Part I

Self-Touching Linkages

Chapter 1

Preliminaries

A *planar geometric embedding* of a planar graph is the mapping of the graph into the plane, possibly with curved edges, without edge crossings. Two geometric planar embeddings are called equivalent if there is a homeomorphism of the plane transforming one into the other. An equivalence class of geometric planar embeddings is called a *planar topological embedding*, or simply *planar embedding*.

A planar embedding of a planar graph induces a cyclic ordering on the edges incident to any fixed vertex, namely the clockwise ordering of the edges around their common endpoint. A graph G together with a cyclic ordering on the edges incident to each vertex is called a *combinatorial embedding*, and it is called a *planar combinatorial embedding* if it is induced by some planar embedding. Different planar embeddings can give rise to the same combinatorial embedding. However, a planar combinatorial embedding of a connected graph uniquely determines its topological embedding on the sphere. In the plane, it determines the topological embedding up to selection of the outer face.

1.1 Linkages and Self-Touching Configurations

We give now some definitions which are taken from [17]. A *linkage* is a graph where edges are rigid bars with fixed length and vertices are flexible joints. A *configuration* of a linkage is a mapping of the vertices to points in $\mathbb{R}^2$. A configuration is *simple* if two bars intersect only at shared endpoints. This notion is also often called "strictly simple".

A *motion* is a continuum of configurations, that is, a continuous function mapping the time interval $[0, 1]$ to configurations; often, each configuration is required to be simple. In addition, topologically, when a linkage moves, the relative positions of its parts must remain *consistent*. For example, a vertex that touches an edge from the left side cannot suddenly move away to the right side of that edge.

The *configuration space* of a given subset of configurations (e.g., simple configurations) is the space in which points correspond to configurations and paths correspond to motions.

We focus here on *planar linkages* embedded in $\mathbb{R}^2$. In this case, the combinatorial planar embedding of the planar linkage is specified, because it cannot change by a motion that avoids bars crossing.

In a *self-touching configuration*, some vertices are infinitesimally close to each other, and bars converge to overlapping configurations, but do not properly cross. Connelly, Demaine and Rote [17] define self-touching configurations formally as follows. We start with a plane straight-line graph. See Figure 1.1 for an example. Each segment (edge) is marked with its *multiplicity*, i. e., how many collinear bars lie along that segment. In addition, for each point p, we add a microscopic magnified view enclosed by a circle. We call this magnified view a *specification disk*, and we denote it by D_p. *Terminals* on the boundary of the disk represent connections to the incident edges. An edge of multiplicity m is adjacent to m terminals. Inside the specification disk D_p, the terminals are connected by a plane graph G_p, not necessarily drawn with straight-line edges, subject to the following rules (we use here the terminology "edge" for the graph G_p):

(1) Every terminal is incident to exactly one edge of G_p.

(2) Every nonterminal vertex is incident to at least one edge of G_p.

(3) There is at least one nonterminal vertex.

(4) An edge of G_p may connect two terminals directly only if the terminals connect to two collinear segments that go in opposite directions.

(5) All other edges of G_p must connect a terminal to a nonterminal vertex. In particular, no edge connects two nonterminal vertices, i.e. there are no edges between points infinitesimally close.

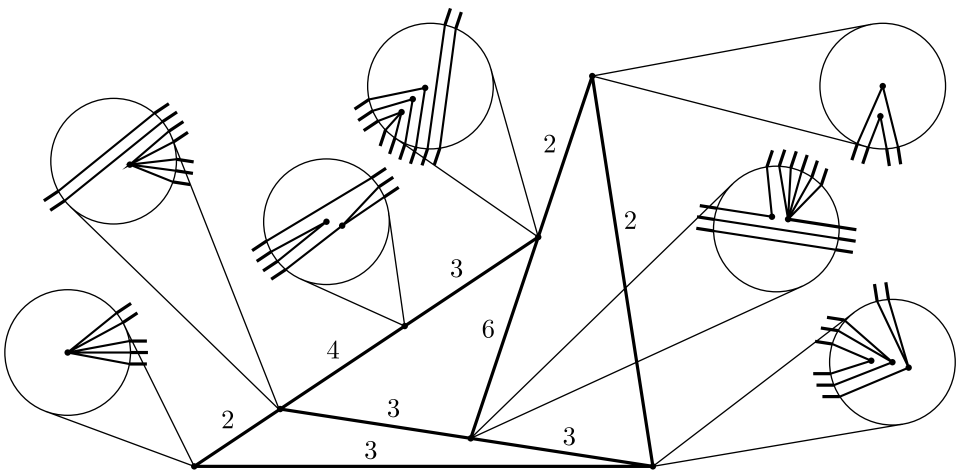

Figure 1.1: From [17]. A self-touching configuration with 14 vertices and 21 bars. Numbers denote edge multiplicities. There are 7 points and 9 edges.

