathbf{p}_k$ must remain on the left side of the line through $\mathbf{p}_i$ and $\mathbf{p}_j$ in order to maintain the combinatorial planar embedding. Note that $\{k; j, i\} \in \mathcal{L}$ means that $\mathbf{p}_k$ pushes against the bar $(\mathbf{p}_i, \mathbf{p}_j)$ and $\mathbf{p}_k$ remains on the *right* side of the line through $\mathbf{p}_i$ and $\mathbf{p}_j$.

Connelly, Demaine and Rote [17] generalized the stresses defined in Section 0.2 for classic configurations to self-touching configurations. Given a self-touching configuration, in addition to the classic stress on the bars, we assign to each triplet $\{k; i, j\}$ a weight $F = \omega_{kij}$, representing the force that $\mathbf{p}_k$ transmits to the bar $(\mathbf{p}_i, \mathbf{p}_j)$. The scalars $F = \omega_{kij}$ are called *self-touching stresses*, and they distribute proportionally as represented in Figure 4.3. We use the representation of $\mathbf{p}_k$ as a convex combination of $\mathbf{p}_i$ and $\mathbf{p}_j$,

$$\mathbf{p}_k = \alpha \mathbf{p}_i + (1-\alpha)\mathbf{p}_j \qquad \forall \{k; i, j\} \in \mathcal{L},$$

where $\alpha = \alpha_{kij}$ is such that $0 < \alpha < 1$. The vertex $\mathbf{p}_k$ feels a force of value F in the direction $(\mathbf{p}_j - \mathbf{p}_i)^{\perp}$, perpendicular to the bar $(\mathbf{p}_i, \mathbf{p}_j)$ and pointing to the left side of it, where $\mathbf{p}_k$ is restricted to move. The vertices $\mathbf{p}_i$ and $\mathbf{p}_j$ feel a proportional force of values $\alpha_{kij}F$ and $(1-\alpha_{kij})F$ respectively, in the opposite direction.

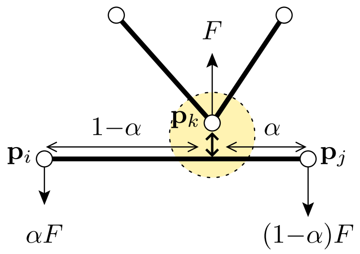


Figure 4.3: A touching incidence (small double arrow) and proportional distribution of self-touching stress F (single arrows). Bold edges denote bars.

We denote by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}^{\perp} = \begin{pmatrix} -y \\ x \\ z \end{pmatrix}$$

a counterclockwise rotation by 90°, parallel to the plane xy.

In general, a vertex v of the configuration can be involved in several self-touching stresses. Let $\mathbf{F}_{ST}(v)$ be the resulting self-touching stress at v, that is, the resulting force of, first, the force that v feels from pushing against other bars, and second, the proportional distribution at v of all the self-touching stresses pushing against both left and right sides of the bars incident to v. This is expressed formally as

$$\mathbf{F}_{ST}(v) = \sum_{i,j:\{v;i,j\}\in\mathcal{L}} \omega_{vij}(\mathbf{p}_j - \mathbf{p}_i)^{\perp}$$
$$+ \sum_{j:(v,j)\in B} \left(\sum_{k:\{k;v,j\}\in\mathcal{L}} -\alpha_{kvj}\omega_{kvj}(\mathbf{p}_j - \mathbf{p}_v)^{\perp} + \sum_{k:\{k;j,v\}\in\mathcal{L}} (1-\alpha_{kjv})\omega_{kjv}(\mathbf{p}_j - \mathbf{p}_v)^{\perp} \right).$$

We say that the configuration is *in equilibrium* if all forces, i. e., classical stresses on the bars and self-touching stresses, add up to zero at every vertex v, that is

$$\mathbf{F}_{ST}(v) + \sum_{j:(v,j)\in B} \omega_{vj}(\mathbf{p}_j - \mathbf{p}_v) = 0\,, \qquad \forall v \in V.$$

4.3 From Stressed Self-Touching Configurations to Polyhedra

We generalize to the self-touching case the approach of Richter-Gebert [48], summarized in Section 0.3, for obtaining the Maxwell-Cremona correspondence for classical configurations.

4.3.1 The vectors $\mathbf{q}_i$

We are given a self-touching configuration in equilibrium stress. Remember that we assume that the configuration is embedded in $\mathbb{R}^3$ in the plane $z = 1$, so that each point $\mathbf{p}_i$ has coordinates $(x_i, y_i, 1)$.

Given an oriented edge (β, τ), of multiplicity λ, there is a unique adjacent cell L to the left of it, and a unique adjacent cell R to the right of it. We call the (ordered) tuple $(\beta, \tau \mid L, R)$ an *oriented patch*. (The letters are chosen as mnemonics as in [48] for $\beta =$ *bottom*, $\tau =$ *top*, $L =$ *left*, $R =$ *right*.) We have an extra information: we have all the overlapping pieces (b_s, t_s) lying along the edge (β, τ), and their corresponding supporting bars $(\bar{b}_s, \bar{t}_s)$, for each $s = 1, \ldots, \lambda$. Assume the pieces are embedded in order from the right cell R to the left cell L, such that (b_1, t_1) is directly adjacent to R and (b_λ, t_λ) is directly adjacent to L.

Now, to each cell c_i we associate a vector $\mathbf{q}_i \in \mathbb{R}^3$ by:

(i) $\mathbf{q}_0 = (0, 0, 0)$;

(ii) $\mathbf{q}_L = \mathbf{q}_R + \sum_{(b,t)\in S_{\beta\tau}} (\omega_{bt}(\mathbf{p}_\beta \times \mathbf{p}_\tau) + \Delta(b,t))$ if $(\beta, \tau \mid L, R)$ is an oriented patch.

where

$$\Delta(b,t) = \Big(\sum_{\substack{k:\{k;\bar{b},\bar{t}\}\in\mathcal{L} \\ k\in[t,\bar{t}]}} -\alpha_{k\bar{b}\bar{t}}\omega_{k\bar{b}\bar{t}} + \sum_{\substack{k:\{k;\bar{t},\bar{b}\}\in\mathcal{L} \\ k\in[t,\bar{t}]}} (1-\alpha_{k\bar{t}\bar{b}})\omega_{k\bar{t}\bar{b}} \Big) \mathbf{p}_{\bar{b}} \times (\mathbf{p}_{\bar{t}} - \mathbf{p}_{\bar{b}})^{\perp}$$
$$- \Big(\sum_{\substack{k:\{k;\bar{b},\bar{t}\}\in\mathcal{L} \\ k\in[\bar{b},b]}} -(1-\alpha_{k\bar{b}\bar{t}})\omega_{k\bar{b}\bar{t}} + \sum_{\substack{k:\{k;\bar{t},\bar{b}\}\in\mathcal{L} \\ k\in[\bar{b},b]}} \alpha_{k\bar{t}\bar{b}}\omega_{k\bar{t}\bar{b}} \Big) \mathbf{p}_{\bar{t}} \times (\mathbf{p}_{\bar{t}} - \mathbf{p}_{\bar{b}})^{\perp}.$$

In other words, $\Delta(b,t)$ is nothing else than the crossproduct of $\mathbf{p}_{\bar{b}}$ by the resultant force which $\bar{b}$ feels from the self-touching stress incident to both sides left and right of the segment

$[t, \bar{t}]$, minus the crossproduct of $\mathbf{p}_{\bar{t}}$ by the resultant force which $\bar{t}$ feels from the self-touching stress incident to both sides of the segment $[\bar{b}, b]$.

To save space, we introduce the following notation: $F_v^{[a,b]}$ denotes the total amount of force that the vertex v feels from the self-touching stress incident to both sides left and right of the segment $[a, b]$ (v, a and b are three vertices on the same bar). Then, $\Delta(b, t)$ can be rewritten as:

$$\Delta(b,t) = F_{\bar{b}}^{[t,\bar{t}]}\mathbf{p}_{\bar{b}} \times (\mathbf{p}_{\bar{t}} - \mathbf{p}_{\bar{b}})^{\perp} - F_{\bar{t}}^{[\bar{b},b]}\mathbf{p}_{\bar{t}} \times (\mathbf{p}_{\bar{t}} - \mathbf{p}_{\bar{b}})^{\perp}. \tag{4.1}$$

One can see in (4.1) that only vertices pushing against the interior of the bar contribute to $\Delta(b, t)$, i. e., there is no contribution to $\Delta(b, t)$ of the self-touching stress coming from vertices touching to $\bar{b}$ or $\bar{t}$.

We classify the pieces into the following types. We say that a piece (b, t) is of *type* (a) when $(b, t) = (\bar{b}, \bar{t})$, that is, when (b, t) is itself a bar with no vertex touching its interior, and hence both b and t are (not auxiliary) vertices of the configuration. We say that a piece (b, t) is of *type* (b) when $b = \bar{b}$ and $t \neq \bar{t}$, that is, when b is a end vertex of the supporting bar and t is an auxiliary vertex (remember that the piece is oriented). We say that a piece (b, t) is of *type* (c) when $b \neq \bar{b}$, that is, b is an auxiliary vertex. The vertex t can be auxiliary (if $t \neq \bar{t}$) or not (if $t = \bar{t}$). See Figure 4.4 for an example.

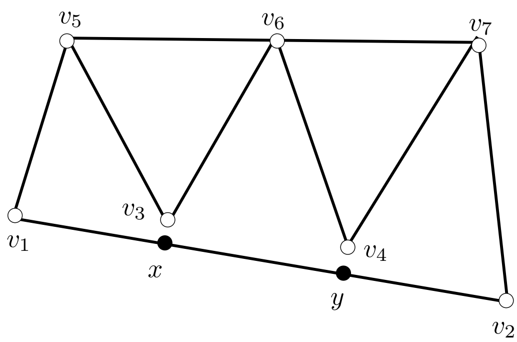


Figure 4.4: Cases for an oriented piece of bar. (v_1, x), (x, y), and (y, v_2) are pieces of the bar (v_1, v_2). The first one is of type (b), the other two are of type (c). The rest of bars are themselves pieces of type (a). With our notation, $(\overline{v_1}, \overline{x}) = (\overline{x}, \overline{y}) = (\overline{y}, \overline{v_2}) = (v_1, v_2)$, and, for example, $(\overline{v_1}, \overline{v_5}) = (v_1, v_5)$.

The next lemma shows how $\Delta(b, t)$ simplifies in the particular cases where (b, t) is a piece of type (a) or (b).

Lemma 4.1. *For a piece of bar (b, t), we have*

1. *If $(b, t) = (\bar{b}, \bar{t})$ is a piece of type (a), then*

$$\Delta(b, t) = 0\,;$$

2. *If $(b, t) = (\bar{b}, t)$ is a piece of type (b), then*

$$\Delta(b,t) = F_{\bar{b}}^{[t,\bar{t}]}\mathbf{p}_{\bar{b}} \times (\mathbf{p}_{\bar{t}} - \mathbf{p}_{\bar{b}})^{\perp}\,.$$

Proof. Part 1 is true by definition, since pieces of type (a) are bars without any self-touching stress incident to its interior. Part 2 follows since $b = \bar{b}$, thus trivially there is no self-touching stress at $[\bar{b}, b]$, so $F_{\bar{t}}^{[\bar{b},b]} = 0$. □

Lemma 4.2. *The vectors $\mathbf{q}_i$ are well-defined.*

For proving Lemma 4.2, we need the following result.

Lemma 4.3. *Any trip around an interior point of the cell decomposition generates a difference vector* $(0, 0, 0)$.

This is a straight forward calculation but very technical, and it is shown in Appendix A in order to avoid technical details here.

Proof of Lemma 4.2. The two oriented patches $(\beta, \tau \mid L, R)$ and $(\tau, \beta | R, L)$ define a consistent relation between $\mathbf{p}_\beta, \mathbf{p}_\tau, \mathbf{q}_L$, and $\mathbf{q}_R$:

$$\mathbf{q}_L = \mathbf{q}_R + \sum_{(b,t)\in S_{\beta\tau}} (\omega_{bt}(\mathbf{p}_\beta \times \mathbf{p}_\tau) + \Delta(b, t))$$
$$\Updownarrow$$
$$\mathbf{q}_R = \mathbf{q}_L + \sum_{(t,b)\in S_{\tau\beta}} (\omega_{tb}(\mathbf{p}_\tau \times \mathbf{p}_\beta) + \Delta(t, b))$$

This is true since $\omega_{bt} = \omega_{tb}$, since $S_{\beta\tau}$ and $S_{\tau\beta}$ contain the same pieces but with inverse orientations, and since

$$\Delta(b, t) = -\Delta(t, b)\,,$$

which follows from the definition (4.1), using that $-\mathbf{x}^\perp = (-\mathbf{x})^\perp$.

As in Richter-Gebert's book [48], we compute the vectors $\mathbf{q}_i$ recursively, choosing a sequence (path) of cells, from $\mathbf{q}_0$ to $\mathbf{q}_i$. Walking along this sequence, $\mathbf{q}_i = \mathbf{q}_L$ is computed from $\mathbf{q}_{i-1} = \mathbf{q}_R$ when we leave the cell R to enter the cell L, crossing the edge (β, τ) (we consider the oriented patch $(\beta, \tau \mid L, R)$).

To show that we obtain the same value for $\mathbf{q}_i$ independently from the path we choose, it suffices to prove that any trip around an interior point of the cell decomposition generates a difference vector $(0, 0, 0)$ [48], which holds by Lemma 4.3. □

4.3.2 The Lifting Function

First we introduce some notation. Given an oriented patch $(\beta, \tau \mid L, R)$, it can happen that some vertices of V converging to $\mathbf{p}_\beta$ or $\mathbf{p}_\tau$ are neither directly incident to c_R nor to c_L, but hidden between pieces of $S_{\beta\tau}$ (see Figure 4.5). Let x be such a vertex. We write $x \in c_R$, $x \in c_L$. We denote by $S^i_{\beta\tau}(x)$ be the set of pieces along (β, τ) crossed when we walk from the cell c_i to x, oriented like (β, τ) when leaving c_i $(i = L, R)$, and by $S^i_{\beta\tau}(x)^{-1}$ the same set of pieces but with inverse orientation. Note that $S^i_{\beta\tau}(v) = \emptyset$ corresponds to the case when v is directly incident to c_i. Note that

$$S_{\beta\tau} = S^R_{\beta\tau}(x) \cup S^L_{\beta\tau}(x)^{-1} \cup S_{\beta\tau}(x)\,, \tag{4.2}$$

where $S_{\beta\tau}(x)$ denotes the set of pieces along (β, τ) incident to x, and $\cup$ denotes here the disjoint union.

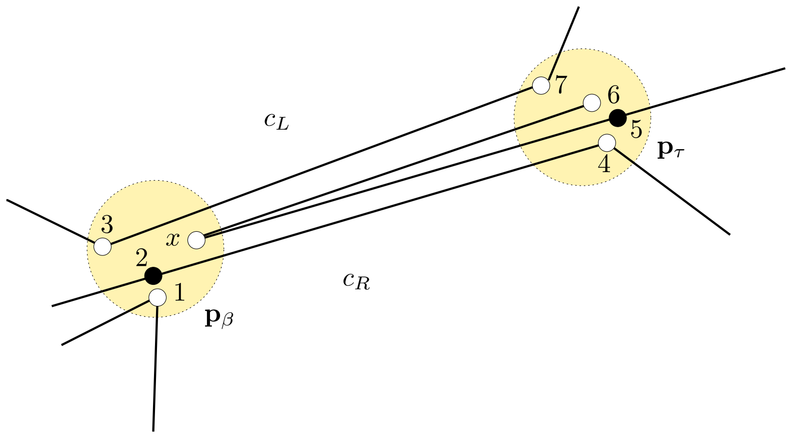


Figure 4.5: In this example, $S^R_{\beta\tau}(x) = \{(2, 4)\}$, $S^L_{\beta\tau}(x) = \{(7, 3)\}$ and $S_{\beta\tau}(x) = \{(x, 5), (x, 6)\}$.

The vectors $\mathbf{q}_i$ are now used to define the *lifting function* h, also called the *height function*. Let $\mathcal{C}$ be the self-touching configuration, composed of bars and vertices. Let $[\mathcal{C}]$ denote the set of points which is the union of all edges of the cell decomposition in $\mathbb{R}^2$ induced by $\mathcal{C}$. Then the height function is defined in the domain $D := \mathbb{R}^2 \setminus [\mathcal{C}] \cup \mathcal{C}$, contrary to the usual lifting for classic configurations, in which the height is defined in $\mathbb{R}^2$. Hence $h\colon D \to \mathbb{R}$ assigns a z–coordinate to each vertex of V, to each point on a bar of $\mathcal{C}$ and to each point of $\mathbb{R}^2 \setminus [\mathcal{C}]$. This means that different vertices or different bars can have different heights at the same geometric location.

The height of a vertex $x \in V$ is defined as

$$h(x) := \left\langle \mathbf{p}_x, \mathbf{q}_i + \sum_{(b,t)\in S^i_{\beta\tau}(x)} (\omega_{bt}(\mathbf{p}_\beta \times \mathbf{p}_\tau) + \Delta(b, t)) \right\rangle , \qquad \text{if } \; x \in c_i \, . \tag{4.3}$$

We obtain the height of the interior points of bars by linear interpolation from their end vertices. The height of the interior points can also be computed directly, by the linearity of the scalar product: if x is interior to a piece (b, t) that lies along an edge (β, τ), we can compute $h(x)$ directly from (4.3) ($S^i_{\beta\tau}(x)$ is analogously defined for interior points). In particular, cells lift to facets in the usual way, and if x belongs the closure of a cell (interior or directly incident to it), then (4.3) translates into

$$h(x) = \langle \mathbf{p}_x, \mathbf{q}_i \rangle \qquad \text{if } \; x \in c_i \, , \tag{4.4}$$

which, as expected, agrees with the height (2) given by the classic Maxwell-Cremona Theorem.

Lemma 4.4. *The function h is well-defined, that is, it defines a unique height for each vertex.*

Proof. Let b_0 be a vertex of V with $\mathbf{p}_{b_0} = \mathbf{p}_\beta$, where $(\beta, \tau \mid L, R)$ is an oriented patch of the configuration. We can reach b_0 walking from c_L or from c_R. We show that h attains the same value at b_0 when walking from both sides. We have

$$
\begin{aligned}
&\mathbf{q}_L + \sum_{(t,b)\in S^L_{\beta\tau}(b_0)} (\omega_{tb}(\mathbf{p}_\tau \times \mathbf{p}_\beta) + \Delta(t,b)) \\
&= \mathbf{q}_R + \sum_{(b,t)\in S_{\beta\tau}} (\omega_{bt}(\mathbf{p}_\beta \times \mathbf{p}_\tau) + \Delta(b,t)) + \sum_{(t,b)\in S^L_{\beta\tau}(b_0)} (\omega_{tb}(\mathbf{p}_\tau \times \mathbf{p}_\beta) + \Delta(t,b)) \\
&= \mathbf{q}_R + \sum_{(b,t)\in S_{\beta\tau}} (\omega_{bt}(\mathbf{p}_\beta \times \mathbf{p}_\tau) + \Delta(b,t)) - \sum_{(b,t)\in S^L_{\beta\tau}(b_0)^{-1}} (\omega_{bt}(\mathbf{p}_\beta \times \mathbf{p}_\tau) + \Delta(b,t)) \\
&= \mathbf{q}_R + \sum_{(b,t)\in S^R_{\beta\tau}(b_0)} (\omega_{bt}(\mathbf{p}_\beta \times \mathbf{p}_\tau) + \Delta(b,t)) + \sum_{(b_0,t)\in S_{\beta\tau}(b_0)} (\omega_{b_0 t}(\mathbf{p}_\beta \times \mathbf{p}_\tau) + \Delta(b_0,t))\,.
\end{aligned}
\tag{4.5}
$$

The first equality is (ii) on page 40. The second equality holds since $(t, b) \in S^L_{\beta\tau}(b_0)$ if and only if $(b, t) \in S^L_{\beta\tau}(b_0)^{-1}$, $\omega_{tb} = \omega_{bt}$, $\mathbf{p}_\tau \times \mathbf{p}_\beta = -\mathbf{p}_\beta \times \mathbf{p}_\tau$ and $\Delta(t, b) = -\Delta(b, t)$. The third equality holds by (4.2).

Since $\mathbf{p}_{b_0} = \mathbf{p}_\beta$, we have

$$\langle \mathbf{p}_{b_0}, \omega_{b_0 t}(\mathbf{p}_\beta \times \mathbf{p}_\tau)\rangle = 0\,,$$

and since $b_0 \in V$, then each piece $(b_0, t) \in S_{\beta\tau}(b_0)$ is of type (a) or (b). Hence, by Lemma 4.1 we have

$$\langle \mathbf{p}_{b_0}, \Delta(b_0, t)\rangle = 0 \qquad \forall (b_0, t) \in S_{\beta\tau}(b_0).$$

Now, applying the scalar product by $\mathbf{p}_{b_0}$ on both sides of (4.5), we obtain

$$
\begin{aligned}
&\left\langle \mathbf{p}_{b_0}, \mathbf{q}_L + \sum_{(t,b)\in S^L_{\beta\tau}(b_0)} (\omega_{tb}(\mathbf{p}_\tau \times \mathbf{p}_\beta) + \Delta(t,b)) \right\rangle \\
&= \left\langle \mathbf{p}_{b_0}, \mathbf{q}_R + \sum_{(b,t)\in S^R_{\beta\tau}(b_0)} (\omega_{bt}(\mathbf{p}_\beta \times \mathbf{p}_\tau) + \Delta(b,t)) \right\rangle,
\end{aligned}
$$

and on the right and left of the equality we have the height of b_0 walking from c_L or from c_R respectively. □

4.3.3 Generalized Polyhedral Terrains

Consider our height function h defined by (4.3), depending on two variables, the two plane coordinates. If the directional limit of h at a point p takes different finite values depending on the direction, the cell or the bar we come from, we say that the function has a *jump discontinuity* at p. The *jump* at p is the difference between two directional limits at p. Note that we can have one, two, three or even more directional limits at p.

The lifting is performed by applying to each vertex of the configuration the height function h. We obtain what we call a three-dimensional *generalized polyhedral terrain.* The difference with the classic polyhedral terrain obtained in the usual lifting is that in the generalized polyhedral terrain we can have concurrent bars and vertices with different heights. The obtained generalized polyhedral terrain has then jump discontinuities at those vertices and bars converging to the same geometric position in the self-touching configuration, as is it shown in this section.

The next lemma shows that each cell lifts to a planar polygon in $\mathbb{R}^3$ and that the self-touching configuration lifts to a generalized polyhedral terrain, with jump discontinuities at those edges affected by self-touching forces.

Lemma 4.5. *The height function h is piecewise linear with jump discontinuities at the bars with incident self-touching stress.*

Proof. By definition, h is piecewise linear, linear on each cell. Given an oriented patch $(\beta, \tau \mid L, R)$ affected by self-touching forces, let us compute the vertical jump discontinuity along (β, τ). First we compute the jump at each of the points $\mathbf{p}_\beta$ and $\mathbf{p}_\tau$.

Let $\delta_h(\beta)$ denote the *jump* δ_h at $\mathbf{p}_\beta$, that is, the difference of heights between two adjacent cells incident to $\mathbf{p}_\beta$ on the lifted polyhedron. For $\mathbf{p}_\beta$ we have

$$\begin{aligned}\langle \mathbf{p}_\beta, \mathbf{q}_L \rangle &= \langle \mathbf{p}_\beta, \mathbf{q}_R \rangle + \sum_{(b,t)\in S_{\beta\tau}} \omega_{bt} \langle \mathbf{p}_\beta, \mathbf{p}_b \times \mathbf{p}_t \rangle + \sum_{(b,t)\in S_{\beta\tau}} \langle \mathbf{p}_\beta, \Delta(b,t) \rangle \\ &= \langle \mathbf{p}_\beta, \mathbf{q}_R \rangle + \sum_{(b,t)\in S_{\beta\tau}} \langle \mathbf{p}_\beta, \Delta(b,t) \rangle .\end{aligned}$$

For the first equation we use the definition of $\mathbf{q}_L$ and the linearity of the scalar product. For the second equation we use that for each piece (b,t) in $S_{\beta\tau}$ we have $\mathbf{p}_b = \mathbf{p}_\beta$, and $\langle \mathbf{x}, \mathbf{x} \times \mathbf{y} \rangle = 0$. This shows that for adjacent cells without self-touching stress, the height agrees along the common bar. Now, for each piece (b,t), consider its supporting bar $(\bar{b}, \bar{t})$ and write $\mathbf{p}_b$ as $\mathbf{p}_b = \alpha \mathbf{p}_{\bar{b}} + (1-\alpha)\mathbf{p}_{\bar{t}}$ with $0 < \alpha < 1$. Then the jump at $\mathbf{p}_\beta$, $\delta_h(\beta)$, is given by

$$\begin{aligned}\delta_h(\beta) &= \langle \mathbf{p}_\beta, \mathbf{q}_L \rangle - \langle \mathbf{p}_\beta, \mathbf{q}_R \rangle \\ &= \sum_{(b,t)\in S_{\beta\tau}} \langle \mathbf{p}_b, \Delta(b,t) \rangle \\ &= -\sum_{(b,t)\in S_{\beta\tau}} \left((1-\alpha) F_{\bar{b}}^{[t,\bar{t}]} + \alpha F_{\bar{t}}^{[\bar{b},b]} \right) \left\langle \mathbf{p}_{\bar{b}}, \mathbf{p}_{\bar{t}} \times (\mathbf{p}_{\bar{t}} - \mathbf{p}_{\bar{b}})^\perp \right\rangle \\ &= -\sum_{(b,t)\in S_{\beta\tau}} \left((1-\alpha) F_{\bar{b}}^{[t,\bar{t}]} + \alpha F_{\bar{t}}^{[\bar{b},b]} \right) \|\mathbf{p}_{\bar{t}} - \mathbf{p}_{\bar{b}}\|^2 . \end{aligned} \tag{4.6}$$

We used $\langle \mathbf{x}, \mathbf{x} \times \mathbf{y} \rangle = 0$ and the fact that, for vectors in $\mathbb{R}^3$ embedded in $z = 0$, we have

$$\left\langle \mathbf{x}, \mathbf{y} \times (\mathbf{y} - \mathbf{x})^\perp \right\rangle = -\left\langle \mathbf{y}, \mathbf{x} \times (\mathbf{y} - \mathbf{x})^\perp \right\rangle = \|\mathbf{y} - \mathbf{x}\|^2 .$$

Similarly, we can compute $\delta_h(\tau)$, the jump at $\mathbf{p}_\tau$.

Any point $\mathbf{p}$ on (β, τ) can be written as the convex combination $\mathbf{p} = \lambda \mathbf{p}_\beta + (1-\lambda)\mathbf{p}_\tau$ with $0 < \lambda < 1$. Since the scalar product is linear, the jump at $\mathbf{p}$ is

$$\delta_h(\mathbf{p}) = \lambda \delta_h(\beta) + (1-\lambda)\delta_h(\tau) .$$

□

In the proof of Lemma 4.5 we have computed the jump between two facets separated by possibly several overlapping pieces of bars. In addition, these separating overlapping pieces are pushing against each other, thus there are jump discontinuities between any two of them if the self-touching stress in between is not zero. The difference of height from a piece (b,t) to the contiguous one is given by its corresponding summand in (4.6). Adding up the jumps between all pieces separating two cells, we obtain the total jump between the two corresponding lifted facets.

Given a bar (a,b) and an auxiliary vertex i on (a,b), let $L_{ab}(i)$ denote the set of touching vertices pushing against i from the left side of (a,b). Analogously, let $R_{ab}(i)$ denote the set of touching vertices pushing against i from the right side of (a,b). Note that, by definition of height (4.3), all vertices in $L_{ab}(i)$ (resp. $R_{ab}(i)$) have the same height $h(L_{ab}(i))$ (resp. $h(R_{ab}(i))$), since we cross the same pieces to reach them from an adjacent cell. That is, touching vertices converging to the same coordinate position lift to the same height. The height of the bar (a,b) at the coordinate position $\mathbf{p}_i$ equals the height $h(L_{ab}(i))$ when the set of vertices $L_{ab}(i)$ transmits to (a,b) no self-touching stress. If the set of vertices $L_{ab}(i)$ pushes against (a,b) with a non-zero self-touching stress, then $h(L_{ab}(i))$ lifts higher than the bar (a,b) at $\mathbf{p}_i$. Analogously, $h(R_{ab}(i))$ lifts higher than the bar (a,b) at $\mathbf{p}_i$ when $R_{ab}(i)$ pushes against (a,b) with a non-zero self-touching stress, otherwise it lifts to the same height as the bar. See Figure 4.6.

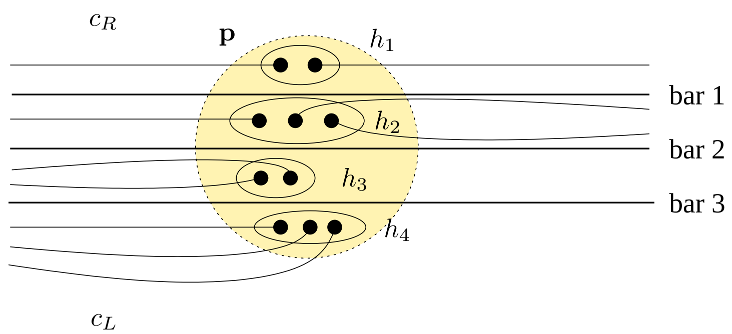


Figure 4.6: Heights of vertices converging to the same point $\mathbf{p}$. All converging vertices lying in the region between c_R and bar 1 have the same height h_1, those lying in the region between bar 1 and bar 2 have height h_2, those lying in the region between bar 2 and bar 3 have height h_3 and those lying in the region between bar 3 and c_L have height h_4. The heights h_1, h_2, h_3 and h_4 can be different. The height of the bar i at $\mathbf{p}$ is at most the minimum between h_i and h_{i+1}, $i = 1, 2, 3$.

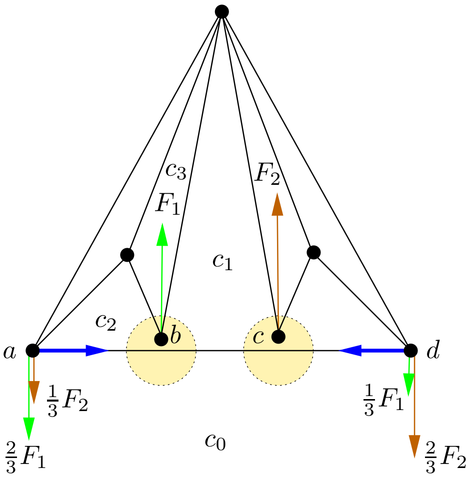
4.3.4 Some Examples of Lifted Self-Touching Configurations

We illustrate in Figure 4.7 and Figure 4.8 a couple of examples to give to the reader a more intuitive idea of how these liftings and jump discontinuities look.

4.3.5 Geometric Interpretation of $\mathbf{q}_L - \mathbf{q}_R$

When we apply the height function h, a cell with associated vector $\mathbf{q} = (q_x, q_y, q_z)$ lifts to a facet F (a planar polygon in $\mathbb{R}^3$) of the polyhedral terrain, contained in the plane

$$q_x x + q_y y - z + q_z = 0\,,$$



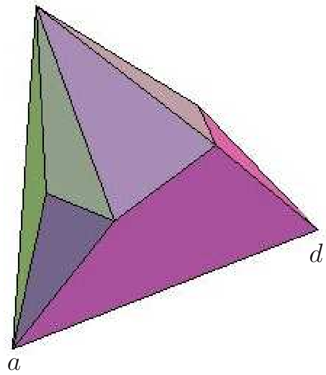


Figure 4.7: A self-touching configuration (left) and its lifting (right). The vertices b and c push against the bar (a, d) with self-touching forces of values $F_1 = \omega_{bad}$ and $F_2 = \omega_{cad}$ respectively. The proportional forces that the vertices a and d feel are indicated. The thick arrows on the bar (a, d) represent the classic stress ω_{ad} on this bar. In the lifting, all vertices incident to the exterior cell c_0 have height zero. The height of b can be computed from the vector $\mathbf{q}$ associated to any of the cells c_1, c_2 or c_3 to which is incident, as $h(b) = \langle \mathbf{p}_b, \mathbf{q}_1 \rangle = \langle \mathbf{p}_b, \mathbf{q}_2 \rangle = \langle \mathbf{p}_b, \mathbf{q}_3 \rangle$. The height of c can for example be computed as $h(c) = \langle \mathbf{p}_c, \mathbf{q}_1 \rangle$. Since F_2 is larger than F_1, c lifts higher than b, as one can see in the lifted polyhedron. The discontinuity at ad is represented in the lifting by a vertical facet $abcd$.

as it can be seen from (4.4). Then $\binom{q_x}{q_y}$ is the gradient vector of F and $\mathbf{n}_F = (-q_x, -q_y, 1)$ is its normal vector.

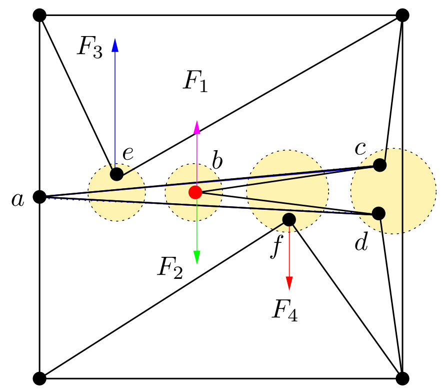
We want to understand geometrically how the vector $\mathbf{q}$ changes when walking from a cell c_R to an adjacent cell c_L. Recall that the difference $\mathbf{q}_L - \mathbf{q}_R$ is given by

$$\mathbf{q}_L - \mathbf{q}_R = \sum_{(b,t) \in S_{\beta\tau}} \left(\omega_{bt}(\mathbf{p}_\beta \times \mathbf{p}_\tau) + F_{\bar{b}}^{[t,\bar{t}]} \mathbf{p}_{\bar{b}} \times (\mathbf{p}_{\bar{t}} - \mathbf{p}_{\bar{b}})^\perp - F_{\bar{t}}^{[\bar{b},b]} \mathbf{p}_{\bar{t}} \times (\mathbf{p}_{\bar{t}} - \mathbf{p}_{\bar{b}})^\perp \right). \tag{4.7}$$

Hence, for each crossed piece (b, t) we have three terms with leading vectors $\mathbf{p}_b \times \mathbf{p}_t = \mathbf{p}_\beta \times \mathbf{p}_\tau$, $\mathbf{p}_{\bar{b}} \times (\mathbf{p}_{\bar{t}} - \mathbf{p}_{\bar{b}})^\perp$ and $\mathbf{p}_{\bar{t}} \times (\mathbf{p}_{\bar{t}} - \mathbf{p}_{\bar{b}})^\perp$ respectively.

The first term of (4.7) is due to the classic stresses on the bars. The vector $\mathbf{p}_b \times \mathbf{p}_t$ causes a rotation about the axis (b, t), i. e., about the bar $(\bar{b}, \bar{t})$. This is represented in Figure 4.9. The difference angle θ between the normals $\mathbf{n}_R$ and $\mathbf{n}_L$ depends on the stress ω_{bt}, and it is positive, negative or zero depending on the sign of ω_{bt}.

The second and third terms of (4.7) are due to the self-touching stresses along the whole supporting bars of the crossed pieces. The vector $\mathbf{p}_{\bar{b}} \times (\mathbf{p}_{\bar{t}} - \mathbf{p}_{\bar{b}})^\perp$ induces a rotation about an horizontal axis through $\bar{b}$ perpendicular to $(\bar{b}, \bar{t})$. The vector $\mathbf{p}_{\bar{t}} \times (\mathbf{p}_{\bar{t}} - \mathbf{p}_{\bar{b}})^\perp$ induces a rotation in the opposite direction about the horizontal axis through $\bar{t}$ perpendicular to $(\bar{b}, \bar{t})$. In both cases it is like "opening a book" with cover F_L, back-cover the plane containing F_R, and spine the mentioned axis. See Figure 4.10. If the self-touching stress is zero, there is no rotation. (The "book remains closed".)



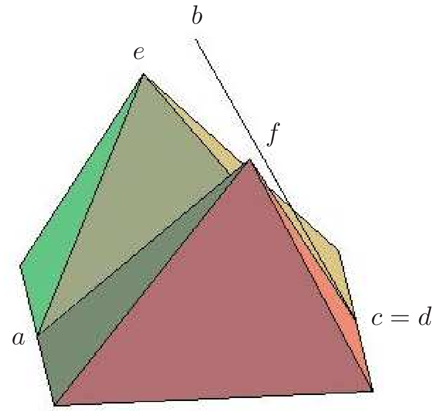


Figure 4.8: Another self-touching configuration (left) and its lifting (right). F_1, F_2, F_3 and F_4 denote the self-touching stresses with which the vertices b, e and f push against the bars (a, c), (a, d), (b, c) and (b, d). The proportional forces at the vertices a, c and d are not represented. Note that F_1 and F_2 must have the same value F so that b is in equilibrium. The amount of force F determines the height of b. Since F_3 is larger than F_4, e lifts to a higher point than f.

4.4 The Correspondence between Self-Touching Configurations and Generalized Polyhedral Terrains

In this section we generalize the Maxwell-Cremona Theorem and establish a correspondence between the set of stresses of a planar self-touching configuration and the set of three-dimensional generalized polyhedral terrains that project onto it.

The direction from the self-touching stresses to the three-dimensional generalized polyhedral terrain is called *lifting* and, as we show below, it is unique up to the addition of a linear function, which is given by $\mathbf{q}_0$. The other direction, from the generalized polyhedral terrain to the self-touching stresses, is called *projection* and it is in general not unique: the lifting has a non-trivial *kernel*, as we prove later.

Theorem 4.1 (The Maxwell-Cremona correspondence for self-touching configurations)**.** *Let $\mathcal{C}$ be a planar self-touching configuration. There is a correspondence between*

(a) set of stresses on $\mathcal{C}$ which are in equilibrium at all vertices.

(b) set of three-dimensional generalized polyhedral terrains that project onto $\mathcal{C}$.

Given a stress ω on $\mathcal{C}$, the lifting is performed by applying the height function h defined by (4.3) *to each vertex of $\mathcal{C}$, obtaining a three-dimensional generalized polyhedral terrain Γ_ω. The correspondence has the following properties:*

1. *Given an equilibrium stress ω on $\mathcal{C}$, there is a unique corresponding three-dimensional generalized polyhedral terrain Γ_ω with the exterior facet lying on $z = 0$ that projects onto it.*

2. *Given a three-dimensional polyhedral terrain Γ with jump discontinuities, its corresponding vertical projection onto the plane has a self-touching equilibrium stress ω_Γ which is*

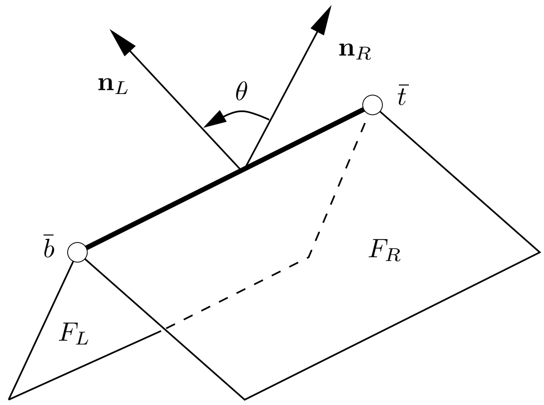


Figure 4.9: The term corresponding to $\mathbf{p}_b \times \mathbf{p}_t$ induces a turn moment about the bar $(\bar{b}, \bar{t})$.

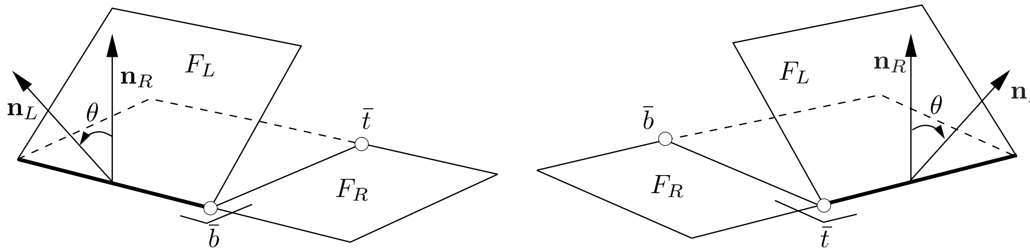


Figure 4.10: Left: rotation induced by $\mathbf{p}_{\bar{b}} \times (\mathbf{p}_{\bar{t}} - \mathbf{p}_{\bar{b}})^{\perp}$. Right: rotation induced by $\mathbf{p}_{\bar{t}} \times (\mathbf{p}_{\bar{t}} - \mathbf{p}_{\bar{b}})^{\perp}$. In both pictures, the rotation axis is drawn in thick lines.

not always unique: the lifting construction is a mapping $\omega \mapsto \Gamma_\omega$ which is not injective in general, i. e, the kernel $\mathcal{K}$ of the lifting can be non-trivial.

Proof of Part 1 of Theorem 4.1. This is true by definition. Since $\mathbf{q}_0$ is fixed to $(0, 0, 0)$, the well-defined vectors $\mathbf{q}_i$ are unique by construction. By Lemma 4.4, h defines, given the stress ω, a unique height for each vertex of the self-touching configuration, obtaining in this way Γ_ω. □

Now we look at the other direction, the projection. For proving Part 2, we need the following lemma.

Lemma 4.6. *Given a three-dimensional polyhedral terrain, we can recover, for each bar independently and at each auxiliary vertex, the total amount of incident self-touching stress pushing against the left and the right side separately.*

Proof. Let $(a, b) \in B$ be a bar and let $1, \ldots, n$ be the (ordered) auxiliary vertices on it. We write the coordinates of the auxiliary vertices as the convex combination $\mathbf{p}_i = \alpha_i \mathbf{p}_a + (1 - \alpha_i)\mathbf{p}_b$, $1 \leq i \leq n$.

We describe how to recover the total amount of self-touching stress incident to an auxiliary vertex from the left side. Analogously we recover the self-touching stresses incident to the right side (or also they can be seen as stresses on the left side of (b, a)).

Consider the auxiliary vertex i on (a,b). Since the height of each vertex in Γ is known, we know the difference $\delta_h(i)$ between the height of i as a point on the bar (a,b) and the height of the set $L_{ab}(i)$ of touching vertices pushing against i from the left side of (a,b). The jump $\delta_h(i)$ is the summand of (4.6) corresponding to the piece $(i,i+1)$, that is

$$\begin{aligned}\delta_h(i) &= \left((1-\alpha_i)F_a^{[i+1,b]} + \alpha_i F_b^{[a,i]}\right)\|\mathbf{p}_b - \mathbf{p}_a\|^2 \\ &= \left(\alpha_i \sum_{s=1}^{i-1}(1-\alpha_s)W_s + (1-\alpha_i)\alpha_i W_i + (1-\alpha_i)\sum_{s=i+1}^{n}\alpha_s W_s\right)\|\mathbf{p}_b - \mathbf{p}_a\|^2,\end{aligned}$$

where

$$W_s = \sum_{k \in L_{ab}(s)} \omega_{kab}\,,$$

i. e., W_s is the total amount of self-touching stress incident to the auxiliary point s from the left side of the bar (a,b). Note that if we write the equation of the jump $\delta_h(i)$ for the piece $(i-1,i)$ instead of the piece $(i,i+1)$, $F_a^{[i+1,b]}$ is substituted by $F_a^{[i,b]}$, $F_b^{[a,i]}$ is substituted by $F_b^{[a,i-1]}$, and the result is exactly the same. In Figure 4.11 we have a representation of a bar and its incident self-touching stresses grouped by coordinates.

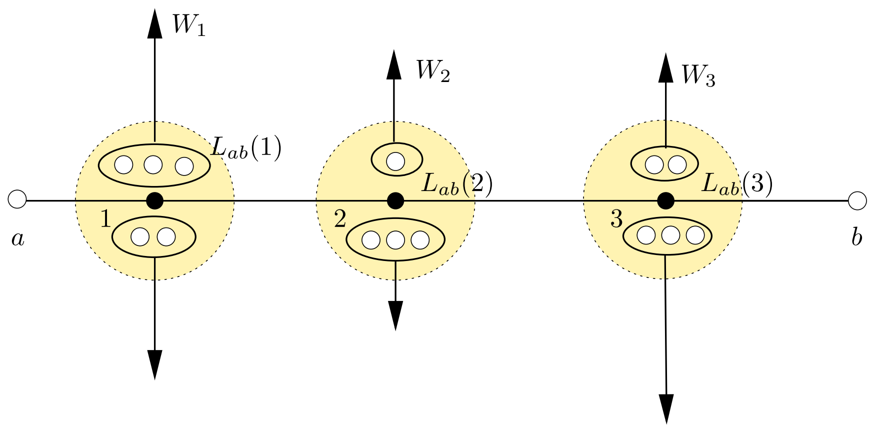


Figure 4.11: This bar (a,b) has three auxiliary vertices $1,2,3$. Each of them receives stress from each side. Look at the left side of the bar. We group the self-touching stresses pushing against the left side according to their coordinates, into the sets $L_{ab}(1), L_{ab}(2), L_{ab}(3)$. We have three linearly independent equations, one for each jump $\delta_h(i) = h(i) - h(L_{ab}(i))$, $i = 1,2,3$, on the variables W_1, W_2, W_3. Hence we can recover uniquely W_1, W_2, W_3. The same reasoning is valid for the stresses pushing against the right side of the bar.

If we consider the equation of the jump $\delta_h(i)$ for each i, $1 \le i \le n$, we obtain the following system of n linear equations and n variables W_i:

$$\frac{\delta_h(i)}{\|\mathbf{p}_b - \mathbf{p}_a\|^2} = \alpha_i \sum_{s=1}^{i-1}(1-\alpha_s)W_s + (1-\alpha_i)\alpha_i W_i + (1-\alpha_i)\sum_{s=i+1}^{n}\alpha_s W_s\,, \qquad 1 \le i \le n\,.$$

We want to show that these equations are linearly independent, i. e., that the system has a unique solution, and hence we can recover $W_1, \ldots, W_n$. For this, we prove that the $n \times n$ matrix $A = (a_{ij})$, with

$$a_{ij} = \begin{cases} \alpha_i(1-\alpha_j) & \text{if } i > j \\ \alpha_i(1-\alpha_i) & \text{if } i = j \\ \alpha_j(1-\alpha_i) & \text{if } i < j \end{cases}$$

is non-singular. Apply the following linear transformations to A: for each i and j, divide the elements of the row i of A by α_i and the elements of the column j by $(1-\alpha_j)$. We obtain a new matrix $A' = (a'_{ij}) = \left(\frac{a_{ij}}{\alpha_i(1-\alpha_j)}\right)$, with

$$a'_{ij} = \begin{cases} 1 & \text{if } i \geq j \\ \frac{(1-\alpha_i)}{\alpha_i}\frac{\alpha_j}{(1-\alpha_j)} & \text{if } i < j \end{cases}$$

Now subtract row $k-1$ from row k, $2 \leq k \leq n$. We obtain an upper triangular matrix A''; its determinant is the product of the elements of the diagonal, that is

$$|A''| = \prod_{i=2}^{n} \left(1 - \frac{(1-\alpha_{i-1})}{\alpha_{i-1}} \frac{\alpha_i}{(1-\alpha_i)}\right)$$

(note that $a''_{11} = 1$). By construction, $\alpha_{i-1} \neq \alpha_i$. Then $\frac{(1-\alpha_{i-1})}{\alpha_{i-1}}\frac{\alpha_i}{(1-\alpha_i)} \neq 1$ for each i, $1 \leq i \leq n$, hence

$$|A''| \neq 0$$

and this implies that the determinant of the original matrix A is also non-zero. □

Proof of Part 2 of Theorem 4.1. Given a polyhedral terrain Γ, we know the height of each vertex. Hence we can compute the vectors $\mathbf{q}$ associated to each proper facet of the configuration, since three independent points determine uniquely a plane in $\mathbb{R}^3$.

By Lemma 4.6 we can recover, for each bar (a, b) and for each auxiliary vertex i on this bar, the amounts

$$\sum_{k \in L_{ab}(i)} \omega_{kab} \tag{4.8}$$

and

$$\sum_{k \in R_{ab}(i)} \omega_{kab} \,. \tag{4.9}$$

But unfortunately, in general we cannot recover each ω_{kab} individually, for a vertex k pushing against i. For each oriented patch $(\beta, \tau \mid L, R)$, we are able to recover $\Delta(b, t)$ from the amounts (4.8) and (4.9), for all $(b, t) \in S_{\beta\tau}$. Hence, from

$$\mathbf{q}_L - \mathbf{q}_R = \sum_{(b,t) \in S_{\beta\tau}} \left(\omega_{bt}(\mathbf{p}_\beta \times \mathbf{p}_\tau) + \Delta(b,t)\right),$$

we can recover

$$\sum_{(b,t) \in S_{\beta\tau}} \omega_{bt} \,. \tag{4.10}$$

But from this amount we cannot, in general, recover the classical stress ω_{bt} on each piece separately.

It is possible to extract some more information about the stresses from the fact that each vertex is in equilibrium, but in general this does not suffice to recover all the classic stresses and self-touching stresses of the configuration. Thus we can recover the stress ω_Γ up to the *kernel* $\mathcal{K}$ of the lifting, which is unfortunately non-trivial: in general, there can be many stresses in equilibrium that keep the amounts (4.8), (4.9) and (4.10) fixed, and that therefore lift to the same polyhedral terrain Γ. This is what happens in Example 1 and Example 2 of Section 4.4.1. □

4.4.1 Characterization of the Kernel $\mathcal{K}$

Let ω be an equilibrium stress of a self-touching configuration, which lifts to a polyhedral terrain Γ_ω. We define the *kernel* $\mathcal{K}$ of the configuration as the set of all assignments of scalars to the bars and touching incidences, such that, being in equilibrium, lift to the flat polyhedron. that is, the configuration with $\omega = \mathbf{0}$ and any assignment $\kappa \in \mathcal{K}$ lifts to the flat polyhedron. Equivalently, all equilibrium stresses of the form

$$\{\omega + \kappa : \kappa \in \mathcal{K}\}$$

lift to the same polyhedron Γ_ω.

An element of the kernel must not necessarily be a stress, in the sense that it has no restrictions on the signs (we give an example later).

Since the addition of an element κ of the kernel to a stress ω does not affect the coordinates of the lifted polyhedron Γ_ω, it must not change these two values:

- $\sum_{(b,t)\in S_{\beta\tau}} \omega_{bt}$, for any edge (β, τ), to keep the angle between the two incident facets.
- The total amount of self-touching stress pushing against any bar (a, b) at any interior point i, from left and right separately (all W_i's, with the notation of Lemma 4.6), to keep the jump discontinuities.

Also each vertex must remain in equilibrium, hence κ itself must be in equilibrium.

Hence, the kernel $\mathcal{K}$ is the set of assignments κ of scalars to the bars and touching incidences satisfying the conditions:

1. $\sum_{(b,t)\in S_{\beta\tau}} \kappa_{bt} = 0$, for any edge (β, τ).
2. $\sum_{v\in L_{ab}(i)} \kappa_{vab} = \sum_{v\in R_{ab}(i)} \kappa_{vab} = 0$, for any bar (a, b) and any point i interior to (a, b).
3. $\mathbf{FK}_{ST}(v) + \sum_{j:(v,j)\in B} \kappa_{vj}(\mathbf{p}_j - \mathbf{p}_v) = 0$, for any vertex v.

where $\mathbf{FK}_{ST}(v)$ is the resulting force at v of the kernel assignments to the touching incidences. $\mathbf{FK}_{ST}(v)$ is defined analogously to $\mathbf{F}_{ST}(v)$:

$$\mathbf{FK}_{ST}(v) = \sum_{i,j:\{v;i,j\}\in\mathcal{L}} \kappa_{vij}(\mathbf{p}_j - \mathbf{p}_i)^\perp + \sum_{j:(v,j)\in B}\left(\sum_{k:\{k;v,j\}\in\mathcal{L}} -\alpha_{kvj}\kappa_{kvj}(\mathbf{p}_j - \mathbf{p}_v)^\perp + \sum_{k:\{k;j,v\}\in\mathcal{L}} (1-\alpha_{kjv})\kappa_{kjv}(\mathbf{p}_j - \mathbf{p}_v)^\perp\right).$$

Next we give some easy cases with a non-trivial kernel.

Example 1.

In this example we have cycles of stresses in equilibrium on overlapping bars. Look at Figure 4.12. Any equilibrium stress ω on this configuration, satisfies $\omega_{ac} = -\omega_{ad} = \omega_1$, $\omega_{bc} = -\omega_{bd} = -\omega_2$ and $\omega_1(\mathbf{p}_a - \mathbf{p}_c) - \omega_2(\mathbf{p}_b - \mathbf{p}_c) = 0$.

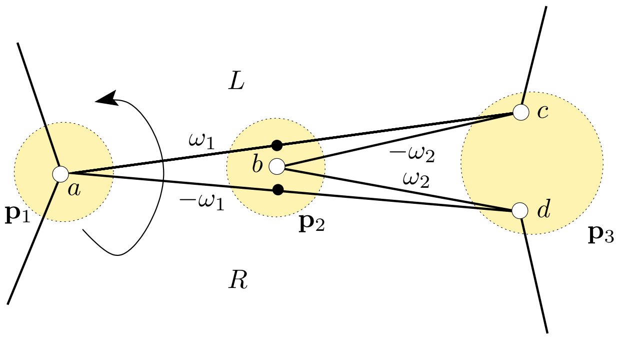


Figure 4.12: Example of non-trivial kernel. First case.

Let κ be a kernel element. Since $\omega + \kappa$ must be in equilibrium, κ must also satisfy $\kappa_{ac} = -\kappa_{ad} = \kappa_1$, $\kappa_{bc} = -\kappa_{bd} = -\kappa_2$ and $\kappa_1(\mathbf{p}_a - \mathbf{p}_c) - \kappa_2(\mathbf{p}_b - \mathbf{p}_c) = 0$. Hence, the following conditions for the pieces between c_R to c_L are also satisfied:

$$\sum_{(b,t)\in S_{12}} \kappa_{bt} = -\omega_1 + \omega_1 = 0$$

if we cross the edge $(\mathbf{p}_1, \mathbf{p}_2)$, and

$$\sum_{(b,t)\in S_{23}} \kappa_{bt} = -\omega_1 + \omega_2 - \omega_2 + \omega_1 = 0$$

if we cross the edge $(\mathbf{p}_2, \mathbf{p}_3)$. Since the amount of self-touching stress pushing against the interior of the bars (a, d) and (c, a) must be maintained, we have $\kappa_{bad} = \kappa_{bca} = 0$. Hence the kernel $\mathcal{K}$ is the one-dimensional subspace defined by the parameters $\kappa_1, \kappa_2 \in \mathbb{R}$ and the equations

$$\{\kappa_{ac} = -\kappa_{ad} = \kappa_1, \quad \kappa_{bc} = -\kappa_{bd} = -\kappa_2, \quad \kappa_1(\mathbf{p}_a - \mathbf{p}_c) - \kappa_2(\mathbf{p}_b - \mathbf{p}_c) = 0\}, \tag{4.11}$$

and κ is zero everywhere else.

Note that this example involves only classical stress and no self-touching stresses. Figure 4.13 shows how a self-touching configuration of this kind looks like.

Example 2

In this example we have cycles of stresses in equilibrium on touching bars converging to several different edges and with self-touching forces between touching vertices implied. This situation is illustrated with an example in Figure 4.14.

Given a polyhedral terrain Γ which projects onto this configuration, from the vectors $\mathbf{q}$ we can extract ω_{ad}, ω_{cd} and ω_{bc}, $\omega_{df} + \omega_{de}$, $\omega_{ce} + \omega_{cg}$, $\omega_{ax} + \omega_{af}$ and $\omega_{bx} + \omega_{bg}$. The heights

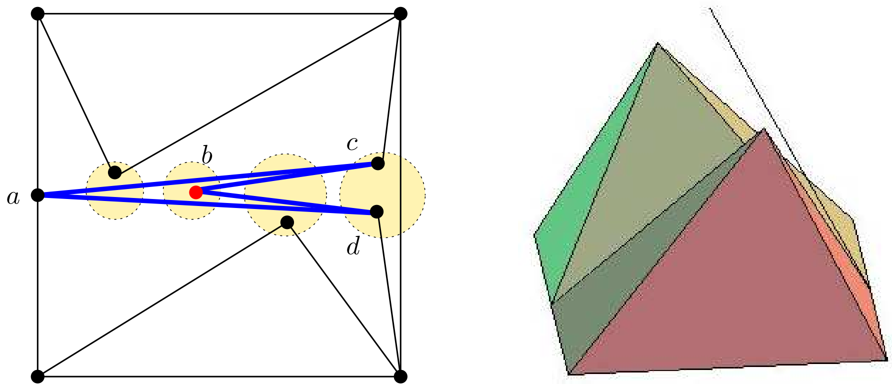


Figure 4.13: The self-touching configuration of Figure 4.8 has a non-trivial kernel. The kernel is zero everywhere except on the bars drawn thick, where it satisfies the conditions (4.11). The height of b depends only on $F = \omega_{bca} = \omega_{bad}$, and it is independent of κ_1.

$h(a)$, $h(b)$, $h(c)$ and $h(d)$ are equal to 0, since they belong to the exterior facet lying on $z = 0$ ($\mathbf{q}_0 = 0$).

From the jump between $h(x) = 0$ and the height $h(L_{ab}(x))$ of any of the vertices of $L_{ab}(x) = \{e, f, g\}$, we can recover by Lemma 4.6 the sum $\omega_{fab} + \omega_{gab}$:

$$\begin{aligned} h(L_{ab}(x)) - h(x) &= (1 - \alpha_x)\, \alpha_x\, W_x \,\|\mathbf{p}_b - \mathbf{p}_a\|^2 \\ &= \frac{3}{4} W_x \\ &= \frac{3}{4} \omega_{fab} + \omega_{gab} \end{aligned}$$

Thus the heights $h(L_{ab}(x)) = h(e) = h(f) = h(g)$ depend only on the sum $W_x = w_{fab} + \omega_{gab}$, and every pair of stresses ω_{fab}, $\omega_{gab} < 0$ adding up to W_x lifts to the same polyhedral terrain. Also, since the vertices e, f, g have the same height we cannot obtain any information about the self-touching stresses between them, w_{fed} and w_{gce}.

The equilibrium equations at each vertex of the configuration are:

$$\begin{aligned}
a:&\quad \omega_{ad}(\mathbf{p}_d - \mathbf{p}_a) - \alpha_x(\omega_{fab} + \omega_{gab})(\mathbf{p}_b - \mathbf{p}_a)^{\perp} = 0 \\
b:&\quad \omega_{bc}(\mathbf{p}_c - \mathbf{p}_b) - (1 - \alpha_x)(\omega_{fab} + \omega_{gab})(\mathbf{p}_b - \mathbf{p}_a)^{\perp} = 0 \\
d:&\quad \omega_{ad}(\mathbf{p}_a - \mathbf{p}_d) + \omega_{cd}(\mathbf{p}_c - \mathbf{p}_d) + (\omega_{df} + \omega_{de})(\mathbf{p}_x - \mathbf{p}_d) = 0 \\
c:&\quad \omega_{bc}(\mathbf{p}_b - \mathbf{p}_c) + \omega_{cd}(\mathbf{p}_d - \mathbf{p}_c) + (\omega_{ce} + \omega_{cg})(\mathbf{p}_x - \mathbf{p}_c) = 0 \\
e:&\quad \omega_{ed}(\mathbf{p}_d - \mathbf{p}_e) + \omega_{ce}(\mathbf{p}_c - \mathbf{p}_e) + \omega_{edf}(\mathbf{p}_f - \mathbf{p}_d)^{\perp} + \omega_{egc}(\mathbf{p}_c - \mathbf{p}_g)^{\perp} = 0 \\
f:&\quad \omega_{df}(\mathbf{p}_d - \mathbf{p}_f) + \omega_{af}(\mathbf{p}_a - \mathbf{p}_f) - \omega_{edf}(\mathbf{p}_f - \mathbf{p}_d)^{\perp} + \omega_{fab}(\mathbf{p}_b - \mathbf{p}_a)^{\perp} = 0 \\
g:&\quad \omega_{gc}(\mathbf{p}_c - \mathbf{p}_g) + \omega_{bg}(\mathbf{p}_b - \mathbf{p}_g) - \omega_{egc}(\mathbf{p}_c - \mathbf{p}_g)^{\perp} + \omega_{gab}(\mathbf{p}_b - \mathbf{p}_a)^{\perp} = 0
\end{aligned}$$

But, unfortunately, there are several sets of stresses which satisfy these equilibrium equations: all stresses of the form $\{\omega_\Gamma + \kappa : \kappa \in \mathcal{K}\}$, where ω_Γ is one possible equilibrium stress that

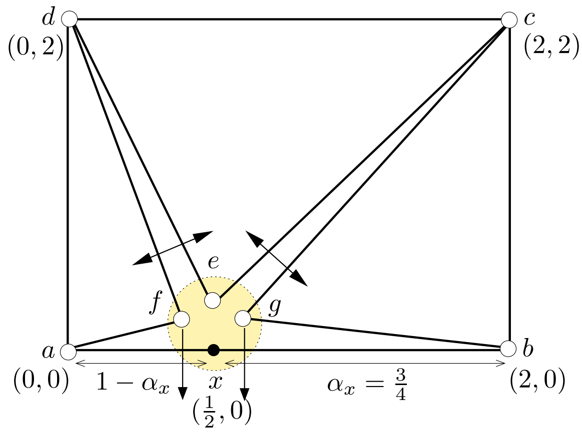


Figure 4.14: Example of non-trivial kernel. Second case.

lifts to Γ and $\mathcal{K}$ is the kernel. The kernel of this configuration is the 2-dimensional subspace defined by the equations below, where κ_{gce} and κ_{fab} are the independent parameters:

$$\begin{aligned}
&\{\,\kappa_{gab} = -\kappa_{fab}\,,\ \kappa_{ad} = \kappa_{cd} = \kappa_{bc} = 0\,,\\
&\kappa_{fed} = \frac{32}{85}\kappa_{fab} + \frac{10\sqrt{17}}{289}\kappa_{gce}\,,\ \kappa_{bg} = -\kappa_{fab} - \frac{25\sqrt{17}}{102}\kappa_{gce}\,,\\
&\kappa_{ab} = \frac{3}{4}\kappa_{fab} + \frac{25\sqrt{17}}{136}\kappa_{gce}\,,\ \kappa_{af} = -3\kappa_{fab} - \frac{25\sqrt{17}}{34}\kappa_{gce}\,,\\
&\kappa_{cg} = -\kappa_{ce} = \kappa_{fab} + \frac{3\sqrt{17}}{34}\kappa_{gce}\,,\ \kappa_{df} = -\kappa_{de} = -\frac{13}{17}\kappa_{fab} + \frac{25\sqrt{17}}{578}\kappa_{gce}\,\}
\end{aligned}$$

Note that $\kappa \in \mathcal{K}$ is not itself a stress because it does not have the correct signs: for example, κ_{fab} and κ_{gab} cannot be both negative, since $\kappa_{gab} = -\kappa_{fab}$. Note also that the addition of any $\kappa \in \mathcal{K}$ to the stress ω_Γ does not modify $W_x = \omega_{fab} + \omega_{gab}$.

Figure 4.15 shows a self-touching configuration with a kernel of this kind.

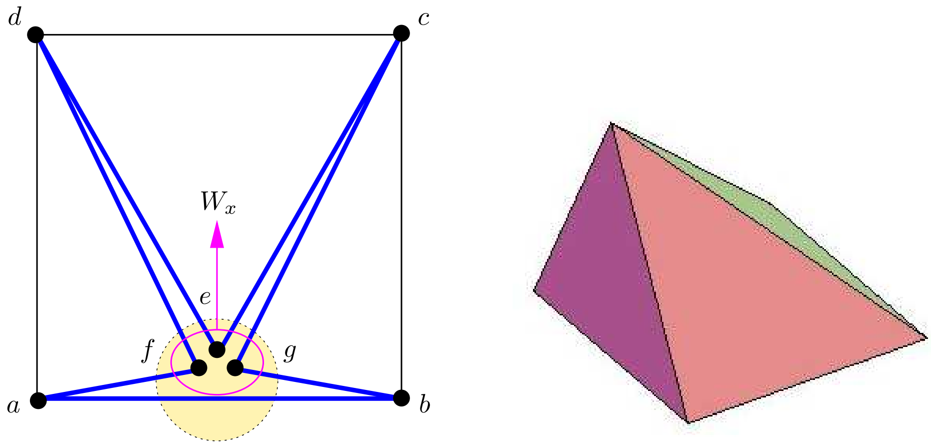


Figure 4.15: A self-touching configuration with a non-trivial kernel and its lifting. The kernel is zero everywhere except on the bars drawn thick. The vertices e, f, and g lift to the same height, and this height depends only on W_x, the total force of e, f, g against the bar (a, b).