Introduction

In this thesis we study different geometric and combinatorial problems for planar graphs and polytopes. The work is organized in three parts which are introduced and discussed below.

Part I of this thesis is about linkages embedded in $\mathbb{R}^2$, planar self-touching configurations and liftings to three-dimensional space.

A *linkage* is a graph where edges are rigid bars with fixed length and vertices are flexible joints. Linkages have many applications in mechanics. Mechanical linkages, in which two or more joints are movable with respect to a fixed joint, are usually designed to take a motion or forces as an input and produce a different output, altering the motion, velocity, acceleration, and applying mechanical advantage. Mechanical linkages are a fundamental part of machine design, and yet many simple linkages were not well understood, or invented until the 19th century. Very complicated and precise motions can be designed into a linkage with only a few parts. During the industrial revolution, which was the golden age of mechanical linkages, mathematical, engineering and manufacturing advances provided both the need and the ability to create new mechanisms. Many simple mechanisms that seem obvious today required some of the greatest minds of the era to create them. Linkages are also used for the design of rigid structures.

The most common linkages have one degree of freedom, meaning that there is one input motion that produces one output motion. Also, in most linkages, the motion takes place in the plane. Spatial linkages, or linkages in which the motion takes place in space, are more difficult to design and therefore not as common.

In this thesis we are considering motions of linkages that avoid crossings of bars. Therefore, the combinatorial planar embedding of the planar linkage is specified, because it cannot change by a motion that avoids bars crossing.

In the last years, there has been much progress in the study of linkages that are *locked* or *stuck*, in the sense that they cannot be moved into some other configuration without crossings and preserving the bar length. The problem is to analyze which kind of linkages are locked. The tools used in this topic are geometric planar properties combined with techniques from rigidity theory, like first-order rigidity, equilibrium stresses and the Maxwell-Cremona Theorem.

It is known that polygonal arcs, polygonal cycles and the disjoint union of non-nested polygonal arcs and cycles, are always unlocked (their configuration space is always connected). In other words, every polygonal arc can be straightened and every polygonal cycle can be convexified by a motion that avoids crossings. This was solved in 2000 independently by Connelly, Demaine and Rote [16] and by Streinu [54], and in particular solves the well-known

"Carpenter's Rule problem", which was open for many years. In Figure 1 we can see the convexification of a linkage.

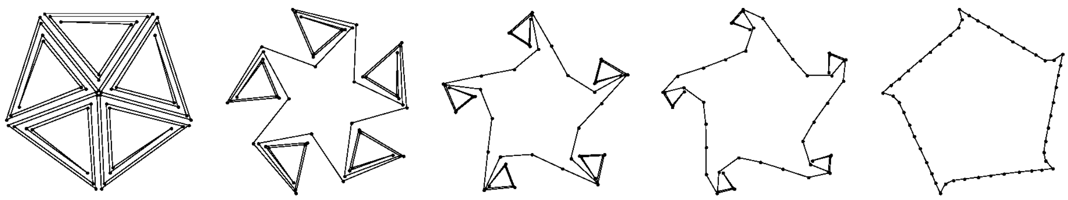


Figure 1: From [16]. The convexification of a polygon that comes from doubling every edge in a locked tree. Snapshots are scaled different to improve visibility; the bar length is maintained throughout the motion.

Also it is known that polygonal trees can lock, or equivalently the configuration space of simple planar configurations of a tree is not always connected, as exemplified by the two trees in Figure 2. The key distinction is that arcs and cycles have maximum degree 2, but a tree may have vertices of higher degree. See [16, 23, 45] for surveys of related results. In fact, a single vertex of degree 3 can prevent opening, as in the example in Figure 2(b).

In the first part of the thesis, we study problems related to *self-touching* or *degenerate* frameworks, in which multiple edges converge to geometrically overlapping configurations [17]. Our motivation is that most existing approaches to locked linkages are based on approximations to self-touching configurations. This part is distributed as follows.

In the preliminary Chapter 1 we give basic definitions and generalities.

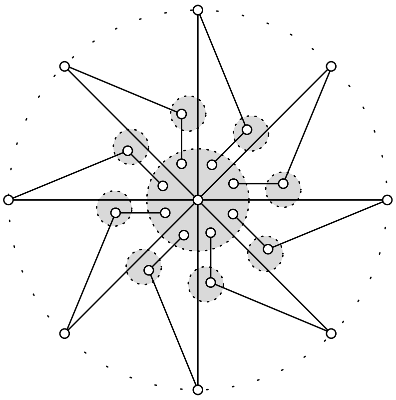
Chapter 2 is a small warm-up chapter about the unfoldability of trees. We show that every monotone tree is unfoldable, using geometric relations between segments with disjoint interiors in the plane. The result was proven independently by Kusakari et al [38].