This structure specifies the combinatorial linkage associated with the configuration as follows. Its vertices are the nonterminal vertices in all circles. Its bars are the connections between those vertices; a single bar is a sequence starting and ending at a connection between a nonterminal and a terminal, and alternating between one or more additional segments and zero or more connections between terminals. We require in addition that the linkage has no multiple bars.

In a self-touching configuration, we find several distinct vertices vertices with the same geometric position. These are the vertices inside the same circle, and we say that they *converge* to the same position (the circles are infinitesimally small). We call *points* the converging positions of the set of vertices. In addition, we say that two vertices are *touching* when they converge to the same point and there is no bar between them. For self-touching configurations, we use the terminology "bars" to refer to the edges of the underlying graph, and "edges" to refer to the segments into which many collinear bars converge. See Figure 1.1.

For representing a self-touching configuration, we use a schematic drawing where parallel bars are slightly separated, and dotted circles surround vertices that converge to the same position, as in Figure 1.2. This representation gives a clearer drawing of the underlying graph, and is closely linked to the concept of a δ-perturbation defined in Section 1.2 below.

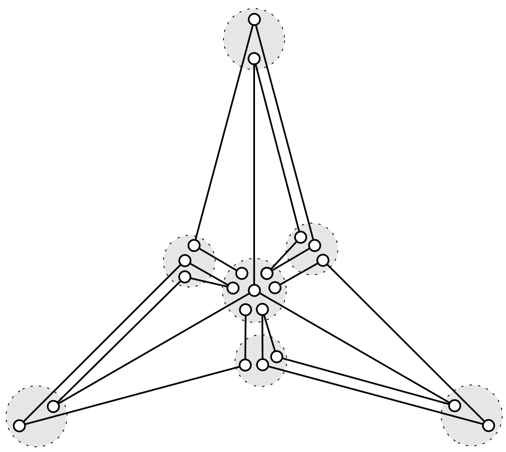

Figure 1.2: From [17]. A self-touching configuration, which is locked. Points in dotted circles are infinitesimally close.

1.2 Locked Linkages

Our motivation for studying, in Chapter 3, perturbations of self-touching configurations comes from the interest in locked linkages.

We say that a self-touching configuration is *locked* if its configuration space has multiple connected components within the class of embeddings with the same combinatorial planar embedding. We call a self-touching configuration *locked within* ε if no path in the configuration space (motion) can get outside of a surrounding ball of radius ε. This last definition is stronger for sufficiently small ε, provided that there are other configurations which represent the same combinatorial embedding.

One example of the second definition is the following: a self-touching configuration is called *rigid* if it is locked within 0, that is, there is no motion to a distinct self-touching configuration. This notion does not make sense for simple configurations of arcs, cycles, and trees, which are always *flexible* (not rigid). One key feature of self-touching configurations of such linkages is that they can be rigid; other examples of rigid configurations that arise throughout rigidity are linkages that form a complex graph structure (consisting of multiple cycles).

To introduce a stronger notion of being locked, we give the following definition. A δ-*perturbation* is a repositioning of the vertices within disks of radius δ that remains consistent with the combinatorial embedding in $\mathbb{R}^2$ defined in Section 1.1. A key aspect of a δ-perturbation is that it allows the bar lengths to change slightly (each by at most 2δ). For particular examples, it is usually easy to see that a simple δ-perturbation exists, but it has been an open problem, conjectured in [17], to see whether this is true in general. In Chapter 3 we go into the definition of δ-perturbation with more detail, and we show that every self-touching configuration can be perturbed within any δ obtaining a simple configuration.

A self-touching configuration is *strongly locked* if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that every δ-perturbation is locked within ε. In particular, all sufficiently small simple perturbations are locked.

We have the following theorem:

Theorem 1.1 ([17])**.** *If a self-touching configuration is rigid, then it is strongly locked.*

We are interested in self-touching configurations since locked linkages are often based on approximations to self-touching configurations.

1.3 Infinitesimal rigidity

A self-touching configuration is *infinitesimally rigid* if it has no *infinitesimal motion*, that is, assignment of feasible velocity vectors $\mathbf{v}_i$ to vertices $\mathbf{p}_i$ that preserves bar lengths to the first order:

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{v}_i - \mathbf{v}_j) = 0 \quad \text{for every bar}\{i, j\}.$$

Not every infinitesimal motion can be extended to a motion. Thus, rigidity does not imply infinitesimal rigidity, but the converse holds:

Theorem 1.2 ([17])**.** *If a self-touching configuration is infinitesimally rigid, then it is rigid.*