Chapter 3 is about perturbations of self-touching configurations. A *δ-perturbation* is a repositioning of the vertices within disks of radius δ that remains consistent with the combinatorial embedding in $\mathbb{R}^2$ (a precise definition is given in Section 3.1). A key aspect of a δ-perturbation is that it allows the bar lengths to change slightly (each by at most 2δ). We prove the following result.

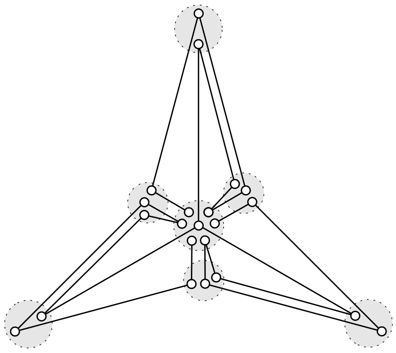
Theorem. *For each self-touching configuration, and for each $\delta > 0$, there exists a simple δ-perturbation.*

The result is proved first for one-dimensional self-touching configurations, that is, self-touching configurations in which all vertices lie initially on a line, and this is used to prove the theorem for the general planar case. As a consequence, we can show that self-touching polygonal chains and cycles are infinitesimally flexible.

In Chapter 4 we present a generalization of the Maxwell-Cremona Theorem for self-touching configurations. The classic Maxwell-Cremona Theorem [20, 21, 22, 48, 56], is a powerful tool that establishes a bijection between the set of classical equilibrium stresses of a planar configuration and the set of three-dimensional polyhedral terrains that project onto it. We study how this theorem translates into the case of self-touching configurations, and we establish a correspondence between the set of stresses of a planar self-touching configuration and the set of three-dimensional generalized polyhedral terrains, that is, polyhedral terrains with jump discontinuities at those edges affected by self-touching forces, that project onto it.



(a) From [5].



(b) From [16].

Figure 2: Locked planar polygonal trees. Points in dotted circles are closer than they appear.

Part II of this thesis consists of two chapters.

In Chapter 5 we study lower and upper bounds for the maximum number of spanning trees of a planar graph. If we add edges to a planar graph, the number of spanning trees grows. Hence we study the case when the graph is triangulated, since they are the graphs with the largest number of spanning trees.

We present a new method based on transfer matrices for computing the asymptotic number of spanning trees of some recursively constructible families of graphs, from which we obtain lower bounds. The main results can be found on page 60.

For upper bounds, apart from the general case, we also study the specific cases when the graph has no triangular faces but at least a quadrangular face, and when the graph has neither triangular nor quadrangular faces. Note that a 2-connected planar graph contains always a triangle, a quadrilateral face or a pentagonal face, because its dual graph has minimum degree at most 5, since it is also a planar graph and hence its average degree is less than 6.

Several techniques for obtaining these upper bounds are discussed, and the obtained results compared. For graphs without triangles, the best results are obtained using a probabilistic method and Suen's inequality. The idea of applying Suen's inequality to this problem was proposed by Andrzej Ruciński and Tomasz Luczak, both at the Department of Discrete Mathematics of the Adam Mickiewicz University, in Poznań. In Section 5.1 we have a summary of the obtained results.

The analysis of the number of spanning trees in planar graphs was initiated in a joint work with Günter Rote and Xuerong Yong, the latter at the Center of Discrete Mathematics and Theoretical Computer Science (DIMACS), in Piscataway, New Jersey, and also at the Department of Computer Science of the Hong Kong University of Science and Technology, in Kowloon, Hong Kong.

The motivation arises from trying to embed polyhedra on small three-dimensional integral grids, using the Maxwell-Cremona lifting. The coordinates of the lifted polytope are bounded in terms of the determinant of the reduced Laplacian matrix. The Matrix-Tree Theorem relates this determinant with the number of spanning forests of the graph of the polytope with three trees, each rooted at one chosen vertex (four trees if the graph has smallest face cycle 4). The number of such spanning forests can be bounded in terms of the number of spanning trees, which is somehow more natural to study. Using the Matrix-Tree Theorem and the bounds obtained in Chapter 5, we work in Chapter 6 on the minimum size of the integral grid in which we can embed all combinatorial types of 3-polytopes. We follow a similar approach as Richter-Gebert [48], based on the lifting given by the Maxwell-Cremona Theorem. The results are summarized in the following theorem:

Theorem. *1. All combinatorial types of 3-polytopes with n vertices with a triangular facet can be realized on an integral grid* $\{0, 1, \ldots, \lfloor 1/6 \cdot n^5\, 28.\bar{4} \rfloor^n\}^3$*;*

2. All combinatorial types of 3-polytopes with n vertices with a quadrilateral facet can be realized on an integral grid $\{0, 1, \ldots, \lfloor 1/3 \cdot n^{15}\, 155.271910 \rfloor^n\}^3$*;*

3. All combinatorial types of 3-polytopes with n vertices can be realized on an integral grid $\{0, 1, \ldots, n^{10n}\, 2^{10n^2+1}\}^3$.

Here $28.\bar{4}$ denotes the periodic number $28.444444\ldots$.

This improves previous results obtained by Onn and Sturmfels [44] and Richter-Gebert [48].

In **Part III** we analyze the growth in the number of polyominoes on a twisted cylinder as the number of cells increases. A twisted cylinder is like an infinite tube, but with a little "twist". These polyominoes are related to classical polyominoes, that is, connected subsets of a square grid that lie in the plane. See Figure 3.

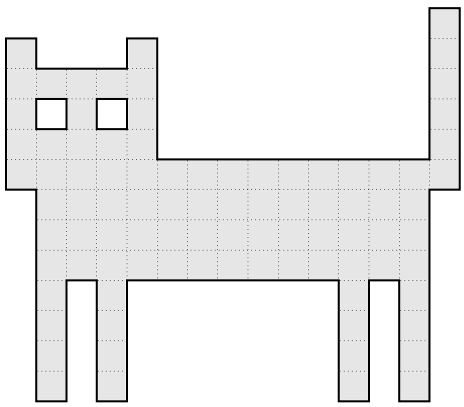


Figure 3: A polyomino in the plane.

We use the transfer-matrix approach, the same technique used in Chapter 5 for computing the asymptotic number of spanning trees of several families of graphs.

The transfer-matrix approach is much easier to apply on the twisted cylinder than on the planar grid, since we can build up the polyomino through a uniform process. We obtain lower bounds on the growth rate of the number of polyominoes on the twisted cylinder. This has also consequences for the number of polyominoes in the plane, and in particular we obtain an improved lower bound of 3.980137 on the growth rate of the number of these polyominoes, which is also known as Klarner's constant [28].

There have been several attempts to lower and upper bound the Klarner's constant, as well as to estimate it, based on enumerating the number of polyominoes constituted by n

adjacent squares, up to certain values of n. The best-known published lower and upper bounds are 3.927378 [33] and 4.649551. However, this lower bound is based on an incorrect assumption, which goes back to the paper of Rands and Welsh [47]. As we point out in Section 7.7, the lower bound should have been corrected to 3.87565.

The content of Chapter 7 has been submitted for publication and it is joint work with Gill Barequet, at the Department of Computer Science of the Technion–Israel Institute of Technology, Micha Moffie, at the Northeastern University, in Boston, and Günter Rote.

This thesis shows how some important techniques can be applied to many different problems. For example, the Tutte embedding is used in Chapter 3 for perturbing self-touching configurations, and in Chapter 6 for deriving equilibrium stresses than can be lifted. These upper bounds are used for analyzing the size of the minimal integer grid in which we can embed a 3-polytope, using the Maxwell-Cremona Theorem. The Maxwell-Cremona Theorem appears also in Chapter 4, establishing a correspondence between self-touching configurations and three-dimensional generalized polyhedral terrains. The concept of stresses in equilibrium is used for the Tutte embedding, and it is also generalized to self-touching configurations in Chapter 4. The concept of stress matrix appears in the Tutte embedding as well as in the Matrix-Tree Theorem, which in Chapter 6 relates the number of forests bounded in Chapter 5 with a determinant used for bounding the size of the minimal grid in which we can embed a 3-polytope. Other tools that we apply are the transfer-matrix method and the Perron-Frobenius Theorem from linear algebra, which are used in Chapter 5 for obtaining the asymptotic number of spanning trees of several recursively constructible families of graphs, and in Chapter 7 for obtaining an improved lower bound on the number of polyominoes in the plane